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Uniformization Theorem: An Introductory Proof

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Abstract

We present a proof of the uniformization theorem for simply connected Riemann surfaces. The proof is based on the construction of harmonic functions via Perron's method, and closely follows the approach in Theodore Gamelin's book 'Complex Analysis' and a summary provided by Sébastian Picard. We reorganize the material and give additional proofs, explanations, and graphical illustrations to make the topic accessible with knowledge equivalent to an undergraduate course in topology and complex analysis.

Contents

1 Introduction

The field of complex analysis offers various essential techniques to approach problems in both physics and other branches of mathematics. As its name indicates, it is the study of functions defined on the complex number plane $\mathbb{C}$. One of the key distinctions from real analysis that makes complex analysis so useful is the rigidity of holomorphic functions. Complex differentiability is a far more restrictive property than real differentiability because $\mathbb{C}$ possesses more structure than $\mathbb{R}^2$. In return, however, holomorphic functions come with additional interesting properties that prove useful for a broad range of tasks.

The Dirichlet problem, for instance, which plays a major role in various physical applications, is the task of finding a harmonic function on a given domain in $\mathbb{C}$ that attains prescribed values on the boundary of the domain. Harmonic functions are real-valued functions on $\mathbb{C}$ that satisfy the two-dimensional Laplace equation; that is, the sum of their second partial derivatives is zero. This definition gives them similarly rigid properties as holomorphic functions possess, which is why in some sense they can be seen as the real equivalent of the latter.

Solving the Dirichlet problem centrally relies on the shape of the given domain and, in particular, on the boundary. It is therefore reasonable to investigate if one can transform one domain to a simpler one, without altering the properties that holomorphic and harmonic functions on this domain have. This could then allow us to first solve the problem on the simpler domain, and then translate the solution back to the original one. Domains that can be transformed to one another in this sense are called *biholomorphic*, and one sufficient condition for the existence of such a transformation to a very simple domain is provided by the Riemann mapping theorem. The latter states that as long as the domain is simply connected, that is, as long as it has no holes, it is biholomorphic to the open unit disc $\mathbb{D}$ or to $\mathbb{C}$.

In many practical situations, however, the domain of a given problem need not be planar. This leads to the notion of a *Riemann surface*, which can be seen as a deformation of the complex plane. Intuitively, Riemann surfaces are objects that locally look like $\mathbb{C}$, but that have different shapes when considering them globally. This local similarity allows to expand the theory of holomorphic and harmonic functions to Riemann surfaces, and to approach Dirichlet problems as well as a wide range of other tasks on them.

It is therefore natural to ask if domains on Riemann surfaces can also be transformed to simpler domains, similarly as for domains in $\mathbb{C}$. This is covered by the *uniformization theorem*, which can be seen as a generalisation of the Riemann mapping theorem to Riemann surfaces. It states that every simply connected Riemann surface is biholomorphic to one of three Riemann surfaces: the Riemann sphere, the complex plane, or the open unit disc. The theorem was first proven independently by Koebe and Poincaré in the year 1907. Today, it can be proven in various ways, one of which we present in this work.

The approach we present relies on the construction of harmonic functions by use of Perron's method. The latter gives certain conditions under which the supremum of a family of subharmonic functions, which is a similar but slightly weaker property than harmonicity, is a harmonic function. The key property of subharmonic and harmonic functions that powers the proof is that they both satisfy the maximum principle, which states that the maximum cannot be attained on the interior of the domain unless the function is constant. Another fact we heavily use is that harmonic functions are locally the real part of holomorphic functions.

In the first chapter, we introduce all those prerequisites that are typically not taught in undergraduate courses in topology and complex analysis. We introduce the concept of a Riemann surface, holomorphic, harmonic, and subharmonic functions on Riemann surfaces, and Perron's method on a Riemann surface. We then introduce a special harmonic function called Green's function, and show how it can be constructed from subharmonic functions, as well as prove some existence and symmetry results. We also define a harmonic function similar to Green's function but with an additional pole, which we call bipolar Green function, and show that it always exists on a simply connected Riemann surface. In the last chapter, we prove the uniformization theorem using our previous results, and provide some consequential characterization for simply connected Riemann surfaces.

The proof in the last chapter follows a two-part structure:

(A) Assuming that a Green's function exists, we extend it to a holomorphic function with certain properties, which we call holomorphic lift. We show that this lift is a biholomorphism onto its image and map the image onto the open unit disc using the Riemann Mapping Theorem.

(B) Assuming that Green's function does not exist, we take a bipolar Green function and use its harmonicity to extend it to a meromorphic map with certain properties. We show that this meromorphic lift is a biholomorphism onto its image and use the Riemann Mapping Theorem, along with the assumption that Green's function does not exist, to conclude that the image is either the Riemann sphere or biholomorphic to the complex plane.

We encourage the reader to first read through the appendix, where we collected some central results about holomorphic, harmonic, and subharmonic functions in the plane that are most frequently used in our proof, and especially in the concepts we introduce in the first chapter.

2 Functions on Riemann Surfaces

2.1 Riemann Surfaces

In this section, we introduce the notion of a Riemann surface and prove some elementary properties. Recall that a n-dimensional manifold is a second countable Hausdorff space which is locally euclidean of dimension n. That is, any two points can be separated by disjoint open sets, there exists a countable basis for its topology, and every point has a neighborhood homeomorphic to $\mathbb{R}^n$.

Definition 2.1. *Let M be a two-dimensional manifold. A* **(complex) chart** *for M is a pair (U, z), where $z : U \to V$ is a homeomorphism from an open subset $U \subseteq M$ onto an open subset $V \subseteq \mathbb{C}$. Two charts (U_1, z_1) and (U_2, z_2) are* **(holomorphically) compatible** *if $U_1 \cap U_2 = \emptyset$ or the transition map*

$$z_2 \circ z_1^{-1} : z_1(U_1 \cap U_2) \to z_2(U_1 \cap U_2)$$

is holomorphic. A collection of compatible charts $\{(U_i, z_i)\}$ such that $\{U_i\}$ covers M is called a **(complex) atlas** *on M. Two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are* **equivalent** *if every chart in $\mathcal{A}_1$ is compatible with every chart in $\mathcal{A}_2$. An atlas that is not strictly contained in any larger atlas is called* **maximal***.*

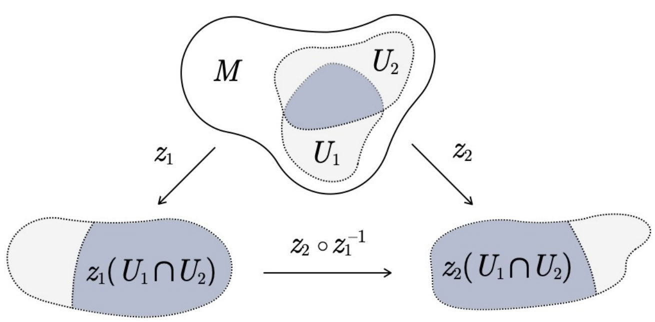

Figure 1. Transition map

The notion of equivalent atlases allows us to consider equivalence classes, and it can be verified that the union of one equivalence class forms a maximal atlas. Intuitively, an atlas extends the structure of the complex numbers to the manifold, and the holomorphicity of the transition maps ensures that the structure is transferred across the entire manifold in a uniform way. This concept is given a name:

Definition 2.2. *A* **Riemann surface** *is a connected, two-dimensional manifold equipped with a maximal atlas.*

We want to emphasize that every Riemann surface is path-connected. In fact, it is connected by definition, and locally path-connected since it is locally homeomorphic to open subsets of $\mathbb{C}$. Now it is a well-known fact from topology that every connected and locally path-connected space is path-connected.

Intuitively, Riemann surfaces can be understood as deformations of the complex plane that preserve the structure of the complex numbers. It is an immediate consequence from the definition that every connected open subset Ω of a Riemann surface R is a Riemann surface in its own right. In fact, we can restrict all charts in an atlas for R to their intersection with Ω to obtain an atlas for Ω. We call such a subset a subsurface or a *domain* on R.

While it is possible to explicitly describe a maximal atlas, it is generally inconvenient to do so. However, the maximal atlas is uniquely determined by a choice of one arbitrary atlas, as the following proposition demonstrates.

Proposition 2.3. *Every atlas for a Riemann surface is contained in a unique maximal atlas.*

Proof. Suppose that an atlas $\mathcal{A}$ is contained in two maximal atlases $\mathcal{A}_1$ and $\mathcal{A}_2$. Now if we take any two charts $(U_1, z_1) \in \mathcal{A}_1$ and $(U_2, z_2) \in \mathcal{A}_2$, then for any $w \in z_1(U_1 \cap U_2)$ there is a chart $(U, z) \in \mathcal{A}$ with $z_1^{-1}(w) \in U$, and we can write

$$z_2 \circ z_1^{-1} = (z_2 \circ z^{-1}) \circ (z \circ z_1^{-1}).$$

Since (U, z) is compatible with both (U_1, z_1) and (U_2, z_2), this map is holomorphic on its domain, in particular at the point w. Hence $z_2 \circ z_1^{-1}$ is holomorphic on $z_1(U_1 \cap U_2)$, or in other words, (U_1, z_1) and (U_2, z_2) are compatible. But $\mathcal{A}_1$ and $\mathcal{A}_2$ are assumed to be maximal, thus we must have $\mathcal{A}_1 = \mathcal{A}_2$. □

This result simplifies the task of describing a Riemann surface, as we can use a collection of charts instead of specifying the maximal atlas. Therefore, from now on, we will simply refer to the manifold as the Riemann surface and implicitly work with the maximal atlas that contains the charts we use. One type of chart that is particularly convenient to work with is the following:

Definition 2.4. *Let R be a Riemann surface and (U, z) a chart for R such that $\overline{\mathbb{D}} \subseteq z(U)$. Then for $D = z^{-1}(\mathbb{D})$, the chart (D, z) is called a* **coordinate disc**. *If $z(p) = 0$, the coordinate disc is said to be* **centered at** *p.*

It may seem unnecessary to require $\overline{\mathbb{D}} \subseteq z(U)$ instead of only $\mathbb{D} \subseteq z(U)$. The first property, however, ensures that the coordinate disc is well-defined on ∂D since z is well-defined on U. Further, since z is a homeomorphism, we must have $z(\partial D) = \partial\mathbb{D}$. The next two propositions justify to use coordinate discs instead of arbitrary charts in various situations.

Proposition 2.5. *Let R be a Riemann surface. Then for every $p \in R$, there exists a coordinate disc centered at p.*

Proof. Let $p \in R$, and let (U, z) be a chart with $p \in U$. Since $z(U)$ is open, there exists some $r > 0$ such that $\mathbb{B}_r^{z(p)} \subseteq z(U)$, where $\mathbb{B}_r^{z(p)} \subseteq \mathbb{C}$ is a closed disc of radius r centered at $z(p)$. If we let $\hat{z} = (z - z(p))r^{-1}$, then $(U, \hat{z})$ is a chart with $\overline{\mathbb{D}} \subseteq \hat{z}(U)$, thus for $D = \hat{z}^{-1}(\mathbb{D})$ we obtain a coordinate disc $(D, \hat{z})$ centered at p. □

Proposition 2.6. *Let R be a Riemann surface. Then R has a countable basis of coordinate discs.*

Proof. Since R is second countable, there exists a countable basis $\hat{\mathcal{B}}$ for the topology on R. Define $\mathcal{B}$ to be the set

$$\mathcal{B} = \{B \in \hat{\mathcal{B}} \mid B \subseteq D \text{ for some coordinate disc } (D, z)\}.$$

We first show that $\mathcal{B}$ is a countable basis. Clearly $\mathcal{B}$ is countable since $\mathcal{B} \subseteq \hat{\mathcal{B}}$. Now let $\Omega \subseteq R$ be any open set in R and $p \in \Omega$. By the previous proposition, there exists a coordinate disc $(\hat{D}, \hat{z})$ centered at p. But $0 = \hat{z}(p) \in \hat{z}(\Omega \cap D)$ and the latter set is open in $\mathbb{C}$, thus there exists some $r > 0$ such that $\mathbb{D}_r^0 \subseteq \hat{z}(\Omega \cap D)$, where $\mathbb{D}_r^0$ is an open disc in $\mathbb{C}$ of radius r centered at 0. But then by letting $D = \hat{z}^{-1}(\mathbb{D}_r^0)$ and $z = \hat{z}/r$, we obtain a coordinate disc (D, z) centered at p with $D \subseteq \Omega$. Since D is open in R and $\hat{\mathcal{B}}$ is a basis, there exists some $B \in \hat{\mathcal{B}}$ with $p \in B \subseteq D$. But then also $B \in \mathcal{B}$ since it is contained in the coordinate disc (D, z), and by construction we have $p \in B \subseteq \Omega$. Hence $\mathcal{B}$ is a countable basis.

Now pick any $B \in \mathcal{B}$, and let (D, z) be a coordinate disc with $B \subseteq D$. The set $z(B)$ is open in $\mathbb{C}$, thus can be written as a countable union of open discs $\mathbb{D}_n$ in $\mathbb{C}$ with rational radius r_n and center p_n of rational coordinates. Now by defining $D_n = z^{-1}(\mathbb{D}_n)$ and $z_n = (z - p_n)r_n^{-1}$, we obtain a countable number of coordinate discs (D_n, z_n) such that $B = \bigcup_n D_n$, so B can be written as a countable union of coordinate discs.

By repeating this for each of the countably many B in the basis $\mathcal{B}$, we obtain a countable basis of coordinate discs. $\square$

2.2 Holomorphic Maps

The requirement for the transition maps to be holomorphic is crucial, as it allows the global definition of holomorphic maps on the Riemann surface. Before reading this section, in which we introduce these concepts and prove some central results, it is recommended to first recall the basic properties of holomorphic functions in the plane, which are summarized in Appendix A.

