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Introduction to Homological Algebra, Up To the Definition $\mathbf{Ext}^{\mathbf{n}}_{\mathbf{R}}(\mathbf{M},\mathbf{N})$

av

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Introduction to Homological Algebra, Up To the Definition $\mathbf{Ext^n_R(M, N)}$

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Abstract

The contravariant functor $\mathrm{Hom}(\square, D)$ is additive, and moreover left exact, but is not exact. In this thesis we define a sequence of functors $\mathrm{Ext}^n_R(\square, D)$ which in some sense measure the failure of the hom functor to be exact. The functors $\mathrm{Ext}^n_R(\square, D)$ are called the derived functors of Hom. They are defined as the cohomology groups of the cochain complex $\mathrm{Hom}(P_\bullet, D)$, where $P_\bullet$ is a choice of projective resolution of the source variable. The definition is independent of the choice of $P_\bullet$, because $P_\bullet$ is unique up to chain homotopy, and cohomology groups are homotopy-invariant.

When the functors $\mathrm{Ext}^n_R(\square, D)$ are applied to a short exact sequence of modules they generate a long exact sequence of cohomology groups. The functors $\mathrm{Ext}^n_R(\square, A)$ are characterized by this long exact sequence of cohomology groups together with the natural isomorphisms $\mathrm{Ext}^0_R(\square, D) \cong \mathrm{Hom}(\square, D)$ and $\mathrm{Ext}^n_R(Q, D) = 0$ when Q is projective and $n > 0$.

The functor $\mathrm{Hom}(\square, D)$ is exact if and only if D is injective. It follows that an R-module D is injective if and only if $\mathrm{Ext}^n_R(B, D) = 0$ for all modules B and $n > 0$.

One of the first applications of Ext groups stems from the fact that there is a bijection between equivalence classes of extensions of A by C and the group $\mathrm{Ext}^1_R(C, A)$. We can define the bijection ψ by using the existence of a chain-map α_n between the projective resolution $\mathbf{P}_C$ of C and the extension A by C. Since a chain-map implies a commuting diagram of the complexes involved, $\alpha_1 d_2 : P_2 \to P_1 \to A$ is equal to $0 : P_2 \to 0 \to A$ and α_1 can be viewed as an element of the kernel of the induced map

$$d_2^* : \mathrm{Hom}(P_1, A) \to \mathrm{Hom}(P_2, A).$$

Since α_1 belongs to the kernel of d_2^*, α_1 is a representative of a coset in $\mathrm{Ext}^1_R(C, A)$. The map ψ is defined by mapping the extension class represented by the extension A by C to the coset of $\mathrm{Ext}^1_R(C, A)$ represented by α_1. The inverse of ψ is defined by choosing a representative of a coset of $\mathrm{Ext}^1_R(C, A)$ and a projective resolution of C. Using these two objects an extension A by C is constructed as a second row in a commutative diagram where all second rows are equivalent extensions. The inverse then maps the coset represented by α_1 to the extension class A by C.

Sammanfattning

Den kontravarianta funktorn $\mathrm{Hom}(\square, D)$ är additiv och vänsterexakt men inte exakt. I den här uppsatsen definierar vi en sekvens funktorer, $\mathrm{Ext}^n_R(\square, D)$, som i viss mån mäter hur mycket $\mathrm{Hom}(\square, D)$ avviker från att vara exakt. Funktorerna $\mathrm{Ext}^n_R(\square, D)$ är definierade som kohomologigrupper av kokedjekomplexet $\mathrm{Hom}(P_\bullet, D)$. Där $P_\bullet$ är valet av projektiv upplösning av argumentet. Definitionen är oberoende av val av $P_\bullet$ då homotopiska kedje-homomorfismer är oberoende av val av upplösning och kohomologigrupper är homotopi-invarianta.

Funktorerna $\mathrm{Ext}^n_R(\square, D)$ mappar korta exakta sviter av moduler till långa exakta sviter av kohomologigrupper. Funktorerna $\mathrm{Hom}(P_\bullet, D)$ karaktäriseras av dessa långa exakta sviter i kombination med den naturliga isomorfismen $\mathrm{Ext}^0_R(\square, D) \cong \mathrm{Hom}(\square, D)$ och $\mathrm{Ext}^n_R(Q, D) = 0$ för en projektiv modul Q och där $n > 0$.

Funktorn $\mathrm{Hom}(\square, D)$ är exakt om och endast om D är injektiv, vilket innebär att en R-modul är injektiv om och endast om $\mathrm{Ext}^n_R(B, D) = 0$ för alla moduler B där $n > 0$.

En av de första tillämpningarna av Ext-grupper kommer från det faktum att det finns en bijektion ψ mellan ekvivalensklasser av korta exakta sviter och gruppen $\mathrm{Ext}^1_R(C, A)$. Låt sviten $0 \to A \to B \to C \to 0$ representera sin ekvivalensklass. Vi kan definiera bijektionen ψ genom att använda kedje-homomorfismen α_n mellan den projektiva upplösningen $\mathbf{P}_C$ av C och den exakta sviten $0 \to A \to B \to C \to 0$. Eftersom kedje-homomorfismer innebär ett kommutativt diagram av komplex får vi att $\alpha_1 d_2 : P_2 \to P_1 \to A$ är samma som $0 : P_2 \to 0 \to A$ vilket innebär att α_1 tillhör kärnan av homomorfismen

$$d_2^* : \mathrm{Hom}(P_1, A) \to \mathrm{Hom}(P_2, A).$$

Då α_1 tillhör kärnan av d_2^*, representerar α_1 en sidoklass i $\mathrm{Ext}^1_R(C, A)$ och ψ definieras sedan genom att ekvivalensklassen $0 \to A \to B \to C \to 0$ mappas till den sidoklass som representeras av α_1. Inversen till ψ definieras genom att först välja en representat av sidoklassen $\mathrm{Ext}^1_R(C, A)$ och därefter en projektiv upplösning av C. Sedan väljs en homomorphism α_1 som representerar den valda sidoklassen av $\mathrm{Ext}^1_R(C, A)$ och utifrån den och den valda resolutionen skapas ett kommutativt diagram av två korta exakta sviter där den andra raden alltid representerar ekvivalensklassen $0 \to A \to B \to C \to 0$. Inversen mappar sedan sidoklassen representerad av α_1 till den korta exakta sviten i andra raden av diagrammet.

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Contents

1 Introduction

The following sections are a compilation of definitions and theorems leading up to the statements characterizing the functors $\operatorname{Ext}^n_R(\square, A)$, the characterization of injective modules by $\operatorname{Ext}^n_R(\square, A)$ and finally the definition of the bijection between the set of equivalent extensions of A by C and the group $\operatorname{Ext}^1_R(C, A)$.

We start by defining modules in section 2, in particular we define direct sums of modules, free modules and projective modules. We also show that Hom-sets under point-wise addition are additive abelian groups. This section might seem a bit too fundamental but the basics are included for two reasons: The first is for reference and the second is for context since basic qualities such as freeness and direct sums of modules or additivity of homomorphisms are central throughout the text.

We move on to exact sequences/extensions in section 3, we show how quotient modules can be written as exact sequences and we define split exact sequences. We define equivalent extensions and we prove the Snake Lemma. We also show how to derive exact sequences of Hom-groups from exact sequences of modules and we define injective modules.

In Section 4 we generalize the concept of exactness and define complexes. We show that complexes share the properties of modules and that there are direct sums and short exact sequences of complexes. We use the Snake Lemma to prove the Horseshoe Lemma which shows how we can derive a short exact sequence of complexes from a short exact sequence of modules. We define projective resolutions of modules and show that there always exist a chain-map of resolutions given homomorphism of modules.

We then define cohomology groups, H^n-groups, from complexes in section 5, and we use the Snake Lemma once again to show that a short exact sequence of cochain complexes induces a long exact sequence of cohomology groups

In section 6 we define additive categories and additive functors and show that they preserve direct sums, complexes and homotopic chain maps.

Finally in section 7 we define the additive contravariant functor $\operatorname{Hom}(\square, B)$ and the additive functor H^n in order to define the derived functor $\operatorname{Ext}^n_R(\square, A)$. We use the Long Exact Sequence of Cohomology to derive the long exact sequence of Ext-groups and we state the axioms characterizing $\operatorname{Ext}^n_R(\square, A)$. We then use $\operatorname{Ext}^n_R(\square, A)$ to characterize injective modules and lastly we define the bijection of equivalent extensions A by C and the group $\operatorname{Ext}^1_R(C, A)$.

2 Modules

2.1 Some Definitions and Properties

A left module or a right module is an abelian group on which a ring acts from the left or from the right respectively. For commutative rings, the notions of left and right module are equivalent and are referred to simply as an R-module. We will not assume that rings are commutative, and adopt the convention that unless said otherwise, "module" means "left module".

Definition 2.1. Let R be a ring with a 1. A **left R-module** or a **left module over** R is a set M together with

(1) a binary operation $+$ on M under which M is an abelian group, and

(2) an action of R on M (that is, a map $R \times M \to M$) denoted rm for all $r \in R$ and for all $m \in M$ which satisfies

  (a) $(r+s)m = rm + sm$, for all $r, s \in R$, $m \in M$
  (b) $r(m+n) = rm + rn$ for all $r \in R$, $m, n \in M$
  (c) $(rs)m = r(sm)$ for all $r, s \in R$, $m \in M$
  (d) $1m = m$ for all $m \in M$.

[1, pg. 337, Definition, Sec. 10.1]

Right modules are defined analogously. Let M be an R-module and let ${}_R0$ be the zero of the ring R and 0_M the zero of M as an abelian group then

(i) ${}_R0m = 0_M$ and

(ii) $(-1)m = -m$

for all $m \in M$. This is due to the distributivity axiom in $2(a)$, since

$$({}_R0 +_R 0)m = \ {}_R0m,$$

and so ${}_R0m = \ 0_M$. With the help of $2(d)$ we then get

$${}_R0m = (1 + (-1))m = m + (-1)m = 0_M$$

and $(-1)m = -m$.

Example 2.2. (i) Abelian groups are $\mathbb{Z}$-modules. Let $r, -r \in \mathbb{Z}$ and let $a \in A$ where A is an additive abelian group. If we define an action of $\mathbb{Z}$ on A by

$$ra \mapsto a + a + ... + a,$$

the axioms of Definition 2.1 (2) are immediately satisfied for all positive integers. It follows that (i) in the text above also holds and if we introduce the negative integers so does (ii) and the action of $-r$ on A becomes

$$-ra = r(-1)a = r(-a) \mapsto (-a) + (-a) + ... + (-a).$$

The axioms of Definition 2.1 (2) are immediately satisfied for all negative integers and A is a $\mathbb{Z}$-module. [3, pg.289, Example B-1.19 (ii), Ch. B-1] [3, pg.132-133, Proposition A-4.20, Ch. A-4]

(ii) Any ring R is an R-module where R acts upon itself by multiplication. Depending on whether R acts on itself from the left or the right it is either a left module or a right module and unless the ring is commutative these modules have different structures. [1, pg. 338, Example (1), Sec. 10.1]

(iii) The **zero module** is the trivial module containing only the zero element, $\{0\}$.

2.1.1 R-module Homomorphisms and Hom-sets

Definition 2.3. Let R be a ring and let M and N be R-modules.

(i) A map $f : M \to N$ is an **R-module homomorphism** if it respects the R-module structures of M and N, i. e.

  (a) $f(x + y) = f(x) + f(y)$ for all $x, y \in M$ and

  (b) $f(rx) = rf(x)$, for all $r \in R$, $x \in M$

(ii) If $f : M \to N$ is an R-module homomorphism, let

  (a) $\ker f = \{m \in M \mid n = \varphi(m) = 0\}$ be the **kernel** of f, let

  (b) $\mathrm{im}\ f = \{n \in N \mid n = f(m) \text{ for some } m \in M\}$ be the **image** of f.

(iii) Define $\mathbf{Hom_R}(M, N)$ to be the set of all R-module homomorphisms from M into N.

[1, pg. 345, Definition (1),(3),(4), Sec. 10.2]

An R-module homomorphism which is both onto and one to one is called an **R-module isomorphism**. We will sometimes refer to R-module homomorphisms as R-maps.

Definition 2.4. Let A and B be R-modules for some ring R. The **zero map**, $0 : A \to B$ is defined by $0 : a \mapsto 0_B$ for all $a \in A$.

The set of all R-module homomorphisms, $\mathrm{Hom}_R(M, N)$, is an abelian group under pointwise addition.

Proposition 2.5. *Let M, N and L be R-modules.*

(i) Let φ, ψ be elements of $Hom_R(M, N)$ define $\varphi + \psi$ by

$$(\varphi + \psi)(m) = \varphi(m) + \psi(m) \text{ for all } m \in M$$

Then $\varphi + \psi \in Hom_R(M, N)$ and with this operation $Hom_R(M, N)$ is an abelian group.

(ii) If $\varphi \in Hom_R(L, M)$ and $\psi \in Hom_R(M, N)$ then $\psi\varphi \in Hom_R(L, N)$.

(iii) Let $Hom_R(M', M)$ and $Hom_R(N, N')$ be Hom-sets equipped with the addition defined in (i). Let $p \in Hom_R(M', M)$ and $q \in Hom_R(N, N')$ then for any $f, g \in Hom_R(M, N)$,

  (a) $(f + g)p = fp + gp$,

  (b) $q(f + g) = qf + qg$.

(iv) With addition as above and multiplication defined as a function composition, $Hom_R(M, M)$ is a ring with a 1.

Proof. (i) Let the additive identity be the zero map, and the inverse of $f(m)$ be $-f(m)$. Then all the axioms of an additive group follows from N being an additive abelian group since $f(m) \in N$.

(ii) It suffices to show $\psi\varphi(rx + y) = r\psi\varphi(x) + \psi\varphi(y)$ since $r = 1$ implies $\psi\varphi$ is a homomorphism and $y = 0$ implies commutativity with R. From *(i)* and the definition of R-module homomorphisms we get,

$$\begin{aligned}\psi\varphi(rx + y) &= \psi(\varphi(rx + y)) = \psi(\varphi(rx) + \varphi(y)) \\ &= \psi(r\varphi(x) + \varphi(y)) = r\psi\varphi(x) + \psi\varphi(y).\end{aligned}$$

(iii) This follows immediately from (i) and the definition of an homomorphism:

(a) Let $b \in M'$ then
$$(f + g)p(b) = f(p(b)) + g(p(b))$$
and so $(f + g)p = fp + gp$.

(b) Let $a \in N'$ then
$$q(f + g)(a) = q(f(a) + g(a)) = q(f(a)) + q(g(a))$$
and so $q(f + g) = qf + qg$.

(iv) Function composition is closed since the domain and codomain are the same for all $f \in \mathrm{Hom}_R(M, M)$ and it is binary according to *(ii)*. The multiplicative identity is the identity homomorphism. Composition is associative in general and distributivity is shown in *(iii)*.

[1, pg. 346, Proposition 2, (2), (3), (4), Sec. 10.2], [2, pg. 39 Lemma 2.3, Ch. 2.1] □

We can mention that there is additional structure to $\mathrm{Hom}_R(M, N)$ besides being an abelian group. Let $f \in \mathrm{Hom}_R(M, N)$ and $z \in Z(R)$, the center of R and define zf by

$$(zf)(m) = zf(m)$$

for all $m \in M$. This action on f by z turns $\mathrm{Hom}_R(M, N)$ into a $Z(R)$-module: zf commutes with $r \in R$ since f and z commutes with r:

$$(zf)(rm) = zf(rm) = zr(f(m)) = rz(f(m)) = r(zf)(m),$$

and zf is a R-homomorphism since f is a R-homomorphism:

$$\begin{aligned}(zf)(rm + m') &= zf(rm + m') \\ &= z(f(rm) + f(m')) \\ &= zf(rm) + zf(m') \\ &= (zf)(rm) + (zf)(m') \\ &= r(zf)(m) + (zf)(m').\end{aligned}$$

Since $(zf) \in \text{Hom}_R(M, N)$, $(zf)(m) \in N$ and the axioms for a module then follows from N being an R-module. Commutativity of $Z(R)$ is the key to $\text{Hom}_R(M, N)$ being a $Z(R)$-module and it follows that if R is commutative $\text{Hom}_R(M, N)$ is an R-module. We will however view the Hom-sets of modules mainly as abelian groups throughout the text. [1, pg. 346-347, Proposition 2 (2), Sec. 10.2], [2, pg.39, Proposition 2.4(ii), Ch. 2.1]

Below are some useful examples regarding Hom-set isomorphisms. Note that the examples applies to finite cyclic groups in general, since all finitely generated cyclic groups are isomorphic to $\mathbb{Z}$ or quotients of $\mathbb{Z}$, [3, pg. 163-164, Example A-4.74, Chapter A-4].

Example 2.6.

(i) The group $\text{Hom}(\mathbb{Z}, D)$ is isomorphic to D. Since $\mathbb{Z} = \langle 1 \rangle$ any $f \in \text{Hom}(\mathbb{Z}, D)$ is completely determined by $f(1)$ since for any $n \in \mathbb{Z}$, $n = 1 + 1 + ... + 1$ n times we have

$$f(n) = f(1 + 1 + ... + 1) = f(1) + f(1) + ... + f(1) = nf(1).$$

And so every f is uniquely defined by an element $d \in D$ for which $f(1) = d$ and $f(n) = nd$. Conversely for every $d \in D$ you can define a homomorphism such that $f(1) = d$ and this gives us an isomorphism $\text{Hom}(\mathbb{Z}, D) \to D$ defined by $f_d \mapsto d$.

(ii) The group $\text{Hom}(\mathbb{Z}/m\mathbb{Z}, D)$ is isomorphic to $mD = \{x \in D \mid mx = 0\}$. Similar to the example above, any $f \in \text{Hom}(\mathbb{Z}/m\mathbb{Z}, D)$ is determined by $f(\bar{1})$ since $\bar{1}$ generates $\mathbb{Z}/m\mathbb{Z}$. But unlike $\mathbb{Z}$, $\mathbb{Z}/m\mathbb{Z}$ has finite order and since $m(\bar{1}) = 0_{\mathbb{Z}/m\mathbb{Z}}$,

$$mf(\bar{1}) = f(\bar{m}) = 0_D$$

and any $x \in D$ for which $f(\bar{1}) = x$ needs to satisfy $mx = 0_D$. It follows that $f(\bar{1}) \in mD$ and $\text{Hom}(\mathbb{Z}/m\mathbb{Z}, D) \cong mD$.

2.1.2 Submodules

A submodule S of M is a subgroup S of M closed under the action of R. A way of creating submodules is to take a subset A of a module M and then forming all R-linear combinations of the elements of A. A submodule N of M may have many different generating sets, N is for instance always generated by itself.

Definition 2.7. Let M be an R-module and let $N_1, N_2, ..., N_n$ be submodules of M.

(i) The **sum** of $N_1, ..., N_n$ is the set of all finite sums of elements from the sets N_i: $\{a_1 + a_2 + ... + a_n \mid a_i \in N_i \text{ for all } i\}$. Denote this sum by $N_1 + ... + N_n$

(ii) For any subset A of M let $RA = \{r_1a_1 + r_2a_2 + ... + r_ma_m \mid r_1, ...r_m \in R,\ a_1, ..., a_m \in A, m \in \mathbb{Z}^+\}$ (where by convention $RA = 0$ if $A = \emptyset$). Call RA the **submodule of** M **generated by** A. If N is a submodule of M (possibly $N = M$) and $N = RA$, for some subset A of M, we call A a **a set of generators** or a **a generating set** for N, and we say N **is generated by**A.

(iii) A submodule N of M (possibly $N = M$) is **finiteley generated** if there is some finite subset A of M such that $N = RA$, that is if N is generated by some finite subset.

[1, pg. 351, Definition (1), (2), (3), Sec. 10.3]

Example 2.8.

(i) Let $f : M \to N$ be a R-homomorphism then the kernel of f is a submodule of M and the image of f is a submodule of N. We will show that they are closed under the action of R: Let $a \in \ker f$ then

$$f(ra) = rf(a) = 0,$$

and so $ra \in \ker f$. Let $b \in M$ then

$$f(rb) = rf(b)$$

and rb is in the image of f.

(ii) A module generated by one element is a cyclic module. For example is the ring R as an R-module over itself cyclic where $R = \langle 1 \rangle$ since $r1 = r$ for all $r \in R$. The principal ideals of R are then cyclic submodules of R as an R-module since $rI = I$ for all $r \in R$.

2.1.3 Quotient Modules

Since R-modules are additive abelian groups every submodule is also a subgroup and every R-module homomorphisms is also a group homomorphism. It follows that there exists an abelian quotient group M/N and also a natural group homomorphism $M \to M/N$ for any module M with any submodule N. By defining an action of the ring R and check that it satisfies the axioms in Definition 2.1 and 2.3 (i)(b) we can extend quotient groups and group homomorphisms to quotient modules and module homomorphisms.

Proposition 2.9. *Let R be a ring, let M be an R-module and let N be a submodule of M. The (additive, abelian) quotient group M/N can be made into an R-module by defining an action of elements of R by*

$$r(x + N) = (rx) + N, \quad \text{for all } r \in R,\ x + N \in M/N.$$

The natural projection map $\pi : M \to M/N$ defined by $\pi(x) = x + N$ is an R module homomorphism with kernel N.

Proof. [1, pg. 348, Proposition 3, Sec. 10.2] □

Definition 2.10. Let $f : M \to N$ be a R-module homomorphism then the **cokernel** of f is the quotient $N/\text{im } f$.

Proposition 2.11. *Let $f : M \to N$ be an R-module then* *(i) f is injective if and only if ker $f = 0$ and*

(ii) f is surjective if and only if coker $f = 0$.

Proof. The proof is the same as for abelian groups. □

By first referring to the corresponding theorems for abelian groups and then checking that the group homomorphism holds under the action of R we can extend all the isomorphism theorems for groups to include modules and module homomorphisms. The first isomorphism theorem for modules will be of importance in the following sections.

Theorem 2.12. *(The First Isomorphism Theorem for Modules) Let M and N be R-modules, and let $f : M \to N$ be an R-module homomorphism, then $M/ker\ f \cong im\ f$.*

Proof. The First Isomorphism Theorem for groups say that given a homomorphism of abelian groups $f : M \to N$,

$$\begin{aligned} \varphi : M/\text{ker } f &\to \text{im } f \\ m + \text{ker } f &\mapsto f(m) \end{aligned}$$

is an isomorphism of abelian groups. If we can show that $\varphi(rx) = r\varphi(x)$ for $x \in M/\text{ker } f$, then φ is an R-mod isomorphism. We have that

$$\varphi(rx) = \varphi r(m + \text{ker } f) = \varphi(rm + \text{ker } f) = f(rm),$$

and since f is an R-homomorphism we get

$$f(rm) = rf(m) = r\varphi(m + \text{ker } f) = r\varphi(x).$$

[1, pg. 349, Theorem 4, Sec. 10.2] □

Let R be an R-module over itself and let I be an ideal of R then the quotient R/I is a cyclic R-module generated by $1 + I$ since

$$r(1 + I) = r + I$$

for all $r \in R$. We can use the First Isomorphism Theorem to show that any cyclic R module M is isomorphic to a quotient R/I.

Example 2.13.
Let M be a cyclic R-module generated by x and let f be a map

$$\begin{aligned} f : R &\to M \\ r &\mapsto rx. \end{aligned}$$

The map f is an R-homomorphism since $f(r'r) = r'rx = r'f(r)$. We have that f is surjective since $M = Rx$ and the kernel of f is an ideal I of R. By the First Isomorphism Theorem we then have $R/I \cong M$. In particular if $R = \mathbb{Z}$ then $I = n\mathbb{Z}$ for some $n \in \mathbb{Z}$ and $M \cong \mathbb{Z}/n\mathbb{Z}$.

2.2 Direct Products and Direct Sums

The direct product of a set of R-modules, finite or infinite, is their Cartesian product together with coordinatewise addition and the action of multiplication of R. Given a finite set of modules, $M_1, M_2, ...M_k$, their (external) direct sum, denoted $M_1 \oplus M_2 \oplus ... \oplus M_k$, is defined as their direct product. In the case of an infinite set of modules, $(A_i : i \in I)$ where I is an index set, their (external) direct sum, $\bigoplus_{i \in I} A_i$, is a submodule of their direct product, $\prod_{i \in I} A_i$, consisting of the I-tuples which have a finite number of nonzero coordinates. In this and in the following sections however direct sums will be of a finite set of modules and therefore we will equate the external direct sum of a set of modules with their direct product.

