SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Rigour in mathematics: A historical perspective

av

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2023 - K2

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Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Rikard Bögvad

2023

Acknowledgements
I would like to thank my supervisor, Rikard Bögvad, for always being patient with me and for bringing fun into the equation when writing this thesis. It has been highly appreciated. I would also like to thank Claudio Bartocci, for inspiring me to write on this topic, as well as helping me with more practical matters.

Abstract

In this essay, we shall be investigating the rigorisation process that occured in Europe during the 19th century, particularly within analysis. The norms of the period regarding the sciences and within the profession of a mathematician will be examined, followed by a deep dive into the mathematical discoveries that brought this change along, namely *continuity, the limit and epsilontics* and *the real numbers*. This will be done by looking at the advancement of these topics, as made by some key mathematicians in France and in Prussia/Germany. Subsequently, the societal factors will be explored to further analyse what brought said development along, mentioning revolutions, the birth of some European countries, the industrial revolution and the technological advancements that followed as well as some other topics.

Keywords: rigour, analysis, Kant, continuity, limit, epsilontics, real numbers, Cauchy, Dedekind, Weiersrass, Cantor, Heine.

> "The rigourisation was not just a question of clarifying a few basic concepts and changing the proofs of a few basic theorems; rather it invaded almost every part of analysis and changed it into the discipline we now learn in high schools and universities. The movement towards rigour can even be seen as a process of creation."

- Lützen (A history of mathematics, 1999)

Contents

1 Introduction

Rigour in mathematics is a topic of great historical curiosity. In part, because every period has a different view on what it actually means, and in part because all ages seem to think that the rigour of the past be non-existant or at least severely lacking. But what is rigour? Cambridge dictionary defines it as the following:

> **rigour** ***noun***
> the quality of being detailed, careful and complete

So what does this mean? What level of detail is enough to be considered rigorous? And whose opinion is valid when deciding if an argument is complete?

According to Jeremy Gray in *Plato's ghost*, in ancient Greece an argument was incomplete if it did not stand on geometrical reasoning (Gray (2008)). If you could not prove your statement using a drawing featuring geometrical figures, your argument would be cast off for being lacking. And yet, even relying on theorems and proofs was a new concept, as the mathematicians - or perhaps rather physicists and astronomers - up until then had done entirely without it. Later on, as we will explore in this essay, a reasoning was incomplete if it did not stand on arithmetic and algebra, which was then seen as the ultimate precision of mathematics.

So who says what rigour in mathematics is? It is fair to assume that the interpretation of the word in a specific period will be that of the mathematicians in that same period, something we ought to have in mind as we delve into the rigour of the past. It is hard, if not impossible, to fully take on the perspective of the people of another age, with another life and another foundation of knowledge at their disposal. Hopefully, this essay will provide at least some context to put this piece of history into.

In this essay, we will explore the changes that occured in how mathematics was written and presented, mainly during the 19th century. Of what did these changes consist? What were the societal aspects that led up to it and what figures played a role in it happening? What were the foundational pieces of mathematics on which the new could be built? These are some of the questions for which I will seek answers in writing this essay.

1.1 Method

I will be using a qualitative method, looking at several sources and examining the reoccurance of certain themes, in order to aquire a bigger picture of the subject at hand. The sources will treat both the general history of the period - that of certain historical events and societal changes - as well as the mathematics that was used, from original sources when possible.

Specifically, the method will be the one described in *Metod: Guide för historiska studier* (Sjöberg (2018/2022), 88-91). Principally, the method can be summerized by the following quote:

> ”Kyles tillvägagångssätt är att bearbeta sitt material och att i det arbetet *lokalisera återkommande områden* och *konstruera samlade kategorier* vari argumenten kan inordnas.”

The categories that will be explored in this essay are *continuity*, *the limit* and *the real numbers*, which were the most recurring topics in the litterature examined. In addition, this essay will delve into general history that does not necessarily include mathematics, in order to gain a better idea of the context in which these changes occured.

1.2 Sources

The sources used can be placed into two main categories: original sources and historical litterature. The original sources, unless stated otherwise, will be read in their original print and language and translated by me. This is in order to diminish the risk of the content being changed or diluted by second-hand sources. The historical litterature will be used mainly to gain an idea of the historical changes that took place during the period, to put everything into context.

2 Immanuel Kant

To start at a point that could be considered a beginning of the story, we need to go back to 1800th century Prussia and Immanuel Kant. Born in 1722, Kant grew up to become one of the most influential thinkers in recent years (Nationalencyklopedin, *Immanuel Kant* (n.a.)). In 1770, he wrote inaugural dissertation - *De mundi sensibilis atque intelligibilis forma et principiis*, or *The Form and Principles of the Sensible and Intelligible World* - and thus became professor in logic and metaphysics at the university of Königsberg.

His philosophy came to become highly regarded within the scientific sphere, not least after the revised publication of his most famous work - *Kritik der reinen Vernunft*, or *Critique of pure reason*, in 1787 (Nationalencyklopedin, *Immanuel Kant* (n.a.)). In this work, he expresses a strong opinion against personal bias, as well as makes a distinction between the different kinds of knowledge (Kant (1781/1902), Chapter I).

First off, he makes a distinction between *a priori* knowledge and *a posteriori* knowledge, with the difference being that *a priori* knowledge is that which holds independently of human experience and *a posteriori* knowledge is that which is instead based on our senses. He then goes on to define *analytic* and *synthetic a priori* knowledge. He defines it as *analytic* if the description of the object is contained in the object itself and thus does not add any new information, while *synthetic* is when the description lies outside of the object. The examples that he makes are that

1. *All bodies are extended* is a piece of *analytic a priori knowledge*, as it is given for all bodies.

2. *All bodies are heavy* is *synthetic*, as it adds information about the body. While it is true that all bodies are in fact heavy, it cannot be derived from the very definition of a body.

In his work he expresses a clear hierarchical order between the types of knowledge, in which *a priori* is raised up as far superior to *a posteriori*, and in which *analytic* is superior to *synthetic*. He goes so far as to talk about the importance ["Dignität"] of the reasoning. He includes philosophy and mathematics into ths superior group of knowledge, while physics - that is conducted through experiments and observations - falls short and is considered a lower form of science (Kant (1781/1902), Chapter II, ¶5).

As the national encyclopedia of Sweden states (Nationalencyklopedin (n.a.), *Immanuel Kant*):

> "Efter honom har varken den rationalistiska eller den empiristiska traditionen någonsin kunnat återvinna sina positioner, och varje försök att komma ur det kunskapsteoretiska dilemma Kant pekat på har visat sig förfelat."

Furthermore, Gray argues the following, in *Plato's ghost* (Gray (2008), 78):

> "Many nineteenth-century arguments about the nature of mathematics can be seen as responses to Kant's philosophy: extensions of it, reinterpretations of it, breaks with it, emancipation from it, rediscovery of it."

As exaplained, Kant's work had an immense influence over the sciences, and it might even have played a key role in the upcoming separation of physics and mathematics.

3 The separation of physics and mathematics

In the 19th century, something fundamental happened with physics: it split up into three parts. On one side of the spectrum, we had physics, the science of matter, based on observations of the world; and on the other we had mathematics, which worked independently from the world around it; with applied mathematics as some sort of middle man.

In Germany one talked of *begriffliche Mathematik* - or conceptual mathematics, as Gray explains - of the newly found independence of mathematics and, one might even argue, its liberation from the other sciences (Gray (2008), 2). When before one might have considered mathematics to be the tool for answering questions in physics, this new take on mathematics leaned instead toward the abstract, with mathematicians such as Gauss, Cayley and not least Noether as its developers. This new mathematics was to stand on its own two legs and exist solely for its own sake, which was furthered by the new mathematical societies that were founded in the period.

4 Critique of past mathematics

In the 1900th century, the idea that the foundational work of mathematics was errated was starting to spread in the mathematical community (Gray (2008), 18, 88). Several mathematicians started going through some of the historical mathematical works on which the newer was founded, only to find error after error in the original texts. How could one build new concepts and find new truths if one reasoned using incorrect arguments?

As some of the mathematicians were going through said works, they recieved some backlash from other scientists, especially from physicists, who claimed that the effort was extravagant and pedantic (Gray (2008), 18-19). However, as Gauss and Crelle, and later on also Pasch and Russell - among others - started finding flaws in the work of Euclid himself, the effort became quite a bit more accepted in the scientific society. What they found was that the work of Euclid was full of gaps in the reasoning, and above all in the definitions on which he had built said reasoning. For example, here is the definition of a point, given in the opening chapter of Elementa (Euclid ($\sim$ 300 B.C./2008), iv):

Definition 4.1 (Point)**.** A *point* is that of which there is no part.

