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Herglotz-Nevanlinna functions and rational approximations

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Abstract

Herglotz-Nevanlinna functions maps the complex upper half plane analytically to itself. The classic theory by Nevanlinna provides an integral representation for such functions in terms of a positive Borel measure on the real line, and establish a one-to-one correspondence between the functions and the class of finite positive Borel measures, $\mathcal{D}(\mathbb{R})$. The first part of this thesis utilize some techniques from functional analysis to prove this in detail. The second part of this thesis is devoted to show an approximation theorem for Herglotz-Nevanlinna functions in terms of rational functions with poles of order at most one. Any such function is again a Herglotz-Nevanlinna function. It turns out that the extreme points of $\mathcal{D}(\mathbb{R})$ are precisely the point mass measures δ_{x_0}, and by establishing compactness and convexity of $\mathcal{D}(\mathbb{R})$ in the weak* sense, the Krein-Milman theorem may be applied to extract a pointwise convergent sequence of such rational functions.

Contents

1 Introduction

In this thesis we are concerned with Herglotz-Nevanlinna functions, that is, analytic functions defined on the complex upper half plane, denoted $\mathbb{C}^+$, with non-negative imaginary part. Here $\mathbb{C}^+$ is defined by $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. This thesis contain two main results. First we state and prove in detail the classic representation theorem by Nevanlinna. That is, a function f is a Herglotz-Nevanlinna function if and only if f is given by the formula

$$f(z) = \alpha + \beta z + \int_{\mathbb{R}} \frac{tz + 1}{t - z} d\nu(t) \tag{1}$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and ν a finite positive Borel measure on $\mathbb{R}$, are all uniquely determined. Secondly we prove that any Herglotz-Nevanlinna function can be pointwise approximated by a sequence of rational functions, $\{q_k\}_{k\in\mathbb{N}}$, each on the form

$$q_k(z) = \alpha_k + \beta_k z + \sum_{n}^{N_k} d_n^k \frac{1}{s_n^k - z} \tag{2}$$

where $\alpha_k \in \mathbb{R}$, $\beta_k > 0$ and $N_k \in \mathbb{N}$ for each $k \in \mathbb{N}$, and $d_n^k > 0$ and $s_n^k \in \mathbb{R}$ for each $k \in \mathbb{N}$ and $0 \leq n \leq N_k$. Any such function is a Herglotz-Nevanlinna function.

The first result by Nevanlinna establishes a correspondence between the set of Herglotz-Nevanlinna functions and the set of finite positive Borel measures. We will denote the set of finite positive Borel measures and complex Borel measures on the real line by $\mathcal{D}(\mathbb{R})$ and $\mathcal{M}(\mathbb{R})$ respectively, where the former is a subset of the later. Taking this view will prove useful since $\mathcal{M}(\mathbb{R})$ is a Banach space, allowing for the use of tools from functional analysis. In particular one can define the weak* topology on $\mathcal{M}(\mathbb{R})$. The set $\mathcal{D}(\mathbb{R})$, while not a vector space, proves to be closed with respect to this topology, the essential property in our proof of equation (1).

For the second result there are two main points. First, defining for each $A > 0$ the set $\mathcal{D}_A(\mathbb{R}) = \{\nu \in \mathcal{D}(\mathbb{R}) : \nu(\mathbb{R}) \leq A\}$, we will see that $\mathcal{D}_A(\mathbb{R})$ as a subset of $\mathcal{M}(\mathbb{R})$ is both convex and compact with respect to the weak* topology. This allows for the use of the Krein-Milman theorem, which roughly states that the set $\mathcal{D}_A(\mathbb{R})$ can be recovered, in terms of weak* convergence, from its extreme points. Secondly we discover explicitly what the extreme points of $\mathcal{D}_A(\mathbb{R})$ are. It turns out that these are precisely the point mass measures, that is measures defined by

$$\delta_{x_0}(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

for measurable sets E, where $x_0 \in \mathbb{R}$. When evaluating the integral in formula (1), this yields the sum of fractions appering in (2).

2 Elements of functional analysis

2.1 Local compactness and the Urysohn lemma

In this section we collect some definitions, a lemma and fix some notation.

Definition 2.1. A topological space X is called **Hausdorff** if for any two points $x, y \in X$ with $x \neq y$ there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 2.2. Let $(X.\mathcal{T})$ be topological space. A subset $K \subset X$ is called **compact** if for every collection $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ of open sets $V_\alpha \in \mathcal{T}$ with $K \subset \bigcup_\alpha V_\alpha$ where $\mathcal{A}$ is some index set, there is a finite sub-collection $\{V_{\alpha_k}\}_{k=1}^n$ such that $K \subset \bigcup_{k=1}^n V_{\alpha_k}$.

If X itself is compact, we say that X is a compact space.

Definition 2.3. A topological space X is called **locally compact** if for each point $x \in X$ there exist an open set U with $x \in U$ and $\overline{U}$ compact.

Definition 2.4. A subset A in a topological space is called σ**-compact** if it is the countable union of compact sets.

Definition 2.5. Let X be a topological space. We define the **support** of a function $f : X \to \mathbb{C}$ by $\operatorname{Supp} f = \{x \in X : f(x) \neq 0\}$, and in case $\overline{\operatorname{Supp} f}$ is compact we says that f has **compact support**. We will denote the set of all continuous functions on X with compact support by $\mathcal{C}_c(X)$.

Clearly, if X is compact we have $\mathcal{C}_c(X) = \mathcal{C}(X)$, simply the continuous functions on X. However we will keep the subscript throughout to keep notation consistent with the presentation in the preliminaries, this since X is sometimes compact and sometimes not.

Definition 2.6. Let X be a locally compact Hausdorff space. A function $f : X \to \mathbb{C}$ is said to **vanish at infinity** if for each $\varepsilon > 0$ there exist a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ hold for each $x \in X \setminus K$. We will denote the set of all continuous functions on X that vanish at infinity by $\mathcal{C}_0(X)$.

It is easy to see that $\mathcal{C}_c(X)$ and $\mathcal{C}_0(X)$ are vector spaces over $\mathbb{C}$. Throughout, we will let the spaces $\mathcal{C}_c(X)$ and $\mathcal{C}_0(X)$ be endowed with **supremum norm** defined by $\|f\| = \sup_{x\in X} |f(x)|$ for f in $\mathcal{C}_c(X)$ or $\mathcal{C}_0(X)$ respectively, turning both spaces into normed vector spaces.

Lemma 2.7. *(Urysohn) Let X be a locally compact Hausdorff space, $V \subset X$ be open, $K \subset X$ compact and $K \subset V$. Then there exist a function $f \in \mathcal{C}_c(X)$ with* $\operatorname{Supp} f \subset U$, $f(x) = 1$ *whenever* $x \in K$ *and* $0 \leq f(x) \leq 1$ *for all* $x \in X$.

Proof. For proof see [4] □

Corollary 2.8. *The space $\mathcal{C}_c(X)$ is dense in $\mathcal{C}_0(X)$.*

Proof. Let $f_0 \in \mathcal{C}_0(X)$ and $\varepsilon > 0$ be given. By definition there exist a compact set $K \subset X$ such that $|f_0(x)| < \varepsilon$ for $x \in X \setminus K$. Let f be a as in Lemma 2.7 for this K and set $g = f_0 f$. Now $g \in \mathcal{C}_c(X)$ and $\|f_0 - g\| < \varepsilon$. □

2.2 Borel measures and regularity

In this section we collect some definitions and properties about measures and fix some notation. Basic knowledge of measure theory is however assumed and this is not an exhaustive treatment. Most of it is directly taken from the textbook [4].

