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The Möbius inversion formulas and common applications

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The Möbius inversion formulas and common applications

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Abstract

In this paper, we will state and prove the Möbius inversion formulas of partially ordered sets, and use them to prove a number of well known mathematical results, such as a formula for Euler's totient function, the Principle of inclusion and exclusion, and Hoffman's formula for multiple harmonic series.

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Contents

Introduction

In this paper, we are looking at the Möbius inversion formulas.

The article will start by reviewing the definition of a partially ordered set, or poset. We will look at some examples of posets, such as the natural numbers or the power set of the natural numbers, and describe a few properties and special cases, such as chains and lattices.

We will then consider the incidence algebra of a poset. We will state and prove the existence of a ring structure for the incidence algebra. Furthermore, we will state and prove the existence of a module which is acted upon by the incidence algebra.

Using this, we will then show the Möbius inversion formulas by considering a special function in the incidence algebra, the zeta function, and how this function acts on this module.

In the rest of the paper, we will look into the application of the formulas. We will state and prove the principle of inclusion and exclusion, a formula for Euler's totient function, and Hoffman's formula for the Riemann zeta function. As a starting example, we will also show a discrete analogue to the fundamental theorem of calculus.

It is recommended that the reader have some prior knowledge in set theory, modular arithmetics, and abstract algebra, particularly ring theory.

0.1 Sources

The main source for the paper is "Enumerative Combinatorics" by Richard P. Stanley[4]. Specifically, chapter 3, and some notation defined earlier in the book, is used extensively. In particular, section 1 relies on concepts discussed in chapters $3.1 - 3.3$.

In section 2, we rely on chapters $3.7 - 3.9$. In this section we will also use the book "Abstract Algebra" by David S. Dummit and Richard S. Foote[5], to recall some basic concepts in abstract algebra.

We use some other sources in the paper, which will be noted when necessary. Since they are only noted once, the page number will be in "References".

1 Posets

We will begin this paper by looking at some of the theory behind the Möbius inversion formulas. Specifically, we will look at posets, some basic examples of posets, and some special cases. This section will rely on $3.1 - 3.3(p.96 - 102)$ in Stanley [4].

1.1 Basic definitions and examples

We will first consider the definition of a poset.

Definition 1.1. A partially ordered set, or poset, $(P, \leq)$, is a set P with a relation $\leq$ such that

1. $\leq$ is reflexive, such that $x \leq x$ for all x in P,
2. $\leq$ is anti-symmetric, such that $x \leq y$ and $y \leq x$ only if $x = y$ in P, and
3. $\leq$ is transitive, such that $x \leq z$ if $x \leq y$ and $y \leq z$, for all x, y, z in P.

We say that $\leq$ is a partial ordering of P, or P is partially ordered by $\leq$. If the ordering is clear, we write $(P, \leq)$ as P.

Before we move on, we review some basic notation:

Remark 1.2. We say that $y \leq x$ if and only if $x \geq y$. We say that x is less than or equal to y if $x \leq y$. Similarly, x is greater than or equal to y if $x \geq y$. We say that $x \leq z \leq y$ if $x \leq z$ and $z \leq y$. We can extend this property inductively to write arbitrarily large chains of inequalities, $x_1 \leq x_2 \leq \ldots \leq x_k$. If $x \leq y$ but $x \neq y$, we say that x is less than y, or $x < y$. We say that x is greater than y if $x > y$. Write $x < z < y$ if $x < z$ and $z < y$. This property can be extended inductively, as above.

We note some basic examples of posets:

Example 1.3. The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, are posets by the usual ordering $\leq$. For example, $2 < 3 < 17$ in $(\mathbb{N}, \leq)$, and $e < \pi$ in $(\mathbb{R}, \leq)$.

Example 1.4. For any natural number n, the n-set of n, denoted by $[n]$, is the set of positive numbers less than or equal to n. In other words, $[n] = \{1, \ldots, n\}$. The n-set is a poset under the usual ordering.

Example 1.5. For a set S, the power set of S, or $P(S)$, is the set of all subsets of S. This forms a poset ordered by inclusion.

Example 1.6. For natural numbers n and m, we say that n divides m, or $n|m$, if there exists a natural number q such that $mq = n$. If this is true, we say that n is a divisor of m. The relation $|$ is called the divisibility relation.

For a natural number n, the divisor set of n is the set of natural numbers dividing n. The divisor poset of n, $\mathbb{N}_{|n}$, is the divisor set ordered by divisibility. The natural numbers, ordered by divisibility, is also a poset, denoted by $(\mathbb{N}, |)$.

1.2 Products and isomorphisms

We now look at two important ways to construct new posets.

Definition 1.7. For two sets P and Q, the product of P and Q is the set of ordered pairs (x, y) for all x in P and all y in Q. For two posets P and Q, the direct product of P and Q is the poset $(P \times Q, \leq)$, where $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$.

Definition 1.8. Two posets P and Q are isomorphic, denoted $P \cong Q$, if there exists a bijective function $\phi : P \to Q$ such that $x \leq y$ if and only if $\phi(x) \leq \phi(y)$. Such a function is called an isomorphism.

We will illustrate the use of these concepts by an example.

Example 1.9. Let S be a set with n elements, and T a subset of S. For every x in S, we have the function θ_x from the power set $P(S)$ to $[2]$, given by

$$\theta_x(T) = \begin{cases} 2 & x \in T \\ 1 & x \notin T. \end{cases}$$

Then, $P(S)$ is isomorphic to the direct product $[2]^n$ through the isomorphism θ_n, given by

$$\theta_n(T) = \{\theta_x(T) | x \in S\}.$$

Example 1.10. Let $S = [4]$, and $T = \{2, 3\}$. Then,

$$\theta_4(T) = \{1, 2, 2, 1\}.$$

1.3 Subposets, ideals and intervals

An important property of posets is the property of having subposets:

Definition 1.11. For a poset $(P, \leq)$, a subposet of P is a poset $(Q, \leq)$ such that Q is a subset of P, and for all x, y in Q, $x \leq y$ in P if $x \leq y$ in Q. If $P = Q$, P is called a refinement of Q. A poset Q is an induced subposet of P if, for all x, y in Q, $x \leq y$ in P if and only if $x \leq y$ in Q.

Example 1.12. Any subset of the real line gives an induced subposet of $(\mathbb{R}, \leq)$ under the usual ordering. This includes the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, and the rational numbers $\mathbb{Q}$. This also includes finite subsets, such as $[n]$.

Example 1.13. The poset $(\mathbb{N}, \leq)$ is a refinement of $(\mathbb{N}, |)$, since, for all natural numbers m and n, $m|n$ only if $m \leq n$. It is, however, not an induced subposet, since for example $2 \leq 3$ but $2 \not| 3$.

Two important subposets are the ideal and the filter:

Definition 1.14. 1. An ideal, denoted by I, is a subposet of a poset P such that y is in I if and only if $y \leq x$ for some element x in I. The principal ideal generated by x, $P_{\leq x}$, is the poset of elements in P less than or equal to x.

2. A filter, denoted by F, is a subposet of a poset P such that y is in I if and only if $y \geq x$ for some element x in F.The principal filter generated by x, $P_{\geq x}$, is the poset of elements in P greater than or equal to x.

Example 1.15. For any natural number n, the divisor poset $\mathbb{N}_{|n}$ is a principal ideal of the poset $(\mathbb{N}, |)$. For any n, the principal filter $\mathbb{N}_{n|}$ is the poset of elements divisible by n. For example, the even numbers is the filter generated by 2.

