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Koszulness of Torelli Lie algebras

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Abstract

In this thesis we will follow the paper "On the Torelli Lie algebra" by Kupers and Randal-Willaims [40] to prove that the Lie algebra $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ is Koszul. The Lie algebra $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ is associated to the Torelli group of a surface and proving the Koszulness answers a conjecture of Richard Hain [21]. We will introduce the auxiliary objects $\mathcal{Z}_n$ and $\mathcal{E}_n$. Using category theory and the theory of $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations we will derive the connection between $\mathcal{E}_1$ and the quadratic dual of $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ which will allows us to study the Koszulness of $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ using $\mathcal{E}_1$. Finally using results on high-dimensional manifolds and results about graphs complexes we prove that $\mathcal{Z}_n$ and $\mathcal{E}_n$ are Koszul implying that $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ is Koszul in a stable range.

Contents

Introduction

Let $\Sigma_{g,1}$ denote a compact oriented surface of genus g and one boundary component. It's mapping class group $\mathcal{M}_{g,1}$ is the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{g,1}$. The mapping class group acts on $H_1(\Sigma_{g,1}; \mathbb{Q})$. We define the Torelli group, $\mathcal{T}_{g,1}$, to be the subgroup of $\mathcal{M}_{g,1}$ which acts trivially on $H_1(\Sigma_{g,1}; \mathbb{Q})$. There is a Lie algebra $\mathfrak{t}_{g,1}$ which is the Mal'cev completion of $\mathcal{T}_{g,1}$. In a beautiful paper Hain [20] proved that $\mathfrak{t}_{g,1}$ is isomorphic to the completion of the associated graded $\mathrm{Gr}^*_{lcs} \mathfrak{t}_{g,1}$ of it's lower central series for $4 \leq g$. Moreover Hain proved that $\mathrm{Gr}^*_{lcs} \mathfrak{t}_{g,1}$ is a quadratic Lie algebra. To do this Hain introduced a relative unipotent completion $\mathfrak{u}_{g,1}$ of $\mathcal{T}_{g,1}$ and proved that $\mathrm{Gr}^*_{lcs} \mathfrak{u}_{g,1}$ is quadratic. Since all Koszul Lie algebras are quadratic, it is a natural question to ask if $\mathrm{Gr}^*_{lcs} \mathfrak{u}_{g,1}$ and $\mathrm{Gr}^*_{lcs} \mathfrak{t}_{g,1}$ are Koszul. In 1997, Hain [20, 21] conjectured that $\mathrm{Gr}^*_{lcs} \mathfrak{u}_{g,1}$ and $\mathrm{Gr}^*_{lcs} \mathfrak{t}_{g,1}$ are Koszul. This conjecture was open until Kupers and Randal-Williams [40] finally gave an answer in their recent paper. The goal of this thesis is to follow Kupers and Randal-Williams and prove the theorem:

Main Theorem. *The Lie algebra* $\mathrm{Gr}_{lcs} \mathfrak{t}_{g,1}$ *is Koszul in weight* $\leq g/3$.

We use the diagonal criteria to define Koszulness. This states that if the weight grading and is not equal to the homological grading then the homology vanishes. We will also explain the tools and some history and results which are used in the proof for this result. Throughout the paper we will work over $\mathbb{Q}$ unless otherwise stated. We will also usually denote the Hom-sets of a category $\mathbf{C}$ by $\mathbf{C}(-,-)$ however we may use the classical $\mathrm{Hom}_{\mathbf{C}}$ or Hom notation as well. We assume the reader has some knowledge of category theory, model categories, representation theory, operads and vector bundles. We do however give references to useful sources on these topics when we introduce some theory which is related to these topics.

Motivation

Other than answering a long standing conjecture. There are a few other reasons why this result is interesting. To talk about the other motivation for this result let $\Sigma_{g,n}^r$ denote a surface of genus g, n boundary components and r marked points and we let $\mathcal{T}_{g,n}^r$ be the Torelli group associated to $\Sigma_{g,n}^r$. Let $\mathfrak{t}_{g,n}^r$ and $\mathfrak{u}_{g,n}^r$ be the Mal'cev completion and the relative unipotent completion of $\mathcal{T}_{g,n}^r$ (see Section 1.4 for detailed definitions).

The first reason why this result is interesting is that if $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ in a stable range then its variants $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_g$, $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_g^1$ and $\mathrm{Gr}^*_{lcs}\,\mathfrak{u}_g$, $\mathrm{Gr}^*_{lcs}\,\mathfrak{u}_g^1$, $\mathrm{Gr}^*_{lcs}\,\mathfrak{u}_{g,1}$ are all Koszul in the same stable range. This answers two of Hain's [21] conjectures.

Another reason is that Garoufalidis and Getzler [15] calculated the stable character of $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_g$ as a $\mathrm{Sp}_{2g}(\mathbb{Z})$-representation. They used the work of Looijenga, Madsen and Wiess to compute the stable character of the quadratic dual of $\mathrm{Gr}^*_{lcs}\,\mathfrak{u}_g$. From this they computed the stable character of $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_g$. Kupers and Randal-Williams [39] give an alternative approach to the computations of Garoufalidis and Getzler as well as a way to extend the work of Garoufalidis and Getzler to the boundary and marked point cases. Using this Koszulness result we can render the computations of characters of the graded algebraic $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_g$, $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_g^1$, $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$, $\mathrm{Gr}^*_{lcs}\,\mathfrak{u}_{g,1}$, $\mathrm{Gr}^*_{lcs}\,\mathfrak{u}_g$, $\mathrm{Gr}^*_{lcs}\,\mathfrak{u}_g^1$ and $\mathrm{Gr}^*_{lcs}\,\mathfrak{u}_{g,1}$ amenable to computer calculation.

Kupers and Randal-Williams [40] also use the Koszulness result to analyse the map of Lie algebras

$$\tau_{g,1}\colon \mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1} \to \mathfrak{h}_{g,1}$$

which is often referred to as the geometric Johnson homomorphism. Morita [53] asked if this map is injective in weight $\neq 2$. Kupers and Randal-Williams [40] also use the Koszulness result to prove that in a stable range the kernel of $\tau_{g,1}$ lies in the center of $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ and consists of trivial $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations.

Finally, the Koszulness of $\mathrm{Gr}^*_{lcs}\,\mathfrak{u}_g$ was simultaneously proved by Felder, Naef and Willwatcher [9] using a different argument to Kupers and Randal-Williams. They use the work of Chan, Payne and Galatius [5, 4] and a different spectral sequence to the one we use with our graph complexes.

Summary of thesis

In a short Chapter 1 we will make explicit some definitions which we need but will not come up in any of the other topics of our chapters. This will include definitions for homology, Koszul algebras and Koszul Lie algebras and explicit definitions for $\mathrm{Gr}^*_{lcs}\, \mathfrak{t}_{g,1}$ and $\mathrm{Gr}^*_{lcs}\, \mathfrak{u}_{g,1}$. We will also explain our grading our grading conventions.

The proof of the main theorem uses the auxiliary objects $\mathcal{Z}_n$ and $\mathcal{E}_n$. In Chapter 2 we will introduce the Downward Brauer category, dB, and the functors $\mathcal{Z}_n$ and $\mathcal{E}_n$. We will need many tools from categories and so we will also introduce enriched monoidal categories and enriched model categories. We will define what is the homology of algebra objects in certain functor categories. This will allow us to define Koszulness for functors as well.

In Chapter 3 our goal is explain the connection between the Lie algebra $\mathrm{Gr}^*_{lcs}\, \mathfrak{t}_{g,1}$ and the functor $\mathcal{E}_1$. We will introduce some theory of representations of groups and categories. This will allow us to realise functors the functors $\mathcal{Z}_n$ and $\mathcal{E}_n$ as $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations. We then explain some previous results on the cohomology of mapping class groups and Torelli groups which we shall use to construct an isomorphism, in a stable range, between the quadratic dual of $\mathrm{Gr}^*_{lcs}\, \mathfrak{t}_{g,1}$ and the realisation of $\mathcal{E}_1$. A corollary of this is that if $\mathcal{E}_n$ is Koszul then $\mathrm{Gr}^*_{lcs}\, \mathfrak{t}_{g,1}$ is Koszul in a stable range.

In Chapter 4 we give a short survey of some results of high-dimensional manifolds. We will draw on a lot of results from Kupers and Randal-Williams [38] to show that the parts of homology of $\mathcal{Z}_n$ not on the diagonal vanishes when evaluated on non-empty sets. Using a long exact sequence the same is true for $\mathcal{E}_n$.

Finally in Chapter 5 we shall explain how to construct graph complexes. We will use two graph complexes to model $\mathcal{Z}_n$ and $\mathcal{E}_n$. Using these models of $\mathcal{Z}_n$ and $\mathcal{E}_n$ we will show that the parts of the homology of $\mathcal{Z}_n$ and $\mathcal{E}_n$ on on the diagonal vanish when evaluated on the empty set. This will conclude the proof of the Koszulness.

Chapter 1

Preliminaries

In this chapter we will briefly introduce some preliminaries which we will use throughout the thesis. We will start by introducing some theory from homological algebra. In particular we will define homology of Lie algebras, homology of Commutative algebras and introduce some theory of Koszul algebras and Koszul duality. Finally we will introduce the Torelli Lie algebra.

1.1 Homology of Lie algebras

We define homology for Lie algebra's using the Chevalley-Eilenberg chain complex.

