SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

An Introduction to Differential Forms and Manifolds

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2023 - K 24

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Självständigt arbete i matematik 15 högskolepoäng, grundnivå

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2023

Abstract

This thesis aims to introduce the concepts of Manifolds and Differential Forms defined on Manifolds. These concepts serve as prerequisites for demonstrating a general result. Namely, the Generalized Stokes' Theorem, for which we will provide a proof. We shall also show how two theorems from Vector Calculus can be seen as special cases of this Generalized Theorem.

Contents

Chapter 1

Manifolds and Tangent Spaces

A manifold can be seen as a generalization of curves and surfaces to higher dimensions. Typical manifolds are things like graphs of a function, or a geometric shape such as a circle. What is special about manifolds is that they are locally Euclidean. If you where to zoom in on a small neighborhood of a circle, it would resemble flat space.
An n-dimensional manifold is a topological space $\mathcal{M}$ for which every point $x \in \mathcal{M}$ there is a neighbourhood homeomorphic to Euclidean space $\mathbb{R}^n$.

For the purpose of this thesis, and the results we wish to show, we are not concerned with abstract manifolds. Rather, subsets of Euclidean space. We call those submanifolds of $\mathbb{R}^m$ (for some m). We shall proceed with defining the manifolds of our interest. We begin by defining the notion of a surface in $\mathbb{R}^n$.

Definition 1.0.1. *Take a subset $\mathcal{M} \subset \mathbb{R}^n$. A function $\phi(u) : \mathcal{U} \to \mathcal{V} \cap \mathcal{M}$, with open subsets $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^m$ and*

$$\phi(u) = \begin{pmatrix} \phi_1(u) \\ \cdot \\ \cdot \\ \phi_m(u) \end{pmatrix} = \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ v_m \end{pmatrix}$$

is called a Local Parametrization (or Local Coordinate Chart) of $\mathcal{M}$ if it satisfies the following properties

1. ϕ is *Smooth* (C^∞) when the codomain is regarded as $\mathbb{R}^n$.

2. $\phi : \mathcal{U} \to \mathcal{V} \cap \mathcal{M}$ is a *Homeomorphism* meaning that ϕ is bijective, and both ϕ and its inverse ϕ^{-1} are continuous.

3. $\text{rank}(D\phi) = n$.

The coordinates $(v_1, ..., v_m)$ are called *Local coordinates* of ϕ. If $\phi : \mathcal{U} \to \mathcal{M}$ is of class C^k for any integer k, then we call it a C^∞ or *Smooth* local parametrization. Throughout this thesis, we assume all parametrizations to be smooth.

Definition 1.0.2. *A subset $\mathcal{M} \subset \mathbb{R}^n$ is called a Smooth (or regular) surface in $\mathbb{R}^n$ if at every point $p \in \mathcal{M}$ there exists an open subset $\mathcal{U} \subset \mathbb{R}^n$, an open subset $\mathcal{V} \subset \mathbb{R}^m$ containing p, a smooth local parametrization $\phi : \mathcal{U} \to \mathcal{V} \cap \mathcal{M}$, and satisfying the three conditions above.*

An example of a parametrization would be the graph of a function. We can consider a smooth function $f(x, y) : \mathcal{U} \to \mathbb{R}$ defined on an open subset $\mathcal{U} \subset \mathbb{R}^2$. The graph of f denoted Γ would be defined as

$$\Gamma = \{(x, y, f(x, y)) : (x, y) \in \mathcal{U} \subset \mathbb{R}^3\}.$$

We can also parametrize the graph by

$$\Phi(x, y) = (x, y, f(x, y)).$$

We have assumed f to be smooth so condition 1 holds. If $\Phi(x_1, y_1) = \Phi(x_2, y_2)$ that will imply that $x_1 = x_2$ and $y_1 = y_2$ and so Φ is injective. We have that Φ has Γ as its image and so is surjective. Also the inverse map $\Phi^{-1} : \Gamma \to \mathcal{U}$ given by

$$\Phi^{-1}(x, y, f(x, y)) = (x, y),$$

is obviously continuous. So Condition 2 holds. Another way to say that $D\Phi$ has full rank, is by saying that for any $(x, y) \in \mathcal{U}$

$$\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} \neq 0.$$

Taking the cross product amounts to

$$D\Phi = \begin{vmatrix} e_x & 1 & 0 \\ e_y & 0 & 1 \\ e_z & \partial f_x & \partial f_y \end{vmatrix} = \begin{pmatrix} -\partial f_x \\ -\partial f_y \\ 1 \end{pmatrix},$$

which is non-zero for all $(x, y) \in \mathcal{U}$. We have shown that condition 3 is satisfied.

An example of a manifold is the unit circle. One representation of the unit circle is

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

We quickly run into the problem that if we wish to cover the entire circle, all points of the circle will not be covered by a single parametrization chart. To cover the entire circle we need four charts. These charts will be given by

$$\phi_1(x) = (x, \sqrt{1 - x^2}) \tag{1.1}$$
$$\phi_2(y) = (-\sqrt{1 - y^2}, y) \tag{1.2}$$
$$\phi_3(x) = (x, -\sqrt{1 - x^2}) \tag{1.3}$$
$$\phi_4(y) = (\sqrt{1 - y^2}, y). \tag{1.4}$$

There are points where these charts (patches) overlap. To show that these charts are compatible, i.e points described by different charts do indeed describe the same points, we must introduce transition maps.

1.1 Transition maps

Definition 1.1.1. *Let $\mathcal{M} \subset \mathbb{R}^n$ be a smooth surface. An atlas on $\mathcal{M}$ is a collection of charts $(\mathcal{U}_a, \phi_a)$ such that $\phi_\alpha(\mathcal{U}_\alpha)$ cover $\mathcal{M}$.*

Definition 1.1.2. *Let $\phi_\alpha : \mathcal{U}_\alpha \to \mathcal{M}$ and $\phi_\beta : \mathcal{U}_\beta \to \mathcal{M}$ be two smooth local parametrizations with overlapping images. Meaning*

$$\phi_\alpha(\mathcal{U}_\alpha) \cap \phi_\beta(\mathcal{U}_\beta) \neq \emptyset.$$

