SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Bondal-Orlov Reconstruction Theorem

av

Luneia Stefano

2023 - M3

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Luneia Stefano

Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Tirabassi Sofia

2023

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Introduction

The concept of derived category were first introduced by J. P. Verdier in 1967 [Ver96], in his doctoral thesis written under the supervision of A. Grothendieck. A few years before P. Gabriel in a seminal article [Gab62] developed some fundamental aspects of the theory of abelian category, which were used and expanded by the Grothendieck-Verdier work.
The main idea that led to the construction of derived categories is that one would like to work with complexes rather than with their (co-)homology [1], since passing directly to the (co-)homology means losing too much information. On the other hand we would like to consider as equivalent complexes that have isomorphic (co-)homology groups. An object of an abelian category should then be considered equivalent to its resolutions, moreover, adopting this point of view, one can obtain a 'correct' definition of functors between abelian categories that in general can be directly defined only on well-behaved objects (e.g. injective, projective and flat modules). To adopt this point of view one has then to consider the category of complexes (where the objects of the abelian category are considered as complexes concentrated in degree zero) from the beginning and extend the identification with the respective resolutions to arbitrary complexes.

The main results of this work is the Reconstruction Theorem proved by A.Bondal and D.Orlov in [BO01], for a smooth projective X with ample or anti-ample canonical bundle, the derived category $\mathrm{D}^b(X)$ of bounded coherent sheaves on X determines up to isomorphism the variety. The idea behind this results is to build a scheme isomorphic to X by constructing objects in the category $\mathrm{D}^b(X)$ which play the role of points and invertible sheaves, thus, using the fundamental hypothesis of ampleness of the canonical bundle, Bondal and Orlov derived an isomorphism between the varieties X and Y from the exact equivalence between their derived categories of coherent sheaves.

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[1] For motivating examples arising from algebraic topology and homological algebra we refer to [Tho01]

Chapter 1

Triangulated categories

In this chapter we are going to introduce two fundamental concepts to develop the theory of derived category. An abelian category is the *categorification* (in a sense that would be clarified later) of the category Ab of abelian group and is the right environment to generalize the standard definition of (co)homology and all the important and structural properties of it. The first non-trivial example of abelian category will be the category R-Mod of R-modules for some ring R, and it is often said that if you need to prove something in an abelian category you can think of it as R-Mod and 'prove the statement there'. More precisely a result by Peter Freyd and Barry Mitchell, known as Freyd-Mitchell embedding, says that for any abelian category $\mathcal{A}$ there exists a ring R such that the category $\mathcal{A}$ can be embedded as a full subcategory of R-Mod. However, following the Johnstone's thesis that in [Joh14] talks about this embedding, if one has to prove a results in a category of modules, one might as well work in an abelian category and the embedding ensures that the results holds in generality, this will help to concentrate only on the essential property of the structure, exactly the ones that characterize an abelian category.

For all the notion from category theory we will use in this work we refer to standard and complete introduction to the subjects, such as the classical [Mac71] and the modern [Rie17]. The interested reader can find there all the basic notions spread out and deeply investigated.

1.1 Abelian categories

Definition 1.1.1. A category $\mathcal{A}$ is an additive category if the following conditions are satisfied

i) For every objects A, B in $\mathcal{A}$ the hom-set $\mathrm{Hom}(A, B)$ is endowed with the structure of an abelian group.

ii) For every objects A, B, C in $\mathcal{A}$ the composition functor $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ is bilinear.

iii) There exists a zero object 0 in $\mathcal{A}$, i.e. 0 is both the trivial and the terminal object.

iv) $\mathcal{A}$ has finite products and coproducts and they are isomorphic, i.e. for every objects A_1, A_2 in $\mathcal{A}$ both exist $A_1 \times A_2$ and $A_1 \sqcup A_2$ and $A_1 \times A_2 \cong A_1 \sqcup A_2$.

Remark 1.1.2. This definition worth some comments. The points (i) and (ii) are equivalent to ask that $\mathcal{A}$ is enriched over Ab, moreover in the Ab-enriched context we can replace (iii) and (iv) by requiring that $\mathcal{A}$ has finite products (or coproducts). More precisely, if $\mathcal{A}$ has finite products in particular has a terminal object 1 then $\mathrm{Hom}(1, 1)$ is the trivial group with one element, but this implies that 1 is also the initial object 0, hence a zero object. Having this particular object we can define the zero morphism in $\mathrm{Hom}(A, B)$ as the unique morphism that factorize

$$\begin{tikzcd}
A \arrow[rr, "0"] \arrow[dr, "@_A"'] & & B \\
& 0 \arrow[ur, "!_B"'] &
\end{tikzcd}$$

Now, if $A_1 \times A_2$ exists, using the zero morphism we can define two 'coprojections'

$$\begin{tikzcd}
& A_1 \arrow[dl, "id_{A_1}"'] \arrow[d, dashed, "i_1"] \arrow[dr, "0"] & \\
A_1 & A_1 \times A_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & A_2
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_2 \arrow[dl, "0"'] \arrow[d, dashed, "i_2"] \arrow[dr, "id_{A_2}"] & \\
A_1 & A_1 \times A_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & A_2
\end{tikzcd}$$

Moreover the following equality $i_1 \circ p_1 + i_2 \circ p_2 = id_{A_1 \times A_2}$ makes $A_1 \times A_2$ (isomorphic to) the coproduct $A_1 \sqcup A_2$. We call this object direct sum and we will denote it $A_1 \oplus A_2$.

Finally let us recall that point (iii) can be stated using representable enriched functors, more precisely having products and coproducts means that for any A, B in $\mathcal{A}$ the following

$$F : \mathcal{A} \longrightarrow \mathsf{Ab}$$

$$X \mapsto \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$$

and

$$G : \mathcal{A} \longrightarrow \mathsf{Ab}$$
$$X \mapsto \mathrm{Hom}(A, X) \times \mathrm{Hom}(B, X)$$

are representable, and by the above discussion, isomorphic functors.

Definition 1.1.3. A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is an additive functor if the action on morphisms

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$$

is a group homomorphism.

Even if the results presented and proved in the following chapters hold for general additive categories, in the practice we will be interested in a little more specific case, since all the geometric construction will be over a fixed field k.

Definition 1.1.4. A k-linear category $\mathcal{A}$ is an additive category enriched over k-vector spaces, i.e. the hom-set $\mathrm{Hom}(A, B)$ is also endowed with the structure of k-vector space and the composition functor $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ is k-linear.
An additive functor $F : \mathcal{A} \to \mathcal{B}$ between two k-linear categories is k-linear if the action on morphisms

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$$

is k-linear.

In an category with an initial object the kernel of a morphism $f : A \to B$ is defined to be the following pullback

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d] & 0 \arrow[d, "@_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Dually, in a category with a terminal object the cokernel of a morphism $f : A \to B$ is defined to be the following pushout

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "!_A"'] & B \arrow[d] \\
1 \arrow[r] & \mathrm{coker}(f)
\end{tikzcd}$$

Equivalently, in a category with zero morphism we can define $\ker(f)$ as the equalizer of the parallel maps $A \underset{0}{\overset{f}{\rightrightarrows}} B$, and dually the $\operatorname{coker}(f)$ as the coequalizer of the same pair $A \underset{0}{\overset{f}{\rightrightarrows}} B$.

Definition 1.1.5. An additive category $\mathcal{A}$ is abelian if for each morphism $f \in \operatorname{Hom}(A, B)$ there exists a kernel and a cokernel. Moreover the natural map induced by f

$$\operatorname{CoIm}(f) \xrightarrow{u} \operatorname{Im}(f)$$

is an isomorphism, where $\operatorname{CoIm}(f) :- \operatorname{coker}(\ker(f) \xrightarrow{i} A)$ and $\operatorname{Im}(f) :- \ker(B \xrightarrow{\pi} \operatorname{coker}(f))$ and the natural map is the dashed arrow in the following diagram induced by the universal properties of $\operatorname{Im}(f)$ and $\operatorname{CoIm}(f)$

$$\begin{tikzcd}
\ker(f) \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "\pi"] \arrow[d] & \operatorname{coker}(f) \\
& \operatorname{CoIm}(f) \arrow[r, dashed, "u"] & \operatorname{Im}(f) &
\end{tikzcd}$$

Remark 1.1.6 (Abelian categories are finitely complete and cocomplte)**.** In an abelian category $\mathcal{A}$ we can compute eqalizer of two morphism $f, g \in \operatorname{Hom}(A, B)$ by taking the kernel of the difference of the two maps $\ker(f - g)$, and dually, for the coequalizer we take $\operatorname{coker}(f - g)$. Hence $\mathcal{A}$ has all finite products and equalizer then $\mathcal{A}$ has all finite limits. Dually, $\mathcal{A}$ has all finite coproducts and coequalizer then $\mathcal{A}$ has all finite colimits.

Example 1.1.7.

- The category of abelian group Ab.
- Given a ring R the category of R-modules R-Mod.
- Given a scheme X the category of sheaves of abelian groups on it $\mathsf{Sh}(X)$ is abelian, as well the categories $\mathsf{Coh}(X)$ and $\mathsf{QCoh}(X)$ of coherent and quasi-coherent sheaves respectively are abelian.
- The category of filtered modules over a ring R is an additive category that is not abelian, in particular it admits kernels and cokernels but the morphism between the image and the coimage is not an isomorphism.

As said before, in the context of abelian categories we are allowed to define chain complexes, moreover almost every result in 'standard' homological algebra will hold in this generalized environment. The last ingredient we need to do so, is the concept of exactness.

Definition 1.1.8. A sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called exact if $\ker(g) = \mathrm{Im}(f)$.

Let us note that if the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact then the composite $g \circ f$ is the zero morphism in $\mathrm{Hom}(A, C)$, since by the epi-mono factorization induced by $\mathrm{Im}(f)$ we obtain the following commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads] & & B \arrow[r, "g"] & C \\
& \mathrm{Im}(f) \arrow[ur, hook] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

Definition 1.1.9. A functor $F : \mathcal{A} \to \mathcal{B}$, between abelian categories, is called left exact if the short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1.1}$$

is mapped to the following exact sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

Similarly F is called right exact if the short exact sequence in (1.1) is mapped to the exact sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

F is called exact if the short exact sequence in (1.1) is mapped to

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

Often the above exactness properties are defined in the following equivalent formulations

Proposition 1.1.10. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories, then

i) F preserves finite limits if and only if F is left exact

ii) F preserves finite colimits if and only if F is right exact

iii) F preserves both finite limits and colimits if and only if F exact

Remark 1.1.11. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories, then if F is additive, any sequence $A \xrightarrow{f} B \xrightarrow{g} C$ such that $g \circ f = 0$ is mapped by F to a sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ such that $F(g) \circ F(f) = F(g \circ f) = F(0) = 0$.
Conversely the exactness hypothesis is stronger, more precisely if F is either left or right exact then F is additive.

Lemma 1.1.12. Let $\mathcal{A}$ be an abelian category then the Hom functor

$$\mathrm{Hom}(A, -) : \mathcal{A} \to \mathsf{Ab}$$

$$\mathrm{Hom}(-, A) : \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$$

are left exact.

Definition 1.1.13. An object P in a category $\mathcal{C}$ is projective if has the left lifting property against epimorphisms, i.e. for any epimorphism $q : X \to Y$ and any morphism $f : P \to Y$ there exist a lift $h : P \to X$ making the diagram commute

$$\begin{tikzcd}
 & X \arrow[d, "q"] \\
P \arrow[ur, dashed, "h"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

Dually, an object I is injective if it has the right extension property against monomorphisms, i.e. for any monomorphism $j : X \to Y$ and any morphism $g : X \to I$ there exists an extension $k : Y \to I$ making the diagram commute

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "j"'] & I \\
Y \arrow[ur, dashed, "k"'] &
\end{tikzcd}$$

Definition 1.1.14. Let $\mathcal{A}$ be an abelian category, an injective resolution of an object A in $\mathcal{A}$ is an exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

where all I^i are injective.
Dually a projective resolution of an object A in $\mathcal{A}$ is an exact sequence

$$\dots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow A \longrightarrow 0$$

In the context of abelian categories we have a nice characterization of projective and injective objects

Proposition 1.1.15. Let $\mathcal{A}$ be an abelian category, then

i) P is projective in $\mathcal{A}$ if and only if the functor
$$\mathrm{Hom}(P, -) : \mathcal{A} \to \mathsf{Ab}$$
is exact.

ii) I is injective in $\mathcal{A}$ if and only if the functor
$$\mathrm{Hom}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$$
is exact.

Note that by Lemma 1.1.12 it is enough to check that the two functors are right exact.

We are now going to introduce a notion that is strictly connected with Serre duality in the sense that generalized the functor given by the duality to an equivalence with an additional natural isomorphism.

Definition 1.1.16. Let $\mathcal{A}$ be a k-linear category, then a Serre functor is a k-linear equivalence $S : \mathcal{A} \to \mathcal{A}$ such that for any objects $A, B \in \mathcal{A}$ there exist an isomorphism
$$\eta_{A,B} : \mathrm{Hom}(A, B) \xrightarrow{\sim} \mathrm{Hom}(B, S(A))^*$$
which is natural in both variables.

As a safe hypothesis we are going to assume that all the Hom-sets are finite dimensional k-vector spaces. A Serre functor has two important properties: it commutes with any k-linear equivalence and given a left (right) adjoint we can construct a right (left) adjoint. More specifically:

Lemma 1.1.17. Let $\mathcal{A}$ and $\mathcal{B}$ be k-linear categories endowed respectively with Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$, then for any k-linear equivalence $F : \mathcal{A} \to \mathcal{B}$ there exists an isomorphism of functors
$$F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$$

Lemma 1.1.18. Let $F : \mathcal{A} \to \mathcal{B}$ a functor between k-linear categories endowed respectively with Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$, if F admits a left adjoint $G \dashv F$, then F has also a right adjoint
$$F \dashv S_{\mathcal{A}} \circ G \circ S_{\mathcal{B}}^{-1}$$
Dually, if F admits a right adjoint $F \dashv H$, then F has also a left adjoint
$$S_{\mathcal{A}}^{-1} \circ H \circ S_{\mathcal{B}} \dashv F$$

1.2 Triangulated categories

[1]

We now introduce the concepts of triangulated categories. The main point is the definition of a class of distinguished triangles that, as said before, will play the role of short exact sequence in an additive category. The axioms for a triangulated category are given in such a way all the 'essential' properties of short exact sequences hold also in this more general structure for the triangles.

Definition 1.2.1. Let $\mathcal{D}$ be an additive category, then the structure of a a triangulated category is the datum of an additive (endo-)equivalence

$$T : \mathcal{D} \longrightarrow \mathcal{D}$$

called the shift (or suspension) functor and a collection of distinguished triangles, i.e. triples of objects and morphisms arranged as follows

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

and subject to the following axioms TR1-TR4.
The name comes from the fact that we can arrange these objects into 'actual triangles'

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"'] & & B \arrow[ul, "v"'] \end{tikzcd}$$

A morphism between two triangles is a triple (f, g, h) such that the following commutes

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & T(A) \arrow[d, "T(f)"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & T(A') \end{tikzcd}$$

TR1

i) Any triangle of the form

$$A \xrightarrow{id_A} A \longrightarrow 0 \longrightarrow T(A)$$

is distinguished.

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[1] For a detailed proof of some technical statements left in this section without proof we refer to standard text on homological algebra as [GM03] and [Wei94]

ii) Any triangle isomorphic to a distinguished triangle is distinguished.

iii) Any morphism $f : A \to B$ can be completed to a distinguished triangle of the form

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow T(A)$$

TR2 (*Closed under rotation*)
A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$$

is distinguished if and only if the triangle

$$B \xrightarrow{g} C \xrightarrow{h} T(A) \xrightarrow{-T(f)} T(B)$$

is distinguished.
TR3
Assume there exists a commutative diagram between two distinguished triangles given by the vertical arrows f and g

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dotted, "h"] & T(A) \arrow[d, "T(f)"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & T(A')
\end{tikzcd}$$

then the diagram can be completed (not necessary uniquely) to a morphism of distinguished triangles by a morphism $h : C \to C'$.
TR4 (*Octahedron axiom*)
Given the following distinguished triangles

$$A \xrightarrow{u} B \xrightarrow{f} C' \xrightarrow{g} T(A)$$

$$B \xrightarrow{v} C \xrightarrow{h} A' \xrightarrow{k} T(B)$$

$$A \xrightarrow{vu} C \xrightarrow{i} B' \xrightarrow{j} T(A)$$

there exists a distinguished triangle

$$C' \xrightarrow{a} B' \xrightarrow{b} A' \xrightarrow{kf} T(C')$$

that complete the octahedron

$$\begin{tikzcd}
 & B \arrow[dr] \arrow[ddl] & \\
A' \arrow[ur] \arrow[d] & & C \arrow[ll] \arrow[ddl] \\
C' \arrow[rr] \arrow[dr, dashed] & & A \arrow[u] \arrow[uul] \\
 & B' \arrow[ur] \arrow[uul, dashed] &
\end{tikzcd}$$

i.e. given the following situation we can complete the lower cap (the diagram on the left) to a octahedron

$$\begin{tikzcd}
A' \arrow[dddd] \arrow[ddr] & & C \arrow[ll] \\
 & \star & \\
 & B \arrow[uur] \arrow[ddl] & \\
 & \star & \\
C' \arrow[rr] & & A \arrow[uuuu] \arrow[uul]
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[dd] & & & & C \arrow[llll] \arrow[dl] \\
 & \star & B' \arrow[ull, dashed] \arrow[drr] & \star & \\
C' \arrow[rrrr] \arrow[urr, dashed] & & & & A' \arrow[uu]
\end{tikzcd}$$

Here the triangles decorated with $\star$ are distinguished and the others are commutative, moreover the two possible composite morphism from B to B' are equal, i.e. $a \circ f = i \circ v$.

Remark 1.2.2. The axiom TR4 can be stated equivalently in a 2-dimensional form by saying that there exists a distinguished triangle

$$C' \xrightarrow{a} B' \xrightarrow{b} A' \xrightarrow{kf} T(C')$$

such that the following diagram with rows distinguished triangles, commutes

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "id_A"] & B \arrow[r, "f"] \arrow[d, "v"] & C' \arrow[r, "g"] \arrow[d, "a"] & T(A) \arrow[d, "id_{T(A)}"] \\
A \arrow[r, "vu"] \arrow[d, "u"] & C \arrow[r, "i"] \arrow[d, "id_C"] & B' \arrow[r, "j"] \arrow[d, "b"] & T(A) \arrow[d, "T(u)"] \\
B \arrow[r, "v"] \arrow[d, "f"] & C \arrow[r, "h"] \arrow[d, "i"] & A' \arrow[r, "k"] \arrow[d, "id_{A'}"] & T(B) \arrow[d, "T(f)"] \\
C' \arrow[r, "a"'] & B' \arrow[r, "b"'] & A' \arrow[r, "kf"'] & T(C')
\end{tikzcd}$$

This axiom in a sense enhance the fact that for a sequence of nested inclusions $A \hookrightarrow B \hookrightarrow C$ of abelian groups there exists a canonical isomorphism

$$C/B \cong (C/A)/(B/A) \tag{1.2}$$

then if we replace the short exact sequences given by the inclusions with distinguish triangles

$$A \hookrightarrow B \longrightarrow B/A \longrightarrow T(A)$$

$$B \hookrightarrow C \longrightarrow C/B \longrightarrow T(B)$$

$$A \hookrightarrow C \longrightarrow C/A \longrightarrow T(A)$$

the axiom says that there exists another distinguished triangle

$$B/A \longrightarrow C/B \longrightarrow C/A \longrightarrow T(B/A)$$

that 'corresponds' to the short exact sequence giving (1.2).

To make more clear the motto 'distinguished triangles play the role of short exact sequence', we are now going to state some properties that can be seen as a generalization of facts holding for short exact sequences. In the following discussion we assume $\mathcal{D}$ a triangulated category with shift functor T.

Proposition 1.2.3. Given a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \to T(A)$, the composite $g \circ f$ is trivial.