The direct product of modules is a module.

Definition 2.14. Let $M_1, ..., M_k$ be a finite collection of R-modules. The collection of k-tuples $m_1, m_2...., m_k$ where $m_i \in M_i$ with addition and action of R defined componentwise is called the **direct product** of $M_1, ..., M_k$, denoted $M_1 \times .... \times M_k$. [1, pg. 353, Definition, Sec. 10.3]

Closely related to the external direct sum is the internal direct sum.

Definition 2.15. Let $S_1, S_2, ...S_k$ be submodules of M. Then M is their **internal direct sum**,

$$M = S_1 \oplus S_2 \oplus ... \oplus S_k, \tag{1}$$

if each $m \in M$ has a unique expression of the form $m = s_1 + s_2 + ... + s_k$, where $s_i \in S_i$ for all $i \in 1, 2, ..., k$. Each submodule S_i in (1) is called a **summand** of M.

A direct product, $M = M_1 \times M_2 \times ... \times M_k$, can be written as an internal direct sum of submodules, $S_1 \oplus S_2 \oplus ... \oplus S_k$, $S_i \in M$. Let

$$S_i = \{(0, 0, .., x_i, .., 0) \mid x_i \in M_i\},$$

S_i is clearly a submodule of M. Let $s_i = (0, 0, .., x_i, .., 0)$ due to the addition on M every sum

$$\sum_{i=1}^{k} s_i = (x_1, 0, .., 0) + ... + (0, .., x_n, .., 0) + ... + (0, .., 0, x_k) \tag{2}$$

is an element of M and since the map $\pi : S_i \to M_i$ defined by $(0, 0, .., x_i, .., 0) \mapsto x_i$ is a bijection, every element $m \in M$ can be written as a sum (2) and we have that $M = \sum_{i=1}^{k} S_i$. The sum (2) is unique for each m, to see this let $m = \sum_{i=1}^{k} s_i$ and $m = \sum_{i=1}^{k} s_i'$, then for all $j \in 1, 2, ..k$,

$$s_j - s_j' = \sum_{i \neq j} (s_i' - s_i) \in S_j \cap S_1 + S_2 + ... + \widehat{S_j} + ... + S_k = \{0\},$$

and $s_j = s_j'$. Note that the bijection π implies there exists an injection $S_i \to M$ and a surjection $M \to S_i$ defined by

$$(x_1, .., x_i, .., x_k) \mapsto (0, ..., x_i, ..., 0).$$

For a module M isomorphic to the direct product of two modules we have the equivalent statements of the proposition below.

Proposition 2.16. *Let M,M_1 and M_2 be left R-modules. The following statements are equivalent :*

(i) $M_1 \times M_2 \cong M$

(ii) *There exists injective R-maps $\iota_1 : M_1 \to M$ and $\iota_2 : M_2 \to M$ such that $M = im\ \iota_1 + im\ \iota_2$ and $im\ \iota_1 \cap im\ \iota_2 = 0$.*

(iii) *There exists R-maps $\iota_1 : M_1 \to M$ and $\iota_2 : M_2 \to M$ such that, for every $m \in M$, there are unique $x_1 \in M_1$ and $x_2 \in M_2$ with $m = \iota_1(x_1) + \iota_2(x_2)$.*

(iv) *There are R-maps $\iota_1 : M_1 \to M$ and $\iota_2 : M_2 \to M$ called* ***injections****, and R-maps $\rho_1 : M \to M_1$, $\rho_2 : M \to M_2$ called* ***projections****, such that*

$$\rho_i \iota_i = 1_{M_i},\ \rho_i \iota_j = 0 \text{ where } i \neq j,\ \iota_1\rho_1 + \iota_2\rho_2 = 1_M,$$

where $i, j \in 1, 2$.

(v) *There exists R-maps $\rho_1 : M \to M_1$, $\rho_2 : M \to M_2$ such that the map $\psi : M \to M_1 \times M_2$, is an isomorphism.*

Proof. For full proof see [2, pg. 48, Proposition 2.20, Sec. 2.1] □

We will refer to internal direct sums only as direct sums from now on.

Definition 2.17. A submodule S of a left R-module M is a **direct summand** of M if there exists a submodule T of M with $M = S \oplus T$. The submodule T is called a **complement** of S.

Let M, M_1 and M_2 be the modules in in Proposition 2.16. The identity map

$$\rho_1\iota_1 = 1_{M_1} : M_1 \to M \to M_1$$

suggest that ρ_1 when restricted to the image of ι_1 is the inverse of ι_1. It follows that there exists a map

$$\begin{aligned} \iota_i\rho_1 : M \to M_1 \to \text{im } \iota_1 \\ m_1 \mapsto m_1 \end{aligned}$$

whose kernel is the image of ι_2 since $\rho_1\iota_2 = 0$ and any other $m \neq 0$ not in the image of ι_2 is in the image of ι_1. We then have that

$$M = \text{im } \iota_1 \oplus \text{im } \iota_2 = \text{im } \iota_1\rho_1 \oplus \text{ker } \iota_1\rho_1.$$

The map $\iota_1\rho_1 : M \to \text{im } \iota_1$ is called a retraction and given a direct sum of two modules there exists two retractions where one implies the other. The Proposition below shows that the opposite is also true. If there exists a retraction $r : M \to S'$ where S' is a submodule of M; $M = \text{im } r \oplus \text{ker } r = S' \oplus \text{ker } r$.

Proposition 2.18. *A submodule S of a left R-module M is a direct summand if and only if there exists an R-map $r : M \to S$ called a* ***retraction****, with $r(s) = s$ for all $s \in S$. The submodule S is then called a* ***retract****.*

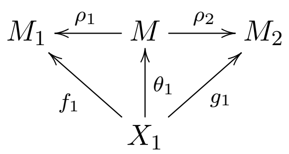
Proof. We have already shown that given a direct sum there exists a retraction. For the opposite, given a retraction r we will show that $M = S + \ker r$ where $S \cap \ker r = \{0\}$. Let $m \in M$ then $m = (m - s) + s$ which we can write

$$m = (m - r(m)) + r(m).$$

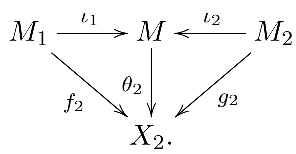
Since $r(m) \in S$ and $r(s) = s$ we have that $r(r(m)) = r(m)$ and so $r(m) - r(r(m)) = 0$, that is, $r(m - r(m)) = 0$ and $m - r(m) \in \ker r$. And we have $M = S + \ker r$. For $s \in S$, $r(s) = 0$ if and only if $s = 0$ and so $S \cap \ker r = \{0\}$. [3, pg. 325, Corollary B-2.15, Ch. B-2] □

Below is a Proposition which shows that the direct sum satisfies the universal property of both a (categorical) product and a (categorical) coproduct which will be defined in section 6.

Proposition 2.19. *Let M_1, M_2 be left R-modules and let $M \cong M_1 \times M_2$. Given any two left R-modules X_1, X_2 together with any two pair of homomorphisms; f_1, g_1 and f_2, g_2, there exists two unique homomorphisms; $\theta_1 = \iota_1 f_1 + \iota_2 g_1$ and $\theta_2 = f_2\rho_1 + g_2\rho_2$ which makes the diagrams below commute:*



(3)



(4)

Proof. We use the equivalent statements (iii), (iv) of Proposition 2.16. To show commutativity in diagram (3) we have for $x \in X_1$,

$$\rho_1\theta_1(x) = \rho_1\iota_1 f_1(x) + \rho_1\iota_2 g_1(x) = 1_{M_1} f_1(x) = f_1(x),$$

and in the same manner is $\rho_2\theta_1(x_1) = g_1(x_1)$ and diagram (3) commutes. Let $\psi : X_1 \to M$, given that diagram (3) commutes we have $\rho_1\psi = f_1$ which gives us $\iota_1\rho_1\psi = \iota_1 f_1$ after composing with ι_1. In the same way $\iota_2\rho_2\psi = \iota_2 g_1$. We then have, since all maps are R-maps,

$$\begin{aligned} (\iota_1\rho_1 + \iota_2\rho_2)\psi &= \iota_1 f_1 + \iota_2 g_1 \\ 1_M\psi &= \iota_1 f_1 + \iota_2 g_1 \\ \psi &= \iota_1 f_1 + \iota_2 g_1 = \theta_1, \end{aligned}$$

and we have shown uniqueness of θ_1. In diagram (4) we get for all $s \in M_1$,

$$\theta_2\iota_1(s) = f_2\rho_1\iota_1(s) + g_2\rho_2\iota_1(s) = f_2 1_{M_1}(s) = f_2(s),$$

and similarly $\theta_2\rho_2(t) = g_2(t)$ for all $t \in M_2$ and the diagram commutes. To show uniqueness, let $\psi : M \to X_2$. Given that diagram (4) commutes we have that $\psi\iota_1 = f_2$ and so $\psi\iota_1\rho_1 = f_2\rho_1$ and in the same way $\psi\iota_2\rho_2 = g_2\rho_2$, and we get

$$\begin{aligned}\psi(\iota_1\rho_1 + \iota_2\rho_2) &= f_2\rho_1 + g_2\rho_2\\ \psi 1_M &= f_2\rho_1 + g_2\rho_2\\ \psi &= f_2\rho_1 + g_2\rho_2 = \theta_2\end{aligned}$$

[2, Proposition 5.1, Proposition 5.8, pg. 214-215, 218-219, Section 5.1] (The proof for Proposition in 5.8 is relating the direct sum $A \oplus B$ to the Cartesian product $A \times B$ with canonical injections $\iota : a \to (a, 0)$ and projections $\rho : (a, b) \to a$, the proof defines $\theta : (a, b) \mapsto f(a) + g(b)$ which is analogous to $\theta_2 : m \mapsto f_2\rho_1(m) + g_2\rho_2(m)$, $m \in M$ in the proof above). □

In Example 2.13 we saw that cyclic R-modules are isomorphic to quotients R/I. According to The Existence Theorem if R is a PID any finitely generated R-module is isomorphic to a direct sum of cyclic modules.

Example 2.20. Let G be a finitely generated R-module then

$$G \cong R^r \oplus R/\langle a_1\rangle \oplus R/\langle a_2\rangle \oplus ... \oplus R/\langle a_n\rangle \tag{5}$$

where $\langle a_i\rangle$ is the ideal in R generated by a_i and where all a_i satisfy the divisibility relation $a_1 \mid a_2 \mid \cdots \mid a_n$ [1, pg. 462-463, Theorem 5, ch. 12]. In particular there exists a decomposition for a finitely generated $\mathbb{Z}$-module H,

$$H \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z},$$

where r and n_i are integers, $r > 0$ and $n_i \geq 2$ and $n_k \mid \cdots \mid n_2 \mid n_1$ [1, p. 158-159, Theoreom 3, ch. 5].

2.3 Free And Projective Modules

Let F be a left R-module generated by a finite set $A = \{a_1, a_2, ..., a_m\}$;

$$F = RA = \{r_1a_1 + r_2a_2 + ... + r_ma_m \mid r_1, ..., r_m \in R, a_1, ..., a_m \in A\}.$$

If every element $x \in F$ can be uniquely written as a sum with respect to the elements of R as well as A then A is a basis for F and F is a free module with basis A, denoted $F(A)$. Since each $x \in F(A)$ can be uniquely written as a sum, $x = r_1a_1 + r_2a_2 + ... + r_ma_m$, $F(A)$ is equal to the direct sum,

$$Ra_1 \oplus Ra_2 \oplus ... \oplus Ra_m,$$

and since each $Ra_i \cong R$ (by the map $ra_i \mapsto r$), $F(A) \cong R^m$ and we can view $F(A)$ as a direct sum of m copies of R. Equivalently we can view F as a direct sum of cyclic modules since Ra_i is the cyclic module generated by the element a_i and we can write F as the direct sum:

$$\langle a_1\rangle \oplus \langle a_2\rangle \oplus ... \oplus \langle a_m\rangle$$

where $a_i \in A$ and each $\langle a_1\rangle \cong R$. Should R be replaced by a field, k, $F(A)$ would be the same as the vector space over k with basis A.

Definition 2.21. An R-module $F(A)$ is said to be **free** on the subset A of F if for every element x of $F(A)$ there exist unique elements $r_1, r_2, ..., r_n$ of R and unique $a_1, a_2, ..., a_n$ in A such that $x = r_1a_1 + r_2a_2 + ... + r_na_n$ for some $n \in \mathbb{Z}^+$. In this situation we say A is a **basis** or a set of **free generators** for $F(A)$. [1, Definition, pg. 354]

Example 2.22.

(i) Direct sums of free modules are free modules. Let F_i be an indexed set of free modules with basis X_i, then a basis for the direct sum $\bigoplus_i F_i$ is the union of the sets X_i.

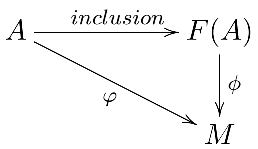
(ii) A ring R is a free module over itself with the identity as a basis since $R1 = R$.

(iii) $\mathbb{Z}/m\mathbb{Z}$ is not free as a $\mathbb{Z}$-module since for any element x, $mx = 0$ and $m \neq 0$ in $\mathbb{Z}$.

(iv) For PIDs a submodule of a free module is always free [3, pg. 331-332, Thm B-2.28, Ch. b-2]. But for a general ring it is not true. For example, $\mathbb{Z}/6\mathbb{Z}$ is a free module over itself, but $\mathbb{Z}/3\mathbb{Z}$ is a submodule of $\mathbb{Z}/6\mathbb{Z}$ that is not free over $\mathbb{Z}/6\mathbb{Z}$.

Given a left R-module M and a set map from a set A to M, there exists an unique R-module homomorphism from the free module with basis A to the module M. Applied to vector spaces V and W over some field, the theorem underlie the unique linear transformation defined by mapping the set of basis elements in V to any set of vectors in W.

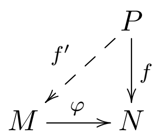
Theorem 2.23. *(The Universal Property of Free Modules) For any set A there is a free R-module $F(A)$ on the set A and $F(A)$ satisfies the following universal property: if M is any R-module and $\varphi : A \to M$ is any map of sets, then there is an unique R-module homomorphism $\phi : F(A) \to M$ such that $\phi(a) = \varphi(a)$, for all $a \in A$, that is, the following diagram commutes.*



(6)

Proof. See [1, pg. 354, Theorem 6, Sec. 10.3] □

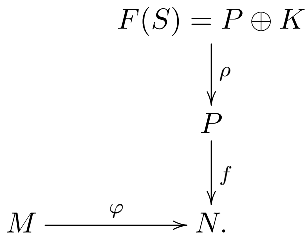
Proposition 2.24. *Let M and N be R-modules and let $\varphi : M \to N$ be a surjective homomorphism. Let P be a direct summand of a free module, then for every R-module homomorphism f from P into N there exists a homomorphism f' from P into M such that the following diagram commutes.*



(7)

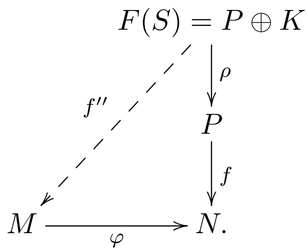
Proof. Let $F(S)$ be a free module generated from a set S and let $P \oplus K = F(S)$. Let ρ be the projection of $F(S)$ onto P so that the composition $f\rho$ becomes a homomorphism from

$F(S)$ to N,



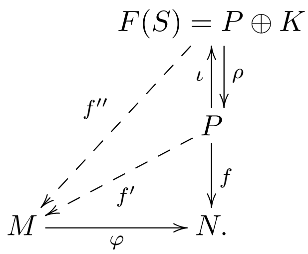
(8)

Let $f\rho(s) = n_s$, for $s \in S$ and some $n_s \in N$. Since φ is surjective there exists an element $m_s \in M$ such that $\varphi(m_s) = n_s$. By the Universal Property of Free Modules there is a unique R-module homomorphism f'' from $F(S)$ to M with $f''(s) = m_s$;



(9)

We get that $\varphi f''(s) = \varphi(m_s) = n_s = f\rho(s)$ and so $\varphi f'' = f\rho$ which makes the diagram above commutative. Define a map $f' : P \to M$ by $f'(x) = f''\iota(x)$, $x \in P$. Since f' is the composite of the injection $\iota : P \to F(S)$ and f'', f' is an R-homomorphism;



(10)

We then have that

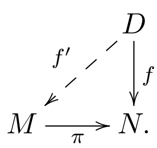
$$\varphi f'(x) = \varphi f''\iota(x) = f\rho\iota(x) = f(x)$$

and the diagram (7) commutes. [1, pg. 389-390, Proposition 30 (4) and (2), Sec. 10.5] $\square$

Free modules and summands of free modules are projective modules.

Definition 2.25. Let π, f, f' and g, g' be R-homomorphisms whose domain and range is indicated in the diagrams below.

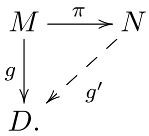
(i) A map f' **lifts** f to M if for $f : D \to N$ and $f' : D \to M$, $\pi f' = f$



(11)

The map f' is called a **lifting** of the map f.

(ii) A map g' **extends** g to N if for $g : M \to D$ and $g' : N \to D$, $g'\pi = g$

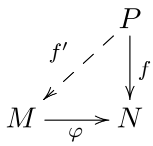


(12)

The map g' is called an **extension** of the map g.

[2, Definition, pg 99, Sec. 3.1], [1, pg 386, Sec. 10.5]

Definition 2.26. A module P is **projective** if, for every surjective homomorphism $\varphi : M \to N$ and every homomorphism $f : P \to N$, f has a lift $f' : P \to M$. That is, there exists a map f' for each map f such that the diagram below commutes.



(13)

[2, Definition, pg 99, Sec. 3.1]

From Theorem 2.23 we have that free modules are projective and from Proposition 2.24 we have that summands of free modules are projective.

Corollary 2.27.

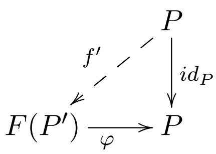
(i) Free modules are projective.

(ii) A finitely generated module is projective if and only if it is a direct summand of a finitely generated free module.

(iii) Every direct sum of projective modules are projective.

Proof. (i) This is a special case of Proposition 2.24 where we let P be a free module, since a free module is always a summand of a free module, see Example 2.22(i).

(ii) We have shown in Proposition 2.24 that a direct summand of a free module is projective. To show that a finitely generated projective module is a summand of a finitely generated free module: Let P be a projective module with generating set P'. Let $F(P')$ be a free module with basis P' and let φ be the surjective map $F(P') \to P$ mapping P' to P'. Since P is projective there exists a lift f' such that the diagram



(14)

commutes. We then have that $\varphi f' = id_P$ which directly implies that f' is injective but also that $f'\varphi : F(P') \to \text{im } f'\varphi$ is a retraction, see the text following Definition 2.17. And so $F(P') = \text{im } f'\varphi \oplus \ker f'\varphi = \text{im } f' \oplus \ker \varphi \cong P \oplus \ker \varphi$.

(iii) We will show this for a sum of two modules, for a proof of an arbitrary sum of modules see [2, pg. 102, Cor. 3.6 (ii), Ch. 3.1]. Let P_1 and P_2 be projective. According to *(ii)* there exists free modules such that $F_1 = P_1 \oplus Q_1$ and $F_2 = P_2 \oplus Q_2$. Let X_1 be a basis for F_1 and X_2 a basis for F_2. We have that $F_1 \oplus F_2$ is free since a basis is $X_1' \cup X_2'$ where $X_1' = \{(x_1, 0) \mid x_1 \in X_1\}$ and $X_2' = \{(0, x_2) \mid x_2 \in X_2\}$. Then we have that

$$\begin{aligned} F_1 \oplus F_2 &= (P_1 \oplus Q_1) \oplus (P_2 \oplus Q_2) \\ &= (P_1 \oplus P_2) \oplus (Q_1 \oplus Q_2) \end{aligned}$$

and so $P_1 \oplus P_2$ is a summand of a free module and as such a projective module.

□

Another consequence of the Universal Property of Free Modules is that every module is a quotient of a free module.

Corollary 2.28. *Every left R-module M is a quotient of a free module F.*

Proof. Let M be a left R-module and let X be a generating set of M, then, according to Theoreom 2.23, the diagram,

$$\begin{array}{ccc} X & \xrightarrow{\textit{inclusion}} & F(X) \\ & \searrow^{\varphi} & \downarrow \phi \\ & & M. \end{array} \tag{15}$$

commutes and $\phi(x) = \varphi(x)$. Since X is a subset of M, φ is an inclusion and $\phi(x) = \varphi(x) = x$ and so X is in the image of ϕ and since X generates M, ϕ is surjective. Using the isomorphism theorem we get that $F(X)/\mathit{ker}\ \phi \cong M$ and M is a quotient of $F(X)$. □

3 Exact Sequences

In the first part of this section we will define exact sequences of modules, also called extensions, and we will define what it means for these sequences to be equivalent. We will then state and prove two lemmas: The first is Lemma 3.9 which we will use in the proof of Theorem 7.12, the final Theorem in this text. The second lemma is the Snake Lemma 3.10 which we will use to prove the Horseshoe Lemma 4.13 and The Long Exact Sequence of Cohomology, Theorem 5.6.

In the second part we define induced exact sequences of Hom-sets. This will later be used to define the contravariant Hom-functor in section 7.1.1.

3.1 Exact Sequences

A surjective R-module homomorphism $\varphi : M \to N$ gives us the relationship $N \cong M/\ker \varphi$ by The First Isomorphism Theorem. If we let i be the injection of the kernel of φ into M and $\bar{\varphi}$ the quotient map we can restate The First Isomorphism Theorem as a sequence

$$\ker \varphi \xrightarrow{i} M \xrightarrow{\bar{\varphi}} M/\ker \varphi \tag{16}$$

where im i = ker $\bar{\varphi}$. The sequence (16) is called an exact sequence or more specifically a sequence exact at M.

Definition 3.1. Let X, Y, Z and L, M, N be left R-modules then:

(i) The pair of homomorphism $X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z$ is **exact** at Y if im α = ker β.

(ii) A sequence $\cdots \longrightarrow X_{n+1} \overset{f_{n+1}}{\longrightarrow} X_n \overset{f_n}{\longrightarrow} X_{n-1} \overset{f_{n-1}}{\longrightarrow} \cdots$ is an **exact sequence** if it is exact at every X_n between a pair of homomorphisms.

(iii) An exact sequence of the form $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is called a **short exact sequence**.

A sequence $0 \longrightarrow L \overset{\psi}{\longrightarrow} M \overset{\varphi}{\longrightarrow} N \longrightarrow 0$ being short exact implies that ψ is injective and φ is surjective.

Proposition 3.2. *The sequence $0 \longrightarrow L \overset{\psi}{\longrightarrow} M$ is exact if and only if ψ is injective and the sequence $M \overset{\varphi}{\longrightarrow} N \longrightarrow 0$ is exact if and only if φ is surjective.*

Proof. The image of the zero homomorphism $0 \to L$ is 0_L, so for the sequence in (i) to be exact *ker* $\psi = 0_L$ and this happens if and only if ψ is injective. The zero homomorphism $N \to 0$ maps all of N to 0 so for the sequence in (ii) to be exact im $\varphi = N$ and this happens if and only if φ is surjective. □

Corollary 3.3.

(i) The sequence $0 \longrightarrow L \overset{\gamma}{\longrightarrow} M \longrightarrow 0$ is exact if and only if γ is an isomorphism.

(ii) The sequence $0 \longrightarrow L \overset{\psi}{\longrightarrow} M \overset{\varphi}{\longrightarrow} N \longrightarrow 0$ is a short exact sequence if and only if ψ is injective, im ψ = ker φ and φ is surjective.

Proof. (i) The sequence in $0 \longrightarrow L \overset{\gamma}{\longrightarrow} M \longrightarrow 0$ is exact when it is exact at L and M. By Proposition 3.2 γ is then both injective and surjective and thus an isomorphism. For the opposite, if γ is an isomorphism it is both injective and surjective and by Proposition 3.2 the sequence is exact.

