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Polynomial recursions for counting big blocks in set partitions

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Abstract

In this paper, we look at polynomials $P_{n,k}(t)$ where the coefficient of t^i counts the number of set partitions of $\{1, 2, \ldots, n\}$ with exactly i big blocks, i.e. blocks of size at least k. We begin by looking at small values of k. Then, we generalize the results to all values of k. Our main focus is to find recurrence relations for the polynomials. We also define type-B set partitions as well as bi-colored set partitions and polynomials generated by these. An important concept we introduce is real-rootedness and interleaving. We present and prove Wagner's lemma, which is important when working with interleaving polynomials. Some examples in the paper suggest that the polynomials $P_{n,k}(t)$ might be real-rooted and interleaving.

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Contents

1 Introduction

In this paper, we consider polynomials of the form

$$P_{n,k}(t) \coloneqq \sum_{\pi \in \mathcal{SP}(n)} t^{\mathrm{bblocks}(\pi,k)}, \tag{1}$$

where $\mathcal{SP}(n)$ is the set of all partitions of the set $1, 2, \dots, n$ and $\mathrm{bblocks}(\pi, k)$ is the number of big blocks (blocks of size at least k) in the set partition π. We will further explain these notations in the next section. We find recurrence formulas for the polynomials, and we look at the properties of these polynomials. Similar problems were studied in e.g., [2] and [3]. In [2], Alexandersson and Nabawanda looked at run-sorted permutations $\mathcal{RSP}(n)$ and polynomials of the form

$$A_n(t) \coloneqq \sum_{\sigma \in \mathcal{RSP}(n)} t^{des(\sigma)}.$$

We will briefly mention how the problem in [2] is linked to this paper in Section 2.2.

2 Prerequisites

2.1 Set partitions

A partitioning of the set S into non-empty subsets $B_1, \dots, B_k$ called blocks, that satisfy the conditions:

1. the union of these blocks should be the whole set, i.e., $B_1 \cup \dots \cup B_k = S$
2. the intersection of any two distinct blocks should be empty, i.e., $B_i \cap B_j = \emptyset$ for any $i \neq j$

is called a partition of S or a set partition. In other words, every element of S appears in exactly one block B_i. The size of a block refers to how many elements are in the block. We will use the term "k-block" to refer to a block of size k. Throughout the text, we assume that S is the set $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. We will introduce the notation $[n] \coloneqq \{1, 2, \dots, n\}$ and we will let $\mathcal{SP}(n)$ denote the set of all partitions of $[n]$.

By convention, when we partition the set $[n]$, we order the blocks such that $\min B_i < \min B_j$, whenever $i < j$. Also, we list the elements of B_i in increasing order. Now, we will introduce two ways to present a set partition. The first way is to write the set of the blocks $\{B_1, B_2, \dots, B_k\}$. This presentation of a set partition is tedious. A more convenient way is to write $b_1 \mid b_2 \mid \cdots \mid b_k$, where b_i denotes the elements of B_i with spaces separating them. We illustrate both ways to present a set partition in the example below.

Example 2.1. Consider the set $[3] = \{1, 2, 3\}$. This set has five different partitions, i.e., $|\mathcal{SP}(n)| = 5$. We list all partitions of the set $[3]$ below:

1. $\{\{1\}, \{2\}, \{3\}\}$, or $1 \mid 2 \mid 3$, partitions $[3]$ into three 1-blocks
2. $\{\{1\}, \{2, 3\}\}$, or $1 \mid 2\ 3$, partitions $[3]$ into a 1-block and a 2-block
3. $\{\{1, 2\}, \{3\}\}$, or $1\ 2 \mid 3$, partitions $[3]$ into a 1-block and a 2-block
4. $\{\{1, 3\}, \{2\}\}$, or $1\ 3 \mid 2$, partitions $[3]$ into a 1-block and a 2-block

5. $\{\{1,2,3\}\}$, or 1 2 3, partitions [3] into a 3-block

Example 2.2. Consider the set $[3] = \{1,2,3\}$. We list some non-examples of set partitions below:

- $\{\{1,\ 3\}\}$ is not a set partition because 2 does not appear in any subset
- $\{\{1,\ 2\},\ \{2,\ 3\}\}$ is not a set partition because 2 appears in two different subsets
- $\{\{1,\ 2\},\ \{1\}\}$ is not a set partition because 1 appears in two different subsets, and 3 does not appear in any subset.

When discussing set partitions, some natural questions that come up are "How many ways can we partition the set?" and "How many ways can we partition the set into a pre-specified number of blocks?" The Bell numbers and Stirling numbers of the second kind give us the answer to each question, respectively. The Bell numbers B_n count the number of ways to partition $[n]$, i.e., $B_n = |\mathcal{SP}(n)|$. As we saw in Example 2.1, $B_3 = 5$. Stirling numbers of the second kind, often denoted $S(n,k)$ or $\left\{ {n \atop k} \right\}$, count the number of ways to partition $[n]$ into k blocks. From Example 2.1, we see that $S(3,1) = S(3,3) = 1$ and $S(3,2) = 3$. Consider what happens when we sum over all possible values of k in the Stirling numbers $\left\{ {n \atop k} \right\}$. We should get the total number of ways to partition $[n]$. Therefore, we have the summation formula

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}. \tag{2}$$

The following lemma gives us a recurrence relation for the Bell numbers.