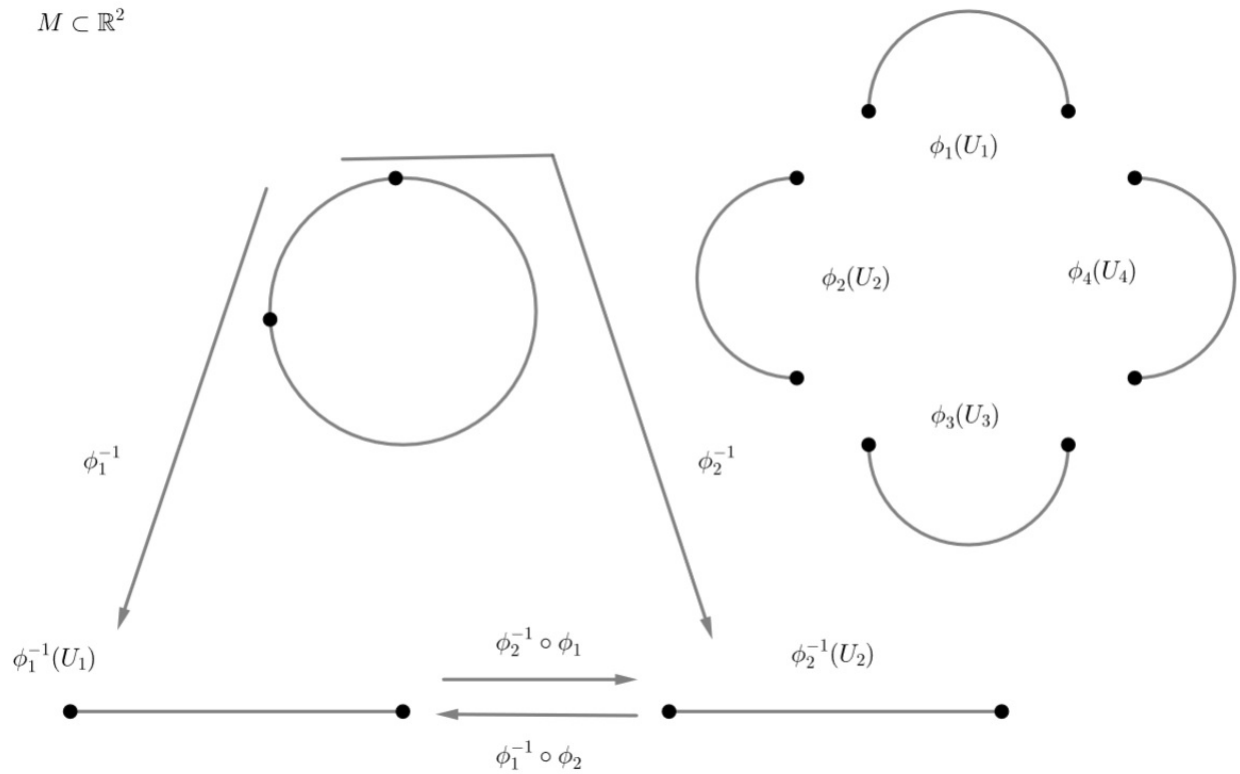
We let

$$\phi_\alpha(\mathcal{U}_\alpha) := \Theta_\alpha = \mathcal{V}_\alpha \cap \mathcal{M}$$
$$\phi_\beta(\mathcal{U}_\beta) := \Theta_\beta = \mathcal{V}_\beta \cap \mathcal{M}.$$

The homeomorphisms

$$\phi_\beta^{-1} \circ \phi_\alpha : \phi_\alpha^{-1}(\Theta_\alpha \cap \Theta_\beta) \to \phi_\beta^{-1}(\Theta_\alpha \cap \Theta_\beta)$$
$$\phi_\alpha^{-1} \circ \phi_\beta : \phi_\beta^{-1}(\Theta_\alpha \cap \Theta_\beta) \to \phi_\alpha^{-1}(\Theta_\alpha \cap \Theta_\beta)$$

we call Transition maps.



Taking the circle in the picture as example we can compute the transition map $\phi_2^{-1} \circ \phi_1$. Let the set $U_2 = \{p \in U_1 \cup U_2 \cup U_3 \cup U_4 : x < 0\}$ and let the inverse map ϕ_2^{-1} be given by $\phi_2^{-1}(y) = y$. Then the local parametrizations given by 1.1 amount to the calculation given by

$$\begin{aligned} \phi_2^{-1} \circ \phi_1(x) &= \phi_2^{-1}(\phi_1(x)) \\ &= \sqrt{1 - x^2} \end{aligned}$$

over the set where $x < 0$. Thus we have indeed shown that the homeomorphism $\phi_2^{-1} \circ \phi_1$ is a transition map.
To generalize this statement to any smooth surface of $\mathbb{R}^n$ we wish to show that transition maps are smooth (regular).

Definition 1.1.3. *A homeomorphism between two local coordinate charts $(\mathcal{U}_\alpha, \phi_\alpha), (\mathcal{U}_\beta, \phi_\beta)$ is called a Diffeomorphism if the transition map from $(\mathcal{U}_\alpha, \phi_\alpha)$ to $(\mathcal{U}_\beta, \phi_\beta)$ is differentiable and has a differentiable inverse.*

So we wish to show the general result that $\phi_\alpha^{-1} \circ \phi_\beta$ is differentiable and its inverse $(\phi_\alpha^{-1} \circ \phi_\beta)^{-1} = \phi_\beta^{-1} \circ \phi_\alpha$ is also differentiable.

Proposition 1.1.3.1. *Let $\mathcal{M} \subset \mathbb{R}^n$ be regular and $\phi_\alpha(u_1, ..., u_n) : \mathcal{U}_\alpha \to \mathcal{M}$ and $\phi_\beta(v_1, ..., v_n) : \mathcal{U}_\beta \to \mathcal{M}$ be smooth local parametrizations with overlapping images. Meaning $\mathcal{W} := \phi_\alpha(\mathcal{U}_\alpha) \cap \phi_\beta(\mathcal{U}_\beta) \neq \emptyset$. Then the transition maps defined as*

$$\begin{aligned} \phi_\beta^{-1} \circ \phi_\alpha &: \phi_\alpha^{-1}(\mathcal{W}) \to \phi_\beta^{-1}(\mathcal{W}) \\ \phi_\alpha^{-1} \circ \phi_\beta &: \phi_\beta^{-1}(\mathcal{W}) \to \phi_\alpha^{-1}(\mathcal{W}), \end{aligned}$$

are also smooth transition maps. We call those maps Smoothly compatible.

We shall show this result for the special case in $\mathbb{R}^3$. A proof of the general case can be found in [Munk91]. We use the following theorem that we state without proof. [Tu11]

Theorem 1.1.4. *Let $\phi : \mathcal{U} \to \mathbb{R}^n$ be a smooth map defined on an open subset $\mathcal{U} \subset \mathbb{R}^n$. For any point $p \in \mathcal{U}$, ϕ is locally invertible at p iff the Jacobian determinant*

$$\det \left[\frac{\partial \phi_i}{\partial u_j} \right] (p) \neq 0.$$

This is called the *Inverse Function Theorem*. We shall also use the lemma without proof

Lemma 1.1.4.1. *The composition of smooth maps is smooth.*

Proof of Proposition. 1.1.3.1 It suffices to show one of them and the other one will automatically follow by symmetry. Differentiability is a local property, so we fix a point $p \in \mathcal{W} \subset \mathcal{M}$ and show that $\phi_\beta^{-1} \circ \phi_\alpha$ is smooth at $\phi_\alpha^{-1}(p)$. Now by Condition 3 as in 1.0.1 we wish to show that

$$\frac{\partial \phi_\alpha}{\partial u_1}(p) \times \frac{\partial \phi_\alpha}{\partial u_2}(p) \neq 0.$$

Meaning it is non-zero at p. Take $\phi_\alpha(u_1, u_2) = (x(u_1, u_2), y(u_1.u_2), z(u_1, u_2))$. By computation of the cross product we obtain

$$\begin{vmatrix} e_x & \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ e_y & \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} \\ e_z & \frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial u_2} \end{vmatrix} = \begin{pmatrix} \det(\frac{\partial(y,z)}{\partial(u_1,u_2)}) \\ \det(\frac{\partial(z,x)}{\partial(u_1,u_2)}) \\ \det(\frac{\partial(x,y)}{\partial(u_1,u_2)}) \end{pmatrix} (p)\,.$$

For this to be a smooth map, at least one of the determinants is never zero. Without loss of generality let us assume that the first component

$$\det(\frac{\partial(y,z)}{\partial(u_1,u_2)})(p)$$

is non-zero. Now we define a map ψ given by

$$\psi(x,y,z) = (y,z).$$

The composition $\psi \circ \phi_\beta(v_1,v_2) = (y(v_1,v_2), z(v_1,v_2))$ by previous assumption has determinant

$$\det(\frac{\partial(y,z)}{\partial(u_1,u_2)})(p) \neq 0$$

at p. By Theorem 1.1.4 there exists a smooth inverse $(\psi \circ \phi_\beta)^{-1}$ at a neighbourhood of p. We may rewrite the composition $\phi_\beta^{-1} \circ \phi_\alpha = (\psi \circ \phi_\beta)^{-1} \circ (\psi \circ \phi_\alpha)$. Since all of them are smooth, then by Lemma 1.1.4.1 we have shown our result. □

1.2 The Tangent space of a Manifold

Definition 1.2.1. *Let $\mathcal{M} \subset \mathbb{R}^n$ be a smooth manifold. Given an open subset $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \cap \mathcal{M} \subset \mathbb{R}^m$. Given a map $\phi : \mathcal{U} \to \mathcal{V} \cap \mathcal{M}$ the derivative map $D\phi$ is linear, i.e*

$$D\phi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = \begin{pmatrix} \frac{\partial \phi_1}{\partial u_1} & .... & \frac{\partial \phi_1}{\partial u_n} \\ . & . & . \\ . & . & . \\ \frac{\partial \phi_m}{\partial u_1} & .... & \frac{\partial \phi_m}{\partial u_n} \end{pmatrix}.$$

Meaning an $m \times n$ matrix.