Definition 1.1. If $\mathfrak{g}$ is a Lie algebra over $\mathbb{Q}$ we define the *Chevelley-Eilenberg complex* $(\mathrm{CE}_*(\mathfrak{g}), d_{\mathrm{CE}})$ of $\mathfrak{g}$ by

$$\mathrm{CE}_p(\mathfrak{g}) = \Lambda^p \mathfrak{g}$$

where $\Lambda^p \mathfrak{g}$ denotes the p-th exterior power. The differential $d_{\mathrm{CE}} \colon \mathrm{CE}_p(\mathfrak{g}) \to \mathrm{CE}_{p-1}(\mathfrak{g})$ defined by

$$d_{\mathrm{CE}}(x_1 \wedge ... \wedge x_p) = \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge ... \wedge \hat{x}_i \wedge ... \wedge \hat{x}_j \wedge ... \wedge x_p$$

for all $x_1 \wedge ... \wedge x_p \in \Lambda^p(\mathfrak{g})$. If $\mathfrak{g} = \bigoplus_{i=0}^{\infty} \mathfrak{g}_i$ is a graded Lie algebra then $\mathrm{CE}_p(\mathfrak{g})$ inherits a grading from $\mathfrak{g}$ by setting

$$\mathrm{CE}_p(\mathfrak{g})_q = \bigoplus_{i_1+...+i_p=q} \mathfrak{g}_{i_1} \wedge ... \wedge \mathfrak{g}_{i_p}$$

for each $0 \leq p$ and $0 \leq q$. We will call the grading inherited from $\mathfrak{g}$ the internal grading. Notice that for a graded Lie algebra $\mathfrak{g}$ we have that $x \wedge y = (-1)^{|x| \cdot |y|} y \wedge x$ for all $x, y \in \mathfrak{g}$.

Sometimes our graded Lie algebras $\mathfrak{g}$ will have a second grading, that is

$$\mathfrak{g} = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} \mathfrak{g}_{i,j}.$$

We will call this second the weight grading and we may say that $x \in \mathfrak{g}_{q,w}$ has weight w. The Chevalley-Eilenberg complex will inherit the weight grading as well. A proof that the Chevelley-Eilenberg complex is indeed a chain complex can be found in [64], the proof carries over the graded case as well.

Definition 1.2. For a Lie algebra $\mathfrak{g}$ and $0 \leq p$ we define

$$H_p(\mathfrak{g}) = H_p(\mathrm{CE}(\mathfrak{g})).$$

If $\mathfrak{g}$ is a graded Lie algebra then the homology inherits the internal grading. We denoted the q-th piece of this grading by

$$H_p(\mathfrak{g})_q$$

and we also call this grading the internal grading. If $\mathfrak{g}$ is graded Lie has an additional weight grading this too is inherited by the homology we denote the w-th piece of the weight grading of the homology by

$$H_p(\mathfrak{g})_{q,w}.$$

We can then define Koszulness of Lie algebras using the diagonal criterion for homology.

Definition 1.3. A graded Lie algebra $\mathfrak{g}$ with a weight grading is Koszul in weight $\leq W$ if $H_p(\mathfrak{g})_{q,w} = 0$ for all $p \neq w$ and $w \leq W$. If $W = \infty$ we say that $\mathfrak{g}$ is absolutely Koszul.

1.2 Homology of Algebras

We define homology of commutative algebras and graded commutative algebras using the Harrison complex. This requires us to define the Bar construction for a commutative algebra. The proofs and definitions of our statements can be found in [55], [42] and [64].

Definition 1.4. The *bar construction* $(B_*(A), d)$ of an commutative algebra A a chain complex where $B_p(A) = A^{\otimes p}$ and the differential $d \colon B_p(A) \to B_{p-1}(A)$ being defined by

$$d(a_1 \otimes ... \otimes a_n) = \sum_{i=1}^{n-1} (-1)^i (a_1 \otimes ... \otimes a_{i-1} \otimes a_i a_{i+1} \otimes ... \otimes a_n)$$

for all $a_1, ..., a_n \in A$. If $A = \bigoplus_{i=0}^{\infty}$ is a graded commutative algebra then $B_*(A)$ inherits a grading from A where

$$B_p(A)_q = \bigoplus_{i_1 + ... + i_p = q} A_{i_1} \otimes ... \otimes A_{i_p}$$

for all $0 \leq p$ and $0 \leq q$. If A is a graded commutative algebra then we will call the grading on $B_*(A)$ inherited from A the internal grading.

We may work with graded commutative algebras A which may have a second grading, that is it may be the case that

$$A = \bigoplus_{i=0}^{\infty} \bigoplus_{j=0}^{\infty} A_{i,j}.$$

We will call the second grading the weight grading. The bar construction also inherits this second grading in the same way. We will denote the w-th piece of the $B_p(A)_q$ by $B_p(A)_{q,w}$. A proof that this is in fact a chain complex can be found in [42] and [55]. We will define a product on $B_*(A)$ using shuffle permutations.

Definition 1.5. A (p, q)-shuffle is a permutation $\sigma \in \mathfrak{S}_{p+q}$ such that

$$\sigma(1) < \sigma(2) < ... < \sigma(p) \text{ and } \sigma(p+1) < \sigma(p+2) < ... < \sigma(p+q).$$

We let $Sh(p, q)$ denote the set of (p, q)-shuffles.

The name of the shuffle permutations comes from the riffle shuffle of cards where the deck is split in two and the two groups are mixed together without changing the order of the cards in each group.

There is a natural action of the permutation group $\mathfrak{S}_k$ on $A^{\otimes k}$ defined by

$$\sigma \cdot (a_1 \otimes ... \otimes a_k) = a_{\sigma^{-1}(1)} \otimes ... \otimes a_{\sigma^{-1}(k)}$$

for all $\sigma \in \mathfrak{S}_k$ and $a_1, ..., a_k \in A$. We then define the shuffle product on $B_*(A)$.

Definition 1.6. Let A be a commutative algebra. We define the *shuffle product* $- \times_{sh} - : B_*(A) \otimes B_*(A) \to B_*(A)$ by

$$(a_1 \otimes ... \otimes a_p) \times_{sh} (a_{p+1} \otimes ... \otimes a_{p+q}) = \sum_{\sigma \in Sh(p,q)} \text{sgn}(\sigma)\sigma \cdot (a_1 \otimes ... \otimes a_{p+q})$$

for all $a_1, ..., a_{p+q} \in A$.

The shuffle product makes $B_*(A)$ into a differential graded algebra. A proof can be found in [42]. We set We can now define the Harrison complex.

Definition 1.7. For a commutative algebra A the *Harrison complex* $(\text{Harr}_*(A), d)$ is defined by

$$\text{Harr}_k(A) = B_k(A) / \left(\bigoplus_{p+q=k} B_p(A) \times_{sh} B_q(A) \right)$$

for all $2 \leq k$. The differential is induced by the differential on $B_*(A)$. When A is a graded commutative algebra then $\text{Harr}_*(A)$ inherits the gradings from $B_*(A)$ and similarly if A has a weight grading.

So for example $\text{Harr}_2(A) = A^{\otimes 2} / \text{span}\{a \otimes b - b \otimes a \mid a, b \in A\}$. We can now define the homology of a commutative algebra.

Definition 1.8. For every commutative algebra A we define the homology of A by

$$\text{Harr}_p(A) = H_p(C_*^h(A))$$

If A is a graded commutative algebra then the homology inherits the grading from A. We denote the q-th part by

$$\text{Harr}_p(A)_q$$

and we shall call this the internal grading. If A is graded and has a weight grading the homology also inherits this weight grading we denote the w-th part by

$$\text{Harr}_p(A)_{q,w}.$$

We call this additional grading on $\text{Harr}_p(A)$ the weight grading.

We can now define Koszulness for commutative algebras.

Definition 1.9. A dg-algebra A with a weight grading is Koszul in weight $\leq W$ if $\text{Harr}_p(A)_{q,w} = 0$ for all $p \neq w$ and $w \leq W$. If $W = \infty$ we say that A is absolutely Koszul.

1.3 Koszul duality

We now describe Koszul duality. To do this we require some notation and we will give the definition of quadratic algebras. A good source for quadartic algebras is the text by Polishuk and Positselski [55].

Definition 1.10. Let $V = \bigoplus_{i=0}^{\infty}$ be a vector space.

i) We denote the free Lie algebra on V by $\mathbb{L}^*(V)$.

ii) We denote the free graded commutative algebra on V by $\Lambda^*(V)$.

iii) We will denote the dual space of V by $V^\vee$, where $V_i^\vee = \mathrm{Hom}(V_{i+1}, \mathbb{Q})$.

We can now define quadratic Lie algebras and commutative algebras.

Definition 1.11. Let A be a commutative algebra and let $\mathfrak{g}$ be a Lie algebra. We say that

i) A is quadratic if there exists a graded vector space V and a subspace $R \subseteq \Lambda^2(V)$ such that $A \cong \Lambda^*(V)/R$,

ii) $\mathfrak{g}$ is quadratic if there exists a graded vector space V and a subspace $R \subseteq \mathbb{L}^2(V)$ such that $\mathfrak{g} \cong \mathbb{L}^*(V)/R$.

We denote the quadratic commutative algebra on V with relations $R \subseteq \Lambda^2(V)$ by $\Lambda(V, R)$. We denote the quadratic Lie algebra on V and $R \subseteq \mathbb{L}^2(V)$ by $\mathbb{L}(V, R)$.

We can then define the quadratic dual.

Definition 1.12. Let V, W be a graded vector space. Let $A = \Lambda(V, R_V)$ be a quadratic commutative algebra and $\mathfrak{g} = \mathbb{L}(W, R_W)$ a quadratic Lie algebra. We define the quadratic dual of A to be the Lie algebra $A^! = \mathbb{L}(V^\vee, R_V^\perp)$ where $R_V^\perp$ is the orthogonal compliment of R_V under the induced pairing

$$\langle \cdot, \cdot \rangle \colon \mathbb{L}^2(V^\vee) \otimes \Lambda^2(V) \to \mathbb{Q},$$

$$\langle [\alpha, \beta], x \wedge y \rangle = (-1)^{|y||\alpha|+|x|+|\alpha|} \alpha(x)\beta(y) - (-1)^{|\alpha||\beta|+|y||\beta|+|x|+|\beta|} \beta(x)\alpha(y).$$

Similarly we define the quadratic dual of $\mathfrak{g}$ to be commutative algebra $\mathfrak{g}^! = \Lambda(W^\vee, R_W^\perp)$ where $R_W^\perp$ is the orthogonal compliment of R_W under the induced pairing

$$\langle \cdot, \cdot \rangle \colon \Lambda^2(W^\vee) \otimes \mathbb{L}^2(W) \to \mathbb{Q},$$

defined by a similar formula as the previous pairing.

