SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

A Journey From The Borsuk-Ulam Theorem

av

Jialing Yu

2023 - M2

A Journey From The Borsuk-Ulam Theorem

Jialing Yu

Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Gregory Arone

2023

Abstract

This article is a journey which starts from the Borsuk-Ulam theorem and ends in the topological Tverberg conjecture. These two topics are the main concerns of this article. We first discuss the Borsuk-Ulam theorem and give some of its applications within and outside of mathematics. Later we present two generalizations of the Borsuk-Ulam theorem and show how they can be used to prove the topological Tverberg conjecture for the prime and prime power situations. In the end, we state the idea about how to disprove the topological Tverberg conjecture for the non prime power situation.

Acknowledgements

For the thesis I would like to thank my professor Gregory Arone who has the patience for my numerous questions, and the capability of detecting where my confusion lies, and clearing it up for me.

I would also like to thank myself for putting so much effect into it, and my twin sister who is always by my side.

Contents

1

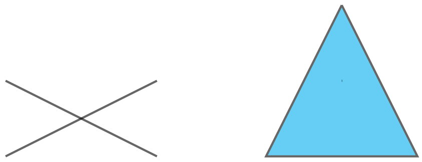
Introduction

This article mainly addresses two topics: the first one regarding the Borsuk-Ulam theorem and its applications; the seond one is about the topological Tverberg conjecture.

The Borsuk-Ulam theorem and its applications Our journey starts from the Boruk-Ulam theorem, which is one of the most useful tool in algebraic topology. The theorem not only is interesting in itself—it has many different equivalent formulations and many different kinds of proofs, but also has lots of generalizations and numerous important applications in various fields within and outside of mathematics.

The most common version of the theorem says that: for any continuous map $f : S^n \to \mathbb{R}^n$, there exists a pair of antipodal points (points that are opposite with respect to the center of S^n) x and $-x$ in S^n, such that $f(x) = f(-x)$. There is a popular interpretation of the theorem which says that, there are always two antipodal locations on earth admitting the same temperature and the same air pressure. However, since the temperature on earth is obviously not continuous, this interpretation is interesting but might not be correct.

Given its importance, we introduce the Borsuk-Ulam theorem and give some of its applications in section 3. More precisely, in subsection 3.1, we give 4 equivalent formulations of the Borsuk-Ulam theorem, and offer a proof of the theorem using more advanced knowledge of algebraic topology. In the subsection 3.2, we state the combinatorial equivalent version of the Borsuk-Ulam theorem—the Tucker's lemma, and give a combinatorial proof of the lemma. This gives another proof of the Boruk-Ulam theorem using pure combinatorical knowledge. In addition, we state a set covering equivalent version of the Boruk-Ulam theorem—the LSB theorem, and show the equivalence between them. We also give an important corollary of the Boruk-Ulam theorem—the Brouwer's fixed point theorem. Similarly with the



(a) case 1 (b) case 2

Figure 1: $r = 2, d = 2$

Boruk-Ulam theorem, we offer a combinatorial equivalent version of the Brouwer's fixed point theorem—the Sperner's lemma, and a set covering equivalent version of the Brouwer's fixed point theorem—the KKM theorem. The Brouwer's fixed point theorem is another important tool in topology. It is used in many different fields including mathematical programming, game theory, and economics. For example, the well-known mathematician John Nash used the Brouwer's fixed point theorem and its generalization—the Kakutani fixed-point theorem to prove the existence of Nash equilibrium.

The topological Tverberg conjecture The second topic of the article is the topological Tverberg conjecture, which was considered to be a central unsolved problem in topological combinatorics. The conjecture says that, for every positive integer r, d, any continuous map $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ of a $(d+1)(r-1)$-dimensional simplex $\Delta_{(d+1)(r-1)}$, there exists r pairwise disjoint faces $F_1, \ldots, F_r \subseteq \Delta_{(d+1)(r-1)}$, such that $f(F_1) \cap \cdots \cap f(F_r) \neq \emptyset$.

The topological Tverberg conjecture comes from the Tverberg theorem, which says that for every positive integer r, d, any $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be decomposed into r disjoint parts, such that the intersection of all the convex hulls of the r parts is non-empty. When $r = 2$, this is called the Radon's theorem.

For $r = 2, d = 2$, the intersection has two situations as in figure 1. The Tverberg's theorem can be reformulated using affine map, which is for every positive integer r, d, for any affine map $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ there exists r pairwise disjoint faces $F_1, \ldots, F_r$ such that the intersection of their images is non-empty, i.e., $f(F_1) \cap f(F_2) \cdots \cap f(F_r) \neq \emptyset$. The topological Tverberg conjecture says that we can generalize the affine map above into continuous map.

The conjecture was proved by Bajmóczy and Bárány in 1981 for r a prime number, and by Özaydin(1987), Volovikov(1996), and Sarkaria(2000) for r a prime power. Whether the conjecture is true for r non prime power remained unsolved for quite a long time. Until 2015, a counterexample of dimension $d \geq 3r+1$ was constructed by Florian Frick using some important technique (the r-fold Van Kampen finger moves and r-fold Whitney trick) developed by Mabillard and Wagner. Thus the topological Tverberg conjecture was shown to be false for r non prime power. The lowest dimension of counterexamples which have been constructed is $d \geq 2r+1$. The problem of whether there exists counterexamples for r non prime power, $d \leq 2r$ is still open.

In this article, we discuss the topological Tverberg conjecture by first introducing two generalizations of the Borsuk-Ulam theorem in subsection 3.3. We later use these two generalizations to prove the topological Tverberg conjecture for r prime and prime power. After that, in subsection 4.1 we give a short history of the conjecture. In subsection 4.2 we prove the conjecture for prime power case by showing the nonexistence of a Σ_r-equivariant map, which gives the non existence of almost r-embedding. In subsection 4.3, we disprove the topological Tverberg conjecture for r non prime power. We do this as follows: we first show the Topological Tverberg conjecture implies the generalized Van Kampen-Flores conjecture in section 4.3.2, and then in section 4.3.3, we show the existence of a Σ_r-equivariant map using equivariant obstruction theory and then show this implies the existence of an almost r-embedding using r-fold Van Kampen finger moves and r-fold Whitney trick, which gives a counterexample for the the generalized Van Kampen-Flores conjecture. However, due to the codimension restriction of the r-fold Whitney trick, the topological Tverberg conjecture is still open for low dimensional cases.

Finally, section 2 is the foundation of this article, which contains some preliminary knowledge and lots of important examples for later discussion.

2

Foundations

2.1 Notations

In this thesis we will adopt the following notations:

- $\mathbb{R}^n$: the n-dimensional Euclidean space
- e: the identity element of a group
- Σ_n: symmetric group of degree n.
- B^n: the unit n-ball.
- S^{n-1}: the unit $n-1$ sphere.
- Δ_n or Δ^n: a standard n-simplex.
- $Id(X)$: the identity map $Id : X \to X$.

2.2 Cell Complexes

In this subsection, we introduce cell complexes and simplicial complexes. Cell complexes are topological spaces built by attaching cells of increasing dimensions along their boundaries. Many topological spaces can be built in this way and the cell structure of the topological space gives important information about the space. Simplicial complexes are a special kind of cell complexes. They are topological spaces built by gluing simplicies of different dimensions along their boundaries under some particular rules.

2.2.1 cell complexes

Definition 2.2.1 (n-cell)**.** A n-cell e^n is a space which is homeomorphic to the unit n-dimensional ball B^n.

Definition 2.2.2 (Cell Complex(CW complex))**.** A cell complex (CW complex) X is a topological space constructed in the following way:

1. start with a discrete set X_0, whose points are regarded as 0-cells, and X_0 is the 0-skeleton of X.

2. form the n-skeleton X_n from the X_{n-1} inductively on n: let $e^n_\alpha \cong B^n_\alpha$ be a n-cell with boundary $\partial e^n_\alpha \cong \partial B^n_\alpha \cong S^{n-1}$ and interior $Int(e^n_\alpha) \cong Int(B^n_\alpha)$, we attach it to X_{n-1} along the boundary ∂B^n_α by the attaching map $\varphi_\alpha : \partial B^n_\alpha \cong S^{n-1} \to X_{n-1}$. And we define X_n to be the quotient space of the disjoint union $X_{n-1} \sqcup_\alpha B^n_\alpha$ of X_{n-1} with a collection of n-cells B^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial B^n_\alpha$. Thus $X_n = X_{n-1} \sqcup_\alpha Int(e^n_\alpha)$ where each $Int(e^n_\alpha)$ is an open n-cell.

3. we can either stop the above procedure at some finite n and have a finite dimensional cell complex(n-dimensional), or we can continue the procedure infinitely and have $X = \cup_n X_n$. We equip X with the weak topology: a set $S \subseteq X$ is open (or closed) if and only if its intersections $S \cap X_n$ with each n-skeleton X_n of X is open (or closed) in X_n for all n.

A cell complex with its top dimensional cells being n-dimensional is a n-dimensional cell complex. A cell complex with finitely many cells is a finite cell complex. Obviously a finite cell complex is finite dimensional but the converse it not true. A cell complex can be finite dimensional but has infinitely many cells.

Each n-cell e^n_α has its characteristic map which is defined by $\phi_\alpha : B^n_\alpha \hookrightarrow X_{n-1} \sqcup_\alpha B^n_\alpha \to X_n \hookrightarrow X$.

A subcomplex of a cell complex X is a closed subspace $A \subseteq X$ which is a union of cells of X. And A itself is a cell complex.

Example 2.2.1 (cell complex structure of real projective space RP^n)*.* Real projective n space RP^n is defined to be the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation $v \sim \lambda v$ for $v \in \mathbb{R}^{n+1} \setminus \{0\}$ and $\lambda \in \mathbb{R}, \lambda \neq 0$. If we restrict to v of length 1, RP^n can also be defined to be $S^n/(v \sim -v)$, i.e., the quotient space of S^n by the equivalence of the antipodal points v and $-v$. This is also the same with saying that RP^n is the quotient space of the hemisphere B^n identifying the antipodal points on its boundary ∂B^n. Since $\partial B^n \cong S^{n-1}$ with its antipodal points identified is again RP^{n-1}, we can see that RP^n is obtained from RP^{n-1} by attaching a n cell e^n along its boundary ∂e^n through the attaching map $\partial e^n \cong S^{n-1} \xrightarrow{q} S^{n-1}/(v \sim -v) \cong RP^{n-1}$

where q is the quotient map. By induction on n we can see that RP^n has a cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^k in each dimension $k \leq n$.

Definition 2.2.3 (cellular map)**.** Let X, Y be two cell complexes, let X_n, Y_n be the n-skeleton of X and Y respectively, a cellular map $f : X \to Y$ is a continuous map which maps the n-skeleton of X to the n-skeleton of Y for all $n = 0, 1, \ldots, n, \ldots$, i.e., $f(X_n) \subseteq Y_n$ for all n.

The following theorem tells us that any continuous map between cell complexes can be deformed to a cellular map.

Theorem 2.2.1 (cellular approximation theorem)**.** *Any continuous map $f : X \to Y$ between two cell complexes is homotopic to a cellular map.*

Proof. The proof can be found in [Hat02, section 4.1]. □

2.2.2 SIMPLICIAL COMPLEX

Definition 2.2.4 (convex combination)**.** A convex combination is a linear combination of points where the coefficients are non negative and sum to 1. More precisely, given a set of finite points $\{v_0, \ldots, v_n\}$ in Euclidean space $\mathbb{R}^m$, a convex combination of these points is any point of the form $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ where $0 \leq \lambda_i \leq 1$ for all i, and $\sum_{i=0}^n \lambda_i = 1$.

Definition 2.2.5 (affine combination)**.** An affine combination is a linear combination of points where the coefficients sum to 1. More precisely, given a set of finite points $\{v_0, \ldots, v_n\}$ in Euclidean space $\mathbb{R}^m$, an affine combination of these points is any point of the form $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ where $\lambda_i \in \mathbb{R}$ for all i, and $\sum_{i=0}^n \lambda_i = 1$.

Definition 2.2.6 (convex hull)**.** Given a subset $S \subseteq \mathbb{R}^m$, the convex hull $conv(S)$ of the subset is the smallest convex set in $\mathbb{R}^m$ containing S, where a subset $C \subseteq \mathbb{R}^m$ is convex if and only if it contains all the convex combination of its points.

Equivalently, the convex hull of S is the set of all convex combination of finite points in S.

Definition 2.2.7 (affine hull)**.** Given a subset $S \subseteq \mathbb{R}^m$, the affine hull of the subset is the smallest affine set in $\mathbb{R}^m$ containing S, where a subset $C \subseteq \mathbb{R}^m$ is affine if and only if it contains all the affine combination of its points.

Equivalently, the affine hull of S is the set of all affine combination of finite points in S.

Definition 2.2.8 (affinely independent points(points in general position))**.** A set of n points $\{v_0, v_1, \dots, v_{n-1}\} \subseteq \mathbb{R}^m$ are affinely independent(in general position) if they are not contained in any affine subspace of dimension less than $n-1$, i.e., the smallest affine subspace containing them has dimension $n-1$.

Equivalently, a set of n points $\{v_0, v_1, \dots, v_{n-1}\} \subseteq \mathbb{R}^m$ are affinely independent if the set of $n-1$ vectors $\{v_i - v_0 | i = 1, \dots, n-1\}$ is linearly independent.

The definition implies $n-1 \leq m$, i.e., there is no more than $m+1$ affinely independent points in $\mathbb{R}^m$.

Definition 2.2.9 (n-simplex)**.** An n-dimensional simplex (n-simplex) σ^n is the convex hull of $n+1$ affinely independent points $\{v_0, \dots, v_n\}$ in Euclidean space $\mathbb{R}^m$. The v_i's are the vertices of σ^n. We say σ^n is spanned by its vertices and write $\sigma^n = conv(v_0, \dots, v_n)$ or $\sigma^n = [v_0, \dots, v_n]$. We can also denote σ^n using barycentric coordinates:

$$\sigma^n = \{\sum_{i=0}^{n} \lambda_i v_i | 0 \leq \lambda_i \leq 1 \text{ for all } i, \text{ and } \sum_{i=0}^{n} \lambda_i = 1\},$$

where λ_i's are the barycentric coordinates of x in σ^n. And $\sum_{i=0}^{n} \frac{1}{n+1} v_i$ is the barycenter of σ^n. Since any subset of affinely independent points are still affinely independent, any subset S of $\{v_0, \dots, v_n\}$ spans a simplex, we call it a face of σ^n. A face is called proper if S is a proper subset of $\{v_0, \dots, v_n\}$. The face of σ^n which contains x as an interior point is the support $supp(x)$ of x in σ^n. The union of all faces of the simplex σ^n of dimension $\leq d$ is the d-skeleton $skel_d \sigma^n$ of σ^n.

Example 2.2.2 (standard n-simplex Δ^n)*.* The standard n-simplex Δ^n in $\mathbb{R}^{n+1}$ is the simplex with vertices at the $n+1$ points $\{e_i, i = 0, \dots, n\}$ of $\mathbb{R}^{n+1}$ where $e_i = (0, \dots, 1, \dots, 0)$(only the i-th component is 1, others are 0.), i.e. the vertices of the standard n-simplex are the unit vectors along the coordinate axes of $\mathbb{R}^{n+1}$. We can denote Δ^n by $conv(e_0, \dots, e_n)$ or $[e_0, \dots, e_n]$. We can also write Δ^n as:

$$\Delta^n = \{(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1} | 0 \leq \lambda_i \leq 1 \text{ for all } i, \text{ and } \sum_{i=0}^{n} \lambda_i = 1\}$$

We can glue the simplicies together to form a geometric simplicial complex:

Definition 2.2.10 (geometric simplicial somplex)**.** A (geometric) simplicial complex $\mathcal{K}$ is a collection of simplices in $\mathbb{R}^m$ which satisfies:

1. any face of a simplex in $\mathcal{K}$ is also a simplex in $\mathcal{K}$.

2. the intersection of any two simplices in $\mathcal{K}$ is a face of both simplices.

The dimension of the simplicial complex is the largest dimension of the simplices of $\mathcal{K}$. A simplicial complex is finite if it contains only finite number of simplices.

Definition 2.2.11 (polyhedron of geometric simplicial complex)**.** Let $\|\mathcal{K}\| = \cup\{\sigma|\sigma \in \mathcal{K}\}$ be the subset of $\mathbb{R}^m$ which is the union of the simplices σ of $\mathcal{K}$, we can define a topology on $\|\mathcal{K}\|$ by giving each simplex σ the natural subspace topology as a subspace of $\mathbb{R}^m$, and then define any subset S of $\|\mathcal{K}\|$ to be open if and only if for all $\sigma \in \mathcal{K}$, the intersection $S \cap \sigma$ is open in σ. This defines a topology on $\|\mathcal{K}\|$ since the collection of open subsets are closed under arbitrary unions and finite intersections. The space $\|\mathcal{K}\|$ is called the polyhedron of $\mathcal{K}$.

Definition 2.2.12 (subcomplex)**.** If $\mathcal{L} \subseteq \mathcal{K}$ is a subcollection of $\mathcal{K}$ that contains all faces of its elements, then $\mathcal{L}$ is itself a simplicial complex, called a subcomplex of $\mathcal{K}$. One subcomplex of $\mathcal{K}$ is the collection of all simplices of $\mathcal{K}$ of dimension at most k, and we call it the k-skeleton of $\mathcal{K}$, and denoted it by $skel_k\mathcal{K}$. The 0-skeleton of $\mathcal{K}$ is called the vertices of $\mathcal{K}$.

Definition 2.2.13 (triangulation of topological space)**.** A triangulation T of a topological space X consists of a geometric simplicial complex $\mathcal{K}$ and a homeomorphism $T : \|\mathcal{K}\| \to X$, where $\|\mathcal{K}\|$ is the polyhedron of $\mathcal{K}$. The vertices and simplices of the geometric simplicial complex $\mathcal{K}$ are said to be the vertices and simplices of the triangulation T. A topological space which admits a triangulation is said to be triangulable.

Definition 2.2.14 (subdivision of simplicial complex)**.** A geometric simplicial complex $\mathcal{K}'$ is a subdivision of a geometric simplicial complex $\mathcal{K}$, denoted as $\mathcal{K}' \prec \mathcal{K}$, if

- $|\mathcal{K}'| = |\mathcal{K}|$
- for any simplex $\sigma \in \mathcal{K}'$, there exists a simplex $\tau \in \mathcal{K}$ such that $\sigma \subseteq \tau$.

Subdivision is a special kind of triangulation.

Definition 2.2.15 (barycentric subdivision of geometric simplicial complex)**.** We first define barycentric subdivision of a simplex: let $[v_0, \ldots, v_n]$ be a simplex, we can define the barycentric subdivision of $[v_0, \ldots, v_n]$ inductively on n as follows: for $n = 0$, the barycentric subdivision of a vertex $[v_0]$ is defined to be $[v_0]$ itself; for $n =$

$k-1, k \geq 1$, suppose we have defined the barycentric subdivision of $[v_0, \dots, v_{k-1}]$; for $n = k$, let $b = \sum_{i=0}^{k} \frac{1}{k+1} v_i$ be the barycenter of $[v_0, \dots, v_k]$, the barycentric subdivision of $[v_0, \dots, v_k]$ is a decomposition of it into k-simplices $[b, w_0, \dots, w_{k-1}]$ where $[w_0, \dots, w_{k-1}]$ is any $k-1$ simplex of each barycentric subdivided $(k-1)$-dimensional face $[v_0, \dots, \overset{\wedge}{v_i}, \dots, v_k]$ of $[v_0, \dots, v_k]$, where $\overset{\wedge}{v_i}$ means the vertex v_i is deleted.

The above definition can be generalized to barycentric subdivision of a simplicial complex by the same process with induction on the n-skeleton of the simplicial complex.

We can approximate any continuous map between simplicial complexes by a simplicial map:

Theorem 2.2.2 (Simplicial Approximation Theorem)**.** *Let $\mathcal{K}$ be a finite simplicial complex, $\mathcal{L}$ be an arbitary simplicial complex, then any continuous map $f : \mathcal{K} \to \mathcal{L}$ is homotopic to a map that is simplicial with respect to some barycentric subdivision of $\mathcal{K}$.*

Proof. The proof can be found in [Hat02, section 2.C]. □

There is a combinatorial equivalent notion of the geometric simplicial complex which is more convenient to use in many situations, i.e., the abstract simplicial complex.

Definition 2.2.16 (abstract simplicial complex)**.** An abstract simplicial complex is a pair (V, K) where V is a set of points, and $\mathsf{K} \subseteq 2^V$ is a hereditary system of subsets of V, which is, if $F \in \mathsf{K}$ and $G \subseteq F$, then $G \in \mathsf{K}$. In particular, $\emptyset \in \mathsf{K}$ if $\mathsf{K} \neq \emptyset$. The points in V are called the vertices of the abstract simplicial complex, and the sets in K are called (abstract) simplicies. We define the dimension of each simplex $F \in \mathsf{K}$ to be $dim(F) = |F| - 1$, and the dimension of K to be $dim(\mathsf{K}) = \max\{dim(F) | F \in K\}$.

We can see from the definition that $V = \cup \mathsf{K}$, i.e., the vertex set V of the abstract simplicial complex is the union of the one-point sets of K, thus we can write an abstract simplicial complex (V, K) simply as K.

Abstract simplicial complex and geometric simplicial complex determine each other: each geometric simplicial complex $\mathcal{K}$ determines an abstract simplicial complex (V, K) as follows: let V be the set of vertices of $\mathcal{K}$, let the sets in K be the vertex sets of the simplices of $\mathcal{K}$, we call $\mathcal{K}$ a geometric realization of K, and call the polyhedron $\|\mathcal{K}\|$ of $\mathcal{K}$ also to be the polyhedron of K. Conversely, each abstract simplicial complex (V, K) determines a geometric simplicial complex $\mathcal{K}$: let σ^n be a n-dimensional simplex where $n = |V| - 1$, let $\mathcal{K}$ to be a subcomplex of σ^n consisting of simplices $conv(F)$ for all $F \in \mathsf{K}$.

Thus in some sense we can say the definition of abstract simplicial complex and geometric simplicial compelx are equivalent.

Definition 2.2.17 (simplicial map)**.** Let K, L be two abstract simplicial complexes, a simplicial map from K to L is a map $f : V(\mathsf{K}) \to V(\mathsf{L})$ sending simplices to simplices, i.e., for each $F \in \mathsf{K}$, $f(F) \in \mathsf{L}$.

A bijective simplicial map whose inverse map is also simplicial is called an isomorphism of abstract simplicial complexes. Two abstract simplicial complexes K and L are said to be isomorphic if there is an isomorphism between them, denoted as $\mathsf{K} \cong \mathsf{L}$.

Definition 2.2.18 (affine extension of simplicial map)**.** Let $\mathcal{K}$, $\mathcal{L}$ be geometric simplicial complexes, let K, L be their associated abstract simplicial complexes, let $f : V(\mathsf{K}) \to V(\mathsf{L})$ be a simplicial map from K to L. We define the affine extension $\|f\|$ of f to be the following map

$$\|f\| : \|\mathcal{K}\| \to \|\mathcal{L}\|$$

by extending f affinely to the relative interiors of the simplices of $\mathcal{K}$: let $\sigma \in \mathcal{K}$ be any simplex with vertices $v_0, \ldots, v_k$, any x in the relative interior of σ can be written as $x = \sum_{i=1}^{k} \lambda_i v_i$, with $\lambda_i \geq 0, 1 \leq i \leq k$, and $\sum_{i=1}^{k} \lambda_i = 1$, and we define $\|f\|(x) = \sum_{i=1}^{k} \lambda_i f(v_i)$.

One can easily show that for every simplicial map f, its affine extension is a continuous map; if f is injective, so does $\|f\|$; if f is an isomorphism, then $\|f\|$ is a homeomorphism.

2.3 Group action

In this subsection, we define group action on a set, on an abelian group, and on a topological space respectively. Moreover, we give some crucial examples which are important for later discussion.

2.3.1 GROUP ACTION ON A SET

Definition 2.3.1 (symmetric group)**.** A symmetric group Σ_X of the set X is the group of all permutations (bijections) from X to itself with function composition as group operation. Since the composition of two bijections is a bijection, the inverse of a bijection is a bijection, and function composition is associative, the group is well defined.

If X is a finite set of n elements, we denote Σ_X as Σ_n and call it the symmetric of degree n.

Definition 2.3.2 (group action on a set)**.** Let G be a group and X be a set, a left action of G on X is a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

which satisfies $e \cdot x = x$ and $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $g_1, g_2 \in G$ and all $x \in X$.

Similarly, a right action of G on X is a map

$$\begin{aligned} X \times G &\to X \\ (x, g) &\mapsto x \cdot g \end{aligned}$$

with the same properties as left action except the composition works in reverse direction: $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$.

The left and right action determine each other by the correspondence $g \cdot x = x \cdot g^{-1}$. Thus they are equivalent and we can use any of them. In this thesis we choose to use left action.

From the above definition we observe that for each fixed $g \in G$ we get a map $\rho_g : X \to X$ defined by $\rho_g(x) = g \cdot x$. And the following two important facts can be shown easily:

- for each fixed $g \in G$, $\rho_g : X \to X$ is a permutation(bijection) of X. Since for each ρ_g, there is an inverse map $\rho_{g^{-1}} : X \to X$.
- the map $\rho : G \to \Sigma_X$ defined by $g \mapsto \rho_g$ is a group homomorphism. Since for all $g_1, g_2 \in G, x \in X$, we have $\rho_{g_1 g_2}(x) = (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) = (\rho_{g_1} \circ \rho_{g_2})(x)$.

Thus we have the following equivalent definition of group action:

Definition 2.3.3 (group action on a set)**.** A left action of G on X is the group homomorphism ρ from G to the symmetric group Σ_X of X:

$$\begin{aligned} \rho : G &\to \Sigma_X \\ g &\mapsto \rho_g : x \mapsto g \cdot x \end{aligned}$$

It is also called the associated permutation representation of the group G on the set X.

There are different kinds of group actions:

- A group action G on X is said to be *fixed point free if no element $x \in X$ is fixed by all $g \in G$, i.e., $\forall x \in X$, there exists $g \in G$, such that $g \cdot x \neq x$.*
- *A group action G on X is said to be* free if no element $x \in X$ is fixed by any nontrivial element $g \neq e, g \in G$, i.e., $\forall x \in X, \forall g \neq e \in G, g \cdot x \neq x$.
- A group action G on X is said to be *transitive if $\forall x_1, x_2 \in X, \exists g \in G$, such that $x_1 = g \cdot x_2$.*
- *A group action G on X is said to be* faithful or effective if distinct group elements corresponds to distinct actions, i.e., if $g_1 \neq g_2$, then $\exists x \in X$ such that $g_1 \cdot x \neq g_2 \cdot x$. Thus the associated permutation representation is injective.

Example 2.3.1 (Σ_X acts on X)*.* Let X be a set, Σ_X be its symmetric group, then Σ_X acts on X by permutations, which is, for all $\sigma \in \Sigma_X$, for all $x \in X$, $\sigma \cdot x = \sigma(x)$. The associated permutation representation is the identity map from Σ_X to itself. This action is obviously transitive since for any $x_1, x_2 \in X$, there exists $\sigma = (x_1 x_2) \in \Sigma_X$, where $(x_1 x_2)$ is the transition of x_1 and x_2, such that $\sigma(x_1) = x_2$.

Definition 2.3.4 (transitive subgroup of Σ_n)**.** If X in the above example is a finite set, denoted as $\{1, 2, \ldots, n\}$, let Σ_n acts on $\{1, 2, \ldots, n\}$ by permuting its order, let $H \subseteq \Sigma_n$ be a subgroup of Σ_n, define the action of H on $\{1, 2, \ldots, n\}$ to be the restriction of the action of Σ_n on $\{1, 2, \ldots, n\}$. If this action is transitive, H is said to be a transitive subgroup of Σ_n.

Example 2.3.2 (any finite group G of n elements can embed into Σ_n as a transitive subgroup)*.* Let G be a group, define a group action of G acting on itself by left multiplication, i.e, for all $g \in G$, for all $x \in G$, $g \cdot x = gx$ where gx is the group multiplication of g and x. This group action is transitive since for any $g_1, g_2 \in G$, there exists $g_2(g_1)^{-1} \in G$ such that $(g_2(g_1)^{-1})g_1 = g_2$; it is free since $gx = x$ gives $g = e$ by multiplying x^{-1} on the right of each side of the equation; it is also faithful since it is free.

Equivalently, this group action is a group homomorphsim $\rho : G \to \Sigma_G, g \mapsto \rho_g$ where $\rho_g : G \to G, x \mapsto gx$. The action being faithful means ρ is injective, and thus $G \cong \rho(G)$. The action being transitive means $\rho(G)$ acts on G transitively. If G is a finite group of n elements, in this case we say G embeds into Σ_n as a transitive subgroup. Thus we have any finite group G of n elements can embed into Σ_n as a transitive subgroup.

In the following we give two lemmas regarding the transitivity of the Sylow p-subgroup of symmetric group. Before that, we first introduce Sylow p-subgroup and Sylow's theorem.

Definition 2.3.5 (p-group)**.** Let p be a prime number, a group of order p^α for some integer $\alpha \geq 0$ is a p-group.

Definition 2.3.6 (p-subgroup)**.** Let G be a group and p be a prime number, a subgroup of G which is a p-group is a p-subgroup of G.

Definition 2.3.7 (sylow p-subgroup)**.** Let G be a group of order $p^\alpha m$ where p is a prime number, $p \nmid m$, and $\alpha \geq 0$ is an integer, then a subgroup of G of order p^α is a Sylow p-subgroup of G.

Remark 2.3.1. For p a prime number not dividing the order $|G|$ of G, we have $\alpha = 0$. In this case, the Sylow p-subgroup has order 1, and it is the trivial subgroup $\{e\}$.

The Lagrange theorem in group theory tells us that the order of any subgroup of a finite group G divides the order $|G|$ of the group G. A natural converse of the question is for any positive integer n which divides the order $|G|$ of G, whether there exists a subgroup of G of the corresponding order n. Cauchy theorem answers partly the question: for any prime number p dividing the order of the group, there exists a cyclic subgroup of the corresponding order p. The following Sylow's theorem is a generalization of the Cauchy theorem, and it tells us that: let p be any prime number, p^α(α is a non negative integer) be its maximal power dividing the order of the group G, then there exists a subgroup of G which has the corresponding order p^α.

Theorem 2.3.1 (Sylow's theorem)**.** *Let G be a finite group, for any prime number p, there exists a Sylow p-subgroup of G. Furthermore, any two Sylow p-subgroups are conjugate and thus isomorphic.*

Proof. A proof can be found in [DF04, section 4.5]. □

Now we give two lemmas regarding the transitivity of the Sylow p-subgroup of the symmetric group Σ_n.

Lemma 2.3.1. *If n is not a power of a prime number, the Sylow p-subgroup of the symmetric group Σ_n acts on the set $\{1, 2, \ldots, n\}$ non transitively.*

Before we prove this lemme, we first give a theorem from number theory without proof, and readers can consult [Fin47, theorem 1] for a proof.