Definition 1.2.2. *Given a point $p \in \mathcal{M}$, let $(\mathcal{U}_\alpha, \phi_\alpha)$ be a coordinate chart such that $p \in \phi_\alpha(\mathcal{U}_\alpha)$. Then we define the Tangent space $T_p\mathcal{M}$ at p as*

$$T_p\mathcal{M} := \text{Span}\left\{\frac{\partial \phi_\alpha}{\partial u_1}, ..., \frac{\partial \phi_\alpha}{\partial u_n}\right\}.$$

From Definition 1.0.1 we know that $\text{rank}(D\phi) = n$ and thus that $\dim(T_p\mathcal{M}) = n$ (the space has trivial kernel). We shall make the following proposition.

Proposition 1.2.2.1. *The definition of Tangent space is independent of our chosen coordinate system.*

Proof. Let us begin by recalling some facts from Linear Algebra. Given an $m \times n$ matrix A

$$\text{A}= \begin{pmatrix} a_1^1 & .... & a_1^n \\ . & . & . \\ . & . & . \\ a_m^1 & .... & a_m^n \end{pmatrix}$$

we define the image of A denoted by $\text{Im}(A)$ as

$$\left\{ x : x = Ac : c = \begin{pmatrix} c_1 \\ .. \\ c_n \end{pmatrix} \in \mathbb{R}^n \right\}.$$

We have that $\text{Im}(A)$ is a vector space (the column space) of dimension n, provided that A has full rank. Now $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, and we are able to construct a change-of-basis transformation $B : \mathbb{R}^n \to \mathbb{R}^n$ where B is some invertible $n \times n$ matrix. Our goal is to show that $\text{Im}(A) = \text{Im}(AB)$. Let $D = AB$. Since B is invertible we have $DB^{-1} = A$. Given an x in $\text{Im}(A)$ there exists a c such that

$$\begin{aligned} x &= Ac \\ \Rightarrow x &= DB^{-1}c \\ \Rightarrow x &= AB(B^{-1}c). \end{aligned}$$

Now there exists some vector $d \in \mathbb{R}^n$ that is accomplished by multiplying c with B^{-1}, such that $x = ABd$. It follows that $x \in \text{Im}(AB)$. Thus we have shown that

$$\text{Im}(A) \subset \text{Im}(AB).$$

We wish to show it in the other direction as well. Take $y \in \text{Im}(AB)$. By definition we have that $y = ABd$. Set $Bd = c$ and we have that

$$y = Ac.$$

We have that $y \in \mathrm{Im}(A)$ and thus that

$$\mathrm{Im}(AB) \subset \mathrm{Im}(A).$$

We conclude that

$$\mathrm{Im}(A) = \mathrm{Im}(AB).$$

Now back to the coordinate charts. Take ϕ_α to be a local parametrization about p. And let ϕ_β be another coordinate chart about p, with domains $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ respectively. We may restrict the domains so that $\mathcal{U}_\alpha = \mathcal{U}_\beta$. Now by the smoothness of transition maps, i.e $\phi_\beta^{-1} \circ \phi_\alpha$ and $\phi_\alpha^{-1} \circ \phi_\beta$ are diffeomorphisms, we may show our result. Again, it suffices to show one of them as there will be an analogous picture with the other one. Now by the chain rule we obtain that

$$d(\phi_\beta \circ \phi_\beta^{-1} \circ \phi_\alpha) = d(\phi_\beta) \circ d(\phi_\beta^{-1} \circ \phi_\alpha) = d\phi_\alpha.$$

Now obviously $\mathrm{Im}(d\phi_\alpha) \subseteq \mathrm{Im}(d\phi_\beta)$. One can switch the arguments and the inclusion will hold in the other direction as well. Thus we have shown that

$$\mathrm{Im}(d\phi_\alpha) = \mathrm{Im}(d\phi_\beta).$$

In other words, the tangent space is independent of coordinate system. $\square$

We will for the purpose of showing later results require the notion of Partitions of unity.

Definition 1.2.3. *Given a smooth function F defined on $\mathbb{R}^m$, then we define the Support of ϕ as*

$$\mathrm{supp}(\phi) := \overline{\{x \in \mathbb{R}^m : \phi(x) \neq 0\}}.$$

That is the closure of the set

$$\{x \in \mathbb{R}^m : \phi(x) \neq 0\}.$$

Definition 1.2.4. *Let A be a union of open sets in $\mathbb{R}^n$. There exists a collection of smooth functions $F_i : \mathbb{R}^n \to \mathbb{R}$ such that*

1. $\mathrm{supp}\, F_i \subset A$ *for all i.*

2. *For any* $p \in A$ *there exists an open set* $\Omega \subset A$ *containing* p *intersecting at least one but only finitely many sets of* $\operatorname{supp} F_i$.

3. $\sum_{i=1}^{\infty} F_i(x) = 1$ *for each* $x \in A$.

4. *The sets* $\operatorname{supp} F_i$ *are compact.*

If the collection $\{F_i\}$ *satisfies the above conditions we call it a Partition of Unity subordinate to* A.

1.3 Manifolds with boundary

For the purpose of this thesis, we require a definition of a manifold with a boundary, and the notion of differentiability on those.

Definition 1.3.1. *We denote the upper half-space of* $\mathbb{R}^n$ *as* $\mathbb{H}^n$ *consisting of those points* $x \in \mathbb{R}^n$ *for which* $x_n \geq 0$. *We denote* $\mathbb{H}^n_+$ *those points for which* $x_n > 0$.

Definition 1.3.2. *Let* S *be a subset of* $\mathbb{R}^m$. *Let* $f : S \to \mathbb{R}^n$. *Then* f *is Smooth (or of Class* $\mathcal{C}^r$*) on* S *if* f *may be extended to a function* $g : \mathcal{U} \to \mathbb{R}^n$ *that is smooth on an open set* $\mathcal{U}$ *of* $\mathbb{R}^m$ *containing* S.

Lemma 1.3.2.1. *Let* $\mathcal{U}$ *be open in* $\mathbb{H}^m$ *but not in* $\mathbb{R}^m$ *and let* $\alpha : \mathcal{U} \to \mathbb{R}^n$ *be smooth. Let* $\beta : \mathcal{U}' \to \mathbb{R}^n$ *be a smooth extension of* α *defined on an open subset* $\mathcal{U}' \subset \mathbb{R}^m$. *The derivative* $D\beta(x)$ *will only depend on the* α *and is independent of* β. *We may denote it* $D\alpha(x)$.

We shall be interested in maps defined on open sets in $\mathbb{H}^n$ but not in $\mathbb{R}^n$. Recall Definition 1.0.1. The same definition of $\mathcal{M}$ holds as a subset of $\mathbb{H}^n$, however with one extension. We require a definition of the boundary of $\mathcal{M}$. We shall do just that.

Definition 1.3.3. *Let* $\mathcal{M}$ *be a smooth submanifold of* $\mathbb{R}^n$ *and let* $p \in \mathcal{M}$. *If there is a local coordinate chart* $\phi : \mathcal{U} \to \mathcal{V}$ *about* p *such that* $\mathcal{U}$ *is open in* $\mathbb{R}^m$, *then we call* p *an Interior point of* $\mathcal{M}$. *Otherwise it is a Boundary point of* $\mathcal{M}$. *We denote the boundary set of* $\mathcal{M}$ *with* $\partial\mathcal{M}$.

Lemma 1.3.3.1. *Let* $\mathcal{M}$ *be a smooth submanifold of* $\mathbb{R}^n$. *Let* $\phi : \mathcal{U} \to \mathcal{V}$ *be a local coordinate chart about* $p \in \mathcal{M}$. *Then we have*

1. *If $\mathcal{U}$ is open in $\mathbb{R}^m$, then p is an interior point of $\mathcal{M}$.*

2. *If $\mathcal{U}$ is open in $\mathbb{H}^m$ and if $p = \phi(x_0)$ for $x_0 \in \mathbb{H}^m_+$, then p is an interior point of $\mathcal{M}$.*

3. *If $\mathcal{U}$ is open in $\mathbb{H}^m$ and if $p = \phi(x_0)$ for $\mathbb{R}^{m-1} \times 0$ and then p is a boundary point of $\mathcal{M}$.*

Proof. (1) follows from the definition, with (2) we have a local coordinate chart $\mathcal{U}_0 = \mathcal{U} \cap \mathbb{H}^m_+$ that avoids the boundary. Given $\phi : \mathcal{U}_0 \to \mathcal{V}_0$ with $\mathcal{V}_0$ being the image of ϕ. This is a homeomorphism and we are done. For (3) one is able to find a proof in [Munk91]. □

Chapter 2

Forms on $\mathbb{R}^n$

We shall introduce the notion of vector forms, and then the differential forms. In so doing, we shall also introduce the operator $\wedge$ denoting the notion of *exterior multiplication*.