We can write the relations more explicitly. If $A = \Lambda(V, S)$ is generated by $\{x_i \mid i \in I\}$ modulo certain relations

$$\sum_{i,j} c_{ij} x_i \wedge x_j = 0$$

where I is some indexing set. Then the quadratic dual $A^! = \mathbb{L}(V^\vee, S^\perp)$ is generate by $\{\alpha_i \mid i \in I\}$ of degree $|\alpha_i| = |x_i| - 1$ modulo the relations

$$\sum_{i,j} \lambda_{ij} [\alpha_i, \alpha_j] = 0$$

such that $\sum_{i,j} (-1)^{|x_i||\alpha_j|} c_{ij} \lambda_{ij} = 0$.

We have the following result given by [50] for absolute Koszulness but the proof can easily be adjusted to Koszulness in a stable range.

Proposition 1.13. *A quadratic algebra $A = \Lambda(V, R)$ on V and R is Koszul if and only if $A^! = \mathbb{L}(V^\vee, R^\perp)$ is Koszul. Similarly, a quadratic Lie algebra $\mathfrak{g} = \mathbb{L}(V, R)$ on V and R is Koszul if and only if $\mathfrak{g}^! = \Lambda(V^\vee, R^\perp)$ is Koszul.*

1.4   Torelli group and Lie algebras

Let $\Sigma^r_{g,n}$ denote an orientable compact surface of genus g with n boundary components and r marked points. If $n = 0$ or $r = 0$ then we shall omit n or r. For example we shall write Σ_g instead of $\Sigma^0_{g,0}$. Recall the mapping class group of the surface is the group

$$\mathcal{M}^r_{g,n} = \pi_0(\mathrm{Diff}^+_\partial(\Sigma^r_{g,n})) = \pi_1(B\,\mathrm{Diff}^+_\partial(\Sigma^r_{g,n}))$$

of isotopy classes of orientation-persevering diffeomorphisms which fix the marked points and the boundary components pointwise. There is a natural action of the mapping class group $\mathcal{M}^r_{g,n}$ on $H_1(\Sigma_g; \mathbb{Z})$ via the inclusion $\mathcal{M}^r_{g,n} \hookrightarrow \mathcal{M}_g$.

Definition 1.14. We define the Torelli group $\mathcal{T}^r_{g,n}$ to be the subgroup of $\mathcal{M}^r_{g,n}$ which acts trivially on $H_1(\Sigma_g; \mathbb{Z})$.

The homology group $H_1(\Sigma_g; \mathbb{Z})$ has a pairing

$$B(\cdot, \cdot) \colon H_1(\Sigma_g; \mathbb{Z}) \times H_1(\Sigma_g; \mathbb{Z}) \to H_2(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}, \quad B(x, y) = \langle i(x) \smile i(y), [\Sigma_g] \rangle$$

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induced by the orientation of $\Sigma_{g,1}$, usual intersection pairing

$$\langle \cdot, \cdot \rangle \colon H_k(\Sigma_g; \mathbb{Z}) \times H^i(\Sigma_g; \mathbb{Z}) \to H_{k-i}(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}$$

and Poincaré duality $i \colon H_1(\Sigma_g; \mathbb{Z}) \cong H^1(\Sigma_g; \mathbb{Z})$, see [22, Chapter 3] for more details. The intersection product is a skew-symmetric and non-degenerate pairing which is preserved by the action of $\mathcal{M}_{g,n}^r$ which gives us a short exact sequence of groups

$$1 \to \mathcal{T}_{g,n}^r \to \mathcal{M}_{g,n}^r \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to 1$$

where $\mathrm{Sp}_{2g}(\mathbb{Z})$ is the integer points of the algebraic group $\mathbf{Sp}_{2g}$.

Now we will define the Mal'cev completion of $\mathcal{T}_{g,n}^r$. Before we can define Torelli Lie algebra we will need to define nilpotent groups and Lie algebras.

Definition 1.15. The lower central series of a group G is defined to be the series
$$\Gamma^0(G) \supseteq \Gamma^1(G) \supseteq \Gamma^2(G) \supseteq \ldots$$
of groups where $\Gamma^0(G) = G$ and $\Gamma^{i+1}(G) = [\Gamma^i(G), G]$ for all $i \in \mathbb{N}$. We say that G is nilpotent if its lower central series vanishes at some finite k, that is $\Gamma^k(G) = 1$ for some $k \in \mathbb{N}$.

Definition 1.16. The lower central series of a Lie algebra $\mathfrak{g}$ is defined to be the series
$$\Gamma^0(\mathfrak{g}) \supseteq \Gamma^1(\mathfrak{g}) \supseteq \Gamma^2(\mathfrak{g}) \supseteq \ldots$$
of groups where $\Gamma^0(\mathfrak{g}) = \mathfrak{g}$ and $\Gamma^{i+1}(\mathfrak{g}) = [\Gamma^i(\mathfrak{g}), \mathfrak{g}]$ for all $i \in \mathbb{N}$. We say that $\mathfrak{g}$ is nilpotent if its lower central series vanishes at some finite k, that is $\Gamma^k(\mathfrak{g}) = 0$ for some $k \in \mathbb{N}$. We also define the associated graded of the lower central series of $\mathfrak{g}$ to be the graded Lie algebra $\mathrm{Gr}_{lcs}^* \mathfrak{g}$ where $\mathrm{Gr}_{lcs}^k \mathfrak{g} = \Gamma^k(\mathfrak{g})/\Gamma^{k+1}(\mathfrak{g})$ for all $0 \le k$.

Now recall the definition of a rationalisation of a nilpotent group.

Definition 1.17. A $\mathbb{Q}$-power group G is a group which has a solution to the equation $x^m = g$ for all $k \in \mathbb{N}$ and $g \in G$.

Definition 1.18. A rationalisation $(G_{\mathbb{Q}}, \phi)$ of a torsion-free nilpotent group G is a group $G_{\mathbb{Q}}$ and an embedding $\phi \colon G \to G_{\mathbb{Q}}$ such that for all $g \in G_{\mathbb{Q}}$ there exists a non-zero $k \in \mathbb{N}$ such that $g^k \in \phi(G)$.

From the work of Mal'cev [47] and Quillen [56] we have the following result. See also [7].

Theorem 1.19. *For every nilpotent group G there exists a rationalisation $(G_{\mathbb{Q}}, \phi)$ and a nilpotent Lie algebra $\mathfrak{l}(G)$ associated to G.*

However it is well-known that for any group G we quotient $G/\Gamma^k(G)$ is nilpotent for all $k > 0$. This allows us to define the following pro-nilpotent Lie algebra of a group.

Definition 1.20. For a group G we define the *Mal'cev completion* of G to be the Lie algebra $\mathfrak{l}(G) = \varprojlim G/\Gamma^k(G))$ given by the inverse limit of

$$\mathfrak{l}(G/\Gamma^0(G)) \subseteq \mathfrak{l}(G/\Gamma^1(G)) \subseteq \mathfrak{l}(G/\Gamma^2(G)) \subseteq \ldots$$

Definition 1.21. We define the Torelli Lie algebra to be the Lie algebra $\mathfrak{t}_{g,n}^r = \mathfrak{l}(\mathcal{T}_{g,n}^r)$. We define $\mathrm{Gr}_{lcs}^* \mathfrak{t}_{g,1}$ to be concentrated in internal grading 0 and define the weight grading of $\mathrm{Gr}_{lcs}^* \mathfrak{t}_{g,1}$ to be given by $(\mathrm{Gr}_{lcs}^* \mathfrak{t}_{g,1})_{0,w} = \mathrm{Gr}_{lcs}^w \mathfrak{t}_{g,1}$.

Hain [19] proved that there is an initial pro-algebraic group $\mathcal{G}_{g,n}^r$ over $\mathbb{Q}$ with a Zariski dense homomorphism to $\mathcal{G}_{g,n}^r \to \mathbf{Sp}_{2g}$ which has a pro-unipotent kernel.

Definition 1.22. We denote the Mal'cev completion of the kernel $\ker(\mathcal{G}_{g,n}^r \to \mathbf{Sp}_{2g})$ by $\mathfrak{u}_{g,n}^r$. We call this the relative pro-unipotent completeion of $\mathcal{T}_{g,n}^r$.

Kupers and Randal-Williams [40] then proved the following result.

Proposition 1.23. *For $g \geq 0$, if $\mathrm{Gr}_{lcs}^* \mathfrak{t}_{g,1}$ is Koszul in weight $\leq W$. Then the Lie algebra's $\mathrm{Gr}_{lcs}^* \mathfrak{t}_g^1, \mathrm{Gr}_{lcs}^* \mathfrak{t}_g, \mathrm{Gr}_{lcs}^* \mathfrak{u}_{g,1}, \mathrm{Gr}_{lcs}^* \mathfrak{u}_g^1$ and $\mathrm{Gr}_{lcs}^* \mathfrak{u}_g$ are Koszul in the same weight.*

Remark. It is not possible for $\mathrm{Gr}_{lcs}^* \mathfrak{t}_g$, $\mathrm{Gr}_{lcs}^* \mathfrak{t}_{g,1}$ and $\mathrm{Gr}_{lcs}^* \mathfrak{t}_g^1$ to be absolutely Koszul. However the absolute Koszulness of $\mathrm{Gr}_{lcs}^* \mathfrak{u}_g$ has not be ruled out. See [40] for more details.

Chapter 2

Homology in Functor Categories

In this chapter, we describe homology for algebra objects in some enriched functor categories. In order to do this we will briefly introduce some theory of enriched monoidal categories and enriched model categories. We will then use this theory to define the homology of algebra objects in functor categories. We will use two notions of homology, André-Quillen Homology and Harrison homology, and prove that they are equivalent. Finally, we will introduce the functors E_n and $\mathcal{Z}_n$ whose homology we will compute in this thesis.

2.1 Model and Monoidal Category theory

In this section, we will introduce some theory of enriched categories. In particular enriched symmetric monoidal category theory and enriched model category theory. The reason for the introduction to these topics is to prove that the category $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ is a symmetric monoidal model category where $\mathbf{I}$ is a small symmetric monoidal $\mathbf{Vec}_{\mathbb{Q}}$-enriched category. Many of the things we introduce can be found in [34] and [60].