Proof. By TR1 we can complete $A \xrightarrow{id_A} A$ to a distinguished triangle

$$A \xrightarrow{id_A} A \xrightarrow{!_A} 0 \longrightarrow T(A)$$

Then

$$\begin{tikzcd}
A \arrow[r, "id_A"] \arrow[d, "id_A"] & A \arrow[r, "!_A"] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dotted, "h"] & T(A) \arrow[d, "id_{T(A)}"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & T(A)
\end{tikzcd}$$

commutes, and by TR3 there exists a map $h : 0 \to C$, i.e. $h = @_C$, completing the diagram, hence $g \circ f$ is the trivial morphism. $\square$

Proposition 1.2.4. Given a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{k} T(A)$,then any object X in $\mathcal{D}$ induces two exact sequences (in Ab)

$$\mathrm{Hom}(X, A) \xrightarrow{f\circ -} \mathrm{Hom}(X, B) \xrightarrow{g\circ -} \mathrm{Hom}(X, C)$$

$$\mathrm{Hom}(C, X) \xrightarrow{-\circ g} \mathrm{Hom}(B, X) \xrightarrow{-\circ f} \mathrm{Hom}(A, X)$$

Using the closure under rotation axiom we can apply the Proposition 1.2.4 to the triangle $B \xrightarrow{g} C \xrightarrow{k} T(A) \xrightarrow{-T(f)} T(B)$ and get other similar exact sequences

$$\mathrm{Hom}(X, B) \xrightarrow{g\circ -} \mathrm{Hom}(X, C) \xrightarrow{k\circ -} \mathrm{Hom}(X, T(A))$$

$$\mathrm{Hom}(T(A), X) \xrightarrow{-\circ k} \mathrm{Hom}(C, X) \xrightarrow{-\circ g} \mathrm{Hom}(B, X)$$

Hence, composing them respectively, what we actually obtain are two long exact sequences.

Proposition 1.2.5. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ a distinguished triangle in $\mathcal{A}$. If f, g or h are zero morphism then the triangle splits, i.e., supposing $h = 0$ without loss of generality, if

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} T(A)$$

is a distinguished triangle then it is isomorphic to

$$A \longrightarrow A \oplus C \longrightarrow C \longrightarrow T(A)$$

Proof. By TR1 both $A \xrightarrow{id_A} A \to 0 \to T(A)$ and $C \xrightarrow{id_C} C \to 0 \to T(C)$, then by TR2, up to modify a sign, we obtain that $0 \to C \xrightarrow{id_C} C \to 0$ is again distinguished. The direct sum of distinguished triangles, which is computed level-wise on objects and on morphisms, is a distinguished triangle, hence we get the following

$$A \longrightarrow A \oplus C \longrightarrow C \xrightarrow{0} TX$$

Now by TR3 there exists a morphism $\alpha : B \to A \oplus C$ making the following commute

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "id_A"'] & B \arrow[r, "g"] \arrow[d, dashed, "\alpha"] & C \arrow[r, "0"] \arrow[d, "id_C"] & T(A) \arrow[d, "id_{T(A)}"] \\
A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r, "0"] & T(A)
\end{tikzcd}$$

Moreover α must be an isomorphism, since all the other vertical morphisms are isomorphisms. □

We are now going to define a right notion of functor between triangulated categories that, as one could expect, should preserve all the triangulated structure, for this reason it is been called 'exact' in analogy with the standard notion of exact functor in a abelian category.

Definition 1.2.6. An additive functor $F : \mathcal{D} \to \mathcal{D}'$ between triangulated categories is called exact if the following are satisfied

i) F commutes with the shift functor, i.e. there exists a natural isomorphism

$$F \circ T_{\mathcal{D}} \cong T_{\mathcal{D}'} \circ F$$

ii) Any distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow T_{\mathcal{D}}(A)$$

is mapped by F to the distinguished triangle

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow T_{\mathcal{D}'}(FA)$$

Not surprisingly, triangulated categories and exact functors of triangulated categories form a category. Moreover this concept will allow us to define a meaningful notion of subcategory of a triangulated category

Definition 1.2.7. A subcategory $\mathcal{C}$ of a triangulated category $\mathcal{D}$ is a triangulated subcategory if it admits a triangulated structure and the inclusion functor $i : \mathcal{C} \hookrightarrow \mathcal{D}$ is an exact functor

Remark 1.2.8. If $\mathcal{C}$ is a full subcategory of a triangulated category $\mathcal{D}$, then it is a triangulated subcategory if and only if the inclusion functor commutes with the shift functor and the following condition holds:
for any distinguished triangle in $\mathcal{D}$

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A)$$

if $A, B \in \mathcal{C}$ then C is isomorphic to an object in $\mathcal{C}$.

We conclude this chapter stating a result due to A. Bondal and M. Kapranov

Proposition 1.2.9. Any Serre functor in a k-linear triangulated category is exact.

Chapter 2

Derived categories

The aim of this chapter is to construct the notion of derived category of an abelian category.

We start our presentation by recalling what a category of complexes of an abelian category is.

Definition 2.0.1. Let $\mathcal{A}$ be an abelian category, we denote with $\mathrm{Ch}(\mathcal{A})$ the category of complexes of $\mathcal{A}$. An object $A^\bullet$ consist of a diagram in $\mathcal{A}$

$$\ldots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \ldots$$

where $d^i \circ d^{i-1} = 0$. A morphism of complexes $f : A^\bullet \to B^\bullet$ consists of a collection of morphism of $\mathcal{A}$ $f^i : A^i \to B^i$ such that the following commutes

$$\begin{tikzcd}
\ldots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \ldots \\
\ldots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \ldots
\end{tikzcd}$$

Remark 2.0.2. It is straightforward to prove that a category of complexes of an abelian category $\mathcal{A}$ is again abelian. Essentially all the objects and property we need can be constructed level-wise, for example the zero object will be the complex with the object 0 at each level, the direct sum of $A^\bullet$ and $B^\bullet$ will be the complex

$$\ldots \longrightarrow A^{i-1} \oplus B^{i-1} \xrightarrow{d_A^{i-1}+d_B^{i-1}} A^i \oplus B^i \xrightarrow{d_A^i+d_B^i} A^{i+1} \oplus B^{i+1} \longrightarrow \ldots$$

and again the kernel of a morphism $f : A^\bullet \to B^\bullet$ will be the complex of kernels

$$\ldots \longrightarrow \ker(f^{i-1}) \xrightarrow{f^{i-1}} \ker(f^i) \xrightarrow{f^i} \ker(f^{i+1}) \longrightarrow \ldots$$

Moreover we can consider $\mathcal{A}$ as a full subcategory of $\mathrm{Ch}(\mathcal{A})$, the embedding is given by the functor the sends an object A of $\mathcal{A}$ in the complex concentrated in the zero level

$$\ldots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \ldots$$

Definition 2.0.3. The category of complexes comes naturally equipped with a shift functor $T : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ that sends and object $A^\bullet$ to $A^\bullet[1]$, the complex defined as $(A[1])^i :- A^{i+1}$ and $d^i_{A[1]} :- -d^{i+1}_A$. A morphism $f : A^\bullet \to B^\bullet$ is mapped to $f[1] : A^\bullet[1] \to B^\bullet[1]$, the morphism defined by $f[1]^i :- f^{i+1}$

Proposition 2.0.4. The shift functor $(-)[1] : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ defined above, is an equivalences of category.

It important to note that this shift functor does not define a triangulated structure on $\mathrm{Ch}(\mathcal{A})$, we need to provide a collection of distinguished triangles. Indeed, since the category of complexes is abelian, in general it will not be triangulated, for example the natural choice of considering a short exact sequence in $\mathrm{Ch}(\mathcal{A})$

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

as a triangle

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{0} A^\bullet[1]$$

will not satisfies the axioms of triangulated category.
The last preliminary ingredient is the fact that we can define the cohomology functor

Definition 2.0.5. Let $A^\bullet$ be in $\mathrm{Ch}(\mathcal{A})$ then its i-th cohomology is

$$\mathrm{H}^i(A^\bullet) :- \ker d^i/\mathrm{Im}(d^{i-1})$$

or more precisely $\mathrm{H}^i(A^\bullet) = \mathrm{coker}(\mathrm{Im}(d^{i-1}) \to \ker(d^i))$. Using the universal property of the coker we can define the action on morphism of the functor

$$\mathrm{H}^i : \mathrm{Ch}(\mathcal{A}) \longrightarrow \mathcal{A}$$

A complex $A^\bullet$ is acyclic if $\mathrm{H}^i(A^\bullet) = 0$ for all $i \in \mathbb{Z}$.

Having now the cohomology functor, we can define a special class of morphism called quasi-isomorphisms. This class will play the central role in the construction of the derived category, we would like to not distinguish complexes that have same cohomology without however passing to the cohomology, since this would mean loose too much information. In order to do so we will invert formally these special morphisms and consider two quasi-isomorphic complexes as isomorphic objects in the derived category.

Definition 2.0.6. A morphism of complexes $f : A^\bullet \to B^\bullet$ is called quasi-isomorphism, or quis, if for all $i \in \mathbb{Z}$ the induced map on the cohomology $\mathrm{H}^i(f) : \mathrm{H}^i(A^\bullet) \to \mathrm{H}^i(B^\bullet)$ is an isomorphism.

Let us provide some examples of quasi-isomorphisms arising in the practice.

Example 2.0.7.

1. Homotopy equivalences are clearly quis.
2. The Excision Theorem provides another example, given topological spaces $Z \subseteq A \subseteq X$ such that $\bar{Z} \subseteq \mathrm{int}(A)$, then the inclusion map

   $$i : (X/Z, A/Z) \hookrightarrow (X, A)$$

   is a quis.
3. An injective resolution $M \xrightarrow{\varepsilon} I^\bullet$

   $$\begin{tikzcd} M \arrow[r] \arrow[d, "\varepsilon"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \dots \end{tikzcd}$$

   is a quis.
   Dually, a projective resolution $P^\bullet \xrightarrow{\eta} M$

   $$\begin{tikzcd} \dots \arrow[r] & P^{-2} \arrow[r] \arrow[d] & P^{-1} \arrow[r] \arrow[d] & P^0 \arrow[d, "\eta"] \\ \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \end{tikzcd}$$

   is a quis.
4. Taking a complex $A^\bullet$ such that there exists a finite $m :- \max\{k \in \mathbb{Z} \mid \mathrm{H}^k(A^\bullet) \neq 0\}$ then there exists a quasi-isomorphism from the truncated complex to the initial one

   $$s : \sigma_{\leq m}(A^\bullet) \to A^\bullet$$

$$\begin{tikzcd}
\dots \arrow[r] & A^{m-2} \arrow[r, "d^{m-2}"] & A^{m-1} \arrow[r, "d^{m-1}"] & A^m \arrow[r, "d^m"] & A^{m+1} \arrow[r] & \dots \\
\dots \arrow[r] & A^{m-2} \arrow[r, "d^{m-2}"'] \arrow[u, "id_{A^{m-2}}"] & A^{m-1} \arrow[r, "d^{m-1}"'] \arrow[u, "id_{A^{m-1}}"] & \ker(d^m) \arrow[r] \arrow[u, hook] & 0 \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

The commutativity is either trivial or comes from the following

$$\begin{tikzcd}
A^{m-1} \arrow[rr, "d^{m-1}"] & & A^m \arrow[r, "d^m"'] & A^{m+1} \\
& \mathrm{Im}(d^{m-1}) \arrow[ur] \arrow[dr] & & \\
A^{m-1} \arrow[uu, "id_{A^{m-1}}"] \arrow[ur] \arrow[rr, "d^{m-1}"'] & & \ker(d^m) \arrow[uu, hook] \arrow[r] \arrow[ur, phantom, "\urcorner", very near start] & 0 \arrow[uu]
\end{tikzcd}$$

5. Similarly, taking a complex $A^\bullet$ such that there exists a finite $m :-$ $\min\{k \in \mathbb{Z} \mid \mathrm{H}^k(A^\bullet) \neq 0\}$ then there exists a quasi-isomorphism from the complex to its truncation

$$t : A^\bullet \to \sigma_{<m}(A^\bullet)$$

$$\begin{tikzcd}
\dots \arrow[r] & A^{m-1} \arrow[r, "d^{m-1}"] \arrow[d] & A^m \arrow[r, "d^m"] \arrow[d, two heads] & A^{m+1} \arrow[r, "d^{m+1}"] \arrow[d, "id_{A^{m+1}}"] & A^{m+2} \arrow[r] \arrow[d, "id_{A^{m+2}}"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \mathrm{coker}(d^{m-1}) \arrow[r] & A^{m+1} \arrow[r, "d^{m+1}"] & A^{m+2} \arrow[r] & \dots
\end{tikzcd}$$

It is instructive to define at first the derived category of $\mathcal{A}$ by its universal property and then provide a precise definition of what objects and morphism actually constitute the category. We will state the universal property as a theorem

Theorem 2.0.8 (Universal property of $\mathrm{D}(\mathcal{A})$)**.** Let $\mathcal{A}$ be an abelian category and $\mathrm{Ch}(\mathcal{A})$ its category of complexes, then there exists a category $\mathrm{D}(\mathcal{A})$ called the derived category of $\mathcal{A}$ and a funtor

$$Q : \mathrm{Ch}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$$

such that

i) each quasi-isomorphism in $\mathrm{Ch}(A)$ is sent to an isomorphism in $\mathrm{D}(\mathcal{A})$

ii) any functor $F : \mathrm{Ch}(\mathcal{A}) \to \mathcal{C}$ satisfying the property (i) factorizes uniquely (up to isomorphism) through Q i.e there exists a functor $G : \mathrm{D}(\mathcal{A}) \to \mathcal{C}$ that fits in the following triangle

$$\begin{tikzcd}
\mathrm{Ch}(\mathcal{A}) \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{C} \\
& \mathrm{D}(\mathcal{A}) \arrow[ur, dashed, "G"'] \arrow[u, phantom, "\cong"] &
\end{tikzcd}$$

The objects in the derived category will be the same as in the respective category of complexes, but to define morphisms in $\mathrm{D}(\mathcal{A})$ we will first need to pass to the homotopy category of complexes, where the morphisms are considered up to homotopy, and then 'localize' it by inverting the quasi-isomorphisms.

Definition 2.0.9. Two morphism in $\mathrm{Ch}(\mathcal{A})$ $f, g : A^\bullet \to B^\bullet$ are called homotopy equivalent, and we denoted it with $f \sim g$, if there exists a collection of maps $h^i : A^i \to B^{i-1}$ such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$$

depicted as follows

$$\begin{tikzcd}
\dots \arrow[r] & A^i \arrow[r, "d_A^i"] \arrow[dl, "h^i"'] \arrow[d, shift right, "f^i"'] \arrow[d, shift left, "g^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[dl, "h^{i+1}"'] \arrow[d, shift right, "f^{i+1}"'] \arrow[d, shift left, "g^{i+1}"] & A^{i+2} \arrow[dl, "h^{i+2}"] \\
B^{i-1} \arrow[r, "d_B^{i-1}"'] & B^i \arrow[r, "d_B^i"'] & B^{i+1} \arrow[r] & \dots
\end{tikzcd}$$

Then we define the homotopy category of complexes $\mathcal{K}(\mathcal{A})$ as the category whose objects are the ones in $\mathrm{Ch}(\mathcal{A})$ and morphisms $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, B^\bullet) :- \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$

Noting that the homotopy equivalence just defined is an equivalence relation is easy to show that identities and compositions are well defined and the above construction is actually a category.Moreover there exists a functor $K : \mathrm{Ch}(A) \to \mathcal{K}(\mathcal{A})$ which is the identity on objects and the quotient map on morphisms, this functor sends in particular homotopy equivalences into isomorphisms. $\mathcal{K}(\mathcal{A})$ then, comes as well with a universal property

Theorem 2.0.10 (Universal Property of $\mathcal{K}(\mathcal{A})$)**.** Let $\mathcal{A}$ be an abelian category and $\mathrm{Ch}(\mathcal{A})$ its category of complexes, then any functor $F : \mathrm{Ch}(\mathcal{A}) \to \mathcal{C}$

that sends homotopy equivalences to isomorphism in $\mathcal{C}$, factors through $\mathcal{K}(\mathcal{A})$ i.e there exists a functor $G : \mathcal{K}(A) \to \mathcal{C}$ fitting in the following triangle

$$\begin{tikzcd}
\mathrm{Ch}(\mathcal{A}) \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathcal{C} \\
& \mathcal{K}(\mathcal{A}) \arrow[ur, dashed, "G"'] &
\end{tikzcd}$$

The intermediate step in the construction of the derived category can then be express as follows, since homotopy equivalences are quasi-isomorphism and those are sent into isomorphisms in $\mathrm{D}(\mathcal{A})$, by 2.0.10 there exists a functor $Q' : \mathcal{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ making the following commute

$$\begin{tikzcd}
\mathrm{Ch}(\mathcal{A}) \arrow[rr, "Q"] \arrow[dr, "K"'] & & \mathrm{D}(\mathcal{A}) \\
& \mathcal{K}(\mathcal{A}) \arrow[ur, dashed, "Q'"'] &
\end{tikzcd}$$

We are now ready to define objects and morphisms in the derived category:

- Objects are the objects of the homotopy category

$$\mathrm{Ob}_{\mathrm{D}(\mathcal{A})} :- \mathrm{Ob}_{\mathcal{K}(\mathcal{A})} = \mathrm{Obj}_{\mathrm{Ch}(\mathcal{A})}$$

- A morphism in $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ is a set of equivalences class of diagram of the form

$$\begin{tikzcd}
& C^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\
A^\bullet & & B^\bullet
\end{tikzcd}$$

where $s : C^\bullet \to A^\bullet$ is a quasi-isomorphism and $f : C^\bullet \to B^\bullet$ is a morphism in $\mathcal{K}(\mathcal{A})$. Two such a diagrams $A^\bullet \leftarrow C_1^\bullet \to B^\bullet$, $A^\bullet \leftarrow C_2^\bullet \to B^\bullet$ are equivalent if exists a third span $C_1^\bullet \leftarrow C^\bullet \to C_2^\bullet$ such that the following commutes in $\mathcal{K}(\mathcal{A})$

$$\begin{tikzcd}
& & C^\bullet \arrow[dl, "r"'] \arrow[dr, "h"] & & \\
& C_1^\bullet \arrow[dl, "s"'] \arrow[drrr, "f"] & & C_2^\bullet \arrow[dlll, "t"'] \arrow[dr, "g"] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}$$

Here the weaker requirements of the commutativity in $\mathcal{K}(\mathcal{A})$ (i.e. up to homotopy) will became clear later in the definition of the triangulated structure, in particular since $s \circ r$ is a quis and is homotopically equivalent to the other leg $t \circ h$, then the latter is a quis as well.

- The identity morphism $A^\bullet \to A^\bullet$ is

$$\begin{tikzcd} & A^\bullet \arrow[dl, "id_{A^\bullet}"'] \arrow[dr, "id_{A^\bullet}"] & \\ A^\bullet & & A^\bullet \end{tikzcd}$$

- Given two morphisms $A^\bullet \xleftarrow{s} C_1^\bullet \xrightarrow{f} B^\bullet$ and $B^\bullet \xleftarrow{t} C_2^\bullet \xrightarrow{g} C^\bullet$, their composition is given by a commutative diagram in $\mathcal{K}(\mathcal{A})$ of the form

$$\begin{tikzcd} & & C_0^\bullet \arrow[dl] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & & C_2^\bullet \arrow[dl, "t"'] \arrow[dr, "g"] & \\ A^\bullet & & B^\bullet & & C^\bullet \end{tikzcd}$$

In order to prove the existence and the uniqueness up to homotopy of the above composition we have to introduce another concept: the mapping cone. This will solve this problem and will be a fundamental ingredient to define the triangulated structure on the derived category.

Remark 2.0.11. As a small remark let us note that the usual composition performed in a category of spans does not work in general, i.e. taking the pullback to complete the desired square is not the right choice. For example, consider the diagram

$$\begin{tikzcd} A^\bullet \arrow[dr, "f"'] & & C^\bullet \arrow[dl, "s"] \\ & B^\bullet & \end{tikzcd}$$

where $B^\bullet :- B^0 \xrightarrow{\alpha} B^1$ such that α is sujective, $A^\bullet :- B^1[-1]$ and $C^\bullet :- C^0 = \ker(\alpha)$; f arises from the identity on B^1 and s from the inclusion $\ker(\alpha) \hookrightarrow B^0$. Moreover $\mathrm{H}^0(B^\bullet) = \ker(\alpha) = \mathrm{H}^0(C^\bullet)$ and the isomorphism is realized exactly by the morphism s, which is then a quis. Completing now the diagram as a pullback (that is defined as the complex with pullback computed level-wise) results in the zero object, thus in general the resulting span is not an admissible morphism of the derived category.