Definition 2.7. *Let R, S be Riemann surfaces. A continuous map $f : R \to S$ is* **holomorphic** *at $p \in R$ if for any chart (U_R, z_R) on R with $p \in U_R$ and any chart (U_S, z_S) on S with $f(p) \in U_S$, the map*

$$z_S \circ f \circ z_R^{-1} : z_R(U_R) \to z_S(U_S \cap f(U_R))$$

is holomorphic at $z_R(p)$. A map $f : R \to \mathbb{C} \cup \{\infty\}$ which is holomorphic at some $p \in R$ is called **meromorphic** *at p.*

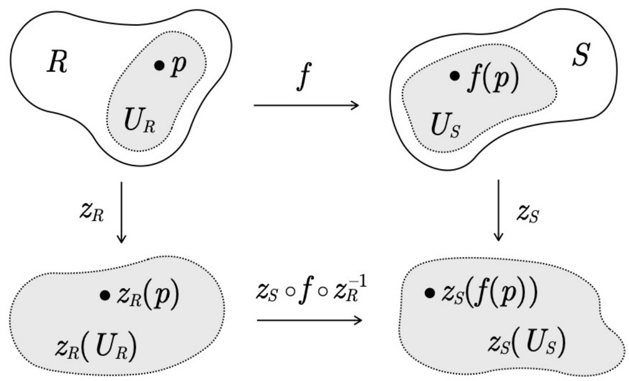

Figure 3. Holomorphic map between Riemann surfaces

It seems cumbersome to verify that a map is holomorphic as the above definition addresses all possible charts in the two atlases. Since the transition maps are holomorphic, however, this definition becomes independent of the charts used. In fact, if $z_S \circ f \circ z_R^{-1}$ is holomorphic at $z_R(p)$ for some charts $(U_R, z_R), (U_S, z_S)$, then for any other pair of charts $(\hat{U}_R, \hat{z}_R), (\hat{U}_S, \hat{z}_S)$ with $p \in \hat{U}_R$ and $f(p) \in \hat{U}_S$, we can write

$$\hat{z}_S \circ f \circ \hat{z}_R^{-1} = (\hat{z}_S \circ z_S^{-1}) \circ (z_S \circ f \circ z_R^{-1}) \circ (z_R \circ \hat{z}_R^{-1}),$$

which is a composition of holomorphic functions and is thus holomorphic at $\hat{z}_R(p)$. Using this, it is straightforward to verify that all local properties of holomorphic functions on the complex plane carry over to Riemann surfaces.

In the special case when $S = \mathbb{C}$, the definition of a holomorphic map reduces to the following: A function $f : R \to \mathbb{C}$ is holomorphic at $p \in R$ if for any chart (U_R, z_R) on R with $p \in U_R$, the map $f \circ z_R^{-1}$ is holomorphic at $z_R(p)$. It is worth noting that the chart maps themselves are such functions, but the fact that they are homeomorphisms gives them additional special properties:

Proposition 2.8. *Let (U, z) be a chart on a Riemann surface R. Then z is holomorphic, and if z has a zero on U, then it is simple.*

Proof. This follows directly from the requirement that the transition maps are holomorphic, and that the charts are homeomorphisms. In fact, if $p \in U$ and $(\hat{U}, \hat{z})$ is any chart on R with $p \in \hat{U}$, then (U, z) and $(\tilde{U}, \tilde{z})$ are compatible, which means that the transition map $\hat{z} \circ z^{-1}$ is holomorphic on its domain and, in particular, at the point $z(p)$. Hence z is holomorphic at p. Now suppose that $z(p) = 0$ for some $p \in U$. Since z is holomorphic and injective at p, we must have $z'(p) \neq 0$, hence the zero at p is simple. □

As explained in the introduction, the uniformization theorem yields the existence of a particular kind of holomorphic map, which we introduce next.

Definition 2.9. *Let R, S be Riemann surfaces. A* **biholomorphism** *is a bijective holomorphic map $f : R \to S$ with holomorphic inverse. If such f exists, R and S are called* **biholomorphic**. *A biholomorphism $f : R \to R$ is called an* **automorphism**.

We emphasize that every biholomorphism is a homeomorphism since every holomorphic map is continuous. Moreover, it turns out that bijectivity of a holomorphic map is sufficient for being a biholomorphism, which is a direct consequence of the complex inverse function theorem. Since this fact will be of of central use for us when proving the uniformization theorem in section 5, we state it as a proposition:

Proposition 2.10. *Let R, S be Riemann surfaces and $f : R \to S$ a bijective holomorphic map. Then f is a biholomorphism.*

Proof. Let $q \in S$, (U_S, z_S) a chart on S with $q \in U_S$, and (U_R, z_R) a chart on R with $f^{-1}(q) \in U_R$. We want to show that the map $z_R \circ f^{-1} \circ z_S^{-1}$ is holomorphic at $z_S(q)$. Since f is injective and z_R, z_S are homeomorphisms, we know that $z_S \circ f \circ z_R^{-1}$ is injective, thus we must have $(z_S \circ f \circ z_R^{-1})' \neq 0$ on its domain $z_R(U_R)$, and in particular at the point $z_R(f^{-1}(q))$. By the complex inverse function theorem, $z_S \circ f \circ z_R^{-1}$ admits a local inverse $z_R \circ f^{-1} \circ z_S^{-1}$ that is holomorphic in some neighborhood of $(z_S \circ f \circ z_R^{-1})(z_R(f^{-1}(q))) = z_S(q)$. □

Another central ingredient when dealing with coordinate discs and for proving the uniformization theorem are the automorphisms of the unit disc:

Proposition 2.11. *For every $\alpha \in \mathbb{D}$, the map*

$$A_\alpha(z) \mapsto \frac{\alpha - z}{1 - \overline{\alpha} z}$$

is an automorphism of $\mathbb{D}$.

Proof. First note that for every $\alpha \in \mathbb{D}$, the function A_α is well-defined and holomorphic on $\mathbb{C} \setminus \{1/\overline{\alpha}\}$. But $1/\overline{\alpha} \notin \overline{\mathbb{D}}$ since $|1/\overline{\alpha}| > 1$, thus A_α is well-defined and holomorphic on $\overline{\mathbb{D}}$. We need to show that A_α maps $\mathbb{D}$ bijectively onto itself. Note that for $z \in \partial\mathbb{D}$, we have

$$|A_\alpha(z)| = \left| \frac{\alpha - z}{1 - \overline{\alpha} z} \right| \cdot |\overline{z}| = \left| \frac{\alpha \overline{z} - 1}{1 - \overline{\alpha} z} \right| = 1.$$

By the maximum modulus principle, the restriction of $|A_\alpha|$ to $\overline{\mathbb{D}}$ can not attain its maximum on $\mathbb{D}$, thus we must have $|A_\alpha| < 1$ on $\mathbb{D}$. Therefore A_α maps from $\mathbb{D}$ to $\mathbb{D}$, and so does the composition $A_\alpha \circ A_\alpha$. Further observe that

$$A_\alpha \circ A_\alpha(0) = 0 \quad \text{and} \quad A_\alpha \circ A_\alpha(\alpha) = \alpha,$$

thus by Schwarz's lemma $A_\alpha \circ A_\alpha$ is the identity on $\mathbb{D}$. Hence A_α is bijective on $\mathbb{D}$, and thus an automorphism of $\mathbb{D}$ by the previous proposition. □

One can even show that the automorphisms of $\mathbb{D}$ are precisely given by the functions of this form, multiplied by a simple rotation. We do not need this stronger fact, but a proof can for instance be found in Chapter 8 of [Gam01].

The uniformization theorem yields the existence of a biholomorphism from a simply connected Riemann surface to one of three Riemann surfaces: the Riemann sphere, the complex plane, or the open unit disc. To reveal the significance of this statement, however, we should first convince ourselves that these three Riemann surfaces are not biholomorphic, which is how we conclude this section.

Proposition 2.12. *Neither two of the Riemann sphere, the complex plane, and the open unit disc are biholomorphic.*

Proof. It is immediate that the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is not biholomorphic to $\mathbb{C}$ or $\mathbb{D}$ since it is not homeomorphic to both of them; in fact, $\mathbb{C} \cup \{\infty\}$ is compact but $\mathbb{C}$ and $\mathbb{D}$ are not, and compactness is preserved under homeomorphisms. That $\mathbb{C}$ and $\mathbb{D}$ are not biholomorphic is a direct consequence of Liouville's theorem: Every holomorphic map from $\mathbb{C}$ to $\mathbb{D}$ is constant, thus can not be bijective. □

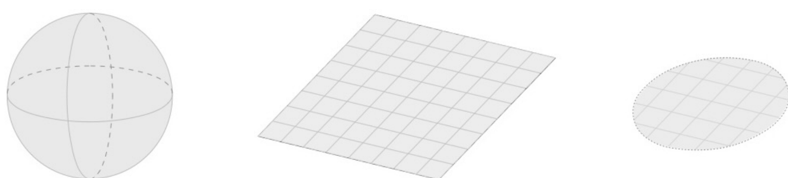

Figure 3. The Riemann sphere, the complex plane, and the open unit disc

2.3 Harmonic and Subharmonic Functions

As mentioned before, harmonic and subharmonic functions possess the central property of satisfying the so-called maximum principle, which we establish in this section. This will later prove useful for deriving upper and lower bounds on functions, and in this way become one of the main ingredients powering the the proof in section 5. Before reading this and the following chapter, however, we recommend to first familiarize with the basic properties of harmonic and subharmonic functions in the plane, some of which we summarized in Appendix B.

From now on, we will state all definitions with coordinate discs since this will facilitate an easier translation to the Dirichlet problem on the unit disc, which has a particularly simple solution via the Poisson integral. Recall that a harmonic function in the plane is defined as a twice continuously differentiable function that satisfies the Laplace equation. We carry over this definition to Riemann surfaces in a similar fashion as for holomorphic maps:

Definition 2.13. *A continuous function $h : R \to \mathbb{R}$ on a Riemann surface R is* **harmonic** *at $p \in R$ if for every coordinate disc (D, z) containing p, the function $h \circ z^{-1} : \mathbb{D} \to \mathbb{R}$ is harmonic at $z(p)$.*

Similarly as in the definition of a holomorphic map, harmonicity is independent of the chosen coordinate disc. In fact, if $h \circ z^{-1}$ is harmonic at $z(p)$ for some coordinate disc (D, z), then for any other coordinate disc $(\hat{D}, \hat{z})$ with $p \in \hat{D}$, the map

$$h \circ \hat{z}^{-1} = (h \circ z^{-1}) \circ (z \circ \hat{z}^{-1})$$

is the composition of a harmonic with a holomorphic function, thus is harmonic at $\hat{z}(p)$. Again, it is straightforward to verify that all local properties of harmonic functions in the plane — and, in particular, the equivalent characterisations given in Proposition B.1 — carry over to Riemann surfaces. In the following, we present some of these properties that are of frequent use for us.

Proposition 2.14. *Let R be a Riemann surface and $f : R \to \mathbb{R}$ holomorphic and nonzero. Then $\log |f|$ is harmonic on R.*

Proof. Let $p \in R$ and (D, z) a coordinate disc containing p. Then $f \circ z^{-1}$ is holomorphic and nonzero, thus $\log(f \circ z^{-1})$ is holomorphic. Now $\log \left| f \circ z^{-1} \right|$ is its real part, thus is harmonic, in particular at the point $z(p)$. □

Proposition 2.15. *Let R be a Riemann surface, (D, z) a coordinate disc on R, and $h : D \to \mathbb{R}$ a harmonic function on D. Then there exists a holomorphic function $\varphi : D \to \mathbb{C}$ with*

$$|\varphi| = e^{-h},$$

which is unique up to a multiplication by $e^{i\theta}$ with $\theta \in \mathbb{R}$.

Proof. Since h is harmonic on D, the function $h \circ z^{-1} : \mathbb{D} \to \mathbb{R}$ is harmonic. Now $\mathbb{D}$ is simply connected and harmonic functions are locally the real part of holomorphic functions, thus there is a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that

$$\mathrm{Re}(f) = h \circ z^{-1},$$

and the harmonic conjugate $\mathrm{Im}(f)$ is uniquely determined up to adding a real constant, as stated in Proposition B.1 (3). We can lift this function to our Riemann surface R by defining $\hat{f} : D \to \mathbb{C}$ as $\hat{f} = f \circ z$, and we have

$$\mathrm{Re}(\hat{f}) = \mathrm{Re}(\hat{f} \circ z) = \mathrm{Re}(f) \circ z = h.$$

Now defining $\varphi : D \to \mathbb{C}$ by $\varphi = e^{-\hat{f}}$ yields

$$|\varphi| = |e^{-\mathrm{Re}(\hat{f}) - i\mathrm{Im}(\hat{f})}| = e^{-h},$$

where we used that $e^{-i\mathrm{Im}(\hat{f})} \in \mathbb{S}^1$ and thus has modulus 1. Furthermore, since $\mathrm{Im}(\hat{f})$ is unique up to an additive real constant, φ is unique up to a multiplication by $e^{i\theta}$ with $\theta \in \mathbb{R}$. □

We now introduce the definition of a subharmonic function on a Riemann surface in a similar fashion. Recall that a subharmonic function in the plane is a function that is pointwise bounded from above by its average value over the boundary of any disc in the domain centered at that point. Equivalently, if a subharmonic function is bounded from above by a harmonic function on the boundary of any disc in the domain, then the bound also holds on the interior of the disc.

Definition 2.16. *A continuous function* $u : R \to \mathbb{R} \cup \{-\infty\}$ *on a Riemann surface* R *is* **subharmonic** *at* $p \in R$ *if for every coordinate disc* (D, z) *containing* p*, the function* $u \circ z^{-1} : \mathbb{D} \to \mathbb{R} \cup \{-\infty\}$ *is subharmonic at* $z(p)$*.*

In general, subharmonic functions are allowed to be only upper semi-continuous; as noted in [Gam01], however, this would lead to no gain for the proof we present, which is why we work exclusively with continuous subharmonic functions.

Intuitively, subharmonic and harmonic functions behave similarly to convex and linear functions on the real line. In fact, if a convex function is bounded from above by a linear function on the boundary of some interval, then the bound also holds on the interior of the interval. As mentioned before, a similar characterization holds for subharmonic and harmonic functions, with discs taking the place of intervals.