Definition 2.9. Let $(X, \mathcal{T})$ be a topological space. The σ-algebra on X generated by $\mathcal{T}$ is called the **Borel σ-algebra** on X with respect to $\mathcal{T}$ and is denoted $\mathcal{B}(X)$. A positive, signed or complex measure defined on $\mathcal{B}(X)$ is called a positive, signed or complex **Borel measure**.

Proposition 2.10. *(Hahn decomposition) Let μ be a signed measure on X. There exist sets A and B with $A \cup B = X$ and $A \cap B = \emptyset$ such that $\mu(E) \geq 0$ holds for each measurable set $E \subset A$ and $\mu(F) \leq 0$ holds for each measurable set $F \subset B$.*

Proof. Proof see [4]. □

Proposition 2.11. *If μ is a complex Borel measure it can be decomposed uniquely in the following way*

$$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4) \tag{4}$$

where each of the measures μ_1, μ_2, μ_3, μ_4 are positive Borel measures. The measures defined by $\mu_{\mathrm{Re}} := \mu_1 - \mu_2$ and $\mu_{\mathrm{Im}} := \mu_3 - \mu_4$ are finite signed Borel measures.

Proof. Proof see [4] □

Definition 2.12. Let μ be a complex Borel measure on X. We define the **total variation** measure, $|\mu|$, by

$$|\mu|(E) = \mu_1(E) + \mu_2(E) + \mu_3(E) + \mu_4(E) \tag{5}$$

for each $E \in \mathcal{B}(X)$.

We note that $|\mu|$ is finite positive Borel measure on X.

Definition 2.13. Let X be a topological space. We denote by $\mathcal{M}(X)$ the set of complex Borel measures on X and by $\mathcal{D}(X)$ the set of finite positive Borel measures on X.

The set $\mathcal{M}(X)$ is a complex vector space in a natural way. Moreover, the mapping $||\cdot||_{\mathcal{M}(X)} : \mathcal{M}(X) \to \mathbb{R}$ defined by

$$||\mu||_{\mathcal{M}(X)} = |\mu|(X) \tag{6}$$

defines a norm on $\mathcal{M}(X)$.

Definition 2.14. Let X be a topological space. For $A > 0$ we define the set $\mathcal{M}_A(X)$ by

$$\mathcal{M}_A(X) = \{\mu \in \mathcal{M}(X) : ||\mu|| \leq A\} \tag{7}$$

and $\mathcal{D}_A(X)$ analogously.

It is clear that $\mathcal{D}(X) \subset \mathcal{M}(X)$ and that $\mathcal{D}_A(X) \subset \mathcal{M}_A(X)$ holds for each $A > 0$.

Definition 2.15. Let X be a locally compact Hausdorff space and let μ be a positive Borel measure. If

$$\mu(E) = \inf\{\mu(V) : E \subset V, V \text{open}\} \tag{8}$$

holds for each set $E \in \mathcal{B}(X)$, we say that μ is **outer regular**. If

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{compact}\} \tag{9}$$

holds for each set $E \in \mathcal{B}(X)$ with either $\mu(E) < \infty$ or E open, we say that μ is **inner regular**. We say that μ is **regular** if it is both outer and inner regular and that a complex Borel measure ν is regular if $|\nu|$ is regular.

For X a locally compact Hausdorff space any linear combination of regular complex measures is again regular. In particular, if $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ is a complex Borel measure and if ν_i is regular for each $1 \leq i \leq 4$, then ν is regular.

Definition 2.16. Let X be a locally compact Hausdorff space. We denote by $\mathcal{R}(X)$ the set of all regular complex measures on X. The sets $\mathcal{R}_A(X)$ for $A > 0$ are defined in the same way as in Definition 2.13.

It is clear that $\mathcal{R}(X) \subset \mathcal{M}(X)$ and that $\mathcal{R}_A(X) \subset \mathcal{M}_A(X)$ for each $A > 0$. Moreover, $\mathcal{R}(X)$ form a vector subspace of $\mathcal{M}(X)$. We will let $\mathcal{R}(X)$ inherit the norm of $\mathcal{M}$ to form a normed vector space. The following lemma gives conditions on a Borel measure μ that guarantees regularity.

Proposition 2.17. *Let X be a locally compact Hausdorff space in which every open set is σ-compact and let μ be a positive Borel measure for which $\mu(K) < +\infty$ holds whenever K is compact. Then μ is regular.*

Proof. Page 48 of [4] □

We also need a version of variables substitution formula. Since we are mostly concerned with Borel measures, we state a variation of this.

Proposition 2.18. *Let X and Y be topological spaces, let μ be a finite positive Borel measure on X and let $\varphi : X \to Y$ be a continuous function. Then the mapping $\nu = \mu \circ \varphi^{-1}$ on $\mathcal{B}(Y)$ defined by the formula*

$$\nu(E) = \mu(\varphi^{-1}(E)) \tag{10}$$

is a finite positive Borel measure on Y. Moreover, a $\mathcal{B}(Y)$-measurable function f is $\mu \circ \varphi$-integrable if and only if $f \circ \varphi$ is μ-integrable and in that case we have the formula

$$\int_Y f \; d(\mu \circ \varphi^{-1}) = \int_X f \circ \varphi \; d\mu. \tag{11}$$

Proof. See Theorem 3.6.1 page 190 [1] □

2.3 Convexity in topological vector spaces

In this section we collect some definitions and results about convexity:

Definition 2.19. Let X be a vector space over $\mathbb{C}$. A subset $S \subset X$ is called a **convex** subset of X if $tx + (1-t)y \in S$ for all $x, y \in S$ and $t \in (0,1)$.

The intuition of this definition should be that any line segment connecting two points of S need be contained in S.

Definition 2.20. Let $S \subset X$ be a set and $\{x_i\}_{i=1}^n$ a collection of points of S. The sum

$$\sum_{i=1}^{n} c_i x_i \tag{12}$$

where each $c_i \geq 0$ and $\sum_{i=1}^n c_i = 1$ is called a **convex combination** of points of S.

Clearly a set S is convex if it includes all its convex combinations, and by an induction argument it is easy to show that the converse holds . We define the **convex hull** of S, $\operatorname{Conv} S$, to be set of all convex combinations of points of S.

Definition 2.21. Let $S \subset X$. A point $x \in S$ is called an **extreme point** of S if the only convex combination of points of S that equal x is the one where all but one constant, c_k, equal zero and $c_k = 1$ and $x_k = x$. We denote the set of extreme points of S by E_S.

Definition 2.22. Let X be a vector space, let X be endowed with a topology $\mathcal{T}$, let $X \times X$ be endowed with the product topology and let $\alpha \in \mathbb{C}$. If each mapping $\alpha : X \to X$ defined by $\alpha(x) = \alpha x$ is continuous with respect to $\mathcal{T}$ and the mapping $\rho : X \times X \to X$ defined by $\rho(x,y) = x + y$ is continuous with respect to the product topology on $X \times X$ we say that X is a **topological vector space**.

Definition 2.23. Let $x \in X$ where $(X, \mathcal{T})$ is topological space and let $\mathcal{J}(x)$ be a collection of neighborhoods of x. We say that $\mathcal{J}(x)$ is a **neighborhood basis** of x if for each neighborhood $U \in \mathcal{T}$ of x there is a member $N \in \mathcal{J}(x)$ with $N \subset U$.

Definition 2.24. Let X be topological vector space. We say say that X is a **locally convex** topological vector space if there exist a neighborhood basis about the origin consisting entirely of convex sets.

Theorem 2.25 (Krein-Milman)**.** *Let X be a locally convex topological vector space where the topology is Hausdorff. If $K \subset X$ is convex and compact and E denotes the extreme points of K then we have that $K = \overline{\operatorname{Conv} E}$.*

Proof. Page 172, Theorem 6.9 of [5]. □

2.4 Dual spaces and the weak* topology

In this section we collect some definitions and results from functional analysis that will be used throughout this thesis. It is mainly a combination of results from the textbooks [4] and [5].