Theorem 2.3.2 (Lucas' theorem)**.** *Let n, m be two integers, p be a prime number, then the binomial coefficient $\binom{n}{m}$ modulo p is*

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod{p}$$

where $n = n_0 + n_1 p + \cdots + n_k p^k$ and $m = m_0 + m_1 p + \cdots + m_k p^k$ are the base p expansions of n and m respectively with k being the maximal integer such that $p^k \leq n$ or $p^k \leq m$, and with the convention $\binom{n}{m} = 0$ if $n < m$.

proof of lemma 2.3.1. Let p be a prime number, we can expand n in the base p as $n = n_0 + n_1 p + n_2 p^2 + \cdots + n_k p^k$, where $0 \leq n_i < p, \forall i = 0, 1, \ldots, k-1, 0 < n_k < p$. If n is not a power of p, we have $p^k < n$, and $S_{p^k} \times S_{n-p^k} \subsetneq S_n$ is a subgroup of S_n which obviously does not act transitively on the set $\{1, \ldots, n\}$. Now we show $S_{p^k} \times S_{n-p^k}$ contains a Sylow p-subgroup of S_n: since

$$[S_n : S_{p^k} \times S_{n-p^k}] = \frac{n!}{p^k!(n-p^k)!} = \binom{n}{p^k},$$

by above Lucas' theorem 2.3.2, we have

$$\binom{n}{p^k} \equiv \binom{n_0}{0}\binom{n_1}{0} \cdots \binom{n_{k-1}}{0}\binom{n_k}{1} \pmod{p} \equiv n_k \pmod{p}$$

where $1 \leq n_k < p$ and thus $[S_n : S_{p^k} \times S_{n-p^k}] \equiv n_k \pmod{p}$. Since $1 \leq n_k < p$, we have $p \nmid [S_n : S_{p^k} \times S_{n-p^k}]$. Assume $n! = p^\alpha c$ where $p \nmid c$, since $p \nmid [S_n : S_{p^k} \times S_{n-p^k}]$

we have $|S_{p^k} \times S_{n-p^k}| = p^\alpha d$ where $d|c$, thus by Sylow's theorem 2.3.1, $S_{p^k} \times S_{n-p^k}$ contains a Sylow p-subgroup of order p^α which is also a Sylow p-subgroup of S_n. Since $S_{p^k} \times S_{n-p^k}$ acts on the set $\{1, \dots, n\}$ non transitively, its subgroup also acts on it non transitively. Furthermore, since all Sylow p-subgroup are conjugate by theorem 2.3.1, we have all Sylow p-subgroup of S_n acts on the set $\{1, \dots, n\}$ non transitively. □

Lemma 2.3.2. *If $n = p^\alpha$ is a power of prime p, the Sylow p-subgroup of the symmetric group Σ_n acts on the set $\{1, 2, \dots, n\}$ transitively.*

Proof. Let $\alpha \geq 0$ be an integer, p be a prime number, $n = p^\alpha$ be a power of p, we can see from example 2.3.2 that the abelian group $(\mathbb{Z}_p)^\alpha$ of rank α can embed into S_{p^α} as a transitive subgroup and thus acts transitively on the set $\{1, \dots, n\}$. Since $(\mathbb{Z}_p)^\alpha$ is a p-subgroup of S_n, it is contained in a Sylow p-subgroup of S_n, which implies the Sylow p-subgroup is a transitive subgroup. Since all Sylow p-subgroups are conjugate, all Sylow p-subgroups acts on $\{1, \dots, n\}$ transitively. □

2.3.2 GROUP ACTION ON AN ABELIAN GROUP

Definition 2.3.8 (G-module)**.** Given a group $(G, \times)$, a G-module is an abelian group $(M, +)$ with a left group action, which is a map

$$\begin{aligned} \rho\colon G \times M &\to M \\ (g, m) &\mapsto g \cdot m := \rho((g, m)). \end{aligned}$$

such that ρ satisfies $e \cdot m = m$ and $(g_1 \times g_2) \cdot m = g_1 \cdot (g_2 \cdot m)$, and the group action is compactible with the operation of the abelian group M, i.e.,

$$g \cdot (m_1 + m_2) = g \cdot m_1 + g \cdot m_2$$

Definition 2.3.9 (morphism of G-modules)**.** Let $(M, +), (N, +)$ be two G-modules, $f : M \to N$ is called a morphism between G-modules if it is a group homomorphism and G-equivariant, i.e.,

$$f(m_1 + m_2) = f(m_1) + f(m_2)$$

$$f(g \cdot m) = g \cdot f(m)$$

The set of all morphisms from M to N is denoted by $Hom_G(M, N)$. It is an abelian group under pointwise addition: $(f+g)(m) = f(m) + g(m), \forall f, g \in Hom_G(M, N), \forall m \in M$.

2.3.3 GROUP ACTION ON A TOPOLOGICAL SPACE

Definition 2.3.10 (group action on a topological space)**.** Let G be a group and let X be a topological space, the group action of G on X is a group action of G on X as a set, with a further requirement that for all $g \in G$, the map $\rho_g : X \to X$ defined by $x \mapsto g \cdot x$ is a continuous map. Since $\rho_{g^{-1}} : X \to X$ is also a continuous map, and by the definition of the group action on a set we have $\rho_g \circ \rho_{g^{-1}} = \rho_{g^{-1}} \circ \rho_g = \rho_e = Id(X)$, it follows that $\rho_g : X \to X$ is a homeomorphism of X for all $g \in G$. We call this action an action by homeomorphism.

Thus we can equivalently define the action of G on a topological space X to be a group homomorphism $\rho : G \to Homeo(X)$ given by $g \mapsto \rho_g : x \mapsto g \cdot x$, where $Homeo(X)$ is a group of all homeomorphisms of X with function composition to be group operation, and the identity map $Id(X)$ to be the identity element.

In later text, all group action on a topological space would adopt the above action without explicit mention.

Definition 2.3.11 (G-Space)**.** A G-space is a pair (X, ρ) consisting of a topological space X with a group action ρ of G on X. We shall usually denote the G-space (X, ρ) only by its underlying topological space X. A free G-space is a G-space with a free G-action; a fixed point free G-space is a G-space with a fixed point free G-action.

Definition 2.3.12 (Orbit Space, G-orbit)**.** Let X be a G-space, the relation $R = \{(x, gx) | x \in X, g \in G\}$ is an equivalence relation on X since G is a group. Denote the set of equivalence classes $X \mod R$ as X/G and define the quotient topology on X/G as: $U \subseteq X/G$ is open if and only if $q^{-1}(U) \subseteq X$ is open, where $q : X \to X/G$ maps $x \in X$ to its equivalence class Gx in X/G. X/G is called the orbit space of the G-space X and the equivalence class Gx of $x \in X$ is called the G-orbit of x.

Definition 2.3.13 (G-subspace)**.** Let X be a G-space, $Y \subseteq X$ be a subspace of X, if $\forall y \in Y, \forall g \in G$, we have $g \cdot y \in Y$, then Y is called G-invariant or a G-subspace of X. In this case the G-action on X induces a G-action on Y, which makes Y also a G-space.

Remark 2.3.2. The orbit Gx are the smallest G-subspace of X, and any G-subspace of X is a union of some orbits Gx.

Definition 2.3.14 (G-map)**.** Let X, Y be two G-spaces, a continuous map $f : X \to Y$ is called a G-map or a G-equivariant map if $\forall x \in X, \forall g \in G$, we have $f(g \cdot x) = g \cdot f(x)$.

Remark 2.3.3. A G-map $f : X \to Y$ induces a map between its orbit spaces: $f/G : X/G \to Y/G, Gx \to Gf(x)$.

Definition 2.3.15 (G-homotopic of G-maps)**.** Two G-maps $f_1, f_2 : X \to Y$ between two G-spaces X and Y are said to be G-homotopic if there exists a G-homotopy between them, i.e., a family of G-maps $h_t : X \to Y, t \in [0, 1]$ such that $h_0 = f_1$, $h_1 = f_2$.

Definition 2.3.16 (fixed point set)**.** Let X be a G-space, let $H \subseteq G$ be a subgroup of G, then $X^H = \{x \in X | h \cdot x = x \text{ for all } h \in H\}$ is called the H-fixed point set of X.

We now give some simple examples regarding G-spaces and G-maps:

Example 2.3.3 (S^n is a free $\mathbb{Z}_2$-space). Let $\mathbb{Z}_2 = \{e, v\}$ be a cyclic group of two elements where e is the identity and $v^2 = e$. Let $\mathbb{Z}_2$ acts on S^n antipodally, i.e., $e \cdot x = x, v \cdot x = -x$. The action is free since for any nontrivial element $v \in \mathbb{Z}_2$, we have $v \cdot x \neq x$ for all $x \in S^n$. Thus S^n is a free $\mathbb{Z}_2$-space under this action.

Example 2.3.4 ($\mathbb{R}^n$ is a non free $\mathbb{Z}_2$-space). let $\mathbb{Z}_2$ acts on $\mathbb{R}^n$ antipodally, i.e., $e \cdot x = x, v \cdot x = -x$. This action is not free since for $x = 0 \in \mathbb{R}^n$, $v \cdot x = -x = x$. $\mathbb{R}^n$ is a non free $\mathbb{Z}_2$-space under this action.

Example 2.3.5 (antipodal map is $\mathbb{Z}_2$-equivariant). Let S^n, $\mathbb{R}^n$ be $\mathbb{Z}_2$-spaces on which $\mathbb{Z}_2$ acts antipodally as above. Let $f : S^n \to \mathbb{R}^n$ defined by $f(-x) = -f(x)$ be the antipodal map. Obviously f is a $\mathbb{Z}_2$-map by definition.

Now we define two special kinds of G-spaces, the cell G-complex and the simplicial G-complex.

Definition 2.3.17 (cell G-complex)**.** A cell G-complex is a cell complex X with a cellular action, which is a group action of G on X such that for each $g \in G$, $\rho_g : X \to X$ is a cellular map, i.e, a continuous map sending cells to cells.

Definition 2.3.18 (simplicial G-complex)**.** A simplicial G-complex is a simplicial complex $\mathcal{K}$ with a simplicial action, which is a group action of G on $\mathcal{K}$ such that for each $g \in G$, $\rho_g : V(\mathcal{K}) \to V(\mathcal{K})$ is a simplicial map.

Remark 2.3.4. By the definition of group action, we have for all $g \in G$, $\rho_g \circ \rho_{g^{-1}} = \rho_{g^{-1}} \circ \rho_g = \rho_e = Id$, and $\rho_{g^{-1}}$ is also a simplicial map. Thus ρ_g is an isomorphism of the abstract simplicial complex $\mathcal{K}$, i.e., a bijective simplicial map whose inverse is also simplicial. Thus the affine extension $\|\rho_g\|$ of ρ_g on the polyhedron $\|\mathcal{K}\|$ of $\mathcal{K}$ is a homeomorphism, i.e.,$\|\rho_g\| : \|\mathcal{K}\| \to \|\mathcal{K}\|$ is a homeomorphism. Thus the above defined group action on a simplicial complex is consistent with the definition of group action on a topological space.

We define a particular kind of G-space called E_nG space.

Definition 2.3.19 (E_nG-space)**.** Let G be a nontrivial finite group, let $n \geq 0$ be an integer, an E_nG space is a n-dimensional, $(n-1)$-connected finite free simplicial G-complex(or cell G-complex.)

An important kind of E_nG-space is given in example 2.4.6 below.

2.4 Product and Deleted product, Join and Deleted join

We define product and deleted product, join and deleted join of topological spaces and simplicial complexes respectively.

2.4.1 product and deleted product

Product and deleted product of topological spaces We first define product and n-fold n-wise deleted product of topological spaces.

Definition 2.4.1 (product of n spaces)**.** Let $X_1, \dots, X_n$ be spaces, define their n-fold product $X_1 \times X_2 \cdots \times X_n$ to be their cartesian product:

$$X_1 \times X_2 \cdots \times X_n = \{(x_1, x_2, \dots, x_n) | x_i \in X_i, \forall i = 1, \dots, n\}$$

with the product topology.

We can also define the product of maps between spaces:

Definition 2.4.2 (product of two maps)**.** Given maps $f : X_1 \to X_2$, $g : Y_1 \to Y_2$, we can define their product to be $f \times g$:

$$f \times g : X_1 \times Y_1 \to X_2 \times Y_2,$$

is given by

$$(x, y) \mapsto (f(x), g(y)).$$

Similarly we can define product of finitely many maps between spaces.

If the above $X_1, \ldots, X_n$ are the same space, we can define the n-fold n-wise deleted product of X:

Definition 2.4.3 (n-fold n-wise deleted product $X^{\times n} - \mathfrak{diag}$ of a space X)**.** Let X be a space, the n-fold n-wise deleted product $X^{\times n} - \mathfrak{diag}$ of a space X is the n-fold product $(X)^{\times n}$ of X minus the diagonal $\mathfrak{diag} = \{(x, \ldots, x) | x \in X\} \subset (X)^{\times n}$ of $(X)^{\times n}$:

$$(X)^{\times n} - \mathfrak{diag} = \{(x_1, x_2, \ldots, x_n) | x_i \in X, \forall i = 1, \ldots, n\} - \{(x, \ldots, x) | x \in X\}$$

For any r-fold r-wise deleted product of a space we can define a standard Σ_r-action on it:

Definition 2.4.4 (standard Σ_r-action on $(X)^{\times r} - \mathfrak{diag}$)**.** Let X be a space, $(X)^{\times r} - \mathfrak{diag}$ be its r-fold r-wise deleted product, then we can define a Σ_r-action on it by permuting its r components and call it the standard Σ_r-action. $(X)^{\times r} - \mathfrak{diag}$ is a Σ_r-space with standard Σ_R-action.

For any subgroup H of Σ_r, we can define the standard H-action on $(X)^{\times r} - \mathfrak{diag}$ to be the restriction of the standard Σ_r-action, and $(X)^{\times r} - \mathfrak{diag}$ is a H-space with this standard H-action. For example, let $\mathbb{Z}_r \subsetneq \Sigma_r$ be a cyclic group generated by the cycle permutation $\mu = (1, \ldots, r)$, we can define the standard $\mathbb{Z}_r$-action on $(X)^{\times r} - \mathfrak{diag}$ to be the restriction of the standard Σ_r-action on $(X)^{\times r} - \mathfrak{diag}$, which is the generator $\mu \in \mathbb{Z}_r$ acts on $(X)^{\times r} - \mathfrak{diag}$ by shifting each of its component to the left by one position.

We now give a theorem regarding the standard Σ_n-action on the n-fold n-wise deleted product of a topological space which would be very useful for our later discussion of the topological Tverberg conjecture.

Before that, we first need a lemma:

Lemma 2.4.1. *[MBZ03, observation 6.1.3] Let p be a prime number, let $(X)^{\times p} - \mathfrak{diag}$ be the p-fold p-wise deleted product of the topological space X with the standard $\mathbb{Z}_p$-action, this action is free if and only if the generator ν of $\mathbb{Z}_p$ has no fixed point.*

proof. If the generator ν has no fixed point, assume the standard $\mathbb{Z}_p$-action on $(X)^{\times p} - \mathfrak{diag}$ is not free, we then have $x_0 \in (X)^{\times p} - \mathfrak{diag}, \nu^m \in \mathbb{Z}_p$ for some $2 \leq m \leq r-1$, such that $\nu^m \cdot x_0 = x_0$. Since p is a prime number, we have m and p coprime, thus there exists integers k_1 and k_2 such that $k_1 p + k_2 m = 1$. Thus we have $x_0 \neq \nu \cdot x_0 = \nu^{k_1 p + k_2 m} \cdot x_0 = \nu^{k_1 p}(\nu^{k_2 m} \cdot x_0) = \nu^{k_1 p} \cdot x_0 = e \cdot x_0 = x_0$, a contradiction. □

Theorem 2.4.1. *[MBZ03, P158 and P161, exercise 1,2.] For $n \geq 2$, let $(X)^{\times n} - \mathfrak{diag}$ be a n-fold n-wise deleted product of a space X with at least two points equipped with a standard Σ_n-action, i.e., Σ_n acts on it by permuting components, this Σ_n-action is not free for $n \geq 3$; let $\mathbb{Z}_n$ be a subgroup of Σ_n, for all $n \geq 2$, $\mathbb{Z}_n$ acts on $(X)^{\times n} - \mathfrak{diag}$ fixed point freely; Furthermore, $\mathbb{Z}_n$ acts on $(X)^{\times n} - \mathfrak{diag}$ freely if and only if n is prime.*

proof. We first show when $n \geq 3$, the Σ_n action on $(X)^{\times n} - \mathfrak{diag}$ is not free: let $(1,2)$ be a permutation in Σ_n which exchanges the first and second coordinates of $(X)^{\times n} - \mathfrak{diag}$, then it fixes those elements in $(X)^{\times n} - \mathfrak{diag}$ which has the same first and second coordinates.

We then show for all integer $n \geq 2$, $\mathbb{Z}_n$ acts on $(X)^{\times n} - \mathfrak{diag}$ fixed point freely: if there exists $x = (x_1, \ldots, x_n) \in (X)^{\times n} - \mathfrak{diag}$ fixed by all elements of $\mathbb{Z}_n$, then it must be fixed by the generator ν of $\mathbb{Z}_n$, which would imply $x_1 = x_2 = \cdots = x_n$, and thus $x \notin (X)^{\times n} - \mathfrak{diag}$.

Finally, we show $\mathbb{Z}_n$ acts on $(X)^{\times n} - \mathfrak{diag}$ freely if and only if n is prime: when n is prime, by lemma 2.4.1 we only need to show the generator ν of $\mathbb{Z}_n$ has no fixed point: if $(x_1, \ldots, x_n) \in (X)^{\times n} - \mathfrak{diag}$ is fixed under the generator ν, then we have $(x_1, x_2, \ldots, x_n) = \nu \cdot (x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n, x_1)$ which gives $x_1 = x_2 = \cdots = x_n$, but $(x, \ldots, x)$ do not lie in $\mathfrak{diag}$. Conversely, if n is not prime, w.l.o.g, we can let $n = a \times b$ where a, b are positive integers ≥ 2. We can take the n components of $(X)^{\times n} - \mathfrak{diag}$ to be a $a \times b$ matrix with the requirement that not all components are the same. If we take a matrix $m \in (X)^{\times n} - \mathfrak{diag}$ whose each column has the same components respectively but not all columns are equal, then each row of m are the same. And m is fixed by an nontrivial element ν^b of $\mathbb{Z}_n$, where ν^b is the b times composition of the generator ν of $\mathbb{Z}_n$. We can see that m is fixed by

ν^b since ν^b cyclically permutes the rows of the matrix m. More precisely, denote $m = (m_{ij})_{i=1,\dots,a;j=1,\dots,b}$, and for $1 \le j \le b$, we have $m_{j1} = m_{j2} = \cdots = m_{ja}$, then ν^b sends the first row $(m_{11}, \dots, m_{1,b})$ of m to the second row $(m_{21}, \dots, m_{2b})$ of m, the second row to the third row, etc. Since each row of m are the same, we have ν^b fixes m.

$\square$

Now we will give an example of r-fold r-wise deleted product $(\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ of the Euclidean space $\mathbb{R}^d$ and show that it is homotopy equivalent to $S^{d(r-1)-1}$. This example is also important for the discussion of topological Tverberg conjecture.

Example 2.4.1 (r-fold r-wise deleted product $(\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ of $\mathbb{R}^d$). The r-fold r-wise deleted product $(\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ of the euclidean space $\mathbb{R}^d$ is

$$(\mathbb{R}^d)^{\times r} - \mathfrak{diag} = \{(x_1, \dots, x_r) | x_i \in \mathbb{R}^d, \forall i = 1, \dots, r\} - \{(x, \dots, x) | x \in \mathbb{R}^d\}$$

where $\mathfrak{diag} = \{(x, \dots, x) | x \in \mathbb{R}^d\} \subset (\mathbb{R}^d)^{\times r}$ is the diagonal of $(\mathbb{R}^d)^{\times r}$.

Furthermore, it is homotopy equivalent to the $d(r-1)-1$ dimensional sphere $S^{d(r-1)-1}$(more precisely, $S^{d(r-1)-1}$ is a deformation retract of $(\mathbb{R}^d)^{\times r} - \mathfrak{diag}$):

we can define a $\Sigma_r(\mathbb{Z}^r)$ equivariant map from $(\mathbb{R}^d)^r - \mathfrak{diag}$ to $S^{d(r-1)-1}$ by the following two steps:

1. project $(\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ to the orthogonal complement $(\mathfrak{diag})^\perp$ of $\mathfrak{diag}$.
2. normalize $(\mathfrak{diag})^\perp \setminus \{0\}$ into its unit sphere.

step 1: $(\mathbb{R}^d)^{\times r}$ is a Σ_r-space with the symmetric group Σ_r acts on it by permuting its components. The subspace $(\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ of $(\mathbb{R}^d)^{\times r}$ is also a Σ_r-space with the same Σ_r-action since it is invariant under this action.

Define $(\mathfrak{diag})^\perp \subset (\mathbb{R}^d)^{\times r}$ to be the orthogonal complement subspace of the diagonal $\mathfrak{diag}$ of $(\mathbb{R}^d)^{\times r}$, we can write down the expression of $(\mathfrak{diag})^\perp$ explicitly:

$$(\mathfrak{diag})^\perp = \{(x_1, x_2, \dots, x_n) | x_i \in \mathbb{R}^d, \forall i = 1, \dots, n, \sum_{i=1}^{n} x_i = \vec{0}\}.$$

We can see from the expression above that $(\mathfrak{diag})^\perp$ is also a Σ_r-invariant subspace of $(\mathbb{R}^d)^{\times r}$, and it is homeomorphic to $\mathbb{R}^{d(r-1)}$.

We define the orthogonal projection p from $(\mathbb{R}^d)^{\times r}$ to the orthogonal complement $(\mathfrak{diag})^\perp$ of $\mathfrak{diag}$:

$$p : (\mathbb{R}^d)^{\times r} \to (\mathfrak{diag})^\perp$$

given by

$$(x_1, x_2, \dots, x_r) \mapsto (x_1 - \frac{1}{r}\sum_{i=1}^{r} x_i, x_2 - \frac{1}{r}\sum_{i=1}^{r} x_i, \dots, x_r - \frac{1}{r}\sum_{i=1}^{r} x_i)$$

which is obviously a Σ_r-equivariant map.

We can restrict the above map p and get another Σ_r-equivariant map:

$$\rho : (\mathbb{R}^d)^{\times r} - \mathfrak{diag} \to (\mathfrak{diag})^{\perp} \setminus \{0\}$$

where $\{0\}$ is the origin of $(\mathbb{R}^d)^{\times r}$.

Furthermore, define the inclusion map $l : (\mathfrak{diag})^{\perp} \setminus \{0\} \to (\mathbb{R}^d)^{\times r} - \mathfrak{diag}$, compose ρ with l we have $l \circ \rho : (\mathbb{R}^d)^{\times r} - \mathfrak{diag} \to (\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ and it is homotopic to the identity map $Id : (\mathbb{R}^d)^{\times r} - \mathfrak{diag} \to (\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ by the following straight line homotopy F and thus $(\mathfrak{diag})^{\perp} \setminus \{0\}$ is a deformation retract of $(\mathbb{R}^d)^{\times r} - \mathfrak{diag}$:

we can define a straight line homotopy F from Id to $l \circ \rho$ as

$$F : ((\mathbb{R}^d)^{\times r} - \mathfrak{diag}) \times [0, 1] \to (\mathbb{R}^d)^{\times r} - \mathfrak{diag}$$

given by

$$(x, t) \mapsto (1 - t)Id(x) + t((l \circ \rho)(x))$$

.

step 2: Now we normalize $(\mathfrak{diag})^{\perp} \setminus \{0\} \cong \mathbb{R}^{d(r-1)} \setminus \{0\}$ to its unit sphere $S^{d(r-1)-1}$. The normalization

$$\mu : (\mathfrak{diag})^{\perp} \setminus \{0\} \to S^{d(r-1)-1}$$

is given by

$$x = (x_1, x_2, \dots, x_r) \mapsto \mu(x) = (\frac{x_1}{\|x\|}, \dots, \frac{x_r}{\|x\|})$$

where $\|x\| = \sqrt{\|x_1\|^2 + \dots \|x_r\|^2}$ is the norm of x and for $x_i = (x_{i_1}, \dots, x_{i_d}) \in \mathbb{R}^d$, $\|x_i\| = \sqrt{(x_{i_1})^2 + \dots + (x_{i_d})^2}$ is the Euclidean norm. μ is well defined since $\|\mu(x)\| = 1$ and $x_1, \dots, x_r$ are not all equal.

We can write down the explicit expression of $S^{d(r-1)-1}$:

$$S^{d(r-1)-1} = \{(x_1, x_2, \dots, x_r) | x_i \in \mathbb{R}^d, i = 1, \dots, r, \sum_{i=1}^{r} x_i = \vec{0}, \sum_{i=1}^{r} \|x_i\|^2 = 1\}$$

Thus, $S^{d(r-1)-1}$ can be made into a Σ_r-space with Σ_r acting on it by permuting components and μ is obviously a Σ_r-equivariant map. Similar with step 1 we can show that $S^{d(r-1)-1}$ is a deformation retract of $(\mathfrak{Diag})^{\perp} \setminus \{0\}$: define the inclusion map $l : S^{d(r-1)-1} \to (\mathfrak{Diag})^{\perp} \setminus \{0\}$, there is a straight line homotopy F from $Id : (\mathfrak{Diag})^{\perp} \setminus \{0\} \to (\mathfrak{Diag})^{\perp} \setminus \{0\}$ to $l \circ \mu : (\mathfrak{Diag})^{\perp} \setminus \{0\} \to (\mathfrak{Diag})^{\perp} \setminus \{0\}$ which is

$$F : ((\mathfrak{Diag})^{\perp} \setminus \{0\}) \times [0,1] \to (\mathfrak{Diag})^{\perp} \setminus \{0\}$$

given by

$$(x,t) \mapsto (1-t)Id(x) + t((l \circ \mu)(x)).$$

Compose ρ and μ we get a Σ_r-equivariant map:

$$\mu \circ \rho : (\mathbb{R}^d)^{\times r} - \mathfrak{Diag} \to S^{d(r-1)-1}.$$

By the above discussion, $S^{d(r-1)-1}$ is a deformation retract of $(\mathbb{R}^d)^{\times r} - \mathfrak{Diag}$ and they are thus homotopy equivalent.

By theorem 2.4.1 we know that when $r \geq 3$, the action Σ_r on $(\mathbb{R}^d)^{\times r} - \mathfrak{Diag}$ and $S^{d(r-1)-1}$ is not free, and $\mathbb{Z}_r$ acts on $(\mathbb{R}^d)^{\times r} - \mathfrak{Diag}$ and $S^{d(r-1)-1}$ freely if and only if r is prime.

In the above example we have encountered another important Σ_r-space—the $d(r-1)-1$-dimensional sphere $S^{d(r-1)-1}$. We give its precise definition and some important properties as follow:

Example 2.4.2 ($S^{d(r-1)-1}$ with standard Σ_r-action). In example 2.4.1 we have give a geometric description of $S^{d(r-1)-1}$, i.e., $S^{d(r-1)-1}$ is the unit sphere of the orthogonal complement $(\mathfrak{Diag})^{\perp}$ of the diagonal $\mathfrak{Diag}$ of $(\mathbb{R}^d)^{\times r}$. And we have given the explicit expression of $S^{d(r-1)-1}$:

$$S^{d(r-1)-1} = \{(x_1, x_2, \dots, x_r) | x_i \in \mathbb{R}^d, i = 1, \dots, r, \sum_{i=1}^{r} x_i = \vec{0}, \sum_{i=1}^{r} \|x_i\|^2 = 1\}$$

where for $i = 1, \dots, r, x_i = (x_{i_1}, \dots, x_{i_d}) \in \mathbb{R}^d$, $\|x_i\| = \sqrt{(x_{i_1})^2 + \cdots + (x_{i_d})^2}$ is the Euclidean norm. We have also defined a standard Σ_r-action on $S^{d(r-1)-1}$ by permuting its r components. In later text, when we think of $S^{d(r-1)-1}$ as a Σ_r-space, it is always equipped with this standard action without explicit saying.

We now give a lemma regarding fixed points of $S^{d(r-1)-1}$ under the standard Σ_r-action.

Lemma 2.4.2. *([Oza87, lemma 2.1]) Let $r \geq 2$, $d \geq 1$ be integers, let Σ_r acts on $S^{d(r-1)-1}$ by permuting its r components as in example 2.4.2, let H be a subgroup of the symmetric group Σ_r, then $S^{d(r-1)-1}$ has a H-fixed point if and only if H is not a transitive subgroup of Σ_r.*

proof. A point $(x_1, \ldots, x_r)$ in $S^{d(r-1)-1}$ is fixed by H if and only if $x_i = x_j$ when i, j are in the same H orbit. If H acts on the set $\{1, \ldots, r\}$ transitively, there is only one H-orbit. Thus we have $x_1 = \cdots = x_r$. Since points in $S^{d(r-1)-1}$ satisfies $\sum_{i=1}^{r} x_i = 0$, we have $x_1 = \cdots = x_r = 0$. But the origin $(0, \ldots, 0)$ is not in $S^{d(r-1)-1}$, a contradiction.

Conversely, if H acts on the set $\{1, \ldots, r\}$ non transitively, we can construct a fixed point in $S^{d(r-1)-1}$: define a point $x = (x_1, \ldots, x_r)$ in $(\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ with the components in the same H-orbit to be the same, while $x_1, \ldots, x_r$ are not all the same. This point is fixed by H. Now we first project this point to the orthogonal complement subspace of $\mathfrak{diag}$ in $(\mathbb{R}^d)^{\times r}$ by projection ρ and then normalize it to the unit sphere $S^{d(r-1)-1}$ by the normalization μ as in example 2.4.1 and get a new point $\mu(\rho(x))$. This is a H-fixed point in $S^{d(r-1)-1}$. □

Remark 2.4.1. The standard Σ_r-action on $S^{d(r-1)-1}$ (more generally on $(X)^{\times r} - \mathfrak{diag}$) as in example 2.4.2 has some special properties depending on whether r is prime, prime power or non prime power. We summarize as follows:

let $H \subseteq \Sigma_r$ be a subgroup,

1. when $H = \Sigma_r$:
   - Σ_r acts on $S^{d(r-1)-1}$ fixed point freely for all r.
   - when $r \geq 3$, Σ_r acts on $S^{d(r-1)-1}$ non freely.
2. when $H = \mathbb{Z}_r$:
   - $\mathbb{Z}_r$ acts on $S^{d(r-1)-1}$ fixed point freely for all r.
   - $\mathbb{Z}_r$ acts on $S^{d(r-1)-1}$ freely if and only if r is a prime number by theorem 2.4.1.
3. when $H = \mathfrak{p}$ is a Sylow p-subgroup:
   - when $r = p^\alpha$ for some positive integer α, then by lemma 2.3.2, any Sylow p-subgroup $\mathfrak{p}$ is a transitive subgroup of Σ_r. Thus by lemma 2.4.2, $S^{d(r-1)-1}$ has no $\mathfrak{p}$-fixed point.

– when r is not prime power, by lemma 2.3.1, the Sylow p-subgroups $\mathfrak{p}$ of Σ_r for all prime numbers p are not transitive. Thus by lemma 2.4.2, $S^{d(r-1)-1}$ has a $\mathfrak{p}$-fixed point for all $\mathfrak{p}$.

Product and deleted product of simplicial complex We now define product and n-fold 2-wise deleted product of simplicial complexes.

Definition 2.4.5 (product of n simplicial complexes)**.** Let $\mathcal{K}_1, \dots, \mathcal{K}_n$ be simplicial complexes, its n-fold product of simplicial complexes $\mathcal{K}_1 \times \cdots \times \mathcal{K}_n$ is a cell complex whose cells are all the products $\sigma_1 \times \cdots \times \sigma_n$ of simplices of $\mathcal{K}_1, \dots, \mathcal{K}_n$ as spaces.

If the above $\mathcal{K}_1, \dots, \mathcal{K}_n$ are the same simplicial complex $\mathcal{K}$, we can define its n-fold 2-wise deleted product $(\mathcal{K})^{\times n}_{\Delta(2)}$ to be the subcomplex of the n-fold product $\mathcal{K}^{\times n}$ of $\mathcal{K}$.

Definition 2.4.6 (n-fold 2-wise deleted product of simplicial complex $\mathcal{K}$)**.** Let $\mathcal{K}$ be a simplicial complex, its n-fold 2-wise deleted product $(\mathcal{K})^{\times n}_{\Delta(2)}$ is the subcomplex of the n-fold product $\mathcal{K}^{\times n}$ of $\mathcal{K}$ whose cells are all the products $\sigma_1 \times \cdots \times \sigma_n$ of n pairwise disjoint simplices of $\mathcal{K}$:

$$(\mathcal{K})^{\times n}_{\Delta(2)} = \{\sigma_1 \times \cdots \times \sigma_n | \sigma_i \text{ a simplex of } \mathcal{K}, \sigma_i \cap \sigma_j = \emptyset \text{ for every } i \neq j\}$$

We now give an important example:

Example 2.4.3 (r-fold 2-wise deleted product $(\Delta_n)^{\times r}_{\Delta(2)}$ of the standard n-simplex Δ_n). The r-fold 2-wise deleted product $(\Delta_n)^{\times r}_{\Delta(2)}$ of the standard n-simplex Δ_n is

$$(\Delta_n)^{\times r}_{\Delta(2)} = \{\sigma_1 \times \cdots \times \sigma_n | \sigma_i \text{ a simplex of } \Delta_n, \sigma_i \cap \sigma_j = \emptyset \text{ for every } i \neq j\}$$

One can show that it is a $(n-r+1)$-dimensional, $(n-r)$-connected free Σ_r-complex(a cell complex with a free and cellular Σ_r-action), which is an important fact for later. In order to show this, we first give some examples to gain some intuition of the structure:

- when $r = 2, n = 1$, the 2-fold 2-wise deleted product $(\Delta_1)^{\times 2}_{\Delta(2)}$ of the standard 1-simplex Δ_1 is two singletons, thus it is 0-dimensional, (-1)-connected (non empty). The $\mathbb{Z}_2$-action acting on it by permuting its components maps one cell to another cell, and thus it is cellular and free.

- when $r = 2, n = 2$, the 2-fold 2-wise deleted product $(\Delta_2)^{\times 2}_{\Delta(2)}$ of the standard 2-simplex Δ_2 is homeomorphic to S^1, and thus it is 1-dimensional, 0-connected. And $\mathbb{Z}_2$ acts on it cellularly and freely, thus it is a 1-dimensional, 0-connected free $\mathbb{Z}_2$-complex.

We now sketch the proof:

$(\Delta_n)^{\times r}_{\Delta(2)}$ is $(n-r+1)$-dimensional since: a general cell in $(\Delta_n)^{\times r}_{\Delta(2)}$ is of the form $\sigma_1 \times \cdots \times \sigma_r$ where $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint. Thus the sum of the number of the vertices of the simplices $\sigma_1, \ldots, \sigma_r$ is no more than the number of the vertices of Δ_n, i.e., $(dim(\sigma_1) + 1) + \cdots + (dim(\sigma_r) + 1) \leq n + 1$, which gives $dim(\sigma_1) + \cdots + dim(\sigma_r) \leq n - r + 1$. When a cell $\sigma_1 \times \cdots \times \sigma_r$ used all the vertices of Δ_n, we have $dim(\sigma_1) + \cdots + dim(\sigma_r) = n - r + 1$. In this case, $dim(\sigma_1 \times \cdots \times \sigma_r) = dim(\sigma_1) + \cdots + dim(\sigma_r) = n - r + 1$. Thus $(\Delta_n)^{\times r}_{\Delta(2)}$ is $(n-r+1)$-dimensional.