It can be shown that this definition is also independent of the coordinate disc since the composition of a subharmonic with a holomorphic function is again subharmonic. A proof of the latter requires a number of techniques from harmonic function theory and can for instance be found in [Ran95].

This has the effect that, similarly as for harmonic functions, all local properties of subharmonic functions in the plane — including the equivalent conditions presented in Proposition B.3 — can be extended to Riemann surfaces. In addition to the fact that the sum and maximum of subharmonic functions is again subharmonic, one property will be of particularly frequent use for us:

Proposition 2.17. *Let* R *be a Riemann surface and* $f : R \to \mathbb{C}$ *a holomorphic function. Then* $\log |f|$ *is subharmonic on* R*.*

Proof. By Proposition 2.14, the function $\log |f|$ is harmonic on $\mathrm{supp}(f)$. But if $f(p) = 0$ and (D, z) is a coordinate disc containing p, then $\log \left| f \circ z^{-1}(z(p)) \right| = -\infty$, thus is subharmonic at $z(p)$ as it trivially satisfies the mean value inequality stated in Proposition B.3. Hence $\log |f|$ is subharmonic on R. □

We next state the maximum principle, which is perhaps *the* central property of subharmonic and harmonic functions that powers the proof in section 5. We state it for subharmonic functions, but it is of course also valid for harmonic functions as every harmonic function is subharmonic. Recall that the maximum principle in the plane states that a subharmonic function on a domain U in $\mathbb{C}$ can not attain a local maximum on U unless it is constant.

Proposition 2.18 (Maximum principle)**.** *Let $u : R \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function on a Riemann surface R. If u attains its maximum at some point of R, then u is constant.*

Proof. Suppose that u attains its maximum at a point $p \in R$, and let $M = u(p)$ be the maximum. If (D, z) is a coordinate disc containing p, then the function $u \circ z^{-1} : \mathbb{D} \to \mathbb{R} \cup \{-\infty\}$ is subharmonic on $\mathbb{D}$, and it attains its maximum at the point $z(p) \in \mathbb{D}$. Thus by the maximum principle for subharmonic functions in the plane we have that $u \circ z^{-1} \equiv M$ on $\mathbb{D}$, and by bijectivity of z we get $u \equiv M$ on D.

For any other coordinate disc $(\hat{D}, \hat{z})$ with $D \cap \hat{D} \neq \emptyset$, we then also have $u(q) \equiv M$ for some $q \in D \cap \hat{D}$, thus we similarly get $u \equiv M$ on $\hat{D}$.

Now let $w \in R$ be any point. Since R is path-connected, there exists a path π in R from p to w. But by use of Proposition 2.6, we can cover the compact set $\Gamma = \pi([0, 1])$ with finitely many coordinate discs. In the same fashion as before, we get that $u \equiv M$ on Γ, and particularly at the point w. Hence $u \equiv M$ on R. □

Note in particular that if h is a harmonic function on R, then both h and $-h$ are subharmonic, thus can not attain a local maximum on R unless h is constant. Hence h can not attain a local extremum on R unless h is constant. We further need the following immediate consequence of the maximum principle:

Proposition 2.19. *Let $u : R \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function on a Riemann surface R. If $u \leq c$ outside of a compact subset of R for some constant $c > 0$, then $u \leq c$ on all of R.*

Proof. Let $u \leq c$ outside of a compact subset K of R. Since u is continuous and K is compact, u attains its maximum on K, say $\max\{u(p) \mid p \in K\} = M$. Now if $M > c$ then $u \leq M$ on all of R, so M would be the maximum of u on all of R. But then by the maximum principle $u \equiv M$ on R, contradicting the assumption that $u \leq c < M$ on $R \setminus K$. Thus $M \leq c$, so $u \leq c$ on K and hence on all of R. □

2.4 Perron's Method

In this section, we prove Perron's theorem, which gives certain conditions under which the supremum of a family of subharmonic functions is harmonic. This will allow us to construct a harmonic function out of subharmonic functions, which is why this procedure is called *Perron's method.*

Recall that the Dirichlet problem on a planar domain is the task of finding a harmonic function that coincides with a specified function on the boundary of the domain. This problem equally applies to Riemann surfaces, and we are particularly interested in one specific Dirichlet problem:

Proposition 2.20. *Let R be a Riemann surface, (D, z) a coordinate disc on R, and $u : R \to \mathbb{R} \cup \{-\infty\}$ a subharmonic function which is finite on ∂D. Then there exists a unique harmonic function h on D satisfying $h = u$ on ∂D.*

Proof. The problem translates to the Dirichlet problem in the unit disc

$$\begin{cases} h \circ z^{-1} & \text{is harmonic on } \mathbb{D} \\ h \circ z^{-1} = u \circ z^{-1} & \text{on } \partial\mathbb{D} \end{cases},$$

which we can solve using the Poisson integral stated in Proposition B.2. This yields a unique harmonic function $\hat{h}$ on $\mathbb{D}$, which extends to a continuous function on $\overline{\mathbb{D}}$ that coincides with g on $\partial\mathbb{D}$. Now $h = \hat{h} \circ z$ is the desired harmonic function. □

This justifies the next definition:

Definition 2.21. *Let R be a Riemann surface, (D, z) a coordinate disc on R, and $u : R \to \mathbb{R} \cup \{-\infty\}$ a subharmonic function which is finite on ∂D. The* **harmonic extension** *of u to D is the subharmonic function $u_D : R \to \mathbb{R} \cup \{-\infty\}$ defined by*

$$u_D = \begin{cases} u & \textit{on } R \setminus D \\ h & \textit{on } D \end{cases},$$

where h is the unique harmonic function on D satisfying $h = u$ on ∂D.

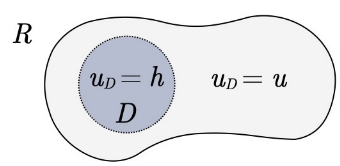

Figure 1. Harmonic extension

We will encounter a situation where the subharmonic function is bounded from above by a constant. It turns out that in this case, the harmonic extension is bounded by the same constant:

Proposition 2.22. *Let R be a Riemann surface, (D, z) a coordinate disc on R, and $u : R \to \mathbb{R} \cup \{-\infty\}$ a subharmonic function which is finite on ∂D. If $u \leq c$ for some constant c, then $u_D \leq c$.*

Proof. Let $p \in D$. Then $z(p) \in \mathbb{D}$, thus the map $A_{z(p)}$ defined in Proposition 2.11 is an automorphism of $\mathbb{D}$, and we have $A_{z(p)}(z(p)) = 0$. By letting $\hat{z} = A_{z(p)} \circ z$, we obtain a coordinate disc $(D, \hat{z})$ centered at p. Recall that in Appendix B, we abbreviated the mean value of a continuous function f on the boundary of some disc $\mathbb{D}_r^a$ in the domain by $M[f, \partial\mathbb{D}_r^a]$. The function $u_D \circ \hat{z}^{-1}$ is harmonic on $\mathbb{D}$, thus satisfies the mean value equality in Proposition B.1 (2) at the point $\hat{z}(p)$. But on $\partial\mathbb{D}$ we have $u_D \circ \hat{z}^{-1} = u \circ \hat{z}^{-1}$ by definition, so we get

$$u_D(p) = u_D \circ \hat{z}^{-1}(\hat{z}(p)) = M[u_D \circ \hat{z}^{-1}, \partial\mathbb{D}] = M[u \circ \hat{z}^{-1}, \partial\mathbb{D}] \leq M[c \circ \hat{z}^{-1}, \partial\mathbb{D}] = c.$$

Hence $u_D \leq c$ on D. On $R \setminus D$ we have $u_D = u \leq c$ by the previous definition. □

It is clear that the harmonic extension is a subharmonic function, and we also know that the maximum of subharmonic functions is again subharmonic. Therefore, the following definition makes sense:

Definition 2.23. *A* **Perron family** $\mathcal{P}$ *on a Riemann surface* R *is a collection of continuous functions* $u : R \to \mathbb{R} \cup \{-\infty\}$ *satisfying:*

(P-1) $\mathcal{P}$ *is nonempty;*

(P-2) *every* $u \in \mathcal{P}$ *is subharmonic;*

(P-3) *if* $u, v \in \mathcal{P}$*, then* $\max(u, v) \in \mathcal{P}$*, and*

(P-4) *if* $u \in \mathcal{P}$ *is finite on* ∂D *for some coordinate disc* (D, z)*, then* $u_D \in \mathcal{P}$*.*

These conditions are sufficient to ensure that the supremum of the family is harmonic unless it is identically infinite, as the following result shows.

Proposition 2.24 (Perron's theorem)**.** *Let* $\mathcal{P}$ *be a Perron family on a Riemann surface* R*. Then* $\sup\{u \mid u \in \mathcal{P}\}$ *is either harmonic or* $\sup\{u \mid u \in \mathcal{P}\} \equiv +\infty$*.*

Proof. Let $h = \sup\{u \mid u \in \mathcal{P}\}$. We will show that if $h(p) < \infty$ for some $p \in R$, then h is harmonic on D for every coordinate disc (D, z) containing p, and then extend harmonicity of h to R by using path-connectedness of R.

Suppose that $h(p) < \infty$, and let (D, z) be a coordinate disc containing p. We can find a sequence $\{\hat{u}_n\}$ of functions in $\mathcal{P}$ such that $\hat{u}_n(p)$ converges to $h(p)$. Now define $u_n = \max\{\hat{u}_1, \ldots, \hat{u}_n\}$ to obtain an increasing sequence $\{u_n\}$ of subharmonic functions. Note that $u_n \in \mathcal{P}$ and also $(u_n)_D \in \mathcal{P}$ for each n by (P-3) and (P-4) of the above definition. Thus $\{(u_n)_D\}$ is an increasing sequence of functions in $\mathcal{P}$ that are harmonic on D. By Harnack's principle, as stated in Proposition B.6, the function

$$u = \sup\{(u_n)_D\}$$

is either harmonic or $u \equiv +\infty$ on D. But since $(u_n)_D \in \mathcal{P}$ for all n, we have $u \leq h$, and in particular

$$u(p) \leq h(p) < \infty.$$

Thus u is harmonic on D. We want to show that $h = u$ on all of D to derive harmonicity of h on D.

For each n, we have $u_n = (u_n)_D$ on ∂D. But u_n is subharmonic and $(u_n)_D$ is harmonic on D, thus property (1) in Proposition B.3 yields $u_n \leq (u_n)_D$ on D, and in particular at the point p. Since $u_n(p)$ is increasing and converges to $h(p)$ by construction, we get

$$u(p) = \sup\{(u_n)_D(p)\} \geq \sup\{u_n(p)\} = h(p).$$

Hence we can conclude that $u(p) = h(p)$.

Now pick any other $q \in D$, and suppose for contradiction that $u(q) < h(q)$. Then we can again find a sequence $\{\hat{g}_n\}$ in $\mathcal{P}$ such that $\hat{g}_n(q)$ converges to $h(q)$. Now if we define $g_n = \max\{\hat{g}_1, \dots, \hat{g}_n, \hat{u}_1, \dots, \hat{u}_n\}$, then $\{g_n\}$ is an increasing sequence in $\mathcal{P}$ with $g_n \geq u_n$. Again by Harnack's principle, the function $g = \sup\{(g_n)_D\}$ is either harmonic or $g \equiv +\infty$ on D. But again $\{(g_n)_D\}$ is a sequence in $\mathcal{P}$, and therefore $g(p) \leq h(p) < +\infty$, which is finite by assumption, so g must be harmonic on D. But we have already shown that $u(p) = h(p)$ and by construction we have $u \leq g \leq h$, thus we must have

$$u(p) = g(p) = h(p).$$

Furthermore, $g(q) \geq h(q)$ by construction of the sequence $\{g_n\}$, so we also have

$$u(q) < g(q) = h(q).$$

But the function $u - g$ is harmonic on D and satisfies $u - g \leq 0$ and $(u - g)(p) = 0$, so $u - g$ attains its maximum at p. By the maximum principle, $u = g$ on D, contradicting the assumption that $u(q) < h(q) = g(q)$. Hence we can conclude that

$$u = h$$

on D, thus h is harmonic and, in particular, finite on D.

The extension to all of R is now similar as in the proof of the maximum principle. In fact, if $(\hat{D}, \hat{z})$ is any other coordinate disc with $D \cap \hat{D} \neq \emptyset$, then $h(x) < \infty$ for some $x \in D \cap \hat{D}$, so we get that h is also harmonic on $\hat{D}$ by the previous.

Now for any other point $w \in R$ and any path π from p to w, we can cover the compact set $\Gamma = \pi([0, 1])$ with finitely many coordinate discs by Proposition 2.6, and conclude that h is harmonic on Γ and, in particular, at the point w. Hence h is harmonic on R. □

3 Green's Function

3.1 Definition and Holomorphic Lift

In this section, we introduce a certain harmonic function called Green's function, and prove that it can in some sense be extended to a holomorphic function if the Riemann surface is simply connected. We call this extension a holomorphic lift, and we will show in Section 5 that it is a biholomorphism onto its image, which will then allow us to prove the first part of the uniformization theorem.