2.1.1 Enriched Symmetric monoidal categories

We will remind the reader of what an enriched category is. We will then describe what an enriched symmetric monoidal category is. Recall that a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbb{1})$ is a category together with

a monoidal product $\otimes\colon \mathbf{V}\times\mathbf{V}\to\mathbf{V}$ and a together with a monoidal unit $\mathbb{1}$. See [45, Chapter VII] for more details.

Definition 2.1. Given a symmetric monoidal category $(\mathbf{V},\otimes,\mathbb{1})$. A $\mathbf{V}$-category $\mathbf{C}$ consists of

i) a class of objects $\mathbf{C}$,

ii) hom-objects $\mathbf{C}(x,y)\in\mathbf{V}$ for each pair x,y of objects in $\mathbf{C}$,

iii) a $\mathbf{V}$-morphism $id_x\colon \mathbb{1}\to\mathbf{C}(x,x)$ for each object $x\in\mathbf{C}$,

iv) a $\mathbf{V}$-morphism $\circ\colon \mathbf{C}(y,z)\otimes\mathbf{C}(x,y)\to\mathbf{C}(x,z)$ for each triple x,y,z of $\mathbf{C}$-objects

such that the following diagrams

$$
\begin{array}{ccc}
\mathbf{C}(z,w)\otimes\mathbf{C}(y,z)\otimes\mathbf{C}(x,y) & \xrightarrow{\ \circ\otimes 1\ } & \mathbf{C}(y,w)\otimes\mathbf{C}(x,y) \\
{\scriptstyle 1\otimes\circ}\big\downarrow & & \big\downarrow{\scriptstyle\circ} \\
\mathbf{C}(z,w)\otimes\mathbf{C}(x,z) & \xrightarrow{\ \ \circ\ \ } & \mathbf{C}(x,w)
\end{array}
$$

$$
\begin{array}{ccccc}
\mathbf{C}(x,y)\otimes\mathbf{C}(x,x) & & & & \mathbf{C}(y,y)\otimes\mathbf{C}(x,y) \\
\big\uparrow & \searrow{\scriptstyle\circ} & & \swarrow{\scriptstyle\circ} & \big\uparrow \\
{\scriptstyle 1\otimes id_x} & & \mathbf{C}(x,y) & & {\scriptstyle id_y\otimes 1} \\
\big| & \nearrow{\scriptstyle\cong} & & \nwarrow{\scriptstyle\cong} & \big| \\
\mathbf{C}(x,y)\otimes\mathbb{1} & & & & \mathbb{1}\otimes\mathbf{C}(x,y)
\end{array}
$$

commute. We call these diagrams the composition and identity diagrams respectively. We will denote the underlying category of $\mathbf{C}$ by $(\mathbf{C})_0$.

The difference between the usual categories which often appear and enriched categories is analogous to the difference between Abelian groups and R-modules. That is one can view Abelian groups as $\mathbb{Z}$-modules. Analogously the usual categories that appear in mathematics are often locally small (i.e. have hom-sets which are actually sets). We can view locally small categories as categories enriched over the category of sets, **Set**. We will give two examples of enriched categories below.

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Example 2.2 (Chain Complexes). Assume R is a unital commutative ring. A chain complex (C_*, d) of R-modules is a collection $\{(C_i, d_i)\}_{i=0}^\infty$ of R-modules $C_0, C_1, C_2, \ldots$ together with linear maps $d_n \colon C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$. We often drop the indices on the differentials $\{d_i\}_{i=0}^\infty$ and to avoid confusion we may also denote the differential on C_* by d_C. We then define $\otimes \colon \mathbf{Ch}_R \times \mathbf{Ch}_R \to \mathbf{Ch}_R$ by

$$(C_* \otimes D_*)_n = \bigoplus_{i+j=n} C_i \otimes_R D_j$$

with the differential given by $d_{C \otimes D} = d_C \otimes id_D + id_C \otimes d_D$ for all $C_*, D_* \in \mathbf{Ch}_R$ and $n \in \mathbb{N}$. The unit $\mathbb{1}$ of $\otimes$ is the chain complex with $\mathbb{1}_0 = R$ and $\mathbb{1}_i = 0$ for all $i \neq 0$.

We can enrich $\mathbf{Ch}_R$ over itself. Define a chain map $f \colon (C_*, d_*) \to (D_*, b_*)$ of degree k to be a collection $\{f_i\}_{i=0}^\infty$ of linear maps $f_n \colon C_n \to D_{n+k}$ such that $f_{n-1} \circ d_C = d_D \circ f_n$. Then we define $\mathbf{Ch}_R(C_*, D_*)_k$ to be the collection of degree k chain maps with R-module structure given by pointwise addition and scalar multiplication. The differential ∂ on $\mathbf{Ch}_R(C_*, D_*)_*$ is given by $\partial(f) = d_D \circ f + (-1)^k f \circ d_C$ for all $f \in \mathbf{Ch}_R(C_*, D_*)_*$. In fact, $\mathbf{Ch}_R$ is an example of a *cosmos*, a term we will define later.

Let $\mathbf{Vec}_\mathbb{Q}$ and $\mathbf{GrVec}_\mathbb{Q}$ denote the category of vector spaces and the category of graded vector spaces respectively. Note that $\mathbf{Vec}_\mathbb{Q}$ is a subcategory of $\mathbf{GrVec}_\mathbb{Q}$ where a vector space is viewed as a graded vector space concentrated in the 0th-degree. The category $\mathbf{GrVec}_\mathbb{Q}$ is a subcategory of $\mathbf{Ch}_\mathbb{Q}$ where we view a graded vector space as a chain complex with zero differential.

Example 2.3 (Simplical Sets). Let us denote the ordered set $\{0, 1, \ldots, n\}$ by $\underline{n}$. We define the *ordinal number* category $\mathbf{\Delta}$ to be the category with objects being the ordered sets $\underline{n} = \{0, 1, 2, \ldots, n\}$ and morphisms being order preserving maps $\phi \colon \underline{n} \to \underline{m}$. In particular for $0 \leq i \leq n-1$ and $0 \leq i \leq n$ the maps

$$d^i \colon \underline{n-1} \to \underline{n}, \quad d^i(x) = \begin{cases} x, & 0 \leq x \leq i-1 \\ x+1, & i \leq x \leq n-1 \end{cases}$$

$$s^j \colon \underline{n} \to \underline{n-1}, \quad s^j(x) = \begin{cases} x, & 0 \leq x \leq j \\ x-1, & j+1 \leq x \leq n \end{cases}$$

generate all other morphisms in $\mathbf{\Delta}$. A simplicial set is then a functor $X\colon \mathbf{\Delta}^{op} \to \mathbf{Set}$. We shall write X_n for $X(\underline{n})$ and $d_i = X(d^i)\colon X_n \to X_{n-1}$ and $s_j = X(s^j)\colon X_{n-1} \to X_n$. These maps are called the face maps and the degeneracy maps respectively. An example Δ^n, which is called the standard n-simplex is the functor $\Delta^n := \mathbf{\Delta}(-, \underline{n})\colon \mathbf{\Delta}^{op} \to \mathbf{Set}$. We denote the category of simplicial sets by $\mathbf{sSet}$. We may enrich $\mathbf{sSet}$ over itself by defining $\mathbf{sSet}(X, Y)_n$ to be the set of natural transformations from $\Delta^n \times X \to Y$. Notice that we have morphisms $\tilde{d}^i\colon \Delta^{n-1} \to \Delta^n$ and $\tilde{s}^j\colon \Delta^n \to \Delta^{n-1}$ for all $0 \le i \le n-1$ and $0 \le j \le n$ given by composition. The face a degeneracy maps of $\mathbf{sSet}(X, Y)_*$ are then given by pre-composing with the morphisms

$$\tilde{d}^i \times 1\colon \Delta^{n-1} \times Y \to \Delta^n \times Y$$

$$\tilde{s}^j \times 1\colon \Delta^n \times Y \to \Delta^{n-1} \times Y$$

for all $0 \le i \le n-1$ and $0 \le j \le n$. The composition of n-simplices $f\colon \Delta^n \times X \to Y$ and $g\colon \Delta^n \times Y \to Z$ is the morphism

$$\Delta^n \times X \xrightarrow{diag \times 1} \Delta^n \times \Delta^n \times X \xrightarrow{1 \times f} \Delta^n \times Y \xrightarrow{g} Z$$

where $diag$ is the diagonal maps, for all simplicial sets X, Y and Z. This define the composition morphism $\mathbf{sSet}(Y, Z)_* \times \mathbf{sSet}(X, Y)_* \to \mathbf{sSet}(X, Z)_*$.

Definition 2.4. Given a $\mathbf{V}$-category $\mathbf{C}$ for every $x \in \mathbf{C}$ we get a functor $\mathbf{C}(x, -)\colon \mathbf{C} \to \mathbf{V}$. If this functor has a left adjoint then we usually denote it by $- \otimes x$. We say that $\mathbf{C}$ is *tensored* if the left adjoint $- \otimes x$ of $\mathbf{C}(x, -)$ exists for all $x \in \mathbf{C}$. Similarly $\mathbf{C}$ is *cotensored* if for all $y \in \mathbf{C}$ there exists a functor $y^{(-)}\colon \mathbf{V} \to \mathbf{C}$ such that

$$\mathbf{C}(a \otimes x, y) \cong \mathbf{V}(a, \mathbf{C}(x, y)) \cong \mathbf{C}(x, y^a).$$

for all $x, y \in \mathbf{C}$ and $a \in \mathbf{V}$. If $\mathbf{C}$ is tensored and cotensored we then say that it is closed.

Remark 2.5. From here on we will assume that all our enriched categories are closed. Moreover, we will also assume that all the categories we work with are complete and cocomplete, i.e all small limits and colimits exist.

There are also definitions for functors which preserve the enrichment structure.