Lemma 2.1. *Let B_n be the nth Bell number. We have the following recurrence:*

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{3}$$

Proof. We look at the block containing the element $n+1$. Let $k+1$ be the size of this block. We can choose the other k elements of this block from $[n]$ in $\binom{n}{k}$ ways. Then, we can partition the remaining $n-k$ elements in B_{n-k} ways. Applying the multiplication principle, we get the term $\binom{n}{k}B_{n-k}$. Since the element $n+1$ being in different-sized blocks are disjoint events, the addition principle gives us

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}.$$

Relabeling the indexes and noting that $\binom{n}{k} = \binom{n}{n-k}$ gives us Equation (3). □

2.1.1 Other types of set partitions

We now introduce two types of set partitions different from the aforementioned "standard" set partitions, namely type-B set partitions and bi-colored set partitions, and how these relate to each other.

We call a partition of the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ a type-B set partition if it satisfies the following conditions:

1. For every block B_i, there exists an opposite block $-B_i$, obtained by negating the elements in B_i.

2. There is at most one zero-block, i.e., at most one block satisfies $B_i = -B_i$.

Interested readers can find more on type-B set partitions in e.g., [7]. To present type-B set partitions, we will use the notation $\{B_0, \pm B_1, \ldots, \pm B_k\}$, where B_0 is the zero-block if it exists, and B_i, for $i = 1, \ldots, k$ are the non-zero blocks. We will use the notation $\mathcal{BSP}(n)$ for the set of all type-B set partitions.

Example 2.3. Let us look at the set $S = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7\}$. Below, we list some of the type-B partitions of S:

1. $\{\{\pm 4, \pm 7\}, \pm\{1, -2, 5\}, \pm\{-3, -6\}\}$
2. $\{\pm\{1, 3, -4\}, \pm\{2, -7\}, \pm\{5, 6\}\}$
3. $\{\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7\}\}$
4. $\{\pm\{1\}, \pm\{2\}, \pm\{3\}, \pm\{4\}, \pm\{5\}, \pm\{6\}, \pm\{7\}\}$

We introduce another type of set partition, called a bi-colored set partition. A bi-colored set partition occurs when we partition the set $[n]$ normally and color each element either red or blue such that the smallest element of each block is colored red. We will use the notation $\mathcal{BCSP}(n)$ for the set of all bi-colored set partitions. Bi-colored set partitions are relevant because we can create a bijection between them and a subset of type-B set partitions, namely type-B set partitions where we don't allow zero-blocks. To see this, let us define the function

$$\varphi : \mathcal{BCSP}(n) \to \mathcal{BSP}(n)$$

by first converting all the red elements to positive integers and the blue elements to negative integers and then creating an opposite block $-B_i$ for every block B_i. The condition that the smallest element must be colored red in the bi-colored set partitions makes this function injective. The image of φ is exactly the set of type-B set partitions without zero blocks. The function φ also has an obvious inverse function φ^{-1}: Take all blocks of a type-B set partition without zero blocks, whose element closest to zero is positive; next, color the positive elements in red and the negative elements in blue. Therefore, bi-colored set partitions are bijective with type-B set partitions without zero blocks.

Example 2.4. Consider the bi-colored set partition $\pi = 1^r\ 3^r\ 4^b \mid 2^r\ 7^b \mid 5^r\ 6^r$. Then $\varphi(\pi) = \{\pm\{1, 3, -4\}, \pm\{2, -7\}, \pm\{6\}\}$.

2.2 Polynomials and real-rootedness

A big part of this paper will be on polynomials generated by set partitions. In this section, we will define properties such as *real-rootedness* and *interleaving*. The main result of this section will be Wagner's lemma. For interested readers, you can find more on this topic in e.g., [1] and [4].

Let $a_i \in \mathbb{R}$ for $i \in \{0, 1, 2, \ldots, n\}$. We call the polynomial $p = \sum_{i=0}^{n} a_i t^i$ *real-rooted* if all the roots of p are real.

Definition 2.1 (Interlacing and alternating, see [9, Section 3]). Let f and g be polynomials with positive leading coefficients and real roots $\{f_i\}$ and $\{g_i\}$, respectively. We say that f *interleaves* g if the roots, as viewed on a number line, alternate between roots of f and roots of g and the largest root is a root of g, i.e.,

$$g_{d+1} \le f_d \le g_d \le \cdots \le f_1 \le g_1,$$

if $\deg(f) = \deg(g) - 1 = d$, and removing g_{d+1} if $\deg(f) = \deg(g) = d$. We say that g *interlaces* f in the first case and g *alternates left of* f in the latter. In both cases, we write $f \ll g$. By convention $0 \ll 0$, $0 \ll h$, and $h \ll 0$ whenever h is a polynomial with a positive leading coefficient.

Example 2.5. Let $f = x^2 - 1$, $g = x^2 - 2x$, and $h = x^3 - 4x$, with the roots

$$h_1 = -2,\ f_1 = -1,\ g_1 = h_2 = 0,\ f_2 = 1,\ g_2 = h_3 = 2$$

Then, f alternates left of g and both f and g interlaces h. We have that $f \ll g$, $f \ll h$, and $g \ll h$.