Definition 3.1. *Let R be a Riemann surface and $q \in R$. A* **Green function** (*with pole at q*) *is a continuous function $g_q : R \setminus \{q\} \to \mathbb{R}$ satisfying:*

- **(G-1)** *g_q is harmonic;*
- **(G-2)** *for some coordinate disc (D, z) centered at q, the function $g_q + \log|z|$ extends to a harmonic function on D;*
- **(G-3)** *$g_q > 0$, and*
- **(G-4)** *g_q is the smallest function that satisfies* (G-1), (G-2), and (G-3)*. That is, if $\hat{g}_q : R \setminus \{p\} \to \mathbb{R}$ satisfies* (G-1), (G-2), and (G-3), *then $g_q \leq \hat{g}_q$.*

Note that condition (G-4) ensures uniqueness, thus it is reasonable to speak of *Green's function* (with pole at q), provided that it exists. Further, Green's function is not defined at the point q; condition (G-2), however, ensures that it has a logarithmic pole at q, in the sense that g_q grows with the same speed when approaching q as $-\log$ grows when approaching 0. In fact, for some coordinate disc (D, z), the function $g_q + \log|z|$ is harmonic on D and, in particular, at the point q. Note that we here implicitly refer to the extension of $g_q + \log|z|$ to all of D since g_q is originally not defined at the pole q. Since a harmonic function is finite, the poles of g_q and $\log|z|$ at q must cancel. We will see in section 3.3 that the location of the pole q is negligible. Fortunately, condition (G-2) is independent of the coordinate disc:

Proposition 3.2. *Let R be a Riemann surface, $q \in R$, and g_q a Green function. Then for every coordinate disc (D, z) centered at q, the function $g_q + \log|z|$ extends to a harmonic function on D. Moreover, if $f : R \to \mathbb{C}$ is holomorphic and $f(q) = 0$, then $g_q + \log|f|$ extends to a subharmonic function on R.*

Proof. By (G-2) of the above definition, there is a coordinate disc $(\hat{D}, \hat{z})$ centered at q such that $g_q + \log|\hat{z}|$ extends to a harmonic function on $\hat{D}$. Now if (D, z) is any other coordinate disc centered at q, then we can write

$$g_q + \log|z| = g_q + \log|\hat{z}| + \log\left|\frac{z}{\hat{z}}\right|.$$

We know that $g_q + \log|\hat{z}|$ is harmonic on $\hat{D}$, and in particular on $D \cap \hat{D}$. Furthermore, by Proposition 2.8, both z and $\hat{z}$ are holomorphic on their domains and have a simple zero at q. The zeroes cancel in $z/\hat{z}$, which thus is holomorphic and nonzero on $D \cap \hat{D}$. Hence $\log|z/\hat{z}|$ is harmonic on $D \cap \hat{D}$ by Proposition 2.14, and since the sum of harmonic functions is again harmonic, it follows that $g_q + \log|z|$ is harmonic on $D \cap \hat{D}$. But $g_q + \log|z|$ is clearly harmonic on $D \setminus \{q\}$ since both g_q and $\log|z|$ are. Hence $g_q + \log|z|$ is harmonic on D.

For the second part, we can equally write

$$g_q + \log|f| = g_q + \log|\hat{z}| + \log\left|\frac{f}{\hat{z}}\right|.$$

Again, $g_q + \log|\hat{z}|$ is harmonic on $\hat{D}$, and $\log|f/\hat{z}|$ is subharmonic on $\hat{D}$ by Proposition 2.17, since the zero of $\hat{z}$ at q cancels in $f/\hat{z}$. Thus $g_q + \log|f|$ is subharmonic on $\hat{D}$. But $g_q + \log|f|$ is clearly subharmonic on $R \setminus \{q\}$ since g_q is harmonic and $\log|f|$ is subharmonic on this set, hence $g_q + \log|f|$ is subharmonic on all of R. □

Assuming that the Riemann surface is simply connected, we next show the existence of a holomorphic function with modulus equal to the inverse exponential of Green's function. We call this function a holomorphic lift as it can in some sense be seen as an extension of Green's function to a holomorphic function. The key for proving the existence of this lift is that harmonic functions are locally the real part of holomorphic functions, which we utilize in the following.

Proposition 3.3. *Let R be a simply connected Riemann surface, $q \in R$, and g_q a Green function. Then there exists a holomorphic function $\varphi : R \to \mathbb{C}$ with*

$$|\varphi| = e^{-g_q}$$

and with a simple zero at q.

Proof. Let $\mathcal{A} = \{(D_\alpha, z_\alpha)\}$ be a basis of coordinate discs for R, which exists by Proposition 2.6. We will show that such a function exists on each D_α, and then use the Monodromy theorem to derive global existence on R. Pick some $(D_\alpha, z_\alpha) \in \mathcal{A}$, and consider the following two cases.

If $q \notin D_\alpha$, then g_q is harmonic on D_α by property (G-1) of Definition 3.1, and thus by Proposition 2.15 there exists a holomorphic function $\varphi_\alpha : D_\alpha \to \mathbb{C}$ with $|\varphi_\alpha| = e^{-g_q}$, which is unique up to a multiplication with $e^{i\theta_\alpha}$ for $\theta_\alpha \in \mathbb{R}$.

If $q \in D_\alpha$, then $A_{z_\alpha(q)}$ is an automorphism of $\mathbb{D}$ by Proposition 2.11, thus by setting $\hat{z}_\alpha = A_{z_\alpha(q)} \circ z_\alpha$ we obtain a coordinate disc $(D_\alpha, \hat{z}_\alpha)$ centered at q. By the previous proposition, the function $f = g_q + \log|\hat{z}_\alpha|$ is harmonic on D_α. Again by Proposition 2.15, there exists a holomorphic function $\hat{\varphi}_\alpha : D_\alpha \to \mathbb{C}$ such that $|\hat{\varphi}_\alpha| = e^{-f}$, which is again unique up to a multiplication with $e^{i\theta_\alpha}$ for $\theta_\alpha \in \mathbb{R}$. Now let $\varphi_\alpha : D_\alpha \to \mathbb{C}$ be defined by $\varphi_\alpha = \hat{z}_\alpha \hat{\varphi}_\alpha$, which is also uniquely defined up to multiplication with $e^{i\theta_\alpha}$, and holomorphic on D_α since both $\hat{\varphi}_\alpha$ and $\hat{z}_\alpha$ are. By construction, this function satisfies

$$|\varphi_\alpha| = |\hat{z}_\alpha|\,|\hat{\varphi}_\alpha| = |\hat{z}_\alpha|\,e^{-g_q}e^{-\log|\hat{z}_\alpha|} = e^{-g_q}.$$

In particular, note here that $f(q)$ is finite since f is harmonic in D_α, thus $|\hat{\varphi}_\alpha(q)|$ is finite, so $|\varphi_\alpha(q)| = 0$ and therefore $\varphi_\alpha(q) = 0$. This zero is independent of the multiplication with $e^{i\theta_\alpha}$, and it is simple since the zero of $\hat{z}_\alpha$ at q is simple by Proposition 2.8.

We have shown that on each D_α, there exists a holomorphic function φ_α with the desired properties. Now suppose that $D_\alpha \cap D_\beta \neq \emptyset$ for two coordinate discs $(D_\alpha, z_\alpha), (D_\beta, z_\beta)$ in $\mathcal{A}$. We want to show that a similar function also exists on $D_\alpha \cup D_\beta$. Since $\mathcal{A}$ is a basis of coordinate discs and $D_\alpha \cap D_\beta$ is open and nonempty, there must be a coordinate disc $(D_\gamma, z_\gamma) \in \mathcal{A}$ such that $D_\gamma \subseteq D_\alpha \cap D_\beta$. By the same reasoning as before, there is a holomorphic function $\varphi_\gamma : D_\gamma \to \mathbb{C}$ with $|\varphi_\gamma| = e^{-g_q}$, which is unique up to a multiplication with $e^{i\theta_\gamma}$. But φ_α and φ_β are such functions when restricting their domains to D_γ, so we must have $\varphi_\gamma = \varphi_\alpha e^{i\theta_\alpha} = \varphi_\beta e^{i\theta_\beta}$ on D_γ for some $\theta_\alpha, \theta_\beta \in \mathbb{R}$, and thus $\varphi_\alpha e^{i\theta_\alpha} = \varphi_\beta e^{i\theta_\beta}$ on $D_\alpha \cap D_\beta$. Therefore the function $\varphi : D_\alpha \cup D_\beta \to \mathbb{C}$ defined by

$$\varphi = \begin{cases} \varphi_\alpha & \text{on } D_\alpha \\ \varphi_\beta e^{i(\theta_\beta - \theta_\alpha)} & \text{on } D_\beta \end{cases}$$

is well-defined and holomorphic, so φ is a holomorphic extension of φ_α to $D_\alpha \cup D_\beta$. Note that still $|\varphi| = e^{-g_q}$, and if $q \in D_\alpha \cup D_\beta$, then φ has a simple zero at q since multiplication with a non-zero constant does not affect zeroes.

Now let $\pi : [0,1] \to R$ be any path in R starting in D_α and $\Gamma = \pi([0,1])$. Since the charts in $\mathcal{A}$ cover R, they also cover Γ, and we can obtain a finite subcover $\{D_i\}$ by compactness of Γ. But then, starting with D_α, we can holomorphically extend φ_α to $\cup_i D_i$, and therefore along π, in the same fashion as before.

Hence we can holomorphically extend φ_α along any path in R starting in D_α. Since R is simply connected, the Monodromy theorem, which is stated in Theorem A.7, yields that the function φ_α can be extended to a holomorphic function defined on all of R, which is the function we wanted. □

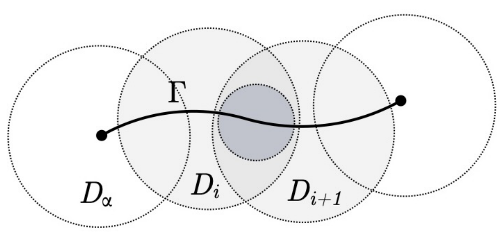

Figure 3. Extending φ_α along Γ

Assuming that Green's function exists, we will show in Section 5 that this holomorphic lift is a biholomorphism onto its image, which we can then map biholomorphically onto the open unit disc using the Riemann mapping theorem.

3.2 Construction via Perron's Method

As we have seen before, being subharmonic is a far less restrictive property than being harmonic, but it still allows us to utilize the maximum principle. By this reason, we will now make use of the powerful Perron‘s method, and prove that Green‘s function is precisely given by the supremum of a certain Perron family. This will then allow us to first consider subharmonic functions in this family, apply the maximum principle, and then pass over to Green's function by taking the supremum. We first define the family:

Definition 3.4. *Let R be a Riemann surface, and $q \in R$. Define $\mathcal{G}_q$ to be the set of all continuous functions $u : R \setminus \{q\} \to \mathbb{R} \cup \{-\infty\}$ satisfying:*

- **(1)** *every $u \in \mathcal{G}_q$ is subharmonic;*
- **(2)** *for every $u \in \mathcal{G}_q$, the set $\operatorname{supp}(u) \cup \{q\}$ is compact in R, and*
- **(3)** *for every $u \in \mathcal{G}_q$, there exists a coordinate disc (D, z) centered at q such that $u + \log|z|$ extends to a subharmonic function on D.*

Similarly as for Green's function, we want to emphasize that the function $u+\log|z|$ in condition (3) is originally not defined at the point q since u is only defined on $R\setminus\{q\}$. In the following, when considering this function on all of D, then we always implicitly refer to its extension to D. Again, condition (3) is independent of the coordinate disc, and it turns out that we can even replace the coordinate disc with any holomorphic map that has a zero at q:

Proposition 3.5. *Let R be a Riemann surface and $f : R \to \mathbb{C}$ a holomorphic function with $f(q) = 0$. Then for every $u \in \mathcal{G}_q$, the function $u + \log|f|$ is subharmonic on R. In particular, $u+\log|z|$ is subharmonic on D for every coordinate disc (D, z) centered at q.*

Proof. We proceed similarly as in the proof of Proposition 3.2. If $u \in \mathcal{G}_q$, then by (3) of the above definition there is a coordinate disc $(\hat{D}, \hat{z})$ centered at q such that $u + \log|\hat{z}|$ is subharmonic on $\hat{D}$. Now we can write

$$u + \log|f| = u + \log|\hat{z}| + \log\left|\frac{f}{\hat{z}}\right|.$$

Now by Proposition 2.8, $\hat{z}$ is holomorphic and has a simple zero at q. The zeroes cancel in $f/\hat{z}$, which thus is holomorphic on $\hat{D}$. Hence $\log|f/\hat{z}|$ is subharmonic on $\hat{D}$, and since the sum of subharmonic functions is again subharmonic, it follows that $u+\log|f|$ is subharmonic on $\hat{D}$. But $u+\log|f|$ is clearly subharmonic on $R\setminus\hat{D}$ since both u and $\log|f|$ are, hence $u + \log|f|$ is subharmonic on R. For the remaining part, simply note that if (D, z) is a coordinate disc centered at q, then $z : D \to \mathbb{D}$ is holomorphic and satisfies $z(q) = 0$. □

As mentioned before, the goal of this section is to construct Green's function as the supremum of the family $\mathcal{G}_q$, which we will prove using Perron's theorem. However, we should first verify that $\mathcal{G}_q$ is a Perron family:

Proposition 3.6. *Let R be a Riemann surface and $q \in R$. Then $\mathcal{G}_q$ is a Perron family on $R \setminus \{q\}$.*

Proof. We show that $\mathcal{G}_q$ satisfies the conditions of Definition 2.23. First note that $R \setminus \{q\}$ is a Riemann surface since it is a connected open subset of R.

Further, each $u \in \mathcal{G}_q$ is subharmonic by definition, and $\mathcal{G}_q$ is nonempty since $0 \in \mathcal{G}_q$. In fact, 0 is subharmonic, $\operatorname{supp}(0) \cup \{q\} = \{q\}$ is compact in R, and $\log|z|$ is subharmonic on D for every coordinate disc (D, z) centered at q.

It is left to prove (P-3) and (P-4). Let $u, v \in \mathcal{G}_q$, and let (D, z) be any coordinate disc in $R \setminus \{q\}$ such that u is finite on ∂D. We need to show that $\max(u, v) \in \mathcal{G}_q$ and $u_D \in \mathcal{G}_q$.