Another important subposet is the interval:

Definition 1.16. Let $x \leq y$ be elements in P. The interval $[x, y]$ in P, or $[x, y]_P$, is the set of all elements z in P such that $x \leq z \leq y$. We say that y covers x if $[x, y] = \{x, y\}$.

Example 1.17. The poset $[n]$ is the interval $[1, n]$ in $\mathbb{N}$.

Definition 1.18. Let P be a poset.

1. The interval set of P, denoted by $Int(P)$, is the set of intervals in P.

2. P is locally finite if every interval in P is finite.

1.4 Chains and lattices

Now, we will briefly consider a few special cases of posets. First, we will look at a chain and an anti-chain:

Definition 1.19. Let x, y be elements in a poset.

1. We say that x, y are comparable if $x \leq y$ or $y \leq x$. A chain $(C, \leq)$ is a poset in which all elements are pairwise comparable.

2. We say that x, y are incomparable if they are not comparable. An anti-chain $(A, \leq)$ is a poset in which all distinct elements are pairwise incomparable.

Example 1.20. The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, are chains under the usual ordering, as well as $[n]$.

Example 1.21. If $k > 1$ and $|P_i| > 1$ for all i, the direct product $P_1 \times \cdots \times P_k$ is not a chain, since, if $x_1 \leq y_1$ and $x_2 \leq y_2$, the pairs (x_1, x_2) and (y_1, y_2) are incomparable.

Example 1.22. The set of prime numbers ordered by divisibility, $(Pr, |)$, is an anti-chain.

Now, we will look at a lattice. First, a definition.

Definition 1.23. If there exists a unique element $\hat{0}$ in P such that $\hat{0} \leq x$ for all x in P, we say that $\hat{0}$ is the zero of P. Similarly, if there exists a unique element $\hat{1}$ in P such that $\hat{1} \geq x$ for all x in P, we say that $\hat{1}$ is the one of P.

Definition 1.24. For two elements x, y in P, $x \wedge y$, or x meet y, is the greatest element z in P such that $x \geq z$ and $y \geq z$, if it exists. Similarly, $x \vee y$, or x join y, is the least element z in P such that $x \leq z$ and $y \leq z$, if it exists.

Definition 1.25. Let P be a poset. If $x \wedge y$ exists for all elements x, y in P, we say that P is a meet semi-lattice. If $x \vee y$ exists for all elements x, y in P, we say that P is a join semi-lattice. If P is both a meet semi-lattice and a join semi-lattice, we say that P is a lattice.

Example 1.26. Let $(\mathbb{N}, \leq)$ be the natural numbers under the usual ordering. For natural numbers m and n, we define $m \vee n = max(m, n)$, and $m \wedge n = min(m, n)$. The natural numbers $\mathbb{N}$ is then a lattice with $\hat{0} = 1$.

2 The Möbius inversion formulas

In this section, we will state and prove the Möbius inversion formulas, and show some examples. To do so, we must first recall the concept of a ring, and a module of a ring. Then, we can move on to consider the incidence algebra of a poset P, which will give us the results.

2.1 Rings and modules

We will now recall some abstract algebra. This section relies on "Abstract Algebra" by Dummit and Foote [5].

First, recall a ring (p. 223 in [5]).

Definition 2.1. A ring $(R, +, \cdot, 0, 1)$ is a set R with addition and multiplication, where $(R, +, 0)$ is an abelian group, and multiplication is associative with identity 1. Furthermore, multiplication distributes addition.

If the context is clear, the ring $(R, +, \cdot, 0, 1)$ is denoted R.

An important concept in ring theory is inverses and invertibility. We will now define these concepts, and prove an important result.

Definition 2.2. Let r be an element in a ring R. Then,

1. A left inverse of r, denoted by s, is an element in R such that $sr = 1$,
2. a right inverse of r, denoted by t, is an element in R such that $rt = 1$, and
3. a two sided inverse r^{-1} of r is a left and right inverse of r.

We say that r is invertible in R if there exists a unique two sided inverse of r.

Proposition 2.3. *Let r be an element in a ring R, with a left inverse s and a right inverse t. Then, r is invertible in R.*

We show this using a lemma:

Lemma 2.4. *In a ring R, let r, p, and q be elements in R. Then,*

1. *if r has a left inverse s in in R, $rp = rq$ only if $p = q$, and*
2. *if r has a right inverse t in R, $rt = st$ only if $r = s$.*

Proof. To show 1, we see that

$$\begin{aligned} p &= 1p = (sr)p = s(rp) \\ &= s(rq) = (sr)q = 1q = q. \end{aligned}$$

2 is shown similarly. ■

Proof of proposition. Suppose there exists a left sided inverse $s' \neq s$ of r in R. Then, $s'r = sr = 1$. But since r has a right inverse, $s' = s$, by Lemma 2.4(2). Which contradicts the assumption. So s is the unique left inverse of r.

By a similar reasoning from Lemma 2.4(1), t is the unique right inverse of r.

But then,

$$s = s1 = s(rt) = (sr)t = 1t = t.$$

Set $s = t = r^{-1}$. Then r^{-1} is the unique two sided inverse of r in R. ■

The other concept we will go over from abstract algebra is a module (p. 337 in [5]).

Definition 2.5. Let $(R, +, \cdot, 0, 1)$ be a ring. A right module $(M, +, \cdot, 0)$ over R is a set M under addition and a right action on R, $\cdot : M \times R \to M$, such that $(M, +, 0)$ is an abelian group, and such that for all x, y in M and r, s in R,

1. $(x + y)r = xr + yr$,
2. $x(r + s) = xr + xs$,
3. $(xs)r = x(sr)$, and
4. $x1 = x$.

A left module of R is a module with a left action.

We have then this result:

Proposition 2.6. *Let r be an invertible element in a ring R, and let x, y be elements in a right module M over R. Then, $xr = y$ if and only if $yr^{-1} = x$.*

Proof. If $xr = y$,

$$yr^{-1} = (xr)r^{-1} = x(rr^{-1}) = x1 = x.$$

If $yr^{-1} = x$,

$$xr = (yr^{-1})r = y(r^{-1}r) = y1 = y.$$

■

A similar result, with a similar proof, holds for left modules.

2.2 The Incidence Algebra

Now, we are ready to define the incidence algebra (3.6.1, p.113 in [4]). Recall from Definition 1.18 that the interval set $Int(P)$ of a poset P is the set of intervals in P, and that a locally finite poset is a poset where every interval is finite.

Definition 2.7. Let P be a locally finite poset, and R a ring. The incidence algebra $I(P, R)$ over P with respect to R is the set of functions

$$\lambda : Int(P) \to R.$$

For $x \le y$ in P, we denote $\lambda([x, y])$ as $\lambda(x, y)$.

We will now consider a ring structure over the incidence algebra. The additive group is the group under pointwise addition.

So it suffices to show the following theorem:

Theorem 2.8. *Let λ, γ be elements in an incidence algebra $I = I(P, R)$. Let $x \le y$ be elements in P.*

1. *Let $\cdot$ be a binary operation over I given by*

$$(\lambda\gamma)(x, y) = \sum_{x \le z \le y} \lambda(x, z)\gamma(z, y). \tag{1}$$

2. *Let δ be the element in I given by*

$$\delta(x, y) = \begin{cases} 1 & x = y \\ 0 & x < y. \end{cases} \tag{2}$$

Then, $\cdot$ is an associative operation over I distributing addition, with the identity δ.

Proof. The distributive properties follows immediately from the distributivity of R. We will show associativity and the identity property of δ.