We gather some fundamental properties of interleaving polynomials in the following lemma, which we also find in [9]:

Lemma 2.2 (Wagner's lemma, see [9, Section 3])**.** *Let $f, g, h \in \mathbb{R}[t]$ be real-rooted polynomials with only non-positive roots and positive leading coefficients. Then*

1. *if $f \ll h$ and $g \ll h$, then $f + g \ll h$.*
2. *if $h \ll f$ and $h \ll g$, then $h \ll f + g$.*
3. *$g \ll f$ if and only if $f \ll tg$.*

Proof. 1. Assume that $f \ll h$ and $g \ll h$, i.e.,

$$\cdots \leq f_2, g_2 \leq h_2 \leq f_1, g_1 \leq h_1,$$

as shown in Figure 1. This, together with f and g having positive leading coefficients implies that both f and g are positive to the left of f_1 and g_1 and negative to the right of f_1 and g_1. This pattern continues, as illustrated in Figure 2, where we have colored the regions where both f and g are positive in a dark grey and the regions where both functions are negative in light grey. We see in the figure that the roots of $f + g$ must be in the white regions, i.e., $f_i \leq (f + g)_i \leq g_i$ or $g_i \leq (f + g)_i \leq f_i$. Since $h_{i+1} \leq f_i, g_i \leq h_i$, we must have that $h_{i+1} \leq (f + g)_i \leq h_i$, ignoring h_{i+1} if it does not exist. If $\deg(f) = \deg(g) = d$, this implies $f + g \ll h$ and we are done. In the case of the degrees differing by 1, let us assume without loss of generality that $\deg(f) = \deg(g) - 1 = d$ and that d is an odd integer. Then, between f_d, g_d and g_{d+1}, both functions f and g are negative. However, to the left of g_{d+1}, the function f is still negative, but the function g will be positive. Since $\deg(g) > \deg(f)$, we have that g increases faster than f decreases. Therefore, the function $f + g$ must have a root to the left of g_{d+1} and we have that

$$(f + g)_{d+1} \leq g_{d+1} \leq h_{d+1} \leq (f + g)_d \leq \cdots \leq (f + g)_1 \leq h_1,$$

i.e. $f + g \ll h$.

2. Assume that $h \ll f$ and $h \ll g$, i.e.,

$$\cdots \leq h_2 \leq f_2, g_2 \leq h_1 \leq f_1, g_1.$$

Similarly to the previous arguments, we have that either $f_i \leq (f+g)_i \leq g_i$ or $g_i \leq (f+g)_i \leq f_i$. Therefore, we have

$$\cdots \leq h_2 \leq (f + g)_2 \leq h_1 \leq (f + g)_1,$$

i.e., $h \ll f + g$.

Figure 1: Interleaving polynomials, $f \ll h$ and $g \ll h$ (f dashed, g dotted, h solid)

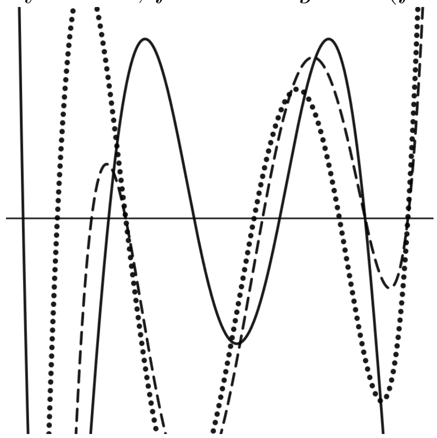
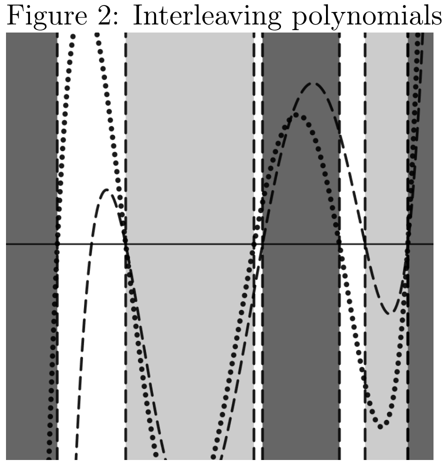


Figure 2: Interleaving polynomials



3. Suppose $g \ll f$, i.e.,

$$\cdots \leq g_2 \leq f_2 \leq g_1 \leq f_1.$$

By multiplying the polynomial g with t, we essentially add an extra root at $t = 0$ to the polynomial, call it g'_0. Let $\{g'_0, g'_1, g'_2, \ldots, g'_d\}$ denote the roots of tg in decreasing order. Then, we would have $g'_1 = g_1$, $g'_2 = g_2$, etc. The polynomial tg preserves the real-rootedness from g and it has non-positive roots (since we only add the root 0) and positive leading coefficient. Also, note that 0 is the largest possible non-positive root. Therefore, we have

$$\cdots \leq g'_2 = g_2 \leq f_2 \leq g'_1 = g_1 \leq f_1 \leq g'_0,$$

and hence, $f \ll tg$.

□

Let $\pi \in \mathcal{SP}(n)$. We will use the notation $\text{bblocks}(\pi, k)$ to count the number of blocks in π of size at least k. We want to study polynomials of the following type:

$$P_{n,k}(t) := \sum_{\pi \in \mathcal{SP}(n)} t^{\text{bblocks}(\pi,k)}. \tag{4}$$

For convenience, we also introduce the notation

$$Q_{n,k}(t) \coloneqq tP_{n,k}(t) = \sum_{\pi \in \mathcal{SP}(n)} t^{\mathrm{bblocks}(\pi,k)+1}. \tag{5}$$

We begin by looking at $P_{n,k}(t)$ when $k = 1$. We want to find a recurrence relation for the polynomials. To do this, let us recall the Stirling numbers of the second kind, which has this well-known recurrence relation (see e.g., [8]):

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad \text{for } 0 < k < n. \tag{6}$$

Note that we have

$$P_{n,1}(t) \coloneqq \sum_{\pi \in \mathcal{SP}(n)} t^{\mathrm{bblocks}(\pi,1)} = \sum_{i=0}^{n} S(n,i)t^i. \tag{7}$$

Therefore, we can get the next result using Equation (6).