Working in the homotopy category $\mathcal{K}(\mathcal{A})$ (and in $\mathrm{D}(\mathcal{A})$) there is, in general, no notion of kernel and cokernel, this is due to the fact that only the additive structure is preserved in both categories. To overcome this problem we introduce the mapping cone. Loosely this object can be thought as the homotopy cofiber C_f of a map f that arise in algebraic topology, this intuition is fruitful in the sense that all structural properties of the homotopy cofiber will hold for the mapping cone, for example taking an inclusion of CW-complexes $f : X \hookrightarrow Y$ we get that C_f is homotopy equivalent to Y/X and denoting with i the inclusion $Y \to C_f$, we obtain that C_i is homotopy equivalent to the suspension of X, ΣX and all this data fits in the sequence

$$X \xrightarrow{f} Y \longrightarrow Y/X \longrightarrow \Sigma X \longrightarrow \dots$$

Passing to the cohomology we obtain the long exact sequence of the pair (Y, X).
Let us then give the precise definition.

Definition 2.0.12. Let $f : A^\bullet \to B^\bullet$ a map between complexes, the mapping cone of f is the complex $\mathrm{cone}(f)$ defined as

$$\mathrm{cone}(f)^i :- A^{i+1} \oplus B^i$$

with differentials

$$d_f^i :- \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}$$

One can be easily check that this is actually a complex and using the universal property of the direct sum that there exist two canonical maps

$$\tau : B^\bullet \to \mathrm{cone}(f) \qquad \text{and} \qquad \pi : \mathrm{cone}(f) \to A^\bullet[1]$$

that are respectively the inclusion and the projection level-wise. Moreover both the composite $A^\bullet \to B^\bullet \to \mathrm{cone}(f)$ and $B^\bullet \to \mathrm{cone}(f) \to A^\bullet[1]$ are homotopic to the respective trivial map.

Remark 2.0.13. We actually obtain something more

$$B^\bullet \xrightarrow{\tau} \mathrm{cone}(f) \xrightarrow{\pi} A^\bullet[1]$$

is a short exact sequence of complexes that gives rise to a long exact cohomology sequence

$$\dots \longrightarrow \mathrm{H}^i(B^\bullet) \longrightarrow \mathrm{H}^i(\mathrm{cone}(f)) \longrightarrow \mathrm{H}^{i+1}(A^\bullet) \longrightarrow \mathrm{H}^{i+1}(B^\bullet) \longrightarrow \dots$$

where we used the canonical isomorphism $\mathrm{H}^i(A^\bullet[1]) \cong \mathrm{H}^{i+1}(A^\bullet)$.
Moreover, this give us the following property, $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if its mapping cone $\mathrm{cone}(f)$ is acyclic.

Remark 2.0.14. To justify the intuition of the mapping cone playing the role of both kernel and cokernel we can consider the following case. Let $A^\bullet = A$ and $B^\bullet = B$ complexes centered in degree zero and a map between them $f : A^\bullet \to B^\bullet$, then $\mathrm{cone}(f)$ is the complex

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

hence, passing to the cohomology we recover exactly $\mathrm{H}^0(\mathrm{cone}(f)) = \ker(f)$ and $\mathrm{H}^1(\mathrm{cone}(f)) = \mathrm{coker}(f)$

Lemma 2.0.15. Given a commutative square in $\mathcal{K}(\mathcal{A})$, it can be completed as following

$$\begin{tikzcd} A_1^\bullet \arrow[r, "f_1"] \arrow[d, "u"] \arrow[dr, phantom, "\circlearrowleft"] & B_1^\bullet \arrow[r, "\tau_1"] \arrow[d, "v"] & \mathrm{cone}(f_1) \arrow[r, "\pi_1"] \arrow[d, dashed, "g"] & A_1^\bullet[1] \arrow[d, "u[1]"] \\ A_2^\bullet \arrow[r, "f_2"] & B_2^\bullet \arrow[r, "\tau_2"] & \mathrm{cone}(f_2) \arrow[r, "\pi_2"] & A_2^\bullet[1] \end{tikzcd}$$

Proof. Let us denote the homotopy between the composites $v \circ f_1$ and $f_2 \circ u$ with $h^i : A_1^i \to B_2^{i-1}$, thus we have $v^i \circ f_1^i - f_2^i \circ u^i = h^{i+1} \circ d_1^i + d_2^{i-1} \circ h^i$. Then we define the required map is given by $g^i : A_1^{i+1} \oplus B_1^i \to A_2^{i+1} \oplus B_2^i$

$$g^i :- \begin{pmatrix} u^{i+1} & 0 \\ h^{i+1} & v^i \end{pmatrix}$$

It is then straightforward to check the commutativity. □

Let us now prove the fundamental result for this construction

Lemma 2.0.16. Let $f : A^\bullet \to B^\bullet$ a morphism of complexes together with its mapping cone, then there exists $g : A^\bullet[1] \to \mathrm{cone}(\tau)$ that is an isomorphism in $\mathcal{K}(\mathcal{A})$ making the following

$$\begin{tikzcd} B^\bullet \arrow[r, "\tau"] \arrow[d, "id_{B^\bullet}"'] & \mathrm{cone}(f) \arrow[r, "\pi"] \arrow[d, "id_{\mathrm{cone}(f)}"'] & A^\bullet[1] \arrow[r, "-f"] \arrow[d, dashed, "g"] & B^\bullet[1] \arrow[d, "id_{B^\bullet[1]}"] \\ B^\bullet \arrow[r, "\tau"] & \mathrm{cone}(f) \arrow[r, "\tau_\tau"] & \mathrm{cone}(\tau) \arrow[r, "\pi_\tau"] & B^\bullet[1] \end{tikzcd}$$

commutes in $\mathcal{K}(\mathcal{A})$.

Proof. Since $\mathrm{cone}(\tau)^i = B^{i+1} \oplus \mathrm{cone}(f)^i = B^{i+1} \oplus A^{i+1} \oplus B^i$, we define $g : A^\bullet[1] \to \mathrm{cone}(\tau)$ by the universal property of the direct sum by the maps $g^i := \langle -f^{i+1}, id_{A^i}, 0\rangle$. The inverse up to homotopy is the given by taking the projection on the middle term, i.e. $(g^{-1})^i :- \pi_{A^{i+1}} : \mathrm{cone}(\tau)^i \to A^{i+1}$. The homotopy between $id_{\mathrm{cone}(\tau)}$ and $g \circ g^{-1}$ is given by the maps $h^i : B^{i+1} \oplus A^{i+1} \oplus B^i \to B^i \oplus A^i \oplus B^{i-1}$ defined as

$$h^i :- \begin{pmatrix} 0 & 0 & id_{B^i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now to check the commutativity of the diagram, we note that

$$\begin{tikzcd} A^\bullet[1] \arrow[r, "-f"] \arrow[d, "g"] & B^\bullet[1] \arrow[d, "id_{B^\bullet[1]}"] \\ \mathrm{cone}(\tau) \arrow[r, "\pi_\tau"] & B^\bullet[1] \end{tikzcd}$$

commutes strictly in $\mathrm{Ch}(\mathcal{A})$ by definition of g. Then

$$\begin{tikzcd} \mathrm{cone}(f) \arrow[r, "\pi"] \arrow[dr, "\tau_\tau"'] & A^\bullet[1] \arrow[d, "g"] \\ & \mathrm{cone}(\tau) \end{tikzcd}$$

commutes only up to homotopy, i.e. in $\mathcal{K}(\mathcal{A})$, by noticing that $\tau_\tau \sim g \circ g^{-1} \circ \tau_\tau$ and that $g^{-1} \circ \tau_\tau = \pi$. ☐

Proposition 2.0.17. Let $f : A^\bullet \to B^\bullet$ be a quis and $g : C^\bullet \to B^\bullet$ a morphism, then there exists a commutative diagram in $\mathcal{K}(\mathcal{A})$

$$\begin{tikzcd} C_0^\bullet \arrow[r, dashed, "s"] \arrow[d, dashed] & C^\bullet \arrow[d, "g"] \\ A^\bullet \arrow[r, "f"] & B^\bullet \end{tikzcd}$$

such that the morphism s is a quasi-isomorphism.

Proof. By Lemma 2.0.16 we obtain an isomorphism $\mathrm{cone}(\tau) \to A^\bullet[1]$ that fits in the commutative diagram

$$\begin{tikzcd}
\operatorname{cone}(\tau \circ g)[-1] \arrow[r, "\sigma"] \arrow[d, dashed] & C^\bullet \arrow[r, "\tau\circ g"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{cone}(f) \arrow[r] \arrow[d, "id_{\operatorname{cone}(f)}"] & \operatorname{cone}(\tau \circ g) \arrow[d, dashed] \\
\operatorname{cone}(\tau)[-1] \arrow[r] \arrow[d, "\cong"] & B^\bullet \arrow[r, "\tau"] \arrow[d, "id_{B^\bullet}"] & \operatorname{cone}(f) \arrow[r] \arrow[d, "id_{\operatorname{cone}(f)}"] & \operatorname{cone}(\tau) \arrow[d, "\cong"] \\
A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "\tau"] & \operatorname{cone}(f) \arrow[r, "\pi"] & A^\bullet[1]
\end{tikzcd}$$

the dashed arrows are obtained by applying Lemma 2.0.15 to the commutative square on top. The only thing to check is if σ is actually a quis. Passing to the long exact cohomology sequence on the top row and using the fact that f is a quis by hypothesis, hence its mapping cone cone(f) is acyclic, we get that for each $i \in \mathbb{Z}$

$$\mathrm{H}^i(\operatorname{cone}(\tau \circ g)[-1]) \xrightarrow{\mathrm{H}^i(\sigma)} \mathrm{H}^i(C^\bullet)$$

is an isomorphism. Thus σ is a quis and we define $C_0^\bullet :- \operatorname{cone}(\tau \circ g)[-1]$. □

Corollary 2.0.18. The composition of morphisms $A^\bullet \overset{s}{\leftarrow} C_1^\bullet \overset{f}{\rightarrow} B^\bullet$ and $B^\bullet \overset{t}{\leftarrow} C_2^\bullet \overset{g}{\rightarrow} C^\bullet$ in the derived category D$(\mathcal{A})$ is then defined as

$$\begin{tikzcd}
& \operatorname{cone}(\tau_t \circ f)[-1] \arrow[dl, "s\circ\sigma"'] \arrow[dr] & \\
A^\bullet & & C^\bullet
\end{tikzcd}$$

2.1 Derived categories are triangulated

We are now ready to define the triangulated structure on the derived category D$(\mathcal{A})$ (the construction we are giving will works also for the homotopy category $\mathcal{K}(\mathcal{A})$). As mention before the shift functor inherited by the one on the category of complexes Ch$(\mathcal{A})$ is the suspension functor on the derived category. Let us then define the collection of distinguished triangles.

Definition 2.1.1. A triangle of the form

$$A_1^\bullet \longrightarrow B_1^\bullet \longrightarrow C_1^\bullet \longrightarrow A_1^\bullet[1]$$

is a distinguished triangle in D$(\mathcal{A})$ (respectively in $\mathcal{K}(\mathcal{A})$) if it is isomorphic in D$(\mathcal{A})$ (respectively in $\mathcal{K}(\mathcal{A})$) to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow \operatorname{cone}(f) \longrightarrow A^\bullet[1]$$

Remark 2.1.2. Even if in the homotopy category (and in the derived category) distinguished triangles play the role of short exact sequences, one could then be tempted to embed a short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ into a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$, but this is not true in general. For example, let us consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

we have then that $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}[1]$ must be isomorphic to $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathrm{cone}(\cdot 2) \longrightarrow \mathbb{Z}/2\mathbb{Z}[1]$, so in particular must exists an isomorphism in $\mathcal{K}(\mathcal{A})$ between $\mathbb{Z}/2\mathbb{Z} \cong \mathrm{cone}(\cdot 2)$, there is no hope to find such isomorphism up to homotopy since the only to possible morphisms of complexes $\mathbb{Z}/2\mathbb{Z} \to \mathrm{cone}(\cdot 2)$ are either the trivial map or the multiplication by 2 and the only possible morphism of complex in the opposite direction is the trivial one.
This problem can be partially solved when we consider the category $\mathrm{D}(\mathcal{A})$, i.e. we have a canonical way to embed a short exact sequence into a distinguished triangle. We can take a projective resolution of $\mathbb{Z}/2\mathbb{Z}$

$$P^\bullet : \qquad 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and, as we will show, there is an isomorphism $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(\mathbb{Z}/2\mathbb{Z}, C^\bullet) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(P^\bullet, C^\bullet)$ where $C^\bullet :- \mathrm{cone}(\cdot 2)$. Now the following is a quis

$$\begin{tikzcd}
P^\bullet : \arrow[d] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] & 0 \\
C^\bullet & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z}/4\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

So the required distinguished triangle is $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \longrightarrow P^\bullet \longrightarrow \mathbb{Z}/2\mathbb{Z}[1]$.

In this section we are going to verify that the above definition gives actually the required triangulated structure on the derived category, we will just sketch the proof and refer for a fully detailed one to [GM03] IV. 2. The only technical difficulty is the fact that we are dealing with morphism expressed as span with the left leg a quis, moreover we are giving for granted the proof of the triangulated structure on the homotopy category $\mathcal{K}(\mathcal{A})$, in this direction Lemma 2.0.16 and Lemma 2.0.15 give essentially a proof for TR2 and TR3 respectively.

Theorem 2.1.3. Let $\mathcal{A}$ be an abelian category, then the derived category $\mathrm{D}(\mathcal{A})$ with the shift functor inherited by the category of complexes and the collection of distinguished triangle in 2.1.1 is a triangulated category.

Proof. (*Sketch of proof*)
TR1
Considering a morphism $f : A^\bullet \to B^\bullet$ in $\mathrm{D}(\mathcal{A})$, i.e. a span of the form $A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f'} B^\bullet$, we can completed f' to a distinguished triangle

$$\begin{tikzcd} C^\bullet \arrow[r, "f'"] & B^\bullet \arrow[r, "g"] & \mathrm{cone}(f') \arrow[r, "h"] & C^\bullet[1] \end{tikzcd}$$

and then using that s is a isomorphism in $\mathrm{D}(\mathcal{A})$ we obtain the following isomorphism of triangles

$$\begin{tikzcd} C^\bullet \arrow[r, "f'"] \arrow[d, "s"'] & B^\bullet \arrow[r, "g"] \arrow[d, "id_{B^\bullet}"] & \mathrm{cone}(f') \arrow[r, "h"] \arrow[d, "id_{\mathrm{cone}(f')}"] & C^\bullet[1] \arrow[d, "s{[1]}"] \\ A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "g"] & \mathrm{cone}(f') \arrow[r, "s{[1]}\circ h"] & A^\bullet[1] \end{tikzcd}$$

TR2
Given a distinguished triangle

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, "\alpha"', "\cong"] & B^\bullet \arrow[r, "g"] \arrow[d, "\beta"', "\cong"] & C^\bullet \arrow[r, "h"] \arrow[d, "\gamma"', "\cong"] & A^\bullet[1] \arrow[d, "\alpha{[1]}"', "\cong"] \\ A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "\tau"] & \mathrm{cone}(f) \arrow[r] & A^\bullet[1] \end{tikzcd}$$

then the following is the rotated distinguished triangle

$$\begin{tikzcd} B^\bullet \arrow[r, "g"] \arrow[d, "\beta"', "\cong"] & C^\bullet \arrow[r, "h"] \arrow[d, "\gamma"', "\cong"] & A^\bullet[1] \arrow[r, "-f{[1]}"] \arrow[d, "i_{A^\bullet[1]}\circ\alpha"', "\cong"] & B^\bullet[1] \arrow[d, "\beta{[1]}"', "\cong"] \\ B^\bullet \arrow[r, "\tau"] & \mathrm{cone}(f) \arrow[r] & \mathrm{cone}(\tau) \arrow[r] & B^\bullet[1] \end{tikzcd}$$

TR3
Taking two distinguished triangles in $\mathrm{D}(\mathcal{A})$ with f, g in $\mathrm{D}(\mathcal{A})$ we have to find a morphism h such that the following commutes

$$\begin{tikzcd} A_1^\bullet \arrow[r, "u"] \arrow[d, "f"] & A_2^\bullet \arrow[r, "v"] \arrow[d, "g"] & A_3^\bullet \arrow[r, "w"] \arrow[d, dashed, "h"] & A_1^\bullet[1] \arrow[d, "f{[1]}"] \\ B_1^\bullet \arrow[r, "u'"] & B_2^\bullet \arrow[r, "v'"] & B_3^\bullet \arrow[r, "w''"] & B_1^\bullet[1] \end{tikzcd}$$

This amounts to find a quis r and a morphism h' in $\mathcal{K}(\mathcal{A})$ such that the following commutes

$$\begin{tikzcd}
 & C_1^\bullet \arrow[r, "u''"] \arrow[dl, "s"'] \arrow[ddl, "f'"] & C_2^\bullet \arrow[rr, "v''"] \arrow[dl, "t"'] \arrow[ddl, "g'"] & & C_3^\bullet \arrow[rr, "w''"] \arrow[dl, dotted, "r"'] \arrow[ddl, dashed, "h'"] & & C_1^\bullet[1] \arrow[dl, "{s[1]}"'] \arrow[ddl, "{f'[1]}"] \\
A_1^\bullet \arrow[r] & A_2^\bullet \arrow[rr] & & A_3^\bullet \arrow[rr] & & A_1^\bullet[1] & \\
B_1^\bullet \arrow[r] & B_2^\bullet \arrow[rr] & & B_3^\bullet \arrow[rr] & & B_1^\bullet[1] &
\end{tikzcd}$$

Up to replace conveniently $C_1^\bullet$ and $C_2^\bullet$ in such a way that the obtained spans still represent the morphisms f and g respectively, we can ensure the existence of a morphism $u'' : C_1^\bullet \to C_2^\bullet$ making the front and the back square commute. Now we can complete u'' to a distinguished triangle

$$C_1^\bullet \xrightarrow{u''} C_2^\bullet \xrightarrow{v''} C_3^\bullet \xrightarrow{w''} C_1^\bullet[1]$$

and use twice TR3 for the homotopy category $\mathcal{K}(\mathcal{A})$ to get both the required morphisms h' and r, where r is a quis since both s and t are.

TR4

Considering an upper cap in $\mathrm{D}(\mathcal{A})$

$$\begin{tikzcd}
A' \arrow[dd] \arrow[dr] & & C \arrow[ll] \\
 & B \arrow[ur] \arrow[dl] \arrow[u, phantom, "\star"] \arrow[d, phantom, "\star"] & \\
C' \arrow[rr] & & A \arrow[uu] \arrow[ul]
\end{tikzcd}$$

where the triangles decorated with a star are distinguished triangles and the remaining ones are commutative, then considering each morphism appearing above as a spans we can again modifying conveniently them (without changing the morphism represented) in order to find an isomorphism between the cap above and a cap in $\mathcal{K}(\mathcal{A})$, i.e. isomorphisms of (distinguished) triangles that preserves the structure. Using now TR4 in the homotopy category we can complete the new upper cap to an octahedron and then extend its lower cap to a lower cap in $\mathrm{D}(\mathcal{A})$. □

Theorem 2.1.4. *Let $\mathcal{A}$ an abelian category, then the functor defined in 2.0.10*

$$Q_{\mathcal{A}} : \mathcal{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$$

is an exact functor of triangulated categories.

The complexes we consider as objects of the categories $\mathcal{K}(\mathcal{A})$ and $\mathrm{D}(\mathcal{A})$ were in general unbounded, but it is important to have a bounded version of this categories. We now define and give a characterization of the derived categories were objects are bounded complexes.

Definition 2.1.5. Let $\mathcal{A}$ be abelian, we denote $\mathrm{Ch}^*(\mathcal{A})$ where $* = b,\ +,\ -$ for the category of complexes where objects are complexes respectively bounded, bounded below or bounded above.
We denote $\mathcal{K}^*(\mathcal{A})$ for the homotopy category of $\mathrm{Ch}^*(\mathcal{A})$ and $\mathrm{D}^*(\mathcal{A})$ for the localization at quasi-isomorphisms of $\mathcal{K}^*(\mathcal{A})$ where $* = b,\ +,\ -$.

