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Stochastics and Its Application in Merton's Problem

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Abstract

In this thesis, we aim to present key concepts in stochastic processes, Itô's calculus, and stochastic differential equations. We will give a brief overview of how stochastic differential equations can be used in stochastic dynamic control problems and how to find explicit solutions to such optimization problems. Our main focus will be to provide explicit solutions for the optimal consumption and investment rules in the case when the risk aversion is constant. Lastly, we will also provide concise economic interpretations and implications of such optimal consumption and investment rules.

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Introduction

This thesis has the goal of providing a mathematical background in stochastic processes, as well as differential equations and dynamic programming using such processes, which will then be used to find the optimal investment and consumption rules for an investor. In Section 3.1 we will provide a comprehensive definition on the probability space where stochastic processes are constructed. Section 3.2 will focus on stochastic processes, the most important for this dissertation being a Brownian motion. Section 3.3 will introduce the Itô's integral, which will be fundamental to the construction of stochastic differential equations, the focus of Section 3.4. Finally, we end the mathematical background in Section 3.5 with the introduction to stochastic dynamic control.

Moreover, this paper will also focus on describing how such mathematical concepts can be applied in financial economics, and how one is able to find explicit solutions to an investor's optimal consumption and investments. Such problems were first introduced by [Merton, 1969], in his paper *Lifetime Portfolio Selection under Uncertainty: The Continuous time Case*, where a model in which stochastic dynamic control is used to determine the individuals optimal investment and consumption levels over their lifetime. This paper is fundamental for financial economics, as it was the first to describe the investors optimal choices in such a way. The proposed model incorporates one risk-free asset and one risky asset, the behavior of which is described using a Brownian motion. Moreover, the paper also provides insight on how the optimal portfolio selection is affected by the investor's time horizon, and their risk preferences when described by constant absolute risk aversion functions. It is important to note that Merton's paper includes several assumptions, such as the behavior of interest rate, specific characteristics of the utility function and risk aversion, and the assumption of a frictionless market. By relaxing these assumptions and slightly reformulating the problem, plenty of new research in the area has led to rich insights into the optimal investment and consumption rules for investors.

One of the first extensions to this Merton's problem was made by Merton himself in [Merton, 1971]. In this paper *Optimal Consumption and Portfolio Rules in a Continuous-Time Model*, Merton builds upon the optimal consumption and investment rules, as he firstly examines a case with n number of assets, then provides a more comprehensive result for a broader type of utility function, different price behavior functions as well as it accounts for non-capital gains sources. One of the key concepts presented in this paper is also the "Mutual Fund Theorem", which, succinctly speaking, states that a pair of funds can be constructed using a linear combination of all the n assets in such a way that investors are indifferent between investing in these funds or all n assets. Lastly, this paper also provides a more comprehensive understanding of the use of Itô's Processes and their application in modelling stock prices.

[Davis and Norman, 1990] reformulate Merton's problem, however with a more realistic approach, in which there are transaction costs for investing in the risk-free asset and the stock. In short, [Davis and Norman, 1990] find that the investments in the risk-free and the risky asset must be treated as two different processes and cannot be merged into a single wealth process. Moreover, another main finding is that on a plot depicting the holdings in each asset, the optimal investment strategy is given by a line. Surrounding such line there is a wedge like shaped region, where the investor should have a minimal trading strategy, while in the region above and below the line the investor should purchase and sell the stock, respectively.

[Korn and Kraft, 2002] on the other hand, provide another interesting expansion to Merton's Problem. By assuming that the interest rate follows an Itô process, the stochastic control problem becomes much more delicate, yet by using two different models for the short rate, it is shown that the investment in a stock, and in a risk-free asset is similar to Merton's original problem. The main finding of this paper is that there is a stochastic correction term for the amount invested in the risky asset, which depends on the short rate model used.

[Steffensen, 2011] provides another interesting extension to the original paper, as the assumption of a risk aversion coefficient is reformulated for the risk aversion being a function of time. That is, the risk aversion behavior of the investor changes in relation to their age. The main findings is that, for a monotonically increasing risk aversion, the investor's optimal consumption rate displays a convex function of wealth. When it comes to the optimal investment rate, the results are more complex due to the influence of higher order terms.

[van Bilsen et al., 2020] expand on Merton's problem by extending the idea of utility gained by the individual from the consumption by introducing a habit formation element to the utility. Despite there not being an analytical solution to multiplicative habit formation, by linearizing the budget equation they are able to approximate it to portfolio problems without the habit formation. Then, this approximation is applied in three different cases, when the investor has time-separable relative consumption and constant relative risk aversion; when there is a stochastic interest rate; as well as combing the habit formation with a stochastic differential utility. The main findings of the three models are following: the risk aversion dictates the yearly change in consumption, while the habit formation dictates the reaction of consumption on stock shocks; the investment in bonds decline over time and the hedging bond portfolio investment over time has a hump shaped curve; and median consumption does not grow at unrealistic rates, respectively.

In Section 4 we will formally pose the original problem as formulated by [Björk, 2020], and find explicit solutions to the optimal consumption and investment rules for an investor that has constant risk aversion and that has the possibility of choosing between 2 assets, one of which is risky and the other risk-free.

Lastly, in section 5 we will provide a concise interpretation of the optimal investment and consumption rules found in the previous section.

3

Mathematical Background

This part of the essay is dedicated to understanding key mathematical concepts which are going to be applied in a financial economic scenario in the later parts of this thesis. It will be heavily based on the book *Stochastic Differential Equations: An introduction with Applications*, by [Øksendal, 2000]. We will first define the probability space and stochastic processes, then understand why Itô's Integral is essential to stochastic calculus. By having a better understanding of Itô's calculus, we will then also introduce the idea of stochastic differential equations, and how they can be used in stochastic dynamic control. All definitions in this section are from [Øksendal, 2000] unless stated otherwise, while examples are provided by the author.

3.1 Preliminaries

We will begin by providing some key concept and building blocks for the construction of stochastic processes, such as random variables, the measurable and probability space, which are defined as follow:

Definition 3.1. We say a σ-algebra of $\mathcal{F}$ on a given set Ω if $\mathcal{F}$ is a family of subsets of Ω (also referred as sample space) which have the following properties:

(i) The empty set $\varnothing$ belongs to $\mathcal{F}$;

(ii) For an element F in $\mathcal{F}$, its complement F^C also belongs in $\mathcal{F}$;

(iii) For the elements $A_1, A_2, \cdots \in \mathcal{F}$, their countable union A is also in $\mathcal{F}$ (i.e. $A \in \mathcal{F}$, $A := \bigcup\limits_{i=1}^{\infty} A_i$.

The pair $(\Omega, \mathcal{F})$ is also usually referred to as measurable space.

Definition 3.2. To define a probability space we need to include a probability measure, P, to such σ-algebra of $\mathcal{F}$ on Ω, as a function $P : \mathcal{F} \to [0, 1]$ such that:

(i) The probability of the empty set is equal to zero ($P(\emptyset) = 0$), and the probability of the whole event space is 1 ($P(\Omega) = 1$),

(ii) For the elements $A_1, A_2, \cdots \in \mathcal{F}$ and a disjoint sequence $\{A_i\}_{i=1}^{\infty}$,the probability of union of such elements is equal to the sum of each probability separately. In other words $P(\bigcup\limits_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Finally, this triple $(\Omega, \mathcal{F}, P)$ is called a probability space and is the environment we wish to construct our stochastic processes on.

Intuitively, this triple $(\Omega, \mathcal{F}, P)$ can be seen as the non-empty sample space Ω, the event space $\mathcal{F}$ and the probability measure P. The probability measure assigns a value which we interpret as the probability of such σ - algebra event F happening.

Example 3.3. To illustrate the triple $(\Omega, \mathcal{F}, P)$ we will use the simple example of determining the probability of getting a number larger than 3 when a unbiased 6-sided dice rolled.

We have that Ω are all possible outcomes, which clearly are 1,2,3,4,5 or 6, in other words $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Since we have three possible outcomes which are larger than three, i.e. 4, 5 or 6, our event space is therefore comprised of $F = \{4, 5, 6\}$.

Lastly, since the dice is unbiased, we have that each outcome has equal probability 1/6. Since our event space comprises of three mutual exclusive events we add up their individual probabilities. Therefore, we can say that $P(\{F\}) = 1/2$.

Definition 3.4. All subsets F of Ω that belong to $\mathcal{F}$ are called $\mathcal{F}$ - measurable sets.

We also note that functions can also be $\mathcal{F}$- measurable. More precisely, we call functions $Y : \Omega \to \mathbf{R}^n$ $\mathcal{F}$- measurable if $Y^{-1}(U) := \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F}$, for all open sets $U \in \mathbf{R}^n$. We are now thus able to define random variables as follows:

Definition 3.5. Using our $(\Omega, \mathcal{F}, P)$ probability space, we say that a random variable X is a $\mathcal{F}$ - measurable function $X : \Omega \to \mathbf{R}^n$.

For more details we refer to page 9 of [Øksendal, 2000].

Another fundamental concept we need to define is the idea of a filtration. Intuitively, a filtration is the history or the development of information up to time t, that is the increasing σ-algrebra families up to time t. We formally define it as:

Definition 3.6. We define a filtration on the measurable space $(\Omega, \mathcal{F})$ as a family $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ of the σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ that satisfy the property: $0 \leq s < \mathrm{t} \Rightarrow \mathcal{F}_s \subset \mathcal{F}_t$.

For more details about filtrations we refer to pages 25 and 31 of [Øksendal, 2000].

3.2 Stochastic Process

In general terms a Brownian motion is a type of Stochastic process, the latter of which is defined as follows.