(ii) This follows directly from Definition 3.1 and Proposition 3.2.

□

So a sequence

$$0 \longrightarrow K \overset{i}{\longrightarrow} M \overset{\varphi}{\longrightarrow} N \longrightarrow 0 \tag{17}$$

being short exact is equivalent to the quotient M/K being isomorphic to N, in particular it implies that there is an extension M of K such that $M/K \cong N$. To highlight the latter the sequence (17) is also called an **extension of** K **by** N and there may exists several extensions K by N.

An exact sequence

$$\cdots \longrightarrow X_{n+1} \overset{f_{n+1}}{\longrightarrow} X_n \overset{f_n}{\longrightarrow} X_{n-1} \overset{f_{n-1}}{\longrightarrow} \cdots$$

can be written as a succession of short exact sequences. Each X_n is an extension of the cokernel of f_{n+1} by the kernel of f_n since for each X_n the sequence

$$0 \longrightarrow \ker f_n \xrightarrow{i} X_n \xrightarrow{\bar{f_n}} X_n/\mathrm{im}\ f_{n+1} \longrightarrow 0$$

is short exact. Free modules and direct sums can always be extended to short exact sequences.

Example 3.4.

(i) According to Corollary 2.28 every left R-module M is a quotient of a free module F,

$$M \cong F(X)/\ker \phi$$

where X is a generating set of M and $\phi : F(X) \to M$ is a surjective homomorphism. The free module $F(X)$ can then be extended to a short exact sequence, $0 \longrightarrow \ker \phi \xrightarrow{i} F(X) \xrightarrow{\phi} M \longrightarrow 0$. As an example consider the short exact sequence

$$0 \longrightarrow n\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

where $\mathbb{Z}/n\mathbb{Z}$ is generated by 1 and we can view $\mathbb{Z}$ as the free module $F(1)$. This sequence is usually written

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

since $n\mathbb{Z}$ being a cyclic module of infinite order is isomorphic to $\mathbb{Z}$.

(ii) The trivial extension of the modules A and C is the sequence

$$0 \longrightarrow A \xrightarrow{\alpha} A \oplus C \xrightarrow{\beta} C \longrightarrow 0,$$

where α and β are the canonical injection and projection maps, respectively, for a direct sum. Equivalently, for a module B where $B \cong A \oplus C$, there exists a short exact sequence,

$$0 \longrightarrow A \xrightarrow{\iota_1} B \xrightarrow{\rho_2} C \longrightarrow 0, \tag{18}$$

where ι_1 and ρ_2 is the injection and projection maps in Proposition 2.16 (*iv*).

In the last example above the existence of a surjective homomorphism $\rho_1 : B \to A$ such that

$$\rho_1\iota_1 = 1_A : A \to B \to A$$

implies a retraction

$$\iota_1\rho_1 : B \to A \to \mathrm{im}\ \iota_1 \cong A. \tag{19}$$

Equivalently an injective homomorphism ι_2 such that

$$\rho_2\iota_2 = 1_C : C \to B \to C,$$

implies a retraction

$$\iota_2\rho_2 : B \to C \to \text{im } \iota_2 \cong C. \tag{20}$$

According to proposition 2.18 equation (19) shows that

$$B = \text{im } \iota_1\rho_1 \oplus \ker \iota_1\rho_1 = \text{im } \iota_1 \oplus \ker \iota_1\rho_1$$

and equation (20) shows that

$$B = \text{im } \iota_2\rho_2 \oplus \ker \iota_2\rho_2 = \text{im } \iota_2 \oplus \ker \iota_2\rho_2.$$

Because the sequence (18) is exact, the image of ι_1 being a summand implies the retraction $\iota_2\rho_2$ since the equivalence

$$\text{im } \iota_1 = \ker \rho_2 = \ker \iota_2\rho_2, \tag{21}$$

suggests that ρ_2 is a retraction (proposition 2.18) meaning there exists an injection $C \to B$ implying the retraction (20). Equivalently if $\iota_2\rho_2$ is a retraction and $\ker \iota_2\rho_2$ is a summand, the equation (21) implies that im ι_1 is a summand and thus there exists a projection ρ_1 such that $\iota_1\rho_1$ is the retraction (19). In conclusion, equations (19) and (20) implies one another and given an exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

if we can find a projection or an injection such that we get the identity map on either A or C then B is isomorphic to their direct sum.

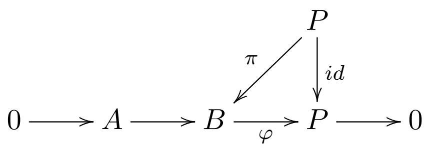
Proposition 3.5. *Let $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$ be a short exact sequence. If there exists a R-map $\pi : N \to M$ with $\varphi\pi = 1_N$ or equivalently, a map $\rho : M \to L$ with $\rho\psi = 1_L$ then $M \cong L \oplus N$ and the short exact sequence is* ***split****.*

Proof. The proof is covered in the text above. □

Example 3.6. Let P be a projective module, then every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

is split. Consider the diagram below, according to Corollary 2.27 (ii) there exists a map $\pi : P \longrightarrow B$ such that $\varphi\pi = id_P$.



and according to Proposition 3.5 the sequence is then split where $B \cong P \oplus \ker \varphi$.

Even though every direct sum implies a split short exact sequence, a short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} A \oplus C \xrightarrow{\beta} C \longrightarrow 0$ is not necessarily split. The R-maps α and β can be defined such that the sequence is exact but not meet the requirements of being split, see example [2, pg 54, Ch. 2.1].

A short exact sequence can be viewed as a single mathematical object, homomorphisms between short exact sequences are then defined as follows.

Definition 3.7. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and $0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$ be two short exact sequences of R-modules. A **homomorphism of short exact sequences** consists of a triple of homomorphisms, α, β, γ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0. \end{array}$$

The homomorphism of short exact sequences above is an **isomorphism of short exact sequences** when α, β, γ are all isomorphisms. The modules B and B' are then called **isomorphic extensions**. [1, pg. 381, Sec. 10.5]

If we let B and B' be isomorphic extensions, then the isomorphism of B and B' as R-modules is restricted to an isomorphism on modules A and A' and inducing an isomorphism on the quotients C and C'. If $\alpha = 1_A$ and $\gamma = 1_C$ the isomorphism of B and B' would be restricted to the identity on A and C and in this case we call B and B' equivalent extensions.

Definition 3.8. Two extensions A by C are **equivalent** if there exists an isomorphism $\beta : B \to B'$ that makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

The modules B and B' are then called **equivalent extensions**. [1, pg. 381, Sec. 10.5]

It is not hard to see that being equivalent is an equivalence relation on extensions and we will in Theorem 7.12 define a bijection on the set of equivalence classes of extensions A by C. The next Lemma will support Theorem 7.12.

Lemma 3.9. *Let $\Xi = 0 \to X_1 \to X_0 \to C \to 0$ be an extension X_1 by C, given a map $h : X_1 \to A$, consider the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & & & \downarrow{\scriptstyle 1_C} & & \\ & & A & & & & C. & & \end{array}$$

(i) There exists a commutative diagram with exact rows which completes the given diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{j} & X_0 & \xrightarrow{\epsilon} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \xrightarrow[i]{} & B & \xrightarrow[\eta]{} & C & \longrightarrow & 0 \end{array} \tag{22}$$

(ii) Any two bottom rows of completed diagrams are equivalent extensions.

Proof. (i) Suppose we define $B = A \oplus X_0$ and define $i(a) = (a, 0)$ and $\beta(b) = (0, b)$. Then i would be an injection but the first square in diagram (22) would fail to commute since for $x_1 \in X_1$ we have

$$ih(x_1) = (h(x_1), 0)$$
$$\beta j(x_1) = (0, j(x_1))$$

and $ih(x_1) \neq \beta j(x_1)$. If we instead define B as the quotient $(A \oplus X_0)/S$ where S is the subgroup of $A \oplus X_1$ consisting of all elements

$$(h(x_1), 0) - (0, j(x_1)) = (h(x_1), -j(x_1)) \text{ for all } x_1 \in X_1$$

and let

$$i(a) = (a, 0) + S \text{ and } \beta(x_0) = (0, x_0) + S$$

for $a \in A$ and $x_0 \in X_0$, then i is an injection and the first square commutes. Define $\eta : B' \to C$ as

$$(a, x_0) + S \mapsto \epsilon(x_0)$$

and the second square commutes since

$$\eta\beta(x_0) = \eta((0, x_0) + S) = \epsilon(x) = 1_C\epsilon(x).$$

We have to show that η is well-defined: Let (a', x'_0) be another representative for the coset $(a_0, x_0) + S$, then

$$(a'_0, x'_0) = (a_0, x_0) + (h(x_1), -j(x_1)) = (a_0 + h(x_1), x_0 - j(x_1))$$

and

$$\eta((a_0 + h(x_1), x_0 - j(x_1))) = \epsilon(x_0 - j(x_1)) = \epsilon(x_0) = \eta(a_0, x_0).$$

Finally we show that the second row is exact, we will first show that $\text{im } i \subseteq \ker \eta$: Since the diagram commutes we have that $\ker \eta\beta = \ker \epsilon = \text{im } j$ which makes $\ker \eta = \beta j$ and since $\beta j = ih$, $\beta j \subseteq \text{im } i$. For the reverse we have that $\eta i(a) = \epsilon(a, 0) = \epsilon(0) = 0_C$ and the second row is exact.

(ii) Let the bottom row of

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{j} & X_0 & \xrightarrow{\epsilon} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle 1_C} & & \\
0 & \longrightarrow & A & \xrightarrow[i']{} & B' & \xrightarrow[\eta']{} & C & \longrightarrow & 0
\end{array}$$

be a second extension completing the diagram. We need to define θ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\eta} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_C} & & \\
0 & \longrightarrow & A & \xrightarrow[i']{} & B' & \xrightarrow[\eta']{} & C & \longrightarrow & 0
\end{array} \tag{23}$$

commutes. Since B is a direct sum of A and X_0 we use the maps i' and β' and define $\theta : B \to B'$ by

$$(a, x_0) + S \mapsto i'(a) + \beta'(x_0)).$$

The map θ is well-defined since for $(a', x_0') = (a, x_0) + (h(x_1), -j(x_1))$ we have

$$\begin{aligned}
\theta(a', x_0') &= \theta(a, x_0) + \theta(h(x_1), -j(x_1)) \\
&= (i'(a) + \beta'(x_0)) + (i'h(x_1) - \beta' j(x_1)) \\
&= i'(a) + \beta'(x_0) = \theta(a, x_0).
\end{aligned}$$

The diagram (23) commutes since:

$$\theta i(a) = \theta((a, 0) + S) = i'(a) = i' 1_A(a)$$

and

$$\begin{aligned}
\eta' \theta((a, x_0) + S) &= \eta' i'(a) + \eta' \beta'(x_0) \\
&= 0 + \epsilon = \eta\beta(x_0) = \eta(0, x_0).
\end{aligned}$$

[2, pg. 423-424, Lemma 7.28, Chapter 7] □

A homomorphims of short exact sequences implies an exact sequence of kernels and cokernels of the homomorphisms involved. The Snake Lemma below will later on be used to prove The Horseshoe Lemma 4.13 and The Long Exact Sequence of Cohomology 5.6.

Lemma 3.10. *(Snake Lemma) Given a commutative diagram of modules with exact rows,*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0,
\end{array} \tag{24}$$

there is an exact sequence

$$0 \to ker\ \alpha \to ker\ \beta \to ker\ \gamma \longrightarrow coker\ \alpha \to coker\ \beta \to coker\ \gamma \to 0$$

[4, pg 268, Corollary C-3.43 ch. C-3]

Proof. Proof of snake lemma . The proof will not be written out in its entirety. We will show the exactness of sequences

$$0 \to \ker\, \alpha \to \ker\, \beta \to \ker\, \gamma,$$

and

$$A'/\text{im}\ \alpha \to B'/\text{im}\ \beta \to C'/\text{im}\ \gamma \to 0$$

and then define the connecting homomorphism $\delta : \ker\, \gamma \to A'/\text{im}\ \alpha$.

(i) Consider the diagram below where the second and third row makes up a commutative diagram,

$$\begin{CD}
0 @>>> \ker\, \alpha @>\bar{j}>> \ker\, \beta @>\bar{p}>> \ker\, \gamma @. \\
@. @VVV @VVV @VVV @. \\
0 @>>> A @>j>> B @>p>> C @>>> 0 \\
@. @V\alpha VV @V\beta VV @V\gamma VV @. \\
0 @>>> A' @>i>> B' @>q>> C' @>>> 0.
\end{CD}$$

Let $a \in \ker\, \alpha$ and $b \in \ker\, \beta$ then the commutativity of the diagram gives us

$$\beta j(a) = i\alpha(a) = i(0_{A'}) = 0_{B'}$$
$$\gamma p(b) = q\beta(b) = q(0_{B'}) = 0_{C'}$$

and we have that $j(a)$ is in the kernel of β and $p(b)$ is in the kernel of γ and there exists a sequence

$$0 \to \ker\, \alpha \xrightarrow{\bar{j}} \ker\, \beta \xrightarrow{\bar{p}} \ker\, \gamma, \tag{25}$$

where $\bar{j}$ and $\bar{p}$ are the homomorphisms j and p restricted to the kernels of α and β. This sequence is exact. Exactness at ker α: Since j is injective so is $\bar{j}$ and the sequence is exact at ker α. Exactness at ker β: Since im j = ker p we have that $\bar{p}\bar{j}(a) = 0_C$ and so im $\bar{j} \subseteq$ ker $\bar{p}$. To show that ker $\bar{p} \subseteq$ im $\bar{j}$, let $\bar{p}(b) = 0_C$. By the exactness at B there exists some $a \in A$, such that $b = j(a)$ and since $i\alpha(a) = \beta j(a) = 0_{B'}$ and i is injective, $\alpha(a) = 0_{A'}$ and so $a \in \ker\, \alpha$ and then $b = j(a) \in \text{im}\ \bar{j}$ and so ker $\bar{p} \subseteq$ im $\bar{j}$.

(ii) Consider the diagram below where the first and second row makes up a commutative diagram,

$$\begin{CD}
0 @>>> A @>j>> B @>p>> C @>>> 0 \\
@. @V\alpha VV @V\beta VV @V\gamma VV @. \\
0 @>>> A' @>i>> B' @>q>> C' @>>> 0 \\
@. @VVV @VVV @VVV @. \\
@. A'/\text{im}\ \alpha @>\bar{i}>> B'/\text{im}\ \beta @>\bar{q}>> C'/\text{im}\ \gamma @>>> 0.
\end{CD}$$

Define $\bar{i}$ and $\bar{q}$ as

$$\bar{i}(a' + \text{im } \alpha) = i(a') + \text{im } \beta$$
$$\bar{q}(b' + \text{im } \beta) = q(b') + \text{im } \gamma$$

for $a' \in A'$ and $b' \in B'$ respectively. The homomorphisms are well-defined. The commutativity of the first two rows in the diagram gives us

$$i\alpha = \beta j$$

and so i maps the image of α to the image of β which means that $\bar{i}$ is well-defined since

$$\begin{aligned} a'_1 + \text{im } \alpha = a'_2 + \text{im } \alpha \in A'/\text{im } \alpha \\ \rightarrow \\ a'_1 - a'_2 \in \text{im } \alpha \\ \rightarrow \\ i(a'_1 - a'_2) \in \text{im } \beta. \end{aligned}$$

And so $i(a'_1) + \text{im } \beta = i(a'_2) + \text{im } \beta \in B'/\text{im } \beta$. Similarly $\bar{q}$ is well-defined since $q\beta = \gamma p$. We will show that the sequence,

$$A'/\text{im } \alpha \xrightarrow{\bar{i}} B'/\text{im } \beta \xrightarrow{\bar{q}} C'/\text{im } \gamma \rightarrow 0, \tag{26}$$

is exact. Since q is surjective, $\bar{q}$ is surjective and the sequence is exact at $C'/\text{im } \gamma$. Exactness at $B'/\text{im } \beta$: Since $qi = 0$ we have that $\bar{q}\bar{i} = 0 + \text{im } \beta$ and so $\text{im } \bar{i} \subset \ker \bar{q}$. To show that $\ker \bar{q} \subset \text{im } \bar{i}$, let $b' + \text{im } \beta \in \ker \bar{q}$, then $q(b') \in \text{im } \gamma$ and so $\gamma(c) = q(b')$ for some $c \in C$. Since p is surjective there exists a $b \in B$ such that $p(b) = c$ and so $\gamma p(b) = q(b')$ and due to the commutativity of the diagram we have

$$\gamma p(b) = q\beta(b) = q(b')$$

and so $b' \in \text{im } \beta$ and $b' + \text{im } \beta \in \text{im } \bar{i}$.

(iii) We will now show that there exists a homomorphism $\ker \gamma \rightarrow \text{coker } \alpha$ defined by

$$\begin{aligned} \delta : \ker \gamma \rightarrow A'/\text{im } \alpha \\ z \mapsto i^{-1}\beta p^{-1}(z) + \text{im } \alpha. \end{aligned} \tag{27}$$

Let $c \in \text{kernel } \gamma$, and let $p(b) = c$, $b \in B$, since p is surjective such b exists. The commutativity of the diagram gives us

$$q\beta(b) = \gamma p(b) = \gamma(c) = 0'_C$$

and so $\beta(b)$ is in the kernel of q and as such in the image of i. We have that i is injective so there exists a unique $a' \in A'$ such that

$$i^{-1}\beta p^{-1}(c) = i^{-1}\beta(b) = a'.$$

and we have a map $c \mapsto a' + \text{im } \alpha$. To show that it is well-defined: Let $c_1 = c_2$, then there exists b_1 and b_2 such that

$$p(b_1 - b_2) = c_1 - c_2 = 0_C,$$

and $(b_1 - b_2)$ is in the kernel of p and as such also in the image of j and there exists an unique $a \in A$ such that $j(a) = (b_1 - b_2)$. From the commutativity of the diagram we then get that

$$i\alpha(a) = \beta j(a) = \beta(b_1 - b_2),$$

and $\beta(b_1 - b_2)$ is in the image of $i\alpha$ and since i is injective,

$$i^{-1}\beta(b_1 - b_2) = \alpha(a),$$

and so $\beta(b_1 - b_2)$ is mapped to the image of α. And so since

$$i^{-1}\beta(b_1 - b_2) = i^{-1}\beta p^{-1}(c_1 - c_2),$$

we have that $c_1 - c_2$ is mapped to the image of α and so

$$i^{-1}\beta p^{-1}(c_1) = i^{-1}\beta p^{-1}(c_2) \in A'/\text{im } \alpha.$$

and (27) is well-defined.

We have shown that there exists a homomorphisms δ connecting two exact sequences (25) and (26) into a sequence

$$0 \to \ker \alpha \xrightarrow{\bar{j}} \ker \beta \xrightarrow{\bar{p}} \ker \gamma \xrightarrow{\delta} A'/\text{im } \alpha \xrightarrow{\bar{i}} B'/\text{im } \beta \xrightarrow{\bar{q}} C'/\text{im } \gamma \to 0.$$

What is left to show is the exactness at $\ker \gamma$ and coker α, these parts will be left out. [4, pg. 269, Excercise C-3.25 (i), (ii), Ch. C-3.5] □

Note that if we change diagram (24) in the Snake Lemma above to

$$\begin{array}{ccccccc} & & A & \xrightarrow{j} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow[q]{} & C' & & \end{array},$$

we would get an exact sequence

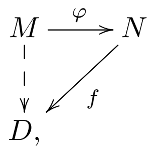
$$\ker \alpha \xrightarrow{\bar{j}} \ker \beta \xrightarrow{\bar{p}} \ker \gamma \xrightarrow{\delta} A'/\text{im } \alpha \xrightarrow{\bar{i}} B'/\text{im } \beta \xrightarrow{\bar{q}} C'/\text{im } \gamma,$$

since the injectivity of $\bar{j}$ and surjectivity of $\bar{q}$ is a direct consequence of j being injective and q being surjective (part (i) and (ii) in the proof above). We will use this to prove The Long Exact Sequence of Cohomology 5.6 in section 5.

We will now look at induced exact sequences.

3.2 Exact Sequences Derived From Hom-sets

Given R-modules M, N, D and a homomorphism φ as below,



we can always construct a composite homomorphism $f\varphi \in \mathrm{Hom}_R(M, D)$, for any $f \in \mathrm{Hom}_R(N, D)$. Since $\mathrm{Hom}_R(M, D)$ and $\mathrm{Hom}_R(N, D)$ are abelian groups, see Proposition 2.5 (i) we can define a homomorphism of groups: $\varphi^* : \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(M, D)$, by $f \mapsto f\varphi$.

Proposition 3.11. *Let M, N and D be R-modules and let $\varphi : M \to N$ be an R-module homomorphism.*

(i) Then the induced map

$$\varphi^* : Hom_R(N, D) \longrightarrow Hom_R(M, D)$$
$$f \longmapsto f\varphi$$

is a homomorphism of abelian groups.

(ii) Moreover if φ is surjective the induced map φ^* is injective.

Proof. Let $m \in M$ and $n \in N$. By Proposition 2.5 are $Hom_R(N, D)$ and $Hom_R(M, D)$ abelian groups. From the addition defined in 2.5*(iv)* we get $(f + g)(\varphi(m)) = f(\varphi(m)) + g(\varphi(m))$ and so $\varphi^*(f + g) = \varphi^*(f) + \varphi^*(g)$ and φ^* is a homomorphism. For the second part, assume φ is surjective, then for all $n \in N$ we have that $n = \varphi(m)$ for some $m \in M$. Let f_1 and f_2 be two homomorphisms from N to D, assume $f_1\varphi(m) = f_2\varphi(m)$, since φ is surjective $f_1\varphi(m) = f_2\varphi(m)$ for all $m \in M$ implies $f_1(n) = f_2(n)$ for all $n \in N$ and so $f_1 = f_2$ and φ^* is injective. □

Part *(ii)* of proposition 3.11 can be written in terms of exact sequences; from the exact sequence of modules,

$$M \xrightarrow{\varphi} N \longrightarrow 0,$$

we get the exact sequence

$$0 \longrightarrow Hom_R(N, D) \xrightarrow{\varphi^*} Hom_R(M, D).$$

of abelian groups. Part *(i)* of proposition 3.11 implies that given a composite,

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0, \tag{28}$$

there exists a sequence

$$0 \to Hom_R(N, D) \overset{\varphi^*}{\to} Hom_R(M, D) \overset{\psi^*}{\to} Hom_R(L, D),$$

where $\psi^* = f'\psi$ for $f' \in \mathrm{Hom}\ (M, D)$. The theorem below then shows that if the extended sequence (28) is exact so is the induced sequence of Hom groups.

Theorem 3.12. *Let D be an R-module and let*

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0 \tag{29}$$

be a sequence of R-modules. Define the homomorphisms $\varphi^* : Hom\ (N, D) \to Hom\ (M, D)$ and $\psi^* : Hom\ (M, D) \to Hom\ (L, D)$ as

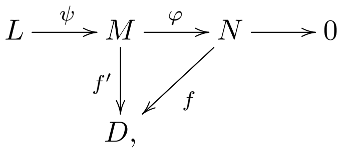
$$\varphi^* : f \mapsto f\varphi,$$
$$\psi^* : f' \mapsto f'\psi.$$

If sequence (29) is exact, is the associated sequence

$$0 \to Hom_R(N, D) \xrightarrow{\varphi^*} Hom_R(M, D) \xrightarrow{\psi^*} Hom_R(L, D)$$

exact.