The categories are also $\mathcal{K}^*(\mathcal{A})$ and $\mathrm{D}^*(\mathcal{A})$ examples of triangulated subcategories. We will denote as well the functor $Q_{\mathcal{A}} : \mathcal{K}^*(\mathcal{A}) \to \mathrm{D}^*(\mathcal{A})$.

Proposition 2.1.6. The inclusion functor $\mathrm{D}^*(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$, where $* = b,\ +,\ -$, defines an equivalence between $\mathrm{D}^*(\mathcal{A})$ and the full subcategory of $\mathrm{D}(\mathcal{A})$ of complexes $A^\bullet$ such that there exists k_0 such that $H^k(A^\bullet) = 0$ for $|k| > k_0$, $k < k_0$ and $k > k_0$ respectively.

Proof. The proof essentially follows from the quasi-isomorphisms defined in 2.0.7, for example taking $A^\bullet$ in $\mathrm{D}(\mathcal{A})$ such that there exists a finite m such that $H^k(A^\bullet) = 0$ for $k > m$ then we can consider the quis $s : \sigma_{\leq m}(A^\bullet) \to A^\bullet$ where $\sigma_{\leq m}(A^\bullet)$ is in $\mathrm{D}^-(\mathcal{A})$. □

Lemma 2.1.7. Let $A^\bullet$ be a complex in $\mathrm{D}(\mathcal{A})$ such that there exists a finite $m :- \max\{k \in \mathbb{Z} \mid \mathrm{H}^k(A^\bullet) \neq 0\}$, then there exists a morphism in $\mathrm{D}(\mathcal{A})$

$$\varphi : A^\bullet \to \mathrm{H}^m(A^\bullet)[-m]$$

such that $\mathrm{H}^m(\varphi)$ is isomorphic to the identity. Dually, if $A^\bullet$ is a complex in $\mathrm{D}(\mathcal{A})$ such that there exists a finite $m :- \min\{k \in \mathbb{Z} \mid \mathrm{H}^k(A^\bullet) \neq 0\}$, then there exists a morphism in $\mathrm{D}(\mathcal{A})$

$$\psi : \mathrm{H}^m(A^\bullet)[-m] \to A^\bullet$$

such that $\mathrm{H}^m(\varphi)$ is isomorphic to the identity.

Proof. Let $A^\bullet$ a complex as in the hypothesis and we have the quis $s : \sigma_{\leq m}(A^\bullet) \to A^\bullet$, then we can define another morphism $\pi : \sigma_{\leq m}(A^\bullet) \to \mathrm{H}^m(A^\bullet)[-m]$

$$\begin{tikzcd}
\dots \arrow[r] & A^{m-1} \arrow[r, "d^{m-1}"] \arrow[d, "0"] & \ker(d^m) \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d, "0"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \ker(d^m)/\mathrm{Im}(d^{m-1}) \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

The require morphism φ in the derived category is the represented by the span

$$\begin{tikzcd} & \sigma_{\leq m}(A^\bullet) \arrow[dl, "s"'] \arrow[dr, "\pi"] & \\ A^\bullet & & \mathrm{H}^m(A^\bullet)[-m] \end{tikzcd}$$

Now $\mathrm{H}^m(s)$ is an isomorphism and $\mathrm{H}^m(\pi)$ is the identity. The dual statement is completely similar. □

We conclude this section with a focus on the interaction between the concept of abelian and triangulated categories. We have explained how, in a loosely way, a distinguished triangle is a generalization of a short exact sequence, although these two notions are competing, i.e. if a triangulated category is in addition abelian then all short exact sequences must split. An abelian category in which every short exact sequence split is called semi-simple. For example the category of finite-dimensional k-vector spaces is semi-simple, instead the category Ab is not semi-simple, e.g. consider the not-splitting short exact sequence in 2.1.2.

Proposition 2.1.8. Let $\mathcal{A}$ be a triangulated category, if $\mathcal{A}$ is abelian then it is semi-simple and any distinguished triangle is of the form

$$X \xrightarrow{f} Y \longrightarrow \ker(f)[1] \oplus \operatorname{coker}(f) \longrightarrow X[1]$$

Proof. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $\mathcal{A}$, then it is easy to check that f is a monomorphism and g is an epimorphism. By TR1 we can complete f to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{\alpha} C \xrightarrow{\beta} T(X)$$

And rotating it by TR2 we obtain

$$T^{-1}(C) \xrightarrow{T^{-1}(\beta)} X \xrightarrow{f} Y \xrightarrow{\alpha} C$$

Since the triangle is again distinguished $f \circ T^{-1}(\beta) = 0$, but f is a mono, hence $T^{-1}(\beta) = 0$ and so $\beta = 0$. We obtained a distinguished triangle with a trivial morphism then by 1.2.5 for the triangulated category $\mathrm{D}(\mathcal{A})$ the triangle splits and in particular $Y \cong X \oplus C$. □

2.2 Derived Functors

An additive functor between abelian categories $F : \mathcal{A} \to \mathcal{B}$ can be naturally extended to an additive functor between the respective categories of complexes $\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$, for the same reasons, F preserves homotopy equivalences between chain maps so it can be extended as well to a functor between the homotopy categories $\mathcal{K}(F) : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$. Moreover F commutes with the shift functor and $F(\mathrm{cone}(A^\bullet \xrightarrow{f} B^\bullet)) \cong \mathrm{cone}(F(A^\bullet) \xrightarrow{F(f)} F(B^\bullet))$, so F sends distinguished triangle in $\mathcal{K}(\mathcal{A})$ to distinguished triangles in $\mathcal{K}(\mathcal{B})$, i.e. $\mathcal{K}(F)$ is an exact functor of triangulated categories. All these assignments are functorial and would then be natural to ask the functionality of the assignment that sends a category of complexes of an abelian category into its derived category. Unfortunately this does not happen in general, when F is exact it easy to extend it to a functor $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$, but many of the functors we would like to consider, e.g. the hom-functor, the tensor product, the global section functor and the direct image, are either just left or right exact. Under the assumption of F being left (or right) exact we will be able to define a functor between the derived categories $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ which will be called right (or left) derived functor.

All the results presented in this section have clearly a dual statement, for the sake of readability we will discuss only one side of the story and will leave the dualizing process to the careful reader.

In order to make precise this discussion we first prove some results that will make calculations on the derived category easier, we give conditions that allows to find an injective (respectively projective) resolution for objects in the derived category and so to work with better-behave objects.

Definition 2.2.1. An abelian category $\mathcal{A}$ has enough injectives if for any object A there exists an injective object I and a monomorphism $A \to I$
Dually, $\mathcal{A}$ has enough projectives if for any object A there exists a projective object P and an epimorphism $P \to A$

If $\mathcal{A}$ contains enough injectives any object A has an injective resolution. Following almost the same strategy we can prove that a similar results holds for an a complex in the homotopy category.

Proposition 2.2.2. Let $\mathcal{A}$ be an abelian category with enough injectives, then for any complex $A^\bullet$ in $\mathcal{K}^+(\mathcal{A})$ there exists a complex of injectives $I^\bullet$ and a quasi-isomorphism $A^\bullet \to I^\bullet$.

Proof. We construct the complex of injectives and the quasi-isomorphism inductively. Since $A^\bullet$ is bounded below we can assume, for simplicity, that

$A^i = 0$ for $i < 0$. Since $\mathcal{A}$ has enough injectives, there exists an injective object I^0 and a monomorphism $i^0 : A^0 \to I^0$, we then construct the next object of the complex by taking the following push-out square and apply the same reasoning to $I^0 \coprod_{A^0} A^1$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A^0 \arrow[r, "d_A^0"] \arrow[d, "i^0"] & A^1 \arrow[r] \arrow[d, dashed, "v^1"] & \dots \\
0 \arrow[r] & I^0 \arrow[r, dashed, "u^1"] & I^0 \coprod_{A^0} A^1 \arrow[dr, "j^1"] & \\
 & & & I^1
\end{tikzcd}$$

where we define $i^1 :- j^1 \circ v^1$ and the differential $d_I^0 :- j^1 \circ u^1$. The inductive step follows the same strategy of the zero step above with a small variant in order to make sure that what we obtain is actually a complex. Given I^i we proceed as follows

$$\begin{tikzcd}
 & A^i \arrow[r, "d_A^i"] \arrow[dl, "i^i"'] \arrow[d, "p"] & A^{i+1} \arrow[d, "v^{i+1}"] & \\
I^i \arrow[r, dashed, "q"] & \operatorname{coker}(d_A^i) \arrow[r, "u^{i+1}"] & \operatorname{coker}(d_A^i) \coprod A^{i+1} \arrow[r, "j^{i+1}"] & I^{i+1}
\end{tikzcd}$$

where the morphism $q : I^i \to \operatorname{coker}(d_A^i)$ is defined by the universal property of I^i, we then define $d_I^{i+1} :- j^{i+1} \circ u^{i+1} \circ q$ and $i^{i+1} j^{i+1} \circ v^{i+1}$. With this definition is now clear that $d_I^{i+1} \circ d_I^i = 0$. To show that $i : A^\bullet \to I^\bullet$ is a quis we use again induction. With the first step we defined a morphism of complex

$$t_0 : A^\bullet \to (\dots \to 0 \to I^0 \to 0 \to \dots)$$

where $\mathrm{H}^k(t_0)$ is a monomorphism by definition for $k = 0$ and it is trivial an isomorphism for $k < 0$. Now, given the morphism

$$t_i : A^\bullet \to (\dots \to I^{i-1} \to I^i \to 0 \to \dots)$$

such that, by inductive hypothesis, $\mathrm{H}^k(t_i)$ is a monomorphism for $k = i$ and an isomorphism for $k < i$, we have constructed I^{i+1} and the morphisms d_I^i and i^{i+1} such that for the resulting morphism

$$t_{i+1} : A^\bullet \to (\dots \to I^{i-1} \to I^i \to I^{i+1} \to 0 \to \dots)$$

holds $\mathrm{H}^k(t_{i+1})$ is a monomorphism for $k = i + 1$ and an isomorphism for $k < i + 1$.

□

Lemma 2.2.3. Let $A^\bullet$ and $B^\bullet$ in $\mathcal{K}^+(\mathcal{A})$ and consider $q : A^\bullet \to B^\bullet$ a quis, then for any complex of injectives $I^\bullet$ in $\mathcal{K}^+(\mathcal{A})$ the induce map on the hom-set

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(B^\bullet, I^\bullet) \xrightarrow{\quad -\circ q \quad} \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, I^\bullet)$$

is a natural isomorphism

As a direct consequence we obatin the following

Lemma 2.2.4. Let $A^\bullet, I^\bullet$ in $\mathrm{Ch}^+(\mathcal{A})$ where $I^\bullet$ is a complex of injectives, then the natural map

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, I^\bullet) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, I^\bullet)$$

is an isomorphism

Proof. The natural map send a morphism $f : A^\bullet \to I^\bullet$ in $\mathcal{K}(\mathcal{A})$ to the span $A^\bullet \xleftarrow{id_{A^\bullet}} A^\bullet \xrightarrow{f} I^\bullet$ in $\mathrm{D}(\mathcal{A})$ is clearly injective. Now taking a morphism in $\mathrm{D}(\mathcal{A})$ represented by the span

$$\begin{tikzcd} & B^\bullet \arrow[dl, "q"'] \arrow[dr, "f"] & \\ A^\bullet & & I^\bullet \end{tikzcd}$$

it corresponds, by Lemma 2.2.3, to a unique morphism $g : A^\bullet \to I^\bullet$ such that $g \circ q \sim f$, i.e. that fills the diagram up to homotopy. □

Let us consider now the full additive subcategories $\mathcal{I}$ of $\mathcal{A}$ of injectives objects, we can now define, following the same procedure, its homotopy category $\mathcal{K}^*(\mathcal{I})$ that is again a triangulated category. Dually we consider the category $\mathcal{K}^*(\mathcal{P})$, where $\mathcal{P}$ is the full additive subcategory of $\mathcal{A}$ of projectives objects. We obtain two obvious functor $\iota : \mathcal{K}^+(\mathcal{I}) \to \mathrm{D}^+(\mathcal{A})$ and $\pi : \mathcal{K}^-(\mathcal{P}) \to \mathrm{D}^-(\mathcal{A})$ given by composing the functor $Q^*_{\mathcal{A}}$ with the inclusions of the homotopy categories of complexes of injectives and projectives respectively. We can now prove the fundamental theorem that allow us the define derived functors.

Theorem 2.2.5. If $\mathcal{A}$ has enough injectives then the functor

$$\iota : \mathcal{K}^+(\mathcal{I}) \to \mathrm{D}^+(\mathcal{A})$$

is an equivalence of categories.

Proof. Let us firstly note that ι is an exact functor of triangulated categories since is the composite of two exact functors. We then need to show that the action of the functor on morphisms

$$\mathrm{Hom}_{\mathcal{K}^+(\mathcal{I})}(I^\bullet, J^\bullet) \to \mathrm{Hom}_{\mathrm{D}^+(\mathcal{A})}(I^\bullet, J^\bullet)$$

is an isomorphism. But this is exactly what we proved in Lemma 2.2.4. By Proposition 2.2.2, for any complex $A^\bullet$ in $\mathcal{K}^+(\mathcal{A})$, hence for any complex in $\mathrm{D}(\mathcal{A})$, there exists a quis to into a complex of injectives $q : A^\bullet \to I^\bullet$, but this gives an isomorphism in the bounded derived category between $A^\bullet$ and a bounded below complex of injectives, thus ι is essentially surjective. □

As noted above, if a functor between abelian categories $F : \mathcal{A} \to \mathcal{B}$ is exact, then we can define in a functorial way an exact functor (of triangulated categories) between the derived categories $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$. Moreover, consider a generic exact functor of triangulated categories $G : \mathcal{K}^*(\mathcal{A}) \to \mathcal{K}^*(\mathcal{B})$, not necessarily coming from a functor between the abelian categories, it can be extended to an exact functor of triangulated categories $\mathrm{D}^*(\mathcal{A}) \to \mathrm{D}^*(\mathcal{B})$ if either sends quis to quis, using the universal properties of $\mathrm{D}^*(\mathcal{A})$

$$\begin{tikzcd}
\mathcal{K}^*(\mathcal{A}) \arrow[rr, "G"] \arrow[dr, "Q_{\mathcal{A}}"'] & & \mathcal{K}^*(\mathcal{B}) \arrow[r, "Q_{\mathcal{B}}"] & \mathrm{D}^*(\mathcal{B}) \\
& \mathrm{D}^*(\mathcal{A}) \arrow[ur, dashed] & &
\end{tikzcd}$$

Alternatively the same result holds if the functor G sends acyclic complexes to acyclic complex, since an acyclic complex $A^\bullet$ in $\mathcal{K}^*(\mathcal{A})$ is quasi-isomorphic to the zero complex, then, in order to obtain the required extension, it must be sent by G to a trivial complex in $\mathrm{D}^*(\mathcal{B})$ i.e. to an acyclic one. Let us collect these results in a Proposition.

Proposition 2.2.6. Let $G : \mathcal{K}^*(\mathcal{A}) \to \mathcal{K}^*(\mathcal{B})$ be an exact functor of triangulated categories, then if one of the following conditions holds

i) G map quasi-isomorphisms to quasi-isomorphisms.

ii) If $A^\bullet$ in $\mathcal{K}^*(\mathcal{A})$ is acyclic, then $G(A^\bullet)$ is acyclic in $\mathcal{K}(\mathcal{B})$.

G induces an exact functor between the derived categories that fits in the following commutative diagram

$$\begin{tikzcd}
\mathcal{K}^*(\mathcal{A}) \arrow[r, "G"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathcal{K}^*(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\
\mathrm{D}^*(\mathcal{A}) \arrow[r, dashed] & \mathrm{D}^*(\mathcal{B})
\end{tikzcd}$$

Let now take a left exact functor between abelian categories $F : \mathcal{A} \to \mathcal{B}$ then the functor obtained by applying point-wise F to the complex of injectives, induced an exact functor of triangulated category $K(F) : \mathcal{K}^+(\mathcal{I}_A) \to \mathcal{K}^+(\mathcal{B})$.

Definition 2.2.7. Let $\mathcal{A}$ an abelian category with enough injectives and $F : \mathcal{A} \to \mathcal{B}$ a left exact functor, we define the right derived functor to be the composite

$$RF :- Q_{\mathcal{B}} \circ \mathcal{K}(F) \circ \iota^{-1} : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$$

given by

$$\begin{tikzcd}
\mathcal{K}(\mathcal{I}_{\mathcal{A}}) \arrow[r, hook] \arrow[dr, "\iota"] & \mathcal{K}^*(\mathcal{A}) \arrow[r, "\mathcal{K}^+(F)"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathcal{K}^*(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\
 & \mathrm{D}^*(\mathcal{A}) \arrow[r, dashed, "RF"] \arrow[ul, bend left, "\iota^{-1}"] & \mathrm{D}^*(\mathcal{B})
\end{tikzcd}$$

where ι^{-1} is the weak inverse of the equivalence defined in 2.2.5.

The right derived functor is clearly exact functor of triangulated categories since $Q_{\mathcal{B}}$ and $\mathcal{K}(F)$ are exact, and ι^{-1} is the weak inverse of an exact functor, thus exact as well. Hence RF is the composite of three exact functors. Dually if we start with a right exact functor $F : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ enough projectives, we can define in a completely similar manner, the left derived functor of F denoted with $LF : \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$.

Definition 2.2.8. Let $F : \mathcal{A} \to \mathcal{B}$ a left exact funcor and $RF : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ its right derived functor, we can define the classical (or higher) derived functor by

$$R^iF(A^\bullet) = \mathrm{H}^i(RF(A^\bullet))$$

for $\mathcal{A}^\bullet$ in $\mathrm{D}^+(\mathcal{A})$.

Remark 2.2.9. If we consider the induced functor at the level of the abelian categories we obtain the classical derived functors

$$R^iF : \mathcal{A} \to \mathcal{B}$$

that sends A in $\mathcal{A}$ into $R^i(A) = \mathrm{H}^i(RF(A)) = \mathrm{H}^i(F(I^\bullet))$ where $A \to I^\bullet$ is an injective resolution. Moreover a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ yields a long exact sequence in $\mathcal{B}$

$$\cdots \to R^iF(A) \to R^iF(B) \to R^iF(C) \to R^{i+1}F(A) \to R^{i+1}F(B) \to \ldots$$

It is natural then to ask how the functorial construction we made behaves with composition, what we expect would be a relation of the form

$$R(G \circ F) \cong RG \circ RF$$

This does not happen straightforward, the problem is that some left exact functors do not take injectives to injectives, but this can be fixed by enlarge the class to objects acyclic for the two composable functors respectively.

Example 2.2.10. Let X, Y and Z noetherian schemes and $X \xrightarrow{f} Y \xrightarrow{g} Z$ morphisms between them. Considering the abelian category of quasi-coherent sheaves and the push-forward functors given by f and g, we know that $g_* \circ f_* \cong (g \circ f)_*$, but f_* dose not take injectives to injectives. Thus without any other assumption we cannot compose $\mathcal{K}(g_*)$ and $\mathcal{K}(f_*)$ since the latter has codomain $\mathcal{K}(\mathsf{QCoh}(Y))$ and cannot be restricted in general to $\mathcal{K}(\mathcal{I}_Y)$.

Definition 2.2.11. Given a left exact functor $F : \mathcal{A} \to \mathcal{B}$, an object A in $\mathcal{A}$ is F-acyclic if $R^i F(A) = 0$ for $i > 0$.

Definition 2.2.12. Let $F : \mathcal{A} \to \mathcal{B}$ a left exact functor then a class of object $\mathcal{I}_F \subset \mathcal{A}$ is called F-adapted if the following conditions are satisfied

i) It is closed under direct sums.

ii) If $A^\bullet$ in $\mathcal{K}^+(\mathcal{A})$ is acyclic with A^i in $\mathcal{I}_F$ for all i, then $F(A^\bullet)$ is acyclic.

iii) Any object of $\mathcal{A}$ can be embedded in an object of the adapted class $\mathcal{I}_F$.

Remark 2.2.13. Given a left exact functor $F : \mathcal{A} \to \mathcal{B}$ and its adapted class $\mathcal{I}_F$, the localization $\mathcal{K}^+(\mathcal{I}_F) \to \mathrm{D}^+(\mathcal{A})$ by inverting quasi-isomorphisms is then an equivalence of categories by condition (iii) of 2.2.12. Moreover, taking a quis f of complexes with elements in $\mathcal{I}$, its mapping cone is again in $\mathcal{K}(\mathcal{I}_F)$ by condition (i), thus cone(f) is acyclic and is mapped by F into an acyclic complex by condition (ii). This then ensures that quasi-isomorphism are mapped by F into quasi-isomorphism and allows us to define the right derived functor in a completely similar way with the respect to the F-adapted class.