Definition 3.7. We define a stochastic process as a parameterized collection of random variables $\{X_t\}_{t \in T}$ on a probability space $(\Omega, \mathcal{F}, P)$, with values in $\mathbf{R}^n$.

The historical origins of the Brownian motion arise from Robert Brown, who studied the movement of pollen particles suspended in liquids. Simply put, a Brownian motion is a stochastic process $\mathrm{B_t}(\omega)$, which interpreted in Brown's application, describes the position at time t of the pollen particle ω. The most common application of Brownian motion we are going to delve into is the price development over time of a given stock, which we do not know the behavior in the future. For the proof of existence and mathematical construction of Brownian motions we refer to pages 10 to 12 on [Øksendal, 2000]. We then have that a stochastic process B_t is a Brownian motion if it has the following properties:

(i) B_t is a Gaussian process, which means that each increment is has a normal distribution with mean equal 0 and variance equal to the length of the increment,

(ii) B_t has independent increments,

(iii) B_t is continuous.

For a proof that a process having these characteristics exists, we refer to page 12 as well as appendix A in [Øksendal, 2000].

Remark 3.8. We note that Brownian motions can be processes in many dimensions, that is, the increments have a multi-normal distribution. For our purposes we will assume a one dimensional process.

Definition 3.9. A geometric Brownian motion (GBM) is a cornerstone for stochastic prices in the financial market. A geometric Brownian motion is a Brownian motion that has the following characteristic: $X_t = X_0 \cdot \mathrm{e}^{(\alpha t + \beta B_t)}$, where α and β are constants.

We refer to page 62 of Øksendal (2000), for more details on geometric Brownian motions.

Remark 3.10. As the field of stochastics advances, so does its applications in other areas of studies. For instance, there have been examples of Brownian motions being used to describe climate change, or more specifically, the change in average temperature over time.

We will use the interpretation that a random variable is $\mathcal{F}_t$-measurable if its values can be decided from the values of a filtration up to a certain time t. For more details we refer to page 25 of [Øksendal, 2000].

An important property of some stochastic processes is that they can be adapted, in other words, the process takes in all information available from a $\mathcal{F}_t$ - measurable function, and incorporates this in the value of the process. More formally we have:

Definition 3.11. A process $X_t(\omega) : [0, \infty) \times \Omega \to \mathbf{R}^n$ is called $\mathcal{F}_t$ - adapted for each $t \geq 0$ if the function:

$$\omega \to X_t(\omega)$$

is $\mathcal{F}_t$ - Measurable, where $\{\mathcal{F}_t\}_{t \geq 0}$ is a filtration.

For more details we refer to page 25 of [Øksendal, 2000].

We now are able to also introduce our last notion for stochastic processes of interest.

Definition 3.12. We define a martingale $\{M_t\}_{t \geq 0}$ to be $n-$dimensional stochastic process on a probability space with regards to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if it satisfies the three following conditions:

(i) For all t, the stochastic process is $\mathcal{F}_t$-measurable,

(ii) The process has finite expectation i.e. $E\left[|M_t|\right] \leq \infty$,

(iii) For all $s \geq t$ the conditional expectation of M_s given M_t is equal to M_t.

Intuitively, the martingale tells us that the expectation of future values, conditional on the present value and the information available at the current time, is equal to the present value. For more details we refer to page 31 of [Øksendal, 2000].

3.3 Itô's Calculus

As previously mentioned in the introduction of this chapter, we are trying to better describe the randomness of stochastic processes. Since we do not know the future behavior of such processes, we are not able to use common integration and differentiation techniques such as the Riemann integral. In short, the main issue we face is that stochastic processes, more specifically Brownian motions, are continuous but not smooth. Therefore, we need to define a new set of rules for integration to understand the behavior of stochastic processes. To do so we need to make several approximations and other assumptions, such as at which point we evaluate the value of the function when defining the integral. Choosing the left hand side to evaluate the value of the integral gives rise to the Itô integral, while choosing the midway point will give rise to the Stratonovich integral. For our purposes we focus on Itô's integral, as we will further have the assumption that in financial markets, we only have information up to the current time and not in the future, which is efficiently captured by Itô's integral.

To define Itô's Integral, we must first go over the class of processes for which they defined. For such we refer to pages 25 and 35 of [Øksendal, 2000].

Definition 3.13. We introduce the class of stochastic processes $\mathcal{V} = \mathcal{V}(S,T)$, where each $X_t(\omega) : [0,\infty) \times \Omega \to \mathbf{R}$ satisfies the following characteristics:

(i) $(t,\omega) \to X_t(\omega)$ is $\mathcal{F}$ -measurable in regards to all open subsets,

(ii) $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t$,

(iii) The expectation of the integral of the square function between two points $(S < T)$ in time is finite i.e. $E\left[\int_S^T X_t(\omega)^2 dt\right] < \infty$.

Moreover, we also introduce the concept of elementary functions, which separate the stochastic part and the time depending part of a process, and their integrals.

Definition 3.14. We define elementary processes $\phi \in \mathcal{V}$ processes as the following:

$$\phi(t,\omega) = \sum_j e_j(\omega) \cdot \chi_{[t_j,t_{j+1})}(t),$$

where χ is the indicator function for the interval $[t_j, t_{j+1})$ and $e_j(\omega)$ is a $\mathcal{F}_{t_j}$ - measurable process.

Definition 3.15. We define the integral for elementary processes as the following:

$$\int_S^T \phi(t,\omega)dB_t(\omega) = \sum_{j\geq 0} e_j(\omega)\left[B_{t_{j+1}} - B_{t_j}\right](\omega).$$

We refer to pages 23 to 26 of [Øksendal, 2000] for more details on elementary functions.

Since we have defined the elementary functions and their integral we are now able to define the Itô integral as follow:

Definition 3.16. Given the process $X_t \in \mathcal{V}(S,T)$, and a sequence of elementary processes $\{\phi_n\}$ such that the expectation of the integral $\int_S^T (X_t(\omega) - \phi_n(t,\omega))^2 \, dt$ tends to zero as $n \to \infty$, the Itô integral is defined as following:

$$\int_S^T X_t(\omega)dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)dB_t(\omega).$$

We note that the limit refers to the limit on the squared expectation, which implies that the sequence of elementary functions converges to a point, regardless of the presence of a stochastic element. For more details on this convergence we refer to Appendix A6 on [Björk, 2020]. For a detailed construction of such integrals we refer to pages 24 to 28 of [Øksendal, 2000]. In short, to construct these integrals we first break down a Brownian motion in smaller sub-intervals, then at each left-hand side of each interval we approximate this to a constant. The next step is to take the limit as the size of each interval approaches zero, and show that by the square square difference between the Brownian motion and the constant converge in expectation. We now introduce an important property of an Itô integral, the Itô isometry, which given by the following corollary:

Corollary 3.17. *For all processes X_t in $\mathcal{V}(S,T)$, the expectation of the squared Itô integral is equal to the integral of the squared integrand, formally:*

$$E\left[\left(\int_S^T X_t(\omega)dB_t\right)^2\right] = E\left[\int_S^T X_t^2(\omega)dt\right].$$

We refer to pages pages 24 to 28 of [Øksendal, 2000] for the proof of this corollary. Nevertheless, the Itô isometry is not the only fundamental characteristic of Itô integrals. The following 4 characteristics are also essential when considering stochastic equations:

Theorem 3.18. *For stochastic processes $X_t, Y_t \in \mathcal{V}(0,T)$ and a constant c under the assumption that $0 \leq S < U < T$, the following properties hold:*

(i) An Itô integral can be broken down into different parts i.e.

$$\int_S^T X_t dB_t = \int_S^U X_t dB_t + \int_U^T X_t dB_t.$$

(ii) An Itô integral of the sum of two functions can be broken down as:

$$\int_S^T (cX_t + Y_t) dB_t = c \cdot \int_S^T X_t dB_t + \int_S^T Y_t dB_t.$$

(iii) The expectation of an Itô integral is equal to zero i.e.

$$E\left[\int_S^T X_t dB_t\right] = 0.$$

(iv) the integral of a Brownian motion is $\mathcal{F}_T -$ measurable.

For the proof of these properties we refer to page 30 of [Øksendal, 2000].

We now wish to define Itô processes, however to do so we need to first relax some requirements of definition 3.13. Firstly, we loosen the requirement (ii) of definition 3.13 to account for the fact that the Brownian motion is a martingale w.r.t a filtration $\mathcal{F}_t$ and that the process X_t is $\mathcal{F}_t$-adapted, for $t \geq 0$. By doing so we let the process depend on more information than just what is provided by the Brownian motion, while the Brownian motion still accounts for the history of the process for $s \leq t$. Secondly, we relax requirement (iii) of definition 3.13 to $P\left[\int_S^T X_s(\omega)^2 ds < \infty\right] = 1$.We say that the class of processes that satisfy the requirements (ii) and (iii) as above and still satisfy requirement (i) from definition 3.13 is denoted by $\mathcal{W}_\mathcal{F}$. For more details see page 34 and 35 of [Øksendal, 2000].

We now define an Itô process as follow:

Definition 3.19. We define Itô process as a stochastic process X_t on the probability space $(\Omega, \mathcal{F}, P)$ that satisfies the following stochastic integral:

$$X_t = X_0 + \int_0^t u(s,\omega) ds + \int_0^t v(s,\omega) dB_s,$$

where B_t is a Brownian motion, $v \in \mathcal{W}_\mathcal{F}$, and u is $\mathcal{F}_t$ adapted, such that the following two characteristics are satisfied for all $t \geq 0$:

(i) The probability of v not diverging is equal to 1, or in other words,

$$P\left[\int_0^t v(s,\omega)^2 ds < \infty\right] = 1.$$

(ii) The probability of u being finite is equal to 1, or in other words,

$$P\left[\int_0^t |u(s,\omega)| ds < \infty\right] = 1$$

It is also common to write this process in the shorter differential form:

$$dX_t = udt + vdB_t.$$

For more details we refer to page 44 of [Øksendal, 2000].