Lemma 2.3. *Let $P_{n,1}(t)$ be defined as in Equation (7). Then, we have the recurrence relation*

$$P_{n,1}(t) = tP_{n-1,1}(t) + tP'_{n-1,1}(t), \tag{8}$$

with starting condition $P_{1,1}(t) = t$. We also have that

$$Q_{n,1}(t) = tQ'_{n-1,1}(t) + tQ_{n-1,1}(t) - Q_{n-1,1}(t), \tag{9}$$

with starting condition $Q_{1,1}(t) = t^2$.

Proof. We have

$$\begin{aligned}
P_{n,1}(t) &= \sum_{i=0}^{n} S(n,i)t^i \\
&= \sum_{i=0}^{n} (S(n-1,i-1) + iS(n-1,i))t^i \\
&= \sum_{i=0}^{n} S(n-1,i-1)t^i + \sum_{i=0}^{n} iS(n-1,i)t^i \\
&= t\sum_{i=0}^{n} S(n-1,i-1)t^{i-1} + t\sum_{i=0}^{n} iS(n-1,i)t^{i-1} \\
&= tP_{n-1,1}(t) + tP'_{n-1,1}(t),
\end{aligned}$$

proving the first part. Now, since $Q_{n,1}(t) = tP_{n,1}(t)$, we have that $Q'_{n,1}(t) = P_{n,1}(t) + tP'_{n,1}(t)$. Combining these with Equation (8), we get

$$\begin{aligned}
Q_{n,1}(t) &= tP_{n,1}(t) \\
&= t^2P_{n-1,1}(t) + t^2P'_{n-1,1}(t) \\
&= tQ_{n-1,1}(t) + t(Q'_{n-1,1}(t) - P_{n-1,1}(t)) \\
&= tQ_{n-1,1}(t) + tQ'_{n-1,1}(t) - Q_{n-1,1}(t).
\end{aligned}$$

□

In Table 1, we see the first five polynomials of $P_{n,1}(t)$. Note that the degree of the polynomial $P_{n,1}(t)$ is n. Also, we find the coefficients of the polynomials in [6, A008277], viewed in a triangular array. They are just the Stirling numbers of the second kind.

n	$P_{n,1}(t)$
1	t
2	$t^2 + t$
3	$t^3 + 3t^2 + t$
4	$t^4 + 6t^3 + 7t^2 + t$
5	$t^5 + 10t^4 + 25t^3 + 15t^2 + t$

Table 1: First five polynomials of $P_{n,1}(t)$

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We prove that the polynomials $P_{n,1}(t)$ are interleaving in the following theorem.

Theorem 2.4. *For $n \in \mathbb{N}$, the polynomial $P_{n,1}(t)$ only has real roots. Furthermore, we have that $P_{n-1,1}(t) \ll P_{n,1}(t)$.*

Proof. From Table 1, we see that $P_{1,1}(t)$ and $P_{2,1}(t)$ are real-rooted. Assume that $P_{n-1,1}(t) \ll P_{n-1,1}(t)$, i.e. that $P_{n-1}(t)$ is real-rooted. Using Item 3 in Lemma 2.2, we get $P_{n-1,1}(t) \ll tP_{n-1,1}(t)$. By Rolle's theorem, we have that $P'_{n-1,1}(t) \ll P_{n-1,1}(t)$. Using Item 3 in Lemma 2.2 again, we get $P_{n-1,1}(t) \ll tP'_{n-1,1}(t)$. Using Item 2 in Lemma 2.2, we get $P_{n-1,1}(t) \ll tP_{n-1,1}(t) + tP'_{n-1,1}(t)$, which gives us $P_{n-1,1}(t) \ll P_{n,1}(t)$ by Lemma 2.3. Therefore, $P_{n,1}(t)$ must also be real rooted. The result follows by the induction hypothesis. □

Let us now look at the polynomial $P_{n,k}(t)$ when $k = 2$ and find a recurrence relation for it. In [2], Alexandersson and Nabawanda found a recurrence for the polynomial

$$R_n(t) \coloneqq \sum_{\pi \in \mathcal{RSP}(n)} t^{\operatorname{des}(\pi)+1}, \tag{10}$$

where $\mathcal{RSP}(n)$ is the set of all run-sorted permutations over $[n]$ and $\operatorname{des}(\pi)$ is the number of descents in π. We will use this to get a recurrence for our polynomial $P_{n,2}(t)$.

Definition 2.2 (See [2] and [5])**.** We define a map **sptorsp** : $\mathcal{SP}(n) \to \mathcal{RSP}(n+1)$. Given $\pi \in \mathcal{SP}(n)$ written in the form $b_1 \mid b_2 \mid \cdots \mid b_k$ (see Section 2.1), we first move the smallest element in each block to the end of the block. Then, all entries are increased by one, and we remove the vertical bars separating the blocks. This creates a word of length n. Finally, we prepend a 1 to the beginning of the word. The result is now an element in $\mathcal{RSP}(n+1)$. In [5], Nabawanda, Rakotondrajao, and Bamunoba proved that **sptorsp** is a bijection.

Interested readers can find examples of **sptorsp** and more about run-sorted permutations in e.g., [2] and [5]. Let us use this bijection and some other lemmas in [2] to find a recurrence for $P_{n,2}(t)$.