Let λ, γ and ϵ be elements in I. We want that

$$\lambda(\gamma\epsilon) = (\lambda\gamma)\epsilon.$$

We note that

$$\begin{aligned}(\lambda(\gamma\epsilon))(x, y) &= \sum_{x \le z \le y} \lambda(x, z)(\gamma\epsilon)(z, y) \\ &= \sum_{x \le z \le t \le y} \lambda(x, z)\gamma(z, t)\epsilon(t, y).\end{aligned}$$

But, on the other hand,

$$\begin{aligned}((\lambda\gamma)\epsilon)(x, y) &= \sum_{x \le t \le y} (\lambda\gamma)(x, t)\epsilon(t, y) \\ &= \sum_{x \le z \le t \le y} \lambda(x, z)\gamma(z, t)\epsilon(t, y).\end{aligned}$$

As needed.

To show that δ is an identity under $\cdot$, we must show that $\lambda\delta = \delta\lambda = \lambda$. We will show that $\delta\lambda = \lambda$. The other direction is similar.

But,

$$\begin{aligned}(\delta\lambda)(x,y) &= \sum_{x \leq z < y} \delta(x,z)\lambda(z,y) \\ &= \lambda(x,y) + 0 + \cdots + 0 \\ &= \lambda(x,y).\end{aligned} \tag{3}$$

■

The following theorem shows that this operation is sometimes invertible:

Theorem 2.9. *In an incidence algebra $I = I(P,R)$, let λ be a function such that $\lambda(x,x)$ is invertible in R, for all x in P. Then, λ is invertible in the ring $(I; \cdot, \delta)$.*

Proof. Set γ in I such that

$$\gamma(x,x) = (\lambda(x,x))^{-1},$$

for $x = y$, and

$$\gamma(x,y) = -(\lambda(y,y))^{-1} \sum_{x \leq z < y} \gamma(x,z)\lambda(z,y)$$

for $x < y$.

Then, by Theorem 2.8,

$$(\gamma\lambda)(x,x) = (\lambda(x,x))^{-1}\lambda(x,x) = 1.$$

For $x < y$, we get that

$$\begin{aligned}(\gamma\lambda)(x,y) &= \sum_{x \leq z \leq y} \gamma(x,z)\lambda(z,y) \\ &= \sum_{x \leq z < y} \gamma(x,z)\lambda(z,y) + \gamma(x,y)\lambda(y,y) \\ &= \sum_{x \leq z < y} \gamma(x,z)\lambda(z,y) - \sum_{x \leq z < y} \gamma(x,z)\lambda(z,y) = 0.\end{aligned}$$

So, it is that $\gamma\lambda = \delta$, meaning that λ has a left inverse.

By a similar reasoning, we can show that λ has a right inverse. By Proposition 2.3, λ has a unique two sided inverse λ^{-1}. ■

Now, we define a module over the incidence algebra.

Definition 2.10. Let P be a poset where every principal ideal is finite, and R a ring. Denote by $M(P, R)$ the set of functions

$$f : P \to R.$$

We will now state and prove that this set can be a module over an incidence algebra. The additive group over $M(P, R)$ is defined as in the incidence algebra, by pointwise addition. It suffices to find a right action. We will need the following theorem:

Theorem 2.11. *Let f, g be elements in $M = M(P, R)$, and λ be an element in the corresponding incidence algebra $I = I(P, R)$.*

Let $\cdot$ be a binary operation from $I \times M$ to M such that

$$(f\lambda)(y) = \sum_{x \leq y} f(x)\lambda(x, y) \text{ for all } x \text{ in } P.$$

Then, $\cdot$ is a right action over $I(P, R)$.

Proof. The first two axioms are satisfied by distributivity of I. So it suffices to show that $(f\lambda)\gamma = f(\lambda\gamma)$ and that $f\delta = f$ for all f in M, and λ, γ in I.

But, for any y in P,

$$\begin{aligned} ((f\lambda)\gamma)(y) &= \sum_{z \leq y} \left(\sum_{x \leq z} f(x)\lambda(x, z) \right) \gamma(z, y) \\ &= \sum_{x \leq z \leq y} f(x)\lambda(x, z)\gamma(z, y) \\ &= \sum_{x \leq y} f(x) \left(\sum_{x \leq z \leq y} \lambda(x, z)\gamma(z, y) \right) \\ &= (f(\lambda\gamma))(y). \end{aligned}$$

That $f\delta = f$ is shown easily;

$$\begin{aligned} (f\delta)(y) &= \sum_{x \leq y} f(x)\delta(x, y) \\ &= f(x) + 0 + \cdots + 0 \\ &= f(x). \end{aligned}$$

■

Remark 2.12. A left action is defined similarly.

Putting Theorems 2.8 and 2.11 together, we get the following:

Theorem 2.13. *Let P be a locally finite poset, and R a ring. Then, the incidence algebra of P is a ring under addition and multiplication. Furthermore, if every principal ideal of P is finite, the set $M(P, R)$, is a right module of the incidence algebra.*

From Theorem 2.9, we get the following result:

Corollary 2.14. *Let λ be an invertible element in an incidence algebra $I = I(P, R)$, and let f, g be functions from P to R. Then,*

$$g(y) = \sum_{x \leq y} f(x)\lambda(x, y) \tag{4}$$

if and only if

$$f(y) = \sum_{x \leq y} g(x)\lambda^{-1}(x, y). \tag{5}$$

Proof. By Definition 2.10, f and g are elements in the module $M(P, R)$ over I. Proposition 2.6 gives that

$$g = f\lambda \Leftrightarrow f = g\lambda^{-1}.$$

■

To demonstrate the concepts considered in this section, we will go over a simple example:

Example 2.15. The integers $\mathbb{Z}$ with ordinary addition and multiplication is a ring with invertible elements 1 and -1.

In the ring $I(\mathbb{N}, \mathbb{Z})$, let $\lambda(m, n) = 2m + n$, and $\gamma(m, n) = (-1)^{m+n}$. In the module $M(\mathbb{N}, \mathbb{Z})$, let $f(n) = n$.

Then,

$$\begin{aligned}(f\lambda)(n) &= \sum_{k=0}^{n} k(2k + n) \\ &= \sum_{k=0}^{n} 2k^2 + n \\ &= n(n + 1) + 2\sum_{k=0}^{n} k^2.\end{aligned}$$

We also have that

$$(f\gamma)(n) = \sum_{k=0}^{n} k((-1)^{k+n}.$$

Furthermore, the element γ is invertible in the ring, so

$$(f\gamma^{-1})(n) = \sum_{k=0}^{n} k\gamma^{-1}(k, n).$$

The inverse γ^{-1} is given by

$$\gamma^{-1}(n,n) = (-1)^{-2m}$$

for $m = n$, and by

$$\gamma^{-1}(m,n) = (-1)^{1-2n} \sum_{m \leq a < n} \gamma^{-1}(m,a)(-1)^{a+n},$$

for $m < n$.

2.3 The Möbius inversion formulas

Now, we are ready to prove our main result, the Möbius inversion formulas. (3.7, p.116, in [4]). First, we need a definition.