The final concept regards Itô calculus we are going to present in this section is of the Itô formula. This formula gives us the idea that when transforming an Itô process through a function, we get a new Itô process. More formally, we describe this using the following theorem.

Theorem 3.20. *For an Itô process X_t and a function $g(t,x) \in C^2([0,\infty) \times R)$, we are able to create a new Itô process $Y_t = g(t, X_t)$, such that:*

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)\, dt + \frac{\partial g}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2 .$$

We also have the following rules:

1. $(dX_t)^2 = (dX_t) \cdot (dX_t)$,
2. $dt \cdot dt = dt \cdot dB_t = dB_t$,
3. $dt = 0$ *and*
4. $dB_t \cdot dB_t = dt$.

We refer to page 46 of [Øksendal, 2000] for a sketch of the proof. Nevertheless, it is highly important to note the perhaps counter intuitive behavior of this last theorem. It is commonly referred to as Itô's Lemma, and the its striking aspect is the quadratic variation, $dB_t \cdot dB_t = dt$. The proof of this is quite complex, however the intuition behind it is that the continuous yet non-differentiable behavior of a Brownian motion gives rise to the quadratic variation. For a heuristic explanation of the quadratic variation we refer to page 54 of [Björk, 2020].

Example 3.21. A simple example to illustrate the usefulness of the previous theorem is when the Itô process is a Brownian motion, (i.e. $dX_t = dB_t$), and the function $Y_t = g(t, X_t) = X_t^2$. We begin by taking the partial derivatives of the g with regards to t and B_t we get:

1. $\partial g/\partial t = 0$,
2. $\partial g/\partial X_t = 2B_t$ and
3. $\partial^2 g/\partial X_t^2 = 2$,

which we are able to substitute back in the equation for dY_t, which yields:

$$dY_t = 0\partial t + 2X_t \cdot dX_t + \frac{1}{2} \cdot 2(dX_t)^2.$$

Since our Itô process is given by $dX_t = dB_t$, we substitute this in the previous equation:

$$dY_t = 2B_t(dB_t) + (dB_t)^2.$$

We now substitute in the Brownian motion, and using the rules 1-4 in the previous theorem, which leads to:

$$dY_t = 2B_t dB_t + dt.$$

3.4 Stochastic Differential Equations

Differential equations is a common area of study within mathematics since it has many useful applications. One of simplest versions of a differential equation has some form:

$$y' = f(x, y(x)). \tag{1}$$

However, when introducing stochastic processes to differential equations we encounter one main issue. More specifically, due to the randomness of a stochastic process, we are not able to use deterministic rules of differentiation and integration to solve such equations.

However, as we have gone over Itô's calculus in the last section alongside its integration and differentiating rules, we can better describe this randomness as a solution to the stochastic equation:

$$X_t = X_0 + \int_0^t b\,(s, X_s)\, ds + \int_0^t \sigma\,(s, X_s)\, dB_s. \tag{2}$$

In this section we will mostly use the differential form of the previous equation, which is as follow:

$$dX_t = b\,(t, X_t)\,dt + \sigma\,(t, X_t)\,dB_t,$$

where B_t is a Brownian motion, and $X_t(\omega) : [0, \infty) \times \Omega \to \mathbf{R}^n$ is a stochastic process. As we can see, this differential equation is quite similar to (1), however we now have another term σ that accounts for the white noise in the process, and as a result the whole equation is aligned to what we have established in definition 3.19. Such differential equations involving stochastic processes are often referred to as stochastic differential equations. We note that to solve a stochastic differential equation we refer to the idea of finding stochastic processes on the right hand side of equation (2) so that the equality is satisfied.

Much like ordinary differential equations, there are situations in which stochastic differential equations have unique solutions. We introduce the following theorem to formalize such situations. For more details on the description and applications of stochastic differential equations we refer to page 61 of [Øksendal, 2000].

Theorem 3.22. *If the measurable functions $b(\cdot,\cdot) : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ and $\sigma(\cdot,\cdot) : [0,T] \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$, and the random variable Z which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by the Brownian motion $B_s(\cdot)$ satisfy the following three conditions, for $T > 0$:*

(i) $|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|)$ for a constant C and $|\sigma|^2 = \sum |\sigma_{i,j}|^2$,

(ii) $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x - y|$, for a constant D and where $x, y \in \mathbf{R}^n$ and $t \in [0,T]$,

(iii) Z has finite second order expectation (i.e. $E\left[|Z|^2\right] < \infty$).

Then the stochastic differential equation

$$dX_t = b\,(t, X_t)\,dt + \sigma\,(t, X_t)\,dB_t, \quad 0 \leq t \leq T, X_0 = Z,$$

has a unique t-continuous solution $X_t(\omega)$. Moreover, we note that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $B_s(\cdot)$; and has the following property: $E\left[\int_0^T |X_t|^2\,dt\right] < \infty$, for $s \leq t$.

For a detailed version of the proof, refer to [Øksendal, 2000], page 67.

The intuition is by having $|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|)$, we ensure that the process does not explode, or in other words does not become unbounded. Moreover, by having $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x - y|$, we ensure that the process has a continuous behavior, in other words, by fixing the time point the change in the differential equation is bounded by the scaled distance between the two points x and y. The fact that the process is adapted to a filtration arises from the fact that we have a unique solution of the equation, which means that at each time t the process takes in all information up to this time t. Lastly, by having $E\left[|Z|^2\right] < \infty$, we ensure that the process is not over volatile.

It is highly important to note that the previous theorem guarantees that a solution exists for b and σ that satisfy such conditions. This however does not mean that the solution is easy to find, or in other words we do not expect to have a closed form solution, instead we only know a solution exists.

Example 3.23. To illustrate the previous theorem we will try to find a solution to the following stochastic differential equation:

$$dX_t = \mu X_t dt + \sigma X_t dB_t,$$

where $X_0 = x$ and μ and σ are constants. We first rearrange the equation as follow:

$$\frac{dX_t}{X_t} = \mu dt + \sigma dB_t.$$

Which we interpret as:

$$\int_0^t \frac{dX_s}{X_s} = \mu t + \sigma B_t. \tag{3}$$

To find a solution to the stochastic differential equation it is enough to show that we are able to differentiate and integrate using Itô's calculus, and that the results match. We then use Itô's Lemma to differentiate the function:

$$g(t,x) = \ln x, x > 0.$$

This yields:

$$\begin{aligned} d\left(\ln X_t\right) &= \frac{\partial\left(\ln X_t\right)}{\partial X_t} dX_t + \frac{1}{2}\frac{\partial^2\left(\ln X_t\right)}{\partial X_t^2}\left(dX_t\right)^2 \\ &= \frac{1}{X_t} dX_t - \frac{1}{2}\left(\frac{1}{X_t^2}\right)\sigma^2 X_t^2 dt. \end{aligned}$$

Rearranging, we have:

$$\frac{dX_t}{X_t} = d\left(\ln X_t\right) + \frac{1}{2}\sigma^2 dt.$$

Which we now substitute in equation (3):

$$\int_0^t d(\ln X_t) + \frac{1}{2}\sigma^2 \int_0^t dt = \mu t + \sigma B_t.$$

Which therefore yields:

$$\ln X_t - \ln X_0 = \mu t - \frac{1}{2}\sigma^2 t + \sigma B_t$$

$$\ln \frac{X_t}{X_0} = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t.$$

We simplify further to:

$$X_t = X_0 \cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t}.$$

If we recall from definition 3.9, this is exactly a geometric Brownian motion such that $\alpha = (\mu - 1/2\sigma^2)$ and $\beta = \sigma$. We refer to page 62 of [Øksendal, 2000] for more details, and for a in depth description of this example.

We note that the solutions of such differential equations can be seen as process themselves. We use the following definition to formalize such processes:

Definition 3.24. We define an Itô diffusion as a stochastic process $X_t(\omega) : [0, \infty) \times \Omega \to \mathbf{R}^n$ which satisfies the following differential equation:

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dB_t,$$

for $t \geq s$ and $X_s = x$, in which B_t is a Brownian motion and we have the drift and diffusion terms as defined in by theorem 3.22.

Remark 3.25. We note that Itô diffusions are solutions to stochastic differential equations.

We now introduce another important characteristic for Itô diffusions. This characteristic is the Markov property, which states that the future behavior of the process only depends on the current state. In a sense, the process is "forgetful".

We define the Markov property formally as follow:

Theorem 3.26. *For a function $f : \mathbf{R}^n \to \mathbf{R}$, and for non negative t, h, the conditional expectation of the function evaluated at time $t + h$ given the filtration is equal to the expectation at time t. More formally we have*

$$E^x \left[f(X_{t+h}) \mid \mathcal{F}_t^{(m)} \right]_{(\omega)} = E^{X_t(\omega)} \left[f(X_h) \right].$$

We refer to page 109 of [Øksendal, 2000] for the proof of this property.

In short, the Markov provides insight that the current behavior of the processes depends only on its current state, and not on how long the process has been going on for.

Remark 3.27. Processes that have the Markov Property are also commonly referred to as Markov processes.

We now introduce the concept of stopping time, which is fundamental for understanding the strong Markov property, to which we refer to page 110 of [Øksendal, 2000].

Definition 3.28. We define stopping time as a function $\tau : \Omega \to [0, \infty]$ with respect to a filtration $\{\mathcal{F}_t\}$ if $\{\omega; \tau(\omega) \leq t\} \in \mathcal{F}_t, \forall t \geq 0$.