2.1 Properties of the exterior product

The following properties hold for exterior products. [Hubb99]

1. Distributivity. For k-forms ϕ, ω_1, ω_2 we have that

$$\phi \wedge (\omega_1 + \omega_2) = \phi \wedge \omega_1 + \phi \wedge \omega_2.$$

2. Associativity. For k-forms $\omega_1, \omega_2, \omega_3$ we have

$$(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3).$$

3. Skew-commutativity. If ϕ is a k-form and ω is an l-form. Then we have that

$$\phi \wedge \omega = (-1)^{kl} \omega \wedge \phi.$$

4. Homogeneity. For k-forms ϕ and ω and a constant $c \in \mathbb{R}$

$$(c\phi) \wedge \omega = c(\phi \wedge \omega) = \phi \wedge (c\omega).$$

2.2 Vector Forms

Definition 2.2.1. *Let $1 \leq k \leq n$. Given vectors $v_1, ..., v_n \in \mathbb{R}^n$ we form an $n \times k$ matrix*

$$V = \begin{pmatrix} v_1^1 & .... & v_1^n \\ . & . & . \\ . & . & . \\ v_k^1 & .... & v_k^n \end{pmatrix}.$$

Now given indices $i_1, ..., i_k \in \{1, ..., n\}$ let $V_{i_1 ... i_k}$ be the matrix formed by choosing the the rows $i_1, ..., i_k$ of V.

Then we define a multilinear and alternating map

$$dx_{i_1} \wedge ... \wedge dx_{i_k} : \mathbb{R}^n \times ... \times \mathbb{R}^n \to \mathbb{R}$$
$$(v_1, ..., v_n) \mapsto (dx_{i_1} \wedge ... \wedge dx_{i_k})(v_1, ..., v_k)$$

given by

$$(dx_{i_1} \wedge ... \wedge dx_{i_k})(v_1, ..., v_k) := \det(V_{i_1, ..., i_k}).$$

The words "multilinear" and "alternating" require some explanation. We shall explain them with the help of a few facts from algebra.

Lemma 2.2.1.1. *[Biggs02] Let S_n be the symmetric group with n elements. Any permutation $\sigma \in S_n$ can be expressed in terms of products of transpositions. That is, cycles of length 2.*

Definition 2.2.2. *We call a permutation σ Odd if it can be expressed in terms of an odd number of transpositions. We call it Even if it can be expressed in terms of an even number of transpositions. The sign function denoted sgn assigns the value $sgn(\sigma) = -1$ if σ is odd and $sgn(\sigma) = 1$ if σ is even.*

Definition 2.2.3. *A definition of the determinant of an $n \times n$ matrix A is given by the following. For a square matrix $A = (a_{ij})$*

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} ... a_{n,\sigma(n)}.$$

Now, by properties of determinants we note the following facts.

1. If $j \neq l$ such that $i_j = i_l$ then $dx_{i_1} \wedge ... \wedge dx_{i_k} = 0$.

   In other words, if any two elements in the exterior product are equal then the whole thing is zero.

2. Take entries i, j in the form. Swapping places of i and j will render

$$\omega(v_1, ..., v_i, v_j, ..., v_k) = -\omega(v_1, ..., v_j, v_i, ...v_k).$$

   Meaning that swapping any two entries picks up a minus sign. This is what alternating means.

3. Suppose we take the i-th entry in the exterior product. Then

$$\omega(v_1, .., v_{i-1}, av + bw, ..., v_k) = a\omega(v_1..., v_{i-1}, v..., v_k) + b\omega(v_1..., v_{i-1}, w..., v_k).$$

   This is what multilinear means. Linear in each coordinate.

Now we may proceed by defining a Form.

Definition 2.2.4. *Let $\omega : \mathbb{R}^n \times ... \times \mathbb{R}^n \to \mathbb{R}$ be a map satisfying the properties that ω is multilinear and alternating. Then we call such a map a k-form.*

Definition 2.2.5. *The set of k-forms denoted*

$$A_c^k(\mathbb{R}^n)$$

constitutes a vector space by defining

$$(a\omega_1 + b\omega_2)(v_1, ..., v_k) = a\omega_1(v_1, ..., v_k) + b\omega_2(v_1, ..., v_k)$$

for all $a, b \in \mathbb{R}$ and $\omega_1, \omega_2 \in A_c^k(\mathbb{R}^n)$.

We shall begin by showing some forms that input vectors, and some examples.

Definition 2.2.6. *Let $I = \{i_1, ..., i_k\}$. And that*

$$dx_I := dx_{i_1} \wedge ... \wedge dx_{i_k}.$$

An elementary (or vector) k-form on $\mathbb{R}^n$ is an expression of the form

$$dx_{i_1} \wedge ... \wedge dx_{i_k}$$

where $1 \leq i_1 < ... < i_k \leq n$.

We denote the set

$$\mathcal{I}_k = \{I = \{i_1, ..., i_k\} : 1 \leq i_1 < ... < i_k \leq n\}.$$

We shall also state and prove the following useful theorem.

Theorem 2.2.7. *A basis for $A_c^k(\mathbb{R}^n)$ is given by*

$$\xi = \{dx_{i_1} \wedge ... \wedge dx_{i_k} : 1 \leq i_1 < ... < i_k \leq n\}.$$

Proof. Recall that for any k-form, the act of swapping two of the vectors picks up a minus sign, implying that if indices coincide, the entire thing is zero. It therefore suffices to consider k-forms with strictly increasing indices. In other words we have

$$dx_{i_1} \wedge ... \wedge dx_{i_k}(e_{j_1}, ..., e_{j_k}) = \begin{cases} 1 & \text{if} \quad i_1 = j_1, ..., i_k = j_k \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Now to show linear independence of our proposed basis elements we may consider a k-form

$$\omega(e_{j_1}, ..., e_{j_k}) = \sum_{1 \leq i_1 < ... < i_k \leq n} \alpha_{i_1,...,i_k} dx_{i_1} \wedge .. \wedge dx_{i_k}(e_{j_1}, ..., e_{j_k}).$$

Then this implies that $\alpha_{i_1,...,i_k} = 0$. Since if we consider

$$\omega(e_{j_1}, ..., e_{j_k}) = \sum_{1 \leq i_1 < ... < i_k \leq n} \alpha_{i_1,...,i_k} \begin{vmatrix} e_{j_1}^{i_1} & .... & e_{j_k}^{i_1} \\ . & . & . \\ . & . & . \\ e_{j_1}^{i_k} & .... & e_{j_k}^{i_k} \end{vmatrix}.$$

Then by 2.1 this says that

$$\omega(e_{j_1}, ..., e_{j_k}) = \begin{cases} \alpha_{i_1,...,i_k} & \text{if} \quad i_1 = j_1, ..., i_k = j_k \\ 0 & \text{otherwise.} \end{cases}$$

proving linear independence. Now we proceed by evaluating ω on k standard basis vectors, not necessarily listed in increasing order, but due to Definition 2.2.2 there is a permutation σ of these elements that results in increasingly ordered elements. In other words, once $i_1, ..., i_k$ are chosen (and they are distinct) there exists σ such that $i_{\sigma(1)} < ... < i_{\sigma(k)}$ is strictly increasing. Setting $\sigma^{-1} = \tau$ we obtain

$$\omega(v_1, ..., v_k) = \sum_{1 \leq i_1 < ... < i_k \leq n} \sum_{\tau \in S_k} \operatorname{sgn}(\tau)(v_{i_1,\tau(1)}, ..., v_{i_k,\tau(k)})\omega(e_{i_1}, ..., e_{i_k}).$$

Note however that due to Definition 2.2.3

$$\begin{aligned}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)(v_{i_1,\sigma(1)}, ..., v_{i_k,\sigma(k)}) &= \det(V_{i_1,...,i_k}) \\ &= (dx_{i_1} \wedge ... \wedge dx_{i_k})(v_1, ..., v_k).\end{aligned}$$