While this is true and quite easily understood by somebody who already knows what a point is, it does not give any information to somebody who does not. What played the larger role in the questioning of the reasoning and the genius of past mathematicians is of course hard to say with any certainty, but if one could contradict Euclid himself and come out correct, then quite obviously even the brightest mathematician could be wrong.

5 Some key discoveries

However important the changes in attitude, the mathematical revolution - as I indeed think that it was - could not have happened were it not for some key definitions and theorems from this time.

5.1 Continuity, the limit and the emergence of epsilontics

While continuity has been present in mathematics since ancient times, it was not until the 17th century that it started being adressed with the importance that it is today. Aristotle wrote, in the fourth century B.C. (Aristotle (350 B. C.), Part 3):

> "The 'continuous' is a subdivision of the contiguous: things are called continuous when the touching limits of each become one and the same and are, as the word implies, contained in each other: continuity is impossible if these extremities are two"

This gives a rather intuitive explanation of the concept, but without the rigour we see in later definitions. The development of its definition can be said to be picked up anew in 1701, when Leibniz introduced his law of continuity (Leibniz (1701)):

> "In any supposed [continuous] transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included"

While this does not add any new ideas to the discussion, it seems that it might have reignited it. On the other hand, already in 1656 we saw an early definition of the limit, set forth by Wallis in his *Arithmetica Infinitorum* (Wallis (1656)):

Definition 5.1 (Limit)**.** But (which for us here suffices) they continually approach more closely to the required ratio, in such a way that at length the difference becomes less than any assignable quantity.

Later on, Euler - having read both The Law of Continuity and *Arithmetica Infinitorum* - added to the discussion stating that the functions that could be represented with a single formula to be continuous (Euler (1748/1961)). While this would not be considered to be correct in present day - consider, for example, the function $f(x) = \frac{1}{x}$ - it still aided in moving the subject along.

In the continuation of the discussion of the limit, we see that the definitions draw quite close to each other. If it goes to show that the mathematicians of the time were all in accord on the matter, or if it was more a question of who would win the fame for defining such a key concept, it is hard to tell; although perhaps we ought to consider the possibility that one would prefer the definition that best match the accompanying mathematics that one is writing.

The discussion - or perhaps, the iteration - goes on with yet another definition, this time by d'Alembert in 1765 (D'Alembert (1765/1789)):

Definition 5.2 (Limit)**.** A value is said to be a limit of another value if the latter can approximate the former nearer than any given value, no matter how small it may be supposed, however, without the approximating value being able to exceed the value it approximates; thus, the difference between such value and its limit is absolutely indeterminable.

After this, we see a novel contibution made by Carnot in 1797. (Yushkevich (1986)) He seeks to unite the method of the limit and that of the infinitesimal, arguing that "procedures of both methods became absolutely identical".

Taking part in the contest set forth by by the Berlin Academy of Sciences, originally proposed as a challenge by Lagrange as a way to arithmetizice analysis, all of these definitions and more were insufficient to win the price. The academy stated that

> "...the principle we need must not be limited to calculation of infinitely small values; it must extend to algebra and geometry as well which render after the manner of the ancients" (Yushkevich (1973))

Indeed, it was not until 1786 that the contest was won, by a Swiss mathematician of the name Simon l'Huilier. In his work *Elementary statement of principles of calculus* introduced the symbol $\lim \frac{\Delta P}{\Delta x}$, that was later on used by Lacroix.

The use of the infinitesimal did not sit right with everyone, among them Lagrange, who himself for a long time avoided using them himself. It was not until 1811 that he openly published a different opinion (Lagrange (1811/1853), iv):

> "We have kept the ordinary notation of differential calculus, because it corresponds to the systen of infinitely small quantities, used in this treatise. When one has well understood the essence of the system, and one has convinced oneself of the exactness of ones results by the geometric method by the first and last reason, or through the use of the infinitely small quantities as an safe and convenient instrument to shorten and simplify the proofs: this is how we shorten the proofs of the ancients, through the method of the indivisibles."

In 1821, Baron Augustin-Louis Cauchy publishes *Cours d'analyse*, in which he drafts a definition of *continuity*, using an early version of the *limit.* He starts off his chapter on continuous functions by defining the infinitesimal (Cauchy (1821), 4):

Definition 5.3 (Infinitely small quantities)**.** We say that a variable quantity becomes *infinitely small*, such that its numerical value decreases indefinitely so as to converge towards the limit zero.

With this definition under his belt, he goes on to define the continuous function (Cauchy (1821), 34-35):

Definition 5.4 (Continuous function)**.** Let f(x) be a function of the variable x, and suppose that, for each value of x in between two given limits, this function constantly admits a unique and finite value. If, starting out with a value of x between these limits, we attribute to the variable x an infinitely small increase α, the function itself will, because of this increase, augment with the difference

$$f(x+\alpha) - f(x), \tag{1}$$

that will depend both on the new variable α and the value of x, a *continuous* function of this variable, if for each value of x intermediate between these limits, the numerical value of the difference

$$f(x+\alpha)-f(x) \tag{2}$$

decreases indefinitely with that of α. In other words, *the function f(x) will be continuous in relation to x between the given limits, if between these limits an infinitely small increase in the variable always produces an infinitely small increase in the function itself.*

Whatever ones opinion of the infinitesimal and its preciseness, Cauchy clearly had a mountainous effect on the mathematics to come. It was, in a backwards manner, he who had laid the groundwork for the $\delta-\epsilon$-definition of the limit. In the same text, he draws the following theorem and proof (Cauchy (1821), 48-50):

Theorem 1. If, for increasing values of x, the difference

$$f(x+1)-f(x) \tag{3}$$

converges towards a certain limit k, then the fraction

$$\frac{f(x)}{x} \tag{4}$$

will at the same time converge towards the same limit.

Proof. First off, suppose that the quantity k has a finite value, and we give ϵ a number as small as we want. Since the the increasing numbers of x makes the difference

$$f(x+1)-f(x) \tag{5}$$

converge towards the limit k, we are able to give the number h a value considerable enough that, x being equal to or larger than h, the difference in question stays within the limits

$$k-\epsilon, \quad k+\epsilon. \tag{6}$$

With that said, if we give n any integer value, then all of the quantities

$$f(h+1)-f(h), f(h+2)-f(h+1), \&c...f(h+n)-f(h+n-1), \tag{7}$$

and as a result their arithmetic mean value, as we know,

$$\frac{f(h+n)-f(h)}{n}, \tag{8}$$

will be found between the limits $k-\epsilon$, $k-\epsilon$. Then we'll have that

$$\frac{f(h+n)-f(h)}{n}=k+\alpha, \tag{9}$$

with α being a quantity between the limits of $-\epsilon$ and ϵ. Now let

$$h + n = x. \tag{10}$$

The previous equation then becomes

$$\frac{f(x) - f(h)}{x - h} = k + \alpha, \tag{11}$$

and we will conclude that

$$f(x) = f(h) + (x - h)(k + \alpha), \tag{12}$$

$$\frac{f(x)}{x} = \frac{f(h)}{x} + (1 - \frac{h}{x})(k + \alpha). \tag{13}$$

Furthermore, in order to make x grow indefinitely it suffices to make the integer n grow indefinitely without altering the value of h. Assume, concsequently, that we will consider h to be constant in (13), and x to be a variable that converges towards the limit ∞. The quantities

$$\frac{f(h)}{x}, \quad \frac{h}{x}, \tag{14}$$

contained in the second part, will converge towards the limit zero, and the second part itself towards a limit of the form

$$k + \alpha, \tag{15}$$

with α still between $-\epsilon$ and ϵ. Consequently, the ratio

$$\frac{f(x)}{x} \tag{16}$$

will be contained between $k-\epsilon$ and $k+\epsilon$. This conclusion must hold, regardsless of how small be the number ϵ, which results in that the limit in question will be precisely k. In other words, we will have

$$lim \frac{f(x)}{x} = k = lim[f(x + 1) - f(x)]. \tag{17}$$

$\square$

What we see here is a structure of much resemblance to the $\delta - \epsilon$-version of the limit, but in the work of Cauchy ϵ is just any arbitrarily small number. Nonetheless, this piece must certainly have been some inspiration in the making of the $\delta - \epsilon$-version that we see today, namely that of Weierstrass - but we will get to him shortly.