Definition 2.6. Given $\mathbf{V}$-categories $\mathbf{C}, \mathbf{D}$ a $\mathbf{V}$-functor $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ consists of an function $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ on objects and for each pair of objects $x, y \in \mathbf{C}$ a morphism $\mathcal{F}_{x,y}\colon \mathbf{C}(x,y) \to \mathbf{D}(\mathcal{F}x, \mathcal{F}y)$ in $\mathbf{V}$. Such that the diagrams

$$
\begin{array}{ccc}
\mathbf{C}(y,z) \otimes \mathbf{C}(x,y) & \xrightarrow{\;\;\circ\;\;} & \mathbf{C}(x,z) \\
{\scriptstyle \mathcal{F} \otimes \mathcal{F}}\big\downarrow & & \big\downarrow {\scriptstyle \mathcal{F}} \\
\mathbf{D}(\mathcal{F}y, \mathcal{F}z) \otimes \mathbf{D}(\mathcal{F}x, \mathcal{F}y) & \xrightarrow{\;\;\circ\;\;} & \mathbf{D}(\mathcal{F}x, \mathcal{F}z)
\end{array}
$$

$$
\begin{array}{ccc}
& & \mathbf{C}(x,x) \\
& {\scriptstyle id_x}\nearrow & \big\downarrow {\scriptstyle \mathcal{F}} \\
* & \xrightarrow[{\scriptstyle id_{\mathcal{F}x}}]{} & \\
& & \mathbf{D}(\mathcal{F}x, \mathcal{F}x)
\end{array}
$$

commute.

Definition 2.7. Given $\mathbf{V}$-functors $\mathcal{F}, \mathcal{G}\colon \mathbf{C} \to \mathbf{D}$ a $\mathbf{V}$-natural transformation $\eta\colon \mathcal{F} \to \mathcal{G}$ is a family of morphisms $\eta_x\colon * \to \mathbf{D}(\mathcal{F}x, \mathcal{G}x)$ in $\mathbf{V}$ such that the diagram

$$
\begin{array}{ccccc}
& & \mathbf{C}(x,y) \otimes * & \xrightarrow{\;\mathcal{G}_{x,y} \otimes \eta_x\;} & \mathbf{D}(\mathcal{G}x, \mathcal{G}y) \otimes \mathbf{D}(\mathcal{F}x, \mathcal{G}x) \\
& {\scriptstyle \cong}\nearrow & & & \searrow {\scriptstyle \circ} \\
\mathbf{C}(x,y) & & & & \mathbf{D}(\mathcal{F}x, \mathcal{G}y) \\
& {\scriptstyle \cong}\searrow & & & \nearrow {\scriptstyle \circ} \\
& & * \otimes \mathbf{C}(x,y) & \xrightarrow[{\;\eta_y \otimes \mathcal{F}_{x,y}\;}]{} & \mathbf{D}(\mathcal{F}y, \mathcal{G}y) \otimes \mathbf{D}(\mathcal{F}x, \mathcal{F}y)
\end{array}
$$

commutes. The composition $\upsilon \circ \eta$ of two $\mathbf{V}$-natural transformations $\eta\colon \mathcal{F} \to \mathcal{G}$ and $\upsilon\colon \mathcal{G} \to \mathcal{H}$ is the family of maps given by the composition

$$
I \xrightarrow{\cong} I \otimes I \xrightarrow{\upsilon_x \otimes \eta_x} \mathbf{D}(\mathcal{G}x, \mathcal{H}x) \otimes \mathbf{D}(\mathcal{F}x, \mathcal{G}x) \xrightarrow{\circ} \mathbf{D}(\mathcal{F}x, \mathcal{H}x)
$$

for all $x \in \mathbf{C}$. The category of $\mathbf{V}$-functors from $\mathbf{C}$ to $\mathbf{D}$ is the category $\mathbf{D}^{\mathbf{C}}$ with objects being $\mathbf{V}$-functors and morphisms being $\mathbf{V}$-natural transformations.

Definition 2.8. A $\mathbf{V}$-enriched symmetric monoidal category or a $\mathbf{V}$-*cosmos* $(\mathbf{C}, \otimes_{\mathbf{C}}, \mathbb{1})$ consists of a $\mathbf{V}$-category $\mathbf{C}$ and a monoidal structure $(\mathbf{C}, \otimes_{\mathbf{C}}, \mathbb{1})$ on the underlying category of $\mathbf{C}$ such that $\otimes_{\mathbf{C}} \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a $\mathbf{V}$-functor. If $\mathbf{C} = \mathbf{V}$ then we simply call $\mathbf{C}$ a cosmos.

From here on we will assume that $\mathbf{V}$ is a cosmos. We will be working with the category of functors between two enriched monoidal categories. We would like to give the category of functors which we will work with a monoidal structure. We will need ends and coends to define an appropriate monoidal product. The treatise "Coend Calculus" by Fosco Loregian [44] provides a good introduction to ends and coends in a variety of categorical settings.

Definition 2.9. Let $\mathbf{C}$ be a $\mathbf{V}$-enriched category and let $\mathcal{F} \colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{V}$ be a $\mathbf{V}$-functor. For any object $v \in \mathbf{V}$ let $const_v$ denote the constant functor $const_v \colon \mathbf{C} \to \mathbf{V}$ with value at v and let $\triangle \colon \mathbf{C} \to \mathbf{C}^{op} \times \mathbf{C}$ denote the diagonal functor. The *end* (w, η) of $\mathcal{F}$ is an object $w \in \mathbf{V}$ and a $\mathbf{V}$-natural transformation $\eta \colon const_w \to \mathcal{F} \circ \triangle$ such that if v is any other object in $\mathbf{V}$ with $\mathbf{V}$-natural transformation $\nu \colon const_v \to \mathcal{F} \circ \triangle$ then there is a unique morphism $f \colon w \to v$ in $\mathbf{V}$ such that the diagram

$$
\begin{array}{ccc}
v & \xrightarrow{\ f\ } & w \\
& {\scriptstyle \nu_c}\searrow & \downarrow {\scriptstyle \eta_c} \\
& & \mathcal{F}(c, c)
\end{array}
$$

commutes for all $c \in \mathbf{C}$. The object of the end will be denoted by $\int_{c \in \mathbf{C}} \mathcal{F}(c, c)$ or by $\int_{\mathbf{C}} \mathcal{F}$ and the associated $\mathbf{V}$-natural transformation will be called the *counit* of $\int_{c \in \mathbf{C}} \mathcal{F}(c, c)$.

If $\mathcal{F}, \mathcal{G} \in \mathbf{D}^{\mathbf{C}^{op} \times \mathbf{C}}$ and $\eta \colon \mathcal{F} \to \mathcal{G}$ is a $\mathbf{V}$-natural transformation then for each $c \in \mathbf{C}$ we can take the composition of the counit with $\eta_{c,c}$

$$
\int_{\mathbf{C}} \mathcal{F} \to \mathcal{F}(c, c) \xrightarrow{\ \eta_{c,c}\ } \mathcal{G}(c, c)
$$

to get a morphism $\int_{\mathbf{C}} \mathcal{F} \to \mathcal{G}(c, c)$. By the definition of the end, there exists a unique morphism

$$
\int_{\mathbf{C}} \mathcal{F} \to \int_{\mathbf{C}} \mathcal{G}
$$

which we will call the end of η and will sometimes denote it by $\int_{\mathbf{C}} \eta$. If $\mathcal{H} \in \mathbf{D}^{\mathbf{C}^{op} \times \mathbf{C}}$ and $\nu \in \mathbf{D}^{\mathbf{C}^{op} \times \mathbf{C}}(\mathcal{G}, \mathcal{H})$ then we can deduce

$$\int_{\mathbf{C}} \nu \circ \eta = \int_{\mathbf{C}} \nu \circ \int_{\mathbf{C}} \eta$$

from the uniqueness property in the definition of the end. This implies that the end is functorial.

The following is a well-known result about ends and a proof can be found in [34] or in [44].

Lemma 2.10. *If $\mathbf{I}$ is a small $\mathbf{V}$-category and $\mathcal{F} \colon \mathbf{I}^{op} \times \mathbf{I} \to \mathbf{V}$ is a $\mathbf{V}$-functor the the end of $\mathcal{F}$ exists and can be expressed as the following coequalizer*

$$\int_{i \in \mathbf{I}} \mathcal{F}(i,i) \xrightarrow{\ \lambda\ } \prod_{j \in \mathbf{I}} \mathcal{F}(j,j) \underset{\sigma_{k,j}}{\overset{\rho_{j,k}}{\rightrightarrows}} \prod_{j,k \in \mathbf{I}} \mathbf{V}(\mathbf{I}(j,k), \mathcal{F}(j,k))$$

where $\rho_{j,k}$ and $\sigma_{j,k}$ are the adjuncts of $\mathcal{F}(j,-)_{j,k}$ and $\mathcal{F}(-,k)_{j,k}$ respectively.

One application of ends is the following lemma.