Intuitively, the stopping time enables us to say if $\tau \leq t$ has occurred or not based on the information $\mathcal{F}_t$. In other words, by looking at all information available at time t we are able to determine if the stopping time has occurred or not.

Moreover, the strong Markov property is useful because it enables us to still have the Markov property for a random time $\tau(\omega)$ instead of a given fixed time t. We note that the strong Markov property holds for Itô diffusions, and for more details on this we refer to page 111 on [Øksendal, 2000]. We define the strong Markov property as follow:

Theorem 3.29. *For a function on $\mathbf{R}^n$, the stopping time τ with regards to $\mathcal{F}_t$, for $\tau < \infty$, the strong Markov property is given by*

$$E^x \left[f\left(X_{\tau+h}\right) \mid \mathcal{F}_\tau^{(m)} \right] = E^{X_\tau} \left[f\left(X_h\right) \right], \forall h \geq 0.$$

The proof is similar to the proof of the Markov property for Itô diffusions and for this we refer to page 111 on [Øksendal, 2000].

The final concept regarding stochastic differential equations that we are introducing in this section is the generator of a diffusion, which we define as follow:

Definition 3.30. We define the generator $\mathcal{A}$ of X_t as

$$\mathcal{A}f(x) = \lim_{t \downarrow 0} \frac{E^x\left[f\left(X_t\right)\right] - f(x)}{t}; x \in \mathbf{R}^n,$$

where $\{X_t\}$ is an Itô diffusion.

We also introduce the notations $\mathcal{D}_A(x)$ as the set of functions $f : \mathbf{R}^n \to \mathbf{R}$ such that the limit exists at x, and $\mathcal{D}_A$ as the set of functions that the limit exists for all $x \in \mathbf{R}^n$.

Finally, a more concrete use of the idea of generator is given by the following theorem, which can be found on page 117 of [Øksendal, 2000].

Theorem 3.31. *For an Itô diffusion (as in definition 3.24), if $f \in C_0^2(\mathbf{R}^n)$ then the generator of such diffusion is given by:*

$$\mathcal{A}f(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}\left(\sigma\sigma^T\right)_{i,j}(x) + \frac{\partial^2 f}{\partial x_i \partial x_j},$$

and $f \in \mathcal{D}_A$.

To prove this theorem we first introduce the following lemma.

Lemma 3.32. *For an Itô process $Y_t = Y_t^x$ given by:*

$$Y_t^x(\omega) = x + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s(\omega),$$

where we have a m-dimensional Brownian motion and the process is in $\mathbf{R}^n$, let a function $f \in C_0^2(\mathbf{R}^n)$, and τ be the stopping time. We also assume that the stopping time has finite expectation ($E_x[\tau] < \infty$), that the processes u, v are bounded on (t,ω) such that $Y(t,\omega)$ belongs to the support of f. Then the expectation of the function being applied on the Itô process is given by:

$$E^x\left[f\left(Y_\tau\right)\right] = f(x) + E^x\left[\int_0^\tau\left(\sum_i u_i(s,\omega)\frac{\partial f}{\partial x_i}\left(Y_s\right) + \frac{1}{2}\sum_{i,j}\left(vv^T\right)_{i,j}(s,\omega)\frac{\partial^2 f}{\partial x_i \partial x_j}\left(Y_s\right)\right)ds\right].$$

The proof for this lemma can also be found on page 116 of [Øksendal, 2000]. In short, we apply Itô's lemma to the Itô diffusion and a function $Z = f\left(Y_t\right)$, then we take the expectation on both sides of the resulting equation. Lastly, it is shown that, for a bounded function $g\left(Y_t\right)$, the expectation of the term depending on dB_t is equal to zero.

We find a short description of the proof for theorem 3.32 on page 118 of [Øksendal, 2000], but we will instead provide a proof sketch.

Proof. We begin this proof sketch by applying Lemma 3.33 for the Itô diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,$$

and a function f.

$$E^x\left[f\left(X_t\right)\right]=f(x)+E^x\left[\int_0^t\left(\sum_i b_i(s,\omega)\frac{\partial f}{\partial x_i}\left(X_s\right)+\frac{1}{2}\sum_{i,j}\left(\sigma\sigma^T\right)_{i,j}(s,\omega)\frac{\partial^2 f}{\partial x_i\partial x_j}\left(X_s\right)\right)ds\right].$$

Which we rearrange and divide by t:

$$\frac{E^x\left[f\left(X_t\right)\right]-f(x)}{t}=\frac{E^x\left[\int_0^t\left(\sum_i b_i(s,\omega)\frac{\partial f}{\partial x_i}\left(X_s\right)+\frac{1}{2}\sum_{i,j}\left(\sigma\sigma^T\right)_{i,j}(s,\omega)\frac{\partial^2 f}{\partial x_i\partial x_j}\left(X_s\right)\right)ds\right]}{t}. \tag{4}$$

We are then able to apply the limit as $t\to 0$ from the right hand side. It becomes clear that the left hand side of (4) becomes our definition of the generator. On the right hand side of the equation we then include $1/t$ and the limit inside the expectation, since the function f satisfies the requirements to monotone convergence, see page 489 of [Björk, 2020]. This yields:

$$\mathcal{A}f(x)=E^x\left[\lim_{t\downarrow 0}\frac{1}{t}\int_0^t\left(\sum_i b_i(s,\omega)\frac{\partial f}{\partial x_i}\left(X_s\right)+\frac{1}{2}\sum_{i,j}\left(\sigma\sigma^T\right)_{i,j}(s,\omega)\frac{\partial^2 f}{\partial x_i\partial x_j}\left(X_s\right)\right)ds\right].$$

Intuitively, as t tends to 0, we are taking the derivative of the integral, which therefore implies we are able to simplify this to the following:

$$\mathcal{A}f(x)=\sum_i b_i(x)\frac{\partial f}{\partial x_i}+\frac{1}{2}\sum_{i,j}\left(\sigma\sigma^T\right)_{i,j}(x)+\frac{\partial^2 f}{\partial x_i\partial x_j},$$

which concludes our proof. □

3.5 Stochastic Dynamic Control

Having defined stochastic processes, delved into studying their behavior using Itô's calculus and stochastic differential equations, the next focus of ours will be understanding how to control such differential equations. Unsurprisingly, this next topic is usually referred to as stochastic dynamic control. Continuing the financial theme throughout this thesis, we want to find consumption and investment rules that dictate what an investor should do. That is, we want to find functions or parameters $u_t\in U\subset\mathbf{R}^k$ such that we can not only describe but also control an Itô process:

$$dX_t=dX_t^u=b(t,X_t,u_t)dt+\sigma(t,X_t,u_t)dB_t, \tag{5}$$

where U represents the set in which we are able to choose the processes u_t at any time t to control the process. In this section we will focus on how we can use (5) as a constraint to maximize a process over time. One way of maximizing a process is by breaking it down into two functions. The first one is the utility rate function and represents how much value is added to the process across its duration. On the other hand, the bequest function is seen as a legacy function, that is, it accounts for the value of the process at its end time. We therefore introduce a utility rate function ($F : \boldsymbol{R} \times \boldsymbol{R}^n \times U \to \boldsymbol{R}$) and a bequest function ($K : \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}$), to understand how they influence the performance of the overall process. Since we are considering a stochastic scenario it is therefore natural to also assume that we would like to maximize the expectation of this process. Then the process we would like to optimize for is given by:

$$E^{s,x}\left[\int_s^{\widehat{T}} |F^{u_t}(t, X_t)|dt + K|(\widehat{T}, X_{\widehat{T}})|\chi_{\{\widehat{T}<\infty\}}\right], \tag{6}$$

where χ is again the indicator function and where we also introduce the first exit time: $\widehat{T} = \widehat{T}^{s,x}(\omega) = \inf\{r > s; (r, X_r^{s,x}(\omega)) \notin G\} \leq \infty$, where G is a fixed domain in $\mathbf{R} \times \mathbf{R}^n$. In economic terms this would represent when an investor goes bankrupt. Noteworthy is that we now also introduce a slight notation change $F(r, X_r, u) = F^u(r, X_r)$. With a formal definition of the process, we emphasize that the utility rate function is the function that gives us the performance from time s up to time t, while the bequest function instead accounts for the overall performance at the time $\widehat{T}$. We then introduce the performance function which is given by:

$$J^u(s,x) = E^{s,x}\left[\int_s^{\widehat{T}} F^{u_r}(r, X_r)dt + K(\widehat{T}, X_{\widehat{T}})\chi_{\{\widehat{T}<\infty\}}\right].$$

By introducing $Y_t = (s+t, X_{s+t}^{s,x})$, which has the initial point $Y_0 = (s,x)$ and for time greater than zero, we can rewrite (5) as:

$$dY_t = dY_t^u = b(Y_t, u_t)dt + \sigma(Y_t, u_t)dB_t.$$

For more details on this change of notation we refer to page 224 of [Øksendal, 2000], where it is also explained in detail that our performance function is thus given by:

$$J^u(y) = E^y\left[\int_0^T F^{u_t}(Y_t)dt + K(Y_T)\chi_{\{T<\infty\}}\right],$$

where T is given by $T := \inf\{t > 0; Y_t \notin G\} = \widehat{T} - s$. To this point we have thus described the problem in terms of a stochastic process that depends over time and a control parameter which dictates the problem. Since our goal is to maximize the overall performance, we want to find a function $\Phi(y)$, such that:

$$\Phi(y) = \sup_{u(t,\omega)} J^u(y) = J^{u^*}(y),$$

where $u^* = u^*(t,\omega) = u^*(y,t,\omega)$. That is, u^* is a control that maximizes the performance function, and if it exists is referred to as optimal control. We are now going to explore one type of control function, namely Markov controls, that for a deeper discussion we refer to page 225 of [Øksendal, 2000].