In 1823, Cauchy's lecture notes from l'Ecole Politechnique were published, from his course in Infinitesimal Calculus, in which he gives a more proper definition of the limit (Cauchy (1823), 1):

Definition 5.5 (Limit)**.** If values attributed to any variable number approximate the value determined so as to finally differ from the latter as small as desired, then these former values are called the limit of all others.

After this definition, he goes on to make a revised definition of the continuous function, with the difference from previously being that he defines it both pointwise and globally, same as we do today. Let us, for study of epsilontics, take a look at the proof that he then gives of the mean value theorem (Cauchy (1823), 27-28).

Theorem 2 (The mean value theorem)**.** Let the function $f(x)$ be continous between the limits $x = x_0$ and $x = X$. Let A be the smallest and B the largest value of the derivative of the function, $f'(x)$, on this interval. Then the quotient of the finite differences

$$\frac{f(X) - f(x_0)}{X - x_0} \tag{18}$$

will necessarily be contained between A and B.

Observation 3. In the following proof, I would for clarity just like to note that the letter i is used as a variable and not as the imaginary unit.

Proof. Let us by δ, ϵ, denote two very small numbers of which the first one be such that for numeric values of i that are less than δ, and for any value of x confined between limits x_0 and X, the relation

$$\frac{f(x+i) - f(x)}{i} \tag{19}$$

will always be greater than $f'(x) - \epsilon$ and smaller than $f'(x) + \epsilon$. Let us introduce $n - 1$ new values of x between aforementioned limits $x = x_0$ and $x = X$

$$x_1, x_2, ..., x_{n-1}, \tag{20}$$

in such a way that the difference $X - x_0$ is divided into the elements

$$x_1 - x_0, x_2 - x_1, ..., X - x_{n-1}, \tag{21}$$

all of which of the same sign and with a numeric value inferior to δ. The fractions

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}, \frac{f(x_2) - f(x_1)}{x_2 - x_1}, ..., \frac{f(X) - f(x_{n-1})}{X - x_{n-1}} \tag{22}$$

will then be contained, the first one between the limits $f'(x_0) - \epsilon$, $f'(x_0) + \epsilon$, the second between the limits $f'(x_1) - \epsilon$, $f'(x_1) + \epsilon$, etc... Furthermore, they will all be superior to the quantity $A - \epsilon$ and inferior to the quantity $B + \epsilon$.

Moreover, with the fractions (22) all having the same sign of the denominator, if we divide the sum of their nominators by the sum of their denominator we obtain a *mean* fraction; in other words it is contained between the smallest and the largest of the those that we are considering (see *l'Analyse algébrique*, note 11, 12th theorem). The expression (19), with which this mean coincides, will thus itself be contained between the limits $A - \epsilon$ and $B + \epsilon$. Hence, since ϵ is as small as we would like, we may conclude that the expression (19) will be between A and B. □

Something of great interest in this proof is the fact that Cauchy uses ϵ and δ, but not in the manner that we do today. It can be argued that he did use epsilontics if one specifically looks at the final sentence of the proof. This does in fact have a certain epsilontic ring to it, but he seems to have made no attempt to explore the role of δ as dependent on ϵ, nor how δ is affected by the difference between x_k and x_{k-1} for $k \in [0, n]$. The relationship between ϵ and δ is pivotal in epsilontics, and hence this proof should be considered to be within the realm of infinitesimal calculus, rather than that of epsilontics. However, there is no denying that this piece of mathematics does start to touch upon the $\epsilon - \delta$-method that is seen in present day.

We move on to the mathematician who is famous for his contribution in epsilontics, Karl Weierstrass. Weierstrass rarely published his own work, his teachings came to be published by his students instead (Sinkevic (2016), 16). It was during his lectures at Königlichen Gewerbeinstitut of Berlin that the $\epsilon - \delta$-method emerged. The earliest report known was recorded by Hermann Schwarz, who was currently a student of Weierstrass' (Dugac (1973), 63-64).

> "By introducing ([A II], 2) the definition of an infinitely small function variable with the use of δ and ϵ, Weierstrass introduces a very important notion which will give to the definition of the limit and of continuity all of the precision and clarity that they have today. Weierstrass thus forms the notion of limit which, until this time, after a decisive step taken by Cauchy, was essentially expressed by saying that when h tends to zero, then $f(x+h) - f(x)$ tends to zero."

Then follows the paramount definition of an infinitely small change set forth by Weierstrass:

Definition 5.6 (Infinitely small change)**.** If it is possible to determine a boundary δ such that for *all* values of h as smaller in absolute value than δ, $f(x+h) - f(x)$ shall be smaller than some quantity ϵ which is as small as we want, then we say that we have made an infinitely small change of the variable correspond to an infinitely small change of the function.

What we see here is a fine display of fully developed epsilontics as we know it today. Dugac calls it "a very important notion that will give to the definitions of limit and of continuity all of the precision and clarity that they have today", and rightly so. It is with a simple application of this definition that he sets forth the definition of a continuous function, because, as we know, a function is continuous in a point x precisely when the function will make only an infinitely small change in a neighborhood of x. Analogously, it is with this definition that he defines the limit that we use still today. Thus, we have now reached the present day use of continuity, the limit and of epsilontics.

5.2 The real numbers

With the appearance of the limit process, it became increasingly obvious that there was a need for what might anachronistically be called a complete set. As the irrational numbers had been known for some time, it was clear that the rational numbers were not in fact complete, a proprty which was necessary for a number of proofs of the time, among others a grand number of those containing the limit. Epple (Epple, Jahnke (1999), 292) describes the change that occurs as

> "[...] the highly complex transition from the traditional concept of quantity to the axiomatic definitions of real numbers and sets."

He goes on to divide the different approaches towards the construction of the real numbers into three camps: one that thought that the real numbers would have to be build upon the intuitive idea of a continuous quantity; one that sought to construct them in a strictly arithmetical manner; and one that thought that the definition should be purely formal. We shall take a look at a few different attempts to construct the aforementioned.

Although several mathematicians had published work about the real numbers before Richard Dedekind did, he might have been the first one to actually make use of them in his teachings (Epple, Jahnke (1999), 297). It was in 1858 he made the discovery, but it was only in 1872 that he published his work *Continuity and Irrational Numbers* (*Stetigkeit und irrazionale Zahlen*), presenting his results publically. In the opening chapter he described the reasoning behind his persuit (Dedekind (1872/1924), 1-2):

> "For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmethic and perfectly rigorous foundation for the principles of infinitesimal analysis. The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given; even the most rigorous expositions of the differential calculus do not base their proofs upon continuity but, with more or less conciousness of the fact, they either appeal to geometric notions or those suggested by geometry, or depend upon theorems which are never established in a purely arithmetic manner."

He goes on to describe his goal with the construction (Dedekind (1924), 9):

> "If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary that the instrument R constructed by the creation of rational numbers be essentially improved by the creation of new numbers such that the domain of numbers shall gain the same completeness, or as we may say at once, the same *continuity*, as the straight line."

Dedekind presented his construction of the real numbers before the beginning students at Zürich Polytechnic, as he had wanted to prove basic theorems of analysis without the use of geometry and found that he could not do it without the real numbers at hand (Epple, Jahnke (1999)). He took inspiration from Eudoxos' definition of the proportionality of four line segments, as he told Lipschitz in a letter (Lipschitz (1986)).

In his publication of 1872, he presents the thought process that brings him to the so-called Dedekind cuts. He starts out by stating that since there exist irrational numbers, infinitely many even, the domain R of rational numbers is discontinuous (Dedekind (1872/1924), 9-13). Thus, if one were only to fill in its gaps, the line would be rendered continuous and complete. He then goes on with the now famous Dedekind cuts.

> "[...] every rational number a effects a separation of the system R into two classes such that every number a_1 of the first class A_1 is smaller than every number a_2 of the second class A_2; the number a is either the greatest number of the class A_1 or the least number of the class A_2. If now any separation of the system R into two classes A_1, A_2, is given which possesses only *this* characteristic property that every number a_1 in A_1 is less than every number a_2 in A_2, then for brevity we shall call such a separation a *cut* [Schnitt] and designate it by (A_1, A_2)."

This is all fine and well, but what does this definition entail? What are the properties of these classes and of the number a? The investigation follows.

Now, for every irrational number a we say that it creates, or that it corresponds to, the cut (A_1, A_2) that it produces. Furthermore, we say that two cuts are equal if and only if they respond to essentially the same cuts. Dedekind goes on to explore the orderly arrangement of the so-called *real* numbers (Dedekind (1872/1924), 15-19.