The adapted class are a fruitful extension of the class of injectives, if we make the assumption of Definition 2.2.7, i.e. that $\mathcal{A}$ contains enough injectives, we obtain that the class of injective objects $\mathcal{I}_\mathcal{A}$ is F-adapted for any left exact functor F, and even further throwing inside $\mathcal{I}_\mathcal{A}$ all the F-acyclic objects yields to a larger F-adapted class.

We can now make precise the discussion above on the composition of derived functors.

Theorem 2.2.14. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ two left exact functors between abelian categories. If there exist adapted classes $\mathcal{I}_F \subset \mathcal{A}$ and $\mathcal{I}_G \subset \mathcal{B}$ respectively for F and G, such that $F(\mathcal{I}_F) \subset \mathcal{I}_G$, then the right derived functor $R(G \circ F) : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ exists and there is a natural isomorphism

$$R(G \circ F) \cong RG \circ RF$$

Proof. By Remark 2.2.13, both derived functors $RF : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ and $RG : \mathrm{D}^+(\mathcal{B}) \to \mathrm{D}^+(\mathcal{C})$ are well defined, the hypothesis $F(\mathcal{I}_F) \subset \mathcal{I}_G$ ensures, for the same reasons, that also the right derived functor of the composition $R(G \circ F) : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{C})$ is well defined. The natural morphism $\alpha : R(G \circ F) \Rightarrow RG \circ RF$ comes from the universal property of the derived functor

$$\begin{tikzcd}
\mathcal{K}^+(\mathcal{I}_F) \arrow[rr, "\mathcal{K}(G\circ F)"] \arrow[dr, "\mathcal{K}(F)"'] & \arrow[d, phantom, "\cong"] & \mathcal{K}^+(\mathcal{C}) \arrow[ddd, "Q_{\mathcal{C}}"] \\
& \mathcal{K}^+(\mathcal{I}_G) \arrow[ur, "\mathcal{K}(G)"'] & \\
& \mathrm{D}^+(\mathcal{B}) \arrow[u, "\iota_G"] \arrow[dr, "RG"] \arrow[d, phantom, "\Uparrow"] & \\
\mathrm{D}^+(\mathcal{A}) \arrow[uuu, "\iota_F"] \arrow[ur, "RF"] \arrow[rr, "R(G\circ F)"'] & & \mathrm{D}^+(\mathcal{C})
\end{tikzcd}$$

(2.1)

Then for a complex $A^\bullet$ in $\mathrm{D}^+(\mathcal{A})$ there exists a quis $A^\bullet \to I^\bullet$, where $I^\bullet$ is a complex in $\mathcal{K}^+(\mathcal{I}_F)$, such that $RF(A^\bullet) \cong \mathcal{K}(F)(I^\bullet)$, then since $\mathcal{I}_F$ is ad adapted class also for the composition $G \circ F$, we calculate as well $R(G \circ F)(A^\bullet) \cong \mathcal{K}(G \circ F)(I^\bullet)$. Now $\alpha_{A^\bullet} : \mathcal{K}(G \circ F)(I^\bullet) \to RG(RF(A^\bullet))$, but $RG(RF(A^\bullet)) \cong RG(\mathcal{K}(F)(I^\bullet)) \cong \mathcal{K}(G)(\mathcal{K}(F)(I^\bullet))$, where the last isomorphism hols since $\mathcal{K}(F)(I^\bullet)$ is in $\mathcal{K}^+(\mathcal{I}_G)$. Thus $\alpha_{A^\bullet} : \mathcal{K}(G \circ F)(I^\bullet) \to \mathcal{K}(G)(\mathcal{K}(F)(I^\bullet))$ is exactly the isomorphism in (2.1). □

If it is required an explicit calculation of the composite of two derived functor there exists a spectral sequence that compute the value $R(G \circ F)(A^\bullet)$. We defer this discussion to Appendix A.

We conclude this section focusing on the connection between the extension classes and the morphisms in the derived category.
Let $\mathcal{A}$ be an abelian category with enough injective then the hom-functor

$\mathrm{Hom}_{\mathcal{A}}(A,-) : \mathcal{A} \to \mathsf{Ab}$ for any A in $\mathcal{A}$ is a left exact functor and we can define its higher right derived functor as

$$\mathrm{Ext}^i(A,-) :- \mathrm{H}^i \circ R\mathrm{Hom}(A,-)$$

As explained in [Căl05], considering a short exact sequence in $\mathcal{A}$

$$0 \to A \to B \to C \to 0$$

thought of as B an extension of C by A, we can construct the following span

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & C \arrow[r] & \dots & C \\
\dots \arrow[r] & A \arrow[r, "f"] \arrow[u] \arrow[d, "id_A"'] & B \arrow[r] \arrow[u, "g"] \arrow[d] & \dots & F^\bullet \arrow[u, "s"'] \arrow[d, "h"] \\
\dots \arrow[r] & A \arrow[r] & 0 \arrow[r] & \dots & A[1]
\end{tikzcd}$$

where s is a quis, hence what we obtained is a morphism $C \to A[1]$ in the derived category, this gives the relation

$$\mathrm{Ext}^1(C,A) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(C, A[1])$$

We can generalize this construction to extension classes of any length in the following statement.

Proposition 2.2.15. Let $\mathcal{A}$ an abelian category with enough injectives, then there exists a natural isomorphism

$$\mathrm{Ext}^i(A,B) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B[i])$$

where A and B are considered as complexes concentrated in zero.

Proof. Consider an element in $\mathrm{Ext}^i(A,B)$ as an extension of length i of A by B

$$\dots \to 0 \to B \xrightarrow{f_i} C_{i-1} \to \dots \to C_0 \xrightarrow{f_0} A \to 0 \to \dots$$

then we can construct the following

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & A \arrow[r] & \dots & A \\
\dots \arrow[r] & B \arrow[r, "f_i"] \arrow[u] \arrow[d, "id_B"] & C_{i-1} \arrow[r] \arrow[u] \arrow[d] & \dots \arrow[r] & C_0 \arrow[r] \arrow[u, "f_0"] \arrow[d] & \dots & \bar{C}^\bullet \arrow[u, "s"'] \arrow[d, "h"] \\
\dots \arrow[r] & B \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & \dots & B[i]
\end{tikzcd}$$

where the morphism s is a quis, hence we obtain an element of $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B[i])$. The assignment is actually an isomorphism and we refer to [Ver96] Chapter 3 Proposition 3.2.2 for a proof of this technical properties. □

Under the assumption that $\mathcal{A}$ contains enough injectives we can generalized the above discussion to any element of the derived category. We can define the functor

$$\mathrm{Hom}^\bullet(A^\bullet, -) : \mathcal{K}^+(\mathcal{A}) \longrightarrow \mathcal{K}(\mathsf{Ab})$$

$$B^\bullet \mapsto \mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$$

where $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ has a natural structure of complex given by

$$\mathrm{Hom}^i(A^\bullet, B^\bullet) :- \bigoplus_k \mathrm{Hom}(A^k, B^{k+i})$$

with differentials

$$d^i(f_k) :- d_B^{k+1} \circ f_k + (-1)^i f_{k+1} \circ d_A^k$$

thus we can define the corresponding derived functor

$$R\mathrm{Hom}^\bullet(A^\bullet, -) : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}(\mathsf{Ab})$$

and the higher derived functor

$$\mathrm{Ext}^i(A^\bullet, B^\bullet) :- \mathrm{H}^i \circ R\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$$

Remark 2.2.16. It is easy to extend the argument in 2.2.15 when we are working with general complex and not just with complex concentrated in degree zero, and thus obtain that the relation holds in this more general case

$$\mathrm{Ext}^i(A^\bullet, B^\bullet) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet[i])$$

2.3 Derived Category of Coherent Sheaves

We now present the derived category we are interested to work in, the bounded derived category of coherent sheaves $\mathrm{D}(\mathsf{Coh}(X))$ for X a scheme. The fundamental idea that lead us to the construction of derived category is that we would like to replace a coherent sheaves with an injective resolution, here arise the first problems, usually the category of coherent sheaves does not contain non-trivial injective object (Example **??**) and where they exists the injective resolutions are rarely bounded. For these reasons usually we

are force to work in bigger abelian as the category of quasi-coherent sheaves $\mathsf{QCoh}(X)$.
Let us briefly recall the definition of coherent and quasi-coherent sheaves, we will just state some properties, for a more detailed presentation of the topic we recommend standard literature as [Har66] and [Har77].

Definition 2.3.1. Let X be a scheme, a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is quasi-coherent if for each $x \in X$ there exists an open U such that the sequence

$$\mathcal{O}|_U^{\oplus I} \longrightarrow \mathcal{O}|_U^{\oplus J} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

is exact, where I and J are possibly infinite sets of indices.

Remark 2.3.2. An equivalent definition for a quasi-coherent sheaf $\mathcal{F}$ is ask that for every affine open $U \subset X$

$$\mathcal{F}|_U \cong \tilde{M}$$

for some $\mathcal{O}_X(U)$-module M.
This concept can be seen as the sheaf-theoretic notion of a modules and look at the exact sequence as a presentation of the module $\mathcal{F}|_U$ where I represents the relations and J the generators.

Requiring additionally the sets of indices to be finite we obtain the following notion.

Definition 2.3.3. Let X be a scheme, a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is coherent if locally is a finitely generated $\mathcal{O}_X$-module, i.e. for each $x \in X$ there exists an open U such that the sequence

$$\mathcal{O}|_U^n \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

is exact, where n is a natural number.

We can arrange these collections of objects into categories. We call the full subcategories of $\mathcal{O}_X$-Mod spanned by coherent and quasi-coherent sheaves $\mathsf{Coh}(X)$ and $\mathsf{QCoh}(X)$ respectively. Those are clearly abelian categories and it is trivial to note that $\mathsf{Coh}(X) \subset \mathsf{QCoh}(X)$.
When $X = \mathrm{Spec}(R)$ is an affine scheme quasi-coherent sheaves and coherent sheaves corresponds, via tilde construction B.2.2, to R-module and finitely generated R-module respectively.

Lemma 2.3.4. Let X be a scheme and $0 \to \mathcal{F} \to \mathcal{F}'' \to \mathcal{F}' \to 0$ a short exact sequence in $\mathsf{QCoh}(X)$. If two of the sheaves are coherent, so is the third one.

Notation 2.3.5. Let X be a scheme, we denote the bounded derived category of coherent sheaves as $\mathrm{D}^b(X) :- \mathrm{D}^b(\mathsf{Coh}(X))$.

The general setting where we are working in can be expressed as follows.

Definition 2.3.6. Let $\mathcal{A}$ a full abelian subcategory of the abelian category $\mathcal{B}$, then $\mathcal{A}$ is a thick subcategories if it is closed under subobjects, quotients and extensions, i.e. if for any short exact sequence in $\mathcal{B}$

$$0 \to M \to M'' \to M' \to 0$$

M'' is in $\mathcal{A}$ if and only if M, M' are in $\mathcal{A}$.

Proposition 2.3.7. Let $\mathcal{A}$ a thick subcategory of $\mathcal{B}$. Suppose that any object A in $\mathcal{A}$ can be embedded in an object A' in $\mathcal{A}$ which is injective as an object in $\mathcal{B}$, then the functor induced by the inclusion $\mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ induces an equivalence between $\mathrm{D}^+(\mathcal{A})$ and $\mathrm{D}^+_{\mathcal{A}}(\mathcal{B})$ the full triangulated subcategory of $\mathrm{D}^+(\mathcal{B})$ of complexes with cohomology in $\mathcal{A}$.

Proof. A fully detailed proof can be founded in [GM03]. $\square$

We will be mostly interested assuming X a smooth projective variety, thus the results in this section will be stated in the more general case where X is a noetherian scheme.

Proposition 2.3.8. Let X a noetherian scheme then $\mathsf{QCoh}(X)$ has enough injectives. Moreover any quasi-coherent sheaf $\mathcal{F}$ admits a resolution $\mathcal{F} \to \mathcal{I}^\bullet$ by quasi-coherent sheaves $\mathcal{I}^i$ which are injectives as object in $\mathcal{O}_X$-Mod.

By the above Proposition the derived category of quasi-coherent sheaves $\mathsf{QCoh}(X)$ satisfies the hypothesis of 2.3.7, hence we obtain the following equivalence.

Proposition 2.3.9. Let X a noetherian scheme, then the inclusion functor induces an equivalence

$$\mathrm{D}^+(\mathsf{QCoh}(X)) \xrightarrow{\cong} \mathrm{D}^+_{\mathrm{qcoh}}(\mathcal{O}_X\text{-}\mathsf{Mod})$$

We cannot apply similarly 2.3.7 when we restrict to the category of coherent sheaves. Even if $\mathsf{Coh}(X)$ is a thick subcategory, it has too few injectives, however we can prove a similar result

Proposition 2.3.10. Let X be a noetherian scheme, then the natural inclusion functor $\mathrm{D}(X) \to \mathrm{D}(\mathsf{QCoh}(X))$ induces an equivalence

$$\mathrm{D}^b(X) \xrightarrow{\cong} \mathrm{D}^b_{\mathrm{coh}}(\mathsf{QCoh}(X))$$

between the derived bounded category of coherent sheaves and $\mathrm{D}^b_{\mathrm{coh}}(\mathsf{QCoh}(X))$ the full triangulated subcategory of bounded complexes of quasi-coherent sheaves with coherent cohomology.

Proof. The proof is based on the fact that for a noetherian scheme and a surjection $f : \mathcal{G} \twoheadrightarrow \mathcal{F}$ of a quasi-coherent sheaf $\mathcal{G}$ into a coherent sheaf $\mathcal{F}$, there exists a sub-presheaf $\mathcal{G}'$ of $\mathcal{G}$ which is coherent and such that the restriction $\mathcal{G}' \twoheadrightarrow \mathcal{F}'$ remains surjective. This can be verified locally, since quasi-coherent and coherent sheaves corresponds to module and finitely generated module respectively and the statement holds in R-Mod.
Taking then a bounded complex of quasi-coherent sheaves $\mathcal{G}^\bullet$ with coherent cohomology, we will produce a coherent sheaf which is quasi-isomorphic to the first one by induction. Since $\mathcal{G}^\bullet$ is bounded exist m such that $\mathcal{G}^m = \ker(d^m) \cong \mathcal{H}^m(\mathcal{G}^\bullet)$ which is then coherent. Let thus assume that there exists j such that $\mathcal{G}^i$ is coherent for $i > j$. We have two sujections $d^j : \mathcal{G}^j \twoheadrightarrow \mathrm{Im}(d^j)$ and $\ker(d^j) \twoheadrightarrow \mathcal{H}^j(\mathcal{G}^\bullet)$, so by the above discussion there exist two sub-presheaves $\mathcal{G}^j_1 \subset \mathcal{G}^j$ and $\mathcal{G}^j_2 \subset \ker(d^j) \subset \mathcal{G}^j$ which are coherent, then let us call $\bar{\mathcal{G}}^j$ the coherent sheaves generated by $\mathcal{G}^j_1$ and $\mathcal{G}^j_2$ and $\bar{\mathcal{G}}^{j-1}$ to be the pre-image of $\bar{\mathcal{G}}^j$ under d^{j-1}. Hence the resulting complex $\bar{\mathcal{G}}^\bullet$ is such that $\bar{\mathcal{G}}^i$ is coherent for $i \geq j$, and the inclusion $\bar{\mathcal{G}}^\bullet \hookrightarrow \mathcal{G}^\bullet$

$$\begin{tikzcd}
\dots \arrow[r] & \bar{\mathcal{G}}^{j-1} \arrow[r] \arrow[d, hook] & \bar{\mathcal{G}}^j \arrow[r, "d^j"] \arrow[d, hook] & \mathcal{G}^j \arrow[r] \arrow[d, Rightarrow] & \dots \\
\dots \arrow[r] & \mathcal{G}^{j-1} \arrow[r, "d^{j-i}"] & \mathcal{G}^j \arrow[r, "d^j"] & \mathcal{G}^{j+1} \arrow[r] & \dots
\end{tikzcd}$$

is a quis by construction. □

2.4 Serre Duality

One of the main reason of the of the introduction of the derived category by Grothendieck and Verdier was the possibility to generalize the Serre duality, and following their philosophy Bondal and Kapranov in [BK90] generalize further this concept giving the definition of Serre funcort 1.1.16.
Considering then a smooth projective variety X over a field, the fundamental object of this construction is the canonical sheaf ω_X which is actually the object that realizes the duality.

Definition 2.4.1. Let X be a smooth projective variety of dimension n, we define the exact functor S_X as the composition

$$\mathrm{D}^*(X) \xrightarrow{\omega_X \otimes -} \mathrm{D}^*(X) \xrightarrow{[n]} \mathrm{D}^*(X)$$

for $* = +, -, b$.

Not surprisingly we will call S_X the Serre functor of X, the sense of this notation is explained by the following result.

Theorem 2.4.2 (Serre Duality)**.** Let X be a smooth projective variety over a field k, then the functor $S_X : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ is a Serre functor in the sense of definition 1.1.16, i.e. for any $\mathcal{E}, \mathcal{F}$ in $\mathrm{D}^b(X)$ there exist an isomorphism natural in both arguments

$$\eta_{\mathcal{E},\mathcal{F}} : \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{F}, \mathcal{E} \otimes \omega_X)^*$$

Chapter 3

Bondal-Orlov Reconstruction Theorem

3.1 Point like and Invertible objects

Let us define two import classes of objects that generalize skyscraper sheaves and invertible sheaves respectively, these objects will play a fundamental role in the reconstruction of the variety. Before giving these definition let us discuss one important feature of the classes of object we want generalize:

Definition 3.1.1. A collection Ω of objects in a triangulated category $\mathcal{D}$ is a spanning class of $\mathcal{D}$ if for all $B \in \mathcal{D}$ the following two condition holds

i) if $\operatorname{Hom}(A, B[i]) = 0$ for all $A \in \Omega$ and $i \in \mathbb{Z}$, then $B \cong 0$

ii) if $\operatorname{Hom}(B[i], A) = 0$ for all $A \in \Omega$ and $i \in \mathbb{Z}$, then $B \cong 0$

Remark 3.1.2. If we are working with a triangulated category endowed with a Serre functor, using the isomorphism $\operatorname{Hom}(A, B) \to \operatorname{Hom}(B, SA)^*$ and the essential surjectivity of S, we get that the two conditions are equivalent.

In the following esposition we will denote the residue field of the local ring at a point x of a scheme X as $k(x) :- \mathcal{O}_{X,x}/\mathsf{m}_x$

Proposition 3.1.3. Let X a smooth projective variety, then the skyscraper sheaves $k(x)$ at a closed points of of X, form a spanning class for $\mathrm{D}^b(X)$.