Lemma 2.5 (See [2, Thm. 2.1])**.** *For any integer $n \geq 0$, if P is a set partition of $[n]$ and σ its corresponding run-sorted permutation over $[n+1]$, then the number of blocks of size greater than 1 in P is equal to the number of descents in σ.*

Remark. I have reworded Theorem 2.1 in [2] since if the number of runs in σ is equal to k, then the number of descents in σ is equal to $k - 1$.

Lemma 2.6 (See [2, Lemma 3.1])**.** *We have that the $R_n(t)$ satisfy the recurrence*

$$R_{n+1}(t) = tR'_n(t) + t(n-1)R_{n-1}(t), \tag{11}$$

with initial conditions $R_1(t) = R_2(t) = t$.

Lemma 2.7. *We have that $Q_{n,2}(t)$ satisfy the recurrence*

$$Q_{n,2}(t) = tQ'_{n-1,2}(t) + t(n-1)Q_{n-2,2}(t), \tag{12}$$

with initial conditions $Q_{1,2}(t) = t$ and $Q_{2,2}(t) = t^2 + t$.

Proof. Note that $Q_{n,2}(t) = R_{n+1}(t)$. This is because a set partition P of $[n]$ corresponds to a run-sorted permutation σ over $[n+1]$ and by Lemma 2.5, $\text{bblocks}(P) = des(\sigma)$. Inserting this into Equation (11), we get Equation (12). $\square$

To get $P_{n,2}(t)$, note that $P_{n,2}(t) = Q_{n,2}(t)/t$ and $Q'_{n,2}(t) = P_{n,2}(t) + tP'_{n,2}(t)$. Therefore, we have

$$\begin{aligned} P_{n,2}(t) &= Q_{n,2}(t)/t \\ &= Q'_{n-1,2}(t) + (n-1)Q_{n-2,2}(t) \\ P_{n,2}(t) &= P_{n-1,2}(t) + tP'_{n-1,2}(t) + t(n-1)P_{n-2,2}(t). \end{aligned} \tag{13}$$

As shown in Table 2, the coefficients of the polynomial $P_{n,2}(t)$ add up to the nth Bell number. We can explain this since the coefficient before t^i in $P_{n,2}(t)$ counts the number of set partitions where i blocks have a size of at least 2. Summing over i should naturally give us the number of set partitions of $[n]$, i.e., the nth Bell number. We also see the first ten polynomials of $P_{n,2}(t)$. We also see that the degree of the $P_{n,2}(t)$ must be $\lfloor n/2 \rfloor$, since $[n]$ can at most have $\lfloor n/2 \rfloor$ blocks of size at least 2. We can generalize this idea: The degree of the polynomial $P_{n,k}(t)$ is $\lfloor n/k \rfloor$.

n	$P_{n,2}(t)$
1	1
2	$t+1$
3	$4t+1$
4	$3t^2+11t+1$
5	$25t^2+26t+1$
6	$15t^3+130t^2+57t+1$
7	$210t^3+546t^2+120t+1$
8	$105t^4+1750t^3+2037t^2+247t+1$
9	$2205t^4+11368t^3+7071t^2+502t+1$
10	$945t^5+26775t^4+63805t^3+23436t^2+1013t+1$

Table 2: First ten polynomials of $P_{n,2}(t)$

We now repeat a central result of [2], but in the language of set partitions.

Theorem 2.8 (See [2, Thm. 3.3])**.** *For $n \in \mathbb{N}$, the polynomial $Q_{n,2}(t)$ only has real roots. Furthermore, we have that $Q_{n,2}(t) \ll Q_{n+1,2}(t)$.*

Proof. We will prove the statement using induction. We have that $Q_{1,2}(t) \ll Q_{2,2}(t)$. Assume now that $Q_{n-2,2}(t) \ll Q_{n-1,2}(t)$. Then, $(n-1)Q_{n-2,2}(t) \ll Q_{n-1,2}(t)$ and using Item 3 in Lemma 2.2, we get $Q_{n-1,2}(t) \ll t(n-1)Q_{n-2,2}(t)$. By Rolle's theorem, we have that $Q'_{n-1,2}(t) \ll Q_{n-1,2}(t)$. Using Item 3 in Lemma 2.2 again, we get that $Q_{n-1,2}(t) \ll tQ'_{n-1,2}(t)$. Using Item 2 in Lemma 2.2, we get $Q_{n-1,2}(t) \ll tQ'_{n-1,2}(t) + t(n-1)Q_{n-2,2}(t)$ and using Lemma 2.7, we get $Q_{n-1,2}(t) \ll Q_{n,2}(t)$. By induction, the statement follows. □

Remark. The statement is also true for $P_{n,2}(t)$, i.e. For $n \in \mathbb{N}$, the polynomial $P_{n,2}(t)$ only has real roots. Furthermore, we have that $P_{n,2}(t) \ll P_{n+1,2}(t)$.

3 Main problem

In this section, we deal with the problem of finding a general recurrence formula for $P_{n,k}(t)$ with larger values of k. We begin by looking at $P_{n,3}(t)$. Our initial approach was to consider a function $S(n,k,i,j)$ that counts the number of ways to partition $[n]$ into k blocks, where i blocks are of size 2 and j blocks are of size at least 3. Note that we can view this function as a modified Stirling's number of the second kind. By finding a recurrence relation for $S(n,k,i,j)$, we can get a recurrence relation for $P_{n,3}(t)$.