$(\Delta_n)^{\times r}_{\Delta(2)}$ is $(n-r)$-connected since: by Hurewicz theorem 2.6.1 below, we only need to show $\pi_1((\Delta_n)^{\times r}_{\Delta(2)}) = 0$ (which implies $H_1((\Delta_n)^{\times r}_{\Delta(2)}) = 0$) and $H_2((\Delta_n)^{\times r}_{\Delta(2)}) = \cdots = H_{n-r}((\Delta_n)^{\times r}_{\Delta(2)}) = 0$. This is the most difficult part of the proof and we omit it here. Readers can consult [BSS81, Lemma 1] and [BZ17, Theorem 3.4].

Furthermore, the symmetric group Σ_r acts on $(\Delta_n)^{\times r}_{\Delta(2)}$ by permuting its components and this action is cellular since permuting the order of the product of a cell $\sigma_1 \times \cdots \times \sigma_r \in (\Delta_n)^{\times r}_{\Delta(2)}$ is still a cell; and the action is free since any cell $\sigma_1 \times \cdots \times \sigma_r$ of $(\Delta_n)^{\times r}_{\Delta(2)}$ is the product of pairwise disjoint simplex. Thus $(\Delta_n)^{\times r}_{\Delta(2)}$ a free Σ_r-complex.

2.4.2 JOIN AND DELETED JOIN

We first define join and n-fold n-wise deleted join of topological spaces.

Join of topological spaces

Definition 2.4.7 (join of two topological spaces)**.** Let X, Y be two spaces, define their join $X * Y$ to be the cartesian product $X \times Y \times [0, 1]$ quotient by the equivalence relation $\sim$, i.e., $X * Y = X \times Y \times [0, 1]/\sim$, where the equivalence relation $\sim$ is defined by

- $(x_1, y, 0) \sim (x_2, y, 0), \forall x_1, x_2 \in X$
- $(x, y_1, 1) \sim (x, y_2, 1), \forall y_1, y_2 \in Y$
- $(x, y, t) \sim (x, y, t), \forall x \in X, y \in Y, 0 < t < 1$

We denote the equivalence class $\overline{(x,y,t)} \in X*Y = X \times Y \times [0,1]/\sim$ as $tx \oplus (1-t)y$ which indicates the equivalence relation: we identify all $x \in X$ when $t = 0$, and identify all $y \in Y$ when $t = 1$. Thus we have

$$X * Y = \{tx \oplus (1-t)y | 0 \leq t \leq 1, x \in X, y \in Y\}.$$

Define the topology on $X * Y$ to be the quotient topology of the map $q : X \times Y \times [0,1] \to X * Y = X \times Y \times [0,1]/\sim$ sending $(x,y,t) \in X \times Y \times [0,1]$ to its equivalence class $tx \oplus (1-t)y$.

When two topological spaces are bounded, there is a geometric interpretation of their join:

Proposition 2.4.1 (geometric join)**.** *Let U and V be two skew affine subspaces of some $\mathbb{R}^n$, i.e., $U \cap V = \emptyset$, and the affine hull of U and V has one more dimension than the sum of the dimensions of U and V. Let $X \subseteq U$ and $Y \subseteq V$ be two bounded subspaces. Then their join $X * Y$ is homeomorphic to the following space:*

$$\{tx + (1-t)y | t \in [0,1], x \in X, y \in Y\} \subseteq \mathbb{R}^n$$

This means that the space consisting of all line segments connecting a point of X to a point of Y is homeomorphic to $X * Y$.

We omit the proof and readers can consult [MBZ03, proposition 4.2.4].

We can define the join of n-spaces by induction:

Definition 2.4.8 (join of n spaces)**.** Let $X_1, \ldots, X_n$ be n spaces, define their join $X_1 * \cdots * X_n$ to be the join of $X_1 * \cdots * X_{n-1}$ and X_n and inductively define $X_1 * \cdots * X_{n-1}$ to be the join of $X_1 * \cdots * X_{n-2}$ and X_{n-1}, etc. Similarly as above we have

$$X_1 * \cdots * X_n = \{t_1x_1 \oplus t_2x_2 \cdots \oplus t_nx_n | t_i \in [0,1], \sum_{i=1}^{n} t_i = 1, x_i \in X_i\}.$$

Example 2.4.4 (join of $(n+1)$-points is a n-simplex). Let $v_0, \ldots, v_n$ be $n+1$ points, then their join is a n-simplex: $v_0 * v_1 = \{t_0v_0 \oplus t_1v_1 | t_i \in [0,1], \sum_{i=1}^{2} t_i = 1\}$, $(v_0 * v_1) * v_2 = \{t_2(t_0v_0 \oplus t_1v_1) \oplus (1-t_2)v_3 | t_i \in [0,1], t_2t_0 + t_2t_1 + (1-t_2) = 1\}$, similarly we have $v_0 * \cdots * v_n = \{t_0v_0 \oplus \cdots \oplus t_nv_n | t_i \in [0,1], \sum_{i=0}^{n} t_i = 1\}$ which is by definition a n-simplex.

We can also define the join of two maps between spaces:

Definition 2.4.9 (join of two maps)**.** Given maps $f : X_1 \to X_2$, $g : Y_1 \to Y_2$, we can define their join to be $f * g$:

$$f * g : X_1 * Y_1 \to X_2 * Y_2,$$

is given by

$$tx \oplus (1-t)y \mapsto tf(x) \oplus (1-t)g(y).$$

We can generate new G-space by joining two G-spaces:

Example 2.4.5 (the join of two G-spaces)*.* Let (X, Φ), (Y, Ψ) be two G-spaces with G-actions $\varphi_g : X \to X$ and $\psi_g : Y \to Y$ respectively, we can make their join $(X * Y, \Phi * \Psi)$ into a G-space by defining a G-action on $X * Y$ as $\varphi_g * \psi_g : X * Y \to X * Y, tx \oplus (1-t)y \mapsto t\varphi_g(x) \oplus (1-t)\psi_g(y)$ where $\varphi_g * \psi_g$ denote the join of two maps. Obviously, the join of two free G-space is a free G-space.

We now give a prototypical example of E_nG-space—the n-fold join $G^{*(n+1)}$ of the group G itself. Before that, we state a theorem about the connectivity of join without proof. Readers can consult [MBZ03, proposition 4.4.3] for more details.

Theorem 2.4.2 (connectivity of join)**.** *Let X, Y be triangulable topological spaces(or CW-complexes), if X is k-connected, Y is l-connected, then their join $X * Y$ is $(k+l+2)$-connected.*

Now we give the example.

Example 2.4.6 ($G^{*(n+1)}$ is a E_nG-space)*.* As a topological space, $G^{*(n+1)}$ is the $(n+1)$-fold join of a $|G|$-points discrete space. It is a n-dimensional simplicial complex since: a general cell of $G^{*(n+1)}$ has the form $t_1 g_1 \oplus \cdots \oplus t_{n+1} g_{n+1}$ where $g_1, \ldots, g_{n+1} \in G$ are singletons. By example 2.4.4, we know that the join of $n+1$ points is a n-simplex. Thus $G^{*(n+1)}$ is a union of n-simplices, and thus it is a n-dimensional simplicial complex. In addition, we have seen in example 2.3.2 that G acts on itself freely by left multiplication. If we equip G with the discrete topology, then G acts on itself freely by homeomorphisms with the same action. Thus G is a free G-space. If we join G with itself $n+1$ times as in example 2.4.5, we get a free G-space $G^{*(n+1)}$. The G-action on $G^{*(n+1)}$ is given by $\forall g \in G$, $\rho_g : G^{*(n+1)} \to G^{*(n+1)}, t_1 x_1 \oplus \cdots \oplus t_{n+1} x_{n+1} \mapsto t_1(gx_1) \oplus \cdots \oplus t_{n+1}(gx_{n+1})$, which is obviously a simplicial map. Thus $G^{*(n+1)}$ is a n-dimensional free simplicial G-complex. Since G is (-1)-connected(non-empty), the $(n-1)$-connectivity of $G^{*(n+1)}$ follows from theorem 2.4.2,

Deleted join of topological space

Definition 2.4.10 (n-fold n-wise deleted join of a topological space)**.** Let X be a space, we can define the n-fold n-wise deleted join $(X)^{*n} - \mathfrak{diag}$ of X to be the n-fold join $(X)^{*n}$ of X minus the diagonal $\mathfrak{diag} = \{\frac{1}{n}x \oplus \frac{1}{n}x \cdots \oplus \frac{1}{n}x | x \in X\}$ of $(X)^{*n}$, i.e.,

$$(X)^{*n} - \mathfrak{diag} =$$
$$\{t_1x_1 \oplus t_2x_2 \cdots \oplus t_nx_n | x_i \in X, t_i \in [0,1], \sum_{i=1}^{n} t_i = 1\} - \{\frac{1}{n}x \oplus \frac{1}{n}x \cdots \oplus \frac{1}{n}x | x \in X\}$$

We now give an important example of r-fold r-wise deleted join $(\mathbb{R}^d)^{*r} - \mathfrak{diag}$ of the Euclidean space $\mathbb{R}^d$, and show that it is homotopy equivalent to $S^{(d+1)(r-1)-1}$.

Example 2.4.7 (r-fold r-wise deleted join $(\mathbb{R}^d)^{*r} - \mathfrak{diag}$ of $\mathbb{R}^d$). The r-fold r-wise deleted join $(\mathbb{R}^d)^{*r} - \mathfrak{diag}$ of $\mathbb{R}^d$ is

$$(\mathbb{R}^d)^{*r} - \mathfrak{diag} =$$
$$\{t_1x_1 \oplus t_2x_2 \cdots \oplus t_rx_r | x_i \in \mathbb{R}^d, t_i \in [0,1], \sum_{i=1}^{r} t_i = 1\} - \{\frac{1}{r}x \oplus \frac{1}{r}x \cdots \oplus \frac{1}{r}x | x \in \mathbb{R}^d\}$$

We can define the Σ_r-action on $(\mathbb{R}^d)^{*r} - \mathfrak{diag}$ by permuting its components, and thus make it a Σ_r-space. We can also define a Σ_r-equivariant map $f : (\mathbb{R}^d)^{*r} - \mathfrak{diag} \to (\mathbb{R}^{d+1})^{\times r} - \mathfrak{diag}$ from the r-fold r-wise deleted join of $\mathbb{R}^d$ to the r-fold r-wise deleted product of $\mathbb{R}^{d+1}$ by the following two steps:

1. embed r times $\mathbb{R}^d$ into $(\mathbb{R}^{d+1})^{\times r}$ as r pairwise skew affine subspaces

2. project and normalize $(\mathbb{R}^{d+1})^{\times r}$ into $S^{(d+1)(r-1)-1}$ as in example 2.4.1.

We illustrate them in detail:

Step 1: embed r times $\mathbb{R}^d$ into $(\mathbb{R}^{d+1})^{\times r}$ as r pairwise skew affine subspaces (Two subspaces $U, V \subset \mathbb{R}^n$ are skew affine if $U \cap V = \emptyset$, and their affine hull has dimension $dim(U) + dim(V) + 1$)by r mappings $\psi_1, \ldots, \psi_r : \mathbb{R}^d \to (\mathbb{R}^{d+1})^{\times r}$ where

$$\psi_i : \mathbb{R}^d \to (\mathbb{R}^{d+1})^{\times r}$$

given by

$$(x_1, \ldots, x_d) \mapsto (0, \ldots, 0, 1, x_1, \ldots, x_d, 0 \ldots, 0)$$

i.e., the i-th component of $\psi_i(x_1, \ldots, x_d) \subset (\mathbb{R}^{d+1})^{\times r}$ has values $(1, x_1, \ldots, x_d)$ and others are 0.

Now define a Σ_r-equivariant map

$$\psi : (\mathbb{R}^d)^{*r} - \mathfrak{diag} \to (\mathbb{R}^{d+1})^{\times r} - \mathfrak{diag}$$

given by

$$t_1 y_1 \oplus \cdots \oplus t_r y_r \mapsto t_1 \psi_1(y_1) + \cdots + t_r \psi_r(y_r)$$

which is well defined since the elements of the diagonal $\mathfrak{diag}$ of $(\mathbb{R}^{d+1})^{\times r}$ satisfy

$$(t_1, t_1 y_{11}, \ldots, t_1 y_{1d}) = (t_2, t_2 y_{21}, \ldots, t_2 y_{2d}) = \cdots = (t_r, t_r y_{r1}, \ldots, t_r y_{rd})$$

where $y_i = (y_{i1}, \ldots, y_{id}) \in \mathbb{R}^d$, and it implies

$$t_1 = \cdots = t_r = \frac{1}{r}, y_1 = \cdots = y_r,$$

but such points $\{\frac{1}{r} y \oplus \cdots \oplus \frac{1}{r} y | y \in \mathbb{R}^d\}$ are contained in the diagonal $\mathfrak{diag}$ of $(\mathbb{R}^d)^{*r}$, thus are not in $(\mathbb{R}^d)^{*r} - \mathfrak{diag}$.

It is continuous since it is a linear combination of continuous maps. It is Σ_r-equivariant since it commutes with the Σ_r-actions on $(\mathbb{R}^d)^{*r} - \mathfrak{diag}$ and $(\mathbb{R}^{d+1})^{\times r} - \mathfrak{diag}$ by permuting its components respectively.

Step 2: define another Σ_r-equivariant map $\mu \circ \rho : (\mathbb{R}^{d+1})^{\times r} - \mathfrak{diag} \to S^{(d+1)(r-1)-1}$ as in example 2.4.1 and compose it with $\psi : (\mathbb{R}^d)^{*r} - \mathfrak{diag} \to (\mathbb{R}^{d+1})^{\times r} - \mathfrak{diag}$ defined above we get the following Σ_r-equivariant map:

$$\mu \circ \rho \circ \psi : (\mathbb{R}^d)^{*r} - \mathfrak{diag} \to S^{(d+1)(r-1)-1}.$$

In fact, $(\mathbb{R}^d)^{*r} - \mathfrak{diag}$ and $S^{(d+1)(r-1)-1}$ are homotopy equivalent([MBZ03, P160]) since the maps defined above are deformation retractions.

We now define join and n-fold 2-wise deleted join of simplicial complexes.

Join of simplicial complexes

Definition 2.4.11 (Join of two abstract simplicial complexes)**.** Let K, L be two abstract simplicial complexes with vertex set $V(\mathsf{K})$ and $V(\mathsf{L})$ respectively, define their join $\mathsf{K} * \mathsf{L}$ to be the abstract simplicial complex with vertex set $V(\mathsf{K} * \mathsf{L}) = V(\mathsf{K}) \sqcup V(\mathsf{L})$

which is the disjoint union of $V(\mathsf{K})$ and $V(\mathsf{L})$, and with simplices to be all the combination of simplices of K and simplices of L, i.e.,

$$\{F \sqcup G | F \in \mathsf{K}, G \in \mathsf{L}\}$$

where $F \sqcup G$ is the disjoint union of F and G, i.e., $F \sqcup G = F \times \{1\} \cup G \times \{2\}$. Thus we have $\mathsf{K} * \mathsf{L} = \{F \sqcup G | F \in \mathsf{K}, G \in \mathsf{L}\}$.

That is, to construct the join of two abstract simplicial complexes K and L, we first take the disjoint union of two vertex sets to be the vertex set of the join, and then we combine each simplex of K and each simplex of L to be the simplices of the join. Observe that in this case, K and L are subcomplex of the join, since the combination of all simplices of K with the empty simplex of L gives K itself, and the combination of all simplices of L with the empty simplex of K gives L itself.

Remark 2.4.2 (the equivalence of the join of simplicial complexes with that of spaces). From the above definition we get the polyhedron $\|\mathsf{K} * \mathsf{L}\|$ of the join $\mathsf{K} * \mathsf{L}$ of two abstract simplicial complexes K and L. From the definition of the join of topological spaces we get the join $\|\mathsf{K}\| * \|\mathsf{L}\|$ of two polyhedra $\|\mathsf{K}\|$ and $\|\mathsf{L}\|$ of two abstract simplicial complexes K and L. We can show that this two definitions are equivalent. This means to show $\|\mathsf{K} * \mathsf{L}\| \cong \|\mathsf{K}\| * \|\mathsf{L}\|$. This can be seen from the geometric interpretation of the join $\|\mathsf{K}\| * \|\mathsf{L}\|$: for each m-simplex σ^m of K and each n-simplex σ^n of L, their join $\sigma^m * \sigma^n$ as abstract simplicial complexes is a simplex σ^{n+m+1} of $n+m+2$ vertices; while their geometric join $\|\sigma^m\| * \|\sigma^n\|$ consists of all line segments from a point of $\|\sigma^m\|$ and a point of $\|\sigma^n\|$, which is $\|\sigma^{n+m+1}\|$.

With the above equivalence, we can write the polyhedron $\|\mathsf{K} * \mathsf{L}\|$ of the join $\mathsf{K} * \mathsf{L}$ of two abstract simplicial complexes K and L as:

$$\|\mathsf{K} * \mathsf{L}\| = \{tx \oplus (1-t)y | t \in [0,1], x \in \|\mathsf{K}\|, y \in \|\mathsf{L}\|\}.$$

We can define the join of n simplicial complexes by induction.

Definition 2.4.12 (join of n abstract simplicial complexes)**.** Let $\mathsf{K}_1, \ldots, \mathsf{K}_n$ be n abstract simplicial complexes with vertex set $V(\mathsf{K}_1), \ldots, V(\mathsf{K}_n)$ respectively, define their join $\mathsf{K}_1 * \cdots * \mathsf{K}_n$ to be the join of $\mathsf{K}_1 * \cdots * \mathsf{K}_{n-1}$ and K_n and inductively define $\mathsf{K}_1 * \cdots * \mathsf{K}_{n-1}$ to be the join of $\mathsf{K}_1 * \cdots * \mathsf{K}_{n-2}$ and K_{n-1}, etc.

Remark 2.4.3 (comparison between product and join of simplicial complexes). while the construction of the product of simplicial complexes is more straight forward than the join, the construction of the join of simplicial complexes has its own advantages: even though the product of two simplicial complexes is no longer a simplicial complex, the join of two simplicial complexes is still a simplicial complex.

Deleted join of simplicial complexes If the simplicial complexes $\mathsf{K}_1, \ldots, \mathsf{K}_n$ are the same simpicial complex K, we can define the n-fold 2-wise deleted join of K.

Definition 2.4.13 (n-fold 2-wise deleted join of a simplicial complex)**.** Let K be a simplicial complex, the n-fold 2-wise deleted join $(\mathsf{K})^{*n}_{\Delta(2)}$ of K has the vertex set $V(\mathsf{K}) \times [n]$, and has simplices which are the combination of the pairwise disjoint simplices of K, i.e.,

$$(\mathsf{K})^{*n}_{\Delta(2)} = \{F_1 \sqcup F_2 \cdots \sqcup F_n | F_i \in \mathsf{K}, F_i \cap F_j = \emptyset \text{ for every } i \neq j\}$$

The polyhedron of $(\mathsf{K})^{*n}_{\Delta(2)}$ can be written as

$$\left\|(\mathsf{K})^{*n}_{\Delta(2)}\right\| = \{t_1x_1 \oplus t_2x_2 \cdots \oplus t_nx_n | x_i \in \|\mathsf{K}\|, supp(x_i) \cap supp(x_j) = \emptyset, t_i \in [0,1], \sum_{i=1}^{n} t_i = 1\}$$

Definition 2.4.14 (standard Σ_n-action on $(\mathsf{K})^{*n}_{\Delta(2)}$)**.** We can define a standard Σ_n-action on $(\mathsf{K})^{*n}_{\Delta(2)}$ by permuting the coordinates of $\left\|(\mathsf{K})^{*n}_{\Delta(2)}\right\|$, i.e., for some $\rho \in \Sigma_n$, $\rho : t_1x_1 \oplus t_2x_2 \cdots \oplus t_nx_n \mapsto t_{\rho(1)}x_{\rho(1)} \oplus t_{\rho(2)}x_{\rho(2)} \cdots \oplus t_{\rho(n)}x_{\rho(n)}$, and this action makes $(\mathsf{K})^{*n}_{\Delta(2)}$ into a simplicial Σ_n-complex. This action is free for all n since the coordinates of $\left\|(\mathsf{K})^{*n}_{\Delta(2)}\right\|$ are pairwise disjoint.

For any subgroup $H \subseteq \Sigma_n$, we can define a standard H-action on $(\mathsf{K})^{*n}_{\Delta(2)}$ to be the restriction of the standard Σ_n-action.

The following example will be useful in later text.

Example 2.4.8 (r-fold 2-wise deleted join $(\Delta_n)^{*r}_{\Delta(2)}$ of Δ_n)*.* The r-fold 2-wise deleted join of the standard n simplex Δ_n is (think of Δ_n as an abstract simplicial complex with vertex set $V(\Delta_n) = \{1, \ldots, n\}$, and the simplices to be all the subsets of $V(\Delta_n)$.):

$$(\Delta_n)^{*r}_{\Delta(2)} = \{\sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_r | \sigma_i \in \Delta^n, \sigma_i \cap \sigma_j = \emptyset \text{ for every } i \neq j\}.$$

Its polyhedron can be written as

$$\left\|(\Delta_n)^{*r}_{\Delta(2)}\right\| = \{t_1x_1 \oplus t_2x_2 \cdots \oplus t_nx_n | x_i \in \|\Delta_n\|, supp(x_i) \cap supp(x_j) = \emptyset, t_i \in [0,1], \sum_{i=1}^{n} t_i = 1\}$$

We have seen before that the standard Σ_r-action on $(\Delta_n)^{*r}_{\Delta(2)}$ is free for all r. We now show $(\Delta_n)^{*r}_{\Delta(2)}$ is a $E_n\Sigma_r$-space, i.e., a n-dimensional, $(n-1)$-connected, free simplicial Σ_r-complex since:

$$(\Delta_n)^{*r}_{\Delta(2)} \cong ((\Delta_0)^{*(n+1)})^{*r}_{\Delta(2)} \cong ((\Delta_0)^{*r}_{\Delta(2)})^{*(n+1)} \cong (\mathbb{Z}_r)^{*(n+1)},$$

and by example 2.4.6, $(\mathbb{Z}_r)^{*(n+1)}$ is a $E_n\mathbb{Z}_r$ space.

2.5 Fundamental groups and Homotopy groups

In the following subsections, we will describe some important algebraic topology machinery that we need later.

Algebraic topology is a subject that study topology questions through algebra, which is, it associates topological spaces and the continuous map between them with some kind of groups and the group homomorphism between them by some kind of relation called 'functor'. If such a functor is constructed good enough, we might be able to get the corresponding group with enough detail so that we can study the topological space by studying the properties of the corresponding group.

For example, the following fundamental group is a such kind of functor.

2.5.1 Fundamental group and covering space

Fundamental group As mention above that the fundamental group π_1 is a functor from the category of pointed topological spaces to the category of groups. For example, let $f : (X, x_0) \to (Y, y_0)$ be a continuous map between two pointed topological spaces (X, x_0) and (Y, y_0), the functor π_1 projects them to the groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, and it also projects f to $f_* := \pi_1(f)$, which gives us a group homomorphsim $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Before we define the fundamental group properly, we need some preparations.

Let X be a topological space, a **path** in X is a continuous map $f(s) : I \to X$ where $I = [0, 1]$ is the unit interval. The **inverse path** of f is defined to be $f^{-1} := f(1 - s)$. Two paths f and g with the same end points (i.e.,$f(0) = g(0), f(1) = g(1)$) are said to be **homotopic**, denoted as $f \simeq g$, if one path can be continuously deformed into another with the end points fixed during the deformation. Mathematically, this means there exists a **homotopy that is stationary on the**

subset $\{0,1\} \subset I$ between these two paths, i.e., there exists a family of maps

$$h_t : I \to X, \forall t \in I$$

such that

- $h_0 = f, h_1 = g$.
- $h_t(0) = f(0) = g(0)$, $h_t(1) = f(1) = g(1), \forall t \in I$, which is, the two end points of the paths are fixed during the deformation.
- The homotopy $H : I \times I \to X$ given by $H(s,t) = h_t(s)$ is continous.

The relation of homotopy of paths with fixed end points defined above are easily seen to be an equivalence relation:

Proposition 2.5.1. *The relation of homotopy of paths with fixed end points defined above is an equivalence relation.*

Proof. We sketch the proof: a path f is obviously homotopic to itself by letting $h_t = f$ be the constant homotopy; if $f \simeq g$ by h_t, then reversing the deformation we have $g \simeq f$ by $h(1-t)$; if $f \simeq g$ by the deformation h_t, $g \simeq l$ by the deformation h'_t, then $f \simeq l$ by $\bar{h}_t$ which is the combination of these two deformation, i.e., $\bar{h}_t = h_{2t}$ for $t \in [0,1/2]$, and $\bar{h}_t = h'_{2t-1}$ for $t \in [1/2,1]$. □

Thus we can denote the equivalence class of the path f under the equivalence relation of homotopy as $[f]$ and call it the **homotopy class** of f.

Now we are ready to define the fundamental group of a topological space X. We consider a special kind of paths of the space X, which are the loops. A **loop** in a space X is a path $f : I \to X$ with the two end points coinside, i.e. $f(0) = f(1)$. Take an arbitrary point $x_0 \in X$ and call it the **base point** of X, similarly as above, we can partite all the loops $f : I \to X$ with the base point x_0 as their end points into homotopy classes $[f]$, and denote the set of all such homotopy classes as $\pi_1(X, x_0)$. We now show that we can make $\pi_1(X, x_0)$ into a group and call it the **fundamental group** of space X:

Proposition 2.5.2. $\pi_1(X, x_0)$ *is a group under a suitably defined group operation.*

Proof. We first define the product on the level of maps, i.e., we define the product of any two loops $f, g : I \to X$ at the base point x_0 to be the their composition, which is

$$(f \cdot g)(s) = \begin{cases} f(2s), & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

i.e., the product $f \cdot g$ is given by: in the first half of I travels through f twice as fast as before, and in the second half of I, travels through g again twice as fast as before. This product is obviously well defined and the product of any two loops is again a loop. The product is not associative since $(f \cdot g) \cdot l \neq f \cdot (g \cdot l)$. But we have $(f \cdot g) \cdot l \simeq f \cdot (g \cdot l)$ since one can be deformed into the other continuously while keeping the end points fixed by adjusting the traveling speed on f, g, l appropriately. We can define the identity element of the product to be the constant loop $c : I \to X, s \mapsto x_0$ which maps all elements of I to the base point x_0, since for any loop f, we have $f \cdot c = c \cdot f = f$. In addition, we can define the inverse element of the loop $f(s)$ to be $f^{-1}(s) = f(1-s)$ since $f \cdot f^{-1} = f^{-1} \cdot f = c$.

This product respects homotopy classes since if $f_0 \simeq f_1$ by h_t, $g_0 \simeq g_1$ by h'_t, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ by their product $h_t \cdot h'_t$. Thus we can define the product on the level of homotopy classes, i.e., $[f] \cdot [g] = [f \cdot g]$, and it is well defined. It is also associative since $(f \cdot g) \cdot l \simeq f \cdot (g \cdot l)$ gives $([f] \cdot [g]) \cdot [l] = [(f \cdot g) \cdot l] = [f \cdot (g \cdot l)] = [f] \cdot ([g] \cdot [l])$. Similarly, we can define the identity element $[c]$ in $\pi_1(X, x_0)$ to be the homotopy class of loops which are homotopic to the constant loop $c : I \to X, s \mapsto x_0$. In addition, we can define the inverse of the homotopy class $[f(s)]$ to be $[f^{-1}(s)] = [f(1-s)]$. Thus we have shown $\pi_1(X, x_0)$ is a group with the above group operation.

□

Since we define fundamental group using a base point, it is natural to think about to what extend is the fundamental group of a space depends on the choice of the base point. And it turns out that for path connected space, the fundamental groups with respect to different base points are isomorphic. It can easily be seen from:

Let x_0, x_1 be two arbitrary points in a path connected space X, let $p(s)$ be a path from x_0 to x_1 and let $p^{-1} := p(1-s)$ be the inverse of the path, then any loop l_0 at x_0 gives a loop $p \cdot l_0 \cdot p^{-1}$ at x_1; conversely, any loop l_1 at x_1 gives a loop $p^{-1} \cdot l_1 \cdot p$ at x_0.

Thus for any path connected space X, **different choice of base point x_0 gives isomorphic fundamental group $\pi_1(X, x_0)$**. We can thus write it simply as $\pi_1(X)$.

Using fundamental group we can define an important kind of space: a space which is path connected and has trivial fundamental group is called **simply connected**.

We can see from the definition that any loop of a simply connected space is homotopic to a constant loop. Any map which is homotopic to a constant map is called **null homotopic**. There is another important property of simply connected space:

Proposition 2.5.3. *Any path between any two points of a simply connected space are homotopic.*

Proof. Let X be a simply connected space and let x_0, x_1 be any two points of X, let f, g be two paths from x_0 to x_1, and let g^{-1} be the inverse path of g, then $f \simeq f \cdot (g^{-1} \cdot g) \simeq (f \cdot g^{-1}) \cdot g \simeq g$ since $g^{-1} \cdot g$ and $f \cdot g^{-1}$ are null homotopic. $\square$

We now show that the n sphere S^n is simply connected when $n \geq 2$:

Proposition 2.5.4. $\pi_1(S^n) = 0$ *if* $n \geq 2$.

Proof. We sketch the proof and the whole proof can be found in [Hat02, proposition 1.14].

we only need to show any loop $l : [0, 1] \to S^n$ is homotopic to a non surjective loop $\tilde{l}$ (which is only possible when $n \geq 2$), say $\tilde{l}$ lies in $S^n \setminus \{x\}$ for some $x \in S^n$. Then any loop l in S^n is in fact homotopic to a loop $\tilde{l}$ in $S^n \setminus \{x\}$. Since $S^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$, and $\mathbb{R}^n$ is obviously simply connected, we have $\tilde{l}$ is null homotopic. Thus any loop l is null homotopic, and S^n is simply connected when $n \geq 2$. $\square$

Despite S^n being simply connected for $n \geq 2$, the circle S^1 is not simply connected. In fact, the fundamental group of S^1 is the infinite cyclic group $\mathbb{Z}$, which means there are infinitely many kinds of nontrivial loops in S^1. We state this important fact [Hat02, Theorem 1.7] without proof:

Proposition 2.5.5. $\pi_1(S^1) = \mathbb{Z}$.

Given any continuous map $\varphi : (X, x_0) \to (Y, y_0)$ between topological spaces taking the base point x_0 of X to the base point y_0 of Y, it induced a group homomorphism between fundamental groups, which is

$$\begin{aligned} \varphi_* : &\pi_1(X) \to \pi_1(Y) \\ &[f] \mapsto [\varphi \circ f] \end{aligned}$$

φ_* is called the **homomorphism induced by** φ. It is well defined since if $f_1 \simeq f_2$ by h_t, then $\varphi \circ f_1 \simeq \varphi \circ f_2$ by $\varphi \circ h_t$. It is a group homomorphism since $\varphi_*([f][g]) = [\varphi \circ (f \cdot g)] = [(\varphi \circ f) \cdot (\varphi \circ g)] = \varphi_*([f]) \cdot \varphi_*([g])$.

Covering Space We now introduce an important tool for computing the fundamental group, which is the covering space.

We begin by definition. A **covering space** of a space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ which satisfies: there exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\widetilde{X}$, each of which is mapped by p homeomorphically onto U_α. The map $p : \widetilde{X} \to X$ is called a **covering map**. Given a covering map $p : \widetilde{X} \to X$, a **lift** of a map $f : Y \to X$ is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p \circ \widetilde{f} = f$. We are interested in the behaviour of covering spaces with respect to the lifting of maps.

We now state some lifting properties of covering spaces and some applications of them without proofs. For their proofs, readers can consult the corresponding citation.

First is the **homotopy lifting property**.

Proposition 2.5.6 (homotopy lifting property)**.** *Given a covering space $p : \widetilde{X} \to X$, a homotopy $f_t : Y \to X$, a map $\widetilde{f}_0 : Y \to \widetilde{X}$ lifting f_0, then there exists a unique homotopy $\widetilde{f}_t : Y \to \widetilde{X}$ of $\widetilde{f}_0$ lifting f_t.*

If we let Y be a point, we have the **path lifting property** for a covering space $p : \widetilde{X} \to X$ which says that: for any path $f : I \to X$, for any lift $\widetilde{x}_0$ of the starting point $x_0 := f(0)$ of the path, there exists a unique path $\widetilde{f} : I \to \widetilde{X}$ lifting f starting at $\widetilde{x}_0$.