And since we established that $\omega(v_1, ..., v_k) = \alpha_{i_1,...,i_k}$. We obtain that

$$\omega(v_1, ..., v_k) = (\sum_{1 \leq i_1 < ... < i_k \leq n} \alpha_{i_1,...,i_k} dx_{i_1} \wedge ... \wedge dx_{i_k})(v_1, ..., v_k)$$

proving the vectors on this form span the set. We have shown that ξ is indeed a basis for $A_c^k(\mathbb{R}^n)$. □

An interesting fact/lemma is that

Lemma 2.2.7.1. *There are $\binom{n}{k}$ elementary k-forms.*

Proof. Let the basis for $A_c^k(\mathbb{R}^n)$ be the set (as per 2.2)

$$\{dx_{i_1} \wedge ... \wedge dx_{i_k} : 1 \leq i_1 < ... < i_k \leq n\}.$$

For any k-form in $A_c^k(\mathbb{R}^n)$ we can make the following argument. We pick k different 1-forms, from a set of n forms in total (so dx_{i_1} up to dx_{i_k}). These are, linearily independent, and we wedge them together. In so doing we produce a non-zero elementary k-form. We may choose k elements from n total ones in $\binom{n}{k}$ ways, and we have shown the result. □

Consequently we have that $\dim(A_c^k(\mathbb{R}^n)) = \binom{n}{k}$. Take the example of $n = 4$ and $k = 2$. There will be six elementary forms arranged in ascending order. Namely

$$\begin{matrix}\{\ v_1 \wedge v_2, & v_1 \wedge v_3, & v_1 \wedge v_4, & \\ & v_2 \wedge v_3, & v_2 \wedge v_4, & . \\ & & v_3 \wedge v_4\ \} & \end{matrix}$$

Evaluating a vector k-form on a set of vectors $v_1...v_k$ will produce a $k \times k$ matrix determinant. So for instance when $n = 2$ and $k = 2$. We have that

$$\omega_1 \wedge \omega_2(v_1, v_2) = \begin{vmatrix} \omega_1(v_1) & \omega_2(v_1) \\ \omega_1(v_2) & \omega_2(v_2) \end{vmatrix}.$$

This will give the area of a paralellogram. For example let $\omega_1 = 3dx+dy+4dz$ and $\omega_2 = dx - 2dy + dz$. And we take vectors $v_1 = (1, 4, 2)$ and $v_2 = (2, 5, 4)$. Our calculation amounts to the following. We get that $\omega_1(v_1) = 3 \cdot 1 + 4 + 4 \cdot 2 = 15$, $\omega_1(v_2) = 3 \cdot 2 + 5 + 4 \cdot 4 = 27$, $\omega_2(v_1) = 1 - 2 \cdot 4 + 2 = -5$ and $\omega_2(v_2) = 2 - 2 \cdot 5 + 4 = -4$. Thus we obtain the calculation

$$\omega_1 \wedge \omega_2(v_1, v_2) = \begin{vmatrix} 15 & -5 \\ 27 & -4 \end{vmatrix} = 75.$$

When $n = 3$ and $k = 3$ we get the area of a paralellopiped. For example, let $\psi = 2dx \wedge dy \wedge dz$. Explicitly meaning $\psi_1 = 2dx, \psi_2 = dy, \psi_3 = dz$. Take the vectors $v_1 = (4, 2, 1)$, $v_2 = (0, 1, 1)$ and $v_3 = (-4, -2, 1)$. We wish to calculate the determinant given by

$$\begin{vmatrix} \psi_1(v_1) & \psi_2(v_1) & \psi_3(v_1) \\ \psi_1(v_2) & \psi_2(v_2) & \psi_3(v_2) \\ \psi_1(v_3) & \psi_2(v_3) & \psi_3(v_3) \end{vmatrix} = \begin{vmatrix} 8 & 2 & 1 \\ 0 & 1 & 1 \\ -8 & -2 & 1 \end{vmatrix}.$$

We can row-reduce this matrix. We simply add the first row to the third to obtain an upper triangular matrix. The determinant of that matrix is just the product of the diagonal entries. We obtain

$$\begin{vmatrix} 8 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 8 \cdot 1 \cdot 2 = 16.$$

2.3 Differential Forms

Now introducing general differential k-forms.

Definition 2.3.1. *A differential k-form on $\mathbb{R}^n$ is given as*

$$\omega = \sum_I f_{i_1,\ldots,i_k} dx_{i_1} \wedge .. \wedge dx_{i_k}$$

where all the $f_{i_n} : (\mathbb{R}^n)^k \to \mathbb{R}$ are continuous and differentiable functions.

With differential forms, we have a function that constantly changes the vector for any point in the tangent space (i.e a vector field). For the sake of demonstration, we consider $\mathbb{R}^3$. Take the example of a differential 2-form $\omega = xzdx \wedge dy + yzdy \wedge dz + (x^2 + y^2)dz \wedge dx$. Now we specify a base point p in the space where we want to evaluate the form. Take $p = (1, 2, 3)$. Then we simply input these values into their corresponding functions, and obtain

$$\omega_{(1,2,3)} = (1)(3)dx \wedge dy + (2)(3)dy \wedge dz + (1^2 + 2^2)dz \wedge dx.$$

Meaning $\omega_p = 3dx \wedge dy + 6dy \wedge dz + 5dz \wedge dx$. Let us pick two vectors. We consider $v_1 = (1, 2, 3)$ and $v_2 = (2, 0, 1)$. The calculation amounts to

$$\omega_{(1,2,3)}(v_1, v_2) = 3 \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 6 \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} + 5 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}.$$

Obviously this just amounts to

$$\omega_{(1,2,3)}(v_1, v_2) = 3(-4) + 6(2) + 5(5) = 25.$$

We can also consider vector fields at an arbitrary point $p = (x, y, z)$ in three-dimensional space. Let us consider a differential form $\omega = y^2dx \wedge dy - 3x^2dy \wedge dz$. Consider also vector fields $v_1 = (xz, yz, x^2 + y^2)$ and $v_2 = (y, xz, z^2)$. Essentially we do exactly as we did with fixed vectors in space. We get

$$\omega_{(x,y,z)}(v_1, v_2) = y^2 \begin{vmatrix} xz & yz \\ y & xz \end{vmatrix} - 3x^2 \begin{vmatrix} yz & xz \\ x^2 + y^2 & z^2 \end{vmatrix}.$$

Of course we simply take the determinants and arrive at

$$w_{(x,y,z)}(v_1, v_2) = (xyz)^2 - y^2z - 3x^2z^3y - 3x^5z + 3x^3y^2z.$$

So essentially, ω is a differential 2-form. It takes in two vector fields and has returned a function in $\mathbb{R}^3$.

2.4 The exterior derivative

Definition 2.4.1. *The exterior derivative denoted with a lower-case d, is a linear operator taking a k-form to a $(k+1)$-form. Given any smooth k-form,* *defined on $\mathbb{R}^n$, we have that*

$$\omega = \sum_{j_1,...j_k=1}^{n} \omega_{j_1,...j_k} du_{j_1} \wedge ... \wedge du_{j_k},$$

and define

$$d\omega := \sum_{j_1,...j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial \omega_{j_1,...,j_k}}{\partial u_i} du_i \wedge du_{j_1} \wedge ... \wedge du_{j_k}.$$

For smooth scalar functions the exterior derivative acts like the chain rule. For example, let $f(x, y, z)$ be a smooth scalar function on $\mathbb{R}^3$. The exterior derivative of f will be

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

We also introduce the following rules for the exterior derivative. For any k-forms ω and ψ

$$\begin{aligned} &1) \quad d(a\omega + b\psi) = ad\omega + bd\psi && \text{(Linearity)} \\ &2) \quad d(\psi \wedge \omega) = d\psi \wedge \omega + (-1)^k \psi \wedge d\omega \\ &3) \quad d(d\omega) = d^2(\omega) = 0. \end{aligned}$$

We can prove each and every one of them.