Definition 2.26. Let X be a normed vector space over $\mathbb{C}$. We will call a linear map $\ell : X \to \mathbb{C}$ that satisfy $\sup_{\|x\| \leq 1} |\ell(x)| < \infty$ a **bounded linear functional** on X. The **dual space** of X, denoted X^*, will be the set of all bounded linear functionals on X. We define the map $\|.\|_{X^*} : X^* \to \mathbb{R}_{\geq 0}$ by $\|\ell\|_{X^*} = \sup_{\|x\| \leq 1} |\ell(x)|$.

It is easy to see that X^* is a vector space over $\mathbb{C}$, moreover, $\|.\|_{X^*}$ defines a norm on X^*. We recall that a complete normed vector space is called a **Banach space**. The fact that a dual space always is complete is the content of the following proposition.

Proposition 2.27. *The pair $(X^*, \|.\|_{X^*})$ is a Banach space over $\mathbb{C}$.*

Proof. Theorem 4.4.4 of [3] □

From now on norm subscripts will be omitted when the meaning is clear from context. Applying Proposition 2.27 to X^* we get the following corollary.

Corollary 2.28. *The space X^{**} is a Banach space.*

For each $x \in X$ we define the map $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(\ell) = \ell(x)$ for $\ell \in X^*$. One can show that $\hat{x}$ is a bounded linear functional on X^* and thus $\hat{x} \in X^{**}$. Moreover, the mapping $j : x \mapsto \hat{x}$ is an injective linear isometry. We will denote the image of X under j by $\hat{X}$ and note that since j is linear, the set $\hat{X} \subset X^{**}$ form a vector subspace. We shall let $\hat{X}$ inherit the norm of X^{**} to form a normed vector space.

Definition 2.29. Let $\mathcal{E}$ be the collection of sets on the form $\hat{x}^{-1}(U)$ where $U \subset \mathbb{C}$ is open. The topology on X^* generated by $\mathcal{E}$ is called the **weak* topology** on X^*.

Definition 2.30. Let $\{\ell_k\}_k$ be a sequence in X^*. If there exist an element $\ell \in X^*$ such that $\lim_{k \to \infty} \ell_k(x) = \ell(x)$ holds for each $x \in X$, we say that that ℓ_k **converges weak*** to ℓ and write $\ell_k \xrightarrow{*} \ell$.

Proposition 2.31. *A sequence converges weak* if and only if it converges with respect to the weak* topology.*

Proof. See section 6.3 of [5] □

One can show that for a locally compact Hausdorff space X and a regular complex measure $\mu \in \mathcal{R}(X)$ on X, the function $\ell_\mu : \mathcal{C}_0(X) \to \mathbb{C}$ defined by

$$\ell_\mu(f) = \int_X f \; d\mu \tag{13}$$

for $f \in \mathcal{C}_0(X)$ is a bounded linear functional on $\mathcal{C}_0(X)$. In other words, $\ell_\mu \in \mathcal{C}_0(X)^*$. The converse statement constitute a version of the Riesz representation theorem.

Theorem 2.32. *(Riesz) Let X be a locally compact Hausdorff space and let $\ell \in \mathcal{C}_0(X)^*$. There exist a unique regular complex measure μ_ℓ such that*

$$\ell(f) = \int_X f \; d\mu_\ell \tag{14}$$

holds for each $f \in \mathcal{C}_0(X)$. Moreover, $\|\mu_\ell\| = \|\ell\|$.

Proof. Theorem 6.19 in [4] □

The uniqueness above tells us for example that the equalities $\ell_{\mu_\ell} = \ell$ and $\mu_{\ell_\mu} = \mu$ holds. In conclusion we have seen the following.

Corollary 2.33. *For X a locally compact Hausdorff space, the mapping $\tau : \mathcal{C}_0(X)^* \to \mathcal{R}(X)$ by $\tau(\ell) \mapsto \mu_\ell$ as in Theorem 2.32 defines a isometric isomorphism of normed vector spaces, $\mathcal{C}_0(X)^* \simeq \mathcal{R}(X)$.*

As a consequence of 2.33 we get that $\mathcal{R}(X)$ is a Banach space and that $\mathcal{C}_0(X)^{**} \simeq \mathcal{R}(X)^*$ as normed vector spaces. We also get that the topology on $\mathcal{R}(X)$ generated by sets on the form $\tau(U)$, where $U \subset \mathcal{C}_0(X)^*$ is weak* open, is precisely the weak* topology on $\mathcal{R}(X)$. We conclude the following.

Corollary 2.34. *The vector spaces $\mathcal{C}_0(X)^*$ and $\mathcal{R}(X)$ endowed with their respective weak* topologies are homeomorphic as topological vector spaces.*

We will also need the following fact

Proposition 2.35. *Let X be a locally compact Hausdorff space. The vector space $\mathcal{M}(X)$ endowed with the weak* topology is a locally convex Hausdorff topological vector space.*

Proof. Section 6.3 of [5] □

Theorem 2.36. *(Alaoglu) The unit ball in X^* is weak* compact.*

Proof. Theorem 6.10 of [5] □

Theorem 2.37. *(Hahn-Banach Corollary) If X is a Banach space and $Y \subset X$ is a dense subspace, then $Y^* \simeq X^*$ as normed vector spaces.*

Proof. Let $\ell \in Y^*$ be non-zero and suppose $\ell', \ell'' \in X^*$ are extensions of ℓ with $\|\ell\| = \|\ell'\| = \|\ell''\|$. By Corollary 4.8.7 of [3], we have that $0 = (\ell' - \ell'')(Y) = (\ell' - \ell'')(\overline{Y}) = (\ell' - \ell'')(X)$, a contradiction. □

3 Herglotz-Nevanlinna functions and their related measures

3.1 Basic properties and examples

In this section we discuss some basic properties and examples of Herglotz-Nevanlinna functions. A natural starting point for this investigation might be to take known analytic functions $\mathbb{C} \to \mathbb{C}$ and simply restrict the domain to $\mathbb{C}^+$.

Definition 3.1. Let $\mathbb{C}^+$ denote the set $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. An analytic function $f : \mathbb{C}^+ \to \mathbb{C}$ that satisfy that $\mathrm{Im}(f(z)) \geq 0$ for all $z \in \mathbb{C}^+$ is called a **Herglotz-Nevanlinna** function.

Remark. In this definition we allow for the function f to attain real values, however this only happens if $f \equiv a$ where $a \in \mathbb{R}$, a real constant. Here we only give an outline of the argument, for a more rigorous treatment we refer the reader to chapter X of [2]. The argument goes as follows. Write $f(z) = u(z) + iv(z)$ and suppose $z_0 \in \mathbb{C}^+$ is such that $v(z_0) = 0$. Recall from elementary complex analysis that v is *harmonic* on $\mathbb{C}^+$ and thus satisfy the *mean value property*. That is,

$$v(w) = \frac{1}{2\pi} \int_{[0,2\pi)} v(w + re^{it})dt \tag{15}$$

whenever the closed disk $\overline{B_r(w)}$ is contained in $\mathbb{C}^+$. Now by a version of the maximum modulus principle (page 255, Theorem 1.10 of [2]) we get that v is constant, and since $v(z_0) = 0$ we have that $v \equiv 0$ on $\mathbb{C}^+$. Finally, the Cauchy-Riemann equations can be applied to find that u is constant, let say $u \equiv a$, and thus the assertion follows.