We start with $\max(u, v)$. Clearly $\max(u, v)$ is subharmonic, and we have

$$\operatorname{supp}(\max(u, v)) \cup \{q\} \subseteq (\operatorname{supp}(u) \cup \{q\}) \cup (\operatorname{supp}(v) \cup \{q\}),$$

where the right hand side is compact in R since it is the union of two compact sets. Recall that $\operatorname{supp}(\max(u, v))$ is closed by definition, so $\operatorname{supp}(\max(u, v)) \cup \{q\}$ is a closed subset of a compact set, thus compact in R. Now let $(\hat{D}, \hat{z})$ be any coordinate disc centered at q. Then both $u + \log|\hat{z}|$ and $v + \log|\hat{z}|$ are subharmonic on $\hat{D}$ by Proposition 3.5, thus also their maximum

$$\max(u + \log|\hat{z}|, v + \log|\hat{z}|) = \max(u, v) + \log|\hat{z}|$$

is subharmonic on $\hat{D}$. Hence $\max(u, v) \in \mathcal{G}_q$.

Next we show that $u_D \in \mathcal{G}_q$. Clearly also u_D is subharmonic, and we have

$$\operatorname{supp}(u_D) \cup \{q\} \subseteq (\operatorname{supp}(u) \cup \{q\}) \cup \overline{D},$$

where the right hand side is again a union of two compact sets, thus compact in R. Hence by the same reasoning as before, $\operatorname{supp}(u_D) \cup \{q\}$ is compact in R. Now let $(\hat{D}, \hat{z})$ be any coordinate disc centered at q. Then $u + \log|\hat{z}|$ is subharmonic on $\hat{D}$ and $u = u_D$ on $\hat{D} \setminus D$, thus $u_D + \log|\hat{z}|$ is subharmonic on $\hat{D} \setminus D$. But since $q \notin D$ and u_D is harmonic on D, $u_D + \log|\hat{z}|$ is clearly subharmonic on $\hat{D} \cap D$, and therefore on all of $\hat{D}$. Hence $u_D \in \mathcal{G}_q$, which completes the proof. □

It is now an immediate consequence of Perron's theorem that the supremum over all functions in $\mathcal{G}_q$ is either identically infinite or harmonic on $R \setminus \{q\}$. In the latter case, conditions (1)–(3) in the definition of $\mathcal{G}_q$ indicate that the supremum could be Green's function, which we verify next.

Proposition 3.7. *Let R be a Riemann surface and $q \in R$. If $\sup\{u \mid u \in \mathcal{G}_q\}$ is finite, then it is Green's function with pole at q. Conversely, if g_q exists, then it is given by $\sup\{u \mid u \in \mathcal{G}_q\}$.*

Proof. Let $h = \sup\{u \mid u \in \mathcal{G}_q\}$ and suppose that $h < \infty$. We show that h satisfies the conditions of Definition 3.1. First note that by Perron's theorem, h is harmonic on $R \setminus \{q\}$ since $\mathcal{G}_q$ is a Perron family.

For (G-2), let (D, z) be any coordinate disc centered at q. We want to show that $f = h + \log|z|$ extends to a harmonic function on D. It is clearly harmonic on $D \setminus \{q\}$ since both h and $\log|z|$ are. We show that f is bounded in a neighborhood of q, since then the singularity at q is removable and f is holomorphically extendable to D by Riemann's removable singularity theorem, which is stated in Proposition A.1. The extension must then also be harmonic on D since the Laplacian is continuous.

Let $M = \max\{h(p) \mid p \in \partial D\}$. Note that z is well-defined on ∂D and satisfies $|z| = 1$ on ∂D, thus for any $u \in \mathcal{G}_q$, we have $u + \log|z| = u \leq h \leq M$ on ∂D. But $u + \log|z|$ is subharmonic on D, thus by the maximum principle also $u + \log|z| \leq M$ on D. Now taking the supremum over all $u \in \mathcal{G}_q$ gives

$$f = h + \log|z| \leq M$$

on D. On the other hand, the function $v : R \setminus \{q\} \to \mathbb{R}_0^+$ defined by

$$v = \begin{cases} -\log|z| & \text{on } D \setminus \{q\} \\ 0 & \text{on } R \setminus D \end{cases}$$

is in $\mathcal{G}_q$. In fact, v is subharmonic on $R \setminus \{q\}$, $\mathrm{supp}(v) \cup \{q\}$ is compact in R since it is a closed subset of the compact set $\overline{D}$, and for the coordinate disc (D, z), the function $v + \log|z|$ is zero and therefore subharmonic on D, when extended by zero. Hence $v \leq h$, and we can conclude that on D we have the bound

$$0 = v + \log|z| \leq f \leq M.$$

Thus f is extendable to a harmonic function on D.

For (G-3), observe that $h \geq 0$ since $0 \in \mathcal{G}_q$, as showed in the proof of Proposition 3.6. The function $-h$ is harmonic, thus by the maximum principle, $-h$ cannot attain zero on $R \setminus \{q\}$ unless $h \equiv 0$. But $0 < v \leq h$ on $D \setminus \{q\}$, thus we must have $h > 0$ on $R \setminus \{q\}$.

It is left to prove the uniqueness property (G-4). Suppose that $\hat{g}_q$ is another function satisfying (G-1), (G-2), and (G-3) of Definition 3.1. Then for $u \in \mathcal{G}_q$, the function $u - \hat{g}_q$ is subharmonic on $R \setminus \{q\}$. Now if (D, z) is a coordinate disc centered at q, then we can write

$$u - \hat{g}_q = (u + \log|z|) - (\hat{g}_q + \log|z|).$$

By Proposition 3.5, the function $u + \log|z|$ is subharmonic on D, and by Proposition 3.2, the function $\hat{g}_q + \log|z|$ is harmonic on D. Thus $u - \hat{g}_q$ is subharmonic on

D, and thus on all of R. But since $\hat{g}_q > 0$ on $R \setminus \{q\}$, we have $u - \hat{g}_q < 0$ outside the compact subset $\mathrm{supp}(u) \cup \{q\}$ of R, and thus $u \leq \hat{g}_q$ on all of R by Proposition 2.19. Now taking the supremum over all $u \in \mathcal{G}_q$ gives $h \leq \hat{g}_q$. Hence h is the Green's function with pole at q.

To prove the reverse direction of the proposition, suppose that g_q is a Green function with pole at q. Then g_q satisfies (G-1), (G-2), and (G-3) of Definition 3.1, thus by the same argument as before we have $u \leq g_q$ for all $u \in \mathcal{G}_q$, and consequently $\sup\{u \mid u \in \mathcal{G}_q\} \leq g_q < \infty$. But we have just shown that if the supremum is finite, then it is the Green's function, and since g_q is unique by property (G-4) we can conclude that $\sup\{u \mid u \in \mathcal{G}_q\} = g_q$. □

Let us take a moment to appreciate the significance of this result. Being a Green's function is a quite restrictive property, and it is crucial to bear in mind that it does not exist on every Riemann surface, which we show in Section 5. In some cases, however, we can derive its existence by picking any subharmonic function in $\mathcal{G}_q$ and proving that it is bounded from above, as then the same bound also holds for the supremum.

3.3 Existence

In this section, we prove that the existence of Green‘s function is independent of the location of the pole; that is, if Green‘s function with pole at q exists for some $q \in R$, then it exists with pole at every point in R. As indicated before, the strategy to accomplish this will be to apply the maximum principle to arbitrary subharmonic functions in the Perron family $\mathcal{G}_q$. This will allow us to show upper bounds, and to construct Green‘s function as the finite supremum of $\mathcal{G}_q$.

We first introduce some abbreviation. If (D, z) is a coordinate disc centered at q and $r \in [0, 1]$, we define

$$B_r^q = z^{-1}(\mathbb{B}_r^0) = \{p \in R \mid |z(p)| \leq r\} \subseteq \overline{D}$$

to be the corresponding closed disc of radius r centered at q. Recall that z is well-defined on ∂D and that $z(\partial D) = \partial \mathbb{D}$, so $B_1^q = \overline{D}$ and $B_0^q = \{q\}$. We shall bear in mind in the following that B_r^q depends on the corresponding coordinate disc (D, z), and in particular on the coordinate map z.

The first result we prove is an easy consequence of the maximum principle:

Proposition 3.8. *Let R be a Riemann surface, $q \in R$, and g_q a Green function for R with pole at q. Then*

$$\inf\{g_q(p) \mid p \in R \setminus \{q\}\} = 0.$$

Moreover, for $r \in (0, 1)$ and any coordinate disc (D, z) centered at q, we have

$$\inf\{g_q(p) \mid p \in R \setminus B_r^q\} = 0.$$

Proof. Let $a = \inf\{g_q(p) \mid p \in R \setminus \{q\}\}$, and let (D, z) be a coordinate disc centered at q. We know that $a \geq 0$ since $g_q > 0$, and for $u \in \mathcal{G}_q$ we can write

$$u - g_q + a = (u + \log|z|) - (g_q + \log|z|) + a,$$

which is subharmonic on D since $u + \log|z|$ is subharmonic and $g_q + \log|z|$ is harmonic on D. But $u - g_q + a$ is clearly subharmonic on $R \setminus D$, hence on all of R. Now outside of the compact subset $\operatorname{supp}(u) \cup \{q\}$ of R we have $u - g_q + a = -g_q + a \leq 0$, and thus $u \leq g_q - a$ on all of R by Proposition 2.19. But then taking the supremum over all $u \in \mathcal{G}_q$ gives $g_q \leq g_q - a$, which implies $a = 0$ since a is non-negative.

Now let $r \in (0, 1)$. Since $g_q + \log|z|$ is harmonic on D and harmonic functions are finite, we must have $g_q(p) \to +\infty$ as $p \to q$, thus there exists an open neighborhood U of q such that $g_q \geq 1$ on $U \setminus \{q\}$ and such that $U \subseteq B_r^q$. Now suppose that $\inf\{g_q(p) \mid p \in R \setminus B_r^q\} > 0$. Since $\inf\{g_q(p) \mid p \in R \setminus \{q\}\} = 0$ by the previous part and $g_q \geq 1$ on $U \setminus \{q\}$, the infimum must lie in the compact annulus $B_r^q \setminus U$, thus must be a minimum, so we have $g_q(w) = 0$ for some $w \in B_r^q \setminus U$. But this contradicts $g_q > 0$, hence $\inf\{g_q(p) \mid p \in R \setminus B_r^q\} = 0$. □

Proposition 3.9. *Let R be a Riemann surface, (D, z) a coordinate disc centered at q, and $r \in (0, 1)$. Define $\mathcal{F}$ to be the following set of functions:*

$$\mathcal{F} = \{v : R \setminus B_r^q \to \mathbb{R} \mid v \text{ subharmonic}, \operatorname{supp}(v) \cup B_r^q \text{ compact in } R, v \leq 1\}.$$

Then $\tilde{v} = \sup\{v \mid v \in \mathcal{F}\}$ is a harmonic function on $R \setminus B_r^q$, and either $\tilde{v} \equiv 1$, or $\tilde{v} \in (0, 1)$ and $\tilde{v}(p) \to 1$ as $p \to \partial B_r^q$.

Proof. For the first part, we show that $\mathcal{F}$ is a Perron family on the Riemann surface $R \setminus B_r^q$; harmonicity of $\tilde{v}$ then follows from Perron's theorem since $\tilde{v} \leq 1$.

First, $\mathcal{F}$ is nonempty since $0 \in \mathcal{F}$. Now if $u, v \in \mathcal{F}$, then similarly as in the proof of Proposition 3.6 we get that both $\max(u, v)$ and u_D are subharmonic and satisfy the compact support condition. It is obvious that $\max(u, v) \leq 1$, and we showed in Proposition 2.22 that $u_D \leq 1$ if $u \leq 1$. Thus $\mathcal{F}$ is indeed a Perron family, and hence $\tilde{v}$ is a harmonic function on $R \setminus B_r^q$ by Perron's theorem.

For the remaining part, consider the function $v_0 : R \setminus B_r^q \to \mathbb{R}$ defined by

$$v_0 = \begin{cases} \log|z| \,/\, \log(r) & \text{on } D \setminus B_r^q \\ 0 & \text{on } R \setminus D \end{cases}.$$

Clearly v_0 is subharmonic, $\operatorname{supp}(v_0) \cup B_r$ is compact in R since it is a closed subset of the compact set $\overline{D}$, and on $D \setminus B_r^q$ we have $r < |z| < 1$, so $0 \leq v_0 < 1$ on all of $R \setminus B_r^q$. Hence $v_0 \in \mathcal{F}$, and we can further see that $v_0(p) \to 1$ as $p \to \partial B_r^q$.

But note that $0 \leq v_0 \leq \tilde{v} \leq 1$ since $\tilde{v}$ is the supremum over all functions in $\mathcal{F}$. Thus we also have $\tilde{v}(p) \to 1$ as $p \to \partial B_r^q$. Now if $\tilde{v}(p) = 0$ for some $p \in R \setminus B_r^q$, then the harmonic function $-\tilde{v}$ attains its maximum at p, thus must be constantly zero by the maximum principle. But if $\tilde{v} \equiv 0$ then it can not tend to 1 as $p \to \partial B_r^q$, which is a contradiction. Hence $\tilde{v} \in (0, 1]$. Once more applying the maximum principle shows that $\tilde{v} \equiv 1$ if $\tilde{v}(p) = 1$ for some $p \in R \setminus B_r$. Hence we can conclude that either $\tilde{v} \in (0, 1)$ or $\tilde{v} \equiv 1$. That $\tilde{v}(p) \to 1$ as $p \to \partial B_r$ was shown during the proof. □

We are now equipped to prove the main result in this section:

Proposition 3.10. *Let R be a Riemann surface. If Green's function with pole at q exists for some $q \in R$, then Green's function with pole at w exists for all $w \in R$.*

Proof. Let $q \in R$, and suppose that Green's function g_q with pole at q exists. Let (D, z) be a coordinate disc centered at q and $r \in (0, 1)$. We want to show that Green's function with pole at y exists for all $y \in \mathrm{Int}(B_r^q)$, and then use path-connectedness of R to derive global existence on R.