Proof of rules for the exterior derivative. 1) Suppose ω and ψ are k-forms on an open set $\mathcal{U} \subset \mathcal{M}$. Then by definition we have

$$\omega = \sum_{j_1,...j_k=1}^{n} \omega_{j_1,...j_k} du_{j_1} \wedge ... \wedge du_{j_k}$$
$$\psi = \sum_{j_1,...j_k=1}^{n} \psi_{j_1,...j_k} du_{j_1} \wedge ... \wedge du_{j_k}.$$

Taking the exterior derivative produces

$$\begin{aligned} d(a\omega + b\psi) &= \sum_{j_1,\dots j_k=1}^{n} d(a\omega_{j_1,\dots j_k} + b\psi_{j_1,\dots,j_k}) du_{j_1} \wedge \dots \wedge du_{j_k} \\ &= a \sum_{j_1,\dots j_k=1}^{n} \frac{\partial \omega_{j_1,\dots,j_k}}{\partial u_i} du_i \wedge du_{j_1} \wedge \dots \wedge du_{j_k} \\ &+ b \sum_{j_1,\dots,j_k=1} \frac{\partial \psi_{j_1,\dots,j_k}}{\partial u_i} du_i \wedge du_{j_1} \wedge \dots \wedge du_{j_k} \\ &= ad\omega + bd\psi. \end{aligned}$$

2) Since d is linear it suffices to assume $\psi = f(du_{i_1} \wedge \dots \wedge du_{i_k})$ and $\omega = g(du_{j_1} \wedge \dots \wedge du_{j_l})$. Then we have that

$$\begin{aligned} d(\psi \wedge \omega) &= \sum_{i=1}^{n} \partial(fg) du_{i_1} \wedge \dots \wedge du_{i_k} \wedge du_{j_1} \wedge \dots \wedge du_{j_l} \\ &= (\sum_{i=1}^{n} \frac{\partial f}{\partial u_i} du_i \wedge du_{i_1} \wedge \dots \wedge du_{i_k} \wedge du_{j_1} \wedge \dots \wedge du_{j_l}) \wedge \omega \\ &+ \sum_{i=1}^{n} \frac{\partial g}{\partial u_i} du_i \wedge du_{j_1} \wedge \dots \wedge du_{j_k} \\ &= d\psi \wedge \omega + (-1)^k \psi \wedge d\omega. \end{aligned}$$

This is by applying the product rule for derivatives for every term in the k-form, and then rearranging the exterior products.

3) We wish to show that applying the operation twice, i.e $d(d\omega)$ gives us zero. Take

$$d\omega = \sum_{i=0}^{n} \sum_{I} \frac{\partial \omega_I}{\partial u_i} du_i \wedge du_I.$$

Then taking the derivative of that produces

$$\begin{aligned}
d(d\omega) &= \sum_{j=0}^{n} \frac{\partial}{\partial u_j} \left(\sum_{i=0}^{n} \sum_{I} \frac{\partial \omega_I}{\partial u_i} du_i \right) du_j \\
&= \sum_{j=0}^{n} \sum_{i=0}^{n} \sum_{I} \frac{\partial^2 \omega_I}{\partial u_j \partial u_i} du_j \wedge du_i \wedge du_I \\
&= \sum_{i<j} \frac{\partial^2 \omega_I}{\partial u_j \partial u_i} - \frac{\partial^2 \omega_I}{\partial u_i \partial u_j} du_i \wedge du_j \wedge du_I \\
&= 0.
\end{aligned}$$

Where $I = (i_1, ...i_k)$, and using the fact that $dx \wedge dy = -dy \wedge dx$ and that mixed partials of the form $\frac{\partial^2 f}{\partial u_j \partial u_i} = \frac{\partial^2 f}{\partial u_i \partial u_j}$ are equal by Clairaut's Theorem. □

2.5 The Pullback

Definition 2.5.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. We define the Pullback of T by*

$$\begin{aligned}
&T^* : A_c^k(\mathbb{R}^m) \to A_c^k(\mathbb{R}^n) \\
&\omega \mapsto T^*(\omega) : \mathbb{R}^n \times ... \times \mathbb{R}^n \to \mathbb{R} : \\
&(v_1, ..., v_k) \mapsto T^*\omega(v_1, ..., v_k).
\end{aligned}$$

By definition we have that

$$T^*(\omega)(v_1, ..., v_k) := \omega(T(v_1), ...T(v_k)).$$

We make the following statement without proof. [Fong18]

Theorem 2.5.2. *Let $f : \mathcal{U} \in \mathbb{R}^n \to \mathcal{V} \in \mathbb{R}^m$ be smooth. Given a k-form $\omega \in A_c^k(\mathcal{V})$ then we have that*

$$d(f^*\omega) = f^*(d\omega).$$

In other words, the exterior derivative and the Pullback commute.
We can make the following proposition.

Proposition 2.5.2.1. *Let* $(y_1, ..., y_i)$ *be coordinates in* $\mathbb{R}^m$ *and* $(x_1, ..., x_j)$ *coordinates in* $\mathbb{R}^n$*. Now let* $F : \mathbb{R}^n \to \mathbb{R}^m$ *be a smooth map. Then we have that*

$$F^*(dx_1 \wedge ... \wedge dx_n) = J(y)dy_1 \wedge ... \wedge dy_n$$

where

$$J(y) = \det \left(\frac{\partial F_j}{\partial y_i} \right)_{i,j=1}^{n}$$

i.e, the determinant of the Jacobian matrix.

Proof. The elements $i_1...i_n$ are a permutation of the numbers $1, ..., n$. There is some permutation σ that orders these elements in increasing order. We write

$$\sum_{\sigma \in S_n} \frac{\partial F_1}{\partial y_{\sigma(1)}} ... \frac{\partial F_n}{\partial y_{\sigma(n)}} dy_{\sigma(1)} \wedge ... \wedge dy_{\sigma(n)}.$$

By Lemma 2.2.1.1 this permutation is expressible in terms of transpositions, by 2.2.2 the number of transpositions is either even or odd. So we obtain

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\partial F_1}{\partial y_{\sigma(1)}} ... \frac{\partial F_n}{\partial y_{\sigma(n)}} dy_1 \wedge ... \wedge dy_n.$$

The term

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\partial F_1}{\partial y_{\sigma(1)}} ... \frac{\partial F_n}{\partial y_{\sigma(n)}}$$

is by 2.2.3 the definition of the determinant of the $n \times n$ matrix consisting of these partial derivatives from F_1 to F_n. Otherwise known as the Jacobian determinant. We conclude that

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\partial F_1}{\partial y_{\sigma(1)}} ... \frac{\partial F_n}{\partial y_{\sigma(n)}} dy_1 \wedge ... \wedge dy_n = J(y) dy_1 ... dy_n.$$

Which is what we wanted to show. □

2.6 Forms on manifolds

In this section we define Forms over submanifolds of Euclidean space.

Definition 2.6.1. *Given a manifold $\mathcal{M}$ of dimension k and an open set $\mathcal{U} \subset \mathbb{R}^m$ such that $\mathcal{M} \subset \mathcal{U}$ and ω an l-form defined on $\mathcal{U}$. Since for every $p \in \mathcal{M} \subset \mathbb{R}^m$, $T_p\mathcal{M}$ is a vector subspace of $\mathbb{R}^m$, we may consider the restriction of ω to $T_p\mathcal{M}$ as*

$$\omega(p)(v_1, ..., v_k) \qquad v_1, ..., v_k \in T_p\mathcal{M}.$$

Given a local coordinate chart $(\mathcal{U}_\alpha, \phi_\alpha)$ the mapping

$$\begin{aligned} &\mathcal{U}_\alpha \to \mathbb{R} \\ &\mathcal{U} \mapsto \omega(\phi_\alpha(u))(\frac{\partial \phi_\alpha}{\partial u_1}, ..., \frac{\partial \phi_\alpha}{\partial u_n}) \end{aligned}$$

is well defined and smooth on $\mathcal{U}_\alpha$.

2.7 Integration of forms

The goal of this section is to explain integration of k-forms. That is an integral such as

$$\int_{\mathcal{M}} \omega.$$

Definition 2.7.1. *We define a parametrization*

$$F(u_1, u_2, ..., u_k) : (p_1, q_1) \times (p_2, q_2) \times ... \times (p_k, q_k) \to \mathcal{M}.$$

Then given a k-form $\omega(u_1, u_2, ..., u_k)du_1 \wedge du_2 \wedge ... \wedge du_k$, the integral of ω over $\mathcal{M}$ is given by

$$\int_{\mathcal{M}} \omega(u_1, u_2, ..., u_k)du_1 \wedge du_2 \wedge ... \wedge du_k := \int_{p_k}^{q_k} ... \int_{p_1}^{q_1} \omega(u_1, u_2, ..., u_k)du_1 du_2 ... du_k.$$

A 1-form must be integrated over a curve, a 2-form over a surface, and a k-form over a region in higher-dimensional space. So we integrate forms over a region in of the same dimension.