Example 1. A power function like $f_1(z) = z^p$, where $p \in \mathbb{R}_{\geq 0}$, is a Herglotz-Nevanlinna function if and only if $p \leq 1$. To see this, we write $z = re^{it}$ and $f_1(re^{it}) = r^p e^{ipt}$ where $0 \leq t < \pi$. Thus, we need the inequality $0 \leq pt < \pi$ to hold, which is always the case when $p \leq 1$. Moreover, when $p > 1$ we can always find a t close to π such that the inequality is fails.

Example 2. By a similar argument as in Example 1, one can show that the function $f_2(z) = -\frac{1}{z^p}$ is a Herglotz-Nevanlinna function if and only of $p \geq -1$.

Example 3. The function $f_3(z) = \frac{a}{b-z}$, where $a \geq 0$ and $b \in \mathbb{R}$ are constants, is a Herglotz-Nevanlinna function. To see this simply write $z = x + iy$, now by the computation

$$\begin{aligned} f_3(x+iy) &= \frac{a}{(b-x) - iy} = \frac{a((b-x)+iy)}{((b-x)-iy))((b-x)+iy))} \\ &= \frac{a(b-x)}{|(b-x)-iy|} + i\frac{ay}{|(b-x)-iy|} \end{aligned} \tag{16}$$

the assertions follows since $ay \geq 0$.

Proposition 3.2. *Let f and g be Herglotz-Nevanlinna functions and suppose $\operatorname{Im}(g) > 0$. Now $f \circ g$ is a Herglotz-Nevanlinna function.*

Proof. This is clear since the range of g is contained in the domain of f and compositions of analytic functions are again analytic. In particular, by Example 2, the function $f_4(z) = -\frac{1}{h(z)}$ is Herglotz-Nevanlinna whenever h is a Herglotz-Nevanlinna function with $\operatorname{Im}(h(z)) > 0$ for all $z \in \mathbb{C}^+$. □

3.2 Some considerations on the spaces at hand

The main representation theorem concerns finite positive Borel measures on the real line, however we will see that by virtue of a transformation we can equivalently consider finite positive Borel measures on the interval $(0, 2\pi)$. We will let $[0, 2\pi]$ and $(0, 2\pi)$ inherit the standard subspace topologies from $\mathbb{R}$, and thus $(0, 2\pi)$ and $[0, 2\pi]$ are both Hausdorff spaces. Since $[0, 2\pi]$ is closed and bounded it is a compact space.

Proposition 3.3. *The space $(0, 2\pi)$ is locally compact.*

Proof. To see that $(0, 2\pi)$ is locally compact let $x \in (0, 2\pi)$ be given and pick an open ball $B_r(x)$ where $r < \min(\operatorname{dist}(0, x), \operatorname{dist}(x, 2\pi))$. The set $\overline{B_r(x)}$ contain x, is closed and bounded and therefore compact. □

Proposition 3.4. *Any open set in $(0, 2\pi)$ is σ-compact.*

Proof. Let $U \subset (0, 2\pi)$ be open. Recall that U can be written as a countable union of open intervals, that is $U = \cup_k (a_k, b_k)$, where (a_k, b_k) is an open interval in $(0, 2\pi)$ for each $k \in \mathbb{N}$. However each interval (a_k, b_k) can be written as a countable union of closed intervals, $(a_k, b_k) = \cup_i [a_k^i, b_k^i]$, each contained in (a_k, b_k). Thus U can be written $U = \cup_k \cup_i [a_k^i, b_k^i]$, a countable union of closed and bounded, and therefore compact, sets. □

Proposition 3.5. *Any open set in $[0, 2\pi]$ is σ-compact.*

Proof. Let $U \subset [0, 2\pi]$ be open. If none of the endpoints belong to U, the same technique as in the proof of Proposition 3.4 can be applied to U. Suppose $0 \in U$. But now the technique above can be applied to $U \setminus \{0\}$ to write $U \setminus \{0\} = \cup_k \cup_i [a_n^i, b_k^i]$. However we have $U = \{0\} \cup (\cup_k \cup_i [a_n^i, b_k^i])$, a countable union of compact sets. The same argument applies in the remaining cases so the assertion follows. □

Proposition 3.6. *Let X and Y be locally compact Hausdorff spaces, the function $\varphi : X \to Y$ a homeomorphism and $\mu \in \mathcal{M}(X)$. Then $\mu \circ \varphi^{-1} \in \mathcal{M}(Y)$. Moreover, the mapping $j_\varphi : \mu \mapsto \mu \circ \varphi^{-1}$ is an isomorphism of normed vector spaces.*

Proof. By Proposition 2.18 we have the inclusion $\mu \circ \varphi^{-1} \in \mathcal{M}(X)$. The computation

$$\begin{aligned} \left\|\mu \circ \varphi^{-1}\right\| &= |\mu \circ \varphi^{-1}|(Y) \\ &= \mu_1(\varphi^{-1}(Y)) + \mu_2(\varphi^{-1}(Y)) + \mu_3(\varphi^{-1}(Y) + \mu_4(\varphi^{-1}(Y)) \\ &= \mu_1(X) + \mu_2(X) + \mu_3(X) + \mu_4(X) \\ &= |\mu|(X) = \|\mu\| \end{aligned} \tag{17}$$

shows that $\mu \circ \varphi^{-1} \in \mathcal{M}(Y)$. For linearity suppose $\mu', \mu'' \in \mathcal{M}_A(X)$, $a, b \in \mathbb{R}$ and $E \in \mathcal{B}(Y)$. By the computation

$$\begin{aligned} ((a\mu' + b\mu'') \circ \varphi)(E) &= (a\mu' + b\mu'')(\varphi^{-1}(E)) \\ &= a\mu'(\varphi^{-1}(E)) + b\mu''(\varphi^{-1}(E)) \\ &= a(\mu' \circ \varphi^{-1})(E) + b(\mu'' \circ \varphi^{-1})(E) \end{aligned} \tag{18}$$

the assertion follows. □

We get the following corollary.

Corollary 3.7. *The restricted map* $j_\varphi|_{\mathcal{D}_A(X)} : \mathcal{D}_A(X) \to \mathcal{D}_A(Y)$ *is a norm preserving linear bijection.*

Since $\mathcal{D}_A(X)$ is not a vector space, the linearity of $j_\varphi|_{\mathcal{D}_A(X)}$ should be understood to be applicable when it makes sense. That is an expression on the form $\sum_k c_k\mu_k$ belong to $\mathcal{D}_A(X)$ if and only if $j_\varphi\left(\sum_k c_k\mu_k\right) = \sum_k c_k\left(\mu \circ \varphi^{-1}\right)$ belong to $\mathcal{D}_A(Y)$.

Returning to complex measures we have the following useful fact.

Corollary 3.8. *Any complex Borel measure on* $(0, 2\pi)$*,* $[0, 2\pi]$ *or* $\mathbb{R}$ *is regular.*

Proof. Let ν be a complex Borel measure on $(0, 2\pi)$, $[0, 2\pi]$ or $\mathbb{R}$ with decomposition $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$. The total variation measure of ν is now given by

$$|\nu| = \nu_1 + \nu_2 + \nu_3 + \nu_4, \tag{19}$$

where Proposition 2.17 can be applied to each component ν_i for $1 \le i \le 4$, showing that $|\nu|$ is a finite sum of regular measures. We conclude that ν is regular. □

Above we have gathered that, in symbols, $\mathcal{R}(0, 2\pi) = \mathcal{M}(0, 2\pi)$, $\mathcal{R}[0, 2\pi] = \mathcal{M}[0, 2\pi]$, and $\mathcal{R}(\mathbb{R}) = \mathcal{M}(\mathbb{R})$. Since by Proposition 2.8 the set $\mathcal{C}_c[0, 2\pi]$ is dense in $\mathcal{C}_0[0, 2\pi]$, we have by Theorems 2.37 and 2.32 that

$$\mathcal{C}_c[0, 2\pi]^* \simeq \mathcal{C}_0[0, 2\pi]^* \simeq \mathcal{R}[0, 2\pi] \simeq \mathcal{M}[0, 2\pi], \tag{20}$$

where the equivalence above is in the sense of Corollary 2.33.