Definition 3.33. We define Markov Controls as functions $\mathbf{u}(t,\omega) = \mathbf{u}_0(t, X_t(\omega))$ that only depend on the current state of the system for a given time t. Formally we have:

$$\mathbf{u} : [0,T) \times \mathbf{R}^{n+1} \to U,$$
$$\mathbf{u}(t, X_t),$$

where U is again the set which we can choose $\mathbf{u}$ at any time t to control the process.

Therefore, we say that the parameters $\mathbf{u}_t$ in (6) are indeed Markov controls.

Remark 3.34. For our purposes it is important to note that these Markov controls are Itô's diffusions and more specifically, also are Markov Processes.

Before we introduce the main part of this section, we first introduce the following definition, which is the generator for a differential equation with Markov controls, hence, very similar to theorem 3.32.

Definition 3.35. For the differential equation $dY_t = b(Y_t, \mathbf{u}(Y_t))\,dt + \sigma(Y_t, \mathbf{u}(Y_t))\,dB_t$, where $Y_t = (s+t, X_{s+t}), Y_0 = (s,x)$, and the Markov controls $\mathbf{u} = \mathbf{u}(t, X_t(\omega))$, we define:

$$(L^v f)(y) := \frac{\partial f}{\partial s}(y) + \sum_{i=1}^{n} b_i(y,v)\frac{\partial f}{\partial x_i} + \sum_{i,j=1}^{n} a_i(y,v)\frac{\partial^2 f}{\partial x_i \partial x_j},$$

where $v \in U$, $f \in C_0^2(\mathbf{R} \times \mathbf{R}^n)$; such that $a_{i,j} = \frac{1}{2}\sigma\sigma^T, y = (s,x)$ and $x = (x_1, \cdots, x_n)$. Therefore, we have an Itô diffusion with generator $\mathcal{A}$ for each choice $Y_t = Y_t^{\mathbf{u}}$ and $f \in C_0^2(\mathbf{R} \times \mathbf{R}^n)$, such that:

$$(\mathcal{A}f)(y) = \left(L^{\mathbf{u}(y)} f\right)(y).$$

More details can be found on page 225 of [Øksendal, 2000]. We also define $F^v(y) = F(y, v)$ for $v \in U$.

The backbone for solving our stochastic dynamic control problem is the Hamilton-Jacobi-Bellman (HJB) equation. The HJB equation is fundamental because it reduces the stochastic control problem to a deterministic one, given a fixed starting point. The HJB equation is central for both the HJB theorem as well as the verification theorem, as the HJB theorem establishes a connection between the control problem and the verification theorem is the other side of this implication.

In brief, according to the HJB theorem if a function Φ is the optimal value function, if $\mathbf{u}^*$ is an optimal Markov control, we are then able to find a maximum to our control problem by satisfying the HJB equation. We note that this maximum is attained by using the control $\mathbf{u}^*(t, x)$. In other words, if the necessary conditions mentioned are fulfilled, then the following equation holds:

$$\begin{cases} \sup_{\mathbf{u} \in U} \{F^v(y) + (L^v \Phi)(y)\} = 0, \forall y \in G \\ \Phi(y) = K(y). \end{cases} \tag{7}$$

Moreover, such supremum is obtained when $v = \mathbf{u}^*$ and is given by:

$$F(y, \mathbf{u}^*(y)) + \left(L^{\mathbf{u}^*(y)} \Phi\right)(y) = 0.$$

Equation (7) is commonly referred to as the HJB equation. We refer to pages 226 and 229 of [Øksendal, 2000] for a formal formulation of the HJB theorem, which includes more details on the boundary conditions in which $\Phi(y) = K(y)$.

Simply put, the HJB theorem states that if we have an optimal value function, an optimal Markov control, then we are able to satisfy the HJB equation. Nevertheless, the central aspect of the HJB theorem for our purposes is that it acts a sufficient condition for the verification theorem.

According to the verification theorem, if we have a (sufficient regular) function that satisfies the HJB equation, we then have optimal Markov controls for some initial starting point and an optimal performance function for our control problem.

This implies that in our purposes it is central to find a function that satisfies the HJB equation. With that, we are then able to use the verification theorem to find an optimal value function, and that we have optimal Markov controls. For more details on the verification theorem we refer to page 229 of [Øksendal, 2000].

We then formulate formulate the verification theorem as follows:

Theorem 3.36. *If the function ϕ in the set $C^2(G) \cap C(G)$ satisfies the following three conditions, for all $v \in U$:*

(i) the sum of the utility rate function and the generator is less than 0:

$$F^v(y) + (L^v \phi)(y) \leq 0 \text{ for } y \in G.$$

(ii) the function $\phi(Y_t)$ approaches the bequest function $K(Y_T) \cdot \chi_{\{T \leq \infty\}}$ as t tends to T with probability 1 with respect to a fixed starting point. In other words, the boundary values satisfy $\lim_{(t \to T)} \phi(Y_T) = K(Y_T) \cdot \chi_{\{T \leq \infty\}}$.

(iii) The function $\phi(Y_\tau)_{\tau \leq T}$ is integrable for all Markov controls and $y \in G$. Then, for all Markov controls $\mathbf{u}$ and $y \in G$

$$\phi(y) \geq J^{\mathbf{u}}(y).$$

Moreover, if for each $y \in G$, we find $u_0(y)$ such that the following equation holds:

$$F^{\mathbf{u}_0(y)}(y) + (L^{\mathbf{u}_0(y)} \phi)(y) = 0,$$

then $\mathbf{u}_0 = \mathbf{u}_0(y)$ is a Markov control which satisfies

$$\phi(y) = J^{\mathbf{u}_0}(y),$$

leading to the fact that $\mathbf{u}_0$ must be an optimal control and $\phi(y) = \Phi(y)$.

We note that this is a slightly simplified version of the verification theorem, for more details and for the proof of the verification theorem we refer to page 229 of [Øksendal, 2000].

In exercise 11.4 of [Øksendal, 2000], the reader is encouraged to deduce Bellman's Principle of Optimality. In short this principle states that the Markov control $\hat{\mathbf{u}}^{tx}$ optimizes the process not only at the initial time interval $[0, t]$ but also for the remaining interval in the process $[t, T]$. Since we are explicitly referring to this theorem in the latter parts of this dissertation, we will borrow from page 339 of [Björk, 2020] to formulate this principle.

Theorem 3.37. *For a fixed initial point (t, x) and the corresponding optimal Markov control $\hat{\mathbf{u}}^{tx}$, the optimal Markov control $\hat{\mathbf{u}}^{tx}$ is also optimal for any subinterval between t and T (i.e. $[r, T]$ where $r \geq t$). More formally we have:*

$$\hat{\mathbf{u}}_s^{tx}(y) = \hat{\mathbf{u}}_s^{r,X_r}(y), \forall s \geq r, y \in \mathbf{R}^n.$$

Proof. To prove this theorem we use contradiction, and for more details we refer to page 339 of [Björk, 2020]. We will instead provide a proof sketch for the case in which $t = 0$. Firstly, we need to make the fundamental assumption that there exists optimal control laws for the problem for every initial point (t, x), which are denoted by $\hat{\mathbf{u}}^{tx}$. For more details on this assumption, we also refer to page Assumption 25.4.1 on page 339 of [Björk, 2020]. Say that for some $t > 0$, there exists a control law $\mathbf{u}^0$ which performs better than our optimal control law $\hat{\mathbf{u}}$ on $[t, T]$. In other words, we are trying to find a control law so that:

$$E_{t,x}\left[\int_t^T F\left(s, X_s^{\mathbf{u}^0}, \mathbf{u}_s^0\right) ds + K\left(X_T^{\mathbf{u}^0}\right)\right] \geq E_{t,x}\left[\int_t^T F\left(s, X_s^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}_s\right) ds + K\left(X_T^{\hat{\mathbf{u}}}\right)\right]. \tag{8}$$

We then introduce the strategy that incorporates the new control law, which incorporates $\hat{\mathbf{u}}$ for time between 0 and t, and $\mathbf{u}^0$ for the rest of the time period. More formally:

$$\overline{\mathbf{u}_s}(y) = \begin{cases} \hat{\mathbf{u}}_s(y), & \text{for } 0 \leq s < t. \\ \mathbf{u}_s^0(y), & \text{for } t \leq s < T. \end{cases}$$

This would imply that that given the same starting point, the new value function will have the same behavior as the one dictated by the optimal control law for the first interval $[0, t]$. However, for the rest of the time period it would behave as the control law $\mathbf{u}^0$, which should outperform the optimal control law. Therefore, for this new strategy the value function then becomes:

$$J_0(x_0, \overline{\mathbf{u}}) = E_{0,x_0}\left[\int_0^T F(s, X_s^{\overline{\mathbf{u}}}, \mathbf{u}_s^*)ds + K(X_T^{\overline{\mathbf{u}}})\right]$$

$$= E_{0,x_0}\left[\int_0^t F(s, X_s^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}_s)ds\right] + E_{0,x_0}\left[\int_t^T F(s, X_s^{\mathbf{u}^0}, \mathbf{u}_s^0)ds + K(X_T^{\mathbf{u}^0})\right].$$

Now using the Markov property of the control law, we are able to adjust the expectation from t to T, which yields:

$$J_0(x_0, \overline{\mathbf{u}}) = E_{0,x_0}\left[\int_0^t F(s, X_s^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}_s)ds\right] + E_{0,x_0}\left[E_{t,X_t}\left[\int_t^T F(s, X_s^{\mathbf{u}^0}, \mathbf{u}^0)ds + K(X_T^{\mathbf{u}_s^0})\right]\right]. \tag{9}$$