Suppose $f(x, y)$ is smooth function. Let the manifold $\mathcal{M}$ be given by the graph

$$\mathcal{M} = \{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2\},$$

that has global parametrization in the form of a field $F : \mathbb{R}^2 \mapsto \mathcal{M}$ explicitly given by

$$F(x, y) = (x, y, f(x, y)).$$

For instance, now let a 2-form be given by

$$\omega = e^{-x^2+y^2} dx \wedge dy.$$

This amounts to the double integral

$$\iint_{\mathbb{R}^2} e^{-x^2+y^2} dxdy.$$

By a change to polar coordinates you can show that this has the value π. Note, how did we go from having wedges to no wedges? It seems tempting in an expression like $dx \wedge dy$ to just erase the wedge to produce $dxdy$ and forget we have a differential form. It seems we could instead write the form

$$\omega = e^{-x^2+y^2} dy \wedge dx.$$

The double integral of this does not coincide with the first one as it would give the result $-\pi$. We can remedy this by simply defining that the form should be in the right order before we integrate.

Chapter 3

Orientation

We shall give a few definitions.

Definition 3.0.1. *Let $g : \mathcal{U} \to \mathcal{V}$ be a diffeomorphism between open sets in $\mathbb{R}^m$. We say that g is Orientation-preserving if $\det Dg > 0$ on $\mathcal{U}$. We call g Orientation-reversing if $\det Dg < 0$ on $\mathcal{U}$.*

Definition 3.0.2. *Let $\mathcal{M}$ be a m-manifold in $\mathbb{R}^n$. Given local coordinate charts $\phi_i : \phi_i(\mathcal{U}_i) \to \phi_i(\mathcal{V}_i)$ on $\mathcal{M}$ for $i = 0, 1$, we say they overlap positively if the transition map $\phi_1^{-1} \circ \phi_0$ is orientation-preserving. If $\mathcal{M}$ can be covered by a collection of local coordinate charts, each pair of which overlap positively, then $\mathcal{M}$ is said to be Orientable. Otherwise, $\mathcal{M}$ is said to be non-orientable.*

Definition 3.0.3. *Let $\mathcal{M}$ be an orientable m-manifold. Given a collection of coordinate charts covering $\mathcal{M}$ that overlap positively, we adjoin this collection all other coordinate charts on $\mathcal{M}$ that overlap these charts positively. This expanded collection is called an Orientation on $\mathcal{M}$. A manifold $\mathcal{M}$ together with an orientation of $\mathcal{M}$, we call an Oriented manifold.*

Certainly, this makes no sense when speaking about a 0-manifold, which is just a set of points. For some dimensions, this becomes more intuitive. In $\mathbb{R}$, we can think about "left" and "right", in $\mathbb{R}^2$ one can imagine "clockwise" and "anti-clockwise", and in $\mathbb{R}^3$ we can imagine "right-handed" and "left-handed". If we consider an $(n-1)$-manifold in $\mathbb{R}^n$, we picture an orientation on $\mathcal{M}$ as a unit normal vector field to $\mathcal{M}$.

Definition 3.0.4. *Let $\mathcal{M}$ be an $(n-1)$ manifold in $\mathbb{R}^{n-1}$. If $p \in \mathcal{M}$, let $(p; n-1)$ be a unit vector in the vector space $T_p(\mathbb{R}^n)$, that is orthogonal to the* *linear subspace of $T_p\mathcal{M}$ of dimension $(n-1)$. Then n is uniquely determined up to a sign. Let a local coordinate chart be given as in 1.2. We specify the sign by requiring that the frame $(\vec{n}, \frac{\partial \phi_\alpha}{\partial x_1}, ..., \frac{\partial \phi_\alpha}{\partial x_{n-1}})$ be right-handed. Meaning that the matrix $[\vec{n}\ D\phi_\alpha]$ has positive determinant.*

A consequence of Definition 3.0.4 is that when integrating an $(n-1)$ form over a manifold with boundary with induced orientation, we also take into account the parity of n. This can be formulated as the following theorem that we state without proof. [Fong18]

Theorem 3.0.5. *Given a positively oriented local coordinate chart $G(u_1, ..., u_n) : \mathcal{V} \to \mathcal{M}$ of boundary type. Then $(u_1, ..., u_{n-1})$ is positively oriented if n is even and negatively oriented if n is odd. Therefore when integrating an $(n-1)$-form $\omega du_1 \wedge ... \wedge du_{n-1}$ on $\partial\mathcal{M}$ we have that*

$$\int_{G(\mathcal{V})\cap\partial\mathcal{M}} \omega du_1 \wedge ... \wedge du_{n-1} = (-1)^n \int_{\mathcal{V}\cap\{u_n=0\}} \omega du_1 \wedge ... \wedge du_{n-1}.$$

Theorem 3.0.6. *If $\mathcal{M}$ is an orientable m-manifold with non-empty boundary, then $\partial\mathcal{M}$ is orientable.*

A proof of this is found in [Munk91].

Chapter 4

Generalized Stokes' Theorem

Our efforts have concluded in showing the following result.

Theorem 4.0.1. *Let $\mathcal{M}$ be an oriented smooth manifold of dimension n. Let ω be a compactly supported smooth $(n-1)$-form on $\mathcal{M}$. We then have that*

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega.$$

This is known as the Generalized Stokes' Theorem.

Proof of Generalized Stokes' Theorem. The proof is by cases. We begin with the case where the form is contained inside a single parametrization of interior type.

We have that ω is a $(n-1)$-form, with its support contained in a single parametrization chart of interior type.
We can write the $(n-1)$-form as

$$\omega = \sum_{i=1}^{n} \omega_i du_1 \wedge ... \wedge du_{i-1} \wedge du_{i+1} \wedge ... \wedge du_n.$$

We have removed the i-th term to obtain an $(n-1)$-form.

We take the exterior derivative to obtain

$$d\omega = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_i}{\partial u_j} du_j \wedge du_1 \wedge ... \wedge du_{i-1} \wedge du_{i+1} \wedge ... \wedge du_n.$$

Now for each i, the product $du_1 \wedge ... \wedge du_{i-1} \wedge du_{i+1} \wedge ... \wedge du_n$ is zero if $j \neq i$. The only sum that will survive as it were, is the one where $j = i$. This reintroduces the du_i to the beginning of the sum. We wish to put it in the order it should be. Thus we make $i-1$ swaps of wedge products and pick up a factor of $(-1)^{i-1}$. Hence we obtain

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial u_i} du_1 \wedge ... \wedge du_{i-1} \wedge du_{i+1} \wedge ... \wedge du_n \\ &= \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial u_i} du_i \wedge du_1 \wedge ... \wedge du_n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du_1 \wedge ... \wedge du_i \wedge ... \wedge du_n. \end{aligned}$$

By Definition 2.7.1 we get

$$\int_{\mathcal{M}} d\omega = \int_{\mathcal{U}} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du_1 ... du_n.$$

The fact that ω has compact support means that there is a number $R > 0$ such that $\operatorname{supp} \omega$ is contained inside a rectangle $[-R, R] ... \times ... \times [-R, R]$ in $\mathbb{R}^n$. Also, outside the support the integral will just be zero. Then using Fubini's theorem we can change the order of integration and summation to produce

$$\begin{aligned} \int_{\mathcal{M}} d\omega &= \int_{-R}^{R} ... \int_{-R}^{R} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du_1 ... du_n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_{-R}^{R} ... \int_{-R}^{R} \frac{\partial \omega_i}{\partial u_i} du_1 ... du_n. \end{aligned}$$

Now take the innermost integral. By the Fundamental Theorem of Calculus we obtain

$$\int_{-R}^{R} \frac{\partial \omega_i}{\partial u_i} du_i = \omega_i \Big|_{-R}^{R}.$$

This pattern will follow for the other integrals. Since $\operatorname{supp}(\omega)$ is contained in a closed box of interior type, the value of all the ω_i's will be zero at $-R$

and R (at the boundary). Thus we have proven that

$$\int_{\mathcal{M}} d\omega = 0.$$

Equivalently we have that on the boundary $\partial\mathcal{M}$ we have $\omega = 0$. Thus we have shown our first result, namely

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega = 0.$$

We may now proceed with showing the next case. Namely when $\mathcal{M}$ is covered by a single parametrization of boundary type. The difference from the previous step consists in knowing that ω_i's may not vanish on the boundary of $\mathcal{M}$. We instead consider the set in the upper half plane with rectangle $A = [-R, R] \times ... \times [-R, R] \times [0, R]$ with $R > 0$ as large so as to contain the support of ω. Since it has compact support we can as in the previous step deduce that $\omega_i(u_1, ..., u_n) = 0$ when $u_n = R$. But not that $\omega_i = 0$ when $u_n = 0$.