Let us also view $\mathcal{M}[0,2\pi]$ endowed with the weak* topology instead of the norm topology. By Corollary 2.34 we get the same chain of equivalences as in equation (20) if we instead consider them as topological vector spaces with the weak* topology.

We now turn to show the very key fact that will be used to prove both main results of this thesis, namely that the set $\mathcal{D}_A[0,2\pi]$ is a compact subset with respect to the weak* topology.

Proposition 3.9. *The set $\mathcal{D}_A[0,2\pi]$ is weak* compact.*

Proof. First note $\mathcal{D}_A[0,2\pi]$ is given by

$$\mathcal{D}_A[0,2\pi] = \mathcal{D}[0,2\pi] \cap \mathcal{M}_A[0,2\pi] \tag{21}$$

and that $\mathcal{M}_A[0,2\pi]$ is compact by Theorem 2.36. Thus it is sufficient to show that $\mathcal{D}[0,2\pi]$ is closed in $\mathcal{M}[0,2\pi]$. For contradiction suppose not, namely that there is exist a weak* convergent sequence $\{\mu_k\}_k$, fully contained in $\mathcal{D}[0,2\pi]$, with weak* limit equal to $\mu \in \mathcal{M}[0,2\pi] \setminus \mathcal{D}[0,2\pi]$. By definition we have that

$$\lim_{k\to\infty} \int_{[0,2\pi]} f \, d\mu_k = \int_{[0,2\pi]} f \, d\mu \tag{22}$$

holds for each $f \in \mathcal{C}_0[0,2\pi]$.

Step 1. First assume that μ is a signed measure. By Theorem 2.10 there exist a set $B \in \mathcal{B}[0,2\pi]$ such that $\mu(B) = -\mu^-(B) < 0$ and $\mu(E) \leq 0$ for each measurable set $E \subset B$. Since μ is regular we can find a compact set $K \subset B$ and an open set $B \subset V$ such that $|\mu|(V \setminus K) < \varepsilon_1 = \frac{1}{2}\mu^-(B)$ and $|\mu(B) - \mu(K)| < \varepsilon_2 = \frac{1}{2}\mu^-(B)$. By Lemma 2.7 there exist a function $g \in \mathcal{C}_c[0,2\pi]$ with $\operatorname{Supp} g \subset V$, $g(x) = 1$ for each $x \in K$ and $0 \leq g(x) \leq 1$. Setting $f = g$ in (22), we have for the right hand side the estimate

$$\begin{aligned}\int_{[0,2\pi]} g \, d\mu &= \int_{V\setminus K} g \, d\mu + \int_K g \, d\mu \\ &\leq \int_{V\setminus K} g \, d|\mu| + \mu(K) \leq |\mu|(V \setminus K) - \mu^-(K) \\ &< \frac{1}{2}\mu^-(B) + \frac{1}{2}\mu^-(B) - \mu^-(B) = 0\end{aligned} \tag{23}$$

which shows that the right hand side of (22) is strictly negative. However each member of the left hand side of (22) is non-negative, a contradiction.

We finish this chapter by showing that $\mathcal{D}_A[0,2\pi]$ is a convex set, a fact that will be useful in section 4

Proposition 3.10. *If $\sum_{i=1}^N c_i\mu_i$ is a convex combinations of measures $\mu_i \in \mathcal{D}_A[0,2\pi]$, then the measure $\sum_{i=1}^N c_i\mu_i \in \mathcal{D}_A[0,2\pi]$.*

Proof. By the following computation,

$$\left(\sum_{i=1}^{N} c_i\mu_i\right)([0,2\pi]) = \sum_{i=1}^{N} c_i\mu_i\left([0,2\pi]\right) \leq A\sum_{i=1} c_i \leq A, \tag{24}$$

the assertion follows. □

Step 2. Next assume μ is a complex measure. By what we have shown, the imaginary part of μ, μ_{Im}, is necessarily non-zero since else μ would be a signed measure. However μ_{Im} is a signed measure, so the same argument as above can be applied to μ_{Im}. In particular there exist a function $h \in \mathcal{C}_c[0, 2\pi]$ such that for $f = h$, the right hand side of (22) have non-zero imaginary part, a contradiction. □

3.3 Poisson Integral Formula

In this section we state and prove the well known Poisson integral formula, the first stepping stone to proving the two main theorems of this thesis. In order to do this we first state and prove two auxiliary lemmas. Here, a **region** will mean an open connected subset of the complex plane.

Lemma 3.11. *For* $|w| < 1$ *the equality* $\text{Re}(\frac{e^{it}+w}{e^{it}-w}) = \frac{1-|w|^2}{|1-e^{-it}w|^2}$ *holds.*

Proof. Write $w = re^{i\theta}$. From the computation

$$\frac{e^{it}+w}{e^{it}-w} = \frac{e^{it}}{e^{it}} \cdot \frac{1+e^{-it}w}{1-e^{-it}w} = \frac{(1+e^{-it}w)(1-e^{it}\overline{w})}{(1-e^{-it}w)(1-e^{it}\overline{w})} \tag{25}$$

$$\tag{26}$$

$$= \frac{1-|w|^2 - e^{it}\overline{w} + e^{-it}w}{|1-e^{-it}w|^2} = \frac{1-|w|^2}{|1-e^{-it}w|^2} + \frac{r(e^{i(\theta-t)} - e^{-i(\theta-t)})}{|1-e^{-it}w|^2} \tag{27}$$

$$= \frac{1-|w|^2}{|1-e^{-it}w|^2} + i\frac{2r\sin(\theta-t)}{|1-e^{-it}w|^2} \tag{28}$$

the assertion follows. □

Lemma 3.12. *If f and g are analytic on a region G and have identical real part they differ by at most an imaginary constant.*

Proof. Let f and g have identical real part. Since the difference, $f-g$, is again analytic it satisfies the Cauchy-Riemann equations. In particular $\frac{\partial}{\partial x}\text{Im}(f-g) = -\frac{\partial}{\partial y}\text{Re}(f-g) = 0$. By computing the derivative,

$$\frac{d}{dz}(f-g) = \frac{\partial}{\partial x}\text{Re}(f-g) + i\frac{\partial}{\partial x}\text{Im}(f-g) = 0 - i\frac{\partial}{\partial y}\text{Re}(f-g) = 0, \tag{29}$$

we see that $f-g$ is constant. □

Theorem 3.13 (Poisson Integral Formula)**.** *Let f be an analytic function on a region G containing the closed unit disk $\overline{\mathbb{D}}$ and let $w \in \mathbb{D}$. Now f is given by the following integral formula*

$$f(w) = i\,\mathrm{Im}(f(0)) + \frac{1}{2\pi}\int_{[0,2\pi)} \mathrm{Re}(f(e^{it}))\frac{e^{it}+w}{e^{it}-w}dt. \tag{30}$$

Proof. We let γ denote the unit circle in $\mathbb{C}$. Since $\frac{1}{\overline{w}} \notin \mathbb{D}$ we have by the Cauchy integral formula

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz \tag{31}$$

and

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-\frac{1}{\overline{w}}}dz \tag{32}$$

and by adding the two equations we get

$$f(w) = \frac{1}{2\pi i}\int_\gamma f(z)\left(\frac{1}{z-w} - \frac{1}{z-\frac{1}{\overline{w}}}\right)dz. \tag{33}$$