Proof. For the previous remark, it is enough to prove that for any non trivial $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$ there exist a closed point $x \in X$ and $i \in \mathbb{Z}$ such that $\operatorname{Hom}(\mathcal{F}^\bullet, k(x)[i]) \neq 0$. We use the spectral sequences (A.5)

$$E_2^{p,q} :- \operatorname{Hom}(\mathcal{H}^{-q}(\mathcal{F}^\bullet), k(x)[p]) \Rightarrow \operatorname{Hom}(\mathcal{F}^\bullet, k(x)[p+q])$$

Since $\mathcal{F}^\bullet \neq 0$ we can consider $m \coloneqq \max\{k \in \mathbb{Z} \mid \mathcal{H}^k(\mathcal{F}^\bullet) \neq 0\}$. Then the differentials in the spectral sequence $d_r^{0,-m} : E_r^{0,-m} \to E_r^{r,-m-r+1}$ are trivial as their targets are trivial in each page. Moreover, as negative Ext groups between coherent sheaves are trivial, for $p < 0$ we obtain

$$0 = \mathrm{Ext}^p(\mathcal{H}^{-q}(\mathcal{F}^\bullet), k(x)) \cong \mathrm{Hom}(\mathcal{H}^{-q}(\mathcal{F}^\bullet), k(x)[p]) = E_2^{p,q}$$

hence $d_r^{-r,-m+r-1} : E_r^{-r,-m+r-1} \to E_r^{0,-m}$ are trivial as well, since their domains are trivial. Thus the objects of the form $E_2^{0,m}$ stabilize after page 2 i.e. $E_2^{0,m} \cong E_\infty^{0,m}$. Summarizing, for each $x \in \mathrm{supp}\mathcal{H}^m(\mathcal{F}^\bullet)$

$$E_2^{0,-m} = \mathrm{Hom}(\mathcal{H}^m(\mathcal{F}^\bullet), k(x)) \neq 0$$

so by the spectral sequence we obtain exactly $\mathcal{H}(\mathcal{F}^\bullet, k(x)[-m]) \neq 0$. □

Definition 3.1.4. A sequence of objects L_i with $i \in \mathbb{Z}$ in a k-linear abelian category $\mathcal{A}$ is called ample if for any objects $A \in \mathcal{A}$ there exist i_0 (depending on A) such that for any $i < i_0$ the following hold

i) the natural morphism $\mathrm{Hom}(L_i, A) \otimes_k L_i \to A$ is surjective

ii) $\mathrm{Hom}(L_i, A[j]) = 0$ for $j \neq 0$

iii) $\mathrm{Hom}(A, L_i) = 0$

Remark 3.1.5. The tensor product in the above definition worth a comment. In order define it we will either assume that $\mathrm{Hom}_{\mathcal{A}}(L, A)$ is a finite dimension k-vector space or that $\mathcal{A}$ has arbitrary direct coproducts. In a k-linear abelian category, i.e. an abelian category enriched over the closed monoidal category k-Vect, the tensor (copower) between an object V of k-Vect and $L \in \mathcal{A}$ is the object $V \otimes L \in \mathcal{A}$ with the natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(V \otimes L, G) \cong \mathrm{Hom}_{k\text{-}\mathsf{Vect}}(V, \mathrm{Hom}_{\mathcal{A}}(L, G))$$

More concretely we can think of $V \otimes L$ as

$$\bigoplus_{v_i} L$$

where v_i are elements of the basis of V. Since we are going to work with the category $\mathsf{Coh}(X)$ with X smooth projective variety, $\mathrm{Hom}_{\mathsf{Coh(X)}}(-,-)$ will be finite dimension k-vector spaces

Proposition 3.1.6. Let L_i with $i \in \mathbb{Z}$ an ample sequence in a $k-$linear abelian category $\mathcal{A}$ of finite homological dimension, then the objects of the ample sequence form a spanning class for $\mathrm{D}^b(\mathcal{A})$.

Proposition 3.1.7. Let X a smooth projective variety over a filed k and let $\mathcal{L}$ be an ample line bundle on X, then the powers $\mathcal{L}^i$ with $i \in \mathbb{Z}$ form an ample sequence in the abelian category $\mathrm{Coh}(X)$

Proof. By definition B.1.2 $\mathcal{L}$ is ample if for any coherent sheaf $\mathcal{F}$ there exists n_0 such that for $n \geq n_0$ $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global section, meaning that the map in (iv) of B.1.3

$$\mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{L}^n) \otimes \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{L}^n \tag{3.1}$$

is surjective. Using that $\mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{L}^n) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{F} \otimes \mathcal{L}^n) \cong \mathrm{Hom}(\mathcal{L}^{-n}, \mathcal{F})$ and tensoring (3.1) with $\mathcal{L}^{-n}$ we obtain a surjective map

$$\mathrm{Hom}(\mathcal{L}^{-n}, \mathcal{F}) \otimes \mathcal{L}^{-n} \to \mathcal{F}$$

and this holds for each $n \geq n_0$, hence for $i_0 :- -n_0$ this fulfills the condition (i).
Let now $j \neq 0$, then

$$\begin{aligned}\mathrm{Hom}(\mathcal{L}^{-n}, \mathcal{F}[j]) &\cong \mathrm{H}^0(X, \mathcal{F}[j] \otimes \mathcal{L}^n)\\ &\cong \mathrm{H}^0(X, (\mathcal{F} \otimes \mathcal{L}^n)[j])\\ &\cong \mathrm{H}^j(X, \mathcal{F} \otimes \mathcal{L}^n)\end{aligned}$$

But by Serre vanishing theorem B.1.7, $\mathrm{H}^j(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for $j > 0$ and $n > n_0$. This proves condition (ii).

Again for $\mathcal{F}$ coherent, it follows straightforward from Serre duality and Lemma 3.2.3

$$\begin{aligned}\mathrm{Hom}(\mathcal{F}, \mathcal{L}^i) &\cong \mathrm{Hom}(\mathcal{F} \otimes \mathcal{L}^{-i}, \mathcal{O}_X)\\ &\cong \mathrm{Hom}(\mathcal{O}_X, (\mathcal{F} \otimes \mathcal{L}^{-i} \otimes \omega_X)[n])\\ &\cong \mathrm{H}^0(X, (\mathcal{F} \otimes \mathcal{L}^{-i} \otimes \omega_X)[n])^*\\ &\cong \mathrm{H}^n(X, \mathcal{F} \otimes \mathcal{L}^{-i} \otimes \omega_X)^*\end{aligned}$$

where the latter term is trivial again by B.1.7, this proves (iii).

□

Since we are working with the category of coherent sheaves on a projective varieties, by taking two coherent sheaves and apply the Serre duality we get

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^{i-n}(\mathcal{F}, \mathcal{G} \otimes \omega_X)^*$$

that is trivial for $i - n < 0$, hence $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for $i > n$, that means the homological dimension of $\mathrm{Coh}(X)$ is n. Hence Proposition 3.1.6 does apply to our case, giving us the expected spanning class.

Definition 3.1.8. Let $\mathcal{D}$ a $k-$linear triangulated category with Serre functor S, an object $P \in \mathcal{D}$ is called point like of codimension d if

i) $S(P) \cong P[d]$

ii) $\mathrm{Hom}(P, P[i]) = 0$ for $i < 0$

iii) $k(P) :- \mathrm{Hom}(P, P)$ is a field

An object P that satisfies the property (iii) is called simple. Moreover is worth to recall that, since we are assuming all Hom's are finite dimensional $k-$vector spaces, $k(P)$ is a finite extension of k, hence if k is algebraically closed $k(P)$ is just k.

Remark 3.1.9. Let X a variety with trivial canonical sheaf $\omega_X \cong \mathcal{O}_X$, e.g. if X is an abelian variety, any simple sheaf is a point like object.

We are now ready to define a generalization of line bundles that live in the derived category.

Definition 3.1.10. Let D a k-linear triangulated category with a Serre functor S, an object L is called invertible if for any point like object $P \in$ D there exist a $n_P \in \mathbb{Z}$, depending on P and L, such that

$$\mathrm{Hom}(L, P[i]) = \begin{cases} k(P) & \text{if } i = n_P \\ 0 & \text{otherwise} \end{cases}$$

It is important to point out that in the case of a smooth projective variety with ample or anti-ample canonical sheaf, point like and invertible objects have a nice (as we could expect) characterization.

Proposition 3.1.11. Let X a smooth projective variety with ω_X ample or anti-ample, then the point like objects in $\mathrm{D}^b(X)$ are isomorphic to $k(x)[m]$ with $x \in X$ a closed point and $m \in \mathbb{Z}$

Proof. Taking $k(x)[m]$ as in the statement, the conditions (i)-(iii) are trivially fulfilled since

$$(\omega_X \otimes k(x)[m])[n] \cong k(x)[m+n]$$

$$\mathrm{Hom}(k(x)[m], k(x)[m+i]) = 0$$

since the skyscraper sheaf is supported in one point, and

$$\mathrm{Hom}(k(x)[m], k(x)[m])$$

has a natural structure of a field.
For the other direction we will need the ampleness hypothesis. Let $P \in \mathrm{D}^b(X)$ a point like object, i.e the condition (i)-(iii) in defined in 3.1.8 hold. By (i) $(P \otimes \omega_X)[n] \cong P[d]$ then $(P \otimes \omega_X)[n-d] \cong P$ and iterating this isomorphism we get $P \cong (P \otimes \omega_X^k)[(n-d)k]$ for any k, but P must be a bounded complex, so the only way this can be satisfied is for $n = d$.
Moreover, considering the cohomology of the complex P, we have the following

$$\mathcal{H}^i(P) \cong \mathcal{H}^i(P) \otimes \omega_X \tag{3.2}$$

We now claim that by (anti)-ampleness of ω_X the support of $\mathcal{H}^i(P)$ is a collection of closed isolated points $\{x_j \mid j \in J\}$.
Iterating (3.2) we obtain $\mathcal{H}^i(P) \cong \mathcal{H}^i(P) \otimes \omega_X^k$, then for a large enough k $\mathcal{H}^i(P)$ is generated by global sections. Hence $\mathcal{H}^i(P)$ can be decomposed as

$$\mathcal{H}^i(P) \cong \bigoplus_{j \in J} \mathcal{G}_j$$

where each $\mathcal{G}_j$ has support of every cohomology sheaves on a single point x_j. This is a contradiction because by (iii) P is indecomposable, since if it was decomposable so would be $\mathrm{Hom}(P, P) = k(P)$, thus $\mathcal{H}^i(P)$ has its support concentrated only in one closed point.
Let us consider the Kunneth spectral sequence that compute $\mathrm{Ext}^m(P, P)$

$$E_2^{p,q} = \bigoplus_{k-j=q} \mathrm{Ext}^p(\mathcal{H}^j(P), \mathcal{H}^k(P)) \Rightarrow \mathrm{Ext}^{p+q}(P, P)$$

and let us recall that for any two sheaves of finite length having their support in the same single closed point, there exists a non trivial morphism between them. Considering $\mathrm{Ext}^m(\mathcal{H}^j(P), \mathcal{H}^k(P))$ with minimal $k-j$ such that $E_2^{0,k-j} = \bigoplus_{k-j=q} \mathrm{Hom}(\mathcal{H}^j(P), \mathcal{H}^k(P)) \neq 0$, by $\mathrm{Hom}(\mathcal{H}^j(P), \mathcal{H}^k(P)) \neq 0$ we get that $k - j \leq 0$.
Moreover $E_2^{0,k-j}$ stabilizes in the spectral sequence i.e. $E_2^{0,k-j} \cong E_\infty^{0,k-j}$. Hence $\mathrm{Ext}^{k-j}(P, P) \neq 0$, so by (iii) we obtain exactly $k - j = 0$. This means that all but one (precisely the one for $k = j$) cohomology sheaves are trivial. Finally (iii) implies that P is a skyscraper sheaf at the closed point x (that is the only point in the support). □

Proposition 3.1.12. Let X a smooth projective variety with ω_X ample or anti-ample, then the invertible objects in $\mathrm{D}^b(X)$ are $L[m]$ where L is a line bundle on X and $m \in \mathbb{Z}$

Proof. Let L an invertible object in $\mathrm{D}^b(X)$, i.e.

$$\mathrm{Ext}^i(L,P) \cong \mathrm{Hom}(L,P[i]) = \begin{cases} k(x) & i = n_P \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

where n_p is an integer depending on P. Let us fix $m := \max\{k \mid \mathcal{H}^k(L) \neq 0\}$, then by Lemma 2.1 there exists a natural morphism in $\mathrm{D}^b(X)$

$$\varphi : L \to \mathcal{H}^m(L)[-m]$$

such that $\mathcal{H}^m(\varphi)$ is the identity, so it is enough to show that $\mathcal{H}^m(L)$ is a line bundle.
Considering now a point $x_0 \in \mathrm{supp}\mathcal{H}^m(L)$, we have a natural non trivial map $\mathcal{H}^m(L) \to k(x_0)$, hence

$$\begin{aligned} 0 \neq \mathrm{Hom}(\mathcal{H}^m(L), k(x_0)) &\cong \mathrm{Hom}(L[m], k(x_0)) \\ &\cong \mathrm{Hom}(L, k(x_0)[-m]) \end{aligned}$$

Thus, by (3.3), $n_{k(x_0)} :- -m$, recalling that skyscraper sheaves are always point like objects. Taking the spectral sequence (A.5)

$$E_2^{p,q} = \mathrm{Hom}(\mathcal{H}^{-q}(L), k(x_0)[p]) \Rightarrow \mathrm{Hom}(L, k(x_0)[p+q])$$

we can notice that by hypothesis $\mathrm{Hom}(L, k(x_0)[1+n_{k(x_0)}]) = 0$, and $E_2^{1,-m} = \mathrm{Hom}(\mathcal{H}^m(L), k(x_0)[1])$ stabilizes, since every morphism with source $E_2^{p,-m}$ is trivial. Hence by the spectral sequence

$$\mathrm{Ext}^1(\mathcal{H}^m(L), k(x_0)) = \mathrm{Hom}(\mathcal{H}^m(L), k(x_0)[1]) = 0$$

To prove that $\mathcal{H}^m(L)$ is free we are going to use the following lemma from commutative algebra

Any finite module M over an arbitrary noetherian local ring (A, m) such that $\mathrm{Ext}^1_A(M, A/\mathsf{m}) = 0$, is free.

We cannot apply the lemma just like that, first we need to use the local-to-global Ext spectral sequence in A.0.16

$$E_2^{p,q} = \mathrm{H}^p(X, \mathcal{E}xt^q(\mathcal{H}^m(L), k(x_0))) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{H}^m(L), k(x_0))$$

in order to transfer the vanish property from $\mathrm{Ext}^1(\mathcal{H}^m(L), k(x_0))$ to $\mathcal{E}xt^q(\mathcal{H}^m(L), k(x_0))$. Now, since the skyscraper $k(x_0)$ is concentrated in x_0, so it is $\mathcal{E}xt^0(\mathcal{H}^m(L), k(x_0))$. Hence $E_2^{2,0} = \mathrm{H}^2(X, \mathcal{E}xt^0(\mathcal{H}^m(L), k(x_0))) = 0$, but this is the target of the differential $d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}$, thus $E_2^{0,1} \cong E_\infty^{0,1}$ stabilizes. Now by $\mathrm{Ext}^1(\mathcal{H}^m(L), k(x_0)) = 0$ applying the local-to-global spectral sequence we obtain

$$0 = E_2^{0,1} = \mathrm{H}^0(X, \mathcal{E}xt^1(\mathcal{H}^m(L), k(x_0))) = \Gamma(X, \mathcal{E}xt^1(\mathcal{H}^m(L), k(x_0)))$$

For the same reason as above $\mathcal{E}xt^1(\mathcal{H}^m(L), k(x_0))$ is concentrated in x_0 as well, so in particular is globally generated, but as its global section are trivial, so it must be i.e. $\mathcal{E}xt^1(\mathcal{H}^m(L), k(x_0)) = 0$.
We can finally apply the Lemma and obtain that $\mathcal{H}^m(L)$ is free in $x_0 \in X$ as $\mathcal{O}_{X,x_0}$-module. By assumption X is irreducible, hence $\mathrm{supp}(\mathcal{H}^m(L))$ must coincide with X. Then for any $x \in X$ we have a non trivial map $\mathcal{H}^m(L) \to k(x)$, so as above

$$0 \neq \mathrm{Hom}(\mathcal{H}^m(L), k(x)) \cong \mathrm{Hom}(L, k(x_0)[-m])$$

Since L is an invertible object $n_{k(x)} = -m$ does not depend on $k(x)$ anymore, moreover

$$k(x) = \mathrm{Hom}(\mathcal{H}^m(L), k(x))$$

which means that the stalks of $\mathcal{H}^m(L)$ are of dimension one.

Conversely, we now show that $L[m]$ with L line bundle and $m \in \mathbb{Z}$ is an invertible object. Here we need the ampleness assumption in order to apply 3.1.11 and consider point like object $P \cong k(x)[n]$ for some integer n. Then

$$\begin{aligned}
\mathrm{Hom}(L[m], P[i]) &\cong \mathrm{Hom}(L[m], k(x)[n+i]) \\
&\cong \mathrm{Hom}(L, k(x)[n+i-m]) \\
&\cong \mathrm{Hom}(\mathcal{O}_X, L^* \otimes k(x)[n+i-m]) \\
&\cong \mathrm{Hom}(\mathcal{O}_X, k(x)[n+i-m]) \\
&\cong \mathrm{H}^0(X, k(x)[n+i-m]) \\
&\cong \mathrm{H}^{n+i-m}(X, k(x))
\end{aligned}$$

where the last there is $k(x) = k(P)$ when $i = m - n$ and 0 otherwise. So it is enough to take $i_P :- m - n$. □

3.2 The Bondal-Orlov Theorem

Lemma 3.2.1. Let X and Y two smooth projective variety over a field k, if there exists an equivalence between their derived categories

$$F : \mathrm{D}^b(X) \xrightarrow{\simeq} \mathrm{D}^b(Y)$$

then

$$\dim(X) = \dim(Y)$$

and their canonical bundle ω_X and ω_Y are of the same order.

Proof. For a detailed proof of this result we refer to [Huy06]. □

Lemma 3.2.2. Let X a smooth projective variety and $\mathcal{L}$ an ample invertible sheaf, then

$$X \cong \operatorname{Proj}(\bigoplus_{m\geq 0} \mathrm{H}^0(X, \mathcal{L}^{\otimes mk}))$$

Proof. Let us first fix some notation, $\mathbb{P}^n = \operatorname{Proj}(S)$ where $S := \bigoplus_m \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(m))$, and $R_{\mathcal{L}^{\otimes k}} :- \bigoplus_m \mathrm{H}^0(X, \mathcal{L}^{\otimes mk})$. By B.1.5 there exists $k > 0$ such that $\mathcal{L}^{\otimes k}$ is very ample i.e. the corresponding $\iota : X \to \mathbb{P}^n$ is a closed immersion with $\mathcal{L}^{\otimes k} \cong \iota^*(\mathcal{O}(1))$. Consider then $\mathcal{I}_X$ the ideal sheaf of X in $\mathbb{P}^n$ this gives rise to the following short exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{L}^{\otimes k} \longrightarrow 0 \tag{3.4}$$

that is obtained by applying the $\tilde{(-)}$ functor, defined in B.2.5, to

$$0 \longrightarrow I_X \longrightarrow S \longrightarrow M \longrightarrow 0$$

with $M = S/I_X$. Then applying $- \otimes \mathcal{O}_{\mathbb{P}^n}(m)$ to (3.4), the sequence remains exact, since we are tensoring with a locally free module that is in particular flat

$$0 \longrightarrow \mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^n}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(m) \longrightarrow \mathcal{L}^{\otimes mk} \longrightarrow 0$$

that naturally extend to

$$0 \longrightarrow \bigoplus_m \mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^n}(m) \longrightarrow \bigoplus_m \mathcal{O}_{\mathbb{P}^n}(m) \longrightarrow \bigoplus_m \mathcal{L}^{\otimes mk} \longrightarrow 0$$

Finally taking the global section functor Γ that is only left exact

$$0 \longrightarrow \Gamma_*(\mathcal{I}_X) \longrightarrow S \longrightarrow R_{\mathcal{L}^{\otimes k}} \longrightarrow \dots$$

and moreover

$$0 \longrightarrow S/\Gamma_*(\mathcal{I}_X) \longrightarrow R_{\mathcal{L}^{\otimes k}} \longrightarrow \dots$$

X is a closed subscheme, so it arises as $\operatorname{Proj}(S/\Gamma_*(\mathcal{I}_X))$, by B.1.7 $\mathrm{H}^i(X, \mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^n}(m)) = 0$ for $i > 0$ and $m >> 0$, thus we have isomorphisms in the higher graded parts of these modules, by [Sta23] section 27.11, this provides the required isomorphism

$$\mathrm{Proj}(S/\Gamma_*(\mathcal{I}_X)) \cong \mathrm{Proj}(R_{\mathcal{L}^{\otimes k}})$$

□

Lemma 3.2.3. Let X a scheme, then for any $\mathcal{F} \in \mathcal{O}_X\text{-}\mathsf{Mod}$

$$\mathrm{H}^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{F})$$

Theorem 3.2.4 (Bondal-Orlov). Let X be a smooth projective variety with ample or anti-ample canonical sheaf. If there exists an exact equivalence $F : \mathrm{D}^b(X) \overset{\simeq}{\to} \mathrm{D}^b(Y)$ for Y a smooth projective variety, then X is isomorphic to Y

It is worth to note that this result is more general than a reconstruction theorem for smooth projective variety with ample or anti-ample canonical bundle, since we are not assuming any positivity on the canonical sheaf of Y, in fact this property will descend just from the equivalence between their derived categories.