Proof. According to Proposition 3.11, φ^* is injective so the sequence is exact at $\mathrm{Hom}_R(N, D)$. We need to show that the sequence is exact at $\mathrm{Hom}_R(M, D)$, that is $\mathrm{Ker}\ \psi^* = \mathrm{Im}\ \varphi^*$. First we show that $\mathrm{Im}\ \varphi^* \subseteq \mathrm{Ker}\ \psi^*$: for $f^* \in \mathrm{Im}\ \varphi^*$ we want to show $\psi^*\varphi^*(f) = 0$. Consider the diagram



we have that $\varphi^*(f) = f\varphi$ and so $\psi^*\varphi^*(f) = (f\varphi)\psi = f(\varphi\psi) = f0 = 0$. Next we show that $\mathrm{Ker}\ \psi^* \subseteq \mathrm{Im}\ \varphi^*$: Let $f' \in \mathrm{Ker}\ \psi^*$ then $f' : M \to D$ and $f'\psi = 0$. For any m and $m' \in M$ where

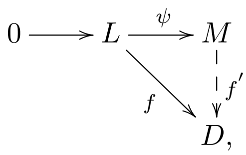
$$m = m' \mod \mathrm{im}\ \psi$$

we have that $f(m) = f(m')$ and so f' is constant on $M/\mathrm{im}\ \psi$, meaning f' is welldefined on $M/\mathrm{im}\ \psi \to D$. Since the sequence (29) is exact and since p is projective we have $M/\mathrm{im}\ \psi = M/\mathrm{ker}\ p \cong N$ and there exists a welldefined map $f : N \to D$ by

$$f(n) = f\varphi(m) = f'(m).$$

Since $f\varphi$ is in the image of φ', f' is in the image of φ^*. For proof of the "only if" part of the statement see [1, pg. 393-394, Thm. 33, ch.10.5]. [3, pg. 468, Thm. B-4.21 (iii), ch.B-4], [6, StackExchange, (1)] □

If the sequence (29) were short exact, would it induce a short exact sequence of Hom groups? Consider the diagram below where $0 \longrightarrow L \xrightarrow{\psi} M$, is an exact sequence



Just like in Proposition 3.11, composing $f^{'} \in Hom_R(M,D)$ with ψ induce a group homomorphism,

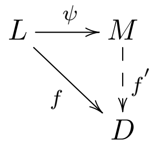
$$\psi^* : Hom_R(M,D) \longrightarrow Hom_R(L,D)$$

defined by

$$f^{'} \longmapsto f^{'}\psi.$$

Suppose ψ^* is surjective, then there would be a lift f' for every $f \in \mathrm{Hom}_R(L,D)$ such that $f = f^{'}\psi$, and this is not in general true, see [1, pg. 393, Sec.10.5]. However, some modules are such that there exists a lift $f' \in \mathrm{Hom}_R(M,D)$ for every $f \in \mathrm{Hom}_R(L,D)$. These modules, analogous to projective modules (Definition 2.26) are called injective modules.

Definition 3.13. Let $\psi : L \longrightarrow M$ be an injective R-map and L, M and D left R-modules, then D is called an **injective** module if there exists a map f' for every map f such that f' to f. That is, there exists a map f' for every map f such that the diagram



commutes.

Given an injective module D, the sequence

$$Hom_R(M,D) \xrightarrow{\psi^*} Hom_R(L,D) \longrightarrow 0,$$

is exact.

Corollary 3.14. *A module D is injective if and only if for every exact sequence*

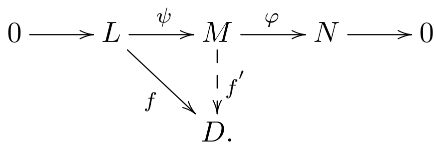
$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$$

the induced sequence

$$0 \to Hom_R(N,D) \xrightarrow{\varphi^*} Hom_R(M,D) \xrightarrow{\psi^*} Hom_R(L,D) \to 0 \tag{30}$$

is short exact.

Proof. Consider the diagram



Assume D is an injective module then there exists a lift f' for each $f \in \mathrm{Hom}_R(N,D)$, the induced map ψ^* is then surjective and the exact sequence,

$$0 \to Hom_R(N,D) \xrightarrow{\varphi^*} Hom_R(M,D) \xrightarrow{\psi^*} Hom_R(L,D),$$

from Theorem 3.12 is extended to a short exact sequence. On the other hand, given a short exact sequence (30) then ψ^* is surjective by definition and $f = \psi^*(f^{'}) = f^{'}\psi$ for all $f \in \mathrm{Hom}_R(N,D)$ and D is an injective module. □

4 Complexes

A complex of modules is a sequence of modules together with a sequence of homomorphisms such that the composition of any two consecutive homomorphisms is zero. Most constructions for modules can be applied to complexes and in the first part of this section we will state some properties for complexes analogue to modules, in particular we will define short exact sequences of complexes. The second and third part describes how we can resolve a module M with an exact sequence of free modules and, by extension, how we can replace a short exact sequence of modules with a short exact sequence of free resolutions.

4.1 Chain and Cochain Complexes

An exact sequence is an example of a more general structure called a complex, where instead of equality of images and kernels of consecutive homomorphisms, the former is a subset of the later.

Definition 4.1. A **chain complex** is a sequence of modules and homomorphisms

$$(\mathbf{C}_\bullet, d_\bullet) = \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots \tag{31}$$

such that the composition of adjacent homomorphisms is 0: $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. The maps d_n are called differentials. [2, pg. 317, Definition, Ch. 5.5]

We will sometimes use $\mathbf{C}_\bullet$ to refer to a chain complex $(\mathbf{C}_\bullet, d_\bullet)$ and we will often say complex to mean chain complex.

Definition 4.2. Define the **zero complex** $0_\bullet = (C_\bullet, d_\bullet)$ as the complex where each $C_n = \{0\}$ and thus $d_n = 0$ for all n.
[4, pg. 257, Example (iv), Ch. C-3.4]

By adding the zero complex to the left and right of any R-module homomorphism it can be viewed as a complex.

Example 4.3. Any R-map $f : A \longrightarrow B$ can be extended to a complex by adding zero-modules and zero maps. Let $A = C_1$, $B = C_0$ and $f = d_1$ then

$$0_\bullet \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0_\bullet$$

is a complex where every module and map except C_0, C_1 and d_1 are zero. Since the differentiations d_2 and d_0 are both zero maps we have that

$$\text{im } d_2 = 0 \subseteq \ker d_1$$

and

$$\text{im } d_1 \subseteq \ker d_0 = C_0.$$

[4, pg. 257, Example (v), Ch. C-3.4]

In the same way as in the example above we can view any exact sequence, $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ as a complex since the equality $\mathrm{Im}\ \alpha = \ker \beta$ implies the inclusion $\mathrm{Im}\ \alpha \subseteq \ker \beta$.

Definition 4.4. (i) A **positive chain complex** is a complex where $C_n = 0$ for all $n < 0$:

$$\longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow ... \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

(ii) A **negative complex** or a **cochain complex** is a complex where $C_n = 0$ for all $n > 0$:

$$0 \longrightarrow C_0 \longrightarrow C_{-1} \longrightarrow ... \longrightarrow C_{-n} \longrightarrow C_{-(n+1)} \longrightarrow ...$$

The negative indices are usually changed to positive and instead put as a superscript:

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow ... \longrightarrow C^n \longrightarrow C^{n+1} \longrightarrow ...$$

[2, pg. 328-329, Definition, Ch. 6.1][2, pg. 329, Ch. 6.1]

Note that the customary notation for cochain complexes do not follow the definition of a complex as the indexes are increasing by one instead of decreasing. However, the negative notation is according to the definition and all statements regarding complexes can be applied to cochain complexes with the appropriate adjustments to the indices.

A homomorphism of complexes $f : \mathbf{C}_\bullet \to \mathbf{C}'_\bullet$ is a sequence of maps $f_n : C_n \to C'_n$ creating a commutative diagram.

Definition 4.5. If $(\mathbf{C}_\bullet, d_\bullet)$ and $(\mathbf{C}'_\bullet, d'_\bullet)$ are complexes, then a **chain map**

$$f = f^\bullet : (\mathbf{C}_\bullet, d_\bullet) \to (\mathbf{C}'_\bullet, d'_\bullet)$$

is a sequence of morphisms $f_n : C_n \to C'_n$ for all $n \in \mathbb{Z}$ making the following diagram commute:

$$\begin{array}{ccccccccc} ... & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & ... \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ ... & \longrightarrow & C'_{n+1} & \xrightarrow[d'_{n+1}]{} & C'_n & \xrightarrow[d'_n]{} & C'_{n-1} & \longrightarrow & .... \end{array} \tag{32}$$

[2, pg. 318, Definition Ch. 5.5]

Complexes inherit the properties of modules. A direct sum of complexes $(C_\bullet, d_\bullet) \oplus (C'_\bullet, d'_\bullet)$ for example, is defined as the complex

$$\longrightarrow C_{n+1} \oplus C'_{n+1} \xrightarrow{\bar{d}_{n+1}} C_n \oplus C'_n \xrightarrow{\bar{d}_n} C_{n-1} \oplus C'_{n-1} \longrightarrow$$

where

$$\bar{d}_n : c \oplus c' \mapsto d_n(c) \oplus d'_n(c).$$

And a chain map $f : \mathbf{C_\bullet} \to \mathbf{C_\bullet'}$ is an isomorphism if and only if each $f_n : C_n \to C_n'$ is an isomorphism in which case the sequence of inverses, f^{-1}, is also a chain map. The First Isomorphism Theorem holds for any chain map $f : \mathbf{C_\bullet} \to \mathbf{C_\bullet'}$ if

$$C_n/\text{Ker } f_n \cong \text{Im } f_n$$

for each homomorphism $f_n : C_n \to C_n'$. [2, pg. 319, Examples (i), (ii), Ch. 5.5] It follows that the notion of exactness can also be applied to chain maps.

Definition 4.6. (i) A complex $(\mathbf{A_\bullet}, \delta_\bullet)$ is a **subcomplex** of $(C_\bullet, d_\bullet)$ if, for every $n \in \mathbb{Z}$, A_n is a submodule of C_n and $\delta_n = d_n|A_n$.

(ii) If $\mathbf{A_\bullet}$ is a subcomplex of $\mathbf{C_\bullet}$ then the **qoutient complex** is defined as

$$\mathbf{C_\bullet}/\mathbf{A_\bullet} = \cdots \to C_n/A_n \overset{\hat{d_n}}{\to} C_{n-1}/A_{n-1} \to$$

where $\hat{d_n} : c_n + A_n \mapsto d_n c_n + A_{n-1}$.

(iii) Let $f_\bullet : (C_\bullet, d_\bullet) \longrightarrow (C_\bullet', d_\bullet')$ be a chain map. Then $\mathbf{ker} f$, defined by

$$\mathbf{ker}\ f = \longrightarrow \text{ker } f_{n+1} \overset{\delta_{n+1}}{\longrightarrow} \text{ker } f_n \overset{\delta_n}{\longrightarrow} \text{ker } f_{n-1} \longrightarrow, \quad \delta_n = d_n \mid \text{ker } f_n,$$

is a subcomplex of $C_\bullet$ and $\mathbf{im}\ f$, defined by

$$\mathbf{im}\ f = \longrightarrow \text{im } f_{n+1} \overset{\Delta_{n+1}}{\longrightarrow} \text{im } f_n \overset{\Delta_n}{\longrightarrow} \text{im } f_{n-1} \longrightarrow, \quad \Delta_n = d_n' \mid \text{ker } f_n,$$

is a subcomplex of $C_\bullet'$.

(iv) A sequence of complexes and chain maps

$$\ldots \longrightarrow \mathbf{C}_\bullet^{n+1} \underset{f_{n+1}}{\longrightarrow} \mathbf{C}_\bullet^{n} \underset{f_n}{\longrightarrow} \mathbf{C}_\bullet^{n-1} \longrightarrow \ldots$$

is **exact** if $\text{im} f_{n+1} = \text{ ker } f_n$ for all $n \in \mathbb{Z}$.

[2, pg. 318 - 320, Ch. 5.5]

To illustrate the above definition of an exact sequence of complexes, if we write each complex as a column, then a short exact sequence of complexes, $0 \longrightarrow \mathbf{C_\bullet'} \overset{f}{\longrightarrow} \mathbf{C_\bullet} \overset{g}{\longrightarrow} \mathbf{C_\bullet''} \longrightarrow 0$, is a commutative diagram where the columns are complexes and the rows are exact:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n+1}' & \xrightarrow{f_{n+1}} & C_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1}'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_{n+1}'} & & \downarrow{\scriptstyle d_{n+1}} & & \downarrow{\scriptstyle d_{n+1}''} & & \\
0 & \longrightarrow & C_n' & \xrightarrow{f_n} & C_n & \xrightarrow{g_n} & C_n'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_n'} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n''} & & \\
0 & \longrightarrow & C_{n-1}' & \xrightarrow{f_{n-1}} & C_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}.$$

We will now show how to construct complexes from any module and short exact complexes from short exact sequences of modules.

4.2 Projective Resolutions of Modules

From a short exact sequence of modules we can construct a short exact sequence of complexes. The following section is based on [2, pg 325-326, ch. 6.1] unless otherwise specified.

In Example 3.4 we saw that for any module M we can create a short exact sequence

$$0 \longrightarrow K \xrightarrow{i} F(X) \xrightarrow{\varphi} M \longrightarrow 0, \tag{33}$$

where X is a generating set of M, $F(X)$ is the free module with basis X and K is the kernel of φ. If we let Y be a generating set of K then (X, Y) is a **presentation** of M. Presentations allows us to to treat equations in M as if they were equations in the free module $F(X)$ [2, pg 325, ch. 6.1]. If K is free and Y is a basis computations in F would become much easier but a sub module of a free module is not necessarily free and so K does not need to be free. However, assuming K is not free, starting from K we can create another short exact sequence as the one above,

$$0 \longrightarrow K_1 \xrightarrow{i_1} F_1(X_1) \xrightarrow{\varphi_1} K \longrightarrow 0, \tag{34}$$

where F_1 is a free module over some generating set X_1 of K and where K_1 is the kernel of φ_1. We can then combine the two sequences (33) and (34) and get

$$0 \longrightarrow K_1 \xrightarrow{i_1} F_1(X_1) \xrightarrow{\varphi_1} K \xrightarrow{i} F(X) \xrightarrow{\varphi} M \longrightarrow 0.$$

By continuing to iterate over K_n in this manner we can construct an exact sequence of free modules from M,

$$\begin{array}{ccccccccccc} \dashrightarrow F_3(X_3) & \overset{f_3}{\dashrightarrow} & F_2(X_2) & \overset{f_2}{\dashrightarrow} & F_1(X_1) & \overset{f_1}{\dashrightarrow} & F(X) & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\ & \overset{\varphi_3}{\searrow} & \uparrow{\scriptstyle i_2} & \overset{\varphi_2}{\searrow} & \uparrow{\scriptstyle i_1} & \overset{\varphi_1}{\searrow} & \uparrow{\scriptstyle i} & & & & \\ & & K_2 & & K_1 & & K & & & & \end{array} \tag{35}$$

where the functions f_n are the composite functions $i_n \circ \varphi_{n+1}$. The sequence is exact by design, since the image of f_n is K_{n-1} and K_{n-1} is the kernel of φ_{n-1}. The sequence (35) is called a free resolution of M and can be seen as a generalized presentation of M and a way of treating equations in M by a sequence of equations in free modules [2, pg 325, Sec. 6.1]. There are however many different generating sets of a module and a single module M can have many different resolutions.

Definition 4.7. Let A be any R-module. A **free resolution** of A is an exact sequence

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \ldots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} A \longrightarrow 0 \tag{36}$$

such that each F_i is a free R-module. Since free modules are projective, this is also a projective resolution.

[1, pg 779, Sec. 17.1]

Example 4.8. The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

from Example 3.4(i) is a projective resolution of $\mathbb{Z}/n\mathbb{Z}$. In general we have that any $\mathbb{Z}$-module M have a free resolution of the form

$$0 \longrightarrow \ker \varphi \longrightarrow F(X) \longrightarrow M \longrightarrow 0.$$

This is because $\ker \varphi$ is free due to $F(X)$ being a $\mathbb{Z}$-module, see Example 2.22.

The module A in sequence (36) is the cokernel of d_1: $A \cong F_0/\mathrm{Ker}\ \epsilon = F_0/\mathrm{Im}\ d_1$ and this allows us to delete A without losing any information.

Definition 4.9. Let A be a module and let the exact sequence

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} A \longrightarrow 0,$$

be a free resolution of A, then the **deleted resolution** of A is the complex

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0. \tag{37}$$

Note that the sequence (37) is no longer exact since at F_0 we have that Ker $(F_0 \to \{0\}) = F_0$ and im $d_1 \neq F_0$.

4.3 Simultaneous Resolution

Free resolutions are exact and as such they are complexes. Given a homomorphism f between modules A and A' there exists homomorphisms $f_n : F_n \to F'_n$ between the modules of some free resolutions of A and A' respectively, constituting a chain map. This chain map is the result of an iteration of lifts, starting with the lift $\hat{f}_0$ of the initial composite function $f\epsilon$,

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & F_0 & \xrightarrow{\epsilon} & A & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \hat{f}_0} & & \Big\downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & F'_0 & \xrightarrow{\epsilon'} & A' & \longrightarrow & 0.
\end{array}$$

Since free modules are projective modules, see Corollary 2.27, we are ensured that a lift $\hat{f}_n$ always exists. The next theorem states that there exist a chain map between the projective resolutions of modules A and A' provided there is a homomorphism $f : A \to A'$ to start the iteration of lifts. Take note in the proof that it suffice for the top row to be projective and the second row to be exact.

Proposition 4.10. *Let $f : A \to A'$ be any homomorphism of R-modules. Take projective resolutions of A and A' respectively. Then for each $n \geq 0$ there is a lift $\hat{f}_n$ of f such that* *the following diagram commutes,*

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \cdots \longrightarrow & P_0 & \xrightarrow{\epsilon} & A & \longrightarrow 0 \\ & & \downarrow{\scriptstyle \hat{f}_{n+1}} & & \downarrow{\scriptstyle \hat{f}_n} & & \downarrow{\scriptstyle \hat{f}_{n-1}} & & & \downarrow{\scriptstyle \hat{f}_0} & & \downarrow{\scriptstyle f} & \\ \cdots & \longrightarrow & P'_{n+1} & \xrightarrow[d'_{n+1}]{} & P'_n & \xrightarrow[d'_n]{} & P'_{n-1} & \xrightarrow[d'_{n-1}]{} & \cdots \longrightarrow & P'_0 & \xrightarrow{\epsilon'} & A' & \longrightarrow 0. \end{array} \tag{38}$$

This makes $\hat{f} : \mathbf{P}_A \to \mathbf{P}_{A'}$ *a chain map, where* $\mathbf{P}_A$ *and* $\mathbf{P}_{A'}$ *are the deleted resolutions of* A *and* A'*.*

Proof. Since P_0 is projective there is a lift $\hat{f}_0$ of the composite map $f\epsilon$ such that $\epsilon' \hat{f}_0 = f\epsilon$. The following lifts can be shown by induction where it suffices to show that Im $\hat{f}_n d_{n+1} \subseteq$ Im d'_{n+1} since P_{n+1} being projective ensures a lift $\hat{f}_{n+1}$ such that $d'_{n+1}\hat{f}_{n+1} = \hat{f} d_{n+1}$. The rows are exact so we will show Im $\hat{f}_n d_{n+1} \subseteq$ Ker d'_n. We have the base step $\hat{f}_0$ which allows us to assume there exists a lift $\hat{f}_n$ such that $d'_n \hat{f}_n = \hat{f}_{n-1} d_n$ and so

$$d'_n \hat{f}_n d_{n+1} = \hat{f}_{n-1} d_n d_{n+1} = 0.$$

Note that there are no lifts required from the second row in the diagram and that it is enough for the second row to be exact, not necessarily projective. [4, pg. 273-274, Theorem C-3.46 (i)] □

Given a homomorphism $f : A \to A'$ of modules there exists a chain map $\hat{f} : \mathbf{P}_A \to \mathbf{P}_{A'}$ of deleted resolutions, moreover $\mathbf{P}_A$ being projective and $\mathbf{P}'_A$ being exact also gives rise to lifts $s_n : P_n \to P'_{n+1}$ such that for any two chain maps $\hat{f}, g : \mathbf{P}_A \to \mathbf{P}_{A'}$ we have $f_n - g_n = d'_{n+1} s_n + s_{n-1} d_n$.

Definition 4.11. Two chain maps $f, g : (\mathbf{A}_\bullet, d_\bullet) \to (\mathbf{A'}_\bullet, d'_\bullet)$ are **homotopic** if,

(i) for chain complexes $(\mathbf{A}_\bullet, d_\bullet)$ and $(\mathbf{A'}_\bullet, d'_\bullet)$;

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \longrightarrow & \cdots \\ & & \downarrow & \swarrow{\scriptstyle s_n} & \downarrow{\scriptstyle f_n - g_n} & \swarrow{\scriptstyle s_{n-1}} & \downarrow & & \\ \cdots & \longrightarrow & A'_{n+1} & \xrightarrow[d'_{n+1}]{} & A'_n & \xrightarrow[d'_n]{} & A'_{n-1} & \longrightarrow & \cdots . \end{array}$$

there are maps $s_n : A_n \to A'_{n+1}$ such that

$$f_n - g_n = d'_{n+1} s_n + s_{n-1} d_n,$$

(ii) and for cochain complexes $(\mathbf{A}_\bullet, d_\bullet)$ and $(\mathbf{A'}_\bullet, d'_\bullet)$;

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{n-1} & \xrightarrow{d_{n-1}} & A^n & \xrightarrow{d_n} & A^{n+1} & \longrightarrow & \cdots \\ & & & \swarrow{\scriptstyle s_n} & \downarrow{\scriptstyle f_n - g_n} & \swarrow{\scriptstyle s_{n+1}} & \downarrow & & \\ \cdots & \longrightarrow & A'^{n-1} & \xrightarrow[d'_{n-1}]{} & A'^n & \xrightarrow[d'_n]{} & A'^{n+1} & \longrightarrow & \cdots , \end{array}$$

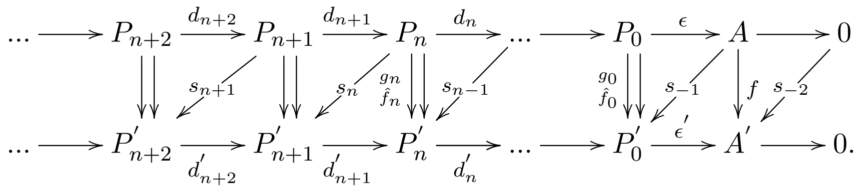
there are maps $s_n : A_n \to A'^{n-1}$ such that

$$f_n - g_n = d'_{n-1}s_n + s_{n+1}d_n.$$

When a chain map f is homotopic to the zero chain map 0 where $0_n : A_n \to 0$, f is called **nullhomotopic**.

Proposition 4.12. *Let $f : A \to A'$ be a homomorphism of modules and let $\mathbf{P}_A$ and $\mathbf{P}'_A$ be deleted projective resolutions of A and A' respectively, then any two chain maps $\hat{f}, g : \mathbf{P}_A \to \mathbf{P}_{A'}$ that are lifts of f are homotopic.*

Proof. The proof will not be written out in its entirety, it is by induction and similar to the one before. Consider the diagram



(39)

We want to show that there exists maps s_n such that $g_n - f_n = d'_{n+1}s_n + s_{n-1}d_n$. For the base step, define $g_{-1} = f = \hat{f}_{-1}$ then by defining $s_{-1} = 0 = s_{-2}$ we get

$$g_{-1} - \hat{f}_{-1} = 0 = d'_0 s_{-1} + s_{-2}d_{-1}.$$

This holds for all d'_0 and d_{-1}, particularly $d'_0 = \epsilon'$ and $d_{-1} = 0$. Continuing with the inductive step, assuming $g_n - \hat{f}_n = d'_{n+1}s_n + s_{n-1}d_n$ we want to show $g_{n+1} - \hat{f}_{n+1} = d'_{n+2}s_{n+1} + s_n d_{n+1}$. To do this we only need to show

$$\text{Im } (g_{n+1} - \hat{f}_{n+1} - s_n d_{n+1}) \subseteq \text{Im } d'_{n+2}, \tag{40}$$

since if (40) is true, P_{n+1} being projective ensures a map s_{n+1} such that $d'_{n+2}s_{n+1} = g_{n+1} - \hat{f}_{n+1} - s_n d_{n+1}$. As in the proposition above, the proof is finalized by showing $d'_{n+1}(g_{n+1} - \hat{f}_{n+1} - s_n d_{n+1}) = 0$ since the exactness of the bottom row of diagram (39) gives im $d'_{n+2} =$ ker d'_{n+1}. For full proof see [4, pg. 273-274, Theorem C-3.46 (ii)]. □

We will now show the Simultaneous Resolution Lemma which says that given a short exact sequence of modules there exists a short exact sequence of resolutions. By constructing a projective resolution of M from the direct product of the free resolutions of modules L and N we get a commutative diagram where the columns are complexes and the rows exact.