Definition 2.16. Let P be a locally finite poset, and R a ring. The zeta function, denoted by ζ, is the element in the incidence algebra $I(P,R)$ such that

$$\zeta(x,y) = 1 \text{ for all } x, y \text{ in P.} \tag{6}$$

Since 1 is an invertible element in R, we have by Theorem 2.9 that the zeta function is an invertible element in the incidence algebra $I(P,R)$. The inverse of ζ is the Möbius function, and is denoted by μ_P, or μ if the poset is clear. The Möbius inversion formula is then an immediate consequence of Corollary 2.14:

Theorem 2.17. *Let P be a poset where every principal ideal is finite, and R a ring. Then, for $f, g : P \to R$,*

$$g(y) = \sum_{x \leq y} f(x)$$

if and only if

$$f(y) = \sum_{x \leq y} g(x)\mu(x,y).$$

By considering the set $M(P,R)$ as a left module over the incidence algebra, and following the same outline as we have done, we can prove the dual Möbius inversion formula:

Theorem 2.18. *Let P be a poset where every principal filter is finite, and R a ring. Then, for $f, g : P \to R$,*

$$g(y) = \sum_{x \geq y} f(x)$$

if and only if

$$f(y) = \sum_{x \geq y} \mu(x,y)g(x).$$

Remark 2.19. For a finite poset P, every subposet is clearly finite. Specifically, every principal ideal and every principal filter is finite, and so, both formulas are satisfied.

For convenience, we will also give a recursive formula for the Möbius function, which follows immediately from the proof of Theorem 2.9:

Proposition 2.20. *The Möbius function over P is given by*

$$\mu_P(x,y) = \begin{cases} 1 & x = y \\ -\sum_{x \leq z < y} \mu(x,z) & x < y \end{cases}. \tag{7}$$

2.4 Basic examples

We will now review some basic examples of Möbius inversions:

Example 2.21. Let $(\mathbb{N}, \leq)$ denote the natural numbers under the usual ordering. The Möbius function is given by

$$\mu(m,n) = \begin{cases} 1 & m = n \\ -1 & n = m+1 \\ 0 & n \geq m+2 \end{cases}. \tag{8}$$

Proof. We will follow the statement of the Möbius function (Proposition 2.20). If $m = n$, it is clear that $\mu(m,n) = 1$. If $n = m+1$,

$$\mu(m,n) = -\mu(m,m) = -1.$$

If $n \geq m+2$, we show by induction that $\mu(m,n) = 0$.

In the base case, $n = m+2$. Then,

$$\mu(m, m+2) = -(\mu(m,m) + \mu(m,m+1)) = -(1-1) = 0.$$

The induction hypothesis is then that $\mu(m,n) = 0$ for some n such that $n = m+k$, where $k \geq 2$. Let $n = m+k+1$. Then,

$$\mu(m,n) = -(1 - 1 + 0 + \ldots + 0) = 0.$$

■

The Möbius inversion formula (Theorem 2.17) then gives that for any functions $f, g : \mathbb{N} \to \mathbb{N}$,

$$g(n) = \sum_{m \leq n} f(m)$$

if and only if

$$f(n) = g(n) - g(n-1).$$

Remark 2.22. Notice that Example 2.21 gives a discrete analogue to the fundamental theorem of calculus(3.8.1, p. 117 in [4]).

2.5 Products and isomorphisms

We will now state and prove two important results concerning Möbius inversion, which will help us calculate Möbius functions for more complicated posets. Let P and Q be posets with Möbius functions.

Recall by Definition 1.7 that the direct product of P and Q is the set of ordered pairs of elements in P and Q, ordered pairwise. Then, we get the following result(3.8.2, p. 118, in [4]):

Theorem 2.23. *Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be pairs in the product $P \times Q$, with $\mathbf{x} \leq \mathbf{y}$. Then,*

$$\mu_{P\times Q}(\mathbf{x}, \mathbf{y}) = \mu_P(x_1, y_1)\mu_Q(x_2, y_2). \tag{9}$$

Proof. We will show the theorem by induction on the interval $[\mathbf{x}, \mathbf{y}]$ in $P \times Q$. Recall by Definition 1.16 that $\mathbf{x}$ covers $\mathbf{y}$ if $\mathbf{x}$ if $[\mathbf{x}, \mathbf{y}] = \{\mathbf{x}, \mathbf{y}$.

For our base case, $\mathbf{x} = \mathbf{y}$. Suppose $\mu_{P\times Q}(\mathbf{x}, \mathbf{x}) \neq \mu_P(x_1, x_1)\mu_Q(x_2, x_2)$. Then,

$$\begin{aligned}
\delta_{P\times Q}(\mathbf{x}, \mathbf{x}) &= \mu_{P\times Q}(\mathbf{x}, \mathbf{x}) \\
&\neq \mu_P(x_1, x_1)\mu_Q(x_2, x_2) \\
&= \delta_P(x_1, x_1)\delta_Q(x_2, x_2).
\end{aligned}$$

Which gives a contradiction. So $\mu_{P\times Q}(\mathbf{x}, \mathbf{y}) = \mu_P(x_1, x_1)\mu_Q(x_2, x_2)$.

The induction hypothesis is then that

$$\mu_{P\times Q}(\mathbf{x}, \mathbf{y}) = \mu_P(x_1, y_1)\mu_Q(x_2, y_2) \tag{10}$$

for some $\mathbf{x} \leq \mathbf{y}$ in $P \times Q$.

Let $\mathbf{y}' = (y_1', y_2')$ be in $P \times Q$ such that $\mathbf{y}'$ covers $\mathbf{y}$. (Definition 1.16)

Suppose $\mu_{P\times Q}(\mathbf{x}, \mathbf{y}') \neq \mu_P(x_1, y_1')\mu_Q(x_2, y_2')$. Then,

$$\begin{aligned}
\delta_{P\times Q}(\mathbf{x}, \mathbf{y}') &= \sum_{\mathbf{x}\leq\mathbf{z}\leq\mathbf{y}'} \mu_{P\times Q}(\mathbf{z}, \mathbf{y}') \\
&= \delta_P(x_1, y_1)\delta_Q(x_2, y_2) + \mu_{P\times Q}(\mathbf{y}, \mathbf{y}') \\
&\neq \delta_P(x_1, y_1)\delta_Q(x_2, y_2) + \mu_P(y_1, y_1')\mu_Q(y_2, y_2') \\
&= \delta_P(x_1, y_1')\delta_Q(x_2, y_2').
\end{aligned}$$

Which gives a contradiction. So $\mu_{P\times Q}(\mathbf{x}, \mathbf{y}') = \mu_P(x_1, y_1')\mu_Q(x_2, y_2')$.. ∎

Now, we will state another important result. Recall that an isomorphism of posets is a bijective function preserving the ordering on the posets.

Theorem 2.24. *Let $\theta : P \to Q$ be an isomorphism. Let x, y be elements in P.*

Then,

$$\mu_P(x, y) = \mu_Q(\theta(x), \theta(y)). \tag{11}$$

Proof. It is clear that the incidence algebra is isomorphic if and only if the poset is isomorphic, and so, the Möbius function preserves the isomorphism. ■

We demonstrate these concepts by a simple example (3.8.3, p.118 in [4]):

Example 2.25. 1. Consider the direct product $[2]^n$. Let $\mathbf{x} = (x_1, \ldots x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be elements in $[2]^n$ such that $\mathbf{x} \leq \mathbf{y}$. Set

$$l(\mathbf{x}, \mathbf{y}) = |\{k \leq n | y_k = x_k + 1\}|.$$

Then,

$$\mu_{\mathbf{2}^n}(x, y) = (-1)^{l(x,y)}.$$

Proof. Recall from Example 2.21 that in $\mathbb{N}$, the Möbius function is given by

$$\mu(m, n) = \begin{cases} 1 & n = m \\ -1 & n = m + 1. \end{cases}$$

Applying Theorem 2.23, the result is clear. ■

2. Let $T \subseteq S$ be sets, with $|S| = n$. In the poset $P(S)$ (Example 1.5), the Möbius function is given by

$$\mu(T, S) = (-1)^{|S-T|}. \tag{12}$$

Proof. By Example 1.9, $P(S)$ is isomorphic to $[2]^n$ trough the isomorphism

$$\theta_n(T) = \{\theta_x(s) | x \in S\},$$

where

$$\theta_x(T) = \begin{cases} 1 & x \in S \\ 0 & x \notin S. \end{cases}$$

It is clear that

$$l(\theta_n(T), \theta_n(S)) = |S - T|.$$

From Theorem 2.24, we get that

$$\begin{aligned} \mu_{P(S)}(T, S) &= \mu_{|2|^n}(\theta_n(T), \theta_n(S)) \\ &= (-1)^{|S-T|}. \end{aligned}$$

■

2.6 Lattices and their Möbius algebras

We will now go over the lattices and their Möbius algebras. First, recall some abstract algebra.