Using (D, z) and r, define $\mathcal{F}$ and $\tilde{v}$ as in the previous proposition. We first prove that $\tilde{v} \in (0, 1)$. If $v \in \mathcal{F}$, then we can continuously extend v to ∂B_r^q since $0 \leq v \leq 1$. Thus for $a = \min\{g_q(p) \mid p \in \partial B_r^q\} > 0$, the function $v - g_q/a$ is well-defined on $R \setminus \mathrm{Int}(B_r^q)$ and subharmonic on $R \setminus B_r^q$. Further, the set

$$K = \mathrm{supp}(v) \cup \partial B_r^q$$

is compact in $R \setminus \mathrm{Int}(B_r^q)$ since $\mathrm{supp}(v) \cup B_r^q$ is compact in R by definition of $\mathcal{F}$. Note that outside of K we have $v - g_q/a = -g_q/a < 0$, and on ∂B_r^q we have $v - g_q/a \leq 1 - g_q/a < 0$ by definition of a. But the function $v - g_q/a$ is subharmonic on $R \setminus B_r^q$, thus by Proposition 2.19 we also have $v - g_q/a \leq 0$ on K and therefore on all of $R \setminus B_r^q$. Rearranging and taking the supremum over all $v \in \mathcal{F}$ now gives

$$\tilde{v} \leq g_q/a$$

on $R \setminus B_r^q$. But since $0 \leq \tilde{v} \leq 1$ and $\inf\{g_q(p) \mid p \in R \setminus B_r^q\} = 0$ by Proposition 3.8, we get that $\inf\{\tilde{v}(p) \mid p \in R \setminus B_r^q\} = 0$. Hence $\tilde{v} \not\equiv 1$, so $\tilde{v} \in (0, 1)$ by the previous proposition.

Now pick any point $y \in \mathrm{Int}(B_r^q)$ and $u \in \mathcal{G}_y$. Then $A_{z(y)}$ is an automorphism of $\mathbb{D}$ by Proposition 2.11, thus by setting $\hat{z} = A_{z(y)} \circ z$ we obtain a coordinate disc $(D, \hat{z})$ centered at y. Therefore the function $u + \log|\hat{z}|$ is subharmonic on D by Proposition 3.5. Now since $y \notin \partial D$, we can choose $C > 0$ large enough such that $|\log|\hat{z}|| \leq C$ on ∂D. Now let $M_u > 0$ be large enough such that $M_u \geq \max\{u(p) \mid p \in \partial D\}$, which exists since ∂D is compact. Then

$$u + \log|\hat{z}| \leq M_u + C$$

on ∂D, and by the maximum principle also on D since $u + \log|\hat{z}|$ is subharmonic on D. But then, after rearranging, on ∂B_r^q we have

$$u \leq M_u + 2C = (M_u + 2C)\tilde{v},$$

where $\tilde{v}$ is extended by 1 on ∂B_r^q. But the function $u - (M_u + 2C)\tilde{v}$ is subharmonic on $R \setminus B_r^q$, and since $\operatorname{supp}(u) \cup B_r^q$ is compact we can use the same argument as before to conclude that $u \leq (M_u + 2C)\tilde{v}$ on $R \setminus B_r^q$. But then taking the maximum on ∂D and rearranging gives

$$M_u \leq 2C(1 - \max\{\tilde{v}(p) \mid p \in \partial D\})^{-1} = L,$$

where L is a constant independent of u, and finite since $\tilde{v} \in (0, 1)$. Thus $u \leq L$ on ∂D, and taking the supremum over all $u \in \mathcal{G}_y$ yields $\sup\{u \mid u \in \mathcal{G}_y\} \leq L < +\infty$ on ∂D. But then the supremum is not identically infinite, thus by Proposition 3.7, $\sup\{u \mid u \in \mathcal{G}_y\}$ is Green's function with pole at y. Hence Green's function with pole at y exists for all $y \in \operatorname{Int}(B_r^q)$.

We have now shown that if Green's function with pole at q exists for some $q \in R$, then for any coordinate disc (D, z) centered at q, Green's function with pole at y exists for all $y \in \operatorname{Int}(B_r^q)$. Now let $w \in R$ be arbitrary, π a path from q to w, and $\Gamma = \pi([0, 1])$. For every point w_i on Γ there is a coordinate disc centered at w_i by Proposition 2.5. Since the radius $r \in (0, 1)$ could be chosen arbitrary and did not depend on the point q, by compactness of Γ we can find a finite collection $\{(D_i, z_i)\}$ of coordinate discs such that the sets $\{\operatorname{Int}(B_r^{w_i})\}$ cover Γ, and such that their centers are mutually intersecting. But starting with $w_{i_0} = q$ and going along Γ, we can see that Green's function with pole at y exists for each $y \in \bigcup_i \operatorname{Int}(B_r^{w_i})$. Hence Green's function with pole at w exists. □

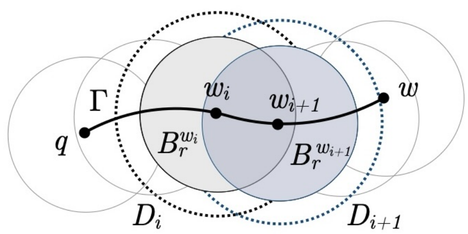

Figure 3. Covering Γ with associated sets $B_r^{w_i}$

We have now shown that the existence of Green's function does not depend on the location of the pole. One special situation where Green's function exists is described by the following:

Proposition 3.11. *Let R be a Riemann surface. If there exists a non-constant and bounded holomorphic function on R, then Green's function exists.*

Proof. Let $q \in R$, and let $g : R \to \mathbb{C}$ be non-constant, holomorphic and bounded by some $C > 0$. Then the function $f : R \to \mathbb{D}$ defined by

$$f = \frac{g - g(q)}{2C}$$

is holomorphic and satisfies $f(q) = 0$. Thus if $u \in \mathcal{G}_q$, then by Proposition 3.5, the function $u + \log|f|$ is subharmonic on R, and we have $u + \log|f| < 0$ outside of the compact set $\text{supp}(u) \cup \{q\}$ since $|f| < 1$. Hence by Proposition 2.19, we have $u \leq -\log|f|$ on all of R. Since g is assumed to be non-constant, $f(w)$ must be nonzero for some $w \in R$, and for this w we have

$$\sup\{u(w) \mid u \in \mathcal{G}_q\} \leq -\log|f(w)| < \infty,$$

so $\sup\{u \mid u \in \mathcal{G}_q\}$ is not identically infinite and thus Green's function with pole at q by Proposition 3.7. Hence Green's function with pole everywhere on R exists by the previous proposition. □

3.4 Symmetry

In the case of simply connected Riemann surfaces, Green‘s function has, assuming that it exists, a remarkable symmetry property:

Proposition 3.12. *Let R be a simply connected Riemann surface and $q, w \in R$ with $q \neq w$. If Green's function exists, then $g_q(w) = g_w(q)$.*

Proof. Fix $q, w \in R$ with $q \neq w$, and let g_q be Green's function with pole at q. By Proposition 3.3, there exists a holomorphic function $\varphi_q : R \to \mathbb{C}$ with $|\varphi_q| = e^{-g_q}$ and a simple zero at q. Note that $\varphi_q(R) \subseteq \mathbb{D}$ since $g_q > 0$ on $R \setminus \{q\}$.

In particular, we have $\varphi_q(w) \in \mathbb{D}$, hence the map $A_{\varphi_q(w)} : \mathbb{D} \to \mathbb{D}$ given by

$$A_{\varphi_q(w)}(z) = \frac{z - \varphi_q(w)}{1 - \overline{\varphi_q(w)}z}$$

is an automorphism of $\mathbb{D}$ by Proposition 2.11. Since $\varphi_q(R) \subseteq \mathbb{D}$, the composition

$$\Phi = A_{\varphi_q(w)} \circ \varphi_q : R \to \mathbb{D}$$

is still well-defined, holomorphic, and has its image in $\mathbb{D}$. Hence if $u \in \mathcal{G}_w$, then $u + \log|\Phi|$ is subharmonic on R by Proposition 3.5. But $u + \log|\Phi| = \log|\Phi| < 0$ outside of the compact subset $\text{supp}(u) \cup \{q\}$ of R, thus $u + \log|\Phi| \leq 0$ on all of R by Proposition 2.19. Note that g_w exists by Proposition 3.10, and that it is given by $\sup\{u \mid u \in \mathcal{G}_w\}$ by Proposition 3.7. Thus taking the supremum over all $u \in \mathcal{G}_w$ yields that

$$g_w + \log|\Phi| \leq 0$$

on R. Now note that by construction of Φ and since φ_q has a zero at q, we have $|\Phi(q)| = |-\varphi_q(w)| = e^{-g_q(w)}$, thus evaluating at q gives

$$g_w(q) + \log|\Phi(q)| = g_w(q) - g_q(w) \leq 0.$$

We can switch q and w and repeat the whole argument to get $g_q(w) - g_w(q) \leq 0$, thus we can conclude that

$$g_q(w) = g_w(q),$$

which is what we wanted to show. □

4 Bipolar Green Function

4.1 Definition and Meromorphic Lift

In this section, we introduce a harmonic function similar to Green's function, but with an additional pole. We prove that on a simply connected Riemann surface, this function can be extended to a meromorphic map in a similar fashion as we extended Green's function to its holomorphic lift.

Definition 4.1. *Let R be a Riemann surface and $q_1, q_2 \in R$ two distinct points. A* **bipolar Green function** *(with poles at q_1 and q_2) is a continuous function $G_{q_1,q_2}(p) : R \setminus \{q_1, q_2\} \to \mathbb{R}$ satisfying:*

(B-1) *G_{q_1,q_2} is harmonic;*

(B-2) *for some coordinate disc (D_1, z_1) centered at q_1, the function $G_{q_1,q_2} + \log|z_1|$ extends to a harmonic function on D_1;*

(B-3) *for some coordinate disc (D_2, z_2) centered at q_2, the function $G_{q_1,q_2} - \log|z_2|$ extends to a harmonic function on D_2, and*

(B-4) *for some coordinate discs $(D_1, z_1), (D_2, z_2)$ centered at q_1, q_2, respectively, the function G_{q_1,q_2} is bounded on $R \setminus (D_1 \cup D_2)$.*

In contrast to Green's function, a bipolar Green function is not unique, and it is defined on the Riemann surface with two points removed, which are both poles. Conditions (B-2) and (B-3) ensure that G_{q_1,q_2} has a positive logarithmic pole at q_1 and a negative logarithmic pole at q_2. As before, this definition is independent of the choice of coordinate discs:

Proposition 4.2. *Let R be a Riemann surface and G_{q_1,q_2} a bipolar Green function. Then for any two coordinate discs $(D_1, z_1), (D_2, z_2)$ centered at q_1, q_2, respectively, $G_{q_1,q_2} + \log|z_1|$ extends to a harmonic function on D_1, $G_{q_1,q_2} - \log|z_2|$ extends to a harmonic function on D_2, and G_{q_1,q_2} is bounded on $R \setminus (D_1 \cup D_2)$.*

Proof. We know by property (B-2) of the above definition that $G_{q_1,q_2} + \log|\hat{z}_1|$ is harmonic on $\hat{D}_1$ for some coordinate disc $(\hat{D}_1, \hat{z}_1)$ centered at q_1. Now similarly as in the proof of Proposition 3.2, we can write

$$G_{q_1,q_2} + \log|z_1| = G_{q_1,q_2} + \log|\hat{z}_1| + \log\left|\frac{z_1}{\hat{z}_1}\right|,$$

which is harmonic on D_1 since the zeroes of z_1 and $\hat{z}_1$ are both simple, and thus cancel in $z_1/\hat{z}_1$. The proof for D_2 is similar. Further, we know that G_{q_1,q_2} is bounded on $R \setminus (\hat{D}_1 \cup \hat{D}_2)$ for some coordinate discs $(\hat{D}_1, \hat{z}_1)$ and $(\hat{D}_2, \hat{z}_2)$ centered at q_1 and q_2, respectively. If we define $W = (\hat{D}_1 \cup \hat{D}_2) \setminus (D_1 \cup D_2)$, then $\overline{W}$ is compact in $R \setminus \{q_1, q_2\}$, thus $|G_{q_1,q_2}|$ attains its maximum on $\overline{W}$. Hence G_{q_1,q_2} is bounded on $\overline{W}$ and thus on all of $R \setminus (D_1 \cup D_2)$. □

We now use harmonicity of a bipolar Green function to lift it to a holomorphic map, similarly as we did for Green's function. Since a bipolar Green function has an additional pole, the lift will, however, be meromorphic:

Proposition 4.3. *Let R be a simply connected Riemann surface, $q_1, q_2 \in R$ with $q_1 \neq q_2$ and G_{q_1,q_2} a bipolar Green function. Then there exists a meromorphic map $\varphi : R \to \mathbb{C} \cup \{\infty\}$ with*

$$|\varphi_{q_1,q_2}| = e^{-G_{q_1,q_2}},$$

a simple zero at q_1, and a simple pole at q_2.

Proof. We proceed similarly as in the proof of Proposition 3.3, with slight modifications and one extra case. Again, let $\mathcal{A} = \{(D_\alpha, z_\alpha)\}$ be a basis of coordinate discs for R, and pick some $(D_\alpha, z_\alpha) \in \mathcal{A}$. Now four cases can occur:

If none of q_1, q_2 is in D_α, or only $q_1 \in D_\alpha$, then the proof is similar as in Proposition 3.3, since away from q_2, the function G_{q_1,q_2} has similar properties as Green's function with pole at q_1 would have.