Lemma 4.13. *(Simultaneous Resolution or the Horseshoe Lemma) Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of R-modules. Let L and N have projective resolutions* *as in the diagram below*

$$\begin{array}{ccccccccc}
 & & \downarrow & & & & \downarrow & & \\
 & & P_1' & & & & P_1'' & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & P_0' & & & & P_0'' & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & L & \xrightarrow[\psi]{} & M & \xrightarrow[\varphi]{} & N & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0. & &
\end{array} \tag{41}$$

Then there exist a projective resolution of M so that the three columns form an exact sequence of complexes:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1' & \longrightarrow & P_1 & \longrightarrow & P_1'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0' & \longrightarrow & P_0 & \longrightarrow & P_0'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \xrightarrow[\psi]{} & M & \xrightarrow[\varphi]{} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{42}$$

Proof. Let $P_n = P_n' \oplus P_n''$, then each row,

$$0 \longrightarrow P_n' \xrightarrow{\iota'} P_n' \oplus P_n'' \xrightarrow{\rho''} P_n'' \to 0,$$

is an exact sequence where ι' and ρ'' are the usual injections and projections. Since $P_n' \oplus P_n''$ and $P_n' \oplus P_n''$ are projective so is $P_n' \oplus P_n''$ (Corollary 2.27 (*iii*)). Let $f' = \psi\epsilon'$ and let f'' be the lift of ϵ'', $\epsilon'' = \varphi f''$, since P_0'' is projective such lift exists.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0' & \xrightarrow{\iota'} & \mathbf{P}_0' \oplus \mathbf{P}_0'' & \xrightarrow{\rho''} & P_0'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon'} & \searrow^{f'} & \downarrow{\scriptstyle \theta} & \swarrow^{f''} & \downarrow{\scriptstyle \epsilon''} & & \\
0 & \longrightarrow & L & \xrightarrow[\psi]{} & M & \xrightarrow[\varphi]{} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
\tag{43}
$$

By Proposition 2.19 there exists a unique homomorphism $\theta : P_0' \oplus P_0'' \to M$ defined by

$$\rho' + \rho'' \mapsto f'\rho' + f''\rho'', \tag{44}$$

where ρ' is the projection $P_0' \oplus P_0'' \to P_0'$. The square with base $M \to N$ then commutes since

$$\epsilon''\rho'' = \varphi f''\rho'' = \varphi(0 + f''\rho'') = \varphi\theta,$$

where $0 : P_0' \oplus P_0'' \to 0_M$. Similarly the square with base $L \to M$ commutes since

$$\psi\epsilon' = f' 1_{P_0'} = f'\rho'\iota' = (f'\rho' + 0)\iota' = \theta\iota',$$

and we have established that the first two rows are commutative. The next step is to use the Snake lemma to get the diagram,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \epsilon' & \longrightarrow & \ker \theta & \longrightarrow & \ker \epsilon'' & - - - - & \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & P_0' & \longrightarrow & \mathbf{P}_0' \oplus \mathbf{P}_0'' & \longrightarrow & P_0'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon'} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \epsilon''} & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & - - \to & \operatorname{coker} \epsilon' & \longrightarrow & \operatorname{coker} \theta & \longrightarrow & \operatorname{coker} \epsilon'' & \longrightarrow & 0,
\end{array}
\tag{45}
$$

where the top and the bottom rows are exact (see proof of lemma 3.10). Since ϵ' and ϵ'' are surjective, coker $\epsilon' =$ coker $\epsilon'' = 0$ and since the bottom row is exact coker $\theta = 0$ and θ

is surjective. We are left with a commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\ \epsilon' & \longrightarrow & \ker\ \theta & \longrightarrow & \ker\ \epsilon'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0' & \longrightarrow & \mathbf{P}_0' \oplus \mathbf{P}_0'' & \longrightarrow & P_0'' & \longrightarrow & 0 \\
 & & {\scriptstyle \epsilon'}\downarrow & & {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \epsilon''} & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{46}$$

Since the resolutions of L and N gives us surjective maps $P_1' \to \ker\ \epsilon'$ and $P'' \to \ker\ \epsilon_1''$ we can construct the following diagram in the same way as the one above,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\ \epsilon_1' & \longrightarrow & \ker\ \theta_1 & \longrightarrow & \ker\ \epsilon_1'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1' & \longrightarrow & \mathbf{P}_1' \oplus \mathbf{P}_1'' & \longrightarrow & P_1'' & \longrightarrow & 0 \\
 & & {\scriptstyle \epsilon_1'}\downarrow & & {\scriptstyle \theta_1}\downarrow & & \downarrow{\scriptstyle \epsilon_1''} & & \\
0 & \longrightarrow & \ker\ \epsilon' & \longrightarrow & \ker\ \theta & \longrightarrow & \ker\ \epsilon'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{47}$$

The proof is by induction on the diagrams and then splicing the $n+1$ diagram to the nth diagram by defining $\theta_{n+1} : P_{n+1} \to P_n$ as the composition $P_{n+1} \to \ker\ \theta_n \to P_n$, this also implies that the constructed resolution of M is exact since the image of θ_{n+1} is the kernel of θ_n. [2, pg. 349-350, Proposition 6.24] □

5 Cohomology and Homology

In this section will we define homology and cohomology of chain and cochain complexes respectively. We will show that homotopic chain maps induce the same map over homology/cohomology. Lastly we will prove the Long Exact Sequence of Cohomology groups using the Snake Lemma.

5.1 Cohomology

Let $(\mathbf{C}_\bullet, d_\bullet)$ be a chain or a cochain complex. Since the image of one map d is a subgroup of the kernel of the following map we can make quotients out of images and kernels. These quotients are called the homology or cohomology of the complex depending on the direction of the complex. Since the difference lies in the direction of the complex, homology and cohomology are basically the same quotient and with the appropriate changes to notation any statements regarding cohomology can be applied to homology.

Definition 5.1. (i) Let $\mathbf{C}_\bullet$ be a complex:

$$\cdots \longrightarrow C_{n+1} \overset{d_{n+1}}{\longrightarrow} C_n \overset{d_n}{\longrightarrow} C_{n-1} \longrightarrow \ldots, \tag{48}$$

then its n^{th} **homology** is the quotient ker d_n / image d_{n+1} and is denoted $H_n(\mathbf{C}_\bullet)$.

(ii) Let $\mathbf{C}_\bullet$ be a cochain complex:

$$0 \longrightarrow C^0 \overset{d_0}{\longrightarrow} C^1 \longrightarrow \cdots \longrightarrow C^{n-1} \overset{d_{n-1}}{\longrightarrow} C^n \overset{d_n}{\longrightarrow} C^{n+1} \longrightarrow \ldots, \tag{49}$$

then its n^{th} **cohomology** is the quotient ker d_n / image d_{n-1} and is denoted $H^n(\mathbf{C}_\bullet)$.

We will focus on cohomology groups from now on.

Example 5.2. A cochain complex is an exact sequence if and only if all of its cohomology groups are $\{0\}$ since Ker d_n/Im $d_{n-1} = \{0\}$ directly implies Im $d_{n-1} =$ Ker d_n for all n.

Let $f : \mathbf{C}_\bullet \to \mathbf{C}'_\bullet$ be a chain map of cochain complexes, the commutativity of the diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & C_{n-1} & \overset{d_{n-1}}{\longrightarrow} & C_n & \overset{d_n}{\longrightarrow} & C_{n+1} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\ \ldots & \longrightarrow & C'_{n-1} & \underset{d'_{n-1}}{\longrightarrow} & C'_n & \underset{d'_n}{\longrightarrow} & C'_{n+1} & \longrightarrow & \ldots . \end{array} \tag{50}$$

ensures that each f_n map the kernel of d_n to the kernel of d'_n and the image of d_{n+1} to the image of d'_{n+1} which implies there exists well-defined homomorphism ker d_n/im $d_{n-1} \to$ ker d'_n/im d'_{n-1}.

Lemma 5.3. *Let $f : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ be a chain map, then for $c \in ker\ d_n$, $f_n(c) \in ker\ d'_n$ and for $c \in im\ d_{n-1}$, $f_n(c) \in im\ d'_{n-1}$.*

Proof. From the commutativity in diagram (50) we have that $f_{n+1}d_n = d'_n f_n$. Let $p \in$ ker d_n then

$$f_{n+1}d_n(p) = d'_n f_n(p) = 0$$

and so $f_n(p) \in$ ker d'_n. Let $a \in$ im d_{n-1} then we get that

$$f_n(a) = f_n d_{n-1}(c) = d'_{n-1} f_{n-1}(c)$$

for some $c \in C_{n-1}$ and so $f_n(a) \in$ im d'_{n-1}. □

Proposition 5.4. *If $f : (\mathbf{C}_\bullet, d_\bullet) \to (\mathbf{C'}_\bullet, d'_\bullet)$ is a chain map of cochain complexes then there are group homomorphisms $H^n(f) : H^n(\mathbf{C}_\bullet) \to H^n(\mathbf{C'}_\bullet)$ defined by*

$$H^n(f) : z + Im\ d_{n-1} \mapsto f_n(z) + Im\ d'_{n-1}$$

where $f_n : C^n \to C'^n$, $z \in ker\ d_n$ and $n \in \mathbb{Z}_{n \geq 0}$

Proof. Let $f : (\mathbf{C}_\bullet, d_\bullet) \to (\mathbf{C'}_\bullet, d'_\bullet)$ be a chain map then there is a commutative diagram,

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d_{n-1}} & C^n & \xrightarrow{d_n} & C^{n+1} & \xrightarrow{d_{n+1}} & \cdots \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\ \cdots & \xrightarrow[d'_{n-1}]{} & C'^n & \xrightarrow[d'_n]{} & C'^{n+1} & \xrightarrow[d'_{n+1}]{} & \cdots . \end{array}$$

By Lemma 5.3 we have that if $z \in \ker\, d_n$ then $f_n(z) \in \ker\, d'_n$ and so we can define a map $z + \text{Im}\ d_{n-1} \mapsto f_n(z) + \text{Im}\ d'_{n-1}$ and since $z \in\ \text{im}\ d_{n-1}$ implies $f_n(z) \in\ \text{im}\ d'_{n-1}$ by the same Lemma, this map is well-defined: Let $z+\text{Im}\ d_{n-1} = y+\text{Im}\ d_{n-1}$, then $z-y \in \text{Im}\ d_{n-1}$ and so $f_n(z-y) \in \text{Im}\ d'_{n-1}$ and we get $f_n(z) + \text{Im}\ d'_{n-1} = f_n(y) + \text{Im}\ d'_{n-1}$. [4, pg. 263, first part of Proof for Proposition C-3-37] □

Since the cohomology groups of a complex are quotients where the zero coset is im d_{n-1}, any two chain maps f and g where

$$(f_n - g_n)(z) \in \text{im}\ d'_{n-1},$$

for all $z \in \ker\, d_n$, induce the same map on cohomology groups. Homotopic chain maps do this and when compositions of chain maps are homotopic to the identity chain map, cohomology groups are isomorphic. Before we show this note that H^n maps the identity map of chain complexes to the identity map of cosets; let $1_{C_\bullet}$ be the identity map between a chain complex and itself, that is the sequence of maps $1_{C_n} : C^n \to C^n$, then $H^n(1_{C_\bullet})$ is the identity map on $H^n(\mathbf{C}_\bullet)$, mapping

$$z + \text{Im}\ d_{n-1} \mapsto 1_{C_n}(z) + \text{Im}\ d_{n-1} = z + \text{Im}\ d_{n-1}$$

for each $H_n(C_\bullet)$, $z \in \ker\, d_n$. Note also that H^n preserves composition; let f and g be two chain maps where

$$(C_\bullet, d_\bullet) \xrightarrow{f} (C'_\bullet, d'_\bullet) \xrightarrow{g} (C''_\bullet, d''_\bullet)$$

then

$$\begin{aligned} H_n(gf)(z + \text{Im}\ d_{n-1}) &= (gf)_n(z) + \text{Im}\ d''_{n-1} \\ &= g_n(f_n(z) + \text{Im}\ d'_{n-1}) \\ &= H_n(g)(f_n(z) + \text{Im}\ d'_{n-1}) \\ &= H_n(g)H_n(f)(z + \text{Im}\ d_{n-1}), \end{aligned}$$

and so $H_n(gf) = H_n(g)H_n(f)$. Let $f, g : \text{A}_\bullet \to \text{A'}_\bullet$ be homotopic chain maps of cochain complexes, then according to Definition 4.11 (ii) we have

$$f_n - g_n = d'_{n-1}s_n + s_{n+1}d_n$$

where s_n are the maps in the below diagram;

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{n-1} & \xrightarrow{d_{n-1}} & A^n & \xrightarrow{d_n} & A^{n+1} & \longrightarrow & \cdots \\ & & & \swarrow{\scriptstyle s_n} & \downarrow{\scriptstyle f_n - g_n} & \swarrow{\scriptstyle s_{n+1}} & \downarrow & & \\ \cdots & \longrightarrow & A'^{n-1} & \xrightarrow[d'_{n-1}]{} & A'^n & \xrightarrow[d'_n]{} & A'^{n+1} & \longrightarrow & \cdots . \end{array}$$

An immediate consequence of the definition is that homotopic chain maps induce the same homomorphisms between cohomoloy groups since for all $z \in \ker d_n$ we get

$$(f_n - g_n)(z) = d'_{n-1}s_n(z) + s_{n+1}d_n(z) = d'_{n-1}s_n(z) \in \text{im } d'_{n-1}$$

which is equivalent to

$$f_n(z) + \text{im } d_{n-1} = g_n(z) + \text{im } d_{n-1},$$

and so if f and g are homotopic then $H^n(f) = H^n(g)$. It follows that given two chain maps

$$h : \mathbf{C}_\bullet \longrightarrow \mathbf{C'}_\bullet, \ \ g : \mathbf{C'_\bullet} \longrightarrow \mathbf{C}_\bullet,$$

if gh is homotopic to $1_{\mathbf{C}_\bullet}$ and hg is homotopic to $1_{\mathbf{C'}_\bullet}$ then $H^n(\mathbf{C})_\bullet \cong H^n(\mathbf{C'})_\bullet$ since

$$H^n(gh) = H^n(1_{\mathbf{C}_\bullet}) \text{ and } H^n(hg) = H^n(1_{\mathbf{C'}_\bullet})$$

is the same as

$$H^n(g)H^n(h) = 1_{H^n(\mathbf{C}_\bullet)} \text{ and } H^n(h)H^n(g) = 1_{H^n(\mathbf{C'}_\bullet)}$$

and this makes $H^n(h)$ an isomorphism with inverse $H^n(g)$. We summarize the above in the proposition below. [4, pg. 265, chapter C-3. Proposition C-3.39], [2, pg. 346-347, Proposition 6.20]

Proposition 5.5.

(i) *Let $1_{C_\bullet}$ be the identity chain map on a cochain complex $\mathbf{C}_\bullet$, then $H^n(1_{C_\bullet}) = 1_{H^n(\mathbf{C}_\bullet)}$, for all $n \in \mathbb{Z}_{n\geq 0}$, where $1_{H^n(\mathbf{C}_\bullet)}$ is the identity map on $H^n(\mathbf{C}_\bullet)$.*

(ii) *Let $f : \mathbf{C}_\bullet \longrightarrow \mathbf{C'}_\bullet$, and $g : \mathbf{C'_\bullet} \longrightarrow \mathbf{C}_\bullet$, be two chain maps then*

$$H^n(gf) = H^n(g)H^n(f).$$

(iii) *If $g, f : (\mathbf{A}_\bullet, d_\bullet) \to (\mathbf{A'}_\bullet, d'_\bullet)$ are two homotopic chain maps between cochain complexes then*

$$H^n(f) = H^n(g) : H^n(\mathcal{A}_\bullet) \to H^n(\mathcal{A'}_\bullet).$$

(iv) *Let $C_\bullet$ and $C'_\bullet$ be two cochain complexes and let $h : \mathbf{C}_\bullet \to \mathbf{C'}_\bullet$ and $g : \mathbf{C'}_\bullet \to \mathbf{C}_\bullet$ be two chain maps. If gh is homotopic to $1_{C_\bullet}$ and hg is homotopic to $1_{C'_\bullet}$ then $H^n(\mathbf{C}_\bullet)$ and $H^n(\mathbf{C'}_\bullet)$ are isomorphic.*

Proof. We have showed the statements in the text above. □

We will now use the Snake Lemma to show that there exists a Long Exact Sequence of cohomology groups given a short exact sequence of complexes.

Theorem 5.6. *(The Long Exact Sequence in Cohomology) Let $0 \to \mathbf{C'}_\bullet \to \mathbf{C}_\bullet \to \mathbf{C''}_\bullet \to 0$ be a short exact sequence of cochain complexes. Then there is a long exact sequence of cohomology groups*

$$\cdots \longrightarrow H^{n-1}(C') \longrightarrow H^{n-1}(C) \longrightarrow H^{n-1}(C'') \longrightarrow H^n(C') \longrightarrow$$

$$\longrightarrow H^n(C) \longrightarrow H^n(C'') \longrightarrow H^{n+1}(C') \longrightarrow H^{n+1}(C) \cdots \dashrightarrow$$

Proof. The first, main part of the proof is showing that given a short exact sequence of cochain complexes,

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C'_{n-1} & \xrightarrow{f_{n-1}} & C_{n-1} & \xrightarrow{g_{n-1}} & C''_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d'_{n-1}} & & \downarrow{\scriptstyle d_{n-1}} & & \downarrow{\scriptstyle d''_{n-1}} & & \\ 0 & \longrightarrow & C'_n & \xrightarrow{f_n} & C_n & \xrightarrow{g_n} & C''_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d'_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d''_n} & & \\ 0 & \longrightarrow & C'_{n+1} & \xrightarrow{f_{n+1}} & C_{n+1} & \xrightarrow{g_{n+1}} & C''_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_{n+1}} & & \downarrow{\scriptstyle d_{n+1}} & & \downarrow{\scriptstyle d_{n+1}} & & \end{array} , \tag{51}$$

there exists a commutative diagram with exact rows,

$$\begin{array}{ccccccccc} & & C'_n/\text{im } d'_{n-1} & \xrightarrow{\bar{f}_n} & C_n/\text{im } d_{n-1} & \xrightarrow{\bar{g}_n} & C''_n/\text{im } d''_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{d'}_n} & & \downarrow{\scriptstyle \bar{d}_n} & & \downarrow{\scriptstyle \bar{d''}_n} & & \\ 0 & \longrightarrow & \ker d'_{n+1} & \xrightarrow{\bar{f}_{n+1}} & \ker d_{n+1} & \xrightarrow{\bar{g}_{n+1}} & \ker d''_{n+1}, & & \end{array} \tag{52}$$

where $\bar{d}_n$ is defined as

$$\bar{d}_n : c + \text{im } d_{n-1} \mapsto d_n(c), \tag{53}$$

$c \in C_n$, and define $\bar{f}_n$ as

$$\bar{f}_n : x' + \text{im } d'_{n-1} \mapsto f_n(x') + \text{im } d_{n-1}, \tag{54}$$

$x' \in C'$. The maps $\bar{d}_n$ and $\bar{d''}_n$ are defined as $\bar{d}_n$ and $\bar{g}_n$ is defined as $\bar{f}_n$.

The second part of the proof is applying the Snake Lemma 3.10 to diagram (52) and observing that the exact sequence which appears is a sequence of cohomology groups.

(i) Since im $d_n \subset$ ker d_{n+1}, the map (53) exists and since im $d_{n-1} \subset$ ker d_n, it is welldefined: Let $x = y$, $x, y \in C_n$ then $x - y \in \text{im } d_{n-1}$ and so

$$\bar{d}_n : x - y + \text{im } d_{n-1} \mapsto d_n(x - y) = 0$$

and $d_n(x) = d_n(y)$. According to Lemma 5.3, $f_n(y') \in \text{im } d_{n-1}$ for any $y' \in \text{im } d'_{n-1}$ and this makes the map (54) welldefined, the proof is the same as the one above. The diagram (52)is commutative since the short exact complex is commutative, for the square

$$\begin{array}{ccc} C'_n/\text{im } d'_{n-1} & \xrightarrow{\bar{f}_n} & C_n/\text{im } d_{n-1} \\ \downarrow{\scriptstyle \bar{d'}_n} & & \downarrow{\scriptstyle \bar{d}_n} \\ \ker d'_{n+1} & \xrightarrow{\bar{f}_{n+1}} & \ker d_{n+1} \end{array} , \tag{55}$$

we have

$$\begin{aligned}&\bar{f}_{n+1}\bar{d'}_n(x') + \text{im } d_{n-1} = f_{n+1}(d_n(x')) + \text{im } d_{n-1} = \\ &d_n f_n(x') + \text{im } d_{n-1} = \bar{d}_n(f_n(x') + \text{im } d_{n-1}) = \\ &\bar{d}_n \bar{f}_n(x' + \text{im } d_{n-1}).\end{aligned}$$

Define $\bar{g}_n$ in the same way as $\bar{f}_n$, the above then applies with the appropriate changes. We will now show exactness of the top row of diagram (52). Since $g_n f_n = 0$, it follows immediately that $\bar{g}_n \bar{f}_n = 0$, that is im $\bar{f}_n \subset$ ker $\bar{g}_n$. To show ker $\bar{g}_n \subset$ im $\bar{f}_n$: If

$$x + \text{im } d_{n-1} \in \text{ker } \bar{g}_n,$$

then $g_n(x) \in$ im d''_{n-1}, since g_{n-1} is surjective there exists an element $y \in C_{n-1}$ such that $d''_{n-1} g_{n-1}(y) = g_n(x)$ and because the complex (51) is commutative we have that $x \in$ im d_{n-1} and since im $d_{n-1} \subset$ ker d_n, $d_n(x) = 0_{C_{n+1}}$. Since f_{n+1} is injective, $f_{n+1}(z) = 0_{C_{n+1}}$ implies $z = 0_{C'_{n+1}}$ and there exists a $x' \in C'_n$ such that $d'_n f_{n+1}(x') = d_n(x)$. Commutativity of (51) then gives us that $x \in$ im f_n and so $x +$ im d_{n-1} is in the image of $\bar{f}_n$ and ker $\bar{g}_n \subset$ im $\bar{f}_n$.

(ii) We have shown that diagram (52) is a commutating diagram where the rows are exact, we can then apply the Snake Lemma 3.10 (and the text following the Snake Lemma) and observe that the exact sequence which appears is a sequence of cohomology groups,

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \text{ker } d'_n/\text{im } d'_{n-1} & \longrightarrow & \text{ker } d_n/\text{im } d_{n-1} & \longrightarrow & \text{ker } d''_n/\text{im } d''_{n-1} & \longrightarrow & & \\
& \downarrow & & \downarrow & & \downarrow & & & \\
& C'_n/\text{im } d'_{n-1} & \xrightarrow{f_n} & C_n/\text{im } d_{n-1} & \xrightarrow{g_n} & C''_n/\text{im } d''_{n-1} & \longrightarrow & 0 & \\
& \downarrow \bar{d'}_n & & \downarrow \bar{d}_n & & \downarrow \bar{d''}_n & & & \\
0 \longrightarrow & \text{ker } d'_{n+1} & \xrightarrow{f_{n+1}} & \text{ker } d_{n+1} & \xrightarrow{g_{n+1}} & \text{ker } d''_{n+1} & & & \\
& \downarrow & & \downarrow & & \downarrow & & & \\
\longrightarrow & \text{ker } d'_{n+1}/\text{im } d'_n & \longrightarrow & \text{ker } d_{n+1}/\text{im } d_n & \longrightarrow & \text{ker } d''_{n+1}/\text{im } d''_n & \longrightarrow & \cdots &
\end{array}$$

The kernel of $\bar{d}_n$ is the cohomology group $H^n(C) =$ ker $d_n/$im d_{n-1} and the cokernel of $\bar{d}_n$ is the cohomology group $H^{n+1}(C) =$ ker $d_{n+1}/$im d_n. The Long exact Sequence then arises from repeating (i) and (ii), starting with the commutative diagram emerging from shifting diagram (51) one step such that the middle row will be indexed $n+1$.