As an application we have the following proposition:

Proposition 2.5.7. *The map $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$ induced by a covering space $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\widetilde{X}$ starting at $\widetilde{x}_0$ are loops.*

Given a covering space $p : \widetilde{X} \to X$ and a general map $f : Y \to X$, it is natural to ask about the existence and uniqueness of lift of f. The following **lifting criterion** answers the question about when does a lift exists:

Proposition 2.5.8 (lifting criterion)**.** *Given a covering space* $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ *and a map* $f : (Y, y_0) \to (X, x_0)$ *with* Y *path connected and locally path connected, then a lift* $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x_0})$ *exists if and only if* $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$*.*

As for the uniqueness, we have the following **unique lifting property**.

Proposition 2.5.9 (unique lifting property)**.** *Given a covering space* $p : \widetilde{X} \to X$ *and a map* $f : Y \to X$ *with two lifts* $\widetilde{f_1}, \widetilde{f_2} : Y \to \widetilde{X}$ *that agree at one point of* Y*, then if* Y *is connected, these two lifts must agree on all of* Y*.*

The fundamental group of real projective space Related with covering space there is a special kind of group action called covering space action, which can give rise to a covering space. We now take a look at these kind of action and use it to calculate the fundamental group of RP^n.

Recall that a group action of the group G on a topological space Y is a group homomorphsim ρ from G to the group $Homeo(Y)$ of all homeomorphisms from Y to itself. Each group element $g \in G$ is associated with a homeomorphism $\rho_g : Y \to Y$. A group action is called a **covering space action** if for any $y \in Y$, there is a neighborhood U_y of y such that for any distinct group elements g_1, g_2, we have $\rho_{g_1}(U_y) \cap \rho_{g_2}(U_y) = \emptyset$. Equivalently, for any $g \neq e$, we have $\rho_g(U_y) \cap U_y = \emptyset$.

The following proposition tells us that if G acts on space Y is a covering space action, then Y is a covering space of the orbit space Y/G:

Proposition 2.5.10. *If a group action of a group* G *on a space* Y *is a covering space action, then*

- *The quotient map* $p : Y \to Y/G$ *which maps each* $y \in Y$ *to its orbit, is a covering space.*
- *If* Y *is path connected and locally path connected, we have* $G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$*.*

Now we can use the above proposition to compute $\pi_1(RP^n)$:

Example 2.5.1 ($\pi_1(RP^n)$)*.* Let $\mathbb{Z}_2$ acts on S^n by antipodality, i.e., for a nontrivial element $v \in \mathbb{Z}_2$, for any $x \in S^n$, define $v \cdot x = -x$. The orbit space of this action is the real projective space RP^n. This action is a covering space action: for any $x \in S^n$, take U_x to be an open neighborhood of x contained in the open hemisphere of S^n in which x is contained. Obviously, $-U_x \cap U_x = \emptyset$. By the above proposition we know that the quotient map $p : S^n \to RP^n$ is a covering space. Since $\pi_1(S^n) = 0$

when $n \geq 2$, we have $\mathbb{Z}_2 \cong \pi_1(RP^n)$ when $n \geq 2$. As for $n = 1$, since $S^1 \cong RP^1$, we have $\pi_1(RP^1) = \pi_1(S^1) = \mathbb{Z}$. In conclusion,

$$\pi_1(RP^n) = \begin{cases} \mathbb{Z}, & n = 1 \\ \mathbb{Z}_2, & n \geq 2 \end{cases}$$

2.5.2 HOMOTOPY GROUPS

Homotopy groups $\pi_n(X)$ are higher dimensional analogs of the fundamental groups $\pi_1(X)$.

Let I^n be the n-dimensional cubes, i.e., the product space of n-copies of the unit interval $[0, 1]$, let ∂I^n be the boundary of I^n, which is the subspace consisting of the points of I^n with at least one coordinates equal to 0 or 1, let X be a topological space with base point x_0, define a map $f : (I^n, \partial I^n) \to (X, x_0)$ sending I^n to X and ∂I^n to x_0, define the **n-th homotopy group** $\pi_n(X, x_0)$ to be the set of homotopy classes of f, and the homotopy $h_t : (I^n, \partial I^n) \to (X, x_0), \forall t \in I$ should be stationary on the subset $\partial I^n \subset I^n$, i.e., for all $t \in I$, $h_t(\partial I^n) = x_0$. For $n = 0$, we can define I^0 to be a point, ∂I^0 to be the empty set, and extend the definition of $\pi_n(X, x_0)$ to $n = 0$: the map $f : (I^0, \emptyset) \to (X, x_0)$ sending the singleton I^0 to a point of X, and any two maps f_1, f_2 with images $f_1(I^0), f_2(I^0)$ lying in the same path component of X are homotopic with the path between $f_1(I^0)$ and $f_2(I^0)$ being the homotopy. Thus $\pi_0(X, x_0)$ is in one to one correspondence with each path component of X.

We now define the group operation on $\pi_n(X, x_0)$ for $n \geq 2$ to be the generalization of the group operation of the fundamental group: for any $f, g : (I^n, \partial I^n) \to (X, x_0)$, define its sum $f + g$ to be

$$(f+g)(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n), & 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, \ldots, s_n), & \frac{1}{2} \leq s_1 \leq 1 \end{cases}$$

The above operation is well defined on the level of homotopy classes of maps, i.e., $[f] + [g] = [f + g]$ since if $f_1 \simeq f_2$, $g_1 \simeq g_2$, then we have $f_1 + g_1 \simeq f_2 + g_2$. Using the same argument as fundamental group, we can also show that $\pi_n(X, x_0)$ is a group with identity with the identity element to be the homotopy class of the constant map sending I^n to x_0, and the inverse of $[f(s_1, s_2, \ldots, s_n)]$ to be $[-f(s_1, s_2, \ldots, s_n)] = [f(1 - s_1, s_2, \ldots, s_n)]$.

Different from the fundamental group $\pi_1(X, x_0)$ being non abelian in general, the homotopy groups $\pi_n(X, x_0)$ are abelian for $n \geq 2$, i.e., $f + g \simeq g + f$. We refer the readers to [Hat02, page 340] for a detailed explanation.

There is also another way to define the homotopy group: the maps $f : (I^n, \partial I^n) \to (X, x_0)$ is the same with the maps $(I^n/\partial I^n, \partial I^n/\partial I^n) \to (X, x_0)$, and if we let $I^n/\partial I^n = S^n$, let $\partial I^n/\partial I^n) = s_0$, we have a map $(S^n, s_0) \to (X, x_0)$. Thus we can view $\pi_n(X, x_0)$ to be the homotopy classes of maps $(S^n, s_0) \to (X, x_0)$. And the sum $f + g$ is the composition $S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X$ where c collasping the equator S^{n-1} of S^n to a point, and we choose the base point s_0 of S^n to lie in this equator.

We now show that similar with the fundamental group, if X is path connected, then its homotopy groups $\pi_n(X, x_0)$ with different base points x_0 are isomorphic. Thus in this case, we can write $\pi_n(X, x_0)$ simply as $\pi_n(X)$. In order to do this, we will define a change-of-base-point isomorphism $\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_0)$ by $\beta_\gamma([f]) = [\gamma f]$ where $\gamma : I \to X$ is a path from a base point $\gamma(0) = x_0$ to another base point $\gamma(1) = x_1$. For any map $f : (I^n, \partial I^n) \to (X, x_1)$ we associate it with $\gamma f : (I^n, \partial I^n) \to (X, x_0)$ using γ as follows: first restrict the domain of f to a smaller cube $\tilde{I}^n$ lying in the center of I^n, and then divide $I^n \setminus \tilde{I}^n$ into line segments connecting ∂I^n and $\partial \tilde{I}^n$. Let γf send ∂I^n to x_0, $\partial \tilde{I}^n$ to x_1, then the line segments can be viewed as paths γ from x_0 to x_1. Since a homotopy of f which is stable on ∂I^n gives a homotopy of γf which is stable on ∂I^n, β_γ is well defined. It can be shown that β_γ is an isomorphism with inverse $\beta_{\gamma^{-1}}$, where $\gamma^{-1}(s) = \gamma(1 - s)$ is the inverse path of γ. For more details readers can see [Hat02, section 4.1]. We have shown that if X is path connected, for any two base points x_0 and x_1, any path γ connecting them gives an isomorphism β_γ between $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$.

Now we define the action of π_1 on π_n using the above β_γ: we restrict γ to be paths with the same start and end points, i.e. to be loops at some base point x_0. Similar with above, the map $\beta_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_0), [f] \mapsto [\gamma f]$ is an automorphism of $\pi_n(X, x_0)$. Observe that a homotopy of γ which is stable on ∂I gives a homotopy of γf which is stable on ∂I^n, we can associate the homotopy class $[\gamma] \in \pi_1(X, x_0)$ to $\beta_\gamma \in Aut(\pi_n(X, x_0))$, i.e., we can define a map $\rho : \pi_1(X, x_0) \to Aut(\pi_n(X, x_0)), [\gamma] \mapsto \beta_\gamma$. It can be shown that $(\gamma_1\gamma_2)f \simeq \gamma_1(\gamma_2 f)$. Thus we have $\beta_{\gamma_1\gamma_2} = \beta_{\gamma_1}\beta_{\gamma_2}$, which gives ρ is a group homomorphism. And we call ρ **the action of** $\pi_1(X, x_0)$ **on** $\pi_n(X, x_0)$.

A space X is called n**-simple** if $\pi_1(x)$ acts on $\pi_n(X)$ trivially. When $n = 1$, this corresponds to $\pi_1(X)$ being abelian. A space is called **simple** if it is n-simple for every n.

A space X with base point x_0 is called **n-connected** if $\pi_i(X, x_0) = 0$ for all $0 \leq i \leq n$. And 0-connected means path connected, 1-connected means simply connected. Since n-connected implies 0-connected, and for path connected space the homotopy groups of different choices of base points are isomorphic, we have $\pi_i(X, x_0) = 0$ for some base point x_0 implies $\pi_i(X, x) = 0$ for any base point x. A space X is called n-connected if $\pi_i(X, x_0) = 0, 0 \leq i \leq n$ for some base point(and thus for all base points) x_0.

For n-connectivity, it can easily be shown that the following three condition are equivalent:

- $\pi_i(X, x_0) = 0$ for all $x_0 \in X$.
- any map $f : S^i \to X$ is homotopic to a constant map.
- any map $f : S^i \to X$ can extend to a map $\tilde{f} : B^{i+1} \to X$ where B^{i+1} is a $(i+1)$ dimensional ball with boundary S^i.

2.6 Homology and Cohomology

2.6.1 singular homology

Introduction The fundamental group that we defined above involves only maps from 1 dimensional space(the 1 dimensional cube I) to topological space. Given this definition, it is natural that the fundamental group can only reflect the low dimensional structure of a topological space. For example, the fundamental group of cw complexes X only depends on the 2-skeleton of X. In order to reflect the properties of higher dimensional topological space, we can define a higher dimensional analog of the fundamental group—the homotopy group $\pi_n(X)$ which invloves maps from the n dimensional cube I^n to topological space X.

However, despite having a straightforward definition, the homotopy group is difficult to compute in general. It would be nice if we can define another kind of group which has connection which the homotopy group on one side, and easier to compute on the other side. The topic in this subsection—the homology group, would be such kind of group.

Singular Homology The price for computability of the homology group is that it has a less straightforward definition. We need some preparations before we give the definition:

A **singular n-simplex** in a space X is a map $\sigma : \Delta^n \to X$ where Δ^n is the stardard n simplex. Let $C_n(X)$ be the free abelian group with basis the set of singular n simplices in X, i.e., $C_n(X) = \{\sum_{i=0}^{k} n_i\sigma_i | n_i, k \in \mathbb{Z}, \sigma_i : \Delta^n \to X\}$. The elements of $C_n(X)$, which is the finite formal sum $\sum_i n_i\sigma_i$ are called n **chains**. We define the **boundary map** $\partial_n : C_n(X) \to C_{n-1}(X)$ on each basis element σ of $C_n(X)$ by the following formula and then extent to the whole group $C_n(X)$ linearly:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, \dots, \overset{\wedge}{v_i}, \cdots, v_n]$$

where $[v_0, \dots, \overset{\wedge}{v_i}, \cdots, v_n]$ is a $(n-1)$ dimensional simplex of Δ^n which lies in the boundary of the n simplex $\Delta^n = [v_0, \cdots, v_n]$ with vertices $\{v_0, \dots, v_{i-1}, v_{i+1}, \cdots, v_n\}$. $\overset{\wedge}{v_i}$ indicates that the vertex v_i is deleted from the sequence $v_0, \dots, v_n$. And $\sigma|[v_0, \dots, \overset{\wedge}{v_i}, \cdots, v_n]$ represents the restriction of σ to the $n-1$ dimensional face $[v_0, \dots, \overset{\wedge}{v_i}, \cdots, v_n]$ of Δ^n, i.e., it is a singular $(n-1)$ simplex $\Delta^{n-1} \to X$.

There is an important property of the boundary map which we state without proof:

Proposition 2.6.1. *For each n, $\partial_n \circ \partial_{n+1} = 0$.*

Thus now we have a sequence of homomorphisms of abelian groups

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

with $\partial_n \circ \partial_{n+1} = 0$ for each n. Such a sequence is called **chain complex**. $\partial_n \circ \partial_{n+1} = 0$ indicates that $Im\partial_{n+1} \subseteq Ker\partial_n$ where Im and Ker represent image and kernel. We define the **n-th singular homology group** of the chain complex to be the quotient group $H_n = Ker\partial_n / Im\partial_{n+1}$. The elements of the kernel are called **cycles** and the elements of the image are called **boundaries**. Elements of H_n are called **homology classes**. Two cycles different by a boundary are in the same homology class and are called **homologous**.

We now show that any map $f : X \to Y$ induces a group homomorphsim $f_* : H_n(X) \to H_n(Y)$:

Let $f : X \to Y$ be a map between two spaces, it induces a group homomorphism $f_\# : C_n(X) \to C_n(Y)$ by first define on each basis $\sigma : \Delta^n \to X$ of $C_n(X)$ to be $f_\#(\sigma) = f \circ \sigma$ and then extend linearly to all $C_n(X)$,.i.e., $f_\#(\sum_i n_i)\sigma_i = \sum_i n_i f_\#(\sigma_i)$.

Furthermore, $f_\#$ commutes with the boundary map ∂, i.e., $f_\# \circ \partial = \partial \circ f_\#$ since

$$f_\# \partial(\sigma) = f_\#(\sum_i (-1)^i \sigma|[v_0, \ldots, \hat{v}_i, \cdots, v_n]) = \sum_i (-1)^i f\sigma|[v_0, \ldots, \hat{v}_i, \cdots, v_n] = \partial f_\#(\sigma)$$

We call $f_\#$ a **chain map** from the singular chain complex of X to that of Y. The fact that $f_\#$ commutes with the boundary map implies that $f_\#$ takes cycles to cycles and takes boundaries to boundaries. Thus $f_\#$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ between the homology groups of the two chain complexes.

Cellular Homology In order to compute the singular homology group of the CW complexes, we introduce another important concept called cellular homology group. It turns out that cellular homology group and singular homology group are isomorphic, which makes cellular homology group an efficient tool to compute the singular homology group of the CW complexes.

We first introduce the relative singular homology group.

Let X be a space and A be a subspace of X, there is a natural inclusion map $C_n(A) \to C_n(X)$. The group of **relative singular chains** is defined as

$$C_n(X, A) = C_n(X)/C_n(A)$$

The boundary map $\partial : C_n(X) \to C_n(X)$ restricts to the boundary $\partial : C_n(A) \to C_n(A)$ and thus induces a boundary map $\partial : C_n(X)/C_n(A) \to C_n(X)/C_n(A)$ on relative chains. We now have the following **singular chain complex** of the pair (X, A):

$$\cdots \to C_{n+1}(X, A) \xrightarrow{\partial_{n+1}} C_n(X, A) \xrightarrow{\partial_n} C_{n-1}(X, A) \cdots \to C_1(X, A) \xrightarrow{\partial_1} C_0(X, A) \xrightarrow{\partial_0} 0$$

Similarly as above, $\partial\partial = 0$ gives the homology group of this chain complex, which are called the **singular homology groups of the pair** (X, A) and are denoted $H_n(X, A)$.

A sequence of homomorphism

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

is said to be **exact** if $Ker\ \alpha_n = Im\ \alpha_{n+1}$ for all n.

There is an important **long exact sequence of the pair** (X, A):

Proposition 2.6.2. *There exists a connecting homomorphism* $\partial_* : H_n(X, A) \to H_{n-1}(A)$ *such that the sequence of homology groups*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$

is exact, where i_* *is the homomorphism induced by the inclusion* $i : C_n(A) \to C_n(X)$ *and* j_* *is the homomorphism induced by the quotient map* $j : C_n(X) \to C_n(X, A)$.

Now we can define the cellular homology of the CW complex:

Let X *be a CW complex, let* $H_n(X^n, X^{n-1})$ *be the singular homology group of the pair* (X^n, X^{n-1}) *where* X^n, X^{n-1} *be the* n *and* $n-1$ *skeleton of* X*. We can show that* $H_n(X^n, X^{n-1})$ *is a free abelian group with basis of the* n *cells of CW complex* X*:*

Proposition 2.6.3. *[Hat02, Lemma 2.34] If* X *is a CW complex, then* $H_k(X^n, X^{n-1})$ *is zero for* $k \neq n$ *and is free abelian for* $k = n$*, with a basis in one to one correspondence with the* n*-cell of* X*.*

Define the ***boundary map*** $d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ *to be the composite*

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial_*} H_{n-1}(X^{n-1}) \xrightarrow{j_*} H_{n-1}(X^{n-1}, X^{n-2})$$

Again we have $d_n \circ d_{n+1} = 0$ *since*

$$H_{n-1}(X^{n-1}) \xrightarrow{j_*} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{\partial_*} H_{p-2}(X^{p-2})$$

is exact. Thus we have the following ***cellular chain complex*** *of* X*:*

$$\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

The homology group $H_n^{cw}(X)$ *given by the above cellular chain complex are called the* ***cellular homology group*** *of* X*, and it can be shown that it is isomorphic to the singular homology of* X*. Thus in order to compute the singular homology group of a CW complex, we only need to compute its cellular homology group.*

Proposition 2.6.4. *[Hat02, Theorem 2.35]* $H_n^{cw}(X) \cong H_n(X)$*.*

In the following, we will give the formula to compute the cellular boundary map. Before that, we first need the definition of degree for maps between spheres:

any map $f : S^n \to S^n$ induces a group homomorphism $f_* : H_n(S^n) \to H_n(S^n)$. Since $H_n(S^n) \cong \mathbb{Z}$, f_* must be of the form $f_* : x \mapsto dx$ for some integer d. Then d is called the ***degree*** *of f.*

The cellular boundary map is given by the following formula:

Proposition 2.6.5. *[Hat02, P140] The boundary map $d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ of the cellular chain complex is given by*

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$$

where e^n_α is a n cell of X and a generator of $H_n(X^n, X^{n-1})$, e^{n-1}_β is a $n-1$ cell of X and a generator of $H_{n-1}(X^{n-1}, X^{n-2})$, $d_{\alpha\beta}$ is the degree of the map

$$S^{n-1}_\alpha \cong \partial e^{n-1}_\beta \xrightarrow{\chi^n_\alpha} X^{n-1} \xrightarrow{q} \frac{X^{n-1}}{X^{n-1} \setminus e^{n-1}_\beta} \cong S^{n-1}_\beta$$

which is the composition of the attaching map $\chi^n_\alpha : \partial e^{n-1}_\beta \to X^{n-1}$ of e^n_α and the quotient map $q : X^{n-1} \to \frac{X^{n-1}}{X^{n-1} \setminus e^{n-1}_\beta}$ collapsing $X^{n-1} \setminus e^{n-1}_\beta$ to a point. The sum is taken over all $n-1$ cells of X.

Homology group of real projective space *Now we compute the singular homology group of the real projective space RP^n using cellular homology.*

Example 2.6.1 (Homology group of RP^n). By example 2.2.1 we know that RP^n has a cw structure with one cell e^k in each dimension $k \leq n$, and the attaching map of e^k is the covering map $\varphi : S^{k-1} \to RP^{k-1}$ identifying the antipodal points of S^{k-1}. In order to apply the above formula for the boundary map, we first need to calculate the degree for the composition $q \circ \varphi S^{k-1} \xrightarrow{\varphi} RP^{k-1} \xrightarrow{q} RP^{k-1}/RP^{k-2} = S^{k-1}$. We compute the degree of $q \circ \varphi$ using local degree: the map $q \circ \varphi$ is a homeomorphsim when restricted to each component of $S^{k-1} - S^{k-2}$, and these two homeomorphisms are obtained from each other by precomposing with the antipodal map of S^{k-1}, which has degree $(-1)^k$. Hence deg $q \circ \varphi = 1 + (-1)^k$. Thus when k is odd, $d_k = 0$; when k is even, $d_k = 2$. Thus we have the following cellular chain complex for RP^n:

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0, \text{ when } n \text{ is even}$$

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0, \text{ when } n \text{ is odd}$$

which induce the homology group of RP^n:

$$H_k(RP^n) = \begin{cases} \mathbb{Z} & k = 0, \text{ or } k = n \text{ and } k \text{ is odd.} \\ \mathbb{Z}_2 & k \text{ is odd and } 0 < k < n \\ 0 & \text{otherwise} \end{cases}$$

Connection between homotopy and homology The following theorem is one of the cornerstones of algebraic topology, and it demonstrates a direct connection between homotopy and homology.

Theorem 2.6.1 (Hurewicz Theorem)**.** *[Hat02, Theorem 4.32] Let X be a space with base point x_0, there is a naturally defined group homomorphism (the Hurewicz map[1])between the homotopy groups $\pi_n(X, x_0)$ and the homology groups $H_n(X)$ given as follows:*

$$\begin{aligned} \rho : \pi_n(X, x_0) &\to H_n(X) \\ [f] &\mapsto f_*(\alpha) \end{aligned}$$

where $f : (S^n, s_0) \to (X, x_0)$ is a map from S^n to X sending the base point s_0 of S^n to the base point x_0 of X, and f induces a map on homology which is $f_* : H_n(S^n) \to H_n(X)$, and $\alpha \in H_n(S^n)$ is a generator of $H_n(S^n) \cong \mathbb{Z}$. Then we have

- *for $n = 1$, if X is 0-connected, i.e., $\pi_0(X, x_0) = 0$, the Hurewicz map ρ induces a group isomorphism*

  $$\tilde{\rho} : \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \to H_1(X)$$

  from the abelianization of $\pi_1(X, x_0)$ to $H_1(X)$, where $[\pi_1(X, x_0), \pi_1(X, x_0)]$ is the commutator subgroup of $\pi_1(X, x_0)$.

- *for $n \geq 2$, if X is $(n-1)$-connected, i.e., $\pi_i(X, x_0) = 0, 0 \leq i \leq n-1$, then the reduced homology groups $\tilde{H}_i(X) = 0, 0 \leq i \leq n-1$, and the Hurewicz map $\rho : \pi_n(X, x_0) \to H_n(X)$ is an isomorphism.*

The above Hurewicz theorem tells us that for a simply connected (1-connected) space X, its first nontrivial homotopy group $\pi_n(X)$ is isomorphic to its first non trivial homology group $H_n(X)$.

[1] The Hurewicz map is well-defined since by [Hat02, Theorem 2.10], if two maps $f, g : X \to Y$ are homotopic, they induces the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$

cohomology group Let X be a space, we have the following singular chain complex

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

and we dualize this chain complex by replacing each chain group $C_n(X)$ by its dual cochain group $C^n(X,G) := Hom(C_n(X),G)$ where $Hom(C_n(X),G)$ is the group of all homomorphisms from $C_n(X)$ to an abelian group G and call it **singular** n **cochains with coefficients in** G. Define the **coboundary map** $\delta : C^n(X,G) \to C^{n+1}(X,G)$ to be the dual ∂^* of the boundary map $\partial : C_{n+1}(X) \to C_n(X)$, i.e., for a cochain $\varphi \in C^n(X,G)$, its coboundary $\delta(\varphi) \in C^{n+1}(X,G)$ is the composition $C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\varphi} G$. The composition $\delta \circ \delta$ of two coboundary maps is a zero map since the composition of two boundary maps is zero. Thus we have the following cochain complex

$$\cdots \leftarrow C^{n+1}(X;G) \xleftarrow{\delta} C^n(X;G) \xleftarrow{\delta} C^{n-1}(X;G) \leftarrow \cdots \leftarrow C^1(X;G) \xleftarrow{\delta} C^0(X) \leftarrow 0$$

Thus we can define the **cohomology group** $H^n(X;G)$ with coefficients in G to be the quotient $Ker\delta/Im\delta$ at $C^n(X;G)$ in the above cochain complex. Elements of $Ker\delta$ are called **cocyles** and elements of $Im\delta$ are called **coboundaries**.

Since cohomology group is obtained from homology group, it is natural to ask about their relationships. The following **universal coefficient theorem for cohomology** tells us that the cohomology group with arbitrary coefficients are determined purely by homology groups with $\mathbb{Z}$ coefficients.

Theorem 2.6.2 (universal coefficient theorem for cohomology)**.** *[Hat02, Theorem 3.2] Let X be a space, G be an abelian group, then the cohomology groups $H^n(X;G)$ of the cochain complex $Hom(C_n(X);G)$ are determined by the exact sequence*

$$0 \to Ext(H_{n-1}(X),G) \to H^n(X;G) \to Hom(H_n(X),G) \to 0$$

Cohomology group of real projective space We can use the above theorem to compute the cohomology group of RP^n with $\mathbb{Z}$ coefficient. Before that, we first give three properties of $Ext(H,G)$ for finitely generated group H:

- $Ext(H \oplus H', G) \cong Ext(H,G) \oplus Ext(H',G)$

- $Ext(H, G) = 0$ if H is free
- $Ext(\mathbb{Z}_n, G) \cong G/nG$

Example 2.6.2 (cohomology group of RP^n). We see from example 2.6.1 the homology group of RP^n is either $\mathbb{Z}$ or $\mathbb{Z}_2$ or 0. When $k > n$, $H_k(RP^n) = 0$, thus $Ext(H_{k-1}(X), G) \cong H^k(X; G)$. Since $H_{k-1}(X)$ is either 0 or $\mathbb{Z}$, we have $H^k(X; G) = 0$. When $k = n$, n is odd, $H_k(X) = \mathbb{Z}$ and $H_{k-1}(X) = 0$. Then we have $Ext(H_{k-1}(X), \mathbb{Z}_2) = 0$, thus $H^k(X; G) \cong Hom(H_k(X), G)$. Furthermore, since $Hom(\mathbb{Z}, \mathbb{Z}_2) = \mathbb{Z}_2$, we have $H^k(X; G) = \mathbb{Z}_2$. Other cases are quite similar, thus we have

$$H^k(PR^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & 0 \leq k \leq n \\ 0, & \text{k>n} \end{cases}$$

cohomology ring We first define a multiplication between cohomology groups, which is the cup product:

consider cohomology with coefficients in a ring R, e.g., $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, for cochain $\varphi \in C^k(X; R)$, $\psi \in C^l(X; R)$, define their **cup product** $\varphi \smile \psi \in C^{k+l}(X; R)$ to be a cochain whose value on a singular simplex $\sigma : \Delta^{k+l} \to X$ is given by the formula

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+l}])$$

where multiplication on the right hand side is the ring multiplication of ring R. The following proposition shows that cup product of cochain induces a cup product on cohomology classes

Lemma 2.6.1. *$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$ for $\varphi \in C^k(X; R)$ and $\psi \in C^l(X; R)$.*

From the above formula we can see that the cup product of two cocycles is a cocycle, and the cup product of a cocyle and a coboundary is a coboundary. Thus the cup product between cochains induces a cup product between cohomology

$$H^k(X; R) \times H^l(X; R) \xrightarrow{\smile} H^{k+l}(X; R)$$

The cup product between cohomology is associative and distributive since the cup product between cochains are obviously associative and distributive. Now we can make the cup product into a multiplication in the following ring structure:

define $H^*(X;R)$ to be the direct sum of the groups $H^n(X;R)$, i.e.,

$$H^*(X;R) = \oplus_n H^n(X;R)$$

Elements in $H^*(X;R)$ are finite sums $\sum_i \alpha_i$ with $\alpha_i \in H^i(X;R)$. We can make $H^*(X;R)$ a ring by defining the product of any two elements $\sum_i \alpha_i, \sum_i \beta_i \in H^*(X;R)$ to be $(\sum_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i\beta_j$.

The cohomology ring is in fact a **graded ring**, i.e., the cohomology ring $H^*(X;R)$ can be decomposed into a sum $\oplus_n H^n(X;R)$ of additive subgroups $H^n(X;R)$ such that the ring multiplication takes elements in $H^k \times H^l(X;R)$ to $H^{k+l}(X;R)$. If an element $x \in H^*(X;R)$ lies in $H^k(X;R)$ we say it has **dimension** k.

We now give an example of cohomology ring without proof.

Example 2.6.3. [Hat02, Theorem 3.12] $H^*(RP^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1})$ with x has dimension 1.

2.7 Equivariant Obstruction theory

Given a relative CW complex (X, A), assume we have a map $f : X_n \to Y$ from the n-skeleton of X to Y, obstruction theory deals with the problem regarding whether f can extend one more dimension to X_{n+1}.

More precisely, we have the following main theorem of obstruction theory:

Theorem 2.7.1 (Main Theorem of Classical Obstruction Theory)**.** *[DK01, chapter 7] Let (X, A) be a relative CW-complex, $n \geq 1$, Y be a path connected n-simple space(i.e. $[S^n, Y] = \pi_n Y$), let $f : X_n \to Y$ be a continuous map, then we have*

1. *f extends to a map $X_{n+1} \to Y$ if and only if the obstruction cocycle $\theta^{n+1}(f) \in Hom(C_{n+1}(X, A), \pi_n Y)$ vanishes.*

2. *the restriction $f|_{X_{n-1}} : X_{n-1} \to Y$ extends to a map $X_{n+1} \to Y$ if and only if the cohomology class $[\theta^{n+1}(f)] \in H^{n+1}(X, A; \pi_n(Y))$ vanishes.*

In order to prove the theorem 2.7.1, we first define the obstruction cocycle:

Let (X, A) be the relative CW-complex as before, let J_n index all the n-cells of (X, A), $C_{n+1}(X, A)$ be the free abelian groups with basis to be all the $(n + 1)$-cells of (X, A), and $C^{n+1}(X, A; \pi_n Y) = Hom(C_{n+1}(X, A), \pi_n Y)$ be the abelian group of all group homomorphism from $C_{n+1}(X, A)$ to $\pi_n Y$. For any $n+1$ cell e_i^{n+1} of (X, A) there is a characteristic map of e_i^{n+1}

$$\phi_i : (B^{n+1}, S^n) \to (e_i^{n+1}, \partial e_i^{n+1}) \subset (X_{n+1}, X_n).$$

Restricting it to S^n we have the attaching map of e_i^{n+1} :

$$\varphi_i = \phi_i|_{S^n} : S^n \to \partial e_i^{n+1} \subset X_n,$$

compose φ_i with $f : X_n \to Y$ we have the following map:

$$S^n \xrightarrow{\varphi_i} X_n \xrightarrow{f} Y,$$

which gives an element $[f \circ \varphi_i] \in [S^n, Y]$. Since we assume Y is n-simple, $[S^n, Y] = \pi_n Y$. Now we have associated to every $n+1$ cell e_i^{n+1} an element $[f \circ \varphi_i]$ in $\pi_n Y$, and we can define the obstruction cochain:

Definition 2.7.1 (obstruction cochain)**.** The obstruction cochain $\theta^{n+1}(f) \in C^{n+1}(X, A; \pi_n Y)$ is a group homomorphism from $C_{n+1}(X, A)$ to $\pi_n Y$, which is defined on each $n+1$ cell e_i^{n+1} of $C_{n+1}(X, A)$ by

$$\theta^{n+1}(f)(e_i^{n+1}) = [f \circ \varphi_i]$$

and extend to all $C_{n+1}(X, A)$ linearly, i.e., for any element $\sum_i n_i e_i^{n+1}$ in $C_{n+1}(X, A)$,

$$\theta^{n+1}(f)(\sum_i n_i e_i^{n+1}) = \sum_i n_i \theta^{n+1}(f)(e_i^{n+1}).$$

A map $S^n \to Y$ is homotopic to a constant map if and only if it can extend to B^{n+1}, and $f \circ \varphi_i : S^n \to Y$ can extend to B^{n+1} if and only if f can extend to X_{n+1}. Thus we have prove the first part of the theorem 2.7.1:

Lemma 2.7.1. *$f : X_n \to Y$ can extend to X_{n+1} if and only if $\theta^{n+1}(f) = 0$.*

Furthermore, the obstruction cochain $\theta^{n+1}(f)$ is in fact a cocycle:

Theorem 2.7.2. *The obstruction cochain $\theta^{n+1}(f)$ is a cocycle.*

Proof. We omit the proof and refer the readers to [DK01, Theorem 7.6]. □

Thus the obstruction cocycle $\theta^{n+1}(f)$ defines a cohomology class $[\theta^{n+1}(f)] \in H^{n+1}(X, A; \pi_n Y)$.