Let (A_1, A_2) be the cut corresponding to any one number α and (B_1, B_2) be the cut corresponding to any one number β. For a number a_1 contained in A_1, the cut has the property that all other numbers that are smaller than a_1 also lie in A_1, and the converse is true for the second class: if $a_2 \in A_2$, then $a_k \in A_2 \ \forall a_k > a_2$.

Dedekind makes a comparison between the first classes A_1 and B_1 and finds the following possibilites:

1. They are perfectly identical; i.e. every element in A_1 is also in B_1 and vice versa. It also means that $\alpha = \beta$.

2. They differ by a single element. Let $a'_1 = b'_2$ be the only number that is contained in A_1 but not in B_1, and which is consequently found in B_2. Then every number that is smaller than a'_1 is contained in B_1, and all other numbers in A_1 are smaller than a'_1 as well.

   Thus, the cut (A_1, A_2) is produced by the rational number $\alpha = a'_1 = b'_2$, with α contained in A_1. However, since every number smaller than a'_1 is contained in B_1 and every number larger than a'_1 is contained in B_2, we know that $\alpha = a'_1 = b'_2 = \beta$. Thus a'_1 defines the cut (B_1, B_2) as well but is not contained in B_1, so a'_1 is the least number contained in B_2. Dedekind describes this as the two cuts being only unessentially different.

3. They differ by at least two elements. Now, let us say that there exist two disjoint numbers $a'_1 = b'_2$ and $a''_1 = b''_2$ such that $a'_1, a''_1 \in A_1$ and $b'_1, b''_1 \in B_2$. Then there are infinitately many numbers by which the cuts differ, as there are infinitely many numbers between a'_1 and a''_1 and between b'_1 and b''_1. The two cuts are essentially different and we say that α and β are *different*.

He makes two more scenarios which are analogous to possibilities 2 and 3, that I choose not to include.

It follows that out of two different (rational or irrational) numbers, one is necessarily greater, and the other is necessarily less, with no possibility of a third option. Thus, the real numbers have the same orderly structure as the rationals do.

Dedekind establishes continuity of the real numbers using the following theorem (Dedekind (1872/1924), 19-21):

Theorem 4. If the system $\mathbb{R}$ of all real numbers break up into two classes U_1, U_2 such that every number a_1 of the class U_1 is less than every number a_2 of the class U_2 then there exists one and only one number α by which this separation is produced.

Proof. Let (U_1, U_2) be the cut in $\mathbb{R}$ produced by α, and let (A_1, A_2) be the cut such that A_1 consists of all of the rational numbers in U_1 and A_2 consists of all the rational numbers in U_2.

Now, if $\beta \in \mathbb{R}$ is different from α, then there are infinitely many rational numbers c lying between the two. If $\beta < \alpha$, then $c < \alpha$, so in that case c is in U_1, as well as in A_1. Since $\beta < c$, we also have that $\beta \in U_1$.

If we instead have that $\beta > \alpha$, then $c > \alpha$ and thus c lies in A_2 and in U_2, while $\beta \in U_2$.

Thus, we have shown that for any real number $\beta \neq \alpha$, β is contained in either U_1 or in U_2, depending on if it is greater than or smaller than α. This means that α is either the greatest number contained in U_1 or the least number contained in U_2, so α is the only possible number by which the cut (U_1, U_2) can be made. $\square$

Having demonstrated this property of the real numbers, he goes on to reveal that arithmetic operations upon these numbers work much in the same way as those over the rational numbers, and he does so with what a startling display of rigour and thought given to detail, always building upon previous statements and definitions. He does not use ambiguous expressions, and when he does he explains what he means. This goes to show that Dedekind played a key part in the increase in rigour in that time, as his texts would even pass the present day norm of completeness. Here is an example to illustrate (Dedekind (1872/1924), 23):

> "It is easy to see that it all reduces to showing that the arithmetic operations possess a certain continuity. What I mean by this statement may be expressed in the form of a general theorem:
>
> **Theorem 5.** 'If the number λ is the result of an operation performed on the numbers $\alpha, \beta, \gamma, ...$ and λ lies within the interval L, then intervals $A, B, C, ...$ can be taken within which lie the numbers $\alpha, \beta, \gamma, ...$ such that the results of the same operation in which the numbers $\alpha, \beta, \gamma, ...$ are replaced by arbitrary numbers of the interval $A, B, C, ...$ is always a number lying within the interval L.'"

Another early bird in the area of the real numbers was Hankel, although his approach was not at all as rigorous as that of Dedekind, most notably in that he used the concept of completeness without specifying what he meant by it (Epple, Jahnke, 294-295). Hankel belonged distinctly to the camp of mathematicians who sought to define the real numbers in a strictly formal sense, and he viewed number systems as systems of signs and operations. For those not familiar with these terms, here are some brief introductions.

Definition 5.7 (Sign, element)**.** A sign, or an element, of a set is any one of the distinct objects belonging to that set.

Example 5.1. The number 1 is a sign of the set of natural numbers.

Definition 5.8 (Operation on a set S)**.** An n-ary operation * from $S \times S \times S \times ... \times S$ [n times] to S is a function *: $S \times S \times S \times ... \times S \to S$ for $a, b, c, ..., n \in S$. We may write the operation over $a, b, c, ..., n$ as $*(a, b, c, ..., n)$.

Example 5.2. Addition is a binary operation $+\colon \mathbb{Q} \to \mathbb{Q}$, which sends $(a, b) \longmapsto c$, for $a, b, c \in \mathbb{Q}$.

Although Hankel had a formal view of the real numbers, his construction holds a rather intuitive edge to it, that of completeness. The general idea is that any element of this set of numbers should be obtainable from certain basic signs, the "units", using only the predecided operations of the system as many times as necessary. He also demands that the system by closed under said operations, but here we need another definition.

Definition 5.9 (Closed under an operation)**.** Let S be a set equipped with an operation *. A subset $X \subset S$ is said to be closed under the operation * if for all $x_1, x_2, ..., x_n$ contained in X, $*(x_1, x_2, ..., x_n)$ is contained in X as well.

Example 5.3. The natural numbers are closed under addition, as any two natural numbers added together will be a natural number. The converse is not true: the natural numbers are not closed under subtraction. For example, $2 - 5 \notin \mathbb{N}$.

As a first step in this construction, Hankel requires that this set be closed under the four arithmetic operations, in other words under addition, subtraction, multiplication and division. He gives a recursive definition of addition and multiplication and in using that he proves the associative, commutative and distributive laws of the natural numbers. When he then introduces subtraction and division, though, the natural numbers are not closed under these operations any longer and Hankel must add signs for negative numbers and for fractions. He thus arrives at the rational numbers, but this is nothing new at that time.

What shakes things up is the question he now asks: is this number system complete or not? It is here that he most obviously lacks rigour as we would consider it in present day, as this question in the twentyfirst century would require a definition of the term.

The idea, however, is that there might exist "higher" operations than the four arithmetic ones, and Hankel requires that the system be complete under these as well.

> "One wonders whether the number system that we have created is complete or not. It is certainly complete insofar as there are no tasks from the four operations that cannot be solved with such a sign. On the other hand, there are tasks for which one cannot find a solution within it [the number system], for example when one seeks x such that $xx = 2$ there is no number to be found, nor when $xx = -1$." (Hankel (1867), 45)

While Hankel understood that this problem was related to the irrational numbers, he did not believe that it was possible to solve formally, as it would be impossible to survey every single operation of the sort.

Hankel did not get much further in his considerations of the topic, as he retreated to the more intuitive notion of the real numbers. Epple makes the argument that Hankel was bound to traditional views on continuous quantities, even though he did take considerable steps outside of those views (Epple, Jahnke (1999), 295). This was in spite of the fact that his teacher, Weierstrass, had already abandoned this approach. As Dedekind did not publish his ideas until a number of years later, Hankel could not make use of Dedekind's work either, and so it was rather his fate to ask some foundational questions for later mathematicians to dive into.

Some would consider Weierstrass to be one of the pioneers of the latter half of the rigorization of analysis, and his notion of the number systems surely contributed to this idea (Lützen, Jahnke (1999), 156). Weierstrass was of the idea that the real numbers should be based on arithmetic, and so this was what he attempted to do (Epple, Jahnke (1999), 295-296). It was his view that numbers were to be considered "aggregates" of specific elements: for example, the positive integers he called those "things identical in thought", while the positive rational numbers were aggregates made from the basic unit - the number 1 - and what he called "exact parts" of them, i.e. some rational number $\frac{1}{a}$, $a \in \mathbb{N}$. Irrational numbers, or "arbitrary number quantities" as he called them, were instead infinite aggregates consisting of the same kind of elements, in other words they could be viewed as infinite series.