Lemma 2.11. *If $\mathbf{I}$ and $\mathbf{C}$ are $\mathbf{V}$-categories where $\mathbf{I}$ is small then the category $\mathbf{C}^{\mathbf{I}}$ can be made into a $\mathbf{V}$-category.*

Proof. For all $\mathcal{F}, \mathcal{G} \in \mathbf{C}^{\mathbf{I}}$ we set $\mathbf{C}^{\mathbf{I}}(\mathcal{F}, \mathcal{G})$ to be

$$\mathbf{C}^{\mathbf{I}}(\mathcal{F}, \mathcal{G}) = \int_{i \in \mathbf{I}} \mathbf{C}(\mathcal{F}(i), \mathcal{G}(i))$$

For all $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbf{C}^{\mathbf{I}}$ and $i \in \mathbf{I}$ we get morphisms

$$\circ_{\mathbf{C},i} \colon \mathbf{C}(\mathcal{G}(i), \mathcal{H}(i)) \otimes \mathbf{C}(\mathcal{F}(i), \mathcal{G}(i)) \to \mathbf{C}(\mathcal{F}(i), \mathcal{H}(i))$$

$$id_{\mathbf{C}, \mathcal{F}(i)} \colon * \to \mathbf{C}(\mathcal{F}(i), \mathcal{F}(i))$$

from the enriched identity and composition morphisms from the enriched structure of $\mathbf{C}$. We then set $\circ$ and $id_{\mathcal{F}}$ to be the morphisms

$$\circ = \int_{\mathbf{C}} \circ_{\mathbf{C}} \colon \mathbf{C}^{\mathbf{I}}(\mathcal{G}, \mathcal{H}) \otimes \mathbf{C}^{\mathbf{I}}(\mathcal{F}, \mathcal{G}) \to \mathbf{C}^{\mathbf{I}}(\mathcal{F}, \mathcal{H})$$

$$id_{\mathcal{F}} = \int_{\mathbf{C}} id_{\mathbf{C}, \mathcal{F}(-)} \colon * \to \mathbf{C}^{\mathbf{I}}(\mathcal{F}, \mathcal{F})$$

for all $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbf{C}^{\mathbf{I}}$. By functoriality of the end, the composition and identity diagrams hold. Therefore $\mathbf{C}^{\mathbf{I}}$ is a $\mathbf{V}$-category. $\square$

Definition 2.12. Let $\mathbf{C}, \mathbf{D}$ be $\mathbf{V}$-enriched categories and let $\mathcal{F}\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$ be a functor. A *coend* (w, ξ) of $\mathcal{F}$ is an object $w \in \mathbf{D}$ together with a $\mathbf{V}$-natural transformation $\xi\colon \mathcal{F} \circ \triangle \to const_w$ such that for all $d \in \mathbf{D}$ we have that

$$\mathbf{D}(w, d) \cong \int_{c \in \mathbf{C}} \mathbf{D}(\mathcal{F}(c, c), d)$$

and the $\mathbf{V}$-natural transformation

$$\mathbf{D}(\xi, d)\colon \mathbf{D}(const_w, d) \to \mathbf{D}(\mathcal{F} \circ \triangle, d)$$

is the counit of $\int_{c \in \mathbf{C}} \mathbf{D}(\mathcal{F}(c, c), d)$. We will denote the object of the coend of $\mathcal{F}$ by $\int^{c \in \mathbf{C}} \mathcal{F}(c, c)$ or by $\int^{\mathbf{C}} \mathcal{F}$. The maps $\mathcal{F}(i, i) \to \int^{\mathbf{C}} \mathcal{F}$ are called the unit of the coend.

We are now ready to define Day convolution.

Definition 2.13. Let $(\mathbf{I}, \square, *)$ be a symmetric monoidal $\mathbf{V}$-category and $\mathbf{C}$ a $\mathbf{V}$-category. *Day convolution*, $\otimes_D$, on $\mathbf{C}^{\mathbf{I}}$ is defined by

$$\mathcal{F} \otimes_D \mathcal{G}(s) = \int^{r, t \in I} \mathbf{I}(r \square t, s) \otimes \mathcal{F}(r) \otimes \mathcal{G}(t)$$

There is another interpretation of Day convolution, which was pointed out in [48, Section 21]. This is will be useful so we describe it below.

Definition 2.14. Let Let $(\mathbf{I}, \square, *)$ be a symmetric monoidal $\mathbf{V}$-category and $\mathbf{M}$ a $\mathbf{V}$-category. For $\mathcal{F}, \mathcal{G} \in \mathbf{C}^{\mathbf{I}}$ define $\mathcal{F} \boxtimes \mathcal{G} = \otimes \circ (\mathcal{F}, \mathcal{G})$, that is

$$(\mathcal{F} \boxtimes \mathcal{G})(i, j) = \mathcal{F}(i) \otimes \mathcal{G}(j)$$

for all $(i, j) \in \mathbf{I} \times \mathbf{I}$.

Proposition 2.15. *For all $\mathcal{F}, \mathcal{G} \in \mathbf{C}^{\mathbf{I}}$ the Day Convolution of $\mathcal{F}$ and $\mathcal{G}$ equivalent to the left Kan extension of $\mathcal{F} \boxtimes \mathcal{G}\colon \mathbf{C}^{\mathbf{I} \times \mathbf{I}} \to \mathbf{C}^{\mathbf{I}}$ along $\square\colon \mathbf{I} \times \mathbf{I} \to \mathbf{I}$. In particular, there is an isomorphism*

$$\mathbf{C}^{\mathbf{I}}(\mathcal{F} \otimes_D \mathcal{G}, \mathcal{X}) \cong \mathbf{C}^{\mathbf{I} \times \mathbf{I}}(\mathcal{F} \boxtimes \mathcal{G}, \mathcal{X} \circ \square).$$

The proof uses the coend formulation of the left Kan extension. For a review of Kan extensions, we recommend [34] and [45]. We can use the colimit definition of Kan extensions to get the following corollary.

Corollary 2.16. *For all $\mathcal{F}, \mathcal{G} \in \mathbf{C}^{\mathbf{I}}$ and $\gamma \in \mathbf{I}$ there is an isomorphism*

$$\mathcal{F} \otimes_D \mathcal{G}(\gamma) \cong \operatorname*{colim}_{\alpha \square \beta \to \gamma \in \mathbf{I}} \mathcal{F}(\alpha) \otimes \mathcal{G}(\beta)$$

where the colimit is indexed by the comma/over category $\square(\mathbf{I} \times \mathbf{I})/\gamma$.

Theorem 2.17. *If $(\mathbf{I}, \square, *)$ is a small $\mathbf{V}$-cosmos and $(\mathbf{C}, \otimes, \mathbb{1})$ is a $\mathbf{V}$-cosmos. Then $(\mathbf{C}^{\mathbf{I}}, \otimes_D, \iota)$ is a $\mathbf{V}$-cosmos, where $\otimes_D$ is the Day convolution product and $\iota \colon \mathbf{I} \to \mathbf{C}$ is the constant functor with value at $\mathbb{1}$.*

Proof. We have already seen that $\mathbf{C}^{\mathbf{I}}$ is a $\mathbf{V}$-category. All that is left to show is that $(\mathbf{C}^{\mathbf{I}}, \otimes_D, \iota)$ is an enriched symmetric monoidal category. We shall prove that the Day convolution is symmetric. The proof of the associativity and the monoidal unit axioms are similar. Let $i \in \mathbf{I}$, $\mathcal{F}, \mathcal{G} \in \mathbf{C}^{\mathbf{I}}$ then for all $j \in \mathbf{I}$ we have that

$$\mathcal{F} \otimes_D \mathcal{G}(j) = \int^{i,k \in \mathbf{I}} \mathbf{I}(i \square k, j) \otimes \mathcal{F}(i) \otimes \mathcal{G}(k)$$

$$\cong \int^{i,k \in \mathbf{I}} \mathbf{I}(k \square i, j) \otimes \mathcal{G}(k) \otimes \mathcal{F}(i)$$

$$= \mathcal{G} \otimes_D \mathcal{F}(j)$$

since $k \square i \cong i \square k$ and $\mathcal{F}(i) \otimes \mathcal{G}(k) \cong \mathcal{G}(k) \otimes \mathcal{F}(i)$, by the symmetry of $\square$ and $\otimes$. Now we need show is that $\otimes_D \colon \mathbf{C}^{\mathbf{I}} \times \mathbf{C}^{\mathbf{I}} \to \mathbf{C}^{\mathbf{I}}$ is a $\mathbf{V}$-functor. Notice for all $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{J} \in \mathbf{C}^{\mathbf{I}}$ and $f \in \mathbf{C}(\mathcal{F}(i), \mathcal{H}(i))$ and $g \in \mathbf{C}(\mathcal{G}(i), \mathcal{J}(i))$ there is a morphism

$$\mathbf{I}(i \square i, i) \otimes \mathcal{F}(i) \otimes \mathcal{G}(i) \xrightarrow{1 \otimes f \otimes g} \mathbf{I}(i \square i, i) \otimes \mathcal{H}(i) \otimes \mathcal{J}(i) \xrightarrow{unit} \mathcal{H} \otimes_D \mathcal{J}(i)$$

This defines a morphism

$$\mathbf{C}(\mathcal{F}i, \mathcal{H}i) \otimes \mathbf{C}(\mathcal{G}i, \mathcal{J}i) \to \mathbf{C}(\mathbf{I}(i \square i, i) \otimes \mathcal{F}i \otimes \mathcal{G}i, \mathcal{H} \otimes_D \mathcal{J}(i))$$

The end of this morphism gives a morphism

$$\mathbf{C}^{\mathbf{I}} \times \mathbf{C}^{\mathbf{I}}((\mathcal{F}, \mathcal{H}), (\mathcal{G}, \mathcal{J})) \to \mathbf{C}^{\mathbf{I}}(\mathcal{F} \otimes_D \mathcal{G}, \mathcal{H} \otimes_D \mathcal{J})$$

This makes $\otimes_D$ into a $\mathbf{V}$-functor. $\square$

2.1.2 Model category theory

In this section, we will describe model categories. We will start with the model categories without any enrichment and then we will introduce enriched model categories.

Definition 2.18. Let $\mathbf{C}$ be a category. A morphism $f\colon A \to B$ is said to be a *retract* of $g\colon X \to Y$ if there is a commutative diagram

$$
\begin{array}{ccccc}
 & & \overset{id}{\overgroup{\hspace{3cm}}} & & \\
A & \longrightarrow & X & \longrightarrow & A \\
f\downarrow & & \downarrow g & & \downarrow f \\
B & \longrightarrow & Y & \longrightarrow & B \\
 & & \underset{id}{\undergroup{\hspace{3cm}}} & &
\end{array}
$$

Definition 2.19. In a category $\mathbf{C}$ a commutative diagram of the form

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
j\downarrow & & \downarrow p \\
B & \longrightarrow & Y
\end{array}
$$

is said to have a *lift* if there exists a morphism $h\colon B \to Y$ such that the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
j\downarrow & \overset{h}{\nearrow} & \downarrow p \\
B & \longrightarrow & Y
\end{array}
$$

commutes.

Definition 2.20. Let $\mathbf{C}$ be a category. A *model structure* on $\mathbf{C}$ is three classes of maps called weak equivalences, fibrations and cofibration such that

(M1) $\mathbf{C}$ is closed under limits and colimits;

(M2) The three classes of maps are closed under retracts;

(M3) (Two out of three Axiom) Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, if any two of $f, g, g \circ f$ is a weak equivalence then so is the third;

(M4) (Lifting Axiom) Given a commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\scriptstyle{j}\downarrow & \nearrow & \downarrow\scriptstyle{p} \\
B & \longrightarrow & Y
\end{array}
$$

where j is a cofibration and p is a fibration a lift (the dotted line) exists if one of j or p is a weak equivalence.