Definition 2.26. A field $(K, +\cdot, 0, 1)$ is a ring where $(K - \{0\}, \cdot, 1)$ is an abelian group. A module over K is called a vector space (p. 337, 224 in [5]).

Now, we can consider the Möbius algebra of a lattice (3.9.1, p.124 in [4]):

Definition 2.27. Let L be a lattice and K a field. The Möbius algebra $A(L, K)$ is a vector space of linear combinations of elements of L, with multiplication defined by $x \cdot y = x \wedge y$.

We will now construct a basis over $A(L, K)$. Recall the definition of a basis:

Definition 2.28. Let K be a field, and V a vector space over K. A basis of K is a set of linearly independent elements in V which spans K (p. 354 in [5]).

Lemma 2.29. *For every $x \in L$, denote by δ_x the element in the Möbius Algebra $A(L, K)$ given by*

$$\delta_x = \sum_{y \le x} \mu(y, x) y. \tag{13}$$

Then, the set

$$\Delta = \{\delta_x | x \text{ in } L\}$$

forms a basis of $A(L, K)$.

Proof. To show that Δ spans K, we simply utilize the Möbius inversion formula (Theorem 2.17) for any x in L:

$$x = \sum_{y \le x} \delta_y. \tag{14}$$

To show linear independence, it is clear that the base exchange matrix of Δ is an upper triangular matrix, since for all $y < x$, $\mu(x, y) = 0$. ∎

Using Δ, we can show an important result.

Theorem 2.30. *Let L be a finite lattice and let K be a field. Then, it is the case that*

$$\delta_x \delta_y = \begin{cases} \delta_x & x = y \\ 0 & x \neq y. \end{cases}$$

Proof. The proof of this theorem is beyond the scope of this paper. See Theorem 3.9.2 on page 124 in Stanley[4]. ∎

Using this theorem, we get this important result:

Corollary 2.31. *Let L be a finite lattice with at least two elements. Let $a \neq \hat{1}$ in L. Then,*

$$\sum_{x \wedge a = 0} \mu(x, \hat{1}) = 0.$$

Proof. Consider $a\delta_{\hat{1}}$. By Theorem 2.30, it is clear that

$$a\delta_{\hat{1}} = \left(\sum_{x \leq a} \delta_x \right) \delta_1 = 0. \tag{15}$$

But, we also get that

$$\begin{aligned} a\delta_{\hat{1}} &= a \left(\sum_{x \in L} \mu(x, \hat{1}) x \right) \\ &= \sum_{x \in L} \mu(x, \hat{1})(x \wedge a) \\ &= \sum_{y \leq a} \left(\sum_{x : x \wedge a = y} \mu(x, \hat{1}) \right) y. \end{aligned} \tag{16}$$

Putting (15) and (16) together, we get that

$$\sum_{x : x \wedge a = 0} \mu(x, \hat{1}) = 0.$$

■

3 The Principle of inclusion and exclusion

A common, and very useful, application of Möbius function, is the Principle of Inclusion and Exclusion. We will state the principle and prove it using the Möbius inversion formula. We will then prove a common application of this formula. We will need a definition:

Definition 3.1. For a set A, let S be a set of properties for elements in A. Set $P(S)$ as the power set of S.

For a ring R, denote the function set $A(S,R)$ as

$$A(S,R) = \{f : P(S) \to R\}.$$

In this function set, define three functions, such that for any $T \subseteq S$;

1. $f_=(T) = |\{x \in A | x \text{ satisfies exactly the properties in } T\}|$,
2. $f_\leq(T) = |\{x \in A | x \text{ satisfies at most the properties in } T\}|$, and
3. $f_\geq(T) = |\{x \in A | x \text{ satisfies at least the properties in } T\}|$.

Now, we can state and prove the principle of inclusion and exclusion:

Proposition 3.2. *Let A be a set, and S be a finite set of properties over A. Then,*

1. $f_=(T) = \sum_{U \supseteq T} (-1)^{|T-U|} f_\leq(U)$, *and*
2. $f_=(T) = \sum_{U \subseteq T} (-1)^{|T-U|} f_\geq(U)$

Proof. We notice that the function ring $A(S,R)$ is just the module $M(P(S),R)$ for the power set $P(S)$ ordered by inclusion. Furthermore, by Example 2.25, we have that the Möbius function of this poset is

$$\mu_{2^S}(U,T) = \begin{cases} (-1)^{|T-U|} & U \leq T \\ 0 & \text{else} \end{cases}. \tag{17}$$

On the other hand, it is clear that

1. $f_\leq(T) = \sum_{U \supseteq T} f_=(U)$, and
2. $f_\geq(T) = \sum_{U \subseteq T} f_=(U)$.

Applying the Möbius inversion formulas (Theorems 2.17 and 2.18), the result is immediate. ■

3.1 Application

The principle of inclusion and inclusion is fairly abstract, in and of itself. To be more direct, we will show a well known application of the principle.

Corollary 3.3. *Let* $A = \{A_1, A_2, \ldots, A_n\}$ *be a finite collection of finite sets. Then,*

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i \leq n} (-1)^{n-i} \sum_{\substack{k_1, \cdots, k_i \leq n \\ distinct}} |A_{k_1} \cap \ldots \cap A_{k_i}|, \tag{18}$$

and

$$|A_1 \cap A_2 \cap \ldots \cap A_n| = \sum_{i \leq n} (-1)^{n-i} \sum_{\substack{k_1, \ldots k_i \leq n \\ distinct}} |A_{k_1} \cup \ldots \cup A_{k_i}|. \tag{19}$$

Proof. Set $B = \{B_1, B_2, \ldots B_m\}$ such that $B \supseteq A$, and let U be the union of B. For every element y in U, and every set $B_i \subseteq B$, the identity property of y, is given such that B_i satisfies the identity property of y if y is in B_i. Then, set S to be the set of identity properties for all elements in B. Then, we get that

1. $f_{\leq}(A) = |A_1 \cup A_2 \cup \cdots \cup A_n|$, and
2. $f_{\geq}(A) = |A_1 \cap A_2 \cap \cdots \cap A_n|$.

Applying Proposition 3.2, the result follows. ■

We illustrate this result using a few simple examples:

Example 3.4. For $n = 2$, the first non-trivial example, we get that

1. $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, and
2. $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2|$.

Example 3.5. For $n = 3$, we get that

1. $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| -$
   $(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|$, and
2. $|A_1 \cap A_2 \cap A_3| = |A_1| + |A_2| + |A_3| -$
   $(|A_1 \cup A_2| + |A_1 \cup A_3| + |A_2 \cup A_3|) + |A_1 \cup A_2 \cup A_3|$.

4 Eulers totient function

In this section, we will look at Euler's totient function, and how to calculate it using the Möbius inversion formula (Theorem 2.17). We will do this in the context of the divisor poset.