Generalizing (8) to the state X_t at time t yields the following:

$$E_{t,X_t}\left[\int_t^T F\left(s, X_s^{\mathbf{u}^0}, \mathbf{u}_s^0\right) ds + K\left(X_T^{\mathbf{u}^0}\right)\right] \geq E_{t,X_t}\left[\int_t^T F\left(s, X_s^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}_s\right) ds + K\left(X_T^{\hat{\mathbf{u}}}\right)\right].$$

Then substituting the right hand side of this generalization in (9) leads to the strict inequality:

$$\begin{aligned} J_0\left(x_0, \overline{\mathbf{u}}\right) > & E_{0,x_0}\left[\int_0^t F\left(x, X_x^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}\right) ds\right] + E_{0,x_0}\left[\int_t^T F\left(s, X_s^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}_s\right) ds + K\left(X_T^{\hat{\mathbf{u}}}\right)\right] \\ = & E_{0,x_0}\left[\int_0^T F\left(s, X_s^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}_s\right) ds + K\left(X_T^{\hat{\mathbf{u}}}\right)\right] = J_0\left(x_0, \hat{\mathbf{u}}\right). \end{aligned}$$

Hence, we get that $J_0\left(x_0, \overline{\mathbf{u}}\right) > J_0(x_0, \hat{\mathbf{u}})$, which contradicts the previous assumption that the optimal control function $\hat{\mathbf{u}}$ exists. Since we are considering such assumption, we see that this result does not hold, and therefore there does not exist a control law $\mathbf{u}^0$ which performs better than $\hat{\mathbf{u}}$. □

4

Optimal Consumption and Investment Rules

In this section we will use our concepts of stochastic calculus and control to derive explicit rules to maximize an investors utility over their lifetime. This implies that we want to find optimal control laws, in other words, investment and consumption rules that bring the most amount of benefit to the investor throughout their lifetime. We will borrow from Chapter 25 of [Björk, 2020] and from [Björk, 2015] to formulate the problem. We first describe the investment options available, then we will describe how we model the investors utility during their lifetime.

Say the investor has the option to invest in a stock and a bond only, where the stock is seen as a risky asset and the bond a risk-free asset. We are then able to describe the price development of a stock using a stochastic differential equation:

$$\frac{dS(t)}{S(t)} = u(t, S(t))dt + v\,(t, S(t))\,dB_t.$$

Where μ and v are the instantaneous conditional expected change and instantaneous conditional variance, respectively. Since in example 3.23 we have shown that both of these terms are constants as a result of the Geometric Brownian motion. We therefore get:

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dB_t.$$

Moreover, since we are assuming that the bond is a risk-free asset, we have a deterministic process that describes its price change. For a bond Y_t we describe its price change as follows:

$$dY_t = rY_t dt,$$

which we know to be a deterministic differential equation, which yields $Y(t) = K \cdot e^{rt}$, for some positive constant K.

Therefore, if we have 2 assets, one involving risk and the other being risk-free, we say that the weight invested in each asset is $w(t)$ and $w'(t)$, respectively. It is therefore natural to have $w(t) = 1 - w'(t)$. We also introduce a deterministic consumption term $c(t)$ which captures the investors consumption over time, which we see as anything that is not destined to investment and hence $c(t) \geq 0, \forall t \geq 0$.

Therefore, we describe the overall wealth process X_t of an investor over time as follow:

$$dX_t = (w(t)(\alpha - r))X_t dt + (rX_t - c(t))dt + w(t)\sigma X_t dB_t, \tag{10}$$

where we have a fixed initial point, i.e. $X_0 = x_0$.

To describe the investor's utility throughout their lifetime we will use two functions. As previously mentioned in Section 3.5, we will have a utility rate function $U(\cdot)$ which in this case represents the benefits and well being the investor gets from consuming throughout their life. We highlight that each investor values a certain level of consumption over investing and vice versa, and that the risk tolerance/ aversion for investing in the risky asset is also an individual characteristic. Hence, these are also accounted for in the utility rate function. Moreover, we also introduce the bequest function $K(\cdot)$, which represents the benefit the investor gets from the money left over at their bank account when the process ends. Therefore, combining these two functions we equate the investors utility throughout their lifetime. Lastly, we also assume that throughout their lifetime, the investor cannot go bankrupt, in other words, the process cannot be equal to zero. This implies that the process goes on until whichever of the two conditions happens first: the investor dies or goes bankrupt. Therefore, to formalize this we use the stopping time: $\tau = \inf\{t \geq 0 | X_t = 0\} \wedge T$, where T is the final time of the wealth process (i.e. when the investor dies).

We now formally introduce the problem we want to solve, as formulated on page 36 of [Björk, 2015]:

Problem 4.1. *The problem is to maximize the overall utility of an investor over their lifetime, which we express as:*

$$\max_{w \in \mathbf{R}, c \geq 0} E^x \left[\int_0^\tau U(c(t), t) dt + K(X_T) \right].$$

Where the wealth dynamics is given by:

$$dX_t = (w(t)(\alpha - r))X_t dt + (rX_t - c(t))dt + w(t)\sigma X_t dB_t,$$

and where $\tau = \inf\{t \geq 0 | X_t = 0\} \wedge T$, *$U$ is the utility rate function and K is the bequest function. Moreover, we have the following constraints:*

$$w(t) + w'(t) = 1, c(t) \geq 0 \forall t; X_0 = x_0.$$

4.1 General Approach to Solving the Problem

Our goal is to describe the benefit the investor gets throughout their lifetime, then use the restrictions of the wealth dynamics above, to find optimal consumption and investment rules. In other words, we would like to describe the problem in a similar way to what we have done in Section 3.5, and use the verification theorem to find explicit solutions for the optimal investment and consumption rules.

To do so we first define generalize the problem, by introducing the control law $\mathbf{u}(t, x)$. Since we have two terms, $c(t, x)$ and $w(t, x)$, that in a sense control the process, that is $c(t, x)$ dictates the investors outflows i.e. expenditures and consumption, while $w(t, x)$ dictates the investment in the risky asset, we use $\mathbf{u}(t, x)$ as a representation to the general combined effect of $c(t,x)$ and $w(t,x)$ on the overall wealth dynamics. We refer to page 335 of [Björk, 2020] for more details. This yields the stochastic differential equation:

$$dX_t = \alpha(t, X_t, \mathbf{u}(t, X_t))dt + \sigma(t, X_t, \mathbf{u}(t, X_t))dB_t. \tag{11}$$

We then introduce the value function $\mathcal{J}_0 : \mathcal{U} \to \mathbf{R}$, which is given by:

$$\mathcal{J}_0(\mathbf{u}) := E\left[\int_0^T U(t, X_t^{\mathbf{u}}, \mathbf{u}_t)ds + K(X_T^{\mathbf{u}})\right],$$

where $\mathcal{U}$ represents the class of admissible controls $\mathbf{u}$ such that (11) has a unique solution for a starting point $X_0 = x$. For more details on the admissible control laws we refer to page 335 of [Björk, 2020]. We now introduce the optimal value function that is given by

$$\hat{\mathcal{J}}_0 = \sup_{\mathbf{u}\in\mathcal{U}} \mathcal{J}_0(\mathbf{u}).$$

Since we have shown by Bellman's principle of optimality, Theorem 3.38, that such problems have an optimal control law, we then note that there exist optimal controls laws such that:

$$\mathcal{J}_0(\hat{\mathbf{u}}) = \hat{\mathcal{J}}_0.$$

We then introduce a optimal value function $v : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}$:

$$V(t, x) := \sup_{\mathbf{u}\in\mathcal{U}} \mathcal{J}(t, x, \mathbf{u}).$$

In Section 3.5 we are able to find explicit solutions to our problem using the verification theorem, in other words we are able to find the optimal control law $\mathbf{u}$.

Reverting back to the control problem we want to solve, we then separate the effect of investment and consumption to the overall wealth process. Therefore, we draw the parallel that we are able to solve the same problem using the constraint (10) instead of the generalized form (11).

Therefore, to maximize the investor's wealth process and find the optimal investment rules we revert back to stochastic dynamic control and the HJB equation. More specifically, the HJB equation for maximizing the investors utility given the invesment and consumption rules in question is:

$$V_t + \sup_{c\geq 0, w\in\mathbf{R}} \{U(t, c) + wx(\alpha - r)V_x + (rx - c)V_x + (\frac{1}{2}w^2\sigma^2x^2)V_{xx}\} = 0, \tag{12}$$

which is also subject to $V(T, x) = 0$ and $V(t, 0) = 0$, and where the derivatives are denoted by the subscripts.

We remind that by the HJB theorem, if we have an optimal value function and optimal investment and consumption rules, then equation (12) is satisfied and its maximum value is attained when using such optimal investment and consumption rules. Moreover, these conditions set by the HJB theorem act as sufficient to show the converse, which according to the verification theorem, states that if a function satisfies equation (12), then we indeed have optimal consumption and investment rules, and an optimal value function for our control problem. For more details on this method of using the HJB equation we refer to page 337 of [Björk, 2020].