In order to prove the second part of the theorem 2.7.1, we first show if $f : X_n \to Y$ can extend to X_{n+1} after modifying f on the n-cells of X_n (if $f|_{X_{n-1}}$ can extend to X_{n+1}), then the obstruction cocycle $\theta^{n+1}(f) \in C^{n+1}(X, A; \pi_n Y)$ is also a coboundary, i.e. $0 = [\theta^{n+1}(f)] \in H^{n+1}(X, A; \pi_n Y)$:

Theorem 2.7.3. *Given $f : X_n \to Y$, the cohomology class $[\theta^{n+1}(f)] \in H^{n+1}(X, A; \pi_n Y)$ vanishes if the restriction $f|_{X_{n-1}}$ can extend to X_{n+1}.*

In order to prove the theorem 2.7.3, we need the following lemma:

Lemma 2.7.2. *Let $f_0, f_1 : X_n \to Y$ be two maps whose restriction to X_{n-1} are homotopic, i.e., $f_0|_{Xn-1} \simeq f_1|_{X_{n-1}}$, then the choice of homotopy $G : X_{n-1} \times I \to Y$ defines a difference cochain $d(f_0, G, f_1) \in C^n(X, A; \pi_n Y)$ such that*

$$\delta d = \theta^{n+1}(f_0) - \theta^{n+1}(f_1)$$

where $\delta : C^n(X, A; \pi_n Y) \to C^{n+1}(X, A; \pi_n Y)$ is the differential of the cochain complex $C^*(X, A; \pi_n Y)$.

Remark 2.7.1. The above lemma shows that for any two maps $f_1, f_2 : X_n \to Y$, if their restriction to X_{n-1} are homotopic, i.e., if $f_1|_{X_{n-1}} \simeq f_2|_{X_{n-1}}$, then their obstruction cocyles differ by a coboundary, i.e., $[\theta^{n+1}(f_1)] = [\theta^{n+1}(f_2)]$.

Proof. We omit the proof and refer readers to [DK01, lemma 7.8]. □

proof of theorem 2.7.3. If $f : X_n \to Y$ can extend to X_{n+1} after modifying the n-cells of X, denote the modified f to be $g : X_n \to Y$, we have g can extend to X_{n+1}, thus by lemma 2.7.1, the obstrution cocycle $\theta^{n+1}(g) = 0$. Since $f|_{X_{n-1}} = g|_{X_{n-1}}$, by lemma 2.7.2, we have $[\theta^{n+1}(f)] = [\theta^{n+1}(g)] = 0$, thus $\theta^{n+1}(f)$ is a coboundary.

□

Now we prove another direction of theorem 2.7.1: if the obstruction cocycle $\theta^{n+1}(f) \in C^{n+1}(X, A; \pi_n Y)$ is a coboundary, then the restriction $f|_{X_{n-1}}$ can extend to X_{n+1}:

Theorem 2.7.4. *Given a map $f : X_n \to Y$, and let its obstruction cocycle $\theta^{n+1}(f) \in C^{n+1}(X, A; \pi_n Y)$ to be a coboundary δd for some cochain $d \in C^n(X, A; \pi_n Y)$, then the restriction $f|_{X_{n-1}}$ can extend to X_{n+1}.*

In order to prove the theorem 2.7.4, we need the following proposition:

Proposition 2.7.1 (realization proposition)**.** *Given a map $f_0 : X_n \to Y$, a homotopy $G : X_{n-1} \times I \to Y$ such that $G(-, 0) = f_0|_{X_{n-1}}$ and an element $d \in C^n(X, A; \pi_n Y)$, there exists a map $f_1 : X_n \to Y$ such that $G(-, 1) = f_1|_{X_{n-1}}$ and $d = d(f_0, G, f_1)$, i.e., $\delta d = \theta^{n+1}(f_0) - \theta^{n+1}(f_1)$.*

Proof. We omit the proof and refer the readers to [DK01, proposition 7.10]. □

proof of theorem 2.7.4. By the above realization proposition 2.7.1, for f, d, and the stationary homotopy $G : X \times I \to Y$ from $f|_{X_{n-1}}$ to itself: $G(x, t) = f|_{X_{n-1}}, \forall t \in I$, there exists a map $f' : X_n \to Y$ which agrees with f on X_{n-1}, and satisfies $\delta d = \theta^{n+1}(f) - \theta^{n+1}(f')$. Then we have $\theta(f') = 0$ so f' extends to X_{n+1}, which is f can extend to X_{n+1} after modifying the n-cells of X. □

Combining all above, we have prove theorem 2.7.1, the main theorem of obstruction theory.

The above classical obstruction theory can be stated in an equivariant setting. The proof and properties of the equivariant obstruction theory are similar with the classical obstruction theory. Readers are refered to [tD11, Chapter II.3] for free G-complex situation, and [Bre06, Chapter II] for general G-complex situation. We state the main theorem of equivariant obstruction theory for general G-complex without proof:

Theorem 2.7.5 (Main Theorem of Equivariant Obstruction Theory)**.** *[Bre06, chapter II.] Let G be a group. let X be a cell G-complex. We require the cellular action of G on X satisfy an additional condition: for any $g \in G$, $\{x \in X | g \cdot x = x\}$ is a subcomplex of X, i.e., if $g \cdot x = x$ for some $g \in G, x \in X$, then g fixes the support(the smallest subcomplex containing x) of x pointwisely. let Y be a G-space, we assume for simplicity that, for each subgroup $H \subseteq G$, the set Y^H of stationary points of H on Y is non-empty, arcwise connected, and n-simple. Assume we have given an G-equivariant map $f : X_n \to Y$. Then we have*

1. *f extends to a map $X_{n+1} \to Y$ if and only if the equivariant obstruction cocycle $\theta^{n+1}(f) \in Hom_G(C_{n+1}(X), \pi_n(Y))$ vanishes.*
2. *the restriction $f|_{X_{n-1}} : X_{n-1} \to Y$ extends to a map $X_{n+1} \to Y$ if and only if the cohomology class $[\theta^{n+1}(f)] \in H^{n+1}_G(X, \pi_n(Y))$ vanishes.*

The following theorem regarding the existence of G-map is a special case of the above theorem which we would use very often later:

Theorem 2.7.6. *[MBZ03, Lemma 6.2.2]*

There is a G-map from any at most n-dimensional free simplicial G-complex $\mathcal{K}$(or a cell G-complex) to any $(n-1)$-connected G-space X. And any two maps $f_1, f_2 : \mathcal{K} \to X$ are G-homotopic when restrict to the $n-1$-skeleton $\mathcal{K}_{n-1}$ of $\mathcal{K}$.

proof. We construct the G-map $f : \mathcal{K} \to X$ inductively using the $(n-1)$-connectivity of X. Suppose we have defined f on the $(d-1)$-skeleton of $\mathcal{K}$ where $d \leq n$. Since $\mathcal{K}$ is a simplicial G-complex, the group action G on $\mathcal{K}$ is a homeomorphism and a simplicial map from $\mathcal{K}$ to itself which maps simplices to simplices by linear maps, we can then partition the d-simplices of $\mathcal{K}$ into equivalence classes under this group action. Since the group action is free, the stablizer of each simplex is the trivial subgroup. Hence the fixed point space $X^{\{e\}} = X$ is $(d-1)$-connected and we can extend f as follow: we first choose one d-simplex σ from each equivalence class and extend f continuously on those simplices using the fact that f is $(d-1)$-connected, after that we extend f on other d-simplices in each equivalence class using the property of G-map(i.e., $f(g \cdot \sigma) = g \cdot f(\sigma)$), so that f becomes a G-map.

For the second statement, we need to construct a G-homotopy $H : \mathcal{K}_{n-1} \times I \to X$ between $f_1|_{\mathcal{K}_{n-1}}$ and $f_2|_{\mathcal{K}_{n-1}}$. Let $H(x, 0) = f_1(x)$, $H(x, 0) = f_2(x)$, we have a G-equivariant map $H : \mathcal{K}_{n-1} \times \{0, 1\} \to X$. We show that we can extend H to the whole $\mathcal{K}_{n-1} \times I$ G-equivariantly: define a G-action on $\mathcal{K}_{n-1} \times I$ by letting G acts on $\mathcal{K}_{n-1}$ as before, and G acts on I trivially, i.e., for $g \in G$, for $(x, t) \in \mathcal{K}_{n-1} \times I$, $g \cdot (x, t) = (g \cdot x, t)$. Since $\mathcal{K}_{n-1}$ is a free G-complex, $\mathcal{K}_{n-1} \times I$ is also a free G-complex. $\mathcal{K}_{n-1} \times I$ is obtained from $\mathcal{K}_{n-1} \times \{0, 1\}$ by attaching cells of at most n-dimensional. Since X is $(n-1)$-connected G space, and $\mathcal{K}_{n-1} \times I$ is a free G-complex, we can extend H to the whole $\mathcal{K}_{n-1} \times I$ cell by cell G-equivariantly as the proof of the first statement: for each cell $e \in H$ of a G-orbit, we have e has dimension at most n. Thus let $\partial : \partial e \to \mathcal{K}_{n-1} \times I$ be the attaching map of e, we can extend the map $H \circ \partial : \partial e \xrightarrow{\partial} \mathcal{K}_{n-1} \times I \xrightarrow{H} X$ continuously to the interior using the $(n-1)$ connectivity of X, which defines $H(e)$, and then extends H to the whole G-orbit of e equivariantly.

□

Remark 2.7.2. The above theorem can in fact be refined as(which is from my director Gregory):

Theorem 2.7.7. *Let X be an n-dimensional cell complex, with a cellular action of a finite group G. We require the cellular action satisfy an additional condition: for* *any* $g \in G$, $\{x \in X | g \cdot x = x\}$ *is a subcomplex of* X, *i.e., if* $g \cdot x = x$ *for some* $g \in G, x \in X$, *then* g *fixes the support(the smallest subcomplex containing* x*) of* x *pointwisely. Suppose that for all* $0 \leq i \leq n$, *for every subgroup* H *of* G *that occurs as a stabilizer of a cell of* X *of dimension* i, *the fixed point space* Y^H *is* $(i-1)$*-connected. Then there exists a* G*-equivariant map from* X *to* Y.

Remark 2.7.3. By lemma 2.7.2 we know that, for any two maps $f_1, f_2 : X_n \to Y$, if their restrictions on X_{n-1} are homotopic, then their corresponding obstruction cocycles differ by a coboundary. By the above theorem we can see that if the target space Y is $(n-1)$-connected, then any two maps $f_1, f_2 : X_n \to Y$ when restricted to X_{n-1} are homotopic, thus $[\theta^{n+1}(f_1)] = [\theta^{n+1}(f_2)]$, i.e., the obstruction cohomology class of extending $f : X_n \to Y$ is independent of f. We call it **the primary obstruction to extending** f.

3

The Borsuk-Ulam theorem

Our journey starts at an important theorem in topology– the Borsuk-Ulam theorem, which is one of the most applied topological theorem and it has lots of applications within and outside of mathematics, including combinatorics, economics and even physics.

The Borsuk-Ulam theorem was conjectured [FRI05] by Ulam at the Scottish Café in Lvov and has many equivalent formulations. One of its form was proven by Borsuk in 1933 in his paper [Bor33a]. Hence we have the name 'Borsuk-Ulam theorem'.

In this section, we will state the theorem and some of its equivalent formulations in the subsection 3.1, and then give some of the applications of the theorem in the subsection 3.2. Finally in the last subsection 3.3, we will give two generalizations of the theorem which would be essential for later discussion of the topological Tverberg theorem.

3.1 The Borsuk-Ulam theorem

We first state the Borsuk-Ulam theorem and some of its equivalent versions.

Theorem 3.1.1 (Borsuk-Ulam theorem and its equivalent formulations)**.** *For $n \geq 0$, the following statements are true and equivalent:*

(BU1.) *For any continuous map $f : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$.* [2]

(BU2.) *For any continuous antipodal map $f : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = 0$. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is antipodal if $\forall x \in \mathbb{R}^n, f(-x) = -f(x)$, where for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, -x = (-x_1, \ldots, -x_n)$.*

[2] $(BU1.)$ is the most common version of the Borsuk-Ulam theorem, and it is also the version that Borsuk has proved.

(*BU3.*) *There is no antipodal continuous map* $f : S^n \to S^m$ *when* $n > m$.

(*BU4.*) *There is no continuous map* $f : B^n \to S^m$ *which is antipodal on the boundary when* $n > m$.

Before we prove the theorem, we first show their equivalence which is easy to show and it would help us gain some intuition of the theorem:

Claim. *The above theorems are equivalent.*

proof of claim.

- $(BU1) \implies (BU2)$: let $f : S^n \to \mathbb{R}^n$ be an antipodal map, by $(BU1)$, there exists $a \in S^n$ such that $f(a) = f(-a)$. Since f is antipodal, we have $f(a) = f(-a) = -f(a)$ which gives $f(a) = 0$.

- $(BU2) \implies (BU3)$: for $m \leq n$, any antipodal map $f : S^n \to S^m$ is an antipodal map $\tilde{f} : S^n \to \mathbb{R}^n$ if we compose f with an inclusion map from S^m to $\mathbb{R}^n$. By $(BU2)$ we know that there exists $a \in S^n$ such that $\tilde{f}(a) = 0$. Since the origin 0 do not lie in any S^m, there is no such map.

- $(BU3) \implies (BU4)$: if when $m < n$, there exists a continuous map $f : B^n \to S^m$ antipodal on the boundary, we can construct an antipodal map $g : S^n \to S^m$ to be: divide S^n into an upper and an lower hemisphere H^+ and H^- which are both closed and include the boundary, denote their boundary as S^{n-1}. define $g : S^n \to S^m$ as $\forall x \in H^+$, let $g(x) = f(x)$, $g(-x) = -f(x)$. g is well defined since f is antipodal on the boundary and g is by definition antipodal.

- $(BU4) \implies (BU1)$: if there exists a map $f : S^n \to \mathbb{R}^n$ such that $\forall x \in S^n$, $f(x) \neq f(-x)$, we can construct a map $h : B^n \to \mathbb{R}^n$ antipodal on the boundary: if we think of the upper hemisphere of S^n as B^n, the map restricts to a map $f : B^n \to \mathbb{R}^n$ which satisfies $\forall x \in B^n$, $f(x) \neq f(-x)$. Define $h : B^n \to \mathbb{R}^n$ as $h(x) = g(x) - g(-x)$, then for all $x \in B^n$, $h(-x) = g(-x) - g(x)$, i.e., h is antipodal on the boundary.

□

The proof of the theorem is however not easy. There are mainly two ways to prove the theorem: either to use combinatorics or to use algebraic topology. We will first give a proof using some advanced knowledge of algebraic topology. We will later give a combinatorial proof of the theorem by proving its combinatorial euqivalent statement — the Tucker's lemma, using pure combinatorics.

We now prove $(BU3)$ in theorem 3.1.1 using algebraic topology. Before that, we first state $(BU3)$ using the language of equivariant topology:

Let $\mathbb{Z}_2$ be a cyclic group of two elements, as in example 2.3.3, 2.3.5, we can view S^n as a free $\mathbb{Z}_2$-space, an antipodal map $f : S^n \to S^m, f(-x) = -f(x)$ as a $\mathbb{Z}_2$-equivariant map, then we can rewrite $(BU3)$ as:

Theorem 3.1.2 (($BU3$), equivariant statement)**.** *There is no $\mathbb{Z}_2$-equivariant map from S^n to S^m when $n > m$.*

proof of $(BU3)$ by algebraic topology. The proof is from [FRI05].

We prove by contradiction. Assume that when $n > m \geq 1$, there exists an $\mathbb{Z}_2$-equivariant map $f : S^n \to S^m$ between two $\mathbb{Z}_2$ spaces, which induces a map $\bar{f} : S^n/\mathbb{Z}_2 \to S^m/\mathbb{Z}_2$ between their orbit spaces. Since the orbit of S^n under the antipodal $\mathbb{Z}_2$ action is the real projective space RP^n, we have $\bar{f} : RP^n \to RP^m$. And the quotient map $q_n : S^n \to RP^n$, $q_m : S^m \to RP^m$ are covering maps with S^n and S^m being the universal cover of RP^n and RP^m. $\bar{f}$ also induces a map between fundamental group $\bar{f}_* : \pi_1(RP^n) \to \pi_1(RP^m), \varphi \mapsto \bar{f} \circ \varphi$.

Let $x \in S^n$ be a base point of S^n, and $f(x) = f(-x), q_n(x) = q_n(-x), q_m(f(x)) = q_m(f(-x))$ be the corresponding base points of S^m, RP^n, RP^m; let $\alpha : [0,1] \to RP^n$ be a loop at the base point $q_n(x)$, i.e., $\alpha(0) = \alpha(1) = q_n(x)$, since $[0,1]$ is simply connected, by lifting criterion, there exists a unique lift $A : [0,1] \to S^n$ of α with $A(0) = x$. Given all above, we have the following commutative diagram:

$$\begin{tikzcd} & S^n \arrow[r, "f"] \arrow[d, "q_n"'] & S^m \arrow[d, "q_m"'] \\ (0,1) \arrow[ur, "\exists! A"] \arrow[r, "\alpha"'] & RP^n \arrow[r, "\bar{f}"'] & RP^m \end{tikzcd}$$

Given $A : [0,1] \to S^n$ a lift of α as above, we know that $A(1)$ is either x or $-x$ since $q_n(A(1)) = \alpha(1) = q_n(x) = q_n(-x)$. When $A(1) = x$, A is a loop at $x \in S^n$, and since S^n is simply connected when $n \geq 2(n > m \geq 1$ implies $n \geq 2)$, A is null homotopic. The composition $q_n \circ A$ is also homotopic equivalent to a constant map, i.e., the homology class $[q_n \circ A] \in \pi_1(RP^n)$ is the trivial element. When $A(1) = -x$, the composition $q_n \circ A$ is not null homotopic and the homology class $[q_n \circ A] \in \pi_1(RP^n)$ is a nontrivial element(If $[q_n \circ A] \in \pi_1(RP^n)$ is a trivial element, then it has to be a image of the group homomorphism $(q_n)_* : \pi_1(S^n) \to \pi_1(RP^n)$, i.e., there exists a loop $A' : [0,1] \to S^n$ with $A(0) = A(1) = x$ which is null homotopic and which satisfies that $q_n \circ A' = \alpha$. Thus A' is also a lift of α. Since $A(0) = A'(0) = x$, by the uniqueness of lift, we should have $A = A'$, which is however not true.) Since $f : S^n \to S^m$ is a $\mathbb{Z}_2$ equivariant map, we have $f(A(1)) = f(-x) = -f(x)$. Similarly with the above argument, $q_m \circ f \circ A$ is not null homotopic and the homology class $[q_m \circ f \circ A] \in \pi_m(RP^m)$ is a non trivial element. Furthermore, since $q_m \circ f \circ A = \bar{f} \circ q_n \circ A$, we have $\bar{f}_*([q_n \circ A]) = [\bar{f} \circ q_n \circ A] = [q_m \circ f \circ A]$ being a group homomorphism from $\pi_n(RP^n)$ to $\pi_m(RP^m)$ sending a nontrivial element to a non trivial element, i.e, $\bar{f}_*$ is not a zero map.

In example 2.5.1 we have seen that the fundamental group of the real projective space RP^n is $\mathbb{Z}$ when $n = 1$, and it is $\mathbb{Z}_2$ when $n > 1$. If $n > m = 1$, by above argument, $\bar{f}_* : \pi_1(RP^n) \to \pi_1(RP^1)$ is a non zero group homomorphism $\bar{f}_* : \mathbb{Z}_2 \to \mathbb{Z}$ which is not possible, since the generator 1 of $\mathbb{Z}_2$ has order 2, but there is no element of $\mathbb{Z}$ has order 2. Thus we have shown that there is no $\mathbb{Z}_2$ equivariant map $f : S^n \to S^1$ when $n > 1$.

If $n > m > 1$, similarly, $\bar{f}_* : \pi_1(RP^n) \to \pi_1(RP^m)$ is a non zero group homomorphism $\bar{f}_* : \mathbb{Z}_2 \to \mathbb{Z}_2$, which can only be the identity map. By Hurewicz theorem, we have the homology group $H_1(RP^n)$ is isomorphic to the abelianization of $\pi_1(RP^n)$. Since $\pi_1(RP^n)$ is an abelian group, we have $H_1(RP^n) \cong \pi_1(RP^n)$. By above we have an identity map between fundamental groups $\bar{f}_* : \pi_1(RP^n) \to \pi_1(RP^m)$, which gives an isomorphism between homology groups $H_1(RP^n) \cong H_1(RP^m)$. And this also induces an isomorphism between cochain groups $Hom(H_1(RP^m), \mathbb{Z}_2) \cong Hom(H_1(RP^n), \mathbb{Z}_2)$. Furthermore, The universal coefficient theorem for cohomology tells us that $Hom(H_1(RP^n), \mathbb{Z}_2) \cong H^1(RP^n, \mathbb{Z}_2)$:

By the universal coefficient theorem for cohomology, there is a short exact sequence

$$0 \to Ext(H_0(RP^n), \mathbb{Z}_2) \to H^1(RP^n, \mathbb{Z}_2) \to Hom(H_1(RP^n), \mathbb{Z}_2) \to 0$$

Since $H_0(RP^n) = \mathbb{Z}$ is a free abelian group, we have $Ext(H_0(RP^n), \mathbb{Z}_2) = 0$. Thus $Hom(H_1(RP^n), \mathbb{Z}_2) \cong H^1(RP^n, \mathbb{Z}_2)$.

Thus we have $\bar{f}^* : H^1(RP^m, \mathbb{Z}_2) \to H^1(RP^n, \mathbb{Z}_2)$ is an isomorphism. We see from example 2.6.2 that for all n, $H^1(RP^n, \mathbb{Z}_2) = \mathbb{Z}_2$, and the only isomorphsim between $\mathbb{Z}_2$ is the identity map. Thus $\bar{f}^* : H^1(RP^m, \mathbb{Z}_2) \to H^1(RP^n, \mathbb{Z}_2)$ is the identity map.

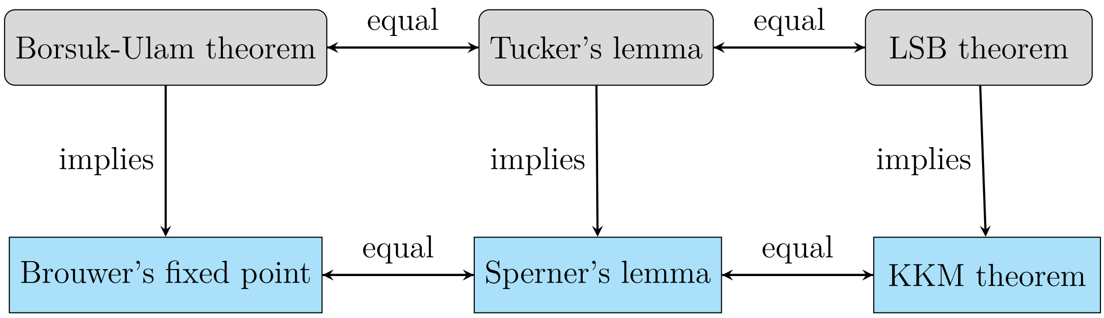


Figure 2: equivalence relations between three domains

We have seen from example 2.6.3 the cohomology ring with $\mathbb{Z}_2$ coefficient of RP^n is $H^*(RP^n, \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1})$.

Thus the ring homomorphism

$$\bar{f}^* : H^*(RP^m, \mathbb{Z}_2) \to H^*(RP^n, \mathbb{Z}_2)$$

is the ring homomorphism

$$\bar{f}^* : \mathbb{Z}_2[x]/(x^{m+1}) \to \mathbb{Z}_2[y]/(y^{n+1}), x \mapsto y$$

which maps x to y since $\bar{f}^* : H^1(RP^m, \mathbb{Z}_2) \to H^1(RP^n, \mathbb{Z}_2)$ is an isomorphism.

However, since $n > m$, we have $0 = \bar{f}^*(x^{m+1}) = (\bar{f}^*(x))^{m+1} = y^{m+1} \neq 0$, which is a contradiction. Thus there is no $\mathbb{Z}_2$-equivariant map $f : S^n \to S^m$ when $n > m > 1$.

Combine the above discussion, we have proved $(BU3)$.

□

3.2 Applications of the Borsuk-Ulam theorem

In this section, we will give some applications of the Borsuk-Ulam theorem. In subsection 3.2.1, we will give two equivalent formulations of the Borsuk-Ulam theorem (a topological theorem) in two different domains — the Tucker's lemma in combinatorics and the LSB theorem in set covering. In subsection 3.2.2, we will first show that the Borsuk-Ulam theorem implies another important theorem — the Brouwer's fixed point theorem, and then similar with before, we gives two equivalent formulations of the Brouwer's fixed point theorem(a topological theorem) in two different domains — the Sperner's lemma in combinatorics and the KKM theorem in set covering. More intuitively, we will show the following relations as in figure 2.

For more applications and combinatorial direction information of the above theorems, [DLGMM19] is a good reference.

3.2.1 Borsuk-Ulam theorem and the Tucker's lemma

Tucker's lemma: a combinatorial equivalent formulation of the Borsuk-Ulam theorem We will first give a combinatorial equivalent formulation of the Borsuk-Ulam theorem, which is the Tucker's lemma. We first state the most common formulation of the Tucker's lemma:

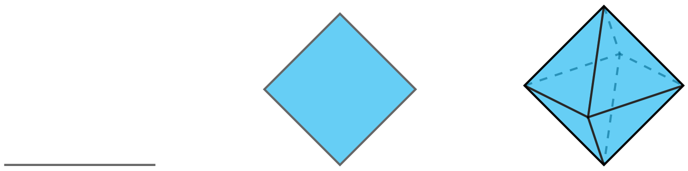
Lemma 3.2.1 (The first statement of Tucker's lemma)**.** *Let T be a triangulation of B^n which is antipodally symmetric on the boundary, which means for any simplex σ, if $\sigma \in T \cap \partial B^n$, the antipodal simplex $-\sigma \in T \cap \partial B^n$. Define a labeling l of the vertices of T to be a map:*

$$l : V(T) \to \{\pm 1, \ldots, \pm n\}$$

which satisfies $l(-v) = -l(v)$ for any $v \in \partial B^n$. For any such labeling l, there exists an edge $e \in T$ such that the two vertices of the edge e are labelled by opposite numbers. We call this edge e a complementary edge.

An algorithmic proof of the above Tucker's lemma was given in [FT81a] and was reformulated in [MBZ03, section 2.3]. We sketch the proof briefly as follows : the idea is to construct a graph using fully labeled simplices as nodes and connect the nodes under some relations of the simplices. Later it is shown that only the nodes of the origin and the simplices which contain a complementary edges has degree 1, other nodes has degree 2. Finally using the property that an undirected graph must have even number of nodes of odd degree we can show that there is at least one fully labeled simplex having a complementary edge(in fact, we have prove there are odd number of fully labeled simplices having a complementary edge.). In addition, this proof can be turned into an algorithm to find out the position of the complementary edge.

It is surprising that the two seemingly unrelated theorem, i.e., the Borsuk-Ulam theroem and the Tucker's lemma are in fact equivalent. The equivalence will be more obvious if we reformulate Tucker's lemma into another equivalent statement. Before that, we first define a geometric object called cross-polytope which will be used in our reformulation.



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$

Figure 3: n-dimensional cross-polytope

Definition 3.2.1 (cross-polytope)**.** The n-dimensional cross-polytope $\diamond^n$ in $\mathbb{R}^n$ is the convex hull

$$\diamond^n = conv(e_1, -e_1, \ldots, e_n, -e_n)$$

of the vectors $\{e_1, e_2, \ldots, e_n\}$ of the standard orthonormal basis of Euclidean space $\mathbb{R}^n$ and their negative vectors $\{-e_1, -e_2, \ldots, -e_n\}$.

Equivalently, the n-dimensional cross-polytope is the unit ball of the l_1-norm:

$$\diamond^n = \{x \in \mathbb{R}^n | \|x\|_1 \leq 1\}.$$

For example, we draw the $1, 2, 3$-dimensional cross-polytope in figure 3.

For the goal of reformulation, we denote the boundary of the cross polytope using abstract simplicial complex as follows: let P^{n-1} be the boundary of n dimensional cross-polytope $\diamond^n$. Then P^{n-1} can be denoted as an abstract simplicial complex with vertex set $V(P^{n-1}) = \{\pm 1, \ldots, \pm n\}$, and with simplices to be the subsets $F \subseteq V(P^{n-1})$ where there is no $i \in \{1, 2, \ldots, n\}$ such that both $i \in F$ and $-i \in F$.

Now we reformulate the Tucker's lemma 3.2.1:

Lemma 3.2.2 (The second statement of Tucker's lemma)**.** *Let T be a triangulation of B^n which is antipodal symmetric on the boundary. Define P^{n-1} to be the boundary of n-dimensional cross-polytope $\diamond^n$, then there is no simplicial map from B^n to P^{n-1} defined by $f : V(T) \to V(P^{n-1})$ which satisfies $f(-v) = -f(v)$ for $v \in \partial B^n$, i.e., f is antipodal on the boundary.*

The equivalence of the two statements of the Tucker's lemma is obvious:

Claim 3.2.1. *Lemma 3.2.1 and lemma 3.2.2 are equivalent statements of the Tucker's lemma.*

proof of claim. We first show lemma 3.2.1 implies lemma 3.2.2: if there exists a simplicial map $f : V(T) \to V(P^{n-1})$ from B^n to P^{n-1} which is antipodal on the boundary, then f gives a labeling of $V(T)$ using $\{\pm 1, \ldots, \pm n\}$ which is antipodal on the boundary and has no complementary edge, contradicting lemma 3.2.1(since a complementary edge having opposite label would be map outside of P^{n-1} under f.).

We now show lemma 3.2.2 implies lemma 3.2.1: if there exists a labeling l of $V(T)$ using $\{\pm 1, \ldots, \pm n\}$ which is antipodal on the boundary and is without complementary edge, then this labeling gives a simplicial map $l : V(T) \to V(P^{n-1})$ from B^n to P^{n-1} antipodal on the boundary, contradicting lemma 3.2.2.

□

We now will show that the Tucker's lemma is equivalent to the Borsuk-Ulam theorem:

Claim 3.2.2. *Tucker's lemma is equivalent to $(BU4)$ of the Borsuk-Ulam theorem 3.1.1.*

proof of claim. $(BU4)$ implies lemma 3.2.2 is obvious: if there exists a simplicial map $f : V(T) \to V(P^{n-1})$ from B^n to P^{n-1} which is antipodal on the boundary, since $P^{n-1} \cong S^{n-1}$, compose f with this homeomorphism we have a continuous map from B^n to S^{n-1} which is antipodal on the boundary, contradicting $(BU4)$.

Lemma 3.2.1 implies $(BU4)$ is more complicated: if there exists a continuous map $f : B^n \to S^{n-1}$ which is antipodal on the boundary, let T be a triangulation of B^n of which the diameters δ of the simplices are sufficiently small(how small will be specified later), we can define a labeling $l : V(T) \to V(P^{n-1}) = \{\pm 1, \ldots, \pm n\}$ using f: since for any $v \in V(T), f(v) \in S^{n-1} \subseteq \mathbb{R}^n$, then there exists $i \in \{1, \ldots, n\}$, such that the i-th component $f(v)_i$ of $f(v)$ satisfies $f(v)_i \geq \frac{1}{\sqrt{n}}$, otherwise we would have the euclidean norm $\|f(v)\|$ of $f(v)$ smaller than 1, contradicting $f(v) \in S^{n-1}$. We can define the labeling l of the vertices of the triangulation T to be: for any $v \in V(T)$,

$$l(v) = \begin{cases} \min\{i : |f(v)_i| \geq \frac{1}{\sqrt{n}}\} & \text{if } f(v)_i > 0 \\ -\min\{i : |f(v)_i| \geq \frac{1}{\sqrt{n}}\} & \text{if } f(v)_i < 0 \end{cases}$$

This labeling is antipodal on the boundary since f is antipodal on the boundary. By lemma 3.2.1, this label l has a complementary edge, say $e = \{v_1, v_2\}$, and $l(v_1) = -l(v_2) = i$ for some $i \in V(P^{n-1})$. By the definition of l we have $|f(v_1)_i - f(v_2)_i| \geq \frac{2}{\sqrt{n}}$, i.e. the maximum norm $\|f(v_1) - f(v_2)\|_\infty \geq \frac{2}{\sqrt{n}}$.

In addition, since f is a continuous function and B^n is a compact set, we have f being uniformly continuous on B^n, i.e., for any $\epsilon \geq 0$, there exists $\delta \geq 0$ such that if $\|x_1 - x_2\| \leq \delta$ we have $\|f(x_1) - f(x_2)\|_\infty < \epsilon$. If we take $\epsilon = \frac{2}{\sqrt{n}}$, we can find the corresponding δ, and if we bound the diameter of the triangulation by δ, we would have $\|v_1 - v_2\| \leq \delta$ but $\|f(v_1) - f(v_2)\|_\infty \geq \frac{2}{\sqrt{n}}$, which is a contradiction.

□

We now give a constructive proof of Tucker's lemmma by pure combinatorics. Since Tucker's lemma is equivalent to the Borsuk-Ulam theorem, this can also be considered as a combinatorial proof of the Borsuk-Ulam theorem:

A constructive proof of the Tucker's lemma. The proof is from [MBZ03] and I specify it more detailedly. It initially comes from [FT81b].