Let $\gamma(t) = e^{it}$ for $t \in [0, 2\pi)$ be a parameterization of the unit circle. Substituting $z = e^{it}$ in the equation above and rewriting the integrand we get

$$f(w) = \frac{1}{2\pi}\int_{[0,2\pi)} f(e^{it})\frac{1-|w|^2}{|1-we^{-it}|^2}dt. \tag{34}$$

Let $h : G \to \mathbb{C}$ be defined by

$$h(w) = \frac{1}{2\pi}\int_{[0,2\pi)} \mathrm{Re}(f(e^{it}))\frac{e^{it}+w}{e^{it}-w}dt \tag{35}$$

and note that h is analytic. Applying Lemma 3.11 to the right hand side of (35) we see that f and h have identical real part, so by Lemma 3.12 f and h differ by at most a purely imaginary constant, say ξ. Thus we can write

$$f(w) = \xi + \frac{1}{2\pi}\int_{[0,2\pi)} \mathrm{Re}(f(e^{it}))\frac{e^{it}+w}{e^{it}-w}dt. \tag{36}$$

Setting $w = 0$ and subtracting ξ from both sides we get

$$f(0) - \xi = \frac{1}{2\pi}\int_{[0,2\pi)} \mathrm{Re}(f(e^{it}))dt, \tag{37}$$

and since the right hand side is real we see that $\xi = i\,\mathrm{Im}(f(0))$. In conclusion we have the formula

$$f(w) = i\,\mathrm{Im}(f(0)) + \frac{1}{2\pi}\int_{[0,2\pi)} \mathrm{Re}(f(e^{it}))\frac{e^{it}+w}{e^{it}-w}dt \tag{38}$$

as desired. □

3.4 Representation Theorem

In this section we prove the classic representation theorem by Nevanlinna. Proving such a well known theorem can surely be done in many ways, and here the details is carried out from the viewpoint of the author, for whom much of the background comes from the courses "Advanced real analysis I + II" given at mathematics department at Stockholm University.

Theorem 3.14. *A function f is Herglotz-Nevanlinna if and only if there exist unique constants $\alpha \in \mathbb{R}$, $\beta \geq 0$ and a unique finite positive Borel measure ν on $\mathbb{R}$ such that*

$$f(z) = \alpha + \beta z + \int_{\mathbb{R}} \frac{sz+1}{s-z} d\nu(s) \tag{39}$$

for each $z \in \mathbb{C}^+$.

In order to prove the statement above, we first consider a closely related class of function.

Definition 3.15. An analytic function $h : \mathbb{D} \to \mathbb{C}$ that satisfy $\text{Re}(h(w)) \geq 0$ for all $w \in \mathbb{D}$ is called a **Carathéodory function**.

Remark. The two classes of functions are related as follows. We note that for $z \in \mathbb{C}^+$ the mapping $z \mapsto \frac{z-i}{z+i}$ is an analytic bijection from $\mathbb{C}^+$ to $\mathbb{D}$ and that for $\zeta \in \{\xi \in \mathbb{C} : \text{Re}(\xi) > 0\}$ the mapping $\zeta \mapsto i\zeta$ is an analytic bijection from $\{\xi \in \mathbb{C} : \text{Re}(\xi) > 0\}$ to $\mathbb{C}^+$. Putting this together we conclude that a function f is a Herglotz-Nevanlinna function if an only if it can be written as

$$f(z) = ih(w) \tag{40}$$

where h is a uniquely determined Carathéodory function, $w = \frac{z-i}{z+i}$ and $z \in \mathbb{C}^+$.

Proposition 3.16. *Any Carathéodory function can be written*

$$h(w) = i\,\text{Im}(h(0)) + \int_{[0,2\pi]} \frac{e^{it}+w}{e^{it}-w} d\mu \tag{41}$$

where $\mu \in \mathcal{D}_A[0, 2\pi]$.

Proof. Define for $0 < r < 1$ the family of functions $\{h_r\}_r$ by $h_r(w) = h(rw)$. Each function h_r is analytic on a region containing the closed unit disk $\overline{\mathbb{D}}$, so Theorem 3.13 may be applied to write

$$\begin{aligned} h_r(w) &= i\,\text{Im}(h(0)) + \frac{1}{2\pi} \int_{[0,2\pi)} \text{Re}(h(re^{it})) \frac{e^{it}+w}{e^{it}-w} dt \\ &= i\,\text{Im}(h(0)) + \frac{1}{2\pi} \int_{[0,2\pi]} \chi_{[0,2\pi)} \text{Re}(h(re^{it})) \frac{e^{it}+w}{e^{it}-w} dt \end{aligned} \tag{42}$$

for $|w| < 1$. Let μ_r be the measure defined by

$$\mu_r(E) = \int_E \chi_{[0,2\pi)} \text{Re}(h(re^{it})) dt \tag{43}$$

for each measurable set $E \in \mathcal{B}[0, 2\pi]$. It is easy to verify that μ_r is a measure and since $\text{Re}(h) > 0$, μ_r is positive. Moreover, by Theorem 3.8 we have that μ_r is regular. Evaluating the expression in (43) with E replaced by $[0, 2\pi]$ we see that $\mu_r([0, 2\pi]) = \text{Re}(h(0))$ holds independent of r, so $\mu_r \in \mathcal{D}_{\text{Re}(h(0))}[0, 2\pi]$ for all r. Define the sequence $\{r_n\}_n$ by $r_n = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$. The sequence of measures μ_{r_n} is fully contained in the weak*-compact set $\mathcal{D}_{\text{Re}(h(0))}[0, 2\pi]$, so by Theorem 3.9 there exist a subsequence $\{\mu_{r_{n_k}}\}_k$, $k \in \mathbb{N}$ and a measure $\mu' \in \mathcal{D}_{\text{Re}(h(0))}[0, 2\pi]$ such that $\mu_{r_{n_k}} \xrightarrow{*} \mu'$. In particular, since each of the functions $g_w(t) := \frac{e^{it}+w}{e^{it}-w}$ belong to $\mathcal{C}_c[0, 2\pi]$, we get that

$$\begin{aligned} h(w) = \lim_{k\to\infty} h_{r_{n_k}}(w) &= i\,\text{Im}(h(0)) + \lim_{k\to\infty} \frac{1}{2\pi} \int_{[0,2\pi]} \frac{e^{it}+w}{e^{it}-w} d\mu_{r_{n_k}}(t) \\ &= i\,\text{Im}(h(0)) + \frac{1}{2\pi} \int_{[0,2\pi]} \frac{e^{it}+w}{e^{it}-w} d\mu'(t). \end{aligned} \tag{44}$$

Setting $\mu = \frac{1}{2\pi}\mu'$ we get that $\mu \in \mathcal{D}_A$ where $A = \frac{\text{Re}(h(0))}{2\pi}$, as well as the desired formula,

$$h(w) = i\,\text{Im}(h(0)) + \int_{[0,2\pi]} \frac{e^{it}+w}{e^{it}-w} d\mu, \tag{45}$$

which concludes the proof. □

We now turn to the proof of the main theorem.