To find the optimal controls $\mathbf{c}^*(t, x)$ and $\mathbf{w}^*(t, x)$, we will provide short description of the methodology. Firstly, we isolate $\mathbf{w}^*(t, x)$ and $\mathbf{c}^*(t, x)$ in our first order conditions of equation (12). Then we are able to write $\mathbf{w}^*(t, x)$ and $\mathbf{c}^*(t, x)$ in terms of the other variables, i.e. V_x, V_t, V_{xx}, x, and t. We then introduce a function that describes the behavior of the utility function, as well as a trial solution to the value function. It is therefore needed to verify that equation (12) holds for these new functions, and if it does we have found the optimal consumption and investment rules due to the verification theorem, since its conditions are satisfied by the HJB theorem as explained above. Lastly, we are able to rewrite $V(x, t)$, $\mathbf{w}^*(t, x)$ and $\mathbf{c}^*(t, x)$ explicitly in terms of x and t.

4.2 Solving the Problem with Constant Risk Aversion

Firstly we introduce a function that relates the investors utility gained with their risk aversion:

$$U(t, c) = e^{-\delta t} \cdot c^{\gamma}, \tag{13}$$

for some δ and where $0 < \gamma < 1$ is a measure of risk aversion. For a more detailed discussion on the investors risk aversion we refer to page 250 of [Merton, 1969].

Therefore, the solutions to Problem 4.1, found on page 39 of [Björk, 2015], are given by the following theorem:

Theorem 4.2. *In the case we have 2 assets, one risky and one risk-free, and assuming that the utility function is given by* (13)*, the optimal consumption and investment* *that solve Problem 4.1 are given by:*

$$\mathbf{c}^*(t,x) = x \cdot h(t)^{(-1/1-\gamma)}$$

and

$$\mathbf{w}^*(t,x) = \frac{(\alpha - r)}{\sigma^2(1-\gamma)}.$$

Moreover, we also have that the optimal value function is given by:

$$V(t,x) = e^{-\delta t} \cdot h(t) \cdot x^\gamma,$$

where γ *and is a parameter of the risk aversion function, and* $h(t)$ *is given by* $h(t) = \left(\frac{1}{A} e^{(t-C)A(z-1)} - B\right)^{1/(1-z)}$ *for* $A = \frac{\gamma(a-r)^2}{\sigma^2(1-\gamma)} + r\gamma - \frac{1\gamma(a-r)^2}{2\sigma^2(1-\gamma)} - \delta, B = 1-\gamma$ *and* $C = T - \frac{ln(B)}{A(z-1)}$ *for* $z = -\gamma/(1-\gamma)$.

We note that deriving explicit solutions to non-linear PDE can be challenging, hence most problems regarding optimal investment and consumption "rig" the problem by selecting utility functions and trial solutions in which asnwers can be found. Hence the similarities between $U(t,c)$ and $V(t,x)$. For more details we refer to Remark 25.6.1 in [Björk, 2020]. In short, by having the function $h(t)$ we are able to separate the effect of time from the current state of the process, which enables us to find explicit solutions later on.

Proof. We will borrow from the methodology earlier in this chapter and from pages 37 to 40 of [Björk, 2015] to solve this problem. We first substitute our utility function in equation (12) and take the first derivatives with regards to c and w. Therefore, we get

$$\gamma \cdot e^{-\delta t} \cdot c^{\gamma-1} - V_x = 0.$$

We then isolate c, which yields

$$\mathbf{c}^*(t,x) = \left(\frac{e^{\delta t}}{\gamma} \cdot V_x\right)^{\frac{1}{\gamma-1}}. \tag{14}$$

Now, focusing on w we get:

$$x(\alpha - r)V_x + w\sigma^2 x^2 V_{xx} = 0.$$

Isolating for w yields:

$$\mathbf{w}^*(t,x) = -\frac{V_x}{x \cdot V_{xx}} \cdot \frac{a-r}{\sigma^2}. \tag{15}$$

We now introduce the trial solution:

$$V(t,x) = e^{-\delta t} \cdot h(t) \cdot x^\gamma, \tag{16}$$

such that $h(T) = 0$, which satisfies the boundary conditions of our bequest function. Now, by Bellman's principle of optimality, it suffices to show that the following trial solution satisfies the HJB equation (12), which implies that the trial solution is indeed the optimal value function. Moreover, by the verification theorem, we therefore get that the investment and consumption controls are thus optimal controls. We begin by taking the partial derivatives of V with regards to x and t, and the second derivative with regards to x.

$$V_t = e^{-\delta t} \cdot h'(t) \cdot x^\gamma - \delta e^{-\delta t} \cdot h(t) \cdot x^\gamma,$$

$$V_x = \gamma \cdot e^{-\delta t} \cdot h(t) \cdot x^{\gamma-1},$$

$$V_{xx} = \gamma \cdot (\gamma - 1) \cdot e^{-\delta t} \cdot h(t) \cdot x^{\gamma-2}.$$

We now substitute back in (14) and (15), which results in:

$$\mathbf{c}^*(t,x) = \left(\frac{e^{\delta t}}{\gamma}\gamma \cdot e^{-\delta t} \cdot h(t) \cdot x^{\gamma-1}\right)^{\frac{1}{\gamma-1}} = x \cdot h(t)^{-1/(1-\gamma)} \tag{17}$$

and

$$\mathbf{w}^*(t,x) = -\frac{\gamma \cdot e^{-\delta t} \cdot h(t) \cdot x^{\gamma-1}}{x \cdot \gamma \cdot (\gamma-1) \cdot e^{-\delta t} \cdot h(t) \cdot x^{\gamma-2}} \cdot \frac{a-r}{\sigma^2} = \frac{a-r}{\sigma^2(1-\gamma)}. \tag{18}$$

The last thing remaining is to explicitly find the function $h(t)$, which we do by substituting our solutions in (12), which gives us:

$$x^\gamma(h'(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)}) = 0,$$

where $A = \frac{\gamma(a-r)^2}{\sigma^2(1-\gamma)} + r\gamma - \frac{1\gamma(a-r)^2}{2\sigma^2(1-\gamma)} - \delta$ and $B = 1 - \gamma$. Which we are able to simplify as:

$$h'(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)} = 0.$$

As on page 40 of [Björk, 2020], this is a Bernoulli differential equation and can easily be solved using the condition that $h(T) = 0$, however no explicit solution is given. Nevertheless, to find such explicit solution we begin by substituting $-\gamma/(1-\gamma) = z$, hence the equation can be rewritten as:

$$h'(t) + Ah(t) + Bh(t)^z = 0.$$

We now rearrange and write in the differential form:

$$\frac{dh}{Ah(t) + Bh(t)^z} = dt.$$

We now integrate both sides, yielding:

$$\frac{ln(Ah(t)^{1-z} + B)}{A(z-1)} + C = t.$$

Now Isolating $h(t)$ we get:

$$ln(Ah(t)^{1-z} + B) = (t - C)A(z-1).$$

Since the right hand side is greater than zero we get:

$$h(t) = \left(\frac{1}{A}e^{(t-C)A(z-1)} - B\right)^{\frac{1}{1-z}}.$$

Substituting $h(T) = 0$, we are able to isolate C, which yields:

$$h(t) = \left(\frac{1}{A}e^{(t-C)A(z-1)} - B\right)^{\frac{1}{1-z}},$$

where $C = T - \frac{ln(B)}{A(z-1)}$.

Since we have now found an explicit expression for the function $h(t)$ such that (12) holds, we have met all conditions for the HJB theorem. Therefore, we also have that the conditions for the verification theorem are met. In other words, for a value function and some given control functions, if the value function satisfies equation (12), which we have just shown, we then have optimal control functions for consumption and investment and an optimal value function. This concludes our proof. □

5

Interpretation of Optimal Consumption and Investment Rules

In this section we will provide a short interpretation of the solutions to the problem posed in the previous section.

Firstly , we note that the optimal investment rule in the risky asset, which is given by equation (18), is a constant as it does not depend on t or x. It instead only depends on market conditions and intrinsic characteristics of the investor. Therefore, we can say that for an average investor, they should allocate a percentage of their wealth in the risky asset depending only on their level of risk aversion, since market conditions are not in control of individual investors.

Figure 1: Optimal investment for different levels of risk aversion

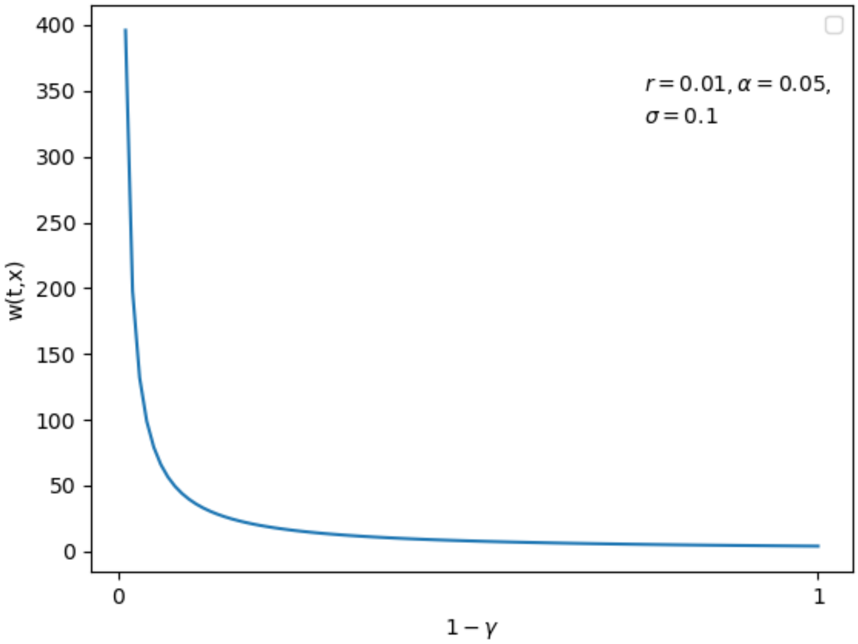
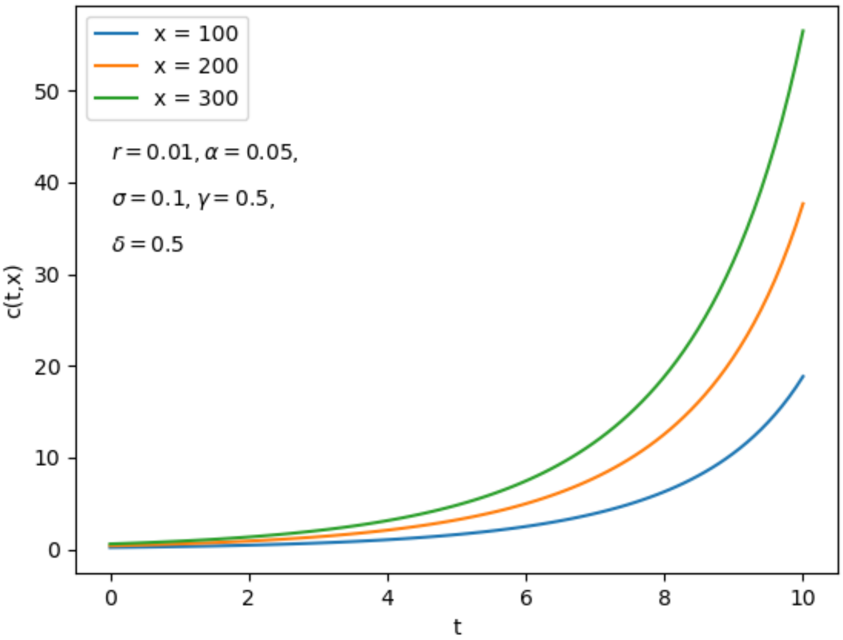