Lemma 3.1. *For relevant n,k,i,j, we have the following recurrence relation:*

$$S(n,k,i,j) = S(n-1,k-1,i,j) + (n-1)S(n-2,k-1,i-1,j) + \binom{n-1}{2}S(n-3,k-1,i,j-1) + jS(n-1,k,i,j), \quad (14)$$

where $S(n,k,i,j) = 0$ otherwise.

Proof. We look at the block containing the element n. There are four relevant situations:

1. The element n is in a 1-block, i.e., we append n into an existing set partition of $[n-1]$ which can be partitioned in $S(n-1,k-1,i,j)$ ways. Hence, we get the first term in Equation (14).
2. The element n is in a 2-block. There are $n-1$ ways to choose the other element in this block from $[n-1]$. We can do this in $(n-1)$ ways. The remaining $n-2$ elements can be partitioned in $S(n-2,k-1,i-1,j)$ ways. Hence, we get the second term in Equation (14).
3. The element n is in a 3-block. There are $\binom{n-1}{2}$ ways to choose the other 2 elements in this block from $[n-1]$. The remaining $n-3$ elements can be partitioned in $S(n-3,k-1,i,j-1)$ ways. Hence, we get the third term in Equation (14).
4. The element n is in a block of size bigger than 3. We insert n into one of the existing j blocks of size at least 3. This gives us the term $jS(n-1,k,i,j)$, which is the fourth term in Equation (14).

The four situations are mutually exclusive. Using the summation formula, we get Equation (14). □

Note that we have

$$P_{n+1,3}(t) = \sum_{\pi \in \mathcal{SP}(n+1)} t^{\mathrm{bblocks}(\pi,3)} = \sum_{i,j,k} S(n+1,i,j,k)t^j. \quad (15)$$

Through some calculations using Equation (15) and Lemma 3.1, we get the recurrence formula

$$P_{n+1,3}(t) = tP'_{n,3}(t) + P_{n,3}(t) + nP_{n-1,3}(t) + t\binom{n}{2}P_{n-2,3}(t), \quad (16)$$

with initial conditions $P_{1,3}(t) = 1$ and $P_{2,3}(t) = 2$.

This approach is similar to how we found a recurrence relation for $P_{n,1}(t)$. However, this method is hard to generalize to bigger blocks. This is because, for each incremental increase in k in $P_{n,k}(t)$, we need to add one more parameter to the modified Stirling's number of the second kind. In Section 3.1, we will use a different method to find a general recurrence formula for $P_{n,k}(t)$.

3.1 Generalization to bigger blocks

So far, we have looked at the polynomials $P_{n,k}(t)$ when k is 1 or 2. A natural question is if we can find a recurrence for larger values of k, i.e., $P_{n,k}(t)$ for $k \geq 3$. One way to construct a recurrence relation for $P_{n,k}(t)$ is to look at what happens with the element $n+1$, similar to how we proved Lemma 2.1. This recurrence is presented and proved in Lemma 3.2.

Lemma 3.2. *Let $P_{n,k}(t)$ be defined as in Equation (4). Then, we have the recurrence*

$$P_{n+1,k}(t) = \sum_{i=0}^{k-2} \binom{n}{i} P_{n-i,k}(t) + t \sum_{i=k-1}^{n} \binom{n}{i} P_{n-i,k}(t) \tag{17}$$

Proof. We look at the block containing the element $n+1$. Let $i+1$ be the size of this block. We can choose the other i elements of this block from $[n]$ in $\binom{n}{i}$ ways. The remaining $n-i$ elements give us the polynomial $P_{n-i,k}(t)$. Therefore, we get the term $\binom{n}{i} P_{n-i,k}(t)$. However, if $i+1 \geq k$, then the block containing the element $n+1$ is a big block. Therefore, we should multiply the term with another t. Since the element $n+1$ being in different-sized blocks are disjoint events, we can use the addition principle and get Equation (17). $\square$

The problem with Equation (17) is that it depends on all n previous polynomials. Also, this recurrence relation is not ideal for finding properties of the polynomials, such as real-rootedness. Therefore, we find another recurrence that depends on fewer previous polynomials. We will do this by noting that

$$P'_{n,k}(t) = \frac{d}{dt}\left(\sum_{\pi \in \mathcal{SP}(n)} t^{\text{bblocks}(\pi,k)} \right) = \sum_{\pi \in \mathcal{SP}(n)} \text{bblocks}(\pi,k) t^{\text{bblocks}(\pi,k)-1}.$$

Multiplying with t on both sides gives us

$$tP'_{n,k}(t) = \sum_{\pi \in \mathcal{SP}(n)} \text{bblocks}(\pi,k) t^{\text{bblocks}(\pi,k)}, \tag{18}$$

which we interpret combinatorically as choosing one of the existing blocks of size at least k and putting the element $n+1$ in it. Using Equation (18), we get the following result:

Theorem 3.3. *The polynomial $P_{n,k}(t)$ has the recurrence relation*

$$P_{n+1,k}(t) = tP'_{n,k}(t) + \sum_{i=0}^{k-2} \binom{n}{i} P_{n-i,k}(t) + t\binom{n}{k-1} P_{n-k+1,k}(t) \tag{19}$$

for $k \geq 2$.