These definitions, understandable as they might be, do not mention the case of infinite aggregates - and by this I specifically mean infinitely large numbers - save in the case of irrational numbers, which would not hold up very well with present day norms of rigour. Perhaps he avoided this kind of speculation to stay close to the more traditional view of the number systems. Either way, these definitions were pivitol in his course, as they enabled him to give rigorous proofs of theorems in the area of theory of analytic functions. Epple describes Weierstrass' way of thinking quite nicely (Epple, Jahnke (1999), 296):

> "Weierstrass's approach reduced the concept of (real and hence also complex) quantity to that of a number. Nominally, Weierstrass continued to use the notion of quantity, but expressions like 'arithmetical quantity' or 'number quantity' made clear what he had in mind: a logical separation of his concepts from their more intuitive counterparts in geometry or physics."

In the lecture notes published in 1878, Weierstrass makes his first definition of the real numbers; and it is precisely a definition, not a construction. Already having defined the irrational numbers as infinite aggregates, he defines the real numbers as the number system for which $(a - b)$ for $a > b$ is defined, for some numbers a and b "with infinitely many elements" ["mit unedlich vielen Elementen" (Weierstrass (1878), 15)]. It is unclear why he does not simply define it as the number system where some number a with infinitely many elements is defined, instead of the difference between two such numbers. Either way, he goes on to prove that the difference does exist in this new number system $\mathbb{R}$, by investigating the expression arithmetically and concluding that it makes sense.

With the knowledge of the different number systems, he manages to make further discoveries. One of these is the proof that he writes on the Bolzano-Weierstrass theorem, as well as the supremum principle that he proves concurrently (Weierstrass (1878), 83-87). It should be noted, however, that the supremum principle follows directly from Dedekind's construction of the real numbers, so with the knowledge of that, Weierstrass' proof is entirely superfluous. Be that as it may, we return to Weierstrass. In order to properly understand his intent we must first delve into his idea of a set, or a domain (Weierstrass (1878), 83-84).

> "An infinitely variable real quantity is one that can take any value between $-\infty$ and $+\infty$; all points of a straight line represent the domain of such a variable. We now imagine a union of infinitely variable quantities. (The following only deals with real quantities; everything that follows can easily be transferred to complex variables.) Each specific system of variables is called a point ["Stelle"] in the domain of variables.
>
> If $x_1, x_2, ..., x_n$ are the variables and $a_1, a_2, ..., a_n$ are points in their neighborhood ["Gebiete"] - which is to be understood in such a way that $x_1 = a_1, x_2 = a_2, ..., x_n = a_n$ is in the value system - then if $|x'_1 - a_1| < \delta, |x'_2 - a_2| < \delta, ..., |x'_n - a_n| < \delta$, then $x'_1, x'_2, ..., x'_n$ is a point in the vicinity δ of the point $a_1, a_2, ..., a_n$. x'_v is between $a_v + \delta$ and $a_v - \delta$.
>
> In the domain of an infinitely variable x, let an infinite number of points be defined in some way; the collection of these points is denoted by x'. Then the x' can be represented either by discretely or by continuously following points of a straight line - in the latter case they are said to form a continuum. This is to be defined analytically as follows: If a is a point in the defined domain x', and all points of this domain in a sufficiently small neighborhood of a are in the domain x', then [the values of] x' form a continuum.
>
> 
>
>
> In a vicinity of a point a of a domain x' there is another point a_1, such that all points of the interval a to a_1 belong to the domain x'; a_2 has the same property with respect to a_1 as a_1 with respect to a;

> as well as a_3 to a_2, a_4 to a_3,...,a_n to a_{n-1} do. Then we say that a continuous transition from a_1 to a_n is possible.
>
> Let a and b be such that there is no continuous transition between the two in the vicinity of x', but such that a whole continuum of values of x' belongs to a as well as to b. Then there exists a place in which continuous transitions from one point to another are possible, from one or more separate continuous bits. What is meant by the limits of a continuous piece is immediately clear.
>
> All of this can without difficulty be transferred to a domain of n variables - over an n-fold manifold. For $n = 3$ it can even be geometrically illustrated what a contintuous transition from one point to another means. The possibility of transitioning continuously from one point to another also implies that it is also possible from the latter to the former.
>
> 
>
>
> This is not obvious as there might for example be a neighborhood of x_1 that contains x_2 but not conversely a neighborhood of x_2 that contains x_1, because the neighborhood of x_2 must not be larger than $\overline{x_2 b}$. But between x_1 and x_2 one only needs to insert points which form such intervals between x_1 and x_2 which are less than or equal to $\overline{x_2 b}$ in order to show that a continuous transition is also possible from b to a if it was from a to b."

Now we are ready for Weierstrass' take on the supremum principle and Bolzano-Weierstrass theorem, after only a brief definition.

Definition 5.10 (Upper and lower limit)**.** g is called the upper limit of a variable quantity if there is no value of the variable greater than g, and if, no matter how small the value of δ, the interval $g - \delta...g$ contains points ["Stellen"] that are in the domain of the variable.

g' is called the lower limit if there is no value of the variable less than g' and if in every interval $g'...g' + \delta$, no matter how small, there are points in the domain.

Whether g and g' themselves belong to the domain or not is immaterial. (g can equal ∞ and g' can equal $-\infty$.)

Now on to the supremum principle and the Bolzano-Weierstrass theorem. Here they both are, with only slight variations made by me to make them a little more easily digestable, and a number of observations throughout.

Theorem 6. Every region of variable magnitude has an upper and a lower limit.

Proof. We assume that the variable is capable of only positive values and cannot equal ∞. The general case can then easily be reduced to this special one. We proceed the proof of our theorem with the following:

Let $a_0, a_1, a_2, ...$ be a sequence ["Reihe"] of numbers that do not decrease and that are smaller than a specifiable (finite) quantity g. Then we form the numbers

$$\begin{aligned} b_1 =& a_1 - a_0, \\ b_2 =& a_2 - a_1, \\ &\vdots \\ b_v =& a_v - a_{v-1} \end{aligned}$$

then $b = b_1 + b_2 + b_3 + ...$ is a finite number.

Observation 7. The word used by Weierstrass to describe what I interpret to be a sequence, is *Reihe*, which means series. However, as the sequence clearly is not a sum, I will correct this error throughout my translation.

The sum of any number of elements of the sequence $b_1,\ b_2,\ b_3, ...$ of which b_n has the highest index is less than or equal to $a_n - a_0$, which in turn is certainly less than g. Thus, $\sum_{i=1}^{\infty} b_i$ is finite.

Observation 8. In this argument, Weierstrass utilises the fact that a bounded series of increasing real numbers is convergent, and furthermore that it has its limit as its supremum.

Let x' be a point in our region and let a be a positive integer. I now consider the number sequence $\frac{1}{a}, \frac{2}{a}, \frac{3}{a}, ...$
We have assumed that x' is consistenty greater than 0 and less than G, where G denotes a positive number.

Observation 9. To clarify, as I think that this piece is not as obvious as it could have been, x' is in the set b_i, and is thus smaller than some fixed positive number, as well as always positive since $a_i - a_{i-1}$ is greater than or equal to zero for all i.

In the above number sequence we therefore arrive at a first term that is greater than or equal to G and therefore also greater than any value that x' can assume.

In the interval $\frac{a_1}{a}, ..., \frac{a_1+1}{a}$ there must then necessarily be one or more points that belong to the region, if we by $\frac{a_1+1}{a}$ mean the first element of the number sequence that exceeds all the values permissible for x' in size. In this way, for every number a there is a number a_1. Let the numbers $a, a^2, a^3, a^4, ..., a^n, ...$ belong to the members of the sequence $a_1, a_2, a_3, ..., a_n, ...$ so that in each of the intervals

$$\frac{a_1}{a}, ..., \frac{a_1+1}{a}$$
$$\frac{a_2}{a^2}, ..., \frac{a_2+1}{a^2}$$
$$\frac{a_3}{a^3}, ..., \frac{a_3+1}{a^3}$$
$$\vdots$$
$$\frac{a_n}{a^n}, ..., \frac{a_n+1}{a^n}$$

there is at least one point of the region x'.