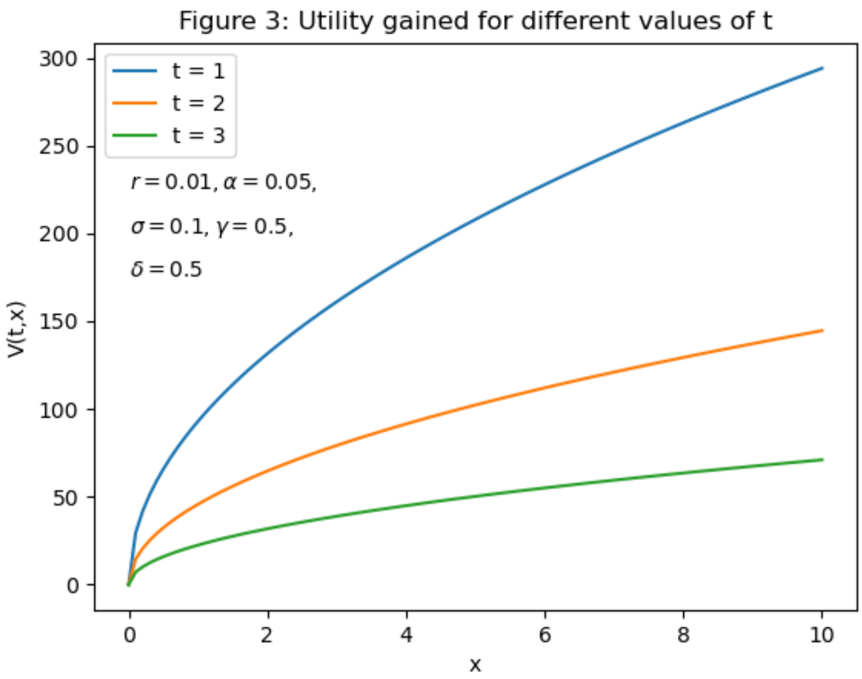
Figure 1 is an illustration of the investment in the risky asset for different values of risk aversion. It is clear that the investors with higher risk aversion should invest less in the stock and more in the bond, while risk tolerant investors on the other hand allocate a higher percentage of their portfolio in the risky asset, irrespective of market conditions. We note that we have plotted $1 - \gamma$ in the x-axis of Figure 1, because this is Pratt's definition of constant relative risk aversion and better describes the investor's investment strategy. For more details on characteristics of this measure we refer to page 250 of [Merton, 1969], as well as page 354 of [Björk, 2020].

In regards to the optimal consumption we have more interesting insights. From equation (17), we note that the consumption has a linear positive relationship with the investor's wealth. Moreover, as illustrated in Figure 2, we notice that for an increasing value of x, the consumption over time has increasing behaviors over time, which is mainly dictated by the term $h(t)$, and that the consumption is a convex function over time.

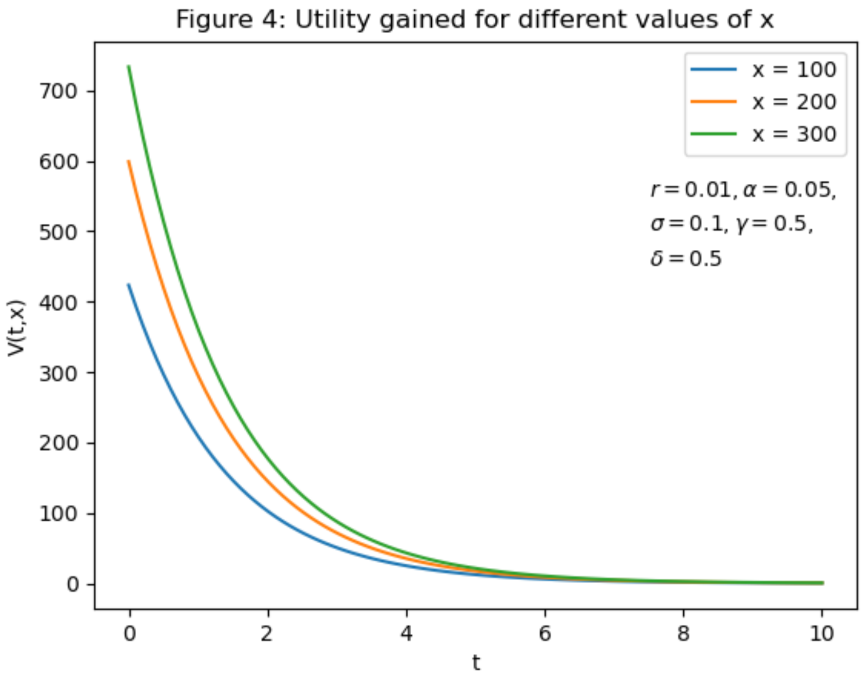
Figure 2: Optimal consumption for different x values



When it comes to the overall utility gained by the investor we are able to analyze the results in two perspectives. Firstly, we note that the overall utility of the investor is increase as the values of x increase, proportionally scaled to a factor of x^γ, as in (16) which is illustrated in Figure 3. It is also important to note that $0 < \gamma < 1$, It is noteworthy to highlight that with an increase in t the optimal value function



decreases, despite displaying similar behaviors. This would imply that the investor gains more utility by having more money at an earlier period in time, which is a reasonable economic interpretation.



On the other hand, we are also able to analyze the performance of the optimal value function $V(t, x)$ when fixing for different values of x, which is illustrated in Figure 4. We see that the function decreases over time regardless of the fixed x value. This behavior becomes clearer when considering the solution to the problem, which has an exponential term that dominates the function $h(t)$. The economic interpretation of such is that an investor gains more utility for having a certain amount of capital earlier in life than later in life, regardless of their risk aversion and their wealth. We also observe that for higher values of wealth, the corresponding utilities are higher across the specified period.

References

[Björk, 2015] Björk, T. (2015). Lecture notes on stochastic optimal control with finance applications. https://www.math.kth.se/matstat/seminarier/Tomas2015/KTH2015.pdf. KTH.

[Björk, 2020] Björk, T. (2020). *Arbitrage Theory in Continuous time* . 4th edition.

[Davis and Norman, 1990] Davis, M. H. A. and Norman, A. R. (1990). Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676–713. 11.

[Korn and Kraft, 2002] Korn, R. and Kraft, H. (2002). A stochastic control approach to portfolio problems with stochastic interest rates. *SIAM Journal on Control and Optimization*, 40(4):1250–1269. doi: 10.1137/S0363012900377791; 11.

[Merton, 1969] Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of economics and statistics*, 51(3):247–257. 07.

[Merton, 1971] Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413. ID: 272399.

[Steffensen, 2011] Steffensen, M. (2011). Optimal consumption and investment under time-varying relative risk aversion. *Journal of Economic Dynamics and Control*, 35(5):659–667. ID: 271650.

[van Bilsen et al., 2020] van Bilsen, S., Bovenberg, A. L., and Laeven, R. J. A. (2020). Consumption and portfolio choice under internal multiplicative habit formation. *Journal of Financial and Quantitative Analysis*, 55(7):2334–2371.

[Øksendal, 2000] Øksendal, B. (2000). *Stochastic Differential Equations: An Introduction with Applications*. 5th edition.

Corrections to Stochastics and Its Application in Merton's Problem

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1. In Theorem 3.20 the rules listed should be corrected to:

1. $(dX_t)^2 = (dX_t) \cdot (dX_t)$,
2. $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$ and
3. $dB_t \cdot dB_t = dt$.

2. In Example 3.21, we should have dt instead of ∂t. This would yield:

$$dY_t = 0dt + 2X_t \cdot dX_t + \frac{1}{2} \cdot 2(dX_t)^2.$$

3. As of page 32 a should be replaced by α.

4. When finding explicit solutions to the optimal investment and consumption rules, the expression for $h(t)$ on page 43 should be given by:

$$h(t) = \left(\frac{1}{A} (e^{(t-C)A(z-1)} - B) \right)^{\frac{1}{1-z}}.$$

5. Typos:

On page 9 σ-algrebra should be corrected to σ-algebra.