The idea of the proof is to construct a graph whose nodes are a special kind of simplexes and connect these nodes in a way such that only the origin and the simplexes with a complementary edge has degree 1 and all other nodes have degree 2. Since a graph has even number of nodes with odd degree, we conclude that there must be at least one simplex with a complementary edge(In fact, we have shown that there are odd number of simplexes with a complementary edge). Furthermore, following the path of the graph from the origin to one of the node with degree 1, we can find out the corresponding way from the origin to one of the simplex with a complementary edge, which gives us an algorithm to compute the location of the complementary edge.

We now give the precise description of the proof. We first replace the n dimensional Euclidean ball B^n by the n dimensional crosspolytope $\diamond^n$, i.e., the unit ball of l_1-norm.

Let $\diamond\!\!\!+^n$ be the natural triangulation of the crosspolytope $\diamond^n$ induced by the coordinate hyperplane of the Euclidean space $\mathbb{R}^n$. Let T be a triangulation of $\diamond^n$ which is a refinement of the natural triangulation $\diamond\!\!\!+^n$, i.e. any simplex of T is contained in a simplex of $\diamond\!\!\!+^n$, we call T a special triangulation of $\diamond^n$. Furthermore, we can construct a special triangulation of arbitrary small diameter. For example, we can repeatedly take the barycentric subdivision of the natural triangulation $\diamond\!\!\!+^n$.

Now let $l : V(T) \to \{\pm 1, \dots, \pm n\}$ be a labeling of the vertices of T which is antipodal on the boundary. For any simplex $\sigma \in T$, define $l(\sigma) := \{l(v) | v \text{ is a vertex of } \sigma\}$, i.e., $l(\sigma)$ is the set of all the labels of the vertices of σ. We now define another set of labels of σ: let $x = (x_1, \dots, x_n) \in \sigma \subset \diamond^n \subset \mathbb{R}^n$ be any point in the relative interior of σ, define

$$S(\sigma) := \{+i | x_i > 0\} \cup \{-i | x_i < 0\}$$

$S(\sigma)$ is well defined, since in our special triangulation T, the Euclidean coordinates of any interior point x of σ have the same sign and thus gives the same $S(\sigma)$.

Now using these two set of labels of simplexes $\sigma \in T$, we specify a special kind of simplexes in T and use them as nodes to construct a graph. These kinds of special simplexes $\sigma \in T$ should satisfy $S(\sigma) \subseteq l(\sigma)$ and we call them happy simplexes.

The happy simplexes σ can be divided into two kinds according to their dimensions: suppose $|S(\sigma)| = k$, then σ is contained in the linear subspace $L_\sigma \subseteq \mathbb{R}^n$ spanned by the coordinates axes of $\{x_{|i|} | i \in S(\sigma)\}$, thus $\dim \sigma \leq k$. On the other hand, $\dim \sigma \geq k - 1$ since $|S(\sigma)| \leq |l(\sigma)|$ which implies σ need to have at least k vertices. If $\dim \sigma = k$, we call such simplex loose; if $\dim \sigma = k - 1$, we call such simplex tight. A happy simplex on the boundary is tight, while a nonboundary happy simplex can be tight or loose. The origin $\{0\}$ is happy($S(\{0\}) = \emptyset$) and loose.

In order to construct a graph, we also need to specify which two nodes are adjacent to each other. We define the following relationships:

Two nodes $\sigma, \tau \in T$ are adjacent (connected by an edge) if they satisfy one of the following conditions:

- σ and τ are antipodal boundary simplexes, i.e., $\sigma, \tau \in \partial \Diamond^n$ and $\sigma = -\tau$.
- σ is a facet of τ(i.e., a $(\dim \tau - 1)$-dimensional face) and $l(\sigma) = S(\tau)$.

The origin $\{0\}$ is a node of degree 1, i.e., with one and only one adjacent node. Since $\{0\}$ has label $l(\{0\})$, it is a facet of a 1 dimensional simplex σ of T with $l(\{0\}) = S(\sigma)$, which means σ lies in the half coordinate axes of $x_{|l(\{0\})|}$: the positive half if $l(\{0\}) > 0$ and the negative half if $l(\{0\}) < 0$. There can not be any other 1 simplex with $\{0\}$ as a facet.

For other happy simplexes, we distinguish them into several cases and show that any happy simplex with no complementary edge has degree 2, and since any graph has to have even number of nodes with odd degree, there must be at least one node except from the origin has odd degree, i.e., there must be at least one happy simplex with a complementary edge.(In fact, we have shown that there must be odd number of happy simplexes with a complementary edge.

1. σ is a tight happy simplex, then it can be a boundary or non boundary simplex.

    1.1. If σ is a boundary happy simplex, then $-\sigma$ is one of its neighbors. W.L.O.G. we can assume that σ is a $k-1$ dimensional boundary of $\Diamond^k$, thus σ can be the facet of exactly one k dimensional simplex τ, and we now show τ is also a happy simplex and thus another neighbor of σ. As a happy simplex σ satisfies $|S(\sigma)| = |l(\sigma)| \leq k$ and as a boundary simplex we have $k = |S(\sigma)|$, thus we have $|S(\sigma)| = |l(\sigma)| = k$ which means each vertex of σ should have distinct labels. Furthermore, combine $S(\sigma) \subseteq l(\sigma)$ and $|S(\sigma)| = |l(\sigma)| = k$, we have $S(\sigma) = l(\sigma)$. This gives $S(\tau) = S(\sigma) = l(\sigma) \subseteq l(\tau)$, which means τ is also happy.

1.2. If σ is not a boundary simplex, W.L.O.G we can assume σ is $k-1$ dimensional and $|S(\sigma)| = k$. Obviously, σ is a facet of two k dimensional simplexes and we now show they are all happy and are neighbors of σ. For any k dimensional simplex τ with σ as a facet, we have $S(\tau) = S(\sigma)$ since $|S(\sigma)| = k$ indicates that σ is a $k-1$ dimensional simplex lies in the linear subspace L_σ spanned by the coordinate axes of $\{x_{|i|} | i \in S(\sigma)\}$ with σ not lying in any hyperplane spanned by the coordinate axes of any proper subset of $\{x_{|i|} | i \in S(\sigma)\}$. Furthermore, since σ is happy, we have $S(\sigma) \subseteq l(\sigma)$. And since σ is $k-1$ dimensional, we have $k = |S(\sigma)| \leq |l(\sigma)| \leq k$, which gives $S(\sigma) = l(\sigma)$. Thus we have $S(\tau) = S(\sigma) = l(\sigma) \subseteq l(\tau)$, which means τ is happy and it is a neighbor of σ.

2. σ is a loose happy simplex and thus a non boundary simplex.

2.1. If $S(\sigma) = l(\sigma)$, i.e., if one of the l labels of σ occurs twice in σ, then there are exactly two facets f_1, f_2 of σ such that $l(f_1) = l(f_2) = l(\sigma)$. And these two facets are exactly the two neighbors of σ: fistly, these two facets are happy simplexes since $S(f_1) \subseteq S(\sigma) = l(f_1)$, same for f_2. Secondly, these two facets satisfy $l(f_1) = l(\sigma) = S(\sigma)$, same for f_2. Furthermore, σ can not be a facet of a happy simplex τ with $l(\sigma) = S(\tau)$ and thus has no more other neighbors: $S(\sigma) = l(\sigma)$ implies that the l labels of σ are given by those $\{i\}$ which satisfy $x_{|i|}$ are among the coordinate axes that spanned σ, which means σ cannot be a facet of one more dimensional happy simplex τ and still satisfies $l(\sigma) = S(\tau)$.

2.2. If $S(\sigma) \subsetneq l(\sigma)$, i.e., there is an extra label $i \in l(\sigma) \setminus S(\sigma)$, if we assume further that $-i \notin S(\sigma)$, then we can see that the x_i coordinate of points of σ is zero, which means σ lie in the hyperplane with $x_i = 0$. Now we show that σ is a facet of a one more dimensional happy simplex τ with $l(\sigma) = S(\tau)$, i.e., τ is a neighbor of σ in the graph that we construct: obviously, σ is a facet of two one more dimensional simplex; if $i > 0$, we let τ be the simplex which lies in the half hyperplane of the positive part of the axis of x_i; if $i < 0$, let τ be the simplex which lies in the other half hyperplane of the negative part of the axis of x_i; τ is happy and a neighbor of σ since $S(\tau) = S(\sigma) \cup \{i\} = l(\sigma) \subseteq l(\tau)$. The other neighbor of σ is the facet $\sigma' \subset \sigma$ with $l(\sigma') = S(\sigma) \subset l(\sigma)$. σ' is happy since $S(\sigma') \subseteq S(\sigma) = l(\sigma)$.

In addition, if $-i \in S(\sigma)$, similar with the argument of 2.1, σ can not be a facet of a happy simplex τ with $l(\sigma) = S(\tau)$ and thus has only one neighbor, which is the facet σ' of σ: $-i \in S(\sigma)$ implies that the l labels of σ are given by those $\{i\}$ which satisfy $x_{|i|}$ are among the coordinate axes that spanned σ, which means σ cannot be a facet of one more dimensional happy simplex τ and still satisfies $l(\sigma) = S(\tau)$.

□

Remark 3.2.1. The above proof is a proof of the Tucker's lemma in a special case instead of the general situation: instead of proving for any general triangulation T of $\diamond^n$, we give some restrictions on T, i.e., T should be a refinement of the natural triangulation $\diamondplus^n$ of $\diamond^n$. Luckily, this special case of the Tucker's lemma that we have proved enough to deduce the Borsuk-Ulam theorem(By the above equivalence of the Tucker's lemma and the Borsuk-Ulam theorem we can see that Tucker's lemma implies Borsuk-Ulam theorem if and only if the triangulation T is small enough which can be achieved in our case by repeatedly taking barycentric subdivision.) Thus we can obtain the proof of the Tucker's lemma by taking the following path: a special case of the Tucker's lemma implies the Borsuk-Ulam theorem which implies the Tucker's lemma the other way around.

LSB theorem: a set covering version of the Borsuk-Ulam theorem In addition to the combinatorial equivalent version of the Borsuk-Ulam theorem — the Tucker's lemma, there is also a set-covering equivalent version of the Borsuk Ulam theorem, which is the Lusternik–Schnirelmann–Borsuk theorem [Bor33b]. We give the statement of the theorem:

Theorem 3.2.1 (Lusternik–Schnirelmann–Borsuk theorem)**.** *If the sphere S^n is covered by $n+1$ closed sets, then one of the set contains a pair of antipodal points x and $-x$ of S^n.*

The LSB theorem can be shown to be equivalent to the Borsuk-Ulam theorem without much difficulty:

Claim 3.2.3. *The LSB theorem and Borsuk-Ulam theorem are equivalent.*

Proof. The proof is from [MBZ03]. One direction is $(BU1)$ of theorem 3.1.1 implies the LSB theorem: Let $C_1, \dots, C_{n+1}$ be subsets of S^n and their union cover S^n, i.e., $S^n = \cup_{i=1}^{n+1} C_i$. We can define a function $f : S^n \to \mathbb{R}^n$, $f(x) = (d(x, C_1), \dots, d(x, C_n))$, where $d(x, C_i)$ is the distance between a point and a set. f is obviously continuous since each of its components are continuous. By $(BU1)$, there exists $x, -x \in S^n$ such that $f(x) = f(-x)$. If there exists $i \in \{1, \dots, n\}$ such that $d(x, C_i) = d(-x, C_i) = 0$, then we have x and $-x$ are both in C_i; if no such i, then $x, -x$ are both in C_{n+1}.

The other direction is LSB theorem implies $(BU3)$ of theorem 3.1.1: for a sphere S^{n-1}, there exists a set of closed subsets $C_1, \dots, C_{n+1}$ of S^{n-1} which cover S^{n-1} but non of the subsets contain an antipodal point of S^{n-1}. For example, we can take C_i to be the projection of the n faces of the standard $n+1$ simplex inscribed in S^n from the center. If there exists a continuous antipodal map $f : S^n \to S^{n-1}$, $f^{-1}(C_i), i \in \{1, \dots, n+1\}$ would be a closed cover of S^n which does not contain antipodal point, contradicting the LSB theorem: if $x, -x \in f^{-1}(C_i)$ for some i, then $f(x), -f(x)$ would be in C_i, but we assumed that $C_1, \dots, C_{n+1}$ is a cover of S^{n-1} without antipodal points. □

Now we have shown the first row of the figure 2, and we will show the second row in the next subsection.

3.2.2 Brouwer's fixed point theorem and Sperner's lemma

Borsuk-Ulam theorem implies the Brouwer's fixed point theorem We first show that the Borsuk-Ulam theorem implies the Brouwer's fixed point theorem, which is due to the Dutch mathematician L. E. J. Brouwer:

Theorem 3.2.2 (Brouwer's fixed point theorem, 1912)**.** *Let C be a nonempty, compact, convex subset of $\mathbb{R}^n$, $f : C \to C$ be a continuous map from C to itself, then there exists an $x \in C$ such that $f(x) = x$, i.e., x is a fixed point of f.*

Remark 3.2.2. We can generalize C to any subset of $\mathbb{R}^n$ which is homeomorphic to a closed unit ball.

proof using (BU4): Since any nonempty, compact, convex subset C of $\mathbb{R}^n$ is homeomorphic to a closed unit ball $B^m \in \mathbb{R}^n$ where $m \leq n$, denote the homeomorphism by $\phi : B^m \to C$, we only need to show any continuous map from B^m to itself has a fixed point: let $g = \phi^{-1} \circ f \circ \phi : B^m \to B^m$, if there exists an $x \in B^m$ such that $g(x) = \phi^{-1}(f(\phi(x))) = x$, then we have $f(\phi(x)) = \phi(x)$, i.e., $\phi(x) \in C$ is a fixed point of f.

Now we show any continuous map from B^m to itself has a fixed point: assume there exists a continuous map $g : B^m \to B^m$ has no fixed point, we define a new map $h : B^m \to S^{m-1}$ by defining $h(x)$ to be the unique intersection of S^{m-1} with the open ray starting from $g(x)$ and go through x, i.e.,

$$h(x) = S^{m-1} \cap \{g(x) + \lambda(x - g(x)) | \lambda \geq 0\}$$

since g has no fixed point, h is well defined, and when $x \in \partial B^m, h(x) = x$, i.e., h is identity on the boundary. Since identity map is also antipodal, h is antipodal on the boundary. Furthermore, since $g(x)$ is continuous and $h(x)$ is uniquely determined by $g(x)$, $h(x)$ is also continuous. Thus we have a continuous map $h : B^m \to S^{m-1}$ which is antipodal on the boundary, contradicting $(BU4)$. □

Remark 3.2.3.

- The above proof is by contradiction and not constructive. It only shows the existence of a fixed point, but does not give us any information about the location of the fixed point. It would be nice to know how to find out the location of the fixed point because it could offer us an algorithm which would be useful for applications. We will see that the following Sperner's lemma can solve the problem and offer us an algorithm to approximate the Brouwer's fixed point.
- Brouwer's fixed point theorem is one of the most important theorem about fixed point and it has lots of applications within and outside of mathematics: John von Neumann use it to prove the Minimax theorem in game theory, which is equivalent to strong duality in linear programming; John Nash use it to prove the existence of the famous Nash equilibria in strategic games [DL16].

Sperner's lemma: a combinatorial version of the Brouwer's fixed point theorem Similarly with the Tucker's lemma being a combinatorial equivalent formulation of the Borsuk-Ulam theorem, there is also a combinatorial equivalent formulation of the Brouwer's fixed point theorem, which is the Sperner's lemma [NS13]. The equivalence is in the sense that two theorems can imply each other, which will be specified later.

Lemma 3.2.3 (Sperner's lemma)**.** *Let S be a n-simplex, T be a triangulation of S(in this case we require $|T| = S$), define a Sperner labeling of S to be a map*

$$l : V(T) \to V(S)$$

where $V(T)$ and $V(S)$ are the vertex set of T and S respectively, such that for any $v \in V(T)$, $l(v) \in V(supp(v))$ where $supp(v)$ is the support of v in S,i.e., any vertex v of the triangulation T of the simplex S can only be labeled by one of the vertices of the face of the simplex S which contains v as its interior point. Then, there is a n-simplex σ of T such that σ is completely labeled, i.e., the vertices of σ are labeled pairwise distinctly with all vertices of S.

More precisely, we can show that there are odd number of completely labeled n-simplices.

proof. The proof is an elementary combinatorial proof by induction on dimension n of the simplex. This proof is from the paper [XU].

Induction base: When $n = 0$, a 0-simplex itself is trivially completely labeled.

When $n = 1$, let $S_1 = (1, 2)$ be a 1-simplex with boundaries vertices denoted by 1 and 2, let T be any triangulation of S_1, there are odd number of 1-simplices of T are labeled completely(call it '1-2'simplex), which is with boundaries vetrices being labeled 1 and 2 respectively since: the boundaries vertices of S are 1 and 2, and only odd number of 1-simplies with distinct boundaries can change 1 to 2(e.g. three '1-2' simplices have effect 1-2-1-2 which changes 1 to 2), while even number of such 1-simplies would keep 1 or keep 2 fixed(e.g, two '1-2' simplices have effect 1-2-1 which keeps 1 fixed).

Induction hypothesis: Assume when $n = k - 1$, the lemma is correct.

Consider when $n = k$, S_k is a k-dimensional simplex with vertices $\{0, \ldots, k\}$, T is any triangulation of S_k, let C be the set of all $k - 1$ simplices of T whose vertices are completely labeled by the vertices $\{0, \ldots, k-1\}$ of S_k, we now count the number $|C|$ of elements c in C in two ways:

way 1. for any $c \in C$, c is either contained in the $k - 1$ dimensional face of S_k with vertices $\{0, \ldots, k - 1\}$ or is contained in the interior of S_k. For those are contained in the interior, we count them twice since they are faces of two k-simplices of T. Thus we have

$$|C| = |c \text{ in the boundary } | + \text{ some even number.}$$

way 2. a k simplex of T is completely labeled by $\{0, \ldots, k\}$ if and only if it contains one and only one $c \in C$. Other k simplices of T which are not completely labeled might contain 0 or 2 such $c \in C$. Thus we have

$$|C| = |\text{ completely labeled } k \text{ simplices}| + \text{ some even number.}$$

By the above two way we have

$$|\text{ completely labeled } k \text{ simplices}| - |c \text{ in the boundary }| = \text{ some even number.}$$

By induction hypothesis, the number $|c$ in the boundary $|$ is odd, thus we have odd numeber of completely labeled k simplices of T.

□

We now show the equivalence of the Brouwer's fixed point theorem and the Sperner's lemma:

Claim 3.2.4. *The Brouwer's fixed point theorem and the Sperner's lemma are equivalent.*

Proof. We first show that the Brouwer's fixed point theorem implies the Sperner's lemma, and the proof is from [Yos74]: the idea is for any triangulation T of any n simplex S, for any Sperner labeling l of T, we construct a continuous function $f_l : S \to S$ associated with l, and then show that if there is no n simplex of T which is completely labeled, then f has no fixed point, contradicting the Brouwer's fixed point theorem.

More precisely, let T be a triangulation of an n simplex S with vertices $V(T) = \{v_1, \ldots, v_m\}$ with $(x_{i0}, \ldots, x_{in})$ being the barycentric coordinates of $v_i, i \in \{1, \ldots, m\}$, let l be the Sperner labeling of $V(T)$, we now construct a continuous function $f : S \to S$ by first defining f on $V(T)$ and then extend to the whole S linearly: Let $l(i)$ be the Sperner label of v_i, let $\epsilon = \min_{i=1}^{n} v_{il(i)}$ where it is the $l(i)$-th coordinate of v_i, define the $l(i)$-th coordinate of $f(v_i), i = \{1, \ldots, m\}$ to be the $l(i)$-th coordinate of v_i minus ϵ, and the other n coordinates of $f(v_i)$ to be the corresponding coordinates of v_i plus $\frac{\epsilon}{n}$, which is:

$$f(v_i) = \begin{cases} v_{ij} - \epsilon & \text{if } j = \lambda(i) \\ v_{ij} + \frac{\epsilon}{n} & \text{if } j = \{0, \ldots, n\} \setminus \{\lambda(i)\} \end{cases}$$

We now extend f linearly to the whole S by extending it linearly to each of the n simplexes of the triangulation T: if $\sigma^n \in T$ is a n simplex with vertices $\{w_0, \ldots, w_n\} \subseteq V(T)$, we can denote it by its barycentric coordinates as

$$\sigma^n = \{x \in S | x = \sum_{i=0}^{n} \lambda_i w_i, 0 \leq \lambda \leq 1, \sum_{i=0}^{n} \lambda_i = 1\}$$

and we can define $f(x)$ to be $f(x) = f(\sum_{i=0}^{n} \lambda_i w_i) = \sum_{i=0}^{n} \lambda_i f(w_i)$.

f is obviously well defined and continuous. We now show that if S has no completely labeled n simplex under the triangulation T and Sperner labeling l, then the function f we construct above will has no fixed point, contradicting the Brouwer's fixed point theorem: by assumption, for any n simplex σ^n of T, there is a label $p \in \{0, \ldots, n\}$ which is not used by any vertex of σ^n. For any $x \in \sigma^n$, the p-th coordinate of $f(x)$ is the p-th coordinate of x plus $\frac{\epsilon}{n}$. Thus for any $x \in S$, the p-th coordinate of x and the p-th coordinate of $f(x)$ can not be equal, i.e., $f : S \to S$ has no fixed point. Thus, S must has a completely labeled simplex.

Another direction is showing the Sperner's lemma implies the Brouwer's fixed point theorem, and the proof is from [Fox09]: Let S be a standard n-simplex in $\mathbb{R}^{n+1}$ with vertices $\{e_1, \ldots, e_{n+1}\}$ where e_i is the i-th unit vector of the i-th coordinate axes of $\mathbb{R}^{n+1}$, assume the Brouwer's fixed point theorem is not true, which is, there exists a continuous map $f : S \to S$ such that $f(x) \neq x, \forall x \in S$, we can draw a contradiction using Sperner's lemma. For all $j \in \mathbb{N}$, let S^j be a triangulation of S such that S^j is a triangulation of S^{j-1}, and the diameter of the triangulation tends to zero as j tends to infinity, let $V(S^j) = \{V^{j,1}, \ldots, V^{j,r}\}$ be the vertices of the triangualtion S^j, denote the k-th barycenteric coordinate of each $V^{j,i}$ as $V_k^{j,i}$, we now give a Sperner labeling L of S^j using f:

$$L : V(S^j) \to \{1, \ldots, n+1\}$$
$$x \mapsto L(x)$$

where $L(V^{j,i})$ should satisfy that it is the smallest coordinate such that $f(V^{j,i})_{L(V^{j,i})} \leq (V^{j,i})_{L(V^{j,i})}$. This is possible since f has no fixed point, and thus $f(V^{j,i}) \neq V^{j,i}$, which is their barycenteric coordinates are not all equal. And since the sum of their barycenteric coordinates are both 1, there exists $k \in \{1, \ldots, n+1\}$ such that $f(V^{j,i})_k \leq (V^{j,i})_k$. This labeling is a Sperner labeling since for any vertices e_k, its i-th barycenteric coordinate is 1 and others are 0, and thus i is the only possible labeling such that $f(e_k)_i \leq (e_k)_i$. Similarly for other vertices of $V(S^j)$: if $V^{j,i} \in conv(e_i, i \in A \subseteq \{1, \ldots, n+1\})$, then $i \in A$ are the only possible labeling of $V^{j,i}$.

By Sperner's lemma, for all j, for any triangulation S^j, there exists a completely labeled n-simplex σ^j with vertices $\{v^{j,i}, i = 1, \ldots, n+1\}$ where vertex $v^{j,i}$ is labeled i. By the above construction of the labeling L, we have $f(v^{j,i})_i \leq (v^{j,i})_i$. Since the diameter of the triangulation S^j tends to zero, we have $\forall i \in \{1, \ldots, n+1\}$, $\lim_{j\to\infty} v^{j,i} = v^*$. Thus by the continuity of f, we have $\forall i \in \{1, \ldots, n+1\}, f(v^*)_i \leq (v^*)_i$. However, since f has no fixed point, we have $f(v^*) \neq v^*$, so there exists an $d \in \{1, \ldots, n+1\}$, such that $f(v^*)_d > (v^*)_d$, a contradiction.

□

Remark 3.2.4. The second direction of the above proof, i.e., the proof showing how Sperner's lemma implies the Brouwer's fixed point theorem in fact gives us an algorithm to approximate the location of the Brouwer's fixed point, which is the limit v^*.

KKM theorem–A set covering version of the Brouwer's fixed point theorem In addition to Sperner's lemma being a combinatorial equivalent version of the Brouwer's fixed point theorem, the follwing KKM theorem can be viewed as a set covering equivalent version of the Brouwer's fixed point theorem.

Before we state the statement of the KKM theorem, let's first take a look at the following example to gain some intuition of the theorem:

Example 3.2.1 (KKM theorem of 2-simplex). Let S be a 2 dimensional simplex with vertex set $I = \{x_0, x_1, x_2\}$, let $\{A_0, A_1, A_2\}$ be the set of closed subsets of S such that for any $i \in I$, each 0-face x_i of S is contained in A_i; for any $i, j \in I, i \neq j$, each 1-face (x_i, x_j) of S is contained in $A_i \cup A_j$; the only 2-face (x_0, x_1, x_2) of S is contained in $A_1 \cup A_2 \cup A_3$. Then we have $A_1 \cap A_2 \cap A_3 \neq \emptyset$.

Now we give the general statement of the theorem:

Theorem 3.2.3 (Knaster-Kuratowski-Mazurkiewicz theorem)**.** *Let S be an n dimensional simplex with vertex set $I = \{x_0, \ldots, x_n\}$, denoted by $S = conv(\{x^i\}_{i\in I})$, let $\{A^i\}_{i\in I}$ be a set of closed subsets of S such that for each $\tilde{I} \subseteq I$, $conv(\{x^i\}_{i\in\tilde{I}}) \subseteq \cup_{i\in\tilde{I}} A^i$, i.e., any face $(x_{i_0}, \ldots, x_{i_k})$ is covered by the union of the set $A_{i_0}, \ldots, A_{i_k}$. Then, $\cap_{i\in I} A^i \neq \emptyset$.*

proof of KKM theorem using Sperner's lemma. Now we proof the KKM theorem using Sperner's lemma, the proof is from [Ale98, P162].

We first state an intuitive lemma which can help our proof.

Lemma 3.2.4 (Lebegue's lemma). *Let $C = \{A_1, \dots, A_n\}$ be a set of closed subsets of a compact set S which covers S, then there is a positive number δ such that: if $M \subseteq S$ has diameter $\leq \delta$ and $\{A_{i_j} | 1 \leq j \leq k\} \subseteq C$ are such that $M \cap A_{i_j} \neq \emptyset$ for all $1 \leq j \leq k$, then $\cap_{j=1}^{k} A_{i_j} \neq \emptyset$.*

The proof of the lemma can be found in [Ale98, P35].

Using the above Lebegue's lemma, the KKM theorem follows from the following theorem:

Theorem 3.2.4. *Let T be any triangulation of the n simplex S whose diameter is arbitrary small, then there is at least one simplex of T which intersects all of the sets $\{A^i\}_{i \in I}$.*

To prove the above theorem, we label each vertex v of T by a vertex $\lambda(v) = x_i$ of the support $supp(v)$ of $v \in S$ such that $v \in A^i$. We can do this because we require the support of v is covered by the union of A^i where x_i is a vertex of $supp(v)$. And this labeling is obviously a Sperner labeling. Now apply Sperner's lemma we can find a completely labeled simplex σ of T. σ intersects all of the sets $\{A^i\}_{i \in I}$ since its vertices which are labeled by distinct $x_i, i \in I$ lies in distinct $A^i, i \in I$.

□

We can also deduce Brouwer's fixed point theorem from the above KKM theorem instead of from the Borsuk-Ulam theorem. Interested reader can consult [GDGJFJ10, Thm 2.13.3].

Now we have shown the second row of figure 2, and have also shown that the first row implies the second row by showing the Borsuk-Ulam theorem implies the Brouwer's fixed point theorem.

For more detailed history of the Brouwer's fixed point theorem, Sperner's lemma, KKM theorem and their applications in mathematical programming problems, in economic equilibrium theory, and in game theoretic problems, one can consult the survey [Par99].

Kakutani fixed point theorem and Nash equilibrium Now we give the following Kakutani fixed point theorem which is a corollary of Brouwer's fixed point theorem and it can be used to show that each strategic game has at least one Nash equilibrium in a quite direct way. Interested reader can consult [GDGJFJ10, Thm 2.2.3].

Theorem 3.2.5 (Kakutani fixed point theorem)**.** *Let $A \subset \mathbb{R}^n$ be a non empty, compact, convex set. Let $F : A \to A$ be an upper hemicontinuous, nonempty-valued, closed-valued, and convex-valued correspondence. Then there is $\tilde{x} \in A$ such that $\tilde{x} \in F(\tilde{x})$, i.e., F has a fixed point.*

3.3 Two generalizations of the Borsuk-Ulam theorem

We state two generalizations of the Borsuk-Ulam theorem which would be useful in later text(in the proof of topological Tverberg theorem for prime power case in section 4.2.) Before that, we first state $(BU3)$ in theorem 3.1.1 using the language of equivariant topology:

Let $\mathbb{Z}_2$ be a cyclic group of two elements, as in example 2.3.3, we can view S^n as a free(fixed point free) $\mathbb{Z}_2$-space with antipodal action; as in example 2.3.5, an antipodal map $f : S^n \to S^m, f(-x) = -f(x)$ is a $\mathbb{Z}_2$-equivariant map, then we can rewrite $(BU3)$ as:

Theorem 3.3.1 (Borsuk-Ulam theorem, equivariant statement)**.** *There is no $\mathbb{Z}_2$-equivariant map from $\mathbb{Z}_2$-space $(S^n, \mathbb{Z}_2)$ to the free(fixed point free) $\mathbb{Z}_2$-space $(S^m, \mathbb{Z}_2)$ when $m < n$.*

We can see from above that the Boruk-Ulam theorem under the language of equivariant topology is about the non-existence of $\mathbb{Z}_2$ map. We will generalize the theorem in the following two directions:

1. generalize the group $\mathbb{Z}_2$ to a general finite group G with a stronger condition on the codomain: the group G acts on the codomain freely.

2. generalize the group $\mathbb{Z}_2$ to a more restricted group $(\mathbb{Z}_p)^n$ with a looser condition on the codomain: $(\mathbb{Z}_p)^n$ only need to acts on the codomain fixed point freely.

The first generalization is generalizing $\mathbb{Z}_2$ to any finite group G while keep requiring the group G acts on codomain freely:

Theorem 3.3.2 (The first generalization of Borsuk-Ulam theorem(Dold's theorem))**.** *Let G be a finite group, there is no G-equivariant map from a n-connected G-space X to an at most n-dimensional free G-complex Y.*

Proof. The proof can be found in Dold's paper [Dol83]. □

Remark 3.3.1. The same as Borsuk-Ulam theorem has a combinatorial version–the Tucker's lemma, the first generalization of Borsuk-Ulam theorem(Dold's theorem) also has its combinatorial version–the $\mathbb{Z}_q$-Fan's lemma. Interested reader can consult [DLGMM19, prop 2.5].

The second generalization generalizes $\mathbb{Z}_2$ to a more restricted group $(\mathbb{Z}_p)^\alpha$ than the above G but only requires $(\mathbb{Z}_p)^\alpha$ acts on the codomain fixed point freely:(which is theorem 3.3.3)

Theorem 3.3.3 (The second generalization of Borsuk-Ulam theorem)**.** *([Oza87, lemma 3.3] or [Vol96a, lemma] or [Sko18, theorem 2.6].) Let p be a prime, α be a positive integer, $(\mathbb{Z}_p)^\alpha$ be a direct product of α copies of the group $\mathbb{Z}_p$(an abelian p-group of rank α), then there is no $(\mathbb{Z}_p)^\alpha$-equivariant map from an n-connected $(n+1)$-dimensional $(\mathbb{Z}_p)^\alpha$-complex to an at most n-dimensional $(\mathbb{Z}_p)^\alpha$-complex without any $(\mathbb{Z}_p)^\alpha$-fixed points.*

Proof. We omit the proof of the theorem which can be found in [Oza87, lemma 3.3] or [Vol96a, lemma] or [Sko18, theorem 2.6].

□

4

Topological Tverberg Conjecture

In this section, for the convenience of notation, we use Δ_n to denote the stardard n-simplex instead of the stardard notation Δ^n.

4.1 Introduction

The goal of this section is to prove that the topological Tverberg conjecture is true for prime power case, i.e., theorem 4.1.4, using the generalised Borsuk-Ulam theorem we have seen before in section 3.3. And then we will show that the topological Tverberg conjecture is false for non prime power case, i.e., prove the theorem 4.1.5.

Before that, we first give a brief history of the topological Tverberg conjecture:

The topological Tverberg conjecture comes from the Tverberg theorem which was initially a conjecture by B. Birch [Bir59] in 1959:

Conjecture(Theorem) 4.1.1 (Tverberg's theorem)**.** For $d \geq 1, r \geq 2$, any $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ admits a partition into r disjoint subsets $A_1, \dots, A_r$ such that the intersection of all the convex hulls of each subsets is non empty, i.e., $conv(A_1) \cap conv(A_2) \dots conv(A_r) \neq \emptyset$.