We now want to show that in the sequence $\frac{a_1}{a}, \frac{a_2}{a^2}, \frac{a_3}{a^3}, ..., \frac{a_n}{a^n}, ...$ each term is greater than the previous one. To do this, we decompose the interval $\frac{a_n}{a^n}, ..., \frac{a_n+1}{a^n}$ into the sequence of intervals:

$$\frac{a_n}{a^n} ... \frac{a_n}{a^n} + \frac{1}{a^{n+1}},$$
$$\frac{a_n}{a^n} + \frac{1}{a^{n+1}} ... \frac{a_n}{a^n} + \frac{2}{a^{n+1}},$$
$$\vdots$$
$$\frac{a_n}{a^n} + \frac{a_n - 1}{a^{n+1}} ... \frac{a_n}{a^n} + \frac{a}{a^{n+1}}$$

Observation 10. Note that $\frac{a_n+1}{a^n} = \frac{a_n}{a^n} + \frac{a}{a^n+1}$.

In at least one of these intervals there must be places of the defined kind (points x') , since there were places in the interval $\frac{a_n}{a^n} ... \frac{a_n+1}{a^n}$. Let the interval

$$\frac{a_n}{a^n} + \frac{m-1}{a^{n+1}} ... \frac{a_n}{a^n} + \frac{m}{a^{n+1}}$$

be the last of the intervals that contains x'. But the interval

$$\frac{a \cdot a_n + (m-1)}{a^{n+1}} ... \frac{a \cdot a_n + m}{a^{n+1}}$$

is identical to the interval

$$\frac{a_{n+1}}{a^{n+1}} ... \frac{a_{n+1}+1}{a^{n+1}},$$

so $a_{n+1} = a \cdot a_n + m - 1$, where $\frac{a_{n+1}}{a^{n+1}} \geq \frac{a_n}{a^n}$, as was claimed. Now let's form the differences

$$\begin{aligned} b_0 =& \frac{a_1}{a}, \\ b_1 =& \frac{a_2}{a^2} - \frac{a_1}{a}, \\ b_2 =& \frac{a_3}{a^3} - \frac{a_2}{a^2}, \\ & \vdots \end{aligned}$$

and the sum

$$b = b_0 + b_1 + b_2 + ... = \frac{a_1}{a} + \frac{m_1 - 1}{a^2} + \frac{m_2 - 1}{a^3} + ...$$

Then b is the upper limit of the region x'.

b is first and foremost a finite quantity. The sum of the n first elements of b is $\frac{a_n}{a^n}$, so $b > \frac{a_n}{a^n}$ (all elements of b are positive numbers); but $b \leq \frac{a_n+1}{a^n}$. There can be no value of x' greater than b; for by choosing n sufficiently large, b can be brought as close to $\frac{a_n+1}{a^n}$ as one wants, and there is no value of x' greater than $\frac{a_n+1}{a^n}$.

Furthermore, since b always lies between $\frac{a_n}{a^n}$ and $\frac{a_n+1}{a^n}$ but at least one value of x' lies between these limits and, as just shown, no value of x' is greater, then it follows that between b and $b - \delta$ there is always at least one value x' (δ is any value, no matter how small). □

Theorem 11. In every discrete domain of a manifold [”Mannigfaltigkeit”][i.e. in $\mathbb{R}^n$] that contains an infinite number of points, there is at least one point that is distinguished by the fact that every neighborhood, no matter how small, contains an infinite number of points.

Observation 12. The term ”discrete” [”discreten”] that Weierstrass uses is not intended in the way we would use it now. In present day, ”discrete” would refer to the property of points being isolated from one another, while Weierstrass rather means a union of intervals.

He clarifies with an example:

Example 5.4. If $a_0 + a_1 + a_2 + ...$ to infinity is a convergent sequence, then the magnitudes $s_0, s_1, s_2, s_3, s_4, ...$, where $s_n = \sum_{i=0}^{n} a_i$, form a discrete domain of infinitely many points. The point in the vicinity of which, no matter how small, there are infinitely many other places in the region, is here s, the sum of the sequence. Namely $s - s_n < \delta$ or $s - \delta < s_n$, if δ is an arbitrarily small chosen quantity and, after choosing δ, n is taken greater than a certain number. So there are between s and $s - \delta$, no matter how small δ is, infinitely many quantities s_n.

A noteworhty difference that we can see between this example and the not-quite epsilontics of Cauchy, is that Weierstrass explicitly states that the number n is chosen *after* the number δ is. Although the difference may look insignificant at a first glance, it marks a stark contrast between the mathematics of the past and that of the present.

Then follows the proof.

Proof. To prove our theorem, we first assume that the defined points are contained between two boundaries g_0 and g_1. Let a be any integer.

Observation 13. In his following argument, Weierstrass assumes that a is in fact a positive integer. The reasoning is correct either way, as one would simply change the sign of every element in the following sequence if a were negative, but it might not pass the present day standard of rigour.

We form the sequence

$$-\frac{m}{a}, -\frac{m-1}{a}, -\frac{m-2}{a}, ..., -\frac{1}{a}, 0, \frac{1}{a}, \frac{2}{a}, ..., \frac{n-2}{a}, \frac{n-1}{a}, \frac{n}{a}. \tag{23}$$

The sequence is continued to the left so far that $-\frac{m}{a} < g_0$, and to the right so far that $\frac{n}{a} > g_1$. Let us now consider all of the intervals of the sequence, $\frac{\mu}{a} ... \frac{\mu+1}{a}$, of which there are only a finite number, since (23) only has a finite number of members. It is clear that there must be at least one among them within which there are an infinite number of points; because there are infinitely many points defined and only finitely many intervals.

Observation 14. Weierstrass assumes that $-\frac{m}{a} < 0 < \frac{n}{a}$, without making any sort of statement of m or n beforehand. It is assumed that $m, n > 0$.

Let $\frac{\mu_1}{a}, ..., \frac{\mu_1+1}{a}$ be the first interval that includes infinitely many points of the sequence; all of the previous intervals contain only finite amounts of isolated points. We may now assign a specific number μ_i to every given number a_i, such that $\frac{\mu_i}{a_i}, ..., \frac{\mu_i+1}{a_i}$ is the first interval with the aforementioned property.

Let $a_i = a^i$. We form the sequence of intervals

$$\begin{gathered} \frac{\mu_1}{a} ... \frac{\mu_1+1}{a} \\ \frac{\mu_2}{a^2} ... \frac{\mu_2+1}{a^2} \\ \vdots \\ \frac{\mu_n}{a^n} ... \frac{\mu_n+1}{a^n} \\ \frac{\mu_{n+1}}{a^{n+1}} ... \frac{\mu_{n+1}+1}{a^{n+1}} \end{gathered}$$

of which we know that each contains infinitely many points in the domain and that below the lower limit of each interval there are only a finite number of points in the domain. From this we can conclude that

$$\frac{\mu_{n+1}+1}{a^{n+1}} > \frac{\mu_n}{a^n} \tag{24}$$

$$\frac{\mu_{n+1}}{a^{n+1}} < \frac{\mu_n+1}{a^n} \tag{25}$$

so we have that

$$\mu_{n+1} + 1 > a \cdot \mu_n, \tag{26}$$

$$\mu_{n+1} < a \cdot \mu_n + a. \tag{27}$$

From (26) we can deduce - since 1 is a unit - that $\mu_{n+1} \geq a \cdot \mu_n$ and thus that $\frac{\mu_{n+1}}{a^{n+1}} \geq \frac{\mu_n}{a^n}$.

Observation 15. Here it is revealed that μ_i is meant to be an integer for all i, even though it was never clearly stated.

The numbers $\frac{\mu_1}{a}, \frac{\mu_2}{a^2}, \frac{\mu_3}{a^3}...$ thus form a sequence of increasing quantities, non of which exceeds the limit over which none of the area can be found. Therefore

$$A = \frac{\mu_1}{a} + \frac{\mu_2 - a\mu_1}{a^2} + \frac{\mu_3 - a\mu_2}{a^3} ... \tag{28}$$

is a finite quantity and I claim that A is such a point in the vicinity of which there are infinitely many points in the domain. For every neighborhood $A - \delta ... A + \delta$

of A, by choosing r sufficiently large, one can find intervals $\frac{\mu_r}{a^r}...\frac{\mu_r+1}{a^r}$ which lie entirely within the interval $A-\delta...A+\delta$ such that, since between $\frac{\mu_r}{a^r}...\frac{\mu_r+1}{a^r}$ there are infinitely many places in the area, this also applies to the interval $A-\delta...A+\delta$. The quantity A is a perfectly de finite one. For example, choosing $a = 10$, we get A by (28) on this page in the form of a decimal. □

Although Weierstrass does not explicitly state that he is working over the real numbers, seeing as he had made the definition in the same work I assume that it was implied. As we now know, not all Cauchy sequences converge in $\mathbb{Q}$. Of course, we must also note that this particular proof is written specifically for $\mathbb{R}$, while the general theorem is for $\mathbb{R}^n$.