Proof. The proof will be composed by various steps, but the idea behind it is quite simple and geometric.

Step 1. Let us assume, for the moment, that $\mathcal{O}_X$ and $\mathcal{O}_Y$ are equivalent under F, i.e. $F(\mathcal{O}_X) = \mathcal{O}_Y$. Since an equivalence commute with both the Serre functor and the shift functor and by 3.2.1 $\dim(X) = \dim(Y)$, using the trivial $\omega_X = S_X(\mathcal{O}_X)[-n]$ we get

$$F(\omega_X^k) = F(S_X^k(\mathcal{O}_X)[-kn]) \cong S_Y^k(F(\mathcal{O}_X))[-kn] \cong$$

$$\cong S_Y^k(\mathcal{O}_Y)[-kn] \cong \omega_Y^k$$

Using now the fully-faithfulness of F and Lemma 3.2.3

$$\mathrm{H}^0(X, \omega_X^k) \cong \mathrm{Hom}(\mathcal{O}_X, \omega_X^k) \cong \mathrm{Hom}(F(\mathcal{O}_X), F(\omega_X^k)) \cong$$

$$\cong \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^k) \cong \mathrm{H}^0(Y, \omega_Y^k)$$

We can now give a graded structure to $\bigoplus_k \mathrm{H}^0(X, \omega_X^k)$, taking $s_k \in \mathrm{Hom}(\mathcal{O}_X, \omega_X^k)$ and $s_h \in \mathrm{Hom}(\mathcal{O}_X, \omega_X^h)$ we define $s_k \cdot s_h$ to be the composite

$$\mathcal{O}_x \xrightarrow{s_k} \omega_X^k \xrightarrow{S_X^k(s_h)[-kn]} \omega_X^{k+h}$$

A completely similar definition can be given for $\bigoplus_k \mathrm{H}^0(Y, \omega_Y^k)$. Moreover, the above isomorphisms between the graded parts extend to an isomorphism between the grade rings

$$\bigoplus_k \mathrm{H}^0(X, \omega_X^k) \cong \bigoplus_k \mathrm{H}^0(Y, \omega_Y^k)$$

and so to their Proj

$$X \cong \mathrm{Proj}(\bigoplus_k \mathrm{H}^0(X, \omega_X^k)) \cong \mathrm{Proj}(\bigoplus_k \mathrm{H}^0(Y, \omega_Y^k))$$

where the first isomorphism comes from Lemma 3.2.2. If we assume now that ω_Y is ample or anti-ample, for the same Lemma, we get the desired isomorphism

$$X \cong \mathrm{Proj}(\bigoplus_k \mathrm{H}^0(X, \omega_X^k)) \cong \mathrm{Proj}(\bigoplus_k \mathrm{H}^0(Y, \omega_Y^k)) \cong Y$$

Step 2. We are now going to remove the assumption we did in the first step, showing that we can reduce to the previous situation just by consider the equivalence F and the ampleness on X.

Since the concepts of point-like and invertible objects depend only on the Serre functor and on the hom-set of the derived category, again by fully-faithfullness and the commutativity with the Serre functor of the equivalence, is trivial to conclude that $F : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ induces bijections between the class of point-like object of $\mathrm{D}^b(X)$ and the class of point-like objects in $\mathrm{D}^b(Y)$ and between the class of invertible objects of $\mathrm{D}^b(X)$ and $\mathrm{D}^b(Y)$ respectively.

Let us now remove the first assumption we made. By 3.1.12 any line bundle on X define an invertible object in $\mathrm{D}^b(X)$, so in particular $\mathcal{O}_X$ is an invertible object, and it is mapped by F to an invertible object in $\mathrm{D}^b(Y)$, i.e. $F(\mathcal{O}_X) = L[m]$ for L line bundle on Y and $m \in \mathbb{Z}$. Thus up to compose F with two equivalences given by tensoring with the line bundle $- \otimes L^*$ and the shift $(-)[-m]$, and with abuse of notation calling this composition F again we obtain $F(\mathcal{O}_X) = \mathcal{O}_Y$.

Step 3. The last thing to prove is that ω_Y is either ample or anti-ample, but before doing this let us check that point-like object in $\mathrm{D}^b(Y)$ are of the form $k(y)[m]$. Without any ampleness assumption on the canonical sheaf of Y, $k(y)[m]$ is a point-like object in $\mathrm{D}^b(Y)$, and since F preserves and reflects point-like objects, there exist a closed point $x_y \in X$ such that $F(k(x_y)[m_y]) \cong k(y)[m]$. Suppose then that P is a point-like in $\mathrm{D}^b(Y)$ which is not of the form $k(y)[m]$, nevertheless there exists $x_P \in X$ and an integer m_P such that $F(k(x_p)[m_P]) \cong P$. For each $y \in Y$ and $m \in (Z)$

$$\mathrm{Hom}(P, k(y)[m]) \cong \mathrm{Hom}(F(k(x_p)[m_P]), F(k(x_y)[m_y + m])) \cong$$

$$\cong \mathrm{Hom}(k(x_P), k(x_y)[m_y + m - m_P]) \cong 0$$

but $k(y)$ form a spanning class for $\mathrm{D}^b(Y)$ by Proposition 3.1.3, hence $P \cong 0$ which is a contradiction.

Step 4. By Proposition B.1.5 ω_Y is ample if and only if ω_Y^k is very ample. Given that ω_X^k is very ample, we are going to prove that ω_Y^k separates closed points and tangent vectors and then apply B.1.8. In the following discussion we add to the assumptions that k is an algebraically closed field.

Given two different points $y_1, y_2 \in Y$, consider the natural restriction map to the fibers

$$\rho_{y_1,y_2} : \omega_Y^k \longrightarrow \omega_Y^k(y_1) \oplus \omega_Y^k(y_2)$$

where $\omega_Y^k(y_i) \cong k(y_i)$ since the canonical sheaf is invertible. Saying that ω_Y^k separates points is equal to require that the induced map on the global section is surjective

$$\Gamma(\rho_{y_1,y_2}) : \Gamma(Y, \omega_Y^k) \longrightarrow \Gamma(Y, k(y_1) \oplus k(y_2))$$

$$s \longmapsto (0, s)$$

where $s \in \mathsf{m}_{y_1}\omega_Y^k(y_1)$ and $s \notin \mathsf{m}_{y_2}\omega_Y^k(y_2)$ (or the contrary, having then $s \mapsto (s, 0)$).

Since the equivalence F carries skyscrapers to skyscrapers and ω_X^k to ω_Y^k, there exists two different closed points $x_1, x_2 \in X$ such that ρ_{y_1,y_2} corresponds to a map

$$\rho_{x_1,x_2} : \omega_X^k \longrightarrow k(x_1) \oplus k(x_2)$$

By hypothesis ω_X^k separates points, i.e. $\Gamma(\rho_{x_1,x_2})$ is surjective, moreover the isomorphism in Lemma 3.2.3 comes from an isomorphism of sheaves, then the following diagram commutes

$$\begin{tikzcd}
\Gamma(Y, \omega_Y^k) \arrow[r, "\Gamma(\rho_{y_1,y_2})"] \arrow[d, "\cong"] & \Gamma(Y, k(y_1) \oplus k(y_2)) \arrow[d, "\cong"] \\
\mathrm{Hom}(\mathcal{O}_Y, \omega_Y^k) \arrow[r, "\rho_{y_1,y_2} \circ -"] \arrow[d, "\cong"] & \mathrm{Hom}(\mathcal{O}_Y, k(y_1) \oplus k(y_2)) \arrow[d, "\cong"] \\
\mathrm{Hom}(\mathcal{O}_X, \omega_X^k) \arrow[r, "\rho_{x_1,x_2} \circ -"] \arrow[d, "\cong"] & \mathrm{Hom}(\mathcal{O}_X, k(x_1) \oplus k(x_2)) \arrow[d, "\cong"] \\
\Gamma(X, \omega_X^k) \arrow[r, "\Gamma(\rho_{x_1,x_2})"] & \Gamma(X, k(x_1) \oplus k(x_2))
\end{tikzcd}$$

Hence the surjectivity of the bottom map implies the surjectivity of the top map.
The last thing to prove is that ω_Y^k actually separates tangent vectors. Let us note that a tangent vector at a closed point $y \in Y$ is equivalent to the datum of a subscheme of length two with structure sheaf concentrated at y. Tangent vectors in $(\mathsf{m}_y/\mathsf{m}_y^2)^*$ at rational point y are just elements of $\mathrm{Hom}(\mathrm{Spec}(k[\varepsilon]), Y)$ with $k[\varepsilon] :- k[\varepsilon]/(\varepsilon)^2$, then for $\varphi \in \mathrm{Hom}(\mathrm{Spec}(k[\varepsilon]), Y)$ choosing the closed point y, we can consider the closed subscheme of Y defined as $Z_y :- \varphi(\mathrm{Spec}(k[\varepsilon]))$, this comes with the short exact sequence

$$0 \longrightarrow \mathcal{I}_{Z_y} \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{Z_y} \longrightarrow 0$$

where $\mathcal{O}_{Z_y}$ is supported in y and $\mathcal{O}_{Z_y,y} \cong \mathcal{O}_{Y,y}/\mathcal{I}_{Z_y}$. Moreover as Z_y is a subscheme of length two we get another short exact sequence

$$0 \longrightarrow k(y) \longrightarrow \mathcal{O}_{Z_y} \longrightarrow k(y) \longrightarrow 0$$

of sheaves concentrated in y. The latter corresponds to an element in $\mathrm{Ext}^1(k(y), k(y))$ that we could call extension class, then by

$$\begin{aligned}
\mathrm{Ext}^1(k(y), k(y)) &\cong \mathrm{Hom}(k(y), k(y)[1]) \\
&\cong \mathrm{Hom}(F(k(y)), F(k(y))[1]) \\
&\cong \mathrm{Hom}(k(x_y), k(x_y)[1]) \cong \mathrm{Ext}^1(k(x_y), k(x_y))
\end{aligned}$$

the above extension class corresponds to a subscheme of length two concentrated in x_y, namely

$$0 \longrightarrow k(x_y) \longrightarrow \mathcal{O}_{Z_{x_y}} \longrightarrow k(x_y) \longrightarrow 0$$

Moreover by the above characterization is straightforward to note that $F(\mathcal{O}_{Z_{xy}}) \cong \mathcal{O}_{Z_y}$, thus

$$\begin{aligned} \mathrm{H}^0(X, \mathcal{O}_{Z_{xy}}) &\cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_{Z_{xy}}) \\ &\cong \mathrm{Hom}(F(\mathcal{O}_X), F(\mathcal{O}_{Z_{xy}})) \\ &\cong \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_{Z_y}) \\ &\cong \mathrm{H}^0(Y, \mathcal{O}_{Z_y}) \end{aligned}$$

Taking the short exact sequence that identifies the subscheme $\mathcal{O}_{Z_{xy}}$

$$0 \longrightarrow \mathcal{I}_{Z_{xy}} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{Z_{xy}} \longrightarrow 0$$

and tensoring it with ω_X^k we obtain again a short exact sequence where the most right map is the restriction map we are interesting in

$$0 \longrightarrow \mathcal{I}_{Z_{xy}} \otimes \omega_X^k \longrightarrow \omega_X^k \xrightarrow{\mathrm{res}_{xy}} \mathcal{O}_{Z_{xy}} \longrightarrow 0$$

The map $\omega_Y^k \xrightarrow{\mathrm{res}_Y} \mathcal{O}_{Z_y}$ arise in a similar way. Considering now the naturality square for F

$$\begin{tikzcd} \mathrm{H}^0(X, \omega_X^k) \arrow[r, two heads] \arrow[d, "\cong"'] & \mathrm{H}^0(X, \mathcal{O}_{Z_{xy}}) \arrow[d, "\cong"] \\ \mathrm{H}^0(Y, \omega_Y^k) \arrow[r] & \mathrm{H}^0(X, \mathcal{O}_{Z_y}) \end{tikzcd}$$

since the top row is surjective, the bottom row must be surjective as well. The above commutative square is given by the fact that $\omega_X^k \xrightarrow{\mathrm{res}_X} \mathcal{O}_{Z_{xy}}$ is mapped to $\omega_Y^k \xrightarrow{\mathrm{res}_Y} \mathcal{O}_{Z_y}$:

This concludes the proof.

□

Remark 3.2.5. The original proof from A.Bondal and D.Orlov in [BO01] is slightly different: after proving, as we did, the correspondence induced by the equivalence F for both the classes of point-like objects and invertible objects in X and Y, they deduce from it a bijection between the underlying sets of the projective schemes. The proof follows by reconstructing the Zarisky topology for X and Y and showing that actually coincide. Taking L_1, L_2 invertible object, $\alpha \in \mathrm{Hom}(L_1, L_2)$ and P a point like object, let us denote the pre-composition map with $\alpha_P^* : \mathrm{Hom}(L_2, P) \to \mathrm{Hom}(L_1, P)$, then it can be defined

$$U_\alpha \coloneqq \{P \in \mathcal{P}_D \mid \alpha_P^* \neq 0\}$$

such that, letting α run over $\mathrm{Hom}(L_1, L_2)$ and L_1, L_2 among the invertible objects, give a basis for the Zariski topology for both X and Y. From this can be deduce that ω_Y is either ample or antiample. The proof is then concluded by given the structure of grade ring to

$$A \coloneqq \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(L_0, L_i)$$

and noticing that is isomorphic to both the canonical graded rings of X and Y.

Appendices

Appendix A

Spectral sequences

Our aim for this appendix is to briefly explain the machinery of the spectral sequences and how they are related to the discussion of this work. The main result of this section is that this framework will provide a direct way to compute the derived functor of the composite. However there is a counterpart of this methods, often the actual computation could become long and nested.

Definition A.0.1. Let $C \coloneqq C^\bullet$ a complex in $\mathrm{Ch}(\mathcal{A})$, where $\mathcal{A}$ is abelian, considered as a graded complex with grading given by the differential $d : C \to C$. A decreasing filtration of C is a collection of subobject $\{F^pC^n\}_p$ for each n such that

i) $F^{p+1}C^n \subset F^pC^n \subset \cdots \subset C^n$

ii) $\bigcup_p F^pC^n = C^n \qquad \text{and} \qquad \bigcap_p F^pC^n = 0$

iii) $d^n(F^pC^n) \subset F^pC^{n+1}$

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1} \arrow[r, "d^{n-1}"] & C^n \arrow[r, "d^n"] & C^{n+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & F^1C^{n-1} \arrow[r, "d^{n-1}"] \arrow[u, hook] & F^1C^n \arrow[r, "d^n"] \arrow[u, hook] & F^1C^{n+1} \arrow[r] \arrow[u, hook] & \cdots \\
\cdots \arrow[r] & F^2C^{n-1} \arrow[r, "d^{n-1}"] \arrow[u, hook] & F^2C^n \arrow[r, "d^n"] \arrow[u, hook] & F^2C^{n+1} \arrow[r] \arrow[u, hook] & \cdots
\end{tikzcd}$$

Definition A.0.2. Given a filtration $\{F^pC^n\}_p$ of a complex C, a graded module associated to the filtration is a graded object defined by

$$\mathrm{gr}^pC^n :- F^pC^n/F^{p+1}C^n$$

Given then a filtration $\{F^pC^n\}_p$ of a complex C, we can define some special graded module associated to it

$$Z_r^{p,q} :- d^{-1}(F^{p+r}C^{p+q+1}) \cap F^pC^{p+q}$$

$$B_r^{p,q} :- Z_{r-1}^{p+1,q-1} + dZ^{p-r+1,q+r-2}$$

$$E_r^{p,q} :- Z_r^{p,q}/B_r^{p,q}$$

where p, q, r are integers, with $r \geq 1$.
Now we claim that the differential of the graded complex $d : C \to C$ induces a complex E_r with differential

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that the chain cohomology of the complex E_r is isomorphic to the complex E_{r+1}.
The construction above take the name of spectral sequence

Definition A.0.3. Let $\mathcal{A}$ an abelian category, a spectral sequence is a collection $\{E_r^{p,q}\}_{p,q,r}$ of objects in $\mathcal{A}$ with p, q, r integers and $r \geq 1$ and morphisms

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

called differentials, such that

i) $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$

ii) there exists an isomorphism

$$E_{r+1}^{p,q} \cong \mathrm{H}^0(E^{p+\bullet r,q-\bullet r+1})$$

The subscript r stands for the number of the page were are collected the objects $\{E_r^{p,q}\}_{p,q}$, hence, roughly speaking we can express the condition (ii) of A.0.3 by saying that the r-th page is the cohomology of the previous one. The power of this machinery is that we can get information on the page for $r = \infty$ knowing the spectral sequence for $r < \infty$. To make this claim precise we need to introduce the notion of convergence a spectral sequence.

Definition A.0.4. Let $\{E_r^{p,q}\}_{p,q,r}$ a spectral sequence, if for each (p, q) there exists r_0 such that for each $r \geq r_0$

$$E_r^{p,q} \cong E_{r_0}^{p,q}$$

we say that the sequence abuts to E_∞, where E_∞ is the page defined by $E_\infty^{p,q} :- E_{r_0}^{p,q}$.

Definition A.0.5. We say that the spectral sequence $\{E_r^{p,q}\}_{p,q,r}$ collapses on an page if there exists $r_s \geq 2$ such that the r_s-page is concentrated only on a column of a row.

Definition A.0.6. A spectral sequence $\{E_r^{p,q}\}_{p,q,r}$ converges to a complex $H^\bullet$ with a decreasing filtration $\{F^pH^n\}_p$ if there exist isomorphisms

$$E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$$

In the practice one does not goes often beyond the second or the third page, i.e. we will obtain some type of convergence by knowing the terms of the first two pages and checking that they stabilize.
Spectral sequences were initially introduced to compute the cohomology of a total complex of some bi-complex. In fact, starting from a double complex, will occur in a natural way a well-behaved filtration on the total complex under some boundedness assumption.

Definition A.0.7. A double complex $K^{\bullet,\bullet}$ consists of a collection of objects $K^{i,j}$ with i, j integers and morphisms

$$d_I^{i,j} : K^{i,j} \to K^{i+1,j} \qquad \text{and} \qquad d_{II}^{i,j} : K^{i,j} \to K^{i,j+1}$$

giving the bi-grading, such that $d_I^2 = 0 = d_{II}^2$ and satisfying the relation

$$d_I^{p,q+1} \circ d_{II}^{p,q} + d_{II}^{p+1,q} \circ d_I^{p,q} = 0$$

d_I and d_{II} are called horizontal and vertical differential respectively.

Definition A.0.8. Given a double complex $K := K^{\bullet,\bullet}$, the total complex associated to it is the complex $\text{tot}(K)$ defined by

$$\text{tot}(K)^n :- \bigoplus_{i+j=n} K^{i,j}$$

with differentials

$$d^n :- \sum_{p+q=n} d_I^{p,q} + d_{II}^{p,q}$$

It is trivial to check that $(\text{tot}(K), d)$ is actually a complex, moreover for the total complex there exists a natural decreasing filtration, to be precise there are two natural decreasing filtration due to the symmetry of the situation. We will define just one of them but all the results will hold for the other one.

Definition A.0.9. Given a double complex $K := K^{\bullet,\bullet}$, we define the filtration on the total complex as

$$F^p \mathrm{tot}(K)^n :- \bigoplus_{j\geq p} K^{n-j,j}$$

i.e. the direct sum of all the elements of the double complex which are on the right side of a column p.

We observe that the graded objects associated to this filtration are of the form

$$\mathrm{gr}^p \mathrm{tot}(K)^n = F^p \mathrm{tot}(K)^n / F^{p+1} \mathrm{tot}(K)^n = \bigoplus_{j=p} K^{n-j,j} = K^{n-p,p}$$

hence we can arrange this objects together with the differential d_I into a complex $K^{\bullet,p}[-p]$. So passing to the cohomology of this complex we obtain

$$\mathrm{H}^k(\mathrm{gr}^p \mathrm{tot}(K)) = \mathrm{H}^k_I(K^{\bullet,p}[-p]) = \mathrm{H}^{k-p}_I(K^{\bullet,p})$$

and passing again to the cohomology with the respect to the remaining differential d_{II} we get $\mathrm{H}^p_{II}(\mathrm{H}^{k-p}_I(K^{\bullet,\bullet}))$.
Now under some boundedness conditions on the shape of the double complex the natural filtration defined above induces a spectral sequence.