(M5) (Factorization) Any map $f \colon X \to Y$ can be factored in two ways:

  i) $X \xrightarrow{i} Y \xrightarrow{q} Z$ where i is a cofibration and q is both a weak equivalence and a fibration.

  ii) $X \xrightarrow{j} Y \xrightarrow{p} Z$ where p is a fibration and j is both a weak equivalence and a cofibration.

We shall refer to a category $\mathbf{C}$ together with a given model structure on it as a model category. If a fibration is also a weak equivalence we call it an *acyclic* or *trivial* fibration. Similarly if a cofibration is also a weak equivalence then we also call it an acyclic or trivial cofibration. Before we give some examples of model categories we will introduce cofibrantly generated model categories. We do this because most model categories are cofibrantly generated and it is an interesting problem to find a model category which is not cofibrantly generated.

Definition 2.21. Let $\mathbf{C}$ be a category and let $\mathbf{F}$ be a class of morphisms in $\mathbf{C}$. Then $A \in \mathbf{C}$ is *small* for $\mathbf{F}$ if whenever

$$
X_1 \to X_2 \to X_3 \to \ldots
$$

is a sequence of morphisms in $\mathbf{F}$, then the natural map

$$
\mathrm{colim}\, \mathbf{C}(A, X_n) \longrightarrow \mathbf{C}(A, \mathrm{colim}\, X_n)
$$

is an isomorphism.

Definition 2.22. A model category $\mathbf{M}$ is *cofibrantly generated* if there are classes of morphisms I and J so that

i) the source (the domain) of every morphism in I is small for the class of all cofibrations and $q\colon X \to Y$ is an acyclic fibration if and only if q has the right lifting property the respect to all morphisms of I;

ii) the source (the domain) of every morphism in J is small for the class of all acyclic cofibrations and $q\colon X \to Y$ is a fibration if and only if q has the right lifting property the respect to all morphisms of J.

We call I and J generating classes of $\mathbf{M}$.

We will now give two examples of cofibrantly generated categories. These are both important examples of model categories.

Example 2.23 (Chain complexes)**.** We may equip $(\mathbf{Ch}_R)_0$ with a model structure given by the following. If $f\colon C_* \to D_*$ is a chain map of degree 0 in $(\mathbf{Ch}_R)_0$ then f is a

i) weak equivalence if $f_*\colon H_*(C) \to H_*(D)$ is an isomorphism,

ii) fibration if $f_n\colon C_n \to D_n$ is surjective for all $n \geq 0$ or

iii) cofibration if $f_n\colon C_n \to D_n$ is injective with projective kernel for all $n \geq 0$.

To see that this is a cofibrantly generated model category define the chain complexes

$$D(n) := (\mathbb{Q}\{z_n\} \oplus \mathbb{Q}\{z_{n-1}\}, |z_n| = n, |z_{n-1}| = n-1, \partial z_n = z_{n-1})$$

$$S(k) := (\mathbb{Q}\{z_k\}, |z_k| = k, \partial = 0)$$

Then $\mathbf{Ch}_{\mathbb{Q}}$ is cofibrantly generated by the set K of the morphisms $0 \to S(0)$ and $S(n-1) \hookrightarrow D(n)$ for $n \geq 0$ and the trivial cofibrantions $0 \hookrightarrow D(n)$ for all $n \geq 0$. We will not prove our claims here but proofs may be found in [8, 17] or [26].

Example 2.24 (Simplicial sets)**.** There exists a functor $|\cdot|\colon \mathbf{sSet} \to \mathbf{Top}$ to the category of topological spaces given by

$$\coprod_n X_n \otimes \Delta^n / \sim$$

where $(\phi^*(x), t) \sim (x, \phi_*(t))$. We define the k-th horn Δ_k^n of Δ^n to be the subcomplex generated by all the faces except the k-th face $d_i(id_{\underline{n}})$. Then there is a model structure on $\mathbf{sSet}$ where a morphism $f\colon X \to Y$ is

i) a weak equivalence if $|f|\colon |X| \to |Y|$ is a topological weak equivalence,

ii) a cofibration if $f_n\colon X_n \to Y_n$ is a monomorphism for all n,

iii) a fibration if f has the left lifting property with respect to the inclusions of horns
$$\Delta_k^n \to \Delta^n, n \geq 1, 0 \leq k \leq n.$$

Later we will study functor categories and will need some results on them. To this end, we will prove that when $\mathbf{I}$ is a small category and $\mathbf{M}$ is a cofibrantly generated model category then the category $\mathbf{M^I}$ of functors from I to $\mathbf{M}$ is also a model category. We adapt the proof from [24].

Lemma 2.25. *Let $\mathcal{G}\colon \mathbf{N} \to \mathbf{M}$ be functor with a left adjoint $\mathcal{F}\colon \mathbf{M} \to \mathbf{N}$. Assume $\mathbf{M}$ is a cofibrantly generated model category. Let I and J be the cofibrant and acyclic cofibrant generating sets of $\mathbf{M}$. We define a morphism f in $\mathbf{N}$ to be a weak equivalence or a fibration if $\mathcal{G}f$ is a weak equivalence or a fibration respectively. If we further assume that*

i) $\mathcal{G}$ commutes with sequential limits,

ii) every cofibration in $\mathbf{N}$ with the left lifting property with respect to fibrations is a weak equivalence.

then $\mathbf{N}$ is a cofibrantly generated model category with generating sets $\{\mathcal{F}i \mid i \in I\}$ and $\{\mathcal{F}j \mid j \in J\}$.

We define Quillen functors as well.

Definition 2.26. Assume $\mathbf{M}$ and $\mathbf{N}$ are model categories. Let $\mathcal{G}\colon \mathbf{N} \to \mathbf{M}$ be functor with a left adjoint $\mathcal{F}\colon \mathbf{M} \to \mathbf{N}$. We say that $(\mathcal{F}, \mathcal{G})$ is a Quillen adjunction if

i) the functor $\mathcal{F}$ preserves cofibrations and weak equivalences between cofibrant objects,

ii) the functor $\mathcal{G}$ preserves fibrations and weak equivalences between fibrant objects.

Proofs for both the above and the next lemmas can be found in [24].

Lemma 2.27. *Let S be a set. For each $s \in S$ let $\mathbf{M}_s$ be a model category with generating classes I_s and J_s then there is a model structure on $\prod_{s \in S} \mathbf{M}_s$ is cofibrantly generated by*

$$I = \bigcup_{s \in S} (I_s \times \prod_{t \neq s} \{id_t\}) \quad \& \quad J = \bigcup_{s \in S} (J_s \times \prod_{t \neq s} \{id_t\})$$

where id_t is the identity morphism on the initial object $*_t$ of $\mathbf{M}_t$.

Theorem 2.28. *Let $\mathbf{I}$ be a small category and $\mathbf{M}$ a model category cofibrantly generated by I and J. Then the category $\mathbf{M}^{\mathbf{I}}$ is a cofibrantly generated model category such that a morphism $\mathcal{X} \to \mathcal{Y}$ is a*

i) *weak equivalence if $\mathcal{X}(\alpha) \to \mathcal{Y}(\alpha)$ is a weak equivalence in $\mathbf{M}$ for all objects α in $\mathbf{I}$;*

ii) *fibration if $\mathcal{X}(\alpha) \to \mathcal{Y}(\alpha)$ is a fibration in $\mathbf{M}$ for all objects α in $\mathbf{I}$.*

Proof. Consider the category $\prod_{\alpha \in \mathbf{I}} \mathbf{M}$. By lemma 2.27 we have that $\prod_{\alpha \in \mathbf{I}} \mathbf{M}$ is a cofibrantly generated model category. The category $\prod_{\alpha \in \mathbf{I}} \mathbf{M} \cong \mathbf{M}^{\mathbf{I}^{disc}}$ where $\mathbf{I}^{disc}$ denotes the discrete subcategory of $\mathbf{I}$. There is a natural inclusion functor $\mathbf{I}^{disc} \to \mathbf{I}$ which induces a functor $\mathbf{M}^{\mathbf{I}} \to \mathbf{M}^{\mathbf{I}^{disc}}$. Taking left Kan extensions gives us an adjoint functor $\mathbf{M}^{\mathbf{I}^{disc}} \to \mathbf{M}^{\mathbf{I}}$. We may then use lemma 2.25 to get the result. $\qquad\square$

In particular, we have that.

Theorem 2.29. *The category $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ has a model structure for all small categories $\mathbf{I}$ where a morphism $\mathcal{X} \to \mathcal{Y}$ in $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ is a*

i) *weak equivalences if $\mathcal{X}(\alpha) \to \mathcal{Y}(\alpha)$ is a weak equivalence in $\mathbf{Ch}_{\mathbb{Q}}$ for all objects α in I,*

ii) *fibrations if $\mathcal{X}(\alpha) \to \mathcal{Y}(\alpha)$ is a fibration in $\mathbf{Ch}_{\mathbb{Q}}$ for all objects α in I,*

Moreover, $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ is cofibrantly generated.

The proof of this theorem is just an application of 2.28.

One benefit of working with model categories is that we can use derived functors.