The final persons that I would like to mention on the topic are Georg Cantor and Eduard Heine. The two were colleueges in Halle and each made his own construction of the real numbers, both however basing it on infinite sets of rationals (Epple, Jahnke (1999), 299). Furthermore, in doing so they strengthened Dedekind's resolve to make his own construction: In the opening chapter of *Continuity and irrational numbers*, Dedekind gives his thanks to both of the aforementioned mathematicians and clearly affirms that they inspired him to make his construction (Dedekind (1872/1924), 3).

Though it was Cantor who made the construction that we will familiarise ourselves with, for accessability reasons it is through the publication of Heine that we will do so. He starts off his first chapter with an important definition (Heine (1872), 174).

Definition 5.11 (Number series)**.** A series of numbers $a_1, a_2,$ etc., $a_n,$ etc., is called a *number series* if for every given non-zero number η, no matter how small, there exists a value n such that $a_n - a_{n+v}$ is less than η for all positive v.

Observation 16. In present day, this is called a *Cauchy sequence.*

Observation 17. It is understood that $a_n - a_{n+v}$ is positive, as the definition would make little sense otherwise. It would be more correct to use the absolute value of the difference, rather than the difference itself.

He will use these so called *number series*, as we too shall call them throughout this next bit, in his construction of the real numbers, and in order to do so he must first investigate how they work arithmetically. He remarks that he will use the word *number* exclusively to convey *rational number*, before he moves on to the next definition.

Definition 5.12 (Elementary series)**.** Every number series in which the numbers a_n, with increasing index n, are less than any given size, is called an elementary series.

Theorem 18. If a_1, a_2, etc. and b_1, b_2, etc. are both number series, then $a_1 + b_1, a_2 + b_2$, etc. and $a_1 b_1, a_2 b_2$, etc. are as well.

Proof. We have that

$$(a_n \pm b_n) - (a_{n+v} \pm b_{n+v}) = (a_n - a_{n+v}) \pm (b_n - b_{n+v}), \tag{29}$$

This expression becomes arbitrarily small with increasing values of n, since the a and the b form number series, i.e. $a_n - a_{n+v}$ and $b_n - b_{n+v}$ become arbitrarily small with increasing values of n.

The same applies to

$$a_n b_n - a_{n+v} b_{n+v} = a_n (b_n - b_{n+v}) + b_{n+v}(a_n - a_{n+v}), \tag{30}$$

since a_n and b_{n+v} remain under a finite value. □

Theorem 19. Under the assumptions of the first theorem, and if, in addition, the a do not form an elementary series, then

$$\frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \ldots \tag{31}$$

is a number series.

Proof. We have that

$$\frac{b_n}{a_n} - \frac{b_{n+v}}{a_{n+v}} = \frac{b_n a_{n+v} - a_n b_{n+v}}{a_n a_{n+v}} = \frac{b_n (a_{n+v} - a_n) + a_n (b_n - b_{n+v})}{a_n a_{n+v}}. \tag{32}$$

Since the numerator of the expression on the right-hand side becomes arbitrarily small as n increases, but the denominator remains above zero, the left-hand side also becomes arbitrarily small as n increases. □

Definition 5.13 (Equal number series)**.** The number series a_1, a_2, etc. and b_1, b_2, etc. is called equal if and only if $a_1 - b_1, a_2 - b_2$, etc. is an elementary number series.

Heine introduces the *number sign* of the number series, which he will later use to construct the real numbers (Heine (1872), 176-177).

Definition 5.14 (Number sign)**.** Add a sign to every number series. The series itself is introduced as a sign, placed in brackets, so that for example the series a, b, c, etc. belong to the sign $[a, b, c, \text{etc.}]$. A *number sign* is the sign belonging to a number series.

Remark. The number sign that belongs to a number series that only contains the same term a is the rational number a itself. Moreover, the corresponding capital letter can also be taken as a number sign belonging to a number series whose members are formed with the same small letters, hence as a sign of $[a_1, a_2, \text{etc.}]$ that would be A, from $[\eta_1, \eta_2, \text{etc.}]$ it would be H.

Definition 5.15. We say that $A > B$ when $a_n - b_n$, for a certain value of n onwards, results in a positive difference, and that $A < B$ when it is negative.

Theorem 20. The sign of the two series

$$b_1, b_2, b_3, ...; \tag{33}$$

$$a_1, a_2, \text{etc.}, a_\rho, b_\mu, b_{\mu+1}, b_{\mu+2}, \text{etc.}, \tag{34}$$

are equal.

Proof. The two series are equal, since the series of the difference

$$a_1 - b_1, a_2 - b_2, \text{etc.}, a_\rho - b_\rho, b_\mu - b_{\rho+1}, b_{\mu+2} - b_{\rho+2}, \text{etc.} \tag{35}$$

is an elementary series. □

Remark. A number sign remains unchanged if one omits any finite number of terms from the series to which it belongs.

He actually does not explicitly define the real numbers, or at least not as the big revelation that it actually is. Instead, he makes these two definitions (Heine (1872), 178):

Definition 5.16 (Numerical value)**.** The *numerical value*, or the *absolute value*, of a sign A is the sign that one obtains when one substitutes a for its value in the number series.

Definition 5.17 (Limit of a number series)**.** If for every (rational) number series a_1, a_2, etc. there is a (rational) number $\mathcal{U}$ such that $\mathcal{U} - a_n$ falls below any given value as n increases, then $\mathcal{U}$ is called the limit of a.

Thereafter, he casually writes

> "If this is the case for every D, then it is also the case for every rational number d, since a rational number is a special case of the number sign." (Heine (1872), 179)

To summerise the construction of Heine (which originates from Cantor) with a more current vocabulary, he starts off defining a Cauchy sequence, then he investigates the properties of these sequences arithmetically, and he defines the limit $\mathcal{U}$ of this sequence. As we know, not all Cauchy sequences converge in $\mathbb{Q}$, but the extention of $\mathbb{Q}$ that includes the limit of all Cauchy sequences is in fact $\mathbb{R}$. Heine even goes so far as to say that these limits, $\mathcal{U}$, are said to be irrational even if they in special cases are rational (Heine (1872), 180). This construction is thus made to complete the rational numbers, simply by using Cauchy sequences.

6 The societal context

Mathematics was not alone in going through large changes during the 19th century. It is important to understand that mathematics does not, and never has, existed in a vacuum, and it has always been affected by aspects such as culture, economics and politics. Although there is no use trying to include all influencial events of the century, some things ought to be highlighted for playing a pivitol role in the development of mathematics.

Although many things occured during this century, much of it is not relevent for the emergence of rigour, as most of that took place in France and Prussia - later Germany - with bits and pieces from other countries such as Italy. In fact, Lützen puts it like this (Lützen, Jahkne (1999), 156):

> "One can divide the rigourisation of analysis into two periods: a French period dominated by Cauchy and a German period dominated by Weierstrass."

In part, as most of these changes occured before the diffusion of electricity in society - and thus communications were still slow - and in part because of the status of certain geografical areas, the more relevant history only takes place in a few European countries. In fact, in the case of Bolzano, who had made significant progress in analysis in the beginning of the 1800s, his work went entirely under the radar as he was from an insignificant corner of the world - Czech Republic.

Thus, we begin our historical overview in the United Kingdom. In the latter half of the 1700s, the industrial revolution emerges, changing the life of the average British worker, as well as of society overall (Nationalencyklopedin (n.a.), *Industriella revolutionen*).

> "The industrial revolution marked a decisive turning point in the development of society and it initiated modern growth. In that sense, the industrial revolution is not just a particular historical process in the United Kingdom, but a valid concept in general. The change means that capital replaces labour within industrial production, that knowledge and science are being used systematically, that production is focused on mass consumption, that the proportion of factory workers increases, that urbanisation increases and that new social conditions are created."

When industrialisation spread to other parts of Europe during the first half of the 19th century, it brought said changes with it. With it came the expansion of the railway system as well. These types of technological advances greatly facilitated the ease of communication and collaboration between scientists, as M.J. Peterson argues in their paper *Roots of Interconnection: Communications, Transportation and Phases of the Industrial Revolution* (Peterson, M.J. (2008), 1,3).