Birch conjectured the theorem and proved it for the case $d = 2$. For the case $r = 2$, it was a lemma (the following Radon's theorem) proved by J. Radon in 1922 [Rad21], which is used by Radon to prove the Helly's theorem:

Theorem 4.1.1 (Radon's theorem)**.** *For $d \geq 1$, any $d+2$ points in $\mathbb{R}^d$ admits a partition into two disjoint subset A_1, A_2 such that the intersection of their convex hulls is non empty, i.e., $conv(A_1) \cap conv(A_2) \neq \emptyset$.*

Tverberg first prove the Tverberg's theorem 4.1.1 for $d = 3$ in 1963 and solve it completely in 1964 [Tve66]. The story about how he found the proof is both sad and encouraging — 'I recall that the weather was bitterly cold in Manchester. I awoke very early one morning shivering, as the electric heater in the hotel room had gone off, and I did not have an extra shilling to feed the meter. So, instead of falling back to sleep, I reviewed the problem once more, and then the solution dawned on me!'

In 1979, Bajmóczy and Bárány reformulated Radon's theorem and Tverberg's theorem using affine map, which are:

Theorem 4.1.2 (Radon's theorem, affine map version)**.** *For any affine map $f : \Delta_{d+1} \to \mathbb{R}^d$ there exists two disjoint faces F_1, F_2 of the $d+1$ simplex Δ_{d+1} such that the intersection of their images is non empty, i.e., $f(F_1) \cap f(F_2) \neq \emptyset$.*

Theorem 4.1.3 (Tverberg's theorem, affine map version)**.** *For $d \geq 1, r \geq 2$, for any affine map $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ there exists r pairwise disjoint faces $F_1, \dots, F_r$ ($F_i \cap F_j = \emptyset$, for all $i \neq j$) such that the intersection of their images is non empty, i.e., $f(F_1) \cap f(F_2) \cdots \cap f(F_r) \neq \emptyset$.*

The affine map version and the original version can easily be show to be equivalent:

Claim 4.1.1. *The affine map reformulation of the Tverberg theorem is equivalent to the original statement.*

Proof. Assuming the affine map version, for any $(d+1)(r-1)+1$ points in $\mathbb{R}^d$, there exists an affine map $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ such that the images of the vertices of $\Delta_{(d+1)(r-1)}$ corresponds to the $(d+1)(r-1)+1$ points. The r pairwise disjoint faces $F_1, \dots, F_r \in \Delta_{(d+1)(r-1)}$ gives a partition of the $(d+1)(r-1)+1$ points into r parts. For the other direction, assuming theorem 4.1.1, for any affine map $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$, the images of its vertices are $(d+1)(r-1)+1$ points in $\mathbb{R}^d$. And the partition of the images into r disjoint subsets gives r disjoint faces of $\Delta_{(d+1)(r-1)}$ whose images under f has non empty intersection. $\square$

Bajmóczy and Bárány also conjectured that the theorems still holds if we generalise the affine map in the theorems to continuous map [BB79], which are the following Topological Radon conjecture and Topological Tverberg conjecture:

Conjecture(Theorem) 4.1.2 (Topological Radon conjecture(theorem))**.** For any continuous map $f : \Delta_{d+1} \to \mathbb{R}^d$ there exists two disjoint faces F_1, F_2 of the $d+1$ simplex Δ_{d+1} such that the intersection of their images is non empty, i.e., $f(F_1) \cap f(F_2) \neq \emptyset$.

Conjecture* (Topological Tverberg conjecture)**.** For $d \geq 1, r \geq 2$, for any continuous map $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ there exists r pairwise disjoint faces $F_1, \dots, F_r$ ($F_i \cap F_j = \emptyset$, for all $i \neq j$) such that the intersection of their images is non empty, i.e., $f(F_1) \cap f(F_2) \cdots \cap F(F_r) \neq \emptyset$.

Obviously, the topological Radon conjecture is a special case of the topological Tverberg conjecture, i.e. it is the topological Tverberg conjecture for $r = 2$. In this section, we will see that the topological Radon conjecture (the topological Tverberg conjecture when $r = 2$) is true. And this can be generalized to the case when r is a prime number and more generally, when r is a power of a prime number. However, when r is not prime power, the topological Tverberg conjecture is no longer true.

We give some equivalent statements of the topological Tverberg conjecture which would be helpful in later text:

Conjecture* (Topological Tverberg conjecture, r-Tverberg point version)**.** For $d \geq 1, r \geq 2$, any continuous map $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ has a r-Tverberg point, which is there exists r points $x_1, \dots, x_r \in \Delta_{(d+1)(r-1)}$ with pairwise disjoint supports, i.e., $supp(x_i) \cap supp(x_j) = \emptyset$ for all $i \neq j$, with the same images, i.e., $f(x_1) = \cdots = f(x_r)$. The image of these r points is called a r-Tverbeg point, which is a point of $\mathbb{R}^d$ with r preimages of disjoint supports.

We can call a map $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ with no r-Tverberg point to be an almost r embedding, and restate the topological Tverberg conjecture simply as:

Conjecture 4.1.1 (Topological Tverberg conjecture, almost r-embedding version)**.** For $d \geq 1, r \geq 2$, there is no almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$.

The topological Tverberg conjecture when $r = 2$(the topological Radon conjecture) is proved by Bajmóczy and Bárány using the Borsuk-Ulam theorem so that it became a theorem. The idea of their proof is firstly to show that there exists a continuous map $g : S^d \to \Delta_{d+1}$ such that for any $x \in S^n$, $g(x), g(-x) \in \Delta_{d+1}$ have disjoint supports. And then for any continuous map $f : \Delta_{d+1} \to \mathbb{R}^d$, composite it with g and get a continuous map $f \circ g : S^d \to \mathbb{R}^d$. Finally, apply Borsuk-Ulam theorem to $f \circ g$ and get $x_0 \in S^n$ such that $f(g(x_0)) = f(g(-x_0))$. Thus $g(x_0), g(-x_0) \in \Delta_{d+1}$ are two points with disjoint supports and have the same image under f.

For r is a prime number, they later(1981) mimicked the above idea and also prove it successful [BSS81]. Their original proof used deleted product and later other proofs using deleted join were also invented by Sarkaria ([Sar90, Sar91a]). We will show the proof in section 4.2.1.

For the case r is a power of a prime number, the topological Tverberg conjecture was first proved (1987) by Özaydin in an unpublished manuscript [Oza87] and much later(1996) by Volovikov [Vol96a] and by Sarkaria(2000) [Sar00]. We will see that proof in section 4.2.2.

Thus for the case r is a prime power, we can now list the topological Tverberg conjecture 4.1.1 as a theorem:

Theorem 4.1.4 (Topological Tverberg theorem)**.** *For $d \geq 1, r \geq 2$, r is a power of a prime number, there is no almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$.*

This is the first theorem we aim to show in this section. We will prove this theorem in section 4.2.

In fact, Özaydin not only has proved the prime power case, but also has shown that for the case when r is not a prime power, there actually exists a Σ_r-equivariant map which makes the configuration space–test map scheme method fail for this situation.

For r a non prime power case, the topological Tverberg conjecture remained open until 2015, and was considered to be one of the most important open problems in topological combinatorics. In 2015, a counterexample was constructed by Florian Frick [Fri15] who combined the work of Mabillard and Wagner [MW14] and the constraint method of Blagojević, Ziegler and himself [BFZ14]. Hence the topological Tverberg conjecture is shown to be false when r is not a power of a prime number. We state this as follows:

Theorem 4.1.5 (Topological Tverberg conjecture failed for non prime power case)**.** *For $d = 3r + 1$, $r \geq 6$ be an integer that is not a prime power, there exists an almost r-embedding $\Delta_{(3r+2)(r-1)} \to \mathbb{R}^{3r+1}$.*

This is the second theorem we aims to show in this section. We will prove this theorem in section 4.3.

4.2 Proof of Theorem 4.1.4

4.2.1 proof of theorem 4.1.4 when r is prime

We will give two proofs of the theorem with the same idea but using different constructions: one with deleted product and the other with deleted join. Each of them has its own advantages and disadvantages over the other.

The proofs use a typical method in topology: the configuration space-test map scheme. The idea is to derive the result from the non existence of some equivariant map (test map) between two topological spaces (configuration space and target space). In our proof we will show the non existence of some Σ_r-equivariant map between some configuration space and target space, from which we can deduce the non existence of any almost r-embedding of $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ when r is prime.

We first recall the theorem 4.1.4 and then we will give two proofs of the theorem for the case when r is a prime number:

Theorem (Topological Tverberg theorem)**.** *For $d \geq 1, r \geq 2$, r is a power of a prime number, there is no almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$.*

proof 1 *using deleted product.* The first proof using deleted product is from [BSS81] and and has the following two steps:

1. If there is an almost r-embedding $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$, then there is a Σ_r equivariant map from $(\Delta_{(d+1)(r-1)})^{\times r}_{\Delta(2)}$ to $S^{d(r-1)-1}$.
2. The first generalization of Borsuk-Ulam theorem 3.3.2 shows there is no such Σ_r-equivariant map. Thus there is no almost r-embedding.

Now we give the details of the proof. For simplicity, we write $N = (d+1)(r-1)$. By computation, we have $N - r = d(r-1) - 1$.

Step 1: If there is an almost r-embedding $f : \Delta_N \to \mathbb{R}^d$, we can define the r-fold product of f:

$$f^{\times r} : (\Delta_N)^{\times r}_{\Delta(2)} \to (\mathbb{R}^d)^{\times r} - \mathfrak{diag}$$

given by

$$(x_1, \dots, x_r) \mapsto (f(x_1), \dots, f(x_r))$$

where $(\Delta_N)^{\times r}_{\Delta(2)}$ is the r-fold 2-wise deleted product as in example 2.4.3, $(\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ is the r-fold r-wise deleted product of $\mathbb{R}^d$ as in example 2.4.1.

$f^{\times r}$ is well defined: since f has no r-Tverberg point, we have for any $x_1, \dots, x_r$ of Δ_N with disjoint supports, i.e., for $(x_1, \dots, x_r) \in (\Delta_N)^{\times r}_{\Delta(2)}$, their images $f(x_1), \dots, f(x_r)$ are not all the same, i.e., $(f(x_1), \dots, f(x_r)) \in (\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ avoids the diagonal $\mathfrak{diag}$ of $(\mathbb{R}^d)^{\times r}$. It is obviously a Σ_r-equivariant map since it commutes with the Σ_r-actions on $(\Delta_N)^{\times r}_{\Delta(2)}$ and $(\mathbb{R}^d)^{\times r} - \mathfrak{diag}$ which acts on the r components of the spaces componentwisely.

Now, compose $f^{\times r}$ and the Σ_r-equivariant map $\mu \circ \rho : (\mathbb{R}^d)^{\times r} - diag \to S^{d(r-1)-1}$ as in example 2.4.1 we get a Σ_r-equivariant map:

$$\mu \circ \rho \circ f^{\times r} : (\Delta_N)^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$$

Step 2: we use the first generalization of Borsuk-Ulam theorem(theorem 3.3.2) to show that there is no Σ_r-equivariant map from $(\Delta_N)^{\times r}_{\Delta(2)}$ to $S^{d(r-1)-1}$.

We recall the theorem:

Theorem (The first generalization of Borsuk-Ulam theorem(Dold's theorem))**.** *Let G be a finite group, there is no G-equivariant map from a n-connected G-space X to an at most n-dimensional free G-complex Y.*

Let $G = \mathbb{Z}_r$ where r is a prime number, from example 2.4.3 we know that $(\Delta_N)^{\times r}_{\Delta(2)}$ is a $(N-r+1)$-dimensional $(N-r)$-connected Σ_r-complex, thus it is a $(N-r+1)$-dimensional $(N-r)$-connected $\mathbb{Z}_r$-space with $\mathbb{Z}_r$ action being the restriction of the Σ_r action. In addition, from example 2.4.2 and theorem 2.4.1 we know that $S^{d(r-1)-1}$ is a $(N-r) = d(r-1)-1$ dimensional free $\mathbb{Z}_r$-space with $\mathbb{Z}_r$ action being the restriction of the Σ_r action. Now apply the above theorem we have shown that there is no $\mathbb{Z}_r$-map from the $\mathbb{Z}_r$-space $(\Delta_N)^{\times r}_{\Delta(2)}$ to the free $\mathbb{Z}_r$-space $S^{d(r-1)-1}$. Thus, there is no Σ_r from the Σ_r-space $(\Delta_N)^{\times r}_{\Delta(2)}$ to the Σ_r-space-$S^{d(r-1)-1}$.

□

proof 2 *using deleted join.* This proof is from [MBZ03, section 6.4] and used the same idea as the first proof. The proof consists of two steps:

1. If there is an almost r-embedding $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$, then there is a Σ_r equivariant map $\mu \circ \rho \circ \psi \circ f^{*r} : (\Delta_{(d+1)(r-1)})^{*r}_{\Delta(2)} \to S^{(d+1)(r-1)-1}$
2. There is no Σ_r-equivariant map by the first generalization of Borsuk-Ulam theorem 3.3.2 and thus no almost r-embedding.

Again denote $N = (d+1)(r-1)$.

step 1: If $f : \Delta_N \to \mathbb{R}^d$ has no r-Tverberg point (is an almost r-embedding), we can define the r-fold join f^{*r} of f:

$$f^{*r} : (\Delta_N)^{*r}_{\Delta(2)} \to (\mathbb{R}^d)^{*r} - \mathfrak{diag}$$

given by

$$t_1 x_1 \oplus \cdots \oplus t_r x_r \mapsto t_1 f(x_1) \oplus \cdots \oplus t_r f(x_r)$$

which is well-defined since: if f has no r-Tverberg point, $f(x_1), \dots, f(x_r)$ are not all equal. Thus the images $\{t_1 f(x_1) \oplus \cdots \oplus t_r f(x_r) | f(x_i) \in \mathbb{R}^d, i = 1, \dots, r, \sum_{i=1}^r t_i = 1\}$ avoids the diagonal $\mathfrak{diag} = \{\frac{1}{r}x \oplus \cdots \oplus \frac{1}{r}x | x \in \mathbb{R}^d\}$ of $(\mathbb{R}^d)^{*r}$. f^{*r} is also Σ_r-equivariant since it commute with the Σ_r-actions on the domain and codomain.

Now define a Σ_r-equivariant map $\mu \circ \rho \circ \psi : (\mathbb{R}^d)^{*r} - \mathfrak{diag} \to S^{N-1}$ as in example 2.4.7 and compose it with $f^{*r} : (\Delta_N)^{*r}_{\Delta(2)} \to (\mathbb{R}^d)^{*r} - \mathfrak{diag}$ above, we can get a Σ_r-equivariant map:

$$\mu \circ \rho \circ \psi \circ f^{*r} : (\Delta_N)^{*r}_{\Delta(2)} \to S^{N-1}$$

Step 2: we now use the first generalized Borsuk-Ulam theorem 3.3.2 again to show no such Σ_r-equivariant map can exist: from example 2.4.8, we know that $(\Delta_N)^{*r}_{\Delta(2)}$ is $(N-1)$-connected $\mathbb{Z}_r$-space. S^{N-1} is a $(N-1)$-dimensional sphere and by theorem 2.4.1 we know that $\mathbb{Z}_r$ acts on S^{N-1} freely when r is prime. Apply the generalized Borsuk-Ulam theorem 3.3.2, there is no $\mathbb{Z}_r$-equivariant map from a $(N-1)$-connected $\mathbb{Z}_r$-space $(\Delta_N)^{*r}_{\Delta(2)}$ to a $(N-1)$-dimensional free $\mathbb{Z}_r$-space $S^{(d+1)(r-1)-1}$. Since $\mathbb{Z}_r$ is a subgroup of Σ_r, no $\mathbb{Z}_r$-map implies no Σ_r-map. $\square$

Remark 4.2.1. We need to be careful that the above two proofs requires r to be a prime number. Since in order to apply Dold's theorem(the first generalized Borsuk-Ulam theorem), $\mathbb{Z}_r$ needs to acts on the target space $S^{d(r-1)-1}$ and $S^{(d+1)(r-1)-1}$ freely. By theorem 2.4.1 we see that this happens if and only if r is a prime number.

4.2.2 PROOF OF THEOREM 4.1.4 WHEN r IS PRIME POWER

proof of topological Tverberg theorem when r is a prime power: The idea of the proof is the same with the case when r is prime: first show that having an Σ_r-equivariant map is a necessary condition for having an almost r-embedding, and then show no such Σ_r-equivariant map can exists using the second generalization of the Borsuk-Ulam theorem. Hence the non-existence of almost r-embedding is proved.

We recall the two steps:

1. If there is an almost r-embedding $f : \Delta_{(d+1)(r-1)} \to \mathbb{R}^d$, then there is a Σ_r equivariant map from $(\Delta_{(d+1)(r-1)})^{\times r}_{\Delta(2)}$ to $S^{d(r-1)-1}$.

2. The second generalization of Borsuk-Ulam theorem 3.3.3 shows there is no such Σ_r-equivariant map.

Denote $N = (d+1)(r-1)$. By computation we have $N - r = d(r-1) - 1$.

step 1: let p be a prime number, α be a positive integer, $r = p^\alpha$ be a power of prime p, if there exists an almost r-embedding $f : \Delta_N \to \mathbb{R}^d$ we can use the same method as the first proof(using deleted product) of theorem 4.1.4 when r is prime in section 4.2.1 to construct a Σ_r-equivariant map from $(\Delta_N)^{\times r}_{\Delta(2)}$ to $S^{d(r-1)-1}$, which is

$$(\Delta_N)^{\times r}_{\Delta(2)} \xrightarrow{f^{\times r}} (\mathbb{R}^d)^{\times r} - diag \xrightarrow{\mu\circ\rho} S^{d(r-1)-1}.$$

step 2: We now show no such equivariant map can exists using the second generalized version of Borsuk-Ulam theorem 3.3.3.

We recall the theorem:

Theorem (The second generalization of Borsuk-Ulam theorem). *([Oza87, lemma 3.3] or [Vol96a, lemma] or [Sko18, theorem* 2.6*]) Let* p *be a prime,* α *be a positive integer,* $(\mathbb{Z}_p)^\alpha$ *be a direct product of* α *copies of the group* $\mathbb{Z}_p$ *(an abelian* p*-group of rank* α*), then there is no* $(\mathbb{Z}_p)^\alpha$*-equivariant map from an* n*-connected* $(n+1)$*-dimensional* $(\mathbb{Z}_p)^\alpha$*-complex to an at most* n*-dimensional* $(\mathbb{Z}_p)^\alpha$*-complex without any* $(\mathbb{Z}_p)^\alpha$*-fixed points.*

We see from example 2.3.2 that $(\mathbb{Z}_p)^\alpha$ can be embedded into $S_r = S_{p^\alpha}$ in a particular way such that $(\mathbb{Z}_p)^\alpha$ acts on the set $\{1, \dots, p^\alpha\}$ transitively.

From example 2.4.3 we know $(\Delta_N)^{\times r}_{\Delta(2)}$ is a $(N - r + 1)$-dimensional, $(N - r)$-connected free Σ_r(thus $(\mathbb{Z}_p)^\alpha$)-complex. In addition, $S^{d(r-1)-1}$ is a $(N - r)$-dimensional Σ_r(thus $(\mathbb{Z}_p)^\alpha$)-complex without $(\mathbb{Z}_p)^\alpha$-fixed point since: $(\mathbb{Z}_p)^\alpha$ acts on $S^{d(r-1)-1}$ by components transitively. By lemma 2.4.2, it has no $(\mathbb{Z}_p)^\alpha$-fixed point in $S^{d(r-1)-1}$.

Now let $n = N - r$ and apply theorem 3.3.3, we get that there is no $(\mathbb{Z}_p)^\alpha$-equivariant map from $(\Delta_{(d+1)(r-1)})^{\times r}_{\Delta(2)}$ to $S^{d(r-1)-1}$ and since $(\mathbb{Z}_p)^\alpha$ is a subgroup of Σ_r, there is no Σ_r-equivariant map from $(\Delta_{(d+1)(r-1)})^{\times r}_{\Delta(2)}$ to $S^{d(r-1)-1}$. $\square$

4.3 Proof of Theorem 4.1.5

4.3.1 Introduction and sketch of proof

Naturally, we want to apply the same method (the configuration space–test map scheme) to try to prove the case when r is not a power of a prime number, which is, we want to show the nonexistence of almost r-embeddings $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ by showing the the nonexistence of Σ_r-equivariant map $(\Delta_{(d+1)(r-1)})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$.

However, this wish failed after Özaydin proved that when r is not a prime power, there actually exists Σ_r-equivariant map from $(\Delta_{(d+1)(r-1)})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$, which is the theorem 4.3.2.

Surprisingly, Özaydin's theorem 4.3.2 combined with two other important results can actually disprove the topological Tverberg conjecture 4.1.1 when r is not a prime power:

The question about whether Topological Tverberg conjecture is true or not when r is not prime power remained unknown for about 50 years. Until 2015, Florian Frick [BFZ19a] rediscovered that the Topological Tverberg conjecture implies the generalized Van Kampen conjecture by constrain method(This has in fact previously been proved by Gromov earlier in 2010 but wasn't recognized.). And he observed that if we combined this result, and Özaydin's theorem 4.3.2, with another important result from Mabillard and Wagner(theorem 4.3.3) we can construct a counterexample for the Topological Tverberg conjecture when r is not prime power, i.e., we can prove theorem 4.1.5.

We first recall theorem 4.1.5:

Theorem (Topological Tverberg conjecture failed for non prime power case)**.** *For $d = 3r + 1$, $r \geq 6$ be an integer that is not a prime power, there exists an almost r-embedding $\Delta_{(3r+2)(r-1)} \to \mathbb{R}^{3r+1}$. Equivalently, there exists a map $f : \Delta_{(3r+2)(r-1)} \to \mathbb{R}^{3r+1}$ without r-Tverberg points.*

We will prove the theorem by constructing counterexample. The construction consists of three important ingredients. We first sketch the main idea of the construction as follows:

step 1. firstly show Topological Tverberg conjecture 4.1.1 implies the generalized Van Kampen-Flores conjecture using constraint methods. This would be shown in section 4.3.2.

step 2. secondly construct counterexample for the generalized Van Kampen-Flores conjecture by two steps: let $r \geq 2, d = 3r, k = \frac{d(r-1)}{r} = 3(r-1)$, $\mathcal{K} = skel_k\Delta_{(d+2)(r-1)}$, we will construct an almost r-embedding $g : \mathcal{K} \to \mathbb{R}^d$ as follows:

step 2.1. first use a result by Özaydin, showing there exists a Σ_r-equivariant map $F : (\mathcal{K})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$ when r is not prime power;

step 2.2. then use a result by Mabillard and Wagner, showing that the existence of the Σ_r-equivariant map $F : (\mathcal{K})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$ implies the existence of an almost r-embedding $g : \mathcal{K} \to \mathbb{R}^d$.

Combine 2.1 and 2.2 we have: there exists a map $f : \mathcal{K} \to \mathbb{R}^d$ without r-tverberg points (there exists an almost r embedding) when r is not prime power. This means that the generalized Van Kampen-Flores conjecture failes for r non prime power. We show this in section 4.3.3.

4.3.2 TOPOLOGICAL TVERBERG IMPLIES GENERALIZED VAN KAMPEN-FLORES

We now give the details of **step** 1 and show that the Topological Tverberg conjecture 4.1.1 implies the generalized Van Kampen-Flores conjecture by constrain method.

We first state the generalized Van Kampen-Flores conjecture:

Conjecture 4.3.1 (The generalized Van Kampen-Flores conjecture [BFZ19b])**.** Let $r \geq 2$, $d \geq 1$ be integers, $k \geq \lceil \frac{r-1}{r} d \rceil$, for any continuous map $f : \Delta_{(r-1)(d+2)} \to \mathbb{R}^d$, there exists r points $x_1, \dots, x_r \in \mathcal{K} := skel_k \Delta_{(r-1)(d+2)}$ with pairwise disjoint supports such that $f(x_1) = f(x_2) = \cdots = f(x_r)$, where $\mathcal{K} := skel_k \Delta_{(r-1)(d+2)}$ is the k-th skeleton of $\Delta_{(r-1)(d+2)}$. It is equivalent to the following two formulations:

1. Any continuous map $f : \mathcal{K} \to \mathbb{R}^d$ has a r-Tverberg point.

2. There is no almost r-embedding $f : \mathcal{K} \to \mathbb{R}^d$.

Now we show that the topological Tverberg conjecture implies the generalized Van Kampen-Flores conjecture.

Claim. *The topological Tverberg conjecture implies the generalized Van Kampen-Flores conjecture, i.e., if for any map $f : \Delta_{(d+2)(r-1)} \to \mathbb{R}^{d+1}$, there exists r points $x_1, \dots, x_r \in \Delta_{(d+2)(r-1)}$ with pairwise disjoint support, such that $f(x_1) = \cdots = f(x_r)$, then for any map $g : skel_k \Delta_{(d+2)(r-1)} \to \mathbb{R}^d$, $k \geq \lceil \frac{r-1}{r} d \rceil$, there exists r points $x_1, \dots, x_r \in skel_k \Delta_{(d+2)(r-1)}$ with pairwise disjoint support, such that $g(x_1) = g(x_2) = \cdots = g(x_r)$.*

proof of claim. Assuming the topological Tverberg conjecture is true, we want to show that for any map $g : skel_k \Delta_{(d+2)(r-1)} \to \mathbb{R}^d$, there exists r points $x_1, \dots, x_r \in skel_k \Delta_{(d+2)(r-1)}$ with pairwise disjoint support, such that $g(x_1) = g(x_2) = \cdots = g(x_r)$. We first extend $g : skel_k \Delta_{(d+2)(r-1)} \to \mathbb{R}^d$ continuously to a map $\tilde{g}$:

$\Delta_{(d+2)(r-1)} \to \mathbb{R}^d$ in some way, which is possible by the Tietze extension theorem. Then we define a new map $f : \Delta_{(r-1)(d+2)} \to \mathbb{R}^{d+1}$ using $\tilde{g}$, which is given by $f(x) = (\tilde{g}(x), dist(x, skel_k\Delta_{(r-1)(d+2)}))$, where the first d components are the same with $\tilde{g}$ and the last component is the Euclidean distance between x and the k-th skeleton $skel_k\Delta_{(r-1)(d+2)}$ of $\Delta_{(r-1)(d+2)}$. Since $\tilde{g}$ and $dist$ are both continuous, f is also continuous. By the topological Tverberg conjecture, f has a r-Tverberg point. This means there exists $x_1, \ldots, x_r \in \Delta_{(r-1)(d+2)}$ with pairwise disjoint support, such that $f(x_1) = \cdots = f(x_r)$. If there exists a point x_i lies in $skel_k\Delta_{(r-1)(d+2)}$, then we have $dist(x_1, skel_k\Delta_{(r-1)(d+2)}) = \cdots = dist(x_r, skel_k\Delta_{(r-1)(d+2)}) = 0$. This means $g : skel_k\Delta_{(d+2)(r-1)} \to \mathbb{R}^d$ has a r-Tverberg point, and we are done. Otherwise, assume all $x_1, \ldots, x_r$ are not in $skel_k\Delta_{(r-1)(d+2)}$. This means the dimensions of all supports of $x_1, \ldots, x_r$ are greater than or equal to $k+1$, i.e., $dim(supp(x_i)) \geq k+1$, for all $1 \leq i \leq r$. Since $x_1, \ldots, x_r$ are pairwise disjoint, the sum of the vertices of the supports of the points $x_1, \ldots, x_r$ is no more than the sum of the vertices of $\Delta_{(d+2)(r-1)}$. Thus we have the inequality $(d+2)(r-1)+1 \geq r(k+2) \geq (r-1)d+2r = (d+2)(r-1)+2$, which is a contradiction. □

Remark 4.3.1. The above proof shows how to construct counterexample for topological Tverberg conjecture for r non prime power: since topological Tverberg conjecture implies the generalized van Kampen-Flores conjecture, it suffices to construct counterexample for the generalized van Kampen-Flores conjecture for r non prime power, i.e., construct $g : skel_k\Delta_{(d+2)(r-1)} \to \mathbb{R}^d$ such that for any r pairwise disjoint faces $F_1, \ldots, F_r$ of $skel_k\Delta_{(d+2)(r-1)}$, we have $g(F_1) \cap \cdots \cap g(F_r) = \emptyset$. And then we extend g continuously arbitrarily to the whole $\Delta_{(d+2)(r-1)}$ and use it to construct $f : \Delta_{(d+2)(r-1)} \to \mathbb{R}^{d+1}$ such that $f(x) = (g(x), dist(x, skel_k\Delta_{(r-1)(d+2)}))$ similarly as above. Now f is a counterexample for the topological Tverberg conjecture since if f has a r-Tverberg point, similar as above this would imply g has a r-Tverberg point, contradicting our definition of g.

Therefore, in order to construct counterexample for topological Tverberg conjecture we only need to prove the following:

Theorem 4.3.1 (The generalized van Kampen-Flores conjecture fails for r non prime power). *Let $r \geq 2$, $d \geq 1$ be integers, $k \geq \lceil \frac{r-1}{r} d \rceil$, there exists a continuous map $f : \Delta_{(r-1)(d+2)} \to \mathbb{R}^d$, such that for any r points $x_1, \ldots, x_r \in \mathcal{K} := skel_k\Delta_{(r-1)(d+2)}$ with pairwise disjoint supports, there images $f(x_1), \ldots, f(x_r)$ are not all equal.*

Remark 4.3.2. We have already shown that the topological Tverberg conjecture is true when r is a power of prime number in section 4.2. This implies the generalized Van Kampen-Flores conjecture is also true for r prime power. Historically, the generalized Van Kampen-Flores conjecture was proved by Sarkaria [Sar91b] for primes and by Volovikov [Vol96b] for prime powers.

However, when r is not a prime power, we will see in section 4.3.2 that the generalized Van Kampen-Flores conjecture is no longer true, which implies the failure of Topological Tverberg conjecture for r non prime power.

4.3.3 COUNTEREXAMPLE OF GENERALIZED VAN KAMPEN-FLORES CONJECTURE

In this section we show **step** 2, i.e., we construct a counterexample for the generalized Van Kampen-Flores conjecture for r non prime power:

let $r \geq 2, d = 3r, k = \frac{d(r-1)}{r} = 3(r-1)$, $\mathcal{K} = skel_k\Delta_{(d+2)(r-1)}$, we will construct an almost r-embedding $g : \mathcal{K} \to \mathbb{R}^{3r}$, i.e., g has no r-Tverberg point. This will disprove the generalized Van Kampen-Flores conjecture for r non prime power and by above discussion in section 4.3.3 thus disprove the topological Tverberg conjecture for non prime power case.

Remark 4.3.3. The above condition $d = 3r, k = 3(r-1)$ is specified in order to satisfy the codimension condition $d-k \geq 3$ so that we can apply the r-fold Whitney trick. We will see it in later text.

We first give details of step 2.1, which is to show the existence of Σ_r-equivariant map $F : (\mathcal{K})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$ when r is not a prime power.

Step 2.1: the construction of Σ_r-equivariant map $F : (\mathcal{K})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$

Let $\mathcal{K}^{\times r}_{\Delta(2)} = \{\sigma_1 \times \sigma_2 \times \cdots \times \sigma_r | \sigma_i \in \mathcal{K}, \sigma_i \cap \sigma_j = \emptyset, \text{ for every } i \neq j\}$ be the r-fold 2-wise deleted product of $\mathcal{K}$, define a Σ_r-action on it by permuting its r components. This action is obvious free and cellular. We now show that when r is not prime power, there is a Σ_r-equivariant map $F : (\mathcal{K})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$ using an important result of Özaydin:

Theorem 4.3.2 (when r is non prime power, exists Σ_r-equivariant map)**.** *([Oza87, theorem 4.2]) Let $d \geq 1$ and $r \geq 2$, if r is not a prime power, let X be an at most $d(r-1)$-dimensional free Σ_r-complex(a cell complex with a free cellular Σ_r-action), then there exists a Σ_r-equivariant map from X to $S^{d(r-1)-1}$.*

Remark 4.3.4.

- For this theorem it is important that Σ_r acts on X freely, cellularly, and the dimension of X is at most two more than the connectivity of $S^{d(r-1)-1}$.

- This theorem is in fact only one direction of [Oza87, theorem 4.2].[3] The other direction of the theorem, which is theorem 3.3.3, can be used to prove the Topological Tverberg conjecture for r being a prime power.

- The existence of the Σ_r-map $\tilde{f} : skel_{d(r-1)-1}X \to S^{d(r-1)-1}$ for any integer $r \geq 2$ is a direct consequence of the theorem 2.7.6, since X is a $d(r-1)-1$-dimensional free Σ_r-complex and $S^{d(r-1)-1}$ is a $(d(r-1)-2)$-connected Σ_r-space. Thus the above theorem in fact states that when r is not a prime power, $\tilde{f}$ can extend G-equivariantly one more dimension, to the whole X.

Since $\mathcal{K}^{\times r}_{\Delta(2)}$ is a free Σ_r-complex of dimension at most $rk = d(r-1)$, apply the above theorem 4.3.2 of Özaydin, there is a Σ_r-equivariant map $F : (\mathcal{K})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$.