Proof of Theorem 3.14. ($\Longrightarrow$) We first show that any expression on the form of the right hand side of (39) is a Herglotz-Nevanlinna function. To show that the right hand side have positive imaginary part, it suffices that the integral expression have positive imaginary part, this since $\beta \geq 0$ and $\text{Im}\, z > 0$. However ν is a positive measure, so it is enough to show that the imaginary part of the integrand is positive. Writing $z = a + bi$, the integrand can be rewritten

$$\begin{aligned} \frac{tz+1}{t-a-bi} = \frac{(tz+1)(t-\overline{z})}{(t-a)^2+b^2} &= \frac{t^2(a+bi)+t-t|z|^2-a+bi}{(t-a)^2+b^2} \\ &= \frac{t^2a+t-t|z|^2-a}{(t-a)^2+b} + i\frac{t^2b+b}{(t-a)^2+b} \end{aligned} \tag{46}$$

where the last expression has positive imaginary part. To show that the expression on the right hand side of (39) is analytic it suffices to show that the integral part is analytic. For some fixed $z_0 \in \mathbb{C}^+$ consider the quotient

$$\frac{1}{z_0-z}\left(\int_{\mathbb{R}} \frac{tz_0+1}{t-z_0} d\nu(t) - \int_{\mathbb{R}} \frac{tz+1}{t-z} d\nu(t) \right) \tag{47}$$

where adding the integrals and simplifying the integrand gives

$$\frac{1}{z_0-z} \int_{\mathbb{R}} \frac{z_0t - zt + z_0 - z}{(t-z_0)(t-z)} d\nu(t) = \int_{\mathbb{R}} \frac{t+1}{(t-z_0)(t-z)} d\nu(t). \tag{48}$$

If we assume that $0 < |z_0 - z| < \frac{\mathrm{Im}(z_0)}{2}$ we get the following estimate for the integrand,

$$\begin{aligned}\left|\frac{t+1}{(t-z_0)(t-z)}\right| &\leq \frac{1}{|t-z|}\max_{t\in\mathbb{R}}\left(\left|\frac{t+1}{t-z_0}\right|\right) \leq \frac{1}{\mathrm{Im}(z)}\max_{t\in\mathbb{R}}\left(\left|\frac{t+1}{t-z_0}\right|\right)\\ &\leq \frac{2}{\mathrm{Im}(z_0)}\max_{t\in\mathbb{R}}\left(\left|\frac{t+1}{t-z_0}\right|\right),\end{aligned} \tag{49}$$

where the maximum above exist since $\mathrm{Im}(z_0) > 0$, the function $t \mapsto \left|\frac{t+1}{t-z_0}\right|$ is continuous and $\lim_{t\to\pm\infty}\left|\frac{t+1}{t-z_0}\right| = 1$. Thus when taking the limit as $z \to z_0$ in the last expression of (48), the Lebesgue dominated convergence theorem is applicable and we get

$$\lim_{z\to z_0}\int_{\mathbb{R}}\frac{t+1}{(t-z_0)(t-z)}d\nu(t) = \int_{\mathbb{R}}\frac{t+1}{(t-z_0)^2}d\nu(t). \tag{50}$$

The last expression above is finite since the integrand is bounded and ν is finite. For uniqueness it is quite clear. Namely that changing α, β or ν will yield a different (Herglotz-Nevanlinna) function.

($\Longleftarrow$) Conversely, let f be a Herglotz-Nevanlinna function. By the remark following Definition 3.15, f can be written as

$$f(z) = ih(w) \tag{51}$$

where h is a Carathéodory function, $w = \frac{z-i}{z+i}$ and $z \in \mathbb{C}^+$. Since h is a Carathéodory function, we can apply Proposition 3.16 to write

$$\begin{aligned}f(z) &= i\left(i\,\mathrm{Im}(h(0)) + \int_{[0,2\pi]}\frac{e^{it}+w}{e^{it}-w}d\mu(t)\right)\\ &= \mathrm{Re}(f(i)) + i\left(\mu(\{0\}) + \mu(\{2\pi\})\right)\frac{1+w}{1-w} + i\int_{(0,2\pi)}\frac{e^{it}+w}{e^{it}-w}\,d\mu(t).\end{aligned} \tag{52}$$

Setting $\alpha = \mathrm{Re}(f(i))$, $\beta = \mu(\{0\}) + \mu(\{2\pi\})$ and noting that $z = i\frac{1+w}{1-w}$ we get

$$f(z) = \alpha + \beta z + i\int_{(0,2\pi)}\frac{e^{it}+w}{e^{it}-w}\,d\mu(t). \tag{53}$$

Secondly, let $\varphi : \mathbb{R} \to \mathbb{D}$ be defined by $\varphi(s) = 2\arctan\left(-\frac{1}{s}\right)$ and define the measure $\nu = \mu \circ \varphi$. Note that φ is a homeomorphism, so by Corollary 3.6 we have that $\nu \in \mathcal{D}_{\frac{\mathrm{Im}(f(i))}{2\pi}}(\mathbb{R})$. Since φ is bijective we have that $\mu = \nu \circ \varphi^{-1}$, and applying Proposition 2.18 to the integral in (53) gives

$$i\int_{(0,2\pi)}\frac{e^{it}+w}{e^{it}-w}\,d(\nu\circ\varphi^{-1})(t) = i\int_{\mathbb{R}}\frac{e^{i\varphi(s)}+w}{e^{i\varphi(s)}-w}\,d\nu(s). \tag{54}$$

Finally, by a tedious but standard computation one can show the equality $\frac{e^{i\varphi(s)}+w}{e^{i\varphi(s)}-w} = \frac{1}{i}\frac{sz+1}{s-z}$, and thus we arrive at the formula

$$f(z) = \alpha + \beta z + \int_{\mathbb{R}} \frac{sz+1}{s-z}\, d\nu(s) \tag{55}$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $\nu \in \mathcal{D}_A(\mathbb{R})$ for $A = \frac{\operatorname{Im}(f(i))}{2\pi}$, a finite positive Borel measure. □

4 Approximation by rational Herglotz-Nevanlinna functions

4.1 Extremal measures of Herglotz-Nevanlinna functions

In this section we show that the extreme points of $\mathcal{D}_A[0, 2\pi]$ is precisely the point mass measures on $[0, 2\pi]$. Without loss of generality, we may assume that $A = 1$ for this section.

Lemma 4.1. *Let $\mu \in \mathcal{D}_1[0, 2\pi]$ and suppose μ is not a point mass measure, then there exist a set $E \in \mathcal{B}[0, 2\pi]$ with $0 < \mu(E) < 1$.*

Proof. Suppose not, namely that for each set $E \in \mathcal{B}(0, 2\pi)$ it holds that $\mu(E) = 1$ or $\mu(E) = 0$. Since μ is not a point mass measure, $\mu(\{x\}) = 0$ holds for each $x \in [0, 2\pi]$. For for $n \in \mathbb{N}$ and $0 \leq k \leq n$ let $I_n^k = (\pi\frac{k}{n}, \pi\frac{k+1}{n})$. Note that for each n only one of the intervals, say $I_n^{k^*}$, have $\mu(I_n^{k^*}) = 1$ and that $\mu((I_n^{k^*})^c) = 0$. Moreover, $I_n^{k^*} \supset I_{n+1}^{k^*}$ holds for each n, and since $\operatorname{diam}(I_n^{k^*}) \to 0$ as $n \to \infty$, only one point, say x^*, belong to the set $\cap_n I_n^{k^*}$. Now the computation

$$0 = \mu(\{x^*\}) = \mu(\cap_n I_n^{k^*}) = \lim_{n\to\infty} \mu(I_n^{k^*}) = \lim_{n\to\infty} 1 = 1 \tag{56}$$

demonstrates the desired contradiction. □

Proposition 4.2. *A measure μ is an extreme point of $\mathcal{D}_1[0, 2\pi]$ if and only if μ is a point mass measure, that is $\mu = \delta_{x_0}$ for some $x_0 \in [0, 2\pi]$.*