4.1 The divisor poset

Recall from Example 1.6 that for any natural number n, the divisor poset $\mathbb{N}_{|n}$ of n is the set of natural numbers dividing n, ordered by divisibility. In this subsection, we will consider a proof of the Möbius function of this poset. But first, a definition:

Definition 4.1. For any natural number n, the prime divisor poset $Pr_{|n} = \{p_1, \ldots, p_k\}$ is the anti-chain of prime numbers dividing n.

Using this definition, we can find an isomorphism from $\mathbb{N}_{|n}$:

Definition 4.2. Let $|Pr_{|n}| = k$. Then, there exists an isomorphism

$$\begin{aligned}\theta_n : &\mathbb{N}_{|n} \to (\mathbb{N}^k)\\ &m \mapsto (\phi_1(m), \ldots, \phi_k(m))\end{aligned}$$

where

$$\phi_j(m) = \max\{l : (p_k)^l | m\}.$$

Example 4.3. For $n = 24$, we have that

$$\mathbb{N}_{|24} = \{1, 2, 3, 4, 6, 12, 24\},$$

and

$$Pr_{|24} = \{2, 3\}.$$

We have that $12 = 2^2 3^1$, so

$$\theta_{24}(12) = (2, 1).$$

Now, we are ready to define the Möbius function:

Proposition 4.4. *For any natural number n, let m, p be elements of $\mathbb{N}_{|n}$. Set $p/m = p_1^{e_1} \cdots p_l^{e_l}$. We say that a natural number is square-free if it's prime factorizaton contains no squares.*

Then, the Möbius function of $\mathbb{N}_{|n}$ is given by

$$\mu_{\mathbb{N}_{|n}}(m, p) = \begin{cases} (-1)^l & p/m \text{ a square-free natural number} \\ 0 & \text{otherwise,} \end{cases}$$

Proof. By Theorem 2.24, we have that

$$\mu_{\mathbb{N}_{|n}}(m,p) = \mu_{(\mathbb{N}^k)}(\theta_n(m), \theta_n(p)).$$

By Theorem 2.23, we have that

$$\mu_{\mathbb{N}^k}(\theta_n(m), \theta_n(p)) = \mu_{\mathbb{N}}(\phi_1(m), \phi_1(p)) \cdots \mu_{\mathbb{N}}(\phi_k(m), \phi_k(p)). \tag{20}$$

Consider each factor in (20). By Example 2.21, there are two cases: If p/m is square-free, there are exactly l factors in (20) which gives -1, so the product is $(-1)^l$. If there exists a square in p/m, at least one factor is 0, and so, the entire product is 0. ■

Example 4.5. In $\mathbb{N}_{|24}$, we have that $24/12 = 2$, so $\mu(12, 24) = -1$, by Proposition 4.4. We can confirm this by following the proof, and see that

$$\begin{aligned}\mu(12,24) &= \mu_{\mathbb{N}}(2,3)\mu_{\mathbb{N}}(1,1)\\ &= -1 \cdot 1 = -1.\end{aligned}$$

4.2 Euler's totient function

Now, we will look at Euler's totient function. Specifically, we will prove, using the Möbius inversion formula, a formula for calculating this function for large input values. We will follow the book "A classical introduction to number theory", by Ireland and Rosen[2].

We begin with two remarks.

Remark 4.6. The divisibility poset $(\mathbb{N}, |)$ is a lattice with the meet operation satisfied by the greatest common divisor. We say that two elements $m \leq q$ are relatively prime if $m \wedge q = \hat{0}$, or equivalently, $\gcd(m, q) = 1$.

Remark 4.7. If $m < q$ is relatively prime to q, m is not a divisor of q.

This leads us to the relative primeness of

Definition 4.8. Euler's totient function ϕ counts, for a number n, the positive integers smaller than n which are relatively prime to n;

$$\phi(n) = |\{m \leq n \text{ and } m \wedge n = 1\}|.$$

Example 4.9. For the first values of n, we get the following values of ϕ:

n	$\phi(n)$
1	1
2	1
3	2
4	2
5	4
6	2
7	6
8	4
9	6
10	5

If we want to calculate ϕ for larger numbers, we need some formula. We will state one of those formulas, and prove it using the Möbius inversion formula (Theorem 2.17):

Proposition 4.10. *Let $n = p_1^{e_1} \dots p_k^{e_k}$. Then,*

$$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right). \tag{21}$$

Before proving it, we will demonstrate its usefulness with some examples:

Example 4.11. For any prime p, Proposition 4.10 gives that

$$\phi(p) = p(1 - 1/p) = p - 1.$$

Example 4.12. We get

$$\begin{aligned} \phi(1000) &= \phi(10^3) \\ &= \phi(2^3 5^3) \\ &= 1000(1 - (1/2))(1 - (1/5)) \\ &= 400. \end{aligned}$$

Now, we will prove the formula. Before applying the Möbius inversion formula, we need this lemma:

Lemma 4.13. *The sum of the values of the totient function for all numbers divided by n, is n;*

$$\sum_{m|n} \phi(m) = n.$$

Proof. Consider the set of rational numbers with denominator n, up to 1, so

$$\left\{ \frac{1}{n}, \frac{2}{n}, \dots, 1 \right\}.$$

Let k be the cardinality of this set. Clearly, $k = n$.

But, we can express each of these numbers as the quotient of relatively prime numbers. For each $m|n$, there are $\phi(m)$ numbers with m as denominator. Adding them all, we notice that $k = \sum_{m|n} \phi(m)$. ■

We can now prove our main result.

Proof of proposition. Recall that for any $m|n$, $n/m = p_1^{e_1} \cdots p_j^{e_j}$. By the Möbius inversion formula, and Lemma 4.13,

$$\begin{aligned} \phi(n) &= \sum_{m|n} \mu(m,n) m \\ &= \sum_{\substack{m/n \\ \text{square free}}} (-1)^j \frac{n}{n/m}, \end{aligned}$$

which can be rewritten as

$$\phi(n) = \sum_{j=0}^{k} (-1)^j \sum_{i_1 < i_2 \ldots < i_j < k} \frac{n}{p_{i_1} \cdots p_{i_j}}, \tag{22}$$

with $\{i_1, \ldots, i_j\}$ being a permutation of $[k]$.

On the other hand, consider the righthand side of (21). We can multiply the product, and get the alternating sum

$$n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \sum_{j=0}^{k} (-1)^j \sum_{i_1 < i_2 \ldots < i_j} \frac{n}{p_{i_1} \cdots p_{i_j}}. \tag{23}$$

Putting (22) and (23) together, we get that

$$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

■

5 Hoffmann's formula

In this section, we will look at Hoffmann's formula. This formula was derived by Michael E. Hoffmann in an article in 1995 [3], and gives a relation between the regular, single valued Riemann zeta function, and the multiple valued Riemann zeta function. In his article, Hoffmann gives a proof of this formula. We will consider a different proof, using the dual Möbius inversion formula (Theorem 2.18). Doing so, we will follow an article by Berglund and Bergström [6]. In order to understand this proof, we need to go over the partition poset.

5.1 The partition poset

Definition 5.1. For a positive integer n, we recall by Example 1.4 that the n-set of n is the set $[n] = \{1, \ldots, n\}$. A partition of $[n]$ is a collection of non-empty subsets of $[n]$ such that every number in $[n]$ is in exactly one of the subsets. In other words, for a partition $\pi = \{\pi_1, \pi_2, \ldots \pi_k\}$ of n, we have that

1. $\pi_i \neq \emptyset$ for all $i \leq k$,
2. $\pi_i \cap \pi_j = \emptyset$ if $i \neq j$, and
3. $\pi_1 \cup \ldots \cup \pi_k = [n]$.