As well as the industrial revolution, the 19th century saw many powerful political changes. There was the unification of Italy in 1861 and that of Germany in 1871; there was the rise of nationalism and there was the wave of revolutions that ran through Europe (Nationalencyklopedin, *Italien* (n.a.); Nationalencyklopedin, *Tyskland* (n.a.); Nationalencyklopedin, *Nationalism*, (n.a.), Britannica (2022), *Revolutions of 1848*). All of these played an important role in the general development in society, and mathematics was no exception.

> “Among the many lasting consequences of the French Revolution was its effect on higher education. Although the École Polytechnique was and is very different from a modern university, its creation marks the decline of the learned academy as a central focus for research, and the start of the system of high-level teaching coupled with the production of new knowledge.“ (Gray (2008), 39)

As Gray explains, the French revolution brought upon a shift in the work of a mathematician, which now came to include teaching. So did the Franco-Prussian war, and coupled with nationalism and a few other favourable conditions, the mathematical societies were born (Gray (2008), 36).

> ”The late nineteenth century also saw the creation of many specialist learned societies. The first of these was the London Mathematical Society, founded in 1865. The Société Mathématique de France (the French Mathematical Society) was founded in 1872 as a response to the disaster of the Franco-Prussian War (with the geometer Michel Chasles as its first president); the Deutsche Mathematiker-Vereinigung, or DMV (the German Mathematical Society) was founded, after earlier attempts had failed, in 1890 with Georg Cantor as its first president. These brought with them new, specialist journals, and, in the French case, a renewed commitment to top-level research, largely seen as catching up with the Germans.”

Furthermore, while in the past mathematicians had often been attached to royal courts, after the revolution teaching was the way in which most of them made their living (Kleiner (1991), 297-298). The financial possibility that a salary provided held new opportunities for many.

> ”They capitalized on, and helped to strengthen, a feeling that had been growing with the nineteenth century that research was a duty of a professor, a feeling that animated both the sciences and the humanities. By providing outlets for the publication of research, the societies helped advance the individual researchers, the universities for whom they worked, and, it was implied, the intellectual needs of the nation” (Gray (2008), 36).

It became custom for a professor to publish his - it was almost always a man - lecture notes for his students, in the case where the content of the course was not entirely standard; in other words when the same content could not readily be found at the library (Kleiner (1991), 298). Kleiner argues that teachers likely thought their work through more closely when they wrote for their students than when they wrote for their colleagues. Furthermore, it lead to several works being made available - in fact, that is how the prodigious *Cours d'analyse* came to be.

One final thing that should be mentioned is the change of mentality in mathematical circles during this time. As Lakatos argues at length in *Proofs and refutations*, a mindset shift was necessary to at all begin the rigorisation process (Lakatos (1976), 147):

> "As long as a counterexample was a blemish not only to a theorem but to the mathematician who advocated it, as long as there were only proofs and non-proofs, but no sound proofs with weak spots, mathematical criticism was barred. It was the infallibist philosophical background of Euclidean method that bred the authoritarian traditional patterns in mathematics, that prevented publication and discussion of conjectures, that made impossible the rise of mathematical criticism."

Additionally, this mindset is not limited to the place of research among mathematicians, but is very much present in the classroom (Lakatos (1976), 151):

> The student of mathematics is obliged, according to the Euclidean ritual, to attend this conjuring act without asking questions either about the background or about how this sleight-of-hand is performed. If the student by chance discovers that some of the unseemly definitions are proof-generated, if he simply wonders how these definitions, lemmas and the theorem can possibly precede the proof, the conjuror will ostracize him for this display of mathematical immaturity.

So what happened? In the 1840s, the method of proofs and refutations was discovered (Lakatos (1976), 135-136). This "pattern of mathematical discovery", as Lakatos expresses it, consists of a number of stages:

1. A primitive conjecture is made. This is the seed of a theorem, which might or might not hold. A hypothesis, one might call it.

2. A rough proof is sketched. This is meant more as an experiment than as a demonstration. Here, the "proof" is decomposed into its parts; into subconjectures or lemmas.

3. Global counterexamples are found. Counterexamples to the conjecture emerge, revealing some weak spots in the original idea.

4. The proof is re-examined. Starting out from the counterexample found, one tries to locate the erroneous assumption in the proof. This way, a

local error is spotted, and said assumption can be added into the original conjecture.

These four steps are then iterated until the theorem is corrected and the proof is sound. Let us make up an example.

Example 6.1. 1. *For all a, b, it is true that $a + b$ is positive.*

2. Proof:
   $a + b > a$, and as a is positive, so is $a + b$.

3. The conjecture does not hold for $a = b = -1$.

4. We find that we had made the assumption that a was positive, which was not originally stated. We thus add $a > 0$ into the primitive conjecture, and begin the loop anew.

1. *For all a, b where $a > 0$, it is true that $a + b > 0$.*

2. Proof:
   $a + b > a$, and as a is positive, so is $a + b$.

3. The conjecture does not hold when $b < 0$ and $|b| > |a|$.

4. We find that there was an assumption that a positive number added to any number would yield a positive sum. We add the condition from (3) into the next iteration of the conjecture.

And so it goes on, until one is satisfied with the validity of the proof. When that point is reached, there are three additional steps that one might take:

5. Other theorems and proofs are inspected to reveal if the faulty lemma or assumption occurs in them as well. If it does, these theorems might be faulty as well.

6. The presently accepted consequences of the original conjecture are put to the question, and may be corrected as well.

7. Counterexamples are made into new examples, which might open up new fields to investigate.

Lakatos describes how Cauchy, having been familiarised with the exception-barring method, takes it upon himself to investigate mathematics that had hitherto gone unproved or that was not entirely obvious (Lakatos (1976), 145).

Like many foundational changes, this one was not appreciated by all. Some held the opinion that while the theorems would now hold, the domain of validity would be so greatly reduced that the theorems would be more or less meaningless (Lakatos (1976), 47, 60-61).

> "An 'intuitionist' counterrevolution began: the frustrating logico-linguistic pedantry of *proof-analysis* was condemned, and new extremist standards of rigour were invented for *proofs*; mathematics and logic were divorced once more." (Lakatos (1976), 59)

Lakatos goes on to describe the shift of mentality in mathematical circles, and that it was very much an ongoing process.

> "By each 'revolution of rigour' proof-analysis penetrated deeper into the proofs and to the *foundational layer* of 'familiar background knowledge' where crystal-clear intuition, the rigour of the proof, reigned supreme and criticism was banned. Thus, *different levels of rigour differ only about where they draw the line between the rigour of proof-analysis and the rigour of proof,* i.e. *about where criticism should stop and justification should start.*" (Lakatos, (1976), 60)

It is with these changes that the emergence of critique among mathematicians was made possible.

7 End discussion

In this essay we have gone through some key causes of the rigorisation of mathematics that occured in the 19th century. Starting out with Kant and the norms that he established surrounding the sciences, we walk through some changes that happened, namely the separation of physics and mathematics and the scrutiny that began of already existing mathematics. We move on to look at some important mathematical discoveries that were prominent in the texts that were examined in the writing of this essay. We finish off with an investigation of what society looked like during the 1800s and how that enabled, and at times made difficult, the development with regards to mathematical progress, and of the shift in mentality that has been noted and which we have referred to as *proofs and refutations.*

It has been made clear that none of the events of the 19th century stood isolated, and that they all worked in a chain of cause and effect. While it is true that, for example, politics was pivotal in the creation of the mathematical discoveries, which in turn had its own effects, it is also made visible the way mathematicians relied on the work and ideas of others in order to achieve what they did. A prominent example is that of the emergence of the real numbers: Dedekind made his construction due to reading the construction made by Cantor and by Heine, and Cantor was in turn student to Weierstrass who himself made a definition - although not a construction - of the real numbers. It is interesting to see the relationships and the conditions that did in fact lead to what we now know as the real numbers, and it was certainly not the only topic in which this occured.

As the method in the beginning of the essay stated, several sources would be examined and the more recurring mathematical topics would be delved into. However, as history is vast and certain topics need more space than others, some have had to be omitted: namely the topic of *the function* and that of *differentiability*. While it was precisely *continuity, the limit and epsilontics* and *the real numbers* that were the most prominent topics of discussion in the texts inspected, the aforementioned ones were discussed often enough to deserve chapters of their own. In the case of future researched, they should be admitted.

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