Now suppose that $q_2 \in D_\alpha$ and $q_1 \notin D_\alpha$. Then for $\hat{z}_\alpha = A_{z_\alpha(q_2)} \circ z$, we obtain a coordinate disc $(D_\alpha, \hat{z}_\alpha)$ centered at q_2. Thus by the previous proposition, the function $f = G_{q_1,q_2} - \log|\hat{z}_\alpha|$ is harmonic on D_α. By Proposition 2.15, there exists a holomorphic function $\hat{\varphi}_\alpha : D_\alpha \to \mathbb{C}$ with $|\hat{\varphi}_\alpha| = e^{-f}$, which is unique up to a multiplication with $e^{i\theta_\alpha}$ for $\theta_\alpha \in \mathbb{R}$. Now let $\varphi_\alpha : D_\alpha \to \mathbb{C} \cup \{\infty\}$ be defined by $\varphi_\alpha = \hat{z}_\alpha^{-1}\hat{\varphi}_\alpha$, which is also uniquely defined up to multiplication with $e^{i\theta_\alpha}$, and meromorphic on D_α since $\hat{\varphi}_\alpha$ is holomorphic and $\hat{z}_\alpha^{-1}$ holomorphic on $D_\alpha \setminus \{q_2\}$ and has a simple pole at q_2. By construction, this function satisfies

$$|\varphi_\alpha| = \left|\hat{z}_\alpha^{-1}\right| |\hat{\varphi}_\alpha| = \left|\hat{z}_\alpha^{-1}\right| e^{-f} = e^{-G_{q_1,q_2}}.$$

Now observe that $f(q_2)$ is finite since f is harmonic on D_α, so $|\hat{\varphi}_\alpha(q_2)|$ is finite, thus φ_α must have a simple pole at q_2. Note that this pole is independent of the multiplication with $e^{i\theta_\alpha}$.

Now suppose that both $q_1, q_2 \in D_\alpha$. The sets $D_\alpha \setminus \{q_1\}$ and $D_\alpha \setminus \{q_2\}$ are open and contain q_2 and q_1, respectively. Since $\mathcal{A}$ is a basis of coordinate discs, there exist coordinate discs $(D_\gamma, z_\gamma), (D_\delta, z_\delta) \in \mathcal{A}$ such that $q_2 \in D_\gamma \subseteq D_\alpha \setminus \{q_1\}$ and $q_1 \in D_\delta \subseteq D_\alpha \setminus \{q_2\}$. But now the previous cases yield our desired functions φ_γ on D_γ and φ_δ on D_δ. Since D_α is simply connected, we can write $D_\alpha \setminus \{q_2\}$ as a union of two simply connected open sets, and apply the Monodromy theorem to each of them in a similar way as in the proof of Proposition 3.3. In this way we obtain a meromorphic map φ_α on D_α with the desired properties.

We have now shown that such a meromorphic map exists on every coordinate disc in $\mathcal{A}$. The extension to all of R by use of the Monodromy theorem is now similar as before, by writing $R \setminus \{q_2\}$ as a union of two simply connected open sets. □

We prove in Chapter 5 that, assuming that Green's function does not exist, this lift is a biholomorphism onto its image, which will then allow us to construct a biholomorphism onto the Riemann sphere or the complex plane.

4.2 Existence

Next to the number of poles, another important difference between Green's function and a bipolar Green function is that the latter always exist on a simply connected Riemann surface, while the former does not. Showing the existence of a bipolar Green function is the aim of this section.

To begin, we need a preparatory result about the existence of Green's function in a special situation; we omitted stating this in Section 3.3 as we exclusively use it in this section for showing the existence of a bipolar Green function.

Proposition 4.4. *Let W, S be simply connected open subsets of a Riemann surface R such that $\overline{W} \subseteq S$, and such that both W and S have a finite atlas of coordinate discs. Then Green's function exists on S, and for any two distinct $q_1, q_2 \in W$ and coordinate discs $(D_1, z_1), (D_2, z_2)$ centered at q_1, q_2, we have*

$$\left| g_{q_1}^{(S)} - g_{q_2}^{(S)} \right| \leq C,$$

on $S \setminus (D_1 \cup D_2)$, where $C > 0$ is a constant independent of S.

Proof. We only sketch the proof. For a more detailed version we refer the reader to [Gam01] or [Pic11].

The first part is to show that Green's function exists for S. To do this, we select a coordinate disc (D, z) on the edge of S, that is, such that $D \cap (R \setminus S) \neq \emptyset$. We fix a point $b \in \partial D \cap \partial S$, and define the function $w : D \to \mathbb{R}$ by

$$\zeta = \mathrm{Re}(\overline{z(b)} \cdot z) - 1,$$

which is subharmonic on D, non-positive, and its extension to ∂D has a zero precisely at b. This function ζ can be extended to the finitely many coordinate discs that cover S, and thus to all of S, such that the extension is still non-positive and has a zero precisely at b. Now by a similar argument as applied in the proof of Proposition 3.10, ζ gives an upper bound for the function $\tilde{v}$ that we introduced in Proposition 3.9. But by construction we have $\zeta(p) \to 0$ as $p \to b$, thus $\tilde{v} \in (0, 1)$, which implies that Green's function exists, as we showed in the proof of Proposition 3.10.

For the second part, we take two distinct points $q_1, q_2 \in W$, disjoint coordinate discs $(D_1, z_1), (D_2, z_2)$ centered at q_1, q_2, respectively, and the corresponding Green functions $g_{q_1}^{(S)}$ and $g_{q_2}^{(S)}$, which exist by the previous. By using several estimates for subharmonic functions, one can show that

$$C_{q_1} - C \leq g_{q_1}^{(S)} \leq C_{q_1}$$

on $\partial B_r^{q_1} \cup B_r^{q_2}$ for some $r \in (0, 1)$, where C_{q_1} is a constant depending on q_1, C a constant independent of S, and $B_r^{q_i}$ the closed ball introduced in section 3.3. A similar result can be shown for $g_{q_2}^{(S)}$. By evaluating these inequalities at q_1 and q_2 and using the symmetry property

$$g_{q_1}^{(S)}(q_2) = g_{q_2}^{(S)}(q_1),$$

which we showed in Proposition 3.12, one then arrives at $|C_{q_1} - C_{q_2}| \leq C$, which is a bound independent of S. Once more applying the maximum principle and rearranging then finally gives

$$\left| g_{q_1}^{(S)} - g_{q_2}^{(S)} \right| \leq 2C$$

on $S \setminus (D_1 \cup D_2)$, with a bound $2C$ that is independent of S. □

We will now construct our simply connected Riemann surface as an increasing sequence of simply connected Riemann surfaces to use this result, and to prove the universal existence of a bipolar Green function:

Proposition 4.5. *Let R be a simply connected Riemann surface. Then for any two distinct points q_1 and q_2 of R, there exists a bipolar Green function with poles at q_1 and q_2.*

Proof. Let $q_1, q_2 \in R$ be distinct, and let $(D_1, z_1), (D_2, z_2)$ be coordinate discs centered at q_1, q_2, respectively. By rescaling, we can choose the discs such that $D_1 \cap D_2 = \emptyset$. By Proposition 2.6 there exists a countable basis of coordinate discs, thus we can write

$$R = \bigcup_{i=1}^{\infty} D_i$$

for some coordinate discs (D_i, z_i). To apply our previous proposition, we aim to write R as a union of simply connected sub-Riemann surfaces S_i that have finite atlases of coordinate discs and satisfy $\overline{S_i} \subseteq S_{i+1}$. We will define this sequence inductively using the D_i. First, let π be a path in R from q_1 to q_2, and $\Gamma = \pi([0,1])$. Since Γ is compact and the D_i are an open cover of Γ, we can pick a finite subcover $\{D_{i_k}\}_{k=1}^{m}$. Now let

$$S_1 = D_1 \cup D_2 \cup \bigcup_{k=1}^{m} D_{i_k}.$$

Since R is simply connected, we can pick our subcover such that S_1 is simply connected, and it has a finite atlas of coordinate discs by design. Now for a given S_i, the set $\overline{S_i}$ is compact and covered by the D_i, thus again admits a finite subcover $\{D_{i_j}\}_{j=1}^{n}$, which we can again pick in a way such that

$$S_{i+1} = S_i \cup \bigcup_{j=1}^{n} D_{i_j}$$

is simply connected. By construction, each S_{i+1} contains at least one coordinate disc that is not contained in S_i. But the D_i are a countable cover of R, thus we must have

$$R = \bigcup_{i=1}^{\infty} S_i.$$

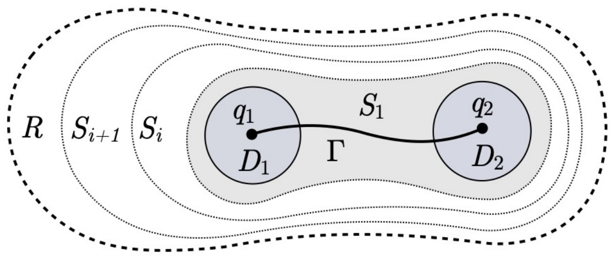

Figure 3. Increasing sequence of Riemann surfaces

We now construct a bipolar Green function for R by use of the previous result and by use of basic convergence properties of uniformly bounded sequences of harmonic functions. We stated the main results we use in Appendix B, but for a more detailed study of harmonic function theory we refer the reader to [ABR01] or [Gam01].

For each S_i with $i \geq 2$, there exists Green functions $(g_{q_1})_i$ and $(g_{q_2})_i$ by the previous proposition. Now define the sequence

$$(G_{q_1,q_2})_i = (g_{q_1})_i - (g_{q_2})_i.$$

Clearly $(G_{q_1,q_2})_i$ is harmonic on $S_i \setminus \{q_1, q_2\}$ since $(g_{q_1})_i$ and $(g_{q_2})_i$ are, and we further have $|(G_{q_1,q_2})_i| \leq C$ on $R \setminus (D_1 \cup D_2)$ with a bound C independent of i by the previous proposition. In other words, $\{(G_{q_1,q_2})_i\}$ is a uniformly bounded sequence of harmonic functions, thus has a subsequence $\{(G_{q_1,q_2})_{i_k}\}$ that converges to a harmonic function G_{q_1,q_2} on $R \setminus (D_1 \cup D_2)$ by Proposition B.5, which is also bounded by C.

It is left to prove that G_{q_1,q_2} satisfies (B-2) and (B-3) of Definition 4.1 and that it can be harmonically extended to $(D_1 \cup D_2) \setminus \{q_1, q_2\}$. We can write

$$(G_{q_1,q_2})_{i_k} + \log|z_1| = (g_{q_1})_{i_k} + \log|z_1| - (g_{q_2})_{i_k},$$

which is harmonic on D_1 by Proposition 3.2 and since $q_2 \notin D_1$. But on ∂D_1 we have $\log|z_1| = 0$, thus we get

$$|(G_{q_1,q_2})_{i_k} + \log|z_1|| = |(g_{q_1})_{i_k} - (g_{q_2})_{i_k}| \leq C,$$

and by the maximum principle the bound also holds on D_1. Similarly as before, by use of Proposition B.5, we can pick a subsequence $\{(G_{q_1,q_2})_{i_l}\}$ of $\{(G_{q_1,q_2})_{i_k}\}$ such that $\{(G_{q_1,q_2})_{i_l} + \log|z_1|\}$ converges to a harmonic function $\hat{G}_{q_1,q_2} + \log|z_1|$ on D_1. On ∂D_1, this sequence converges to G_{q_1,q_2} as the logarithm is zero and since a subsequence of a convergent sequence has the same limit. Thus we can continuously extend G_{q_1,q_2} to $D_1 \setminus \{q_1\}$ by defining it to be the function $\hat{G}_{q_1,q_2}$ on $D_1 \setminus \{q\}$, which is harmonic since $\log|z_1|$ is harmonic on $D_1 \setminus \{q_1\}$. For the coordinate disc (D_2, z_2) we can write

$$(G_{q_1,q_2})_{i_k} - \log|z_2| = (g_{q_1})_{i_k} - ((g_{q_2})_{i_k} + \log|z_2|),$$

and now harmonically extend G_{q_1,q_2} to $D_2 \setminus \{q_2\}$ in a similar way as for D_1. Our extension G_{q_1,q_2} is now a bipolar Green function with poles at q_1 and q_2. □

5 Uniformization Theorem

We are now equipped with all prerequisites to give a proof of the uniformization theorem, which states that every simply connected Riemann surface is biholomorphic to either the Riemann sphere, the complex plane, or the open unit disc. As explained in the introduction, the proof is divided into two parts.

First, we assume that Green‘s function exists, and show that its holomorphic lift, whose existence we proved in Section 3.1, is a biholomorphism onto its image. We then use the Riemann mapping theorem to map the image biholomorphically onto the open unit disc:

Proposition 5.1. *Let R be a simply connected Riemann surface for which Green's function exists. Then R is biholomorphic to the open unit disc.*

Proof. We start similarly as in the proof of Proposition 3.12. Fix $q, w \in R$ with $q \neq w$, and let $\varphi_q : R \to \mathbb{C}$ be a holomorphic function with $|\varphi_q| = e^{-g_q}$. Consider the map $\Phi : R \to \mathbb{D}$ given by

$$\Phi = \frac{\varphi_q - \varphi_q(w)}{1 - \overline{\varphi_q(w)}\varphi_q}.$$

We have already shown in the proof of Proposition 3.12 that Φ is well-defined and holomorphic on R, and that $g_w + \log|\Phi| \leq 0$. We used this inequality in Proposition 3.12 to prove symmetry of Green's function. Recall here that g_w exists by Proposition 3.10, and that $g_q(w) = g_w(q)$ by its symmetry property. But similarly as in Proposition 3.12, we then get

$$g_w(q) + \log|\Phi(q)| = g_w(q) - g_q(w) = 0,$$

so the function $g_w + \log|\Phi|$ attains its maximum at q. But $g_w + \log|\Phi|$ is subharmonic on R by Proposition 3.2, hence by the maximum principle $g_w + \log|\Phi| = 0$ on all of R. Rearranging and taking exponential gives

$$|\varphi_w| = e^{-g_w} = |\Phi|.$$

Now φ_w only has a zero w by Proposition 3.3, thus so has Φ. But by construction, $\Phi(p) = 0$ if and only if $\varphi_q(p) = \varphi_q(w)$; thus φ_q is injective.