Following the same calculation as in step 1 we obtain

$$\begin{aligned}
\int_{\mathcal{M}} d\omega &= \int_0^R \int_{-R}^R ... \int_{-R}^R \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du_1 ... du_n \\
&= \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R ... \int_{-R}^R \frac{\partial \omega_i}{\partial u_i} du_1 ... du_n \\
&+ (-1)^{n-1} \int_0^R \int_{-R}^R ... \int_{-R}^R \frac{\partial \omega_n}{\partial u_n} du_1 ... du_n.
\end{aligned}$$

The first term is as we obtained in the first step, namely

$$\sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R ... \int_{-R}^R \frac{\partial \omega_i}{\partial u_i} du_1 ... du_n = 0,$$

so we need to consider the term

$$(-1)^{n-1} \int_{-R}^R \int_{-R}^R ... \int_0^R \frac{\partial \omega_n}{\partial u_n} du_1 ... du_n.$$

Integration is linear, so computing the innermost integral produces

$$(-1)^n \int_{-R}^{R} ... \int_{-R}^{R} \omega_n(u_1, ..., u_{n-1}, 0) du_1 ... du_{n-1}.$$

We now wish to relate this to

$$\int_{\partial \mathcal{M}} \omega$$

for the purpose of showing that the two integrals are indeed equal. On the boundary $\partial \mathcal{M}$ are points where $u_n = 0$. Hence $du_n = 0$ across $\partial \mathcal{M}$. We then have that

$$\begin{aligned} \omega &= \sum_{i=1}^{n} \omega_i(u_1, ..., u_{n-1}, 0) du_1 \wedge ... \wedge du_{i-1} \wedge du_{i+1} \wedge ... \wedge du^n \\ &= \omega_n(u_1, ..., u_{n-1}, 0) du_1 \wedge ... \wedge du_{n-1}. \end{aligned}$$

So consequently we obtain that

$$\begin{aligned} \int_{\partial \mathcal{M}} \omega &= \int_{A \cap \partial \mathcal{M}} \omega_n(u_1, ..., u_{n-1}, 0) du_1 \wedge ... \wedge du_{n-1} \\ &= \int_{A \cap \{u_n = 0\}} \omega_n(u_1, ..., u_{n-1}, 0) du_1 ... du_{n-1} \\ &= \int_{-R}^{R} ... \int_{-R}^{R} \omega_n(u_1, ..., u_{n-1}, 0) du_1 ... du_{n-1}. \end{aligned}$$

We have gotten that the two sides are exactly equal apart from the factor of $(-1)^n$. This factor only takes into account the orientation as per Theorem 3.0.5. We have thus proven the result of step 2 that is

$$\int_{\partial \mathcal{M}} \omega = \int_{\mathcal{M}} d\omega.$$

Our final case is where we use partitions of unity to prove a more general case of Stokes' theorem.

We take

$$A = \{F_a : U_a \to \mathcal{M}\}$$

to be an atlas of $\mathcal{M}$, by assumption with all positively oriented coordinates. So A contains both interior and boundary types of local parametrizations. Suppose now that

$$\{\rho_a : \mathcal{M} \to [0,1]\}$$

is a partition of unity subordinate to A. We then obtain

$$\omega = \sum \rho_a \omega$$

since by Definition 1.2.4 $\sum \rho_a = 1$. Following the same procedure as before we obtain

$$\begin{aligned}
\sum_a \int_{\partial\mathcal{M}} \rho_a\omega &= \sum_a \int_{\mathcal{M}} d(\rho_a\omega) \\
&= \sum_a \int_{\mathcal{M}} d(\rho_a \wedge \omega + \rho_a d\omega) && \text{(By 2.4)} \\
&= \int_{\mathcal{M}} d\left(\sum_a \rho_a\right) \wedge \omega + \left(\sum_a \rho_a\right) d\omega \\
&= \int_{\mathcal{M}} 0 \wedge \omega + 1 d\omega \\
&= \int_{\mathcal{M}} d\omega.
\end{aligned}$$

Since we have that

$$\sum_a \int_{\partial\mathcal{M}} \rho_a\omega = \int_{\partial\mathcal{M}} \omega,$$

we finally obtain that

$$\int_{\partial\mathcal{M}} \omega = \int_{\mathcal{M}} d\omega.$$

Which is what we wanted to show, and the proof is complete. □

4.1 Applications of the Generalized Stokes' Theorem in Vector Calculus

In this section we wish to derive two examples of special cases of the Generalized Stokes' Theorem. Namely Green's Theorem and the Divergence Theorem.

Theorem 4.1.1. *Let R be a closed, bounded and smooth submanifold of $\mathbb{R}^2$. Let $C = \partial R$ be the boundary of R, such that C is a simple closed curve in the plane, and R is a region contained by C. Let*

$$F(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$$

be a smooth vector field defined in R. Then

$$\int_C P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

This is known as Green's Theorem from Vector Calculus. We may proceed by proving this with the use of differential forms and the Generalized Stokes' Theorem.

Proof. Consider the 1-form $\omega = P(x,y) \wedge dx + Q(x,y) \wedge dy$. By the definition of exterior derivatives we obtain

$$\begin{aligned} d\omega &= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \right) \wedge dx + \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy \right) \wedge dy. \\ &= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial y} dx \wedge dy. \end{aligned}$$

We have used that $dx \wedge dx = 0$. Now using that $dy \wedge dx = -dx \wedge dy$ we may rearrange this to obtain

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

Now using Stokes' Theorem which says that

$$\int_{\partial R} \omega = \int_C Pdx + Qdy,$$

and

$$\int_R d\omega = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

We then obtain the final result of

$$\int_C P dx + Q dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy. \qquad \text{(By 2.7.1)}$$

Which is what we wanted to show. □

The three-dimensional version of this is the Divergence Theorem due to Gauss.

Theorem 4.1.2. *Let D be a closed, bounded and smooth submanifold of $\mathbb{R}^3$ and ∂D its boundary. Let*

$$F(x, y, z) = \begin{pmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{pmatrix}$$

be a smooth vector field defined in D. Then

$$\iint_{\partial D} F \cdot \nu dS = \iiint_D div(F) dxdydz$$

where $div(F) = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$, and ν is an outward-pointing normal of ∂D.

So in a similar way as with Green's we want to prove this.

Proof. Consider the 2-form $\omega = P \wedge dy \wedge dz + Q \wedge dz \wedge dx + R \wedge dx \wedge dy$. The choice of this form itself comes from the ways to attaching two differentials wedged together to each one of the vector inputs. Taking the exterior derivative gives

$$d\omega = \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \wedge dy \wedge dz$$

$$+ \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy + \frac{\partial Q}{\partial z} dz \right) \wedge dz \wedge dx$$

$$+ \left(\frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \frac{\partial R}{\partial z} dz \right) \wedge dx \wedge dy.$$

We take the first term with respect to P. Obviously after simplification this will produce

$$\left(\frac{\partial P}{\partial x}\right) \wedge dx \wedge dy \wedge dz.$$

A similar pattern will obviously follow for Q and R. We will get

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \wedge dx \wedge dy \wedge dz.$$

Now using Stokes' Theorem we have that

$$\iint_{\partial \mathcal{D}} F \cdot \nu dS = \int_{\partial \mathcal{D}} \omega$$

and that

$$\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \wedge dx \wedge dy \wedge dz = \int_D d\omega$$

By the definition of integrals of differential forms we obtain

$$\iint_{\partial \mathcal{D}} F \cdot \nu dS = \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz.$$

Which is what we wanted to show. □

Acknowledgements

I would like to extend my gratitude to my supervisor Salvador Rodriguez Lopez. For his patience, support, his deep knowledge of mathematics, and good mood during this time. His help has been an indispensable part in the writing of this thesis.

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