Proof. We look at the block containing the element $n+1$. Let $i+1$ be the size of this block. If $i+1 > k$, we can get this block by choosing one of the existing blocks of size at least k and appending $n+1$. This gives us the term $tP'_{n,k}(t)$, as mentioned above. If $i+1 \le k$, we get the remaining terms analogous to how we did when we proved Lemma 3.2. $\square$

Corollary 3.3.1. *The polynomial $Q_{n,k}(t)$ has the recurrence relation*

$$Q_{n+1,k}(t) = tQ'_{n,k}(t) + \sum_{i=1}^{k-2} \binom{n}{i} Q_{n-i,k}(t) + t\binom{n}{k-1} Q_{n-k+1,k}(t) \tag{20}$$

for $k \ge 3$.

Proof. We have that $Q'_{n,k}(t) = tP_{n,k}(t)' + P_{n,k}(t)$. We get the result by multiplying both sides in Equation (19) with t and simplifying the expression. $\square$

3.1.1 The polynomials $P_{n,k}(t)$, for $k=3$ and $k=4$

Let us study the polynomials $P_{n,3}(t)$ in greater detail. We will also look at a brief example of the polynomials $P_{n,4}(t)$. As shown in Table 3, the degree of $P_{n,3}(t)$ is $\lfloor n/3 \rfloor$, confirming our statement in section 2.2. Also, we can find the coefficients of the polynomials in [6, A355144].

n	$P_{n,3}(t)$
1	1
2	2
3	$t+4$
4	$5t+10$
5	$26t+26$
6	$10t^2+117t+76$
7	$105t^2+540t+232$
8	$931t^2+2445t+764$
9	$280t^3+6909t^2+11338t+2620$
10	$4900t^3+48546t^2+53033t+9496$

Table 3: First ten polynomials of $P_{n,3}(t)$

Using Lemma 3.2, we get the the recurrence formula

$$P_{n+1,3}(t) = P_{n,3}(t) + nP_{n-1,3}(t) + t\sum_{i=2}^{n-1} \binom{n}{i} P_{n-i,3}(t). \tag{21}$$

Using Theorem 3.3, we again get the recurrence formula in Equation (16). Using Corollary 3.3.1, we get

$$Q_{n+1,3}(t) = tQ'_{n,3}(t) + nQ_{n-1,3}(t) + t\binom{n}{2} Q_{n-2,3}(t). \tag{22}$$

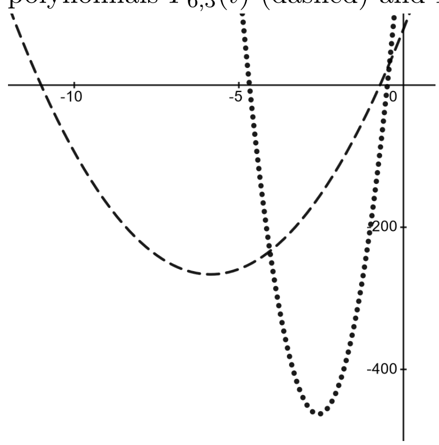
Example 3.1. From Table 3, we have $P_{6,3}(t) = 10t^2+117t+76$ and $P_{7,3}(t) = 105t^2+540t+232$.

We call the polynomials f and g respectively. The roots of the polynomials are calculated to be

$$f_1 = \frac{-117 - \sqrt{10649}}{20} \approx -11.010,\ f_2 = \frac{-117 + \sqrt{10649}}{20} \approx -0.69030,$$
$$g_1 = \frac{-270 - 2\sqrt{12135}}{105} \approx -4.6697,\ g_2 = \frac{-270 + 2\sqrt{12135}}{105} \approx -0.47316.$$

The polynomials are plotted in Figure 3. We can draw the conclusion that $P_{6,3}(t) \ll P_{7,3}(t)$. Through some computer calculations, we find that $P_{n,3}(t) \ll P_{n+1,3}(t)$ for all the polynomials listed in Table 3.

Figure 3: The polynomials $P_{6,3}(t)$ (dashed) and $P_{7,3}(t)$ (dotted)



Example 3.2. From Table 4, we have $P_{7,4}(t) = 225t + 652$ and $P_{8,4}(t) = 35t^2 + 1325t + 2780$. We call the polynomials f and g respectively. The roots of the polynomials are calculated to be

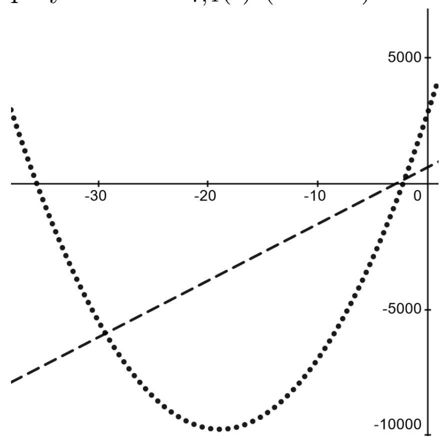
$$g_1 = \frac{-265 - 3\sqrt{6073}}{14} \approx -35.628,\ f_1 = -\frac{652}{225} \approx -2.8978,\ g_2 = \frac{-265 + 3\sqrt{6073}}{14} \approx -2.2294.$$

The polynomials are plotted in Figure 4. We can draw the conclusion that $P_{7,4}(t) \ll P_{8,4}(t)$. Through some computer calculations, we find that $P_{n,4}(t) \ll P_{n+1,4}(t)$ for all the polynomials listed in Table 4.

n	$P_{n,4}(t)$
1	1
2	2
3	5
4	$t + 14$
5	$6t + 46$
6	$37t + 166$
7	$225t + 652$
8	$35t^2 + 1325t + 2780$
9	$441t^2 + 8062t + 12644$
10	$4746t^2 + 50093t + 61136$

Table 4: First ten polynomials of $P_{n,4}(t)$

Figure 4: The polynomials $P_{7,4}(t)$ (dashed) and $P_{8,4}(t)$ (dotted)



Conjecture. *The polynomials $P_{n,k}(t)$ interleaves, i.e., $P_{n,k}(t) \ll P_{n+1,k}(t)$, for all integers $k \geq 1$.*

3.2 Polynomials generated by type-B set partitions

Let us look at polynomials generated by type-B set partitions and bi-colored set partitions. We begin with bi-colored set partitions.

$$R_{n,k}(t) := \sum_{\pi \in \mathcal{BCSP}(n)} t^{\text{bblocks}(\pi,k)}. \tag{23}$$

The method used to derive a recurrence relation for Equation (23) will be similar to what we used in Section 3.1.