Proof. First suppose μ is a point mass measure, $\mu = \delta_{x_0}$, for some $x_0 \in [0, 2\pi]$. For contradiction suppose μ can be written as a non trivial convex combinations of measures $\mu_1, \mu_2 \in \mathcal{D}_1[0, 2\pi]$. That is

$$\delta_{x_0} = c_1\mu_1 + c_2\mu_2 \tag{57}$$

where $c_1, c_2 > 0$ and $c_1 + c_2 = 1$. However the computation

$$0 = \delta_{x_0}\Big([0, 2\pi] \setminus \{x_0\}\Big) = c_1\mu_1\Big([0, 2\pi] \setminus \{x_0\}\Big) + c_2\mu_2\Big([0, 2\pi] \setminus \{x_0\}\Big) \tag{58}$$

shows that

$$\mu_1\Big([0, 2\pi] \setminus \{x_0\}\Big) = \mu_2\Big([0, 2\pi] \setminus \{x_0\}\Big) = 0, \tag{59}$$

implying that

$$\mu_1\big(\{x_0\}\big) = \mu_2\big(\{x_0\}\big) = 1, \tag{60}$$

and thus we have that $\mu_1 = \mu_2 = \delta_{x_0}$, a contradiction. Conversely, suppose that μ is an extreme point of $\mathcal{D}_1[0, 2\pi]$, and suppose for contradiction that μ is not a point mass measure. By Lemma 4.1 there exist a set $E \in \mathcal{B}[0, 2\pi]$ such that $0 < \mu(E) < 1$. Define the measures μ_1 and μ_2 by $\mu_1(C) = \mu(C \cap E)$ and $\mu_2(C) = \mu(C \cap E^c)$ for each $C \in \mathcal{B}[0, 2\pi]$. Now μ is given by

$$\mu = \frac{1}{\mu(E)}\mu_1 + \frac{1}{\mu(E^c)}\mu_2, \tag{61}$$

a non-trivial convex combination of measures $\mu_1, \mu_2 \in \mathcal{D}_1[0, 2\pi]$, a contradiction. □

4.2 Approximation by rational functions

In this section we present the second main theorem of this thesis. That is, any Herglotz-Nevanlinna function f can be pointwise approximated by sequence of rational Herglotz-Nevanlinna functions $\{q_k\}_k$, all with a finite number of poles of order at most one, such that $q_k(z) \to f(z)$ for all $z \in \mathbb{C}^+$. By virtue of conclusions made in Section 3.1, it is quite simple to infer that any such rational function is Herglotz-Nevanlinna. The main proof is carried out in a similar way to the proof of Theorem 3.14, utilizing many of the same techniques, as well as the content of Theorem 3.14 itself.

Definition 4.3. Denote by $\mathcal{Q}$ the set of functions q on $\mathbb{C}^+$ on the form

$$q(z) = \alpha + \beta z + \sum_{n=1}^{N} d_n \frac{1}{t_0 - z} \tag{62}$$

where $\alpha, t_0 \in \mathbb{R}$, $\beta, d_n > 0$ and $N \in \mathbb{N}$.

The set Q do indeed consist entirely of Herglotz-Nevanlinna functions. To see this, one can combine the examples of section 3.1, Proposition 3.2 to note that each term of a function on the form as in equations (62) is Herglotz-Nevanlinna. Moreover it is easy to see that a sum of Herglotz-Nevanlinna functions is again a Herglotz-Nevanlinna function.

Theorem 4.4. *Let f be a Herglotz-Nevanlinna function. There exist a sequence $\{q_k\}_k$ contained in $\mathcal{Q}$ such that $q_k(z) \to f(z)$ for each $z \in \mathbb{C}^+$.*

Proof. We write f as in equation (52), that is

$$f(z) = \mathrm{Re}(f(i)) + i \int_{[0,2\pi]} \frac{e^{it} + w}{e^{it} - w} d\mu(t) \tag{63}$$

where again $\mu \in \mathcal{D}_{\frac{\mathrm{Im}(f(i))}{2\pi}}[0, 2\pi]$, $w = \frac{z-i}{z+i}$ and $z \in \mathbb{C}^+$. We abbreviate $A = \frac{\mathrm{Im}(f(i))}{2\pi}$ and let $E_{\mathcal{D}_A[0,2\pi]}$ denote the set of extreme points of $\mathcal{D}_A[0, 2\pi]$. By

Proposition 4.2, $E_{\mathcal{D}_A[0,2\pi]}$ consist entirely of point mass measures. Consider $\mathcal{D}_A[0,2\pi] \subset \mathcal{M}[0,2\pi]$, both endowed with the weak* topology. We have seen by Proposition 2.35 that $\mathcal{M}[0,2\pi]$ is locally convex, and by Propositions 3.10 and 3.9 that $\mathcal{D}_A[0,2\pi]$ is convex and compact, thus by Theorem 2.25 we have that $\overline{\mathrm{Conv}(E_{\mathcal{D}_A[0,2\pi]})} = \mathcal{D}_A[0,2\pi]$. That is, there exist a sequence of measures on the form

$$\mu_k = \sum_{n=1}^{N_k} c_n^k \delta_{t_n^k} \tag{64}$$

where each $t_n^k \in [0,2\pi]$, $c_n^k > 0$, $\sum_{n=1}^{N_k} c_n^k = 1$ and $N_k \in \mathbb{N}$, such that $\mu_k \xrightarrow{*} \mu$. We define the functions $q_k : \mathbb{C}^+ \to \mathbb{C}^+$ by

$$\begin{aligned} q_k(z) &= \mathrm{Re}(f(i)) + i \int_{[0,2\pi]} \frac{e^{it} + w}{e^{it} - w} d\mu_k(t) \\ &= \mathrm{Re}(f(i)) + i\left(\mu_k(\{0\}) + \mu_k(\{2\pi\})\right) \frac{1+w}{1-w} + i \int_{(0,2\pi)} \frac{e^{it}+w}{e^{it}-w}\, d\mu_k(t). \end{aligned} \tag{65}$$

Set $\beta_k = \mu_k(\{0\}) + \mu_k(\{2\pi\})$. We use the same variable substitutions as in the proof of Theorem 3.14, that is let φ be defined by $\varphi(s) = 2\arctan\left(-\frac{1}{s}\right)$ and $\nu_k = \mu_k \circ \varphi$. This yields the formula

$$q_k(z) = \mathrm{Re}(f(i)) + \beta_k z + \int_{\mathbb{R}} \frac{sz+1}{s-z}\, d\nu_k(s). \tag{66}$$

However, Corollary 3.7 allows us to calculate ν_k, and in turn the whole integral in equation (66), explicitly. We get

$$\nu_k = \mu_k \circ \varphi = \left(\sum_{n=1}^{N_k} c_n^k \delta_{t_n^k}\right) \circ \varphi = \sum_{n=1}^{N_k} c_n^k (\delta_{t_n^k} \circ \varphi) = \sum_{n=1}^{N_k} c_n^k \delta_{s_n^k} \tag{67}$$

where we define each $s_n^k = \varphi^{-1}(t_n^k)$. Applying this to the integral in equation (66) we get

$$\begin{aligned} q_k(z) &= \mathrm{Re}(f(i)) + \beta_k z + \sum_{n=1}^{N_k} c_n^k \frac{s_n^k z + 1}{s_n^k - z} \\ &= \mathrm{Re}(f(i)) - \sum_{n=1}^{N_k} c_n^k s_n^k + \beta_k z + \sum_{n=1}^{N_k} c_n^k \frac{(s_n^k)^2 + 1}{s_n^k - z} \end{aligned} \tag{68}$$

and setting $\alpha_k = \mathrm{Re}(f(i)) - \sum_{n=1}^{N_k} c_n^k s_n^k$ and $d_n^k = c_n^k((s_n^k)^2 + 1)$ we get

$$q_k(z) = \alpha_k + \beta_k z + \sum_{n=1}^{N_k} d_n^k \frac{1}{s_n^k - z}. \tag{69}$$

Since the coefficients β_k and d_n^k are non-negative we for all k and n, we get that $q_k \in \mathcal{Q}$ for each k, which concludes the proof. □

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