The next commutative diagram with cohomology groups would then be

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & \ker d'_{n+1}/\text{im } d'_n & \longrightarrow & \ker d_{n+1}/\text{im } d_n & \longrightarrow & \ker d''_{n+1}/\text{im } d''_n & \longrightarrow & & \\
& \downarrow & & \downarrow & & \downarrow & & & \\
& C'_{n+1}/\text{im } d'_n & \xrightarrow{f_{n+1}} & C_{n+1}/\text{im } d_n & \xrightarrow{g_{n+1}} & C''_{n+1}/\text{im } d''_n & \longrightarrow & 0 & \\
& \downarrow{\scriptstyle \bar{d}'_{n+1}} & & \downarrow{\scriptstyle \bar{d}_{n+1}} & & \downarrow{\scriptstyle \bar{d}''_{n+1}} & & & \\
0 \longrightarrow & \ker d'_{n+2} & \xrightarrow{f_{n+2}} & \ker d_{n+2} & \xrightarrow{g_{n+2}} & \ker d''_{n+2} & & & \\
& \downarrow & & \downarrow & & \downarrow & & & \\
\longrightarrow & \ker d'_{n+2}/\text{im } d'_{n+1} & \longrightarrow & \ker d_{n+2}/\text{im } d_{n+1} & \longrightarrow & \ker d''_{n+2}/\text{im } d''_{n+1} & \longrightarrow \cdots & &
\end{array}
$$

and the proof is finalized with an induction proof on the last two diagrams.
[4, pg. 269-270, Exercise C-3.25 (ii), (iii), Ch. C-3.5]

□

6 Additive Categories and Additive Functors

Chain complexes, modules and groups are all objects belonging to different mathematical systems. Category mathematics is used to compare mathematical structures shared by different systems or categories, it consists of a general mathematical language where statements often are depicted by diagrams of arrows and objects. Two examples are the diagrams in Proposition 2.19 defining products and coproducts which in the category of R-modules both are direct sums. A transformation of objects from one category into another is called a functorial map if it also transforms the maps between the objects. The mapping of cohomology groups, $H^n(C_\bullet) \to H^n(C'_\bullet)$ derived from chain maps of complexes $f : \mathbf{C}_\bullet \to \mathbf{C'}_\bullet$ in section 5 can be seen as a functorial transformation from the category of complexes to the categroy of groups and the mapping of Hom-groups, $\varphi^* : \text{Hom}(N, D) \to \text{Hom}(M, D)$ derived from the R-module homomorphism $\varphi : M \to N$ in section 3 is also a functorial mapping. Both H^n and Hom will be defined as functors in the next section but in order to do that we need to define categories and functors.

6.1 Categories

A category is defined as a class of objects together with sets of morphisms on which composition is defined. It is possible to define a category solely as a set of morphisms since each object can be uniquely identified by an identity morphism but it is more natural to think of categories as containing both objects and morphisms [2, pg. 17].

Definition 6.1. A category $\mathcal{C}$ consists of

(i) A class of objects,

(ii) a set of morphisms $\text{Hom}_{\mathcal{C}}(A, B)$ for every ordered pair of objects A, B in $\mathcal{C}$

(iii) and composition for every ordered triple of objects A, B, C denoted by

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(A,B) \times \text{Hom}_{\mathcal{C}}(B,C) &\to \text{Hom}_{\mathcal{C}}(A,C) \\ (f,g) &\mapsto gf. \end{aligned}$$

The objects and morphism must satisfy the following three axioms for objects $A, B, C, D \in \mathcal{C}$:

(1) Each object B has an **identity morphism** $1_B \in \text{Hom}_{\mathcal{C}}(B,B)$ such that $1_B f = f$ and $g 1_B = g$ for $f \in \text{Hom}_{\mathcal{C}}(A,B)$ and $g \in \text{Hom}_{\mathcal{C}}(B,C)$.

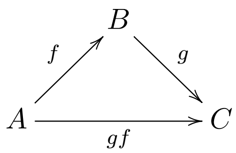
(2) Each $f \in \text{Hom}_{\mathcal{C}}(A,B)$ has a unique **domain** A and a unique **codomain** B. It follows that if $(A,B) \neq (C,D)$, $\text{Hom}_{\mathcal{C}}(A,B)$ and $\text{Hom}_{\mathcal{C}}(C,D)$ are disjoint sets.

(3) Composition is associative. For $f \in \text{Hom}_{\mathcal{C}}(A,B)$, $g \in \text{Hom}_{\mathcal{C}}(B,C)$ and $h \in \text{Hom}_{\mathcal{C}}(C,D)$,

$$h(gf) = (hg)f.$$

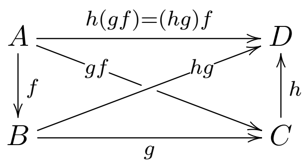
[2, pg. 8, Definition, Sec. 1.2]

A diagram in a category $\mathcal{C}$ is a directed multigraph where the vertices is a set of objects of $\mathcal{C}$ and the arrows are morphisms. Using diagrams we can effectively depict categorical statements, we may picture composition, Definition 6.1 (iii), as the commutative diagram



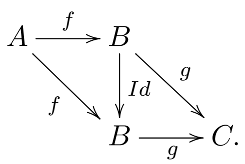
(56)

where it is clearly illustrated that the domain of g is the codomain of f and that the domain and codomain of gf is the domain of f and the codomain of g respectively. The associativity of composition, axiom three of Definition 6.1 is displayed by the commutative diagram



(57)

and the first axiom is pictured by the commutative diagram



(58)

Hom-sets are allowed to be empty in general, one exception is the set of morphisms $\text{Hom}(A,A)$ which must contain the identity morphism 1_A. That each object can be identified by its identity morphism follows from the first axiom which implies that the identity morphism is unique for each object, since if we let 1_A and $1'_A$ both be the identity morphism on $A \in \mathcal{C}$ then $1'_A 1_A = 1_A$ and $1'_A 1_A = 1'_A$ and so $1_A = 1'_A$.

Example 6.2. (i) The category **Ab** has as its objects abelian groups, as its morphisms homomorphisms, and composition is the usual composition.

(ii) The category ${}_R\mathbf{Mod}$ has as its objects left R-modules over a ring R, as its morphisms R-homomorphisms, and as its composition the usual composition of functions. If R is $\mathbb{Z}$, then ${}_{\mathbb{Z}}\mathbf{Mod} = \mathbf{Ab}$ since $\mathbb{Z}$-modules are abelian groups.

(iii) The category ${}_R\mathbf{Comp}$ has as its objects complexes $(\mathbf{C_\bullet}, \mathbf{d_\bullet})$, where C_n are left R-modules and d_n are R-module homomorphisms and as its morphisms chain maps $f = f_\bullet : (\mathbf{C_\bullet}, \mathbf{d_\bullet}) \to (\mathbf{C'_\bullet}, \mathbf{d'_\bullet})$.
[3, pg. 444, Example B-4.1 (iii), (vi)], [4, pg.259, Defintion, Ch. C-3.4]

Definition 6.3. Let $\mathcal{C}$ be a category and let $C \in \mathcal{C}$.

(i) An object $A \in \mathcal{C}$ is called **initial** if, for every object C there exists a unique morphism $A \to C$.

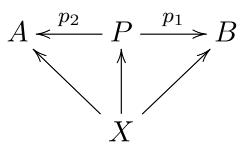
(ii) An object $\Omega \in \mathcal{C}$ is called **terminal** if, for every object C there exists a unique morphism $C \to \Omega$.

(iii) A **zero object** in a category $\mathcal{C}$ is an object which is both initial and terminal.

[2, pg. 216, Defintion, Ch. 5.1], [2, pg. 218, Definition, Ch. 5.1], [2, pg. 226, Exercise 5.2, Ch. 5.1]

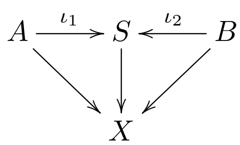
Definition 6.4. Let $\mathcal{C}$ be a category and let $A, B \in \mathcal{C}$.

(i) An object $P \in \mathcal{C}$ is a **product** of A and B if there exists maps $p_1 : P \to A$ and $p_2 : P \to B$ such that for every $X \in \mathcal{C}$ and every pair of morphisms $X \to A$ and $X \to B$ there is a unique morphism $X \to P$ such that the diagram below commutes,



(59)

(ii) an object $S \in \mathcal{C}$ is a **coproduct** of A and B if there exists maps $\iota_1 : A \to S$ and $\iota_2 : B \to S$ such that for every $X \in \mathcal{C}$ and every pair of morphisms $A \to X$ and $B \to X$ there is a unique morphism $S \to X$ such that the diagram below commutes,



(60)

Lemma 6.5. *In ${}_R\mathbf{Mod}$ we have that* *(i) the zero object is the zero module* $\{0\}$ *and*

(ii) the product and the coproduct of any two left R*-modules* A, B *are isomorphic and is their direct sum,* $A \oplus B$*.*

Proof. (i) Assume B is the zero object in ${}_R\mathbf{Mod}$ then there exists a unique morphism $B \to B$. Since $\text{Hom}(B, B)$ is a ring, see Proposition 2.5*(iv)*, it must contain both the identity homomorphism on B, 1_B, and the zero homomorphism 0 and so $1_B = 0$ and B needs to be the zero module. [2, pg. 226, Exercise 5.2, Ch. 226]

(ii) See Proposition 2.19 and the commutative diagrams (3) and (4).

□

6.1.1 Additive Categories

In additive categories Hom-sets are abelian groups under point wise addition where the distributive law holds in regard to composition.

Definition 6.6. A category $\mathcal{A}$ is **pre-additive** if

(i) $\mathbf{Hom}_{\mathcal{A}}(A, B)$ is an (additive) abelian group for every $A, B \in \text{obj}\mathcal{A}$,

(ii) the distributive laws hold: given morphisms

$$X \xrightarrow{k} A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} Y$$

where X and $Y \in \text{obj}\mathcal{A}$, then

$$h(f + g) = hf + hg \text{ and } (f + g)k = fk + gk.$$

A category $\mathcal{A}$ is **additive** if it is pre-additive and

(iii) $\mathcal{A}$ has a zero object,

(vi) $\mathcal{A}$ has finite products and finite coproducts: for all objects A, B in $\mathcal{A}$, both their product and their coproduct exist in $\text{obj}\mathcal{A}$.

[2, pg. 303, Definition, Section 5.5]

Proposition 6.7. *The category* ${}_R\mathbf{Mod}$ *is an additive category.*

Proof. Let $A, B \in_R \mathbf{Mod}$, according to Proposition 2.5 $\text{Hom}_R(A, B)$ is an abelian group and the distributive laws in (ii) hold for any $f, g \in \text{Hom}_R(A, B)$ according to Proposition 2.5 *(iii)*. The zero object is the zero module according to Lemma 6.5*(i)* and the product and coproduct is the direct sum according to Lemma 6.5*(ii)*. □

Proposition 6.8. *The category* ${}_R\mathbf{Comp}$ *of* ${}_R Mod$ *complexes and chain maps, is a pre-additive category if we define, for chain maps* f, g

$$(f + g)_n = f_n + g_n$$

for each $n \in \mathbb{Z}$*.*

[4, pg. 259, Ch. 3.4]

Proof. This follows from chain maps consisting of homomorphisms of modules. The group structure of an additive abelian group then follows as do the distributive laws. □

6.2 Functors

A functor is a structure preserving map between categories mapping both objects and morphisms, it maps identity to identity and it preserves composition of morphisms. An additive functor is a functor between pre-additive categories preserving the point-wise addition of the Hom-sets and since Hom sets in additive categories are abelian groups under pointwise addition the additive functor acts as a homomorphism over the Hom sets. As such additive functors maps zero to zero as well as identity to identity and preserves thus complexes, direct sums and homotopic maps. Functors are either covariant or contravariant, where the last reverses the direction of the morphisms between the mapped objects.

Definition 6.9. A **functor** $T : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are categories, is a function such that

(i) if $A \in \text{obj}(\mathcal{C})$ then $T(A) \in \text{obj}(\mathcal{D})$,

(ii) $T(1_A) = 1_{T(A)}$ for every $A \in \text{obj}(\mathcal{C})$.

(iii) A **covariant functor** $T : \mathcal{C} \to \mathcal{D}$

  (a) maps $f : A \to A^{'}$ to $T(f) : T(A) \to T(A^{'})$ and

  (b) $A \xrightarrow{f} A' \xrightarrow{g} A''$ to $T(A) \xrightarrow{T(f)} T(A^{'}) \xrightarrow{T(g)} T(A^{''})$

  (c) and if gf is a composition of morphisms in $\mathcal{C}$ then $T(gf) = T(g)T(f)$ is a composition of morphisms in $\mathcal{D}$.

(iv) A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$

  (a) maps $f : A \to A^{'}$ to $F(f) : F(A') \to F(A)$ and

  (b) $A \xrightarrow{f} A' \xrightarrow{g} A''$ to $F(A'') \xrightarrow{F(g)} F(A') \xrightarrow{F(f)} F(A)$

  (c) and if gf is a composition of morphisms in $\mathcal{C}$ then $F(gf) = F(f)T(g)$ is a composition of morphisms in $\mathcal{D}$.

[2, pg. 17, 19, Definition, Ch. 1.2]

Since a functor preservers composition of morphisms it also preserves commutative diagrams. For example, let $f : X \to P$, $g : A \to X$ and $h : X \to B$ be the morphisms from diagram (4) in Definition 6.4, defining a categorical product in $\mathcal{C}$. Let $T : \mathcal{C} \to \mathcal{D}$ be a functor into some category $\mathcal{D}$, then we have

$$T(h) = T(p_1 f) = T(p_1)T(f)$$

and

$$T(g) = T(p_2 f) = T(p_2)T(f)$$

and so T preserves the commutativity of diagram (4), but not necessarily the product. An additive functor however does.

Definition 6.10. Let $\mathcal{C}$, $\mathcal{D}$ be additive categories and let $C, C' \in \text{obj}\,(\mathcal{C})$. A functor $T : \mathcal{C} \to \mathcal{D}$ of either variance is called an **additive functor** if, for every pair of morphisms $f, g : C \to C'$, we have

$$T(f+g) = T(f) + T(g). \tag{61}$$

The definition of an additive functor T turns T into an homomorphism over the Hom-sets and as such T will map zero maps to zero maps and subsequently zero objects to zero objects.

Proposition 6.11. *Let $T : {}_R\textbf{\textit{Mod}} \to {}_R\textbf{\textit{Mod}}$ be an additive functor of either variance, then $T(0) = 0$ where 0 is the zero map or the zero object in ${}_R\textbf{\textit{Mod}}$.*

Proof. Since the Hom-sets of ${}_R\textbf{Mod}$ are abelian groups, the equation (61) turns T into a homomorphism over the Hom-sets and as such T preserves the zero homomorphism. Let A be the zero object in ${}_R\textbf{Mod}$ then there exists a unique homomorphism $f : A \to A$ which, according to the proof of Lemma 6.5*(i)*, is the zero homomorphism and so $T(f)$ is the zero homomorphism and $T(A)$ is the zero object. [3, pg. 474, Excercise B-4.18, Ch. B-4] □

Using Proposition 6.11 we can show that additive functors preserves direct sums, complexes and homotopic maps.

Proposition 6.12. *Let $T : {}_R\textbf{\textit{Mod}} \to_R \textbf{\textit{Mod}}$ be an additive functor then*

(i) $T(A_1 \oplus A_2) \cong T(A_1) \oplus T(A_2)$ where $A_1, A_2 \in {}_R\textbf{\textit{Mod}}$,

(ii) If $(\boldsymbol{A}_\bullet, d_\bullet) \in {}_R\textbf{\textit{Mod}}$ is a chain complex the functored sequence $(T\mathbf{A}_\bullet, Td_\bullet)$ is a chain complex.

(iii) If $f, g : (\boldsymbol{A}_\bullet, d_\bullet) \to (\boldsymbol{B}_\bullet, d_\bullet)$ are homotopic chain maps in ${}_R\textbf{\textit{Mod}}$, the functored maps $T(f), T(g)$ are also homotopic.

Proof. (i) A direct sum $M = A_1 \oplus A_2$ is characterized by the equations in Proposition 2.16 *(iv)*:

$$\rho_i \iota_i = 1_{M_i},\ \rho_i \iota_j = 0 \text{ where } i \neq j,\ \iota_1 \rho_1 + \iota_2 \rho_2 = 1_M,$$

where $\iota_n : A_n \to M$ and $\rho_n : M \to A_n$. Since T is a functor it preserves composition and the identity map and since T is an additive functor it preserves addition and the zero map and so T will preserve the above equations. [3, pg. 466, Prop. B-4.18, Ch B-4]

(ii) Let T be a covariant functor then for a complex

$$(\mathbf{A}_\bullet, d_\bullet) = \ \cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \cdots, \tag{62}$$

we get the functored sequence

$$(T\mathbf{A}_\bullet, Td_\bullet) = \ \cdots \to T(A_{n+1}) \overset{T(d_{n+1})}{\longrightarrow} T(A_n) \overset{T(d_n)}{\longrightarrow} T(A_{n-1}) \to \cdots .$$

Since T is an additve functor we then have

$$T(d_n)T(d_{n+1}) = T(d_n d_{n+1}) = T(0) = 0.$$

Let T be a contravariant functor then the functored sequence will be

$$(T\mathbf{A}_\bullet, Td_\bullet) = \ \cdots \to T(A_{n-1}) \overset{T(d_n)}{\longrightarrow} T(A_n) \overset{T(d_{n+1})}{\longrightarrow} T(A_{n+1}) \to \cdots . \tag{63}$$

However, the sequence (63) have increasing indices which is in conflict with the definition of a complex. We can fix this by setting $X_{-n} = T(A_n)$ and $\delta_{-n+1} = T(d_n)$ to get the sequence

$$(T\mathbf{A}_\bullet, Td_\bullet) = \ \cdots \to X_{-n+1} \overset{\delta_{-n+1}}{\longrightarrow} X_{-n} \overset{\delta_{-n}}{\longrightarrow} X_{-n-1} \to \cdots ,$$

and just as in the definition of a cochain complex, Definition 4.4 (*ii*), we change the indices to positive and put them as a superscript;

$$(T\mathbf{A}_\bullet, Td_\bullet) = \ \cdots \to X^{n-1} \overset{\delta^{n-1}}{\longrightarrow} X^n \overset{\delta^n}{\longrightarrow} X^{n+1} \to \cdots . \tag{64}$$

We then get

$$\delta^n \delta^{n-1} = \delta_{-n}\delta_{-n+1} = T(d_{n+1})T(d_n) = T(d_n d_{n+1}) = T(0) = 0.$$

and (64) is a cochain complex. [4, pg.258, Ex. C-3.34 (ix, x) Ch. C-3.4]

(iii) Let T be an additive covariant functor, let $(\mathbf{A}_\bullet, d_\bullet)$ be the complex (62) and $(\mathbf{B}_\bullet, d'_\bullet)$ be a similar positive complex and

$$(\mathbf{A}_\bullet, d_\bullet) \to (\mathbf{B}_\bullet, d'_\bullet) \tag{65}$$

be a chain map. According to the definition of a covariant functor

$$(T\mathbf{A}_\bullet, Td_\bullet) \to (T\mathbf{B}_\bullet, Td_\bullet)$$

is a chain map where $(T\mathbf{A}_\bullet, Td_\bullet)$ are (positive) chain complexes according to (ii). According to definition 4.11 homotopic chainmaps $f, g : (\mathbf{A}_\bullet, d_\bullet) \to (\mathbf{B}_\bullet, d'_\bullet)$ are defined by the equation

$$g_n - f_n = d'_{n+1}s_n + s_{n-1}d_n, \tag{66}$$

where $s_n : A_n \to B_{n+1}$. Since T is an additive covariant functor it will map equation (66) to

$$T(g_n) - T(f_n) = T(d'_{n+1})T(s_n) + T(s_{n-1})T(d_n),$$

where $T(s_n) : T(A_n) \to T(B_{n+1})$. And this makes $T(f), T(g)$ homotopic chain maps. Let F be an additive contravariant functor. According to the definition of a contravariant functor F maps (65) to

$$(F\mathbf{B}_\bullet, Fd'_\bullet) \to (F\mathbf{A}_\bullet, Fd_\bullet)$$

where, according to (ii), $(F\mathbf{B}_\bullet, Fd'_\bullet)$ and $(F\mathbf{A}_\bullet, Fd_\bullet)$ are cochain complexes. Since F is additive equation (66) is mapped to

$$F(f_n) - F(g_n) = F(s_n)F(d'_{n+1}) + F(d_n)F(s_{n-1})$$

where $F(s_n) : F(B_{n+1}) \to F(A_n)$ and so, according to Definition 4.11, $F(f)$ and $F(g)$ are homotopic.

□

Additive Functors preserve complexes but not necessarily exactness, when they do they are called exact functors.

Definition 6.13. (i) A covariant functor is called **left exact** if exactness of

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C$$

implies exactness of

$$0 \to T(A) \xrightarrow{T(i)} T(B) \xrightarrow{T(p)} T(C).$$

(ii) A contravariant functor is called **left exact** if exactness of

$$A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

implies exactness of

$$0 \to T(C) \xrightarrow{T(p)} T(B) \xrightarrow{T(i)} T(A).$$

(iii) A functor of either variance is called exact if all the above sequences are extended to short exact sequences.

6.3 Products of Categories and Bifunctors

There exists products of categories. Let $\mathcal{B}$ and $\mathcal{C}$ be two categories, let the sequence

$$B \xrightarrow{f} B' \xrightarrow{f'} B''$$

of objects and morphisms belong to $\mathcal{B}$ and let the sequence

$$C \xrightarrow{g} C' \xrightarrow{g'} C''$$

belong to $\mathcal{C}$. The product $\mathcal{B} \times \mathcal{C}$ is a category whose objects are pairs $\langle B, C\rangle$ and whose morphisms are pairs $\langle f, g\rangle$. The composition,

$$\langle B, C\rangle \xrightarrow{\langle f', g'\rangle} \langle B', C'\rangle \xrightarrow{\langle f, g\rangle} \langle B'', C''\rangle,$$

in $\mathcal{B} \times \mathcal{C}$ is defined as

$$\langle f, g\rangle\langle f', g'\rangle = \langle ff', gg'\rangle. \tag{67}$$

As per Definition 6.4 (i), $\mathcal{B} \times \mathcal{C}$ is a product of categories if there exists a category $\mathcal{D}$ and functors P_1 and P_2 such that the diagram

$$\begin{array}{ccccc} & & \mathcal{D} & & \\ & \swarrow^{L} & \downarrow F & \searrow^{R} & \\ \mathcal{B} & \xleftarrow{P_2} & \mathcal{B} \times \mathcal{C} & \xrightarrow{P_1} & \mathcal{C} \end{array}$$

commutes and in which F is a unique functor, mapping any morphism h in $\mathcal{D}$ to $\langle Lh, Rh\rangle \in \mathcal{B} \times \mathcal{C}$.

A **bifunctor** S is a functor $\mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{D}$ of two arguments determined by the set of one-argument functors S_B and S_C fixing B and C respectively for all objects $\langle B, C\rangle$: Define functors

$$S_B : \mathcal{C} \longrightarrow \mathcal{D}$$
$$S_C : \mathcal{B} \longrightarrow \mathcal{D}$$

for all $C \in \mathcal{C}$ and all $B \in \mathcal{B}$ where

$$S_B(C) = S_C(B).$$

Let $f : B \to B' \in \mathcal{B}$ and $g : C \to C' \in \mathcal{C}$ then there exists a bifunctor $S : \mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{D}$ such that

$$\begin{aligned} S_B(C) &= S(B, C) \\ S_B(g) &= S(id_B, g), \end{aligned} \tag{68}$$

$$\begin{aligned} S_C(B) &= S(B, C) \\ S_C(f) &= S(f, id_C) \end{aligned} \tag{69}$$

if and only if

$$S_{B'}(g)S_C(f) = S_{C'}(f)S_B(g) \in \mathcal{D}. \tag{70}$$

Assume S is a bifunctor. Using the definition (67) of composition in $\mathcal{B} \times \mathcal{C}$ we have

$$\langle id_{B'}, g\rangle\langle f, id_C\rangle = \langle id_{B'}f, gid_C\rangle = \langle f, g\rangle = \langle fid_B, id_{C'}g\rangle = \langle f, id_{C'}\rangle\langle id_B, g\rangle$$

which gives us

$$S(id_{B'}, g)S(f, id_C) = S(f, id_{C'})S(id_B, g)$$

in $\mathcal{D}$ which is the same as (70). Conversely, given functors S_B and S_C for all $B \in \mathcal{B}$ and all $C \in \mathcal{C}$, for every pair $\langle f, g \rangle$ we have

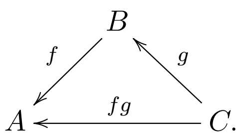
$$\begin{aligned} S_{B'}(g)S_C(f) = S(id_{B'}, g)S(f, id_C) &= S((id_{B'}, g)(f, id_C)) \\ &= S(id_{B'}f, gid_C) \\ &= S(f, g). \end{aligned}$$

Similarly we have $S_{C'}(f)S_B(g) = S(f, g)$ and so equation (70) implies the functor S.