Theorem A.0.10. Let $K^{\bullet,\bullet}$ be a double complex such that for any n there exist integers p^+_n and p^-_n such that $K^n - p, p = 0$ for $p > p^+_n$ and $p > p^-_n$. Then the filtration defined in $A.0.8$ induces a natural spectral sequence

$$E^{p,q}_2 = \mathrm{H}^p_{II}(\mathrm{H}^{k-p}_I(K^{\bullet,\bullet})) \Rightarrow \mathrm{H}^{p+q}(\mathrm{tot}(K))$$

Remark A.0.11. The boundedness condition ensures that for a fixed diagonal of the double complex the elements composing it will eventually become zero, for example this condition is trivially satisfied for a double complex in the first or in the third quadrant or for a double complex with a finite number of rows and columns.
The results in A.0.10 holds actually in a more general form, considering a filtered complex that not necessary comes from a double complex. Moreover yields another spectral sequence that that starts from the page 1

$$E^{p,q}_1 :- \mathrm{H}^{p+q}(F^p C^\bullet / F^{p+1} C^\bullet)$$

We can now state the main result of this section that give us a procedure to compute the exact value of the derived functor of the composite.

Theorem A.0.12 (Grothendieck Spectral Sequence)**.** Let $F : \mathcal{K}^+(\mathcal{A}) \to \mathcal{K}^+(\mathcal{B})$ and $G : \mathcal{K}^+(\mathcal{B}) \to \mathcal{K}^+(\mathcal{C})$ be two exact functor. If $\mathcal{A}$ and $\mathcal{B}$ contain enough injectives and a complex of injectives in $\mathcal{K}^+(\mathcal{I}_{\mathcal{A}})$ is mapped by F into a G-adapted triangulated, then for any complex $A^\bullet$ in $\mathrm{D}^+(\mathcal{A})$ there exists a spectral sequence

$$E_2^{p,q} = R^pG(R^qF(A^\bullet)) \Rightarrow R^{p+q}(G \circ F)(A^\bullet)$$

Remark A.0.13. This general results is actually used more often considering the the first functor the identity, then we obtain a spectral sequence of the form

$$E_2^{p,q} = R^pF(\mathrm{H}^q(A^\bullet)) \Rightarrow R^{p+q}F(A^\bullet) \tag{A.1}$$

Moreover using (A.1) we can deduce the general statement as follows. Taking $A^\bullet$ in $\mathrm{D}^+(\mathcal{A})$ and a complex $I^\bullet$ in $\mathcal{K}^+(\mathcal{I}_F)$ which is quasi-isomorphic to $A^\bullet$ then $RF(A^\bullet) = F(\mathcal{I}^\bullet) -\colon B^\bullet$. Then by (A.1)

$$R^pG(\mathrm{H}^q(B^\bullet)) \Rightarrow R^{p+q}G(B^\bullet)$$

We also have $R^{p+q}(G \circ F)(A^\bullet \cong \mathrm{H}^n(R(G \circ F)(A^\bullet)) \cong \mathrm{H}^n(RG(RF(A^\bullet))) \cong \mathrm{H}^n(RG(B^\bullet))$, so putting everything together we get the general case of the spectral sequence.

We conclude this section by recalling some useful spectral sequence used in this work that are all instances of the Grothendieck spectral sequence in Theorem A.0.12.

Proposition A.0.14. Let $A^\bullet$ in $\mathrm{D}(\mathcal{A})$, where $\mathcal{A}$ contains enough injectives, then we can consider the functor $\mathrm{Hom}(A^\bullet, -) : \mathcal{K}^+(\mathcal{A}) \to \mathcal{K}(\mathsf{Ab})$ and its higher derived functor $R^p\mathrm{Hom}(A^\bullet, -) -\colon \mathrm{Ext}^p(A^\bullet, -)$. Then there exists a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(A^\bullet, \mathrm{H}^q(B^\bullet)) \Rightarrow \mathrm{Ext}^{p+q}(A^\bullet, B^\bullet) \tag{A.2}$$

Dually, if $\mathcal{A}$ contains enough projective we can define the derived functor of the contravariant hom-functor $\mathrm{Hom}(-, B^\bullet) : \mathcal{K}^-(\mathcal{A}) \to \mathcal{K}(\mathsf{Ab})$. Then there exists

$$E_2^{p,q} = \mathrm{Ext}^p(H^{-q}(A^\bullet), B^\bullet) \Rightarrow \mathrm{Ext}^{p+q}(A^\bullet, B^\bullet) \tag{A.3}$$

Remark A.0.15. Recalling the isomorphism in 2.2.16

$$\mathrm{Ext}^i(A^\bullet, B^\bullet) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet[i])$$

we can express the two spectral sequence above using the hom-set of the derived category

$$E_2^{p,q} = \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, \mathrm{H}^q(B^\bullet)[p]) \Rightarrow \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet[p+q]) \tag{A.4}$$

$$E_2^{p,q} = \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(\mathrm{H}^{-q}(A^\bullet), B^\bullet[p]) \Rightarrow \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet[p+q]) \tag{A.5}$$

Let us now consider the abelian category of quasi-coherent sheaves $\mathsf{QCoh}(X)$ on a noetherian scheme X, taking two sheaves $\mathcal{F}$ and $\mathcal{E}$ we can define the so called 'local Hom' sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{E})$ on the opens as

$$U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{E}|_U)$$

It easy to check that this is actually a sheaf, moreover if $\mathcal{F}$ and $\mathcal{E}$ are (quasi-)coherent, the local Hom sheaf is (quasi-)coherent as well. It is worth to note that the global section of the local Hom coincides with the standard hom functor $\Gamma(\mathcal{H}om(\mathcal{F}, \mathcal{E}), X) = \mathrm{Hom}(\mathcal{F}, \mathcal{E})$.
For a quasi-coherent sheaf $\mathcal{F}$ we define the left exact functor

$$\mathcal{H}om(\mathcal{F}, -) : \mathsf{QCoh}(X) \to \mathsf{QCoh}(X)$$

Hence since for a noetherian scheme $\mathsf{QCoh}(X)$ has enough injectives there exists the right derived functor

$$R\mathcal{H}om(\mathcal{F}, \,) : \mathrm{D}^+(\mathsf{QCoh}(X)) \to \mathrm{D}^+(\mathsf{QCoh}(X))$$

and its higher derived functors are called 'local Ext'

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{E}) :- R^i\mathcal{H}om(\mathcal{F}, \mathcal{E})$$

Proposition A.0.16 (Local to Global Ext)**.** Let $\mathcal{F}$ and $\mathcal{E}$ in $\mathsf{QCoh}(X)$, for a noetherian scheme X, then there exists a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(X; \mathcal{E}xt^q(\mathcal{F}, \mathcal{F})) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{E})$$

Appendix B

Standard results in Scheme Theory

B.1 Ample and anti-ample canonical bundle

In these section we want to introduce the concept of ampleness for an invertible sheaf. This feature will be an crucial hypothesis on the canonical sheaf for the Bondal-Orlov theorem.
Let us recall that give a morphism from $X \to \mathbb{P}^n_Y$, where X is a scheme over Y, can be characterized by giving an invertible sheaf $\mathcal{L}$ on X and a suitable set of generating global section, or more precisely the functor

$$\mathrm{Sch} \to \mathrm{Set}$$

that sends a scheme X to the set of $(n+1)$-decorated sheaves on X up to isomorphisms, i.e. invertibles sheaves with a set $n+1$ generating global sections, is represented by $\mathbb{P}^n_Y$.
Then an invertible $\mathcal{L}$ sheaf is said to be very ample relative to Y if there is an immersion $i : X \to \mathbb{P}^n_Y$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. In the special case of $Y = \mathrm{Spec}A$ this is the same as asking that $\mathcal{L}$ admits a set of $n+1$ global section, such that the corresponding morphism $X \to \mathbb{P}^n_A$ under the above natural isomorphism is an immersion. Moreover we have the following

Proposition B.1.1. If $\mathcal{L}$ is a very ample invertible sheaf on a projective scheme X over a noetherian ring A, then for any coherent sheaf $\mathcal{F}$ there exists $n_0 > 0$ such that for all $n \geq n_0$ $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections.

We are going to use this property as a definition for a more general concept.

Definition B.1.2. An invertible sheaf $\mathcal{L}$ on a noetherian scheme X is ample if for any coherent sheaf $\mathcal{F}$ there exists $n_0 > 0$ such that for all $n \geq n_0$ $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections.

Let us briefly recall what means to be generated by global sections.

Definition B.1.3. The collection $\{s_i \mid i \in I\}$ of global sections generates the sheaf $\mathcal{L} \in \mathcal{O}_X\text{-}\mathsf{Mod}$ if one of the following equivalent conditions hold

i) $\forall x \in X$ $\{s_{i,x} \mid i \in I\}$ generates $\mathcal{L}_x$ as $\mathcal{O}_{X,x}$-module

ii) $\forall x \in X$ $\exists i \in I$ such that $s_{i,x}$ generates $\mathcal{L}_x$ as $\mathcal{O}_{X,x}$-module

iii) The collection of

$$X_{s_i} :- \{x \in X \mid s_{i,x} \text{ generates } \mathcal{L}_x \text{ as } \mathcal{O}_{X,x}\text{-module}\}$$

for $i \in I$ form an open cover of X

iv) The map

$$\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{L}$$

determined by the maps $\mathcal{O}_X \xrightarrow{\cdot s_i} \mathcal{L}$, is surjective.

Proposition B.1.4. Let $\mathcal{L}$ be an invertible sheaf on a noetherian scheme X, then the following are equivalent

i) $\mathcal{L}$ is ample

ii) $\mathcal{L}^m$ is ample for $m > 0$

iii) $\mathcal{L}^m$ is ample for some $m > 0$

In general ample sheaves need not to be very ample, but under suitable assumptions these two definitions coincide

Proposition B.1.5. Let X be a scheme of finite type over a noetherian ring A and let $\mathcal{L}$ be an invertible sheaf on X, then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample over Spec$A$ for some $m > 0$.

Remark B.1.6. If $X = \mathbb{P}^n_k$, $\mathcal{O}(1)$ is ample by definition, moreover for $d > 0$ also $\mathcal{O}(d)$ is ample since is the twisting sheaf of the Proj construction on shifted graded ring, moreover those are the only ample ones, since for $d < 0$ $\mathcal{O}(d)$ has no global sections, so it cannot be ample.

Ample invertible sheaves can also be characterized in term of vanishing of higher cohomology groups. The following result is due to Serre.

Proposition B.1.7 (**Serre Vanishing Theorem**)**.** Let X be a projective scheme over a noetherian ring A and let $\mathcal{L}$ be an invertible sheaf on X, then the following are equivalent

i) $\mathcal{L}$ is ample

ii) for each $\mathcal{F} \in \mathsf{Coh}(X)$ there exists $n_0 > 0$ depending on $\mathcal{F}$, such that for $i > 0$ and $n \geq n_0$

$$\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$$

The following is a criterion for a morphism to a projective space to be a closed immersion, i.e. for the corresponding invertible sheaf to be very ample.

Proposition B.1.8. Let X be a scheme over an algebraically closed field k, $\varphi : X \to \mathbb{P}^n_k$ a morphism with corresponding invertible sheaf $\mathcal{L}$ and $s_0, \ldots, s_n$ generating global sections and $V \subseteq \Gamma(X, \mathcal{L})$ a subspace spanned by the s_i's. Then φ is a closed immersion if and only if the following hold

i) elements of V separates points, i.e. for any two distinct closed point $P, Q \in X$, there exists $s \in V$ such that $s \in \mathsf{m}_P\mathcal{L}_P$ and $s \notin \mathsf{m}_Q\mathcal{L}_Q$

ii) elements of V separates tangent vectors, i.e. for each closed point $P \in X$ the set $\{s \in V | s_P \in \mathsf{m}_P\mathcal{L}_P\}$ spans the k-vector space $\mathsf{m}_P\mathcal{L}_P/\mathsf{m}_P^2\mathcal{L}_P$

We are now going to define the canonical sheaf associated to a scheme X as the n-th external power of the cotangent sheaf, in order to do so let us recall the following definition and basic properties.

Definition B.1.9. Let X be a scheme over Y with $f : X \to Y$ and $\Delta : X \to X \times_Y X$ be the corresponding diagonal map, we define the cotangent sheaf $\Omega_{X/Y} :- \Delta^*(\mathcal{I}/\mathcal{I}^2)$ where $\mathcal{I}$ is the shaef of ideaL associated to Δ.

At first sight this could seem a not completely immediate definition, but it reduces to a more 'reasonable' concept when we consider the affine case, in fact taking $X = \mathrm{Spec}B$ and $Y = \mathrm{Spec}A$ using the standard equivalence $\mathsf{AffSch} \simeq \mathsf{Ring}^{op}$ the following diagram

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

corresponds to

$$\begin{tikzcd}
B & & \\
& B \otimes_A B \arrow[ul] & B \arrow[l] \\
& B \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

then, setting $I = \ker(B \otimes_A B \to B)$, we get that $\Omega_{B/A} :- I/I^2$ is the module of relative differential form of B over A defined in [Har77] Chapter II section 8, that comes with the A-derivation

$$d : B \to I/I^2$$

$$b \mapsto b \otimes 1 - 1 \otimes b$$

and the universal property that for any other B-module M and A-derivation $d' : B \to M$ there exists a unique module morphism $I/I^2 \to M$ such that the following commutes

$$\begin{tikzcd}
B \arrow[r, "d'"] \arrow[dr, "d"'] & M \\
& I/I^2 \arrow[u, dashed]
\end{tikzcd}$$

Proposition B.1.10. Let $f : X \to Y$ be a morphism of scheme and Z a closed subscheme of X with corresponding sheaf of ideals $\mathcal{J}$. Then there is a short exact sequence in $\mathsf{Sh}(Z)$

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0 \tag{B.1}$$

Proposition B.1.11. Let $Y = \mathrm{Spec}(A)$ and $X = \mathbb{P}^n_A$, then there is a short exact sequence in $\mathsf{Sh}(X)$

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \bigoplus_{n+1} \mathcal{O}(-1) \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{B.2}$$

Proposition B.1.12. If X is a non singular projective variety over a field k, then $\Omega_{X/k}$ is a locally free sheaf of rank $n = \dim(X)$.

Definition B.1.13. Let X be a non-singular variety over k, we define the canonical sheaf of X to be $\omega_X :- \Lambda^n \Omega_{X/k}$ where $n = dim(X)$.

Since we are going to work with singular projective variety, by Proposition B.1.12 we get that ω_X is locally free of rank one, i.e. an invertible sheaf.

Example B.1.14. Let $X = \mathbb{P}^n_A := \mathbb{P}^r_{\mathbb{Z}} \times \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(A)$, taking the exterior power of the exact sequence in (B.2) with degree the rank of each term respectively, we get

$$\Lambda^n(\Omega_{\mathbb{P}^n_A}) \otimes \Lambda^1(\mathcal{O}_{\mathbb{P}^n_A}) \cong \Lambda^{n+1}(\bigoplus_{n+1} \mathcal{O}(-1))$$

but the n-th exterior power of n copies of one object is the tensor product n times of that object, so $\omega_{\mathbb{P}^n_A} \cong \mathcal{O}_{(} - n - 1)$.
Similarly taking Z a non singular hypersurface of degree d in X and applying the same reasoning to (B.1) we get $\omega_Y \cong \mathcal{O}(d - n - 1)$.

B.2 Tilde construction

We now briefly present the so called 'tilde construction' in the affine and the projective case. The tilde construction is a procedure that allow us to define a large variety of well-behaved $\mathcal{O}_X$-module, more precisely an $\mathcal{O}_X$-module that locally is of the form the tilde construction is quasi-coherent.
Let $X = \mathrm{Spec}(R)$ be an affine scheme and let us recall that for the structure sheaf we have $\mathcal{O}_X(X_f) = R_f$ and $\mathcal{O}_{X,p} = R_p$ with p a prime ideal of R and $f \in R$.

Definition B.2.1. Let $X = \mathrm{Spec}(R)$ be an affine scheme and M an R-module, we can define

$$\tilde{M}(X_f) :- M \otimes_R R_f \quad \text{and} \quad \tilde{M}_p := M \otimes_R R_p$$

Now the resulting sheaf, defined on the basis as $\tilde{M}(X_f)$ and with stalks $\tilde{M}_p$, for an open set $U \subseteq \mathrm{Spec}(R)$ is the set $\tilde{M}(U)$ of sections $s \in \prod_{P \in U} M_p$ such that holds the gluing condition: for all $p \in U$ there exist $V \subseteq U$ with $p \in V$, $m \in M$ and $f \in R$ such that for all $q \in V$ with $f \neq q$ the restriction is given by $s(q) = \frac{m}{f}$ in M_q.

Proposition B.2.2. Let $X = \mathrm{Spec}(R)$ be an affine scheme, then the tilde functor

$$R\text{-}\mathsf{Mod} \longrightarrow \mathcal{O}_X\text{-}\mathsf{Mod}$$
$$M \mapsto \tilde{M}$$

is fully-faithful and exact. Moreover it has a right adjoint given by the global section functor

$$\begin{tikzcd} R\text{-}\mathsf{Mod} \arrow[r, bend left, "(\tilde{-})"] \arrow[r, phantom, "\perp"] & \mathcal{O}_X\text{-}\mathsf{Mod} \arrow[l, bend left, "\Gamma(X{,}-)"] \end{tikzcd}$$

This adjunction becomes an equivalence of categories if we restrict to $\mathsf{QCoh}(X)$.

Let A be a ring and $r \in \mathbb{N}$, we consider $X = \mathbb{P}^r_A$ and the polynomial ring $S = A[X_1, \dots, X_r]$ which is naturally positively graded by considering the homogeneous polynomials of degree d.

Definition B.2.3. Let S be a graded ring, then a graded S-module M is an S-module with a decomposition $M = \bigoplus_{e \in \mathbb{Z}} M_e$ such that $S_d \cdot M_e \subseteq M_{d+e}$. Let then M be a graded S-module and $f \in S$ an homogeneous element, we define the homogeneous localization of M at the element f as

$$(M_f)_o :- \left\{ \frac{m}{f^e} \mid e \in \mathbb{Z},\ m \in M_{e \cdot \deg(f)} \right\}$$

which is a module over the homogeneous localization $(S_f)_o$.

Definition B.2.4. Considering now the standard affine covering of X given by $U_i = \operatorname{Spec}(R_i)$, where $R_i := A[\dots X_{ji} \dots]_{j \neq i}$ for $i = 0, \dots, r$, we define a quasi-coherent $\mathcal{O}_X$-module $\tilde{M}$ by setting

$$\tilde{M}(U_i) :- (\tilde{M_{X_i}})_o$$

with the gluing conditions on the overlaps $U_i \cup U_j$ given, at the ring level, by the isomorphisms

$$(M_{X_i})_{o, X_j/X_i} \cong (M_{X_j})_{o, X_i/X_j}$$

Let S graded ring as above and consider the graded ring $S(d)$ shifted by d, defined by $S(d)_m := S_{m+d}$ which has a trivial structure of graded S-module, then applying the tilde construction to it we re-obtain the standard twisted sheaf

$$\tilde{S(d)} \cong \mathcal{O}_X(d)$$

Proposition B.2.5. Let $X = \mathbb{P}^r_A$, then the tilde functor

$$\mathsf{Graded}\ S\text{-}\mathsf{Mod} \longrightarrow \mathsf{QCoh}(X)$$

$$M \mapsto \tilde{M}$$

is fully-faithful and exact. Moreover it is an equivalence of categories with (waek) right inverse functor

$$\Gamma_* : \mathsf{QCoh}(X) \longrightarrow \mathsf{Graded}\ S\text{-}\mathsf{Mod}$$

$$\mathcal{F} \mapsto \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathcal{F} \otimes \mathcal{O}_X(d))$$

where $\Gamma_*(\mathcal{F})$ is naturally a graded S-module considering $S = \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(d))$.

Proposition B.2.6. Let $S \coloneqq \bigoplus_m S_m$ be a graded ring generated by S_1 as S_0-algebra, and let $X = \operatorname{Proj}(S)$ the corresponding projective scheme. Let now $S(d)$ the grade ring shifted by d, then there exists an isomorphism

$$X \cong \operatorname{Proj}(S(d))$$

Moreover the invertible sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}(S(d))$ corresponds under this isomorphism to the sheaf $\mathcal{O}_X(d)$.

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