Theorem 3.4. *The polynomial $R_{n,k}(t)$, defined in Equation* (23) *has the recurrence relation*

$$R_{n+1,k}(t) = 2tR'_{n,k}(t) + \sum_{i=0}^{k-2} 2^i \binom{n}{i} R_{n,k}(t) + 2^{k-1} t \binom{n}{k-1} R_{n-k+1,k}(t). \tag{24}$$

Proof. We look at the block containing the element $n+1$. Let $i+1$ be the size of this block. If $i+1 > k$, we can get this block by choosing one of the existing blocks of size at least k and appending $n+1$. This gives us the term $tR'_{n,k}(t)$, akin to how we did in Section 3.1. However, we can color the element $n+1$ either red or blue, so we need to add a coefficient 2, which gives us the term $2tR'_{n,k}(t)$. Now, we consider when the size of the block containing the element $n+1$ is less than or equal to k, i.e., $i+1 \leq k$. We choose the other i elements of this block from $[n]$. Then, we need to color every either red or blue, with the exception of the smallest element, i.e., we must color i elements, and we obtain the coefficient 2^i. The remaining $n-i$ terms give us the polynomial $R_{n-1,k}(t)$. From this, we obtain the terms $2^i \binom{n}{i} R_{n,k}(t)$ for $i = 0, \ldots, k-2$. However, when $i+1 = k$, the block containing the element $n+1$ is a big block. Therefore, we need to multiply $2^{k-1} R_{n-k+1,k}(t)$ with t, which gives us the last term in Equation (23). $\square$

We now turn our attention to type-B set partitions and the polynomials defined by

$$T_n(t) := \sum_{\pi \in \mathcal{BSP}(n)} t^{\text{bblocks}(\pi,1)}. \tag{25}$$

Although we did not find a recurrence relation to $T_n(t)$, we used Mathematica to generate the first four polynomials, which we see in Table 5. Note that the degree of $T_n(t)$ is $2n$. This is because the most number of blocks in a type-B set partition of $\{\pm 1, \ldots, \pm n\}$ is $2n$, achieved by putting each element in separate blocks. Also, by adding the coefficients in $T_n(t)$, we obtain the Dowling numbers, see [6, A007405]. This means that we can interpret the Dowling numbers as the number of type-B set partitions of $\{\pm 1, \ldots, \pm n\}$. From the table, we also observe that $T_2(t) = t^4 + 2t^3 + 2t^2 + t$, and through some computer calculations, we find that $T_2(t)$ has the roots $t_1 = -1$, $t_2 = 0$, $t_3 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$, and $t_4 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$. Therefore, we know that the polynomials $T_n(t)$ can not be interleaving, and they are not even real-rooted.

n	$T_n(t)$
1	$t^2 + t$
2	$t^4 + 2t^3 + 2t^2 + t$
3	$t^6 + 3t^5 + 6t^4 + 9t^3 + 4t^2 + t$
4	$t^8 + 4t^7 + 12t^6 + 30t^5 + 28t^7 + 32t^3 + 8t^2 + t$

Table 5: First four polynomials of $T_n(t)$

4 Discussion

In the previous sections, we have looked at polynomials that count big blocks. We have looked at the polynomials generated by set partitions, polynomials generated by type-B set partitions, and polynomials generated by bi-colored set partitions. The main results regarding normal set partitions are Theorem 2.8 and Theorem 3.3. Theorem 2.8 is related to one of the main results in [2] but applied to set partitions instead of run-sorted permutations. Theorem 3.3 gives us an intuitive recursion relation for the polynomials $P_{n,k}(t)$. However, the recursion formula in Corollary 3.3.1 depends on fewer previous polynomials; therefore, it might be better. We could not find a recursion formula for type-B set partitions, but with the help of a computer, we discovered that $R_{n,k}(t)$ is generally not real-rooted. We will further discuss the topic of real-rootedness in Section 4.1. We found a recursion formula for the bi-colored set partitions in Theorem 3.4.

4.1 Real-rootedness

A topic of interest is whether or not the different polynomials we have looked at interleave. We already saw that $P_{n,2}(t) \ll P_{n+1,2}(t)$ and we can similarly show that $R_{n,2}(t) \ll R_{n+1,2}(t)$. In Example 3.1, we saw that at least the first ten polynomials interleave and the presumption is that $P_{n,3}(t) \ll P_{n+1,3}(t)$ for all $n \geq 1$. We found similar results for $P_{n,4}(t)$ in Example 3.2. This leads us to the idea of whether or not $P_{n,k}(t)$ interleave for all $k \geq 1$. However, proving this was deemed too complex of a task for this paper. For the polynomials $T_n(t)$, we found counter-examples showing that the polynomials are neither interleaving nor real-rooted.

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