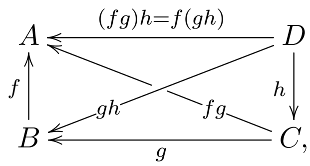
[5, pg. 36-38, Proposition 1, Sec. II Construction on Categories, Subsec. 3. Products of Categories]

6.4 Opposite Categories

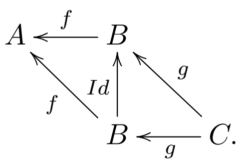
Given a categorical statement, its **dual statement** is a statement arising from reversing directions of morphisms. If a statement is depicted by a commutative digram its dual is a commutative diagram where all the arrows are reversed. For example is the dual of a categorical product P in Definition 6.4(i) the coproduct S in 6.4(ii) and the dual of the initial object A in 6.3 (i) is the terminal object Ω in 6.3 (ii). The statements defining a category all have duals. The dual of Definition 6.1(iii) regarding composition is the reversal of the commutative diagram (56), which is the commutative diagram



The duals of axioms 6.1 (3) and 6.1 (1) regarding associativity of composition and identity are the reversals of the commutative diagrams (57) and (58) which are the commutative diagrams



and



The above diagrams are as valid as the original which implies that any dual of a categorical statement based on Definition 6.1 is as valid as the original. This is called the **duality principle**. The duality principle can be applied to statements involving several categories and functors between them where statements of each category are simultaneously dualized. [5, pg. 31-32, Duality, Construction on Categories]

The duality principle allows us to associate each category $\mathcal{C}$ with an opposite category $\mathcal{C}^{op}$ where the morphisms are reversed and where each statement is a dual to a statement in $\mathcal{C}$.

Definition 6.14. Let $\mathcal{C}$ be a category, then the associated **opposite category** $\mathcal{C}^{op}$ is a category whose objects are the objects of $\mathcal{C}$ and whose morphisms are in one-to-one correspondence to the morphisms of $\mathcal{C}$: For each morphism

$$f : A \to B$$

in $\mathcal{C}$, there is an opposite, reversed, morphism

$$f^{op} : B \to A$$

in $\mathcal{C}^{op}$. Let $g : B \to C$ be a morphism in $\mathcal{C}$. The composite $f^{op}g^{op}$ is defined

$$f^{op}g^{op} = (gf)^{op},$$

and exists exactly when gf is defined in $\mathcal{C}$.

Any categorical statement about $\mathcal{C}$ generate a dual statement in $\mathcal{C}^{op}$ and a dual statement is true in $\mathcal{C}^{op}$ if and only if the original statement is true in $\mathcal{C}$. [5, pg. 33, Contravariance and opposites, Construction on Categories]

Example 6.15. The opposite of ${}_R\mathbf{Mod}$, ${}_R\mathbf{Mod}^{op}$, is an additive category. According to Definition 6.14 if $\varphi \in \mathrm{Hom}(A, B)$ there exists a homomorphism $\varphi^{op} \in \mathrm{Hom}(B, A)$. It follows immediately from Definition 2.5 that $\mathrm{Hom}(B, A)$ is an additive group and that the distributive law holds. We also want to show that the zero-object, products and coproducts exists in ${}_R\mathbf{Mod}^{op}$: Since we have that the initial object and the terminal object are duals the zero-object exists in ${}_R\mathbf{Mod}^{op}$ and is the same as in ${}_R\mathbf{Mod}$. For the same reason do products and coproducts exist in ${}_R\mathbf{Mod}^{op}$: Since coproducts and products are duals and since they exist and coincide in ${}_R\mathbf{Mod}$ they exist and coincide in ${}_R\mathbf{Mod}^{op}$.

Contravariant functors $\mathcal{C} \to \mathcal{D}$ are often defined as covariant functors $\mathcal{C}^{op} \to \mathcal{D}$. Define a covariant functor $S : \mathcal{C}^{op} \to \mathcal{D}$ we then have that

$$f^{op} : D \to C \in \mathcal{C}^{op}$$

is mapped covariantly

$$S(f^{op}) : S(D) \to S(C) \in \mathcal{D}.$$

and compositions are mapped

$$S(f^{op}g^{op}) = S(f^{op})S(g^{op})$$

given that $f^{op}g^{op}$ exists in $\mathcal{C}^{op}$. Since we have defined the functor S on $\mathcal{C}^{op}$ we may also define it on $\mathcal{C}$. Let $\bar{S} : \mathcal{C} \to \mathcal{D}$ be the functor S defined on $\mathcal{C}$ then

$$\bar{S}(f) = S(f^{op})$$

and

$$\bar{S}(f)\bar{S}(g) = S(f^{op})S(g^{op})$$

which makes $\bar{S}$ a contravariant functor since

$$f : C \to D \in \mathcal{C}$$

is mapped

$$\bar{S}(f) : \bar{S}(D) \to \bar{S}(C) \in \mathcal{D}$$

and composition is mapped

$$\bar{S}(gf) = \bar{S}(f)\bar{S}(g).$$

[5, pg. 33, Contravariance and opposites, Construction on Categories]

We can use the above and define an additive contravariant functor ${}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$ 'covariantly' as an additive functor ${}_R\mathbf{Mod}^{op} \to {}_R\mathbf{Mod}$.

Definition 6.16. Let $\bar{T} : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$ be an additive contravariant functor. Define an additive covariant functor

$$T : {}_R\mathbf{Mod}^{op} \to {}_R\mathbf{Mod},$$

where T maps elements $A \in {}_R\mathbf{Mod}^{op}$ to elements $\bar{T}(A) \in {}_R\mathbf{Mod}$ and homomorphisms

$$f^{op} : A \to B$$

in ${}_R\mathbf{Mod}^{op}$ to homomorphisms

$$\bar{T}(f) : \bar{T}(A) \to \bar{T}(B)$$

in ${}_R\mathbf{Mod}$. Composition is mapped

$$T(f^{op}g^{op}) = \bar{T}(f)\bar{T}(g).$$

The properties of T follows from the properties of $\bar{T}$. Even though T technically is a covariant functor, it is regarded as and referred to as an contravariant functor. In the next section we will define the contravariant Hom functor as a functor ${}_R\mathbf{Mod}^{op} \to \mathbf{Ab}$.

7 Functors Hom, H^n and Derived Functors Ext

We will begin this section by defining the additive contravariant bifunctor $\mathrm{Hom}(\square, D)$ and the cohomology functor H^n. Then, using these definitions, we will continue to define the Right Derived Functors of Hom: $\mathbf{Ext}^n_R(\square, D)$. We will show and then state the axioms characterizing $\mathbf{Ext}^n_R(\square, D)$ which we will use to show some computations of Ext-groups. As an application of $\mathbf{Ext}^n_R(\square, D)$ we will characterize injective modules and we will finally define the bijection between $\mathrm{Ext}^1_R(C, A)$ and the set of extension classes of the extension A by C.

7.1 Functors Hom and $\mathbf{H}^n$

In section 3.2 we defined homomorphisms of hom-sets $\varphi^* : \mathrm{Hom}(N, D) \to \mathrm{Hom}(M, D)$ derived from R-module homomorphisms $\varphi : M \to N$ and in section 5 we defined homomorphisms of cohomology groups, $H^n(C_\bullet) \to H^n(C'_\bullet)$ derived from chain maps of complexes $f : \mathbf{C}_\bullet \to \mathbf{C'}_\bullet$. We will now show that these definitions constitutes functorial mappings.

7.1.1 The Contravariant Hom functor

Using Proposition 3.11 we can define a bifunctor in ${}_R\mathbf{Mod}$. Let $\varphi : M \longrightarrow N$ be an homomorphism in ${}_R\mathbf{Mod}$ and let D be an R-module. Define a bifunctor

$$T : {}_R\mathbf{Mod} \times {}_R\mathbf{Mod} \to \mathbf{Ab}$$

by

$$T(M, D) = \mathrm{Hom}(M, D),$$
$$T(\varphi, id_D) = \varphi^*$$

where φ^* is the homomorphism defined in Proposition 3.11. If we fix the module D, Proposition 3.11 and the text following shows that T maps the sequence

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N$$

to the sequence

$$Hom_R(N, D) \xrightarrow{\varphi^*} Hom_R(M, D) \xrightarrow{\psi^*} Hom_R(L, D)$$

making T a bifunctor contravariant in the first argument.

Since D is fixed we can view T as a contravariant functor $T^D : {}_R\mathbf{Mod} \to \mathbf{Ab}$ of one argument, similar to the functor (69) in the section before. Alongside T^D there exist a functor

$$T_D : {}_R\mathbf{Mod} \to \mathbf{Ab}$$

similar to (68) for which the equality (70) holds making T a bifunctor. This is the covariant Hom-functor $\mathrm{Hom}(B, \Box)$ which we are not including in this text. [5, pg. 38, Sec. II Construction on Categories, Subsec. 3. Products of Categories]

Proposition 7.1. *Let $B \in {}_R\boldsymbol{Mod}$ then*

(i) $Hom(\Box, B)$ is a contravariant additive functor, $T^B : {}_R\boldsymbol{Mod}^{\boldsymbol{op}} \to \boldsymbol{Ab}$, defined for all objects $C \in {}_R\boldsymbol{Mod}$ by

$$T^B : {}_R\boldsymbol{Mod}^{\boldsymbol{op}} \to \boldsymbol{Ab}$$
$$C \mapsto Hom(C, B),$$

and defined for $f : C \to C'$ in ${}_R\boldsymbol{Mod}$ by

$$T^B(f^{op}) : Hom(C', B) \to Hom(C, B)$$
$$h \mapsto hf.$$

We will denote $T^B(f^{op})$ by f^*.

(ii) *Hom*$(\Box, B)$ *is a left exact functor; if the sequence*

$$L \to M \to N \to 0$$

is exact in $_R$***Mod*** *is the sequence*

$$0 \to Hom\ (N, B) \to Hom\ (M, B) \to Hom\ (L, B)$$

exact.

(iii) *Hom*$(\Box, B)$ *is an exact functor if and only if* B *is an injective module: The sequence*

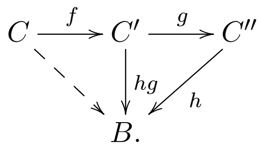
$$0 \to Hom\ (N, B) \to Hom\ (M, B) \to Hom\ (L, B) \to 0$$

is a short exact sequence whenever

$$0 \to L \to M \to N \to 0,$$

is a short exact sequence in $_R$***Mod*** *and if and only if* B *is injective.*

Proof. (i) The mappings of objects and morphisms are shown in Proposition 3.11 and its proof. We need to show that composition is preserved and that identity is mapped to identity and finally that additivity is preserved. Let C, C', C'' and B be R-modules and let h be any map in Hom (C'', B). Consider the diagram of R-module homomorphisms below,



(71)

We have that

$$g^* : h \to hg$$

and so

$$f^*g^* : h \to hg \to (hg)f$$

and since

$$(gf)^* : h \to h(gf) = (hg)f,$$

$(gf)^* = f^*g^*$ and we have shown composition. For the identity mapping, let $1_C : C \to C$ be the identity map then $1_C^* : h \to h$ for any map $h \in$ Hom (C, B) since,

$$1_C^* : h \to h1_C = h,$$

and we have shown that Hom$(\Box, B)$ is a functor. What is left to show is additivity. Let $f, g : C \to C'$ and let $h \in$ Hom (C', B) be any homomorphism. From Proposition 2.5 *(iii)* we get that

$$h(f + g) = hf + hg$$

that is

$$(f + g)^* = f^* + g^*$$

and so Hom$(\Box, B)$ is an additive functor. [2, pg. 20, Example. 1.10, Ch 1.2], [2, pg. 40, Proposition 2.5, Ch 2.1]

(ii) - (iii) Follows from Theorem 3.12 and Corollary 3.14.

□

7.1.2 The Cohomology Functor H^n

We will now use Propositions 5.4 and 5.5 to define the functor H^n.

Proposition 7.2. *The cohomology functor, denoted H^n, is an additive functor H^n : $\mathbf{Comp}({}_R\mathbf{Mod}) \to \mathbf{Ab}$ for each $n \in \mathbb{Z}$, defined for all objects $\mathbf{C}_\bullet = (C_\bullet, d_\bullet) \in {}_R\mathbf{Comp}$ by*

$$H^n : \mathbf{Comp}({}_R\mathbf{Mod}) \to \mathbf{Ab}$$
$$C \mapsto ker\ d_n / Image\ d_{n-1}$$

for every C_n in $\mathbf{C}_\bullet$. And defined for all chain maps $f : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ where $f_n : C_n \to C'_n$ by

$$H^n(f) : z_n + Im\ d_{n-1} \mapsto f_n(z_n) + Im\ d'_{n-1}$$

for every $z_n \in C^n$.

Proof. The morphisms are welldefined according to Proposition 5.4. In Proposition 5.5*(i)* and *(ii)* we have shown that identity and composition are preserved. Lastly H^n is additive since group homomorphisms are additive: Let $f, g : C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ be two chain maps, then we have

$$\begin{aligned} H_n(f+g) : z + \text{Im } d_{n-1} \mapsto & (f_n + g_n)z + \text{Im } d'_{n-1} \\ &= (f_n(z) + g_n(z)) + \text{Im } d'_{n-1} \\ &= (H_n(f) + H_n(g))(z + \text{Im } d'_{n-1}), \end{aligned}$$

for each $n \in \mathbb{Z}$. [4, pg. 263-264, Proposition C-3.37, Ch. C-3.5] □

7.2 Ext Groups and Right Derived Functors Ext

The contravariant functors $\mathbf{Ext}^n_R(\Box, D)$ are a set of functors ${}_R\mathbf{Mod} \to \mathbf{Ab}$, called the right derived functors for $\text{Hom}_R(\Box, D)$. They are a composition of functors $\text{Hom}_R(\Box, D)$ and H^n applied on the projective resolution of a module A, that is $\mathbf{Ext}^n_R(\Box, D)(A) = \mathbf{Ext}^n_R(A, D)$ generate cohomology groups derived from a complex of Hom-groups which in turn are derived from the projective resolution of the module A.

Let us start with the projective resolution of an R-module A and let

$$\mathbf{P}_A = \cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} 0 \tag{72}$$

be the deleted projective resolution of A. Apply $\text{Hom}_R(\Box, D)$, since $\text{Hom}_R(\Box, D)$ is an additive contravariant functor, see Proposition 7.1, and the sequence (72) is a chain complex the functored sequence is the cochain complex,

$$0 \longrightarrow \text{Hom}_R(P_0, D) \xrightarrow{d_1^*} \ldots \xrightarrow{d_{n-1}^*} \text{Hom}_R(P_{n-1}, D) \xrightarrow{d_n^*} \text{Hom}_R(P_n, D) \longrightarrow \ldots.$$

The n^{th} cohomology group of the sequence above is called $\text{Ext}^n_R(A, D)$ indicating that it is derived specifically from a complex emerging from the projective resolution of A after being mapped by the functor $\text{Hom}_R(\Box, D)$.

Definition 7.3. Let A and D be R-modules. Let $\mathbf{P}_A$ be the deleted resolution (72) and let

$$0 \longrightarrow \mathrm{Hom}_R(P_0, D) \xrightarrow{d_1^*} \dots \xrightarrow{d_{n-1}^*} \mathrm{Hom}_R(P_{n-1}, D) \xrightarrow{d_n^*} \mathrm{Hom}_R(P_n, D) \longrightarrow ..., \tag{73}$$

be the cochain complex of abelian groups resulting from applying the functor $\mathrm{Hom}_R(\square, D)$ to each $P_n \in \mathbf{P}_A$. Define

$$\mathbf{Ext}_R^n(A, D) = H^n(\mathrm{Hom}_R(\mathbf{P}_A, D)) = \ker\ d_{n+1}^*/\mathrm{im}\ d_n^*$$

to be the cohomology group derived from the n^{th} element of (73).

Example 7.4. Let A and B be $\mathbb{Z}$-modules, we showed in Example 4.8 that a free resolution of A is of the form

$$0 \to \ker\ \varphi \to F(X) \xrightarrow{\varphi} A \to 0$$

which gives us a deleted resolution

$$0 \to \ker\ \varphi \to F(X) \to 0. \tag{74}$$

Let $F = F(X)$ and $K = \ker\ \varphi$, if apply $\mathrm{Hom}_{\mathbb{Z}}(\square, B)$ to (74) we get

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F, B) \xrightarrow{d_1^*} \mathrm{Hom}_{\mathbb{Z}}(K, B) \xrightarrow{d_2^*} 0$$

which gives us the groups $\mathrm{Ext}_{\mathbb{Z}}^0(A, B) = \ker\ d_1^*$ and $\mathrm{Ext}_{\mathbb{Z}}^1(A, B) = \ker\ d_2^*/\mathrm{im}\ d_1^*$ and we get $\mathrm{Ext}_{\mathbb{Z}}^n(A, B) = 0$ for all $n > 1$.

Since $\mathrm{Hom}_R(\square, D)$ is a left exact functor it maps the first modules of the resolution of A, which is the exact sequence

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0,$$

to the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(A, D) \xrightarrow{\epsilon^*} \mathrm{Hom}_R(P_0, D) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1, D).$$

The exactness of the last sequence then gives us that $\mathrm{Hom}_R(A, D) \cong \mathrm{im}\ \epsilon^* = \ker\ d_1^*$. According to definition we have that

$$\mathrm{Ext}_R^0(A, D) = \ker\ d_1^*/0 = \ker\ d_1^*$$

and we get that $\mathrm{Ext}_R^0(A, D) \cong \mathrm{Hom}_R(A, D)$. We state the result in the Proposition below.

Proposition 7.5. *For any R-module A and D we have that $Ext_R^0(A, D) \cong Hom_R(A, D)$*

Proof. This follows from the text above. □

Since any two chain maps $f, \hat{f} : \mathbf{P}_A \to \mathbf{P'}_{A'}$ over the same homomorphism $f : A \to A'$ are homotopic, see Proposition 4.12, the induced chain maps $f^*, \hat{f}^*$ are also homotopic due to additive functors preserving homotopy, see Proposition 6.12. A consequence of this is that the cohomology groups $\mathrm{Ext}^n_R(A, D)$ do not depend on the choice of resolution of A, that is if $\mathbf{P}_A$ and $\mathbf{P'}_A$ are two different projective resolutions of A, $H^n(\mathrm{Hom}_R(\mathbf{P}_A, D))$ and $H^n(\mathrm{Hom}_R(\mathbf{P'}_A, D))$ are canonically isomorphic.

Proposition 7.6. *The groups $Ext^n_R(A, D)$ are independent of the choice of projective resolution of A.*

Proof. Let $\mathbf{P}$ and $\mathbf{P'}$ be two projective resolutions of the the module A and let $g : \mathbf{P}_A \to \mathbf{P'}_A$ and $h : \mathbf{P'}_A \to \mathbf{P}_A$ be two chain maps over 1_A and consider the diagram below,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P^2 & \xrightarrow{d_2} & P^1 & \xrightarrow{d_1} & P^0 & \longrightarrow & A \\ & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle 1_A} \\ \cdots & \longrightarrow & P'^2 & \xrightarrow{d_2} & P'^1 & \xrightarrow{d_1} & P'^0 & \longrightarrow & A \\ & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle 1_A} \\ \cdots & \longrightarrow & P^2 & \xrightarrow{d_2} & P^1 & \xrightarrow{d_1} & P^0 & \longrightarrow & A. \end{array}$$

The compositions hg and the identity chain map $1_{\mathbf{P}_A}$ are both chain maps $\mathbf{P}_A \to \mathbf{P}_A$ over 1_A and as such they are homotopic according to Proposition 4.12. In the same way are gh and $1_{\mathbf{P'}_A}$ homotopic chain maps $\mathbf{P'}_A \to \mathbf{P'}_A$. Since $\mathrm{Hom}(\Box, D)$ is a contravariant additive functor we have that

$$(hg)^* = g^* h^* : \mathrm{Hom}_R(\mathbf{P}_A, D) \to \mathrm{Hom}_R(\mathbf{P}_A, D)$$

is homotopic to $1^*_{\mathbf{P}_A}$ and that

$$(gh)^* = h^* g^* : \mathrm{Hom}_R(\mathbf{P'}_A, D) \to \mathrm{Hom}_R(\mathbf{P'}_A, D)$$

is homotopic to $1^*_{\mathbf{P'}_A}$ and, according to Proposition 5.5, $H^n(\mathrm{Hom}_R(\mathbf{P}_A, D)) \cong H^n(\mathrm{Hom}_R(\mathbf{P'}_A, D))$. $\Box$

Proposition 7.7. *Let P be a projective R-module and D any R-module then $Ext^n_R(P, D) = 0$ for $n \geq 1$.*

Proof. Due to Proposition 7.6 we only need to show that the statement is true for some resolution of P. Since P is projective a projective resolution of P is

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P \xrightarrow{1_P} P \longrightarrow 0$$

which gives us the deleted resolution

$$\mathbf{C}_P = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P \longrightarrow 0$$

where every $C_n = 0$ for $n \geq 1$. After applying $\text{Hom}(\Box, D)$ we get the complex

$$0 \to \text{Hom}(P, D) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where every $H^n(\text{Hom}_R(\mathbf{C}_P, D)) = H^n(0) = 0$ for $n \geq 1$. And $\text{Ext}^n_R(P, D) = 0$ for $n \geq 1$. □

We will now define the right derived functors for Hom.

Proposition 7.8. *Let B be an R-module. The* ***right derived functors for Hom****, $\textbf{\textit{Ext}}^n_R(\Box, B)$, $n \geq 0$, is a set of additive contravariant functors ${}_R\mathbf{Mod} \to \mathbf{Ab}$.*

$\textbf{\textit{Ext}}^n_R(\Box, B)$ is defined on objects $A \in {}_R\mathbf{Mod}$ by

$$A \mapsto Ext^n_R(A, B)$$

where $Ext^n_R(A, B) = H^n(Hom_R(\mathbf{P}_A, B))$ are the cohomology groups in Definition 7.3.

$\textbf{\textit{Ext}}^n_R(\Box, B)$ is defined on morphisms $f : A \to A' \in {}_R\mathbf{Mod}$ by

$$f \mapsto H^n(\hat{f}^*_n),$$

$$H^n(\hat{f}^*_n) : H^n(Hom_R(\mathbf{C}_{A'}, B)) \to H^n(Hom_R(\mathbf{P}_A, B))$$

where H^n is the cohomology functor in Proposition 7.2 and where $\hat{f}^*_n : Hom(C_n, B) \to Hom(P_n, B)$ is the map induced by the lift $\hat{f}_n : P_n \to C_n$ in the chainmap $\hat{f} : \mathbf{P}_A \to \mathbf{C}_{A'}$ where $\mathbf{C}_{A'}$ is the deleted resolution of the module A'.