We say that each set π_i is a block of π (p. 33 in [4]).

Example 5.2. One partition of [3] would be $\pi = \{\{1\}, \{2, 3\}\}$, which we write as $\pi = [1 - 23]$. Similarly, one partition of [7] is written $[12 - 345 - 67]$.

We can now define the partition poset.

Definition 5.3. If ρ, π are partitions of n, we say that $\rho \preceq \pi$, or π refines ρ, if there exists a partition P of $|\pi|$ such that

$$\pi_i = \bigcup_{j \in P_i} \rho_j.$$

If this is the case, we write that $\pi = P(\rho)$. The partition set of n, denoted Π_n, is the set of partitions of n. The partition poset of n is the partition set of n ordered under the refinement relation $\preceq$.

We can immediately see the following important fact about the partition poset:

Remark 5.4. The partition poset is a lattice, where the blocks of $\rho \wedge \pi$ are given by pairwise intersections of blocks of ρ and π, and the blocks of $\rho \vee \pi$ are given by pairwise unions of blocks of ρ and π. Furthermore, $\hat{1} = [n]$, and $\hat{0} = [1 - 2 - - - n]$.

Example 5.5. The partition poset over 3 is given by $\Pi_3 = \{[1 - 2 - 3], [1 - 23], [2 - 13], [3 - 12], [123]\}$. By remark 5.4, $[1 - 2 - 3] \leq \pi \leq [123]$ for all partitions π.

Example 5.6. In Π_7, set $\pi = [12 - 345 - 6 - 7]$, and $\rho = [123 - 456 - 7]$. Then, we have that

$$\pi \wedge \rho = [12 - 3 - 45 - 6 - 7],$$

and

$$\pi \vee \rho = [123456 - 7].$$

5.2 The Möbius function

Now, we look at the Möbius function of the partition poset. We will first look at the special case where $[\rho, \pi] = [\hat{0}, \hat{1}]$:

Lemma 5.7. *Denote* $\mu_{\Pi_n}(\hat{0}, \hat{1})$ *as* μ_n. *Then,*

$$\mu_n = (-1)^{n-1}(n-1)! \tag{24}$$

Proof. We will show this by induction on n.

For our base case, set $n = 1$. Then, $\Pi_1 = \{\hat{0}\}$, and

$$\mu_1 = \mu(\hat{0}, \hat{0}) = 1 = (-1)^0 0!.$$

Our induction hypothesis is then that

$$\mu_k = (-1)^{k-1}(k-1)!$$

for $k < n$, for some n.

For our induction step, consider the case for n. Let $\pi = \{[n] - \{n\}, \{n\}\}$. Look at the equation $\rho \wedge \pi = \hat{0}$. This equation has n solutions: one solution is $\rho = 0$, and $n - 1$ solutions for $\rho = \{[n] - \{i, n\}, \{i, n\}\}$. Each of these solutions are isomorphic to the partition $\hat{0}$ in Π_{n-1}. By Remark 5.4, Π_n is a lattice with a $\hat{1}$. So, by corollary 2.31, we have that

$$0 = \mu_n + (n-1)\mu_{n-1}, \tag{25}$$

giving that

$$\mu_n = -(n-1)\mu_{n-1} = (-1)^{n-1}(n-1)!$$

■

We are now ready to state and prove the general case:

Proposition 5.8. *For any natural number n, let* $\rho \preceq \pi$ *be two elements in the partition poset* $(\Pi_n, \leq)$. *Then, their Möbius function is given by*

$$\mu(\rho, \pi) = (-1)^{|\pi| - |\rho|} \prod_{0 < i \leq l} (|P_i| - 1)! \tag{26}$$

Proof. Since $\rho \preceq \pi$, by Definition 5.3, we can write $\pi = P(\rho)$ for some partition P in $\Pi_{|\rho|}$ with $|P| = |\pi|$.

Now, for every $i \leq |\pi|$, set

$$\rho_{P_i} = \{\rho_j \in \rho | j \in P_i\}.$$

Then,

$$[\rho, \pi] = \prod_{i \leq |\pi|} [\rho_{P_i}, \pi_i],$$

since $[\rho, \pi]$ is made out of of elements in the intervals $[\rho_{P_i}, \pi_i]$.

But each of these intervals is isomorphic to the interval $[\hat{0}, \hat{1}]$ in the set $\Pi_{|P_i|}$, so

$$[\rho, \pi] \cong \prod_{i \leq |\pi|} [0, 1]_{\Pi(P_i)}.$$

From this, follows by Theorem 2.23 and Theorem 2.24, and Lemma 5.7 that

$$\mu(\rho, \pi) = \prod_{i \leq |\pi|} \mu_{|P_i|}$$

From this, the result follows (3.10.3, p.127 − 128 in [4]). ■

5.3 The Riemann zeta function

In a few sections, we will apply Theorem 2.18 to prove Hoffman's formula. But before that, we need to traverse some background.

First, we recall the Riemann zeta function:

Definition 5.9. For a real number s greater than 1, the Riemann zeta function ζ of s is given by

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

Remark 5.10. If $s \leq 1$, the series does not converge [1].

Example 5.11. If $s = 2$ we have

$$\zeta(2) = \sum_{n \in \mathbb{N}} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots .$$

Example 5.12. For rational numbers s, the zeta function gives a sum of reciprocals of roots. For example,

$$\zeta\left(\frac{3}{2}\right) = 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots .$$

5.4 The generalized Riemann zeta function

Now, consider the generalized Riemann zeta function:

Definition 5.13. Let $s_1, ..., s_k$ be real numbers larger than 1. Then, the generalized Riemann zeta function is given by

$$\zeta(s_1, ..., s_k) = \sum_{1 \le n_1 < \cdots < n_k} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$

Example 5.14. For $k = 1$ we have the regular zeta function, as described in subsection 5.3.

Example 5.15. For $k = 2$ we get a double sum:

$$\zeta(s_1, s_2) = \sum_{1 \le n_1 < n_2} \frac{1}{n_1^{s_1} n_2^{s_2}} = \sum_{1 \le n_1} \frac{1}{n_1^{s_1}} \sum_{n_1 < n_2} \frac{1}{n_2^{s_2}}.$$

For example,

$$\zeta(2,3) = \left(\frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots\right) + \frac{1}{4}\left(\frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots\right) + \cdots .$$

We can prove an interesting result about the case for $k = 2$, showing the connection between the Riemann zeta function of one variable, and the generalized Riemann zeta function:

Proposition 5.16. *For s_1, s_2 real numbers greater than 1,*

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2). \tag{27}$$

Proof. We take $\zeta(s_1)\zeta(s_2)$, and we use basic rules for manipulating infinite series:

$$\zeta(s_1)\zeta(s_2) =$$

$$\left(\sum_n \frac{1}{n^{s_1}}\right)\left(\sum_n \frac{1}{n^{s_2}}\right) = \sum_{1 \le n} \frac{1}{n^{s_1}} \left(\sum_{1 \le n} \frac{1}{n^{s_2}}\right) =$$

$$\sum_{1 \le n_1} \frac{1}{n_1^{s_1}} \sum_{n_1 \le n_2} \frac{1}{n^{s_2}} + \sum_{1 \le n_1} \frac{1}{n_1^{s_2}} \sum_{n_1 \le n_2} \frac{1}{n_2^{s_1}} + \sum_{1 \le n} \frac{1}{n^{s_1 + s_2}} =$$

$$\zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$

■

5.5 Hoffmann's formula

We will now consider Hoffman's formula, which generalizes proposition 5.16. We will prove it using the Möbius inversion formula. To do this, we need some terms.