Now we give a proof of the theorem 4.3.2. In order to prove the theorem, we need a lemma:

Lemma 4.3.1. *[Oza87, lemma 4.1] Let G be a finite group, G_p be a Sylow p-subgroup for each prime p dividing the order of G, n be a positive integer, X be a $(n+1)$-dimensional free G-complex, and Y be a $(n-1)$-connected G-complex (if $n = 1$ we also assume that $\pi_1 Y$ is abelian). There is a G-map from X to Y if and only if there is a G_p-map from X to Y for each p.*

The proof of the lemma requires a lot of work. We postpone it until later and first prove theorem 4.3.2 assuming the lemma.

proof of theorem 4.3.2. When r is non prime power, let p be a prime number dividing the order of Σ_r, by lemma 2.3.1, any Sylow p-subgroup $\mathfrak{p}$ of Σ_r acts on $\{1, \ldots, r\}$ non transitively. By lemma 2.4.2, this implies $S^{d(r-1)-1}$ has a $\mathfrak{p}$-fixed point for all Sylow p-subgroup $\mathfrak{p}$. Let X be an at most $d(r-1)$-dimensional free Σ_r-complex,

[3] Theorem 4.2 of Özaydin's paper states that: Let $d \geq 1$ and $r \geq 2$, there exists an Σ_r-equivariant map

$$f : E_{d(r-1)}\Sigma_r \xrightarrow{\Sigma_r} S^{d(r-1)-1}$$

if and only if r is not a prime power. The theorem has two directions: one is when r is not a prime power, there exists a Σ_r-equivariant map; the other is when r is a power of prime, there is no Σ_r-equivariant map. In fact, Özaydin has proved more in both directions and we state them as theorem 4.3.2 and theorem 3.3.3 respectively in this article.

for any $\mathfrak{p}$, we can define a $\mathfrak{p}$-map from X to $S^{d(r-1)-1}$ to be a constant map which maps X to the $\mathfrak{p}$-fixed point. Thus by lemma 4.3.1, there is a Σ_r-map from X to $S^{d(r-1)-1}$. □

Now we prove the lemma 4.3.1:

proof of lemma 4.3.1. One direction is obvious: any G-map is also a H-map for any subgroup H of G.

We prove the other direction using theorem 2.7.5, the main theorem of equivariant obstruction theory:

$C_{n+1}(X)$ is a free abelian group with basis of all $n+1$ cells of X, observe that it is a G-module for all n: X is a free G-complex with a free cellular G-action on X, which induces a free G-action on the set of all $n+1$ cells of X, i.e., on the basis of $C_{n+1}(X)$. Thus we can define a free G-action on $C_{n+1}(X)$ by extend the free G-action on the set of all $n+1$ cells of X linearly.

Since we have assumed that Y is $(n-1)$-connected, and for $n=1$, we assume further that $\pi_1 Y$ is abelian, we can apply Hurewicz theorem 2.6.1, and get that the Hurewize homomorphism $h_* : \pi_n Y \to H_n(Y)$ is an isomorphism. Since Y is a G-space, the G-action on Y induces a well defined G-action on $H_n(Y)$, given by:

$$\begin{aligned} G \times H_n(Y) &\to H_n(Y) \\ (g, [f]) &\mapsto g \cdot [f] = [g \cdot f] \end{aligned}$$

where for an n-cell σ^n of Y, $(g \cdot f)(\sigma^n) = g \cdot f(\sigma^n)$ and the latter $\cdot$ is the G-action on the G-space Y. This gives a well defined G-action on $\pi_n Y$ through the above Hurewicz isomorphism $h_* : \pi_n Y \to H_n(Y)$. Thus we can also regard the abelian groups $\pi_n Y$ as a G-module with the this G-action as the scalar multiplication.

Denote the group of all morphisms between the G-modules $C_{n+1}(x)$ and $\pi_n Y$ as $Hom_G(C_{n+1}(x), \pi_n Y)$. We have the following cochain complex:

$$\cdots Hom_G(C_{n+1}(X), \pi_n Y) \xleftarrow{\delta} Hom_G(C_n(X), \pi_n Y) \xleftarrow{\delta} Hom_G(C_{n-1}(X), \pi_n Y) \cdots,$$

the coboundary map $\delta : Hom_G(C_n(X), \pi_n Y) \to Hom_G(C_{n+1}(X), \pi_n Y)$ is defined by $\delta(f) = f \circ \partial$ where $\partial : C_{n+1}(X) \to C_n(X)$ is the boundary map of the cellular chain complex. Since ∂ and f are both G-equivariant, we have $\delta(f)$ is G-equivariant. Hence δ is well defined. Since $\delta \circ \delta = 0$, this cochain complex induces a cohomology group $H^n_G(X, \pi_n Y)$.

Similarly for all p-sylow subgroup G_p of G, $C_{n+1}(x)$ and $\pi_n Y$ are also G_p-modules and $Hom_{G_p}(C_{n+1}(x), \pi_n Y)$ denotes the group of all morphisms between these two G_p-modules. We also have the following cochain complex:

$$\cdots Hom_{G_p}(C_{n+1}(X), \pi_n Y) \xleftarrow{\delta} Hom_{G_p}(C_n(X), \pi_n Y) \xleftarrow{\delta} Hom_{G_p}(C_{n-1}(X), \pi_n Y) \cdots$$

which also induces a cohomology group $H^n_{G_p}(X, \pi_n Y)$.

Since X is a $(n+1)$-dimensional free G-complex, and Y is a $(n-1)$-connected G-complex, by theorem 2.7.6, we already have a G-map $f : X_n \to Y$ from the n-th skeleton X_n of X to Y. We now show that there exists G-map $g : X \to Y$ if there exists G_p-map $f_p : X \to Y$. By obstruction theory 2.7.5, it suffices to show $f|_{X_{n-1}} : X_{n-1} \to Y$ can extends to X G-equivariantly from X_{n-1}, i.e., to show the obstruction cocycle $[\theta^{n+1}_G(f)] \in H^{n+1}_G(X, \pi_n Y)$ vanishes.

We first define obstruction cocycle $\theta^{n+1}_G(f)$ as follows: since X is a free G-complex, we have G acts freely and cellularly on the set of all the $n+1$ cells of X. This G-action gives a partition of the set of all $n+1$ cells into equivalence classes, called G-orbits. For a $n+1$ cell e^{n+1}_i of each G-orbit $G \cdot e^{n+1}_i$, we can associate it with a map $f \circ \varphi_i : S^n \xrightarrow{\varphi_i} \partial e^{n+1}_i \subset X_n \xrightarrow{f} Y$ where ∂e^{n+1}_i is the boundary of e^{n+1}_i and φ_i is the attaching map of e^{n+1}_i. And $f \circ \varphi_i$ defines an element $[f \circ \varphi_i] \in [S^n, Y] = \pi_n Y$. Let $C_{n+1}(X)$ be the free abelian group with basis consisting of all $n+1$ cells of X, since we have defined a map $\theta_i(f) : e^{n+1}_i \mapsto [f \circ \varphi_i]$ for all e^{n+1}_i, we can extend $\theta_i(f)$ G-equivariantly to their G-orbits $G \cdot e^{n+1}_i$ and now $\theta_i(f)$ is defined on all $n+1$ cells of $C_{n+1}(X)$. Since each element of $C_{n+1}(X)$ is a finite linear combination of $n+1$ cells, we can extend $\theta_i(f)$ linearly to the whole $C_{n+1}(X)$. Now we have a G-equivariant map $\theta^{n+1}_G(f) : C_{n+1}(X) \to \pi_n Y, \sum n_i e^{n+1}_i \mapsto \sum n_i [f \circ \varphi_i]$. Thus we have defined an G-equivariant obstruction cochain $\theta^{n+1}_G(f) \in C^{n+1}_G(X, \pi_n Y) = Hom_G(C_{n+1}(X), \pi_n Y)$. This obstruction cochain is also a cocycle since X is $(n+1)$-dimensional which makes $C^{n+2}(X, \pi_n(Y)) = \{0\}$. Thus the obstruction cocycle $\theta^{n+1}_G(f)$ represents a cohomology class $[\theta^{n+1}_G(f)] \in H^{n+1}_G(X, \pi_n Y)$.

Now we only need to show the cohomology class $[\theta^{n+1}_G(f)] \in H^{n+1}_G(X, \pi_n Y)$ vanishes, i.e., $\theta^{n+1}_G(f)$ is a G-equivariant coboundary: since Y is a $(n-1)$-connected G_p-complex, by theorem 2.7.6, $f|_{X_{n-1}}$ and $f_p|_{X_{n-1}}$ are G_p-homotopic. Thus by lemma 2.7.2, their corresponding obstruction cocycles of extending $f|_{X_{n-1}}$ and $f_p|_{X_{n-1}}$ G_p-equivariantly to X differs by a coboundary, i.e., $[\theta^{n+1}_{G_p}(f)] = [\theta^{n+1}_{G_p}(f_p)] \in H^{n+1}_{G_p}(X, \pi_n Y)$. Since $f_p|_{X_{n-1}}$ can extend to X G_p-equivariantly, we have $[\theta^{n+1}_{G_p}(f)] = [\theta^{n+1}_{G_p}(f_p)] = 0$

in $H^{n+1}_{G_p}(X, \pi_n Y)$.

We now define an restriction homomorphism $i^* : H^{n+1}_G(X, \pi_n Y) \to H^{n+1}_{G_p}(X, \pi_n Y)$, where $G_p \subseteq G$ is a p-sylow subgroup of G for each p divides $|G|$ as follows:

since any G-equivariant map are also G_p-equivariant, $Hom_G(C_{n+1}(x), \pi_n Y)$ is a subgroup of $Hom_{G_p}(C_{n+1}(x), \pi_n Y)$. Let

$$i : Hom_G(C_{n+1}(X), \pi_n Y) \to Hom_{G_p}(C_{n+1}(X), \pi_n Y)$$

denote the inclusion map, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots Hom_G(C_{n+1}(X), \pi_n(Y)) \arrow[d, "i"'] & Hom_G(C_n(X), \pi_n(Y)) \arrow[l, "\delta"'] \arrow[d, "i"'] & Hom_G(C_{n-1}(X), \pi_n(Y)) \cdots \arrow[l, "\delta"'] \arrow[d, "i"'] \\
\cdots Hom_{G_p}(C_{n+1}(X), \pi_n(Y)) & Hom_{G_p}(C_n(X), \pi_n(Y)) \arrow[l, "\delta"'] & Hom_{G_p}(C_{n-1}(X), \pi_n(Y)) \cdots \arrow[l, "\delta"']
\end{tikzcd}$$

which satisfies $\delta \circ i = i \circ \delta$, i.e., the inclusion map i maps G-equivariant cocycles to G_p-equivariant cocyles, G-equivariant coboundaries to G_p-equivariant coboundaries. Thus i induces a map on cohomology $i^* : H^{n+1}_G(X, \pi_n Y) \to H^{n+1}_{G_p}(X, \pi_n Y)$, which is the restriction map we need. And we have $i^*([\theta^{n+1}_G(f)]) = [\theta^{n+1}_{G_p}(f)] = 0$.

Now we define a map between two cochain complexes:

$$\begin{aligned} t : Hom_{G_p}(C_{n+1}(X), \pi_n Y) &\to Hom_G(C_{n+1}(X), \pi_n Y) \\ x &\mapsto \sum_{k=1}^{m} g_k x \end{aligned}$$

where $g_k, k = 1, \cdots, m$ are the representatives of the left coset of G_p in G.

We can see that the map t is independent of the choice of the representatives and $t(x) \in Hom_G(C_{n+1}(X), \pi_n Y)$ since $\forall g \in G, g \cdot (\sum_{k=1}^m g_k x) = \sum_{k=1}^m (g \cdot g_k)x = \sum_{k=1}^m g_k x$.

Thus we have the following commutative diagram:

$$\begin{tikzcd}
\cdots Hom_{G_p}(C_{n+1}(X), \pi_n(Y)) \arrow[d, "t"'] & Hom_{G_p}(C_n(X), \pi_n(Y)) \arrow[l, "\delta"'] \arrow[d, "t"'] & Hom_{G_p}(C_{n-1}(X), \pi_n(Y)) \cdots \arrow[l, "\delta"'] \arrow[d, "t"'] \\
\cdots Hom_G(C_{n+1}(X), \pi_n(Y)) & Hom_G(C_n(X), \pi_n(Y)) \arrow[l, "\delta"'] & Hom_G(C_{n-1}(X), \pi_n(Y)) \cdots \arrow[l, "\delta"']
\end{tikzcd}$$

which induced a map on the level of cohomology and we call it the transfer map:

$$\begin{aligned} tr : H^{n+1}_{G_p}(X, \pi_n Y) &\to H^{n+1}_G(X, \pi_n Y) \\ x &\mapsto \sum_{k=1}^{m} g_k x \end{aligned}$$

The composition of restriction map i^* and the transfer map tr is multiplication by the index $[G : G_p]$:

$$\begin{array}{ccccc} H^{n+1}_G(X, \pi_n Y) & \xrightarrow{i^*} & H^{n+1}_{G_p}(X, \pi_n Y) & \xrightarrow{tr} & H^{n+1}_G(X, \pi_n Y) \\ \\ x \longmapsto & & x \longmapsto & & \sum_{k=1}^{m} g_k x = [G : G_p]x \end{array}$$

Since $i^*([\theta^{n+1}_G(f)]) = [\theta^{n+1}_{G_p}(f)] = 0$, we have $(tr \circ i^*)([\theta^{n+1}_G(f)]) = [G : G_p][\theta^{n+1}_G(f)] = 0$ for all prime p dividing $|G|$. For all prime p_i dividing $|G|$, $i = 1, \ldots, n$, we have $\gcd([G : G_{p_1}], \ldots, [G : G_{p_i}], \ldots) = 1$ since: denote the order of p_i in $|G|$ to be e_i, we have $|G| = p_1^{e_1} \cdot p_2^{e_2} \cdots p_n^{e_n}$. Thus $[G : G_{p_i}] = p_1^{e_1} \cdots p_{i-1}^{e_{i-1}} \cdot p_{i+1}^{e_{i+1}} \cdots p_n^{e_n}$, and $[G : G_{p_i}]$ for $i = 1, \ldots, n$ has no common divisor except 1. Thus there exists integers $s_1, \ldots, s_n$ such that $s_1[G : G_{p_1}] + \cdots + s_n[G : G_{p_n}] = 1$.

Hence we have $[\theta^{n+1}_G(f)] = 1 \cdot [\theta^{n+1}_G(f)] = \gcd([G : G_{p_1}], \ldots, [G : G_{p_n}])[\theta^{n+1}_G(f)] = (s_1[G : G_{p_1}] + \cdots + s_n[G : G_{p_n}])[\theta^{n+1}_G(f)] = 0$ in $H^{n+1}_G(X, \pi_n Y)$.

□

Now we show that the existence of the equivariant map $F : (\mathcal{K})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$ can prove theorem 4.3.1, which is, it implies the generalized van Kampen-Flores conjecture is not true for r non prime power, and thus disprove the topological Tverberg conjecture for r non prime power.

Step 2.2. the existence of equivariant map $F : (\mathcal{K})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$ implies map $g : \mathcal{K} \to \mathbb{R}^d$ with no r-fold Tverberg points, which is, there exists r pairwise disjoint faces of $\mathcal{K}$ such that their images has no common intersection.

Theorem 4.3.3 (Sufficiency of the Deleted Product Criterion for Tverberg Points)**.** *([MW15, theorem 7],[MW14, theorem 3]) Suppose $r \geq 2$, $(r-1)d = (\geq ?)rk$, and $d - k \geq 3$. If $\mathcal{K}$ is a finite k-dimensional simplicial complex, then there exists a map $f : \mathcal{K} \to \mathbb{R}^d$ without r-Tverberg point if and only if there exists an Σ_r-equivariant map $F : \mathcal{K}^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$.(Equivalently, if and only if the primary obstruction of $\mathcal{K}^{\times r}_{\Delta(2)}$ vanishes, i.e., $\mathfrak{o}(\mathcal{K}^{\times r}_{\Delta(2)}) = 0$.)*

Remark 4.3.5.

- In our case, we can let $d = 3r$, let $\mathcal{K}$ be the $3(r-1)$ skeleton of $\Delta_{(d+2)(r-1)}$, thus $d - k = 3$ and we can apply the theorem. Thus we have there exists a map $f : \mathcal{K} \to \mathbb{R}^{3r}$ without r-Tverberg point if and only if there exists an Σ_r-equivariant map $F : \mathcal{K}^{\times r}_{\Delta(2)} \to S^{3r(r-1)-1}$.

- We only need one direction of above theorem in our proof of theorem 4.3.1, i.e. the existence of Σ_r-equivariant map $F : \mathcal{K}^{\times r}_{\Delta(2)} \to S^{3r(r-1)-1}$ implies the existence of a map $f : \mathcal{K} \to \mathbb{R}^{3r}$ without r-Tverberg point.

proof of theorem 4.3.3. One direction is obvious: if there exists a map $f : \mathcal{K} \to \mathbb{R}^d$ without r-Tverberg point, we can construct its r-fold product $f^{\times r} : (\mathcal{K})^{\times r}_{\Delta} \to (\mathbb{R}^d)^{\times r} - \mathfrak{diag}$, which is obviously a Σ_r-equivariant map. And then, as in example 2.4.1, we can compose it with the projection $\rho : (\mathbb{R}^d)^{\times r} - \mathfrak{diag} \to (\mathfrak{diag})^{\perp} \setminus \{0\}$ and the normalization $\mu : (\mathfrak{diag})^{\perp} \setminus \{0\} \to S^{d(r-1)-1}$, and get a Σ_r-equivariant map $\bar{F} = \mu \circ \rho \circ f^{\times r} : (\mathcal{K})^{\times r}_{\Delta(2)} \to S^{d(r-1)-1}$.

The other direction is to show that the existence of Σ_r-equivariant map $F : \mathcal{K}^{\times r}_{\Delta(2)} \to S^{3r(r-1)-1}$ implies the existence of a map $f : \mathcal{K} \to \mathbb{R}^{3r}$ without r-Tverberg point. This is highly nontrivial, and it involves lots of advance techniques developed by Mabillard and Wagner in their paper [MW15] and [MW14]. We will only sketch the proof:

We need to apply the r-fold Whitney trick and the r-fold Van Kampen finger moves(which are the generalizations of the 2-fold Whitney trick and 2-fold Van Kampen finger moves) and show the following:

recall that $\mathcal{K} = skel_k\Delta_{(d+2)(r-1)}$ where $k = \frac{d(r-1)}{r} = 3(r-1)$, $d = 3r$, dim $\mathcal{K}^{\times r}_{\Delta(2)} = rk = (r-1)d = 3r(r-1)$,

1. let $f : \mathcal{K} \to \mathbb{R}^{3r}$ be any map in general position, let $f^{\times r}$ be its r-fold product $f^{\times r} : \mathcal{K}^{\times r}_{\Delta(2)} \to (\mathbb{R}^{3r})^{\times r}$, since f is in general position, only the images of top dimensional ($3(r-1)$-dimensional) simplices of $\mathcal{K}$ can intersect, i.e., only $3r(r-1)$ dimensional simplices $\sigma_1 \times \cdots \times \sigma_r \in \mathcal{K}^{\times r}_{\Delta(2)}$ where $dim(\sigma_i) = 3(r-1)$ can have $F(\sigma_1 \times \cdots \times \sigma_r) \cap \mathfrak{diag} \neq \emptyset$. Thus if we restrict $f^{\times r}$ to $(3r(r-1)-1)$-skeleton of $\mathcal{K}^{\times r}_{\Delta(2)}$, its image avoid the $\mathfrak{diag}$, and we have a Σ_r-equivariant map $f^{\times r}_{-1} : skel_{3r(r-1)-1}\mathcal{K}^{\times r}_{\Delta(2)} \to (\mathbb{R}^{3r})^{\times r} - \mathfrak{diag}$.

2. let $\varphi_{f^{\times r}_{-1}}$ be the obstruction cocycle of $f^{\times r}_{-1}$ representing the primary obstruction $\mathfrak{o}(\mathcal{K}^{\times r}_{\Delta(2)})$ of $\mathcal{K}^{\times r}_{\Delta(2)}$, i.e., for any $3r(r-1)$-cell $\sigma = \sigma_1 \times \cdots \times \sigma_r$ of $\mathcal{K}^{\times r}_{\Delta(2)}$, $\varphi_{f^{\times r}_{-1}}$ associate it with the degree of the map which is the composition of its attaching

map and $f_{-1}^{\times r}$: $\partial\sigma(\simeq S^{3r(r-1)-1}) \to skel_{3r(r-1)-1}\mathcal{K}_{\Delta(2)}^{\times r} \xrightarrow{f_{-1}^{\times r}} (\mathbb{R}^{3r})^{\times r} - \mathfrak{diag}(\simeq S^{3r(r-1)-1})$, i.e., $\varphi_{f_{-1}^{\times r}}(\sigma) = deg(f_{-1}^{\times r}|_{\partial\sigma}) \in \mathbb{Z}$

3. the degree of the map $f_{-1}^{\times r}|_{\partial\sigma}$ is equal to the sum of the signs of the intersection points of $f^{\times r}(\sigma = \sigma_1 \times \cdots \times \sigma_r)$ and $\mathfrak{diag}$, i.e., $deg(f_{-1}^{\times r}|_{\partial\sigma}) = \sum_y sgn_y$ where sgn_y is the (r-fold) intersection sign of $y \in f(\sigma_1) \cap \cdots \cap f(\sigma_r)$.

4. by Özadyin's theorem 4.3.2, we know that there exists Σ_r-equivariant map $F : \mathcal{K}_{\Delta(2)}^{\times r} \to (\mathbb{R}^d)^{\times r} - \mathfrak{diag}$. Thus by equivariant obstruction theory, the primary obstruction $\mathfrak{o}(\mathcal{K}_{\Delta(2)}^{\times r})$ vanishes. Thus we have $[\varphi_{f_{-1}^{\times r}}] = \mathfrak{o}(\mathcal{K}_{\Delta(2)}^{\times r}) = 0$, i.e., $\varphi_{f_{-1}^{\times r}}$ is a Σ_r-equivariant coboundary, which is the finite sum of elementary coboundary.

5. modifying f by an r-fold van Kampen finger move has the effect that $\varphi_{f_{-1}^{\times r}}$ changes by an elementary coboundary. Thus by applying finitely many r-fold van Kampen finger moves on f we get a modified map $\tilde{f} : \mathcal{K} \to \mathbb{R}^{3r}$ such that $\varphi_{\tilde{f}_{-1}^{\times r}} = 0$.

6. since for any $3r(r-1)$-dimensional cell $\sigma = \sigma_1 \times \cdots \times \sigma_r$ of $\mathcal{K}_{\Delta(2)}^{\times r}$, $\varphi_{\tilde{f}_{-1}^{\times r}}(\sigma) = deg(\tilde{f}_{-1}^{\times r}|_{\partial\sigma}) = \sum_y sgn_y$ is the sum of the signs of the intersection points y of $\tilde{f}(\sigma_1) \cap \cdots \cap \tilde{f}(\sigma_r)$, $\sum_y sgn_y = 0$ means the intersection points of $\tilde{f}(\sigma_1) \cap \cdots \cap \tilde{f}(\sigma_r)$ can be divided into pairs of opposite signs.

7. since for all $i = 1, \ldots, r$, $3r - dim\sigma_i = 3r - 3r(r-1) = 3$ satisfies the codimension condition of the r-fold Whitney trick, and the intersection points of their images $\tilde{f}(\sigma_1) \cap \cdots \cap \tilde{f}(\sigma_r)$ can be paired up into pairs of opposite signs, we can thus apply the r-fold Whitney trick on $\tilde{f}$ to each pair locally to eliminate the intersection points pair by pair without introducing new intersection point during the process. Repeat the same process to all $3r(r-1)$-dimensional cells $\sigma = \sigma_1 \times \cdots \times \sigma_r$ of $\mathcal{K}_{\Delta(2)}^{\times r}$, we have $\tilde{f}(\sigma_1) \cap \cdots \cap \tilde{f}(\sigma_r) = \emptyset$ for all σ. Denote the modified map as $g : \mathcal{K} \to \mathbb{R}^{3r}$, since g is a map in general position and only the images of top dimensional cells of $\mathcal{K}$ can intersect, it is a map without r-Tverberg point, which is exactly the map we need.

□

In conclusion, by constructing the map $g : \mathcal{K} \to \mathbb{R}^{3r}$ without r-Tverberg point, we have disprove the generalized van Kampen-Flores conjecture for r non prime power, i.e., we have prove theorem 4.3.1. Furthermore, using the procedure in the remark 4.3.1, we can use $g : \mathcal{K} \to \mathbb{R}^{3r}$ to construct a map $f : \Delta_{(3r+2)(r-1)} \to \mathbb{R}^{3r+1}$ without r-Tverberg point, and thus give a counterexample for the topological Tverberg conjecture 4.1.1 for r non prime power, $d = 3r + 1$. Thus we have prove the theorem 4.1.5.

Remark 4.3.6 (failure of topological Tverberg conjecture of lower dimension implies failure of higher dimension)*.* By [dL01, proposition 2.5.], if the topological Tverberg theorem holds for any $r \geq 2, d+1 \geq 2$, then it also holds for $r \geq 2, d \geq 1$. Thus the failure of the topological Tverberg conjecture for d implies its failure for $d+1$. Since we have already construct a counterexample for the topological Tvereberg conjecture for r non prime power and $d = 3r + 1$, we now have the failure of the topological Tverberg conjecture for r non prime power and $d \geq 3r + 1$, which is the following theorem.

Theorem 4.3.4. *If r is not a prime power and $d \geq 3r + 1$, there exists an almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$.*

Further developments In conclusion, using the above techniques, we have constructed counterexamples of topological Tverberg conjecture for non prime power r and for dimensions $d \geq 3r + 1$. The restriction on d has been further improved to $d \geq 2r + 1$: the lowest dimensional counterexample is an almost 6-embedding $\Delta_{70} \to \mathbb{R}^{13}$, which is given in [AMSW21, theorem 1.1]. The problem of whether there exists counterexamples for r non prime power, $d \leq 2r$ is still open.

References

[Ale98] Pavel S Aleksandrov. *Combinatorial topology*, volume 1. Courier Corporation, 1998.

[AMSW21] Sergey Avvakumov, Isaac Mabillard, Arkadiy B Skopenkov, and Uli Wagner. Eliminating higher-multiplicity intersections. iii. codimension 2. *Israel Journal of Mathematics*, 245(2):501–534, 2021.

[BB79] Ervin G Bajmóczy and Imre Bárány. On a common generalization of borsuk's and radon's theorem. *Acta Mathematica Hungarica*, 34(3-4):347–350, 1979.

[BFZ14] Pavle VM Blagojević, Florian Frick, and Günter M Ziegler. Tverberg plus constraints. *Bulletin of the London Mathematical Society*, 46(5):953–967, 2014.

[BFZ19a] Pavle VM Blagojević, Florian Frick, and Günter M Ziegler. Barycenters of polytope skeleta and counterexamples to the topological tverberg conjecture, via constraints. *Journal of the European Mathematical Society*, 21(7):2107–2116, 2019.

[BFZ19b] Pavle VM Blagojević, Florian Frick, and Günter M Ziegler. Barycenters of polytope skeleta and counterexamples to the topological tverberg conjecture, via constraints. *Journal of the European Mathematical Society*, 21(7):2107–2116, 2019.

[Bir59] Bryan John Birch. On 3n points in a plane. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 55, pages 289–293. Cambridge University Press, 1959.

[Bor33a] Karol Borsuk. Drei sätze über die n-dimensionale euklidische sphäre. *Fundamenta Mathematicae*, 20(1):177–190, 1933.

[Bor33b] Karol Borsuk. Drei sätze über die n-dimensionale euklidische sphäre. *Fundamenta mathematicae*, 20(1):177–190, 1933.

[Bre06] Glen E Bredon. *Equivariant cohomology theories*, volume 34. Springer, 2006.

[BSS81] Imre Bárány, Senya B Shlosman, and András Szücs. On a topological generalization of a theorem of tverberg. *Journal of the London Mathematical Society*, 2(1):158–164, 1981.

[BZ17] Pavle VM Blagojević and Günter M Ziegler. Beyond the borsuk–ulam theorem: the topological tverberg story. *A Journey Through Discrete Mathematics: A Tribute to Jiří Matoušek*, pages 273–341, 2017.

[DF04] David Steven Dummit and Richard M Foote. *Abstract algebra*, volume 3. Wiley Hoboken, 2004.

[DK01] James Frederic Davis and Paul Kirk. *Lecture notes in algebraic topology*, volume 35. American Mathematical Soc., 2001.

[dL01] Mark de Longueville. Notes on the topological tverberg theorem. *Discrete Mathematics*, 241(1-3):207–233, 2001.

[DL16] JA De Loera. Geometric and topological combinatorics in economics, game theory, & optimization. 2016.

[DLGMM19] Jesús De Loera, Xavier Goaoc, Frédéric Meunier, and Nabil Mustafa. The discrete yet ubiquitous theorems of carathéodory, helly, sperner, tucker, and tverberg. *Bulletin of the American Mathematical Society*, 56(3):415–511, 2019.

[Dol83] Albrecht Dold. Simple proofs of some borsuk-ulam results. *Contemp. math*, 19:65–69, 1983.

[Fin47] Nathan J Fine. Binomial coefficients modulo a prime. *The American Mathematical Monthly*, 54(10):589–592, 1947.

[Fox09] Jacob Fox. Lecture 3: Sperner's lemma and brouwer's theorem, 2009.

[FRI05] TYPESET BY ML FRIES. The borsuk-ulam theorem and applications presented by alex suciu and marcus fries. *Preprint*, 2005.

[Fri15] Florian Frick. Counterexamples to the topological tverberg conjecture. *arXiv preprint arXiv:1502.00947*, 2015.

[FT81a] Robert M Freund and Michael J Todd. A constructive proof of tucker's combinatorial lemma. *Journal of Combinatorial Theory, Series A*, 30(3):321–325, 1981.

[FT81b] Robert M Freund and Michael J Todd. A constructive proof of tucker's combinatorial lemma. *Journal of Combinatorial Theory, Series A*, 30(3):321–325, 1981.

[GDGJFJ10] Julio González-Dıaz, Ignacio Garcıa-Jurado, and M Gloria Fiestras-Janeiro. An introductory course on mathematical game theory. *Graduate studies in mathematics*, 115, 2010.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[MBZ03] Jiří Matoušek, Anders Björner, and Günter M Ziegler. *Using the Borsuk-Ulam theorem: lectures on topological methods in combinatorics and geometry*, volume 2003. Springer, 2003.

[MW14] Isaac Mabillard and Uli Wagner. Eliminating tverberg points, i. an analogue of the whitney trick. In *Proceedings of the thirtieth annual symposium on Computational geometry*, pages 171–180, 2014.

[MW15] Isaac Mabillard and Uli Wagner. Eliminating higher-multiplicity intersections, i. a whitney trick for tverberg-type problems. *arXiv preprint arXiv:1508.02349*, 2015.

[NS13] Kathryn L Nyman and Francis Edward Su. A borsuk–ulam equivalent that directly implies sperner's lemma. *The American Mathematical Monthly*, 120(4):346–354, 2013.

[Oza87] Murad Ozaydin. Equivariant maps for the symmetric group. 1987.

[Par99] Sehie Park. Ninety years of the brouwer fixed point theorem. *Vietnam J. Math*, 27(3):187–222, 1999.

[Rad21] Johann Radon. Mengen konvexer körper, die einen gemeinsamen punkt enthalten. *Mathematische Annalen*, 83(1):113–115, 1921.

[Sar90] Karanbir S Sarkaria. A generalized kneser conjecture. *Journal of Combinatorial Theory, Series B*, 49(2):236–240, 1990.

[Sar91a] Karanbir S Sarkaria. A generalized van kampen-flores theorem. *Proceedings of the American Mathematical Society*, 111(2):559–565, 1991.

[Sar91b] Karanbir S Sarkaria. A generalized van kampen-flores theorem. *Proceedings of the American Mathematical Society*, 111(2):559–565, 1991.

[Sar00] Karanbir Singh Sarkaria. Tverberg partitions and borsuk–ulam theorems. *Pacific Journal of Mathematics*, 196(1):231–241, 2000.

[Sko18] Arkadiy Borisovich Skopenkov. A user's guide to the topological tverberg conjecture. *Russian Mathematical Surveys*, 73(2):323, 2018.

[tD11] Tammo tom Dieck. *Transformation groups*, volume 8. Walter de Gruyter, 2011.

[Tve66] Helge Tverberg. A generalization of radon's theorem. *Journal of the London Mathematical Society*, 1(1):123–128, 1966.

[Vol96a] A Yu Volovikov. On a topological generalization of the tverberg theorem. *Mathematical Notes*, 59(3):324–326, 1996.

[Vol96b] Aleksei Yu Volovikov. On the van kampen-flores theorem. *Mathematical Notes*, 59(5):477–481, 1996.

[XU] MOOR XU. Sperner's lemma. *Available in: https://math. berkeley. edu/˜ moorxu/misc/equiareal. pdf.*

[Yos74] Mark Yoseloff. Topologic proofs of some combinatorial theorems. *Journal of Combinatorial Theory, Series A*, 17(1):95–111, 1974.