Proof. That $\text{Ext}^n_R(\Box, B)$ is additive follows from Hom and H^n being additive. That $\text{Ext}^n_R(\Box, B)$ is well defined follows from Hom and H^n being well-defined functors together with Proposition 7.6 since by Proposition 7.6 the $\text{Ext}^n_R(A, B)$ groups are the same (canonically isomorphic) regardless of the choice of resolution of A. That $\text{Ext}^n_R(\Box, B)$ is contravariant follows from $\text{Hom}(\Box, D)$ being contravariant. Let

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{d_2} & P_2 & \xrightarrow{d_1} & P_1 & \xrightarrow{d_0} & P_0 & \xrightarrow{\epsilon} & A & \longrightarrow & 0 \\ \downarrow & & \downarrow \hat{f}_2 & & \downarrow \hat{f}_1 & & \downarrow \hat{f}_0 & & \downarrow f & & \\ \cdots & \xrightarrow{d_2'} & C_2 & \xrightarrow{d_1'} & C_1 & \xrightarrow{d_0'} & C_0 & \xrightarrow{\epsilon'} & A' & \longrightarrow & 0. \end{array} \tag{75}$$

be the commutative diagram of projective resolutions constructed from $f : A \to A'$, see Proposition 4.10. Since $\text{Hom}(\Box, B)$ is an additive contravariant functor, applying Hom on diagram (75) generates a diagram where the rows and columns are reversed making it a chain map of cochain complexes:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \text{Hom}(A', B) & \xrightarrow{\epsilon'^*} & \text{Hom}(C_0, B) & \xrightarrow{d_0'^*} & \text{Hom}(C_1, B) & \xrightarrow{d_1'^*} & \text{Hom}(C_2, B) & \xrightarrow{d_2'^*} & \cdots \\ & & \downarrow f^* & & \downarrow \hat{f}_0^* & & \downarrow \hat{f}_1^* & & \downarrow \hat{f}_2^* & & \\ 0 & \longrightarrow & \text{Hom}(A, B) & \xrightarrow{\epsilon^*} & \text{Hom}(P_0, B) & \xrightarrow{d_0^*} & \text{Hom}(P_1, B) & \xrightarrow{d_1^*} & \text{Hom}(P_2, B) & \xrightarrow{d_2^*} & \cdots . \end{array}$$

By Proposition 7.2 we can apply H^n and get a set of homomorphisms

$$H^n(\hat{f}^*) : \ker d_n'^* / \operatorname{im} d_{n-1}'^* \to \ker d_n^* / \operatorname{im} d_{n-1}^*$$

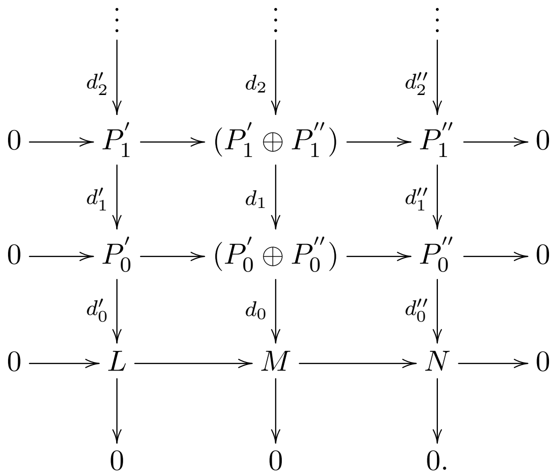
which according to Definition 7.3 is the same as

$$H^n(\hat{f}^*) : \operatorname{Ext}_R^n(A', B) \to \operatorname{Ext}_R^n(A, B)$$

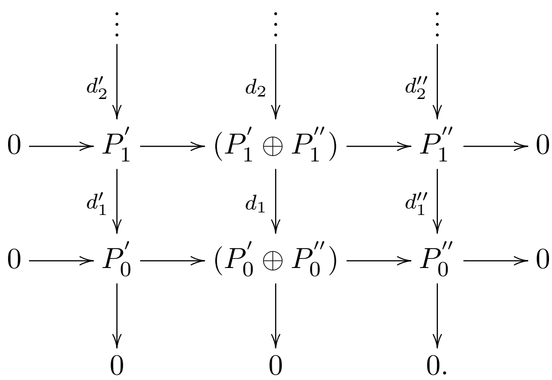
and $\operatorname{Ext}_R^n(\square, B)$ are contravariant functors. □

7.2.1 The Long Exact Sequence of Ext Groups

Next we will use the Horseshoe Lemma (Lemma 4.13) and the Long Exact Sequence in Cohomology (Theorem 5.6) to derive a long exact sequence of Ext groups. Take the projective resolution of modules L and N in the short exact sequence $0 \to L \to M \to N \to 0$, according to the Horseshoe Lemma there exists a short exact sequence of complexes whose columns are complexes and rows are split,



If we replace the first row with zeroes, turning the columns into deleted resolutions, we still have a short exact sequence of complexes since deleted resolutions are still complexes and the remaining rows are still short exact sequences:



Next we apply $\mathrm{Hom}_R(\Box, D)$ to the diagram above and get the diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(P_2^{''}, D) & \longrightarrow & \mathrm{Hom}((P_2^{'} \oplus P_2^{''}), D) & \longrightarrow & \mathrm{Hom}(P_2^{'}, D) & \longrightarrow & 0 \\
 & & \uparrow {\scriptstyle d''^2} & & \uparrow {\scriptstyle d^2} & & \uparrow {\scriptstyle d'^2} & & \\
0 & \longrightarrow & \mathrm{Hom}(P_1^{''}, D) & \longrightarrow & \mathrm{Hom}((P_1^{'} \oplus P_1^{''}), D) & \longrightarrow & \mathrm{Hom}(P_1^{'}, D) & \longrightarrow & 0 \\
 & & \uparrow {\scriptstyle d''^1} & & \uparrow {\scriptstyle d^1} & & \uparrow {\scriptstyle d'^1} & & \\
0 & \longrightarrow & \mathrm{Hom}(P_0^{''}, D) & \longrightarrow & \mathrm{Hom}((P_0^{'} \oplus P_0^{''}), D) & \longrightarrow & \mathrm{Hom}(P_0^{'}, D) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}
\tag{76}
$$

where we have left out the stars on induced maps to simplify notation. By Proposition 6.12 the diagram (76) is also a short exact complex: The columns are cochain complexes since $\mathrm{Hom}_R(\Box, D)$ turns chain complexes into cochain complexes, and the rows are split exact since

$$\mathrm{Hom}(P_n^{'} \oplus P_n^{''}, D) = \mathrm{Hom}(P_n^{'}, D) \oplus \mathrm{Hom}(P_n^{''}, D)$$

due to $\mathrm{Hom}_R(\Box, D)$ preserving direct sums. By Theorem 5.6 there now exists a long exact sequence of cohomology groups,

$$0 \longrightarrow \ker\, d''^1/0 \longrightarrow \ker\, d^1/0 \longrightarrow \ker\, d'^1/0 \longrightarrow \ker\, d''^2/\mathrm{im}\, d''^1 \longrightarrow$$

$$\longrightarrow \ker\, d^2/\mathrm{im}\, d^1 \longrightarrow \ker\, d'^2/\mathrm{im}\, d'^1 \longrightarrow \ker\, d''^3/\mathrm{im}\, d''^2 \longrightarrow \ker\, d^3/\mathrm{im}\, d^2 \longrightarrow \cdots$$

which, according to Definition 7.3 and Proposition 7.5, is the same as

$$0 \longrightarrow \mathrm{Hom}_R(N, D) \longrightarrow \mathrm{Hom}_R(M, D) \longrightarrow \mathrm{Hom}_R(L, D) \longrightarrow \mathrm{Ext}^1_R(N, D) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_R(M, D) \longrightarrow \mathrm{Ext}^1_R(L, D) \longrightarrow \mathrm{Ext}^2_R(N, D) \longrightarrow \mathrm{Ext}^2_R(M, D) \longrightarrow \cdots .$$

We will summarize the above in the next theorem which also characterizes the right derived functors $\mathbf{Ext}^n_R(\Box, D)$ [2, pg. 373, Theorem 6.64, Ch. 6.2].

Theorem 7.9. *Let* $Ext^n_R(\Box, D) : {}_R\boldsymbol{Mod} \to \boldsymbol{Ab}$ *be the right derived functor of Hom defined in Proposition 7.8. Then*

(i) for every short exact sequence

$$0 \to A \to B \to C \to 0$$

of left R-modules there is a long exact sequence and a connecting homomorphisms

$$\cdots \to Ext^n_R(C,D) \to Ext^n_R(B,D) \to Ext^n_R(A,D) \xrightarrow{\delta_n} Ext^{n+1}_R(C,D) \to \cdots$$

(ii) where $Ext^0_R(\square, D)$ and $Hom(\square, D)$ are equivalent and where

(iii) $Ext^n_R(P,D) = 0$ for all modules D and all projective modules P.

Proof. The first statement follows from the text above, the second and third is Proposition 7.5 and Proposition 7.7 □

Theroem 7.9 can be used in determining Ext-groups whose arguments are finitely generated $\mathbb{Z}$-modules. Let A and B be $\mathbb{Z}$-modules. In Example 7.4 we showed that the only non-zero $\text{Ext}^n_{\mathbb{Z}}(A,B)$-groups are $\text{Ext}^0_{\mathbb{Z}}(A,B)$ and $\text{Ext}^1_{\mathbb{Z}}(A,B)$ where $\text{Ext}^0_{\mathbb{Z}}(A,B) = \text{Hom}_{\mathbb{Z}}(A,B)$ by statement *(iii)* in the Theorem above. What is left to determine is $\text{Ext}^1_{\mathbb{Z}}(A,B)$. In Example 2.20 we stated that any finitely generated module over a PID can be decomposed into a direct sum of cyclic groups, in particular finitely generated $\mathbb{Z}$-modules have a unique decomposition of cyclic modules. We assume that A is finitely generated and B is cyclic. Since $\mathbf{Ext}^n_R(\square, D)$ preserves direct sums we can compute $\text{Ext}^1_{\mathbb{Z}}(A,B)$ by decomposing A into a direct sum of cyclic submodules C, the group $\text{Ext}^1_{\mathbb{Z}}(A,B)$ is then the direct sum of groups $\text{Ext}^1_{\mathbb{Z}}(C,B)$ which we can determine using the statements below.

Example 7.10.

(i) $\text{Ext}^n_{\mathbb{Z}}(\mathbb{Z}, B) = 0$, $n > 0$ and where B is any $\mathbb{Z}$-module.

(ii) $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$.

(iii) $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}$ where $g = \gcd(m,n)$.

(i) This follows from Theorem 7.9 (iii), since $\mathbb{Z}$ is a free module and as such projective.

(ii) The projective resolution of $\mathbb{Z}/n\mathbb{Z}$ is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

(Example 4.8) where n is the homomorphisms multiplying each element in $\mathbb{Z}$ by n. By Theorem 7.9 (i) there exists an exact sequence,

$$0 \longrightarrow \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \longrightarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{n^*} \text{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \longrightarrow 0$$

where $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$ due to (i). In particular is

$$\text{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{n^*} \text{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \longrightarrow 0$$

exact. Since $\text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ and n^* is multiplication by n, there exists an exact sequence

$$\mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow 0.$$

Using the First Isomorphism Theorem we then get $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$.

(*iii*) This example follows the one above. The projective resolution of $\mathbb{Z}/m\mathbb{Z}$ is a short exact sequence
$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0.$$
Theorem 7.9 (i) gives us an exact sequence of Ext-groups:
$$\text{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{m^*} \text{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow 0$$
where $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ is zero due to (i). Using Example 2.6(i) we get an exact sequence
$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z} \longrightarrow \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \longrightarrow 0$$
and so
$$\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})/m(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/g\mathbb{Z}.$$
To show the last equality: Let $G = \mathbb{Z}/n\mathbb{Z}$, and $mG = m(\mathbb{Z}/n\mathbb{Z})$ and let t be the order of mG. Let $m = lg$ and $n = kg$ where $g = \text{gcd}(m, n)$ where $k, l \in \mathbb{Z}$. We will show that $t = k$ which leads to $|G/mG| = |G|/|mG| = g$ implying that G/mG is a cyclic group of order g. First we show that $t|k$: Since $\text{gcd}(l, k) = 1$ we have that $kgl = lcm(m, n)$ and since $kgl = km$, we get that $k(mx) = 0$ and so $t|k$. Next we show that $k|t$: We have that $t(mx) = 0_{\mathbb{Z}/n\mathbb{Z}}$ and so $n|tm$ which implies $k|tl$ and since $gcd(k, l) = 1$ we get that $k|t$. And so $t = k$.

We can use $\text{Ext}^n_R(\Box, D)$ to characterize injective modules. Consider Diagram (76), if we let D be an injective module then according to Proposition 7.1(*iii*) the columns of the diagram would be not only complexes but also exact at every group $\text{Hom}(P_n, D)$ where $n \geq 1$. In other words, given an injective module D the $\text{Ext}^n_R(\Box, D)$ groups where $n \geq 1$ are all zero. Proposition 7.1(*iii*) is an if and only if statement which means that the converse is also true: if $\text{Ext}^1_R(N, D) = 0$ for *all* R-modules N then D is an injective module and if D is injective then the following $\text{Ext}^n_R(\Box, D)$ groups are also zero. We have the equivalent statements of the Proposition below characterizing injective modules.

Proposition 7.11. *The following are equivalent.*

(*i*) *D is injective.*

(*ii*) *$Ext^1_R(N, D) = 0$ for all R-modules N.*

(*iii*) *$Ext^n_R(N, D) = 0$ for all R-modules N and $n \geq 1$.*

[1, pg. 784, Proposition 9, Sec. 17.1]

In the last part of this text we will show that there is a bijection between the cohomology group $\text{Ext}^1_R(C, A)$ and the set of equivalence classes of extensions A by C.

7.2.2 Bijection of $\mathrm{Ext}^1_R(C, A)$ and $e(C, A)$

In section 3 we defined equivalent extensions A by C as two extensions making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{77}$$

commute. We will now show, using Lemma 3.9 that there is a bijection between the set of equivalence classes of extensions A by C and the group $\mathrm{Ext}^1_R(C, A)$.

Theorem 7.12. *For any R-modules C and A there is a bijection between the group $Ext^n_R(C, A)$ and the set of equivalence classes of extensions A by C.*

Proof. Define

$$e(C, A) = \{[\xi] \mid \xi \text{ is an extension of } A \text{ by } C\}.$$

We will first define the map

$$\psi : e(C, A) \to \mathrm{Ext}^1_R(C, A),$$

and then construct its inverse

$$\theta : \mathrm{Ext}^1_R(C, A) \to e(C, A).$$

Let

$$\mathbf{P} = \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow C \to 0$$

be a projective resolution of a module C and remember that

$$\mathrm{Ext}^1_R(C, A) = \ker\, d_2^* / \mathrm{im}\, d_1^*$$

where d_2^* is the homomorphism

$$\begin{aligned} \mathrm{Hom}(P_1, A) &\to \mathrm{Hom}(P_2, A) \\ \alpha_1 &\mapsto \alpha_1 d_2. \end{aligned}$$

Let $[\xi] = 0 \to A \to B \to C \to 0$ be an extension class A by C and consider the diagram

$$\begin{array}{ccccccccccc} \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & C & \longrightarrow & 0 \\ & {\scriptstyle 0}\downarrow & & {\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_0}\downarrow & & \downarrow{\scriptstyle 1_C} & & \\ \longrightarrow & 0 & \longrightarrow & A & \xrightarrow[i]{} & B & \xrightarrow[p]{} & C & \longrightarrow & 0. \end{array} \tag{78}$$

From Proposition 4.10 we get that there exists a chain map α_n over 1_C since the first row is projective and the second row is exact. Since α_n is a chain map the diagram commutes and we have that $\alpha_1 d_2 = 0$, that is $\alpha_1 \in \ker\, d_2^*$ and we can now define ψ :

$$\begin{aligned} \psi : e(C, A) &\to \mathrm{Ext}^1_R(C, A) \\ [\xi] &\mapsto \alpha_1 + \mathrm{im}\, d_1^*. \end{aligned}$$

Suppose $\mathbf{Q} = \cdots \to P_2 \to P_1 \xrightarrow{t_1} P_0 \to C$ is another resolution of C we would then have a chain map $\beta : \mathbf{Q} \to \xi$ and a homomorphis $\beta_1 : Q_1 \to A$ and we could define ψ as

$$[\xi] \mapsto \beta_1 + \text{im } t_1^*.$$

However, since any two resolutions of C gives rise to isomorphic groups $\text{Ext}_R^n(C, A)$, see Proposition 7.6, we have an isomorphism π such that

$$\pi(\beta_1 + \text{im } t_1^*) = \alpha_1 + \text{im } d_1^*$$

and the map ψ is independent of the resolution. We need to show that ψ does not depend on the chain map α_n. Proposition 4.12 states that any two chain maps over 1_C are homotopic, this means that given a chain map α'_n,

$$\begin{array}{ccccccccccc}
\longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & C & \longrightarrow & 0 \\
& {\scriptstyle 0}\downarrow & & {\scriptstyle \alpha'_1}\downarrow & & {\scriptstyle \alpha'_0}\downarrow & & \downarrow{\scriptstyle 1_C} & & \\
\longrightarrow & 0 & \longrightarrow & A & \xrightarrow[i]{} & B & \xrightarrow[p]{} & C & \longrightarrow & 0,
\end{array}$$

there exists maps $s_1 : P_1 \to 0$ and $s_0 : P_0 \to A$ such that

$$\alpha' - \alpha = 0s_1 + s_0 d_1 = s_0 d_1$$

and $\alpha - \alpha' \in \text{im } d_1^*$ and so $\alpha + \text{im } d_1^* = \alpha'_1 + \text{im } d_1^*$ and we have shown that ψ does not depend on the chain map α_n. We also need to show that ψ does not depend on the extension ξ. Consider the diagram below where the two bottom rows are equivalent extensions,

$$\begin{array}{cccccccccccc}
\mathbf{P} = & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow & 0 \\
& & {\scriptstyle 0}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1_C} & & \\
\xi = & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& & \downarrow & & {\scriptstyle 1_A}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1_C} & & \\
\xi' = & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

Since equivalent extensions implies commutating diagrams we can view the equivalence of the rows as a chain map and the chain map $\mathbf{P} \to \xi'$ as the composition $\mathbf{P} \to \xi \to \xi'$ and so $\psi(\xi') = 1_A \alpha + \text{im } d_1^* = \alpha + \text{im } d_1^* = \psi(\xi)$.

We will now construct the inverse of ψ :

$$\theta : \text{Ext}_R^1(C, A) \to e(C, A).$$

Let $u \in \text{Ext}_R^1(C, A)$, choose a projective resolution to C and choose a homomorphism $\alpha_1 : P_1 \to A$, such that $\alpha_1 + \text{im } d_1^*$ represents the coset u:

$$\begin{array}{cccccccccc}
\mathbf{P} = & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & C & \longrightarrow 0 \\
& & & & {\scriptstyle \alpha_1}\downarrow & & & & & \\
& & & & A. & & & & &
\end{array} \tag{79}$$

Since $\alpha_1 \in \ker d_2^*$ we have that $\alpha_1 d_2 = 0_A$ and α_1 induces a homomorphism

$$\begin{aligned} \bar{\alpha}_1 : P_1/\text{im } d_2 &\to A \\ x_1 + \text{im } d_2 &\mapsto \alpha_1(x_1) \end{aligned}$$

where $x_1 \in P_1$. Given the resolution $\mathbf{P}$ in diagram (79) we have by the First Isomorphism Theorem that $P_1/\text{im } d_2 \cong \text{im } d_1$ and by the exactness of $\mathbf{P}$ that $\text{im } d_1 = \ker d_0$ and there exists a short exact sequence

$$\Xi = 0 \longrightarrow P_1/\text{im } d_2 \xrightarrow{\bar{d_1}} P_0 \xrightarrow{d_0} C \longrightarrow 0,$$

where $\bar{d_1}(x_1 + \text{im } d_2) = d_1(x_1)$. By Lemma 3.9 there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1/\text{im } d_2 & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{\alpha}_1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \xrightarrow[i]{} & B & \longrightarrow & C & \longrightarrow & 0. \end{array} \tag{80}$$

We can now define the map θ where the cosets of $\text{Ext}^1_R(C, A)$ maps to the equivalence class, $[\bar{\alpha}_1 \Xi]$ which are all extensions C by A in the bottom row of (80) completing the diagram. Define θ as

$$\begin{aligned} \theta : \text{Ext}^1_R(C, A) &\to e(C, A) \\ \alpha_1 + \text{im } d_1^* &\mapsto [\bar{\alpha}_1 \Xi]. \end{aligned}$$

We need to show that θ does not depend on the choice of homomorphism representing the coset $\alpha_1 + \text{im } d_1^*$, first note that the commutativity of diagram

$$\begin{array}{ccccccccccc} \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & C & \longrightarrow & 0 \\ & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_C} & & \\ \longrightarrow & 0 & \longrightarrow & A & \xrightarrow[i]{} & B & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{81}$$

implies diagram (80) and vice versa. We have that

$$\alpha_1 + \text{im } d_1^* = \{\alpha_1 + s d_1 \mid s \in \text{Hom}(P_0, A)\}$$

and so any other representative for the coset is of the form $\alpha_1' = \alpha_1 + s d_1$. Consider the diagram

$$\begin{array}{ccccccccccc} \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & C & \longrightarrow & 0 \\ & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \alpha_1 + s d_1} & & \downarrow{\scriptstyle \beta + i s} & & \downarrow{\scriptstyle 1_C} & & \\ \longrightarrow & 0 & \longrightarrow & A & \xrightarrow[i]{} & B & \xrightarrow[p]{} & C & \longrightarrow & 0. \end{array} \tag{82}$$

where the top row is the resolution chosen initially and the bottom row is the extension in diagram (80). This diagram commutes:

$$(\alpha_1 + s d_1) d_2 = \alpha_1 d_2 + s d_1 d_2 = 0_A$$

and

$$(\beta + is)d_1 = \beta d_1 + isd_1 = i\alpha_1 + isd_1 = i(\alpha_1 + sd_1)$$

and finally

$$p(\beta + is) = p\beta + pis = d_0 + pis = d_0.$$

Since diagram (82) commutes we have that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1/\text{im } d_2 & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \bar{\alpha}_1+sd_1} & & \Big\downarrow{\scriptstyle \beta+is} & & \Big\downarrow{\scriptstyle 1_C} & & \\
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0,
\end{array} \tag{83}$$

commutes and so $[\bar{\alpha}_1'\Xi] = [\bar{\alpha}_1\Xi]$ according to Lemma 3.9. We also have to show that $\theta\psi$ and $\psi\theta$ are identity maps. To show that $\psi\theta$ is an identity map, let $\alpha_1 + \text{im } d_1^* \in \text{Ext}_R^1(C, A)$ then $\theta(\alpha_1 + \text{im } d_1^*)$ is the extension class consisting of the bottom rows of

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1/\text{im } d_2 & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \bar{\alpha}_1} & & \Big\downarrow & & \Big\downarrow{\scriptstyle 1_C} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

which are the same sequences as the bottom row of the chain map α_n over 1_C starting with α_1,

$$\begin{array}{ccccccccc}
P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \alpha_1} & & \Big\downarrow{\scriptstyle \alpha_0} & & \Big\downarrow{\scriptstyle 1_C} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

And so, from the definition of ψ we get that $\psi\theta(\alpha_1 + \text{im } d_1^*) = \alpha_1 + \text{im } d_1^*$. For the reversed map, let ξ be an extension of C by A then $\psi(\xi) = \alpha + \text{im } d_1^*$ where α is a map

$$\begin{array}{cccccccccc}
 & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow & 0 \\
 & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow & & \Big\downarrow{\scriptstyle 1_C} & & \\
\xi = & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

making the diagram commute. This implies there exists a diagram

$$\begin{array}{cccccccccc}
\Xi = & 0 & \longrightarrow & P_1/\text{im } d_2 & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow & 0 \\
 & & & \Big\downarrow{\scriptstyle \bar{\alpha}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle 1_C} & & \\
\xi = & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0.
\end{array}$$

and so $\xi \in [\alpha\Xi]$ and $\theta\psi(\xi) = [\alpha\Xi]$ is an identity map. [2, pg. 421-426, Theorem 7.30, Ch. 7.2] □

References

[1] Dummit, Foote Abstract Algebra,

[2] Rotman An Introduction to Homological Algebra,

[3] Rotman Advanced Modern Algebra, Part 1

[4] Rotman Advanced Modern Algebra, Part 2

[5] Saunders Mac Lane Categories for the Working Mathematician,

[6] StackExchange, https://math.stackexchange.com/questions/1200021/m-to-m-to-m-to-0-exact-implies-0-to-texthomm-n-to-texthomm?rq=1