Definition 5.17. For a set S, a permutation over S is a bijection from S to itself. For a positive integer k, the permutation set of k, denoted Σ_k, is the set of permutations over $[k]$.

We can also show some definitions related to a partition:

Definition 5.18. For a partition π on k, we denote ζ_π as

$$\zeta_\pi = \prod_{\pi_i \in \pi} \zeta\left(\sum_{j \in \pi_i} s_j\right) \tag{28}$$

Example 5.19. For $\pi = [1-23]$ in Π_3,

$$\zeta_\pi = \zeta(s_1)\zeta(s_2+s_3).$$

If $(s_1, s_2, s_3) = (2, 4, 5)$, we get that

$$\zeta_\pi = \zeta(2)\zeta(9).$$

Definition 5.20. For a partition π on k, we denote by $c(\pi)$ the power series

$$c(\pi) = (-1)^{(k-|\pi|)} \prod_{P_i \in \pi} (|P_i| - 1)! \tag{29}$$

We can now state Hoffman's formula:

Proposition 5.21. *Let $s_1, \dots s_k$ be real numbers larger than 1. Then,*

$$\sum_{\sigma \in \Sigma_k} \zeta(s_{\sigma(1)}, \dots s_{\sigma(k)}) = \sum_{\pi \in \Pi_k} \zeta_\pi c(\pi). \tag{30}$$

Before diving into the proof of the formula, consider some examples of the formula for small k:

Example 5.22. For $k = 2$, the permutation set $\Sigma_2 = \{(), (12)\}$ and the partition set $\Pi_2 = \{1-2, 12\}$ gives

$$\zeta(s_1, s_2) + \zeta(s_2, s_1) = \zeta(s_1)\zeta(s_2) - \zeta(s_1 + s_2),$$

which we saw in Proposition 5.16.

Example 5.23. For $k = 3$, Proposition 5.21 gives that

$$\begin{aligned}\zeta(s_1, s_2, s_3) + \zeta(s_1, s_3, s_2) + \zeta(s_2, s_1, s_3) + \\ \zeta(s_2, s_3, s_1) + \zeta(s_3, s_1, s_2) + \zeta(s_3, s_2, s_1) \\ = \zeta(s_1 + s_2)\zeta(s_3) + \zeta(s_1 + s_3)\zeta(s_2) + \zeta(s_2 + s_3)\zeta(s_1) + \\ \zeta(s_1)\zeta(s_2)\zeta(s_3) + \zeta(s_1 + s_2 + s_3).\end{aligned}$$

To begin proving the proposition, consider two important power series related to the partition poset and Riemann's function. Let π be a partition of $[k]$, with $|\pi| = r$.

Definition 5.24. The p-sum over π is the series

$$p_\pi = \sum_{n_1,\dots,n_r} \prod_{j\in\pi_1} \frac{1}{n_1^{s_j}} \cdots \prod_{j\in\pi_r} \frac{1}{n_r^{s_j}},$$

Definition 5.25. The m-sum over π is the series

$$m_\pi = \sum_{\substack{n_1\dots n_r \\ \text{distinct}}} \prod_{j\in\pi_1} \frac{1}{n_1^{s_j}} \cdots \prod_{j\in\pi_r} \frac{1}{n_r^{s_j}}.$$

We can illustrate these sums using a few examples:

Example 5.26. Under the partition poset $\Pi_2 = \{[1-2], [12]\}$, the p-sum of $\hat{0} = [1-2]$ is given by

$$\begin{aligned}p_{\hat{0}} &= \sum_{n_1,n_2} \frac{1}{n_1} \frac{1}{(n_2)^2} \\ &= \sum_n \frac{1}{n^3} + \sum_{n_1\le n_2} \frac{1}{n_1}\frac{1}{(n_2)^2} + \sum_{n_2\le n_1} \frac{1}{n_1}\frac{1}{n_2}.\end{aligned}$$

The p-sum of $\hat{1} = [12]$ is given by

$$p_{\hat{1}} = \sum_n \frac{1}{n^3}.$$

The m-sum of $\hat{0}$ is given by

$$\begin{aligned}m_{\hat{0}} &= \sum_{n_1\le n_2} \frac{1}{n_1}\frac{1}{(n_2)^2} + \sum_{n_2\le n_1} \frac{1}{(n_1)^2}\frac{1}{n_2} \\ &= p_{\hat{0}} - p_{\hat{1}}.\end{aligned}$$

We can easily see this lemma, related to the p-sum and m-sum;

Lemma 5.27. *For any number k, and for any partition π over k, we have that*

$$p_\pi = \sum_{\rho\ge\pi} m_\rho.$$

We can now prove Hoffmann's formula:

Proof of Proposition. Recall that $\pi = \{\pi_1, \dots, \pi_r\}$. Lemma 5.27, and Theorem 2.18, gives that

$$m_\pi = \sum_{\rho \geq \pi} p_\rho \mu(\rho, \pi).$$

We notice that $c(\pi) = \mu(\hat{0}, \pi)$, so to show (30) it suffices to show that

$$m_{\hat{0}} = \sum_{\sigma \in \Sigma_k} \zeta(s_{\sigma(1)}, \dots s_{\sigma(k)}), \tag{31}$$

and

$$p_\pi = \prod_{P_i \in \pi} \zeta \left(\sum_{j \in P_i} s_j \right). \tag{32}$$

To show (31), it is clear that,

$$\sum_{\sigma \in \Sigma_k} \zeta(s_{\sigma(1)}, \dots s_{\sigma(k)}) = \sum_{\sigma \in \Sigma_k} \sum_{1 \leq n_{\sigma(1)} < \dots < n_{\sigma(k)}} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}. \tag{33}$$

By adding up all of the terms, we get that

$$\sum_{\sigma \in \Sigma_k} \sum_{1 \leq n_{\sigma(1)} < \dots < n_{\sigma(k)}} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} = \sum_{\substack{n_1, \dots, n_k \\ \text{distinct}}} \frac{1}{n_1^{s_1}} \cdots \frac{1}{n_k^{s_k}} = m_{\hat{0}}. \tag{34}$$

To show (32), we note that

$$\prod_{P_i \in \pi} \zeta \left(\sum_{j \in P_i} s_j \right) = \prod_{\pi_i \in \pi} \sum_n \prod_{j \in \pi_i} \frac{1}{n^{s_j}}.$$

Multiplying out the sums gives

$$\prod_{\pi_i \in \pi} \sum_n \prod_{j \in \pi_i} \frac{1}{n^{s_j}} = \sum_{n_1, \dots, n_r} \prod_{j \in \pi_1} \frac{1}{n_1^{s_j}} \cdots \prod_{j \in \pi_r} \frac{1}{n_r^{s_j}} = p_\pi.$$

■

6 Conclusion

In this paper, we have stated and proved the Möbius inversion formulas of posets, and applied these formulas to show a number of well known results.

We have studied the definition of a poset, and considered some special cases of posets, namely chains and lattices.

Then, we considered the incidence algebra on a poset. We proved that this was a ring, and that this ring had a module.

Finally, we looked at the zeta function in the incidence algebra, and found the Möbius inversion formulas by applying the actions on the module.

By applying the Möbius inversion formulas on various known posets, we considered some proofs of some important and well known results in mathematics, namely the principle of inclusion and exclusion, a formula for Euler's totient function, and Hoffmann's formula.

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