Hence $\varphi_q : R \to \varphi_q(R)$ is a bijective holomorphic map, thus a biholomorphism by Proposition 2.10. Further, since φ_q is a homeomorphism, $\varphi_q(R)$ is a non-trivial simply connected open subset of $\mathbb{C}$. By the Riemann mapping theorem, there exists a biholomorphism f from $\varphi_q(R)$ onto $\mathbb{D}$. The composition $f \circ \varphi_q$ is now our desired biholomorphism from R onto $\mathbb{D}$. □

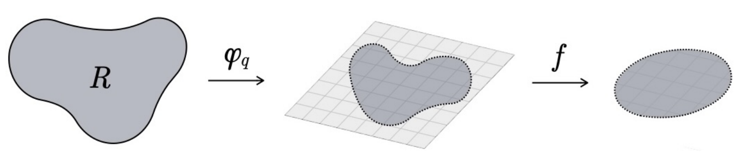

Figure 1. Uniformization via Green's function

We shall emphasize here that this result also proves that Green's function does not exist on every Riemann surface, which we mentioned before without giving a proof. In fact, Green's function cannot exist on $\mathbb{C} \cup \{\infty\}$ or $\mathbb{C}$ since they are not biholomorphic to $\mathbb{D}$, as we showed in Proposition 2.12.

For the remaining case, we assume that Green's function does not exist, and show that the meromorphic lift of any bipolar Green function, whose existence we showed in Section 4.1, is a biholomorphism onto its image. The image is then either the Riemann sphere or biholomorphic to the complex plane:

Proposition 5.2. *Let R be a simply connected Riemann surface for which Green's function does not exist. Then R is biholomorphic to the Riemann sphere or the complex plane.*

Proof. Fix some distinct $q_1, q_2 \in R$ and let G_{q_1,q_2} be a bipolar Green function, which exists by Proposition 4.5. By Proposition 4.3 there exists a meromorphic function $\varphi_{q_1,q_2} : R \to \mathbb{C} \cup \{\infty\}$ with $|\varphi_{q_1,q_2}| = e^{-G_{q_1,q_2}}$. Similarly as in the previous case, we will show that this function is a biholomorphism onto its image.

Let $q_0 \in R$ be distinct from q_1, q_2, and let $\Phi : R \to \mathbb{C}$ be defined by

$$\Phi = \frac{\varphi_{q_1,q_2} - \varphi_{q_1,q_2}(q_0)}{\varphi_{q_0,q_2}}.$$

We aim to show that Φ is a nonzero constant to derive injectivity of φ_{q_1,q_2}. To do this, we prove that Φ is holomorphic and bounded, and then use Proposition 3.11 together with our assumption that Green's function does not exist.

We first prove that Φ is holomorphic. The points q_0 and q_2 are the only critical points to investigate, since everywhere else φ_{q_0,q_2} and $\varphi_{q_1,q_2} - \varphi_{q_1,q_2}(q_0)$ have neither zeroes nor poles by Proposition 4.3. Note that φ_{q_0,q_2} only has a simple zero at q_0 and $\varphi_{q_1,q_2} - \varphi_{q_1,q_2}(q_0)$ also has a simple zero at q_0, and that both $\varphi_{q_1,q_2} - \varphi_{q_1,q_2}(q_0)$ and φ_{q_0,q_2} only have a simple pole at q_2. Thus the zeroes and poles cancel in Φ, so Φ is indeed holomorphic.

Next, we show that Φ is bounded. We know by property (B-4) of Definition 4.1 that G_{q_1,q_2} is bounded away from q_1 and q_2, and that G_{q_0,q_2} is bounded away from q_0 and q_2, thus Φ is bounded away from q_0, q_1 and q_2. But we have just shown that Φ is holomorphic, so it must also be bounded at the points q_0, q_1 and q_2; thus Φ is bounded on all of R.

Now Φ is a bounded holomorphic function on R and Green's function does not exist on R by assumption. Hence Φ must be constant by Proposition 3.11, and nonzero since $\Phi(q_1) = -\varphi_{q_1,q_2}(q_0)/\varphi_{q_0,q_2}(q_1) \neq 0$. But then on $R \setminus \{q_0\}$ we have $\varphi_{q_1,q_2} - \varphi_{q_1,q_2}(q_0) \equiv \Phi \cdot \varphi_{q_0,q_2} \neq 0$. Hence φ_{q_1,q_2} is injective and therefore a biholomorphism from R onto its image by Proposition 2.10.

We now show that the set $S = (\mathbb{C} \cup \{\infty\}) \setminus \varphi_{q_1,q_2}(R)$ can contain at most one point. Suppose for contradiction that S has more than one point, and let $w \in S$. Note that the complement of every simply connected subset of $\mathbb{C} \cup \{\infty\}$ is simply connected;

hence so is S, as it is the complement of the simply connected set $\varphi_{q_1,q_2}(R)$. Now let $\hat{f}_w : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be the fractional transformation

$$\hat{f}_w(z) = \frac{z+w}{z-w},$$

and set $f_w = P_N \circ \hat{f}_w$, where P_N is the stereographic projection from the north pole of $\mathbb{C} \cup \{\infty\}$ onto $\mathbb{C}$. Then f_w is a biholomorphism, and the set $(f_w \circ \varphi_{q_1,q_2})(R)$ is a simply connected open subset of $\mathbb{C}$. Now since S contains more than one point by assumption, $(f_w \circ \varphi_{q_1,q_2})(R)$ is non-empty and not all of $\mathbb{C}$, thus there exists a biholomorphism $\hat{\varphi}$ onto the open unit disc by the Riemann mapping theorem. But then $\hat{\varphi} \circ f_w \circ \varphi_{q_1,q_2} : R \to \mathbb{D}$ is a non-constant, bounded and holomorphic function on R, thus by Proposition 3.11, Green's function exists for R, contradicting our assumption. Hence S can contain at most one point.

If S contains precisely one point, say w, then $f_w \circ \varphi_{q_1,q_2}$ is a biholomorphism from R onto $\mathbb{C}$. If $S = \emptyset$, then φ_{q_1,q_2} is a biholomorphism from R onto $\mathbb{C} \cup \{\infty\}$. □

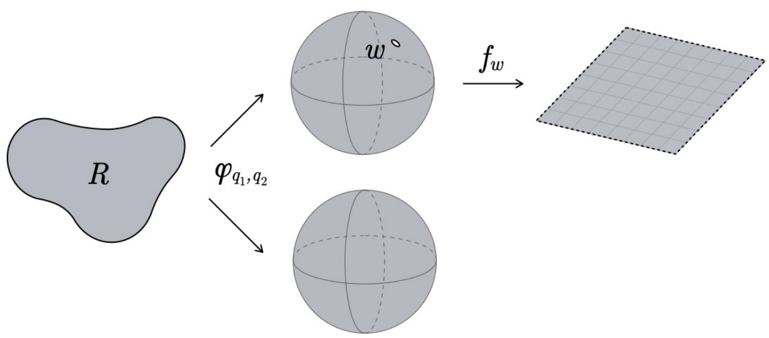

Figure 1. Uniformization via a bipolar Green function

It is important to note that the meromorphic lift of a bipolar Green function is only injective if we assume that Green's function does not exist. Therefore, the division into these two cases is crucial to make the proof work.

This two-part structure, however, automatically yields a characterisation of simply connected Riemann surfaces. In fact, we have the following:

Proposition 5.3. *Let R be a simply connected Riemann surface. Then R is biholomorphic to the open unit disc if and only if Green's function exists for R, and R is biholomorphic to the Riemann sphere if and only if R is compact.*

Proof. We already showed the first part with Proposition 5.1 and Proposition 3.11. For the second part, simply note that the Riemann sphere is compact while the complex plane and the open unit disc are not, and compactness is preserved under biholomorphisms. □

Those simply connected Riemann surfaces that are biholomorphic to the open unit disc are also called *hyperbolic*, those biholomorphic to the Riemann sphere are called *elliptic*, and those biholomorphic to the complex plane are called *parabolic*.

A Holomorphic Functions in the Plane

We provide here some elementary properties of holomorphic functions in the plane that we frequently use. Throughout the whole chapter, let U denote a domain in $\mathbb{C}$, that is, a connected open subset of $\mathbb{C}$.

Proposition A.1 (Riemann's removable singularity theorem). *Let $f : U \setminus \{w\} \to \mathbb{C}$ be a holomorphic function. Then the following are equivalent:*

(1) *f is continuously extendable to U;*

(2) *f is holomorphically extendable to U, and*

(3) *f is bounded on some neighbourhood of w.*

If the conditions are satisfied, then w is called a **removable singularity***.*

Proposition A.2. *Let $f : U \to \mathbb{C}$ be an injective holomorphic function. Then $f'(z) \neq 0$ for all $z \in U$.*

Proposition A.3 (Complex inverse function theorem). *Let $f : U \to \mathbb{C}$ be a holomorphic function. If $f'(w) \neq 0$ for some $w \in U$, then there exists an open neighbourhood V of w such that f is injective on V, and such that $f^{-1} : f(V) \to V$ is holomorphic.*

Proposition A.4 (Maximum modulus principle). *Let $f : U \to \mathbb{C}$ be holomorphic and non-constant. Then $|f|$ can not attain its maximum on U. In particular, for every compact set $K \subseteq U$ the maximum of f on K must be attained on ∂K.*

Proposition A.5 (Schwarz's lemma). *Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$ and $|f| \leq 1$ on $\mathbb{D}$. Then*

$$|f(z)| \leq |z| \quad \text{and} \quad \left|f'(0)\right| \leq 1$$

for all $z \in \mathbb{D}$. Furthermore, if $|f(z)| = |z|$ for some $z \in \mathbb{D} \setminus \{0\}$ or if $|f'(0)| = 1$, then f is a rotation, that is, $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$.

Proposition A.6 (Liouville's theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded holomorphic function. Then f is constant.*

Theorem A.7 (Monodromy theorem). *Let U be a simply connected open subset of $\mathbb{C}$, and $f : U \to \mathbb{C}$ holomorphic at z_0. If f can be holomorphically extended to any curve in U starting at z_0, then f can be holomorphically extended to all of U.*

Theorem A.8 (Riemann mapping theorem). *Every nontrivial simply connected open subset of $\mathbb{C}$ is biholomorphic to the open unit disc.*

B Harmonic and Subharmonic Functions in the Plane

We provide here some elementary properties of harmonic and subharmonic functions in the plane that we frequently use. Again, U denotes a domain in $\mathbb{C}$.

If $f : U \to \mathbb{R}$ is a continuous function and $\mathbb{D}_r^a$ is an open disc in U of radius r centered at a, we abbreviate the mean value of f on $\partial\mathbb{D}_r^a$ by

$$M[f, \partial\mathbb{D}_r^a] = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a + re^{i\theta}) d\theta.$$

Proposition B.1. *Let $h : U \to \mathbb{R}$ be a twice continuously differentiable function. Then the following conditions are equivalent:*

(1) *h is a solution to the Laplace equation, that is, $\Delta h = 0$;*

(2) *f satisfies the mean value equality, that is, $h(a) = M[h, \partial\mathbb{D}_r^a]$ for every disc $\mathbb{D}_r^a$ in U centered at a;*

(3) *h is locally the real part of a holomorphic function, whose imaginary part is unique up to an additive real constant.*

A function that satisfies these properties is called **harmonic***.*

Especially by the third property, harmonic functions can be thought of as the real equivalent to holomorphic functions. Holomorphic functions have far more rigid properties than real differentiable functions; harmonic functions, however, show similar rigidity as we shall see shortly with the maximum principles.

Proposition B.2 (Poisson Integral)**.** *Let $g : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function, and for $r \in [0, 1)$ define the function $P_r : [0, 2\pi] \to \mathbb{R}$ by*

$$P_r(\theta) = \operatorname{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}.$$

Then the function $h : \mathbb{D} \to \mathbb{R}$ defined by

$$h(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) g(e^{it}) dt$$

is the unique harmonic function on $\mathbb{D}$ such that its extension to $\partial\mathbb{D}$ coincides with g almost everywhere on $\partial\mathbb{D}$. The function P_r is called the **Poisson kernel***.*

Note here that it is possible that the extension does not coincide with g at isolated points of $\partial\mathbb{D}$. These points, however, have measure zero and do not affect our definition of the harmonic extension in Section 2.4, as harmonicity and subharmonicity are properties characterised by integrals. We thus neglect this fact in the text, and for a more detailed study of the Poisson integral we refer the reader to [Gam01].

Proposition B.3. *Let $u : U \to \mathbb{R} \cup \{-\infty\}$ be a continuous function. Then the following conditions are equivalent:*

(1) *If h is a harmonic function with $u \leq h$ on $\partial\mathbb{D}_r^a$ for some open disc $\mathbb{D}_r^a$ in U of radius r and center a, then $u \leq h$ on $\mathbb{D}_r^a$;*

(2) *u satisfies the mean value inequality, that is, $u(a) \leq M[u, \partial\mathbb{D}_r^a]$ for every open disc $\mathbb{D}_r^a$ in U of radius r and center a;*

(3) *If u is twice continuously differentiable, then $\Delta u \geq 0$;*

A function that satisfies these properties is called **subharmonic***.*

It is an immediate consequence of this definition that both the sum and the maximum of subharmonic functions is again subharmonic.

The following result is the main property why subharmonic functions are so useful for the proof of the uniformization theorem. It also shows similarities between being subharmonic or harmonic and holomorphic, when comparing this proposition to the maximum modulus principle. We shall emphasize here that the maximum principle also holds for harmonic functions, as every harmonic function is automatically subharmonic. Further, when restricting to harmonic functions, a similar statement holds for minima.

Proposition B.4. *Let $u : U \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function. Then u can not attain its maximum on U unless u is constant. In particular, for every compact set $K \subseteq U$ the maximum of u on K must be attained on ∂K.*

Proposition B.5. *Let $\{u_n\}$ be a sequence of harmonic functions on U that is uniformly bounded on each compact subset of U. Then there exists a subsequence of $\{u_n\}$ that converges uniformly to a harmonic function on each compact subset of U.*

Proposition B.6 (Harnack's principle)**.** *Let $\{u_n\}$ be an increasing sequence of subharmonic functions on U. Then either $u_n \to \infty$ as $n \to \infty$, or $\{u_n\}$ converges uniformly on compact subsets of U to a harmonic function.*

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