Definition 2.30. Let $\mathbf{M}$ be a model category. The *homotopy category* of $\mathbf{M}$, denoted by $\mathrm{Ho}(\mathbf{M})$, is the category obtained from inverting weak equivalences that is changing them into isomorphisms. This category can be characterised by the property that there is a functor $\mathcal{H}_{\mathbf{M}} \colon \mathbf{M} \to \mathrm{Ho}(\mathbf{M})$ which takes weak equivalences to isomorphisms and if $\mathcal{F} \colon \mathbf{M} \to \mathbf{N}$ is a functor then there exists a unique functor $\mathcal{F}' \colon \mathrm{Ho}(\mathbf{M}) \to \mathbf{N}$ such that

$$
\begin{array}{ccc}
\mathbf{M} & \xrightarrow{\;\mathcal{H}_{\mathbf{M}}\;} & \mathrm{Ho}(\mathbf{M}) \\
{\scriptstyle \mathcal{F}}\downarrow & \swarrow {\scriptstyle \mathcal{F}'} & \\
\mathbf{N} & &
\end{array}
$$

commutes. If $\mathbf{N}$ is also a model category we let $\mathrm{Ho}(\mathcal{F})$ denote the composition

$$
\mathrm{Ho}(\mathbf{M}) \xrightarrow{\;\mathcal{F}'\;} \mathbf{N} \xrightarrow{\;\mathcal{H}_{\mathbf{N}}\;} \mathrm{Ho}(\mathbf{N})
$$

There is a way to define a homotopy relation $\sim_h$ on morphisms in $\mathbf{M}$ such that $\mathrm{Ho}(\mathbf{M})(x,y) = \mathbf{M}(x,y)/\sim$. If $\mathbf{M}_c$ denotes the full subcategory of cofibrant objects in $\mathbf{M}$ we can then define a natural functor $\mathrm{Ho}(\mathbf{M}) \to \mathrm{Ho}(\mathbf{M}_c)$ which sends an object to a cofibrant replacement and it sends a morphism $f \colon x \to y$ to the class represented by a lift for the diagram

$$
\begin{array}{ccc}
* & \longrightarrow & x_c \\
\downarrow & & \downarrow \\
y_c & \longrightarrow & y
\end{array}
$$

where $x, y \in \mathbf{M}$ and $x_c \to x$ and $y_c \to y$ are cofibrant replacements and the right vertical morphism in the diagram is the composition of $* \to x_c \to x \xrightarrow{f} y$. Let use denote this functor by $(-)_c \colon \mathrm{Ho}(\mathbf{M}) \to \mathrm{Ho}(\mathbf{M}_c)$. We can do a similar construction to define the fibrant replacement functor $(-)_f \colon \mathrm{Ho}(\mathbf{M}) \to \mathrm{Ho}(\mathbf{M}_f)$ where $\mathbf{M}_f$ is the category of cofibrant objects. More details can be found in [8] and [26].

Definition 2.31. Given a model categories $\mathbf{M}$ and $\mathbf{N}$. Let $\mathcal{F} \colon \mathbf{M} \to \mathbf{N}$ be a functor. We define the *(total) derived functor* of $\mathcal{F}$ to be the functor $L\mathcal{F} \colon \mathrm{Ho}(\mathbf{M}) \to \mathrm{Ho}(\mathbf{N})$ which is the composition of

$$
\mathrm{Ho}(\mathbf{M}) \xrightarrow{\;(-)_c\;} \mathrm{Ho}(\mathbf{M}_c) \xrightarrow{\;\mathrm{Ho}(\mathcal{F})\;} \mathrm{Ho}(\mathbf{N}).
$$

Similarly we define the right derived functor $R\mathcal{F}$ of $\mathcal{F}$ to be the composition of

$$\text{Ho}(\mathbf{M}) \xrightarrow{(-)_f} \text{Ho}(\mathbf{M}_f) \xrightarrow{\text{Ho}(\mathcal{F})} \text{Ho}(\mathbf{N}).$$

There is a version of this for fibrant replacements which gives rise to the right derived functor, see [26]. In homotopy theory it is always a bit awkward trying to define homotopy limits and colimits. Model categories and derived functors give us a way to do this.

Example 2.32. If $\mathbf{I}$ is a small category and $\mathbf{M}$ is a model category then the homotopy limit is the right derived functor

$$\text{holim} = R\lim\colon \text{Ho}(\mathbf{M}^{\mathbf{I}}) \to \text{Ho}(\mathbf{M})$$

and the homotopy colimit is

$$\text{hocolim} = L\operatorname{colim}\colon \text{Ho}(\mathbf{M}^{\mathbf{I}}) \to \text{Ho}(\mathbf{M}).$$

Examples of such limits are homotopy pushouts and pullbacks.

There is an explicit way to construct homotopy limits. This uses the bar construction of functors.

Definition 2.33. Let $\mathbf{M}$ be a model category enriched $\mathbf{V}$. Let $\mathbf{I}$ be a small $\mathbf{V}$-category. Let $\mathcal{F}\colon \mathbf{I}^{op} \to \mathbf{V}$ and $\mathcal{G}\colon \mathbf{I} \to \mathbf{M}$ be $\mathbf{V}$-functors. Define $B_*(\mathcal{F}, \mathbf{I}, \mathcal{G})$ to be the simplicial object

$$B_n(\mathcal{F}, \mathbf{I}, \mathcal{G}) = \coprod_{a_0,\ldots,a_n \in \mathbf{I}} \mathcal{F}(a_n) \otimes \mathbf{I}(a_{n-1}, a_n) \otimes \ldots \otimes \mathbf{I}(a_0, a_1) \otimes \mathcal{G}(a_0)$$

Where the face maps are given by composing morphisms, the evaluation $\mathcal{F}(a_n) \otimes \mathbf{I}(a_{n-1}, a_n) \to \mathcal{F}(a_{n-1})$ and the evaluation $\mathbf{I}(a_0, a_1) \otimes \mathcal{G}(a_0) \to \mathcal{G}(a_1)$. This is called the two-sided bar construction.

The bar construction is, as Riehl [60] put it, unreasonably effective. We can use the two-sided bar construction to calculate homotopy colimits.

Lemma 2.34. *Let $\mathbf{I}$ be a small category, $\mathbf{M}$ a model category and $\mathcal{G}\colon \mathbf{I} \to \mathbf{M}$ a functor. There is are isomorphisms of functors*

$$\operatorname*{hocolim}_{\mathbf{I}} \mathcal{G} \cong \operatorname*{colim}_{\mathbf{\Delta}} B_*(\mathbf{t}, \mathbf{I}, \mathcal{G})$$

where $\mathbf{t}$ is the constant functor which sends everything to the terminal object of $\mathbf{M}$.

A proof can be found in [60]. We will use the bar construction later in the thesis but it has a number of important properties and uses for example $B_*(\mathbf{t}, \mathbf{I}, \mathbf{t})$ is the nerve of $\mathbf{I}$. There is also an enriched version. For more information see [60]. There will be a homotopy pushout which we will need.

Definition 2.35. Let $\mathbf{M}$ be a model category. If $f \colon M \to N$ is a morphism in $\mathbf{M}$ we define the cofibre of f to be the homotopy pushout of the diagram $* \leftarrow M \xrightarrow{f} N$. This is also sometimes called the cone of f, due to this we will denote the cofibre of f by $\mathrm{cone}(f)$.

We will now turn to enriched model categories. To do this we need enriched monoidal categories.

Definition 2.36. Given a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbb{1})$ and two morphisms $f \colon v \to w$ and $g \colon x \to y$ in $\mathbf{V}$ we define $(v \otimes y) \sqcup_{v \otimes x} (w \otimes x)$ to be the pushout of the diagram

$$
\begin{array}{ccc}
v \otimes x & \xrightarrow{\ f \otimes id_x\ } & w \otimes x \\
{\scriptstyle id_v \otimes g}\big\downarrow & & \big\downarrow \\
v \otimes y & \longrightarrow & (v \otimes y) \sqcup_{v \otimes x} (w \otimes x)
\end{array}
$$

The morphism $f \sqcup_{v \otimes x} g \colon (v \otimes y) \sqcup_{v \otimes x} (w \otimes x) \to w \otimes y$ induced by the morphisms $f \otimes id_y$ and $id_w \otimes g$ and the universal property of the pushout.

Definition 2.37. A monoidal model category $(\mathbf{V}, \otimes, \mathbb{1})$ consists of a symmetric monoidal category together with a model structure such that

i) If $f \colon v \to w$ and $g \colon x \to y$ are cofibrant then the pushout-product map $f \sqcup_{v \otimes x} g$ is cofibrant. Moreover, the if f or g is a trivial cofibration then so is $f \sqcup_{v \otimes x} g$;

ii) for every cofibrant replacement $q \to \mathbb{1}$ the induced morphism

$$
q \otimes x \to \mathbb{1} \otimes x \xrightarrow{\cong} x
$$

is a weak equivalence for all $x \in \mathbf{V}$.

Definition 2.38. Given a closed symmetric monoidal model category $\mathbf{V}$, we say that $\mathbf{M}$ is a $\mathbf{V}$-model category if the underlying category $\mathbf{M}_0$ of $\mathbf{M}$ is

equipped with a model structure such that for any fibration $x \to y$ and any cofibration $w \to v$ in $\mathbf{M}_0$ the pullback-power map

$$\mathbf{M}(y, w) \to \mathbf{M}(y, v) \times_{\mathbf{M}(x,v)} \mathbf{M}(x, w)$$

is a fibration in $\mathbf{V}$, which is a weak equivalence if either $x \to y$ or $w \to v$ is a weak equivalence.

Theorem 2.39. *Let $(\mathbf{I}, \square, \mathbb{1})$ be a small monoidal $\mathbf{V}$-category and $\mathbf{M}$ a symmetric monoidal model $\mathbf{V}$-category cofibrantly generated by I and J. Then the category $\mathbf{M}^{\mathbf{I}}$ is a symmetric monoidal model $\mathbf{V}$-category.*

We can use a similar argument as in the un-enriched setting to prove this. A proof can be found in [3].

From what we have just mentioned Day convolution gives a monoidal structure on $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ with the unit $\mathbb{1} \colon I \to \mathbf{Ch}_{\mathbb{Q}}$ given by $\mathbb{1}(*) = \mathbb{Q}$ in degree 0 and $\mathbb{1}(s)$ the zero complex for all $* \neq s \in I$. The day convolution also respects the model structure on $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$. Lastly, we will need the following result.

Lemma 2.40. *The Day convolution product on $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ preserves weak equivalences.*

A proof can be found in [27].

2.2 Algebra objects and regular sequences in $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$

We are now ready to extend the story to a more general and less familiar setting. Throughout this section take $(\mathbf{I}, \square, *)$ to be a symmetric monoidal category enriched over $\mathbb{Q}$-vector spaces, closed under all limits and colimits. We denote be the category of enriched functors from I to $\mathbf{Ch}_{\mathbb{Q}}$ by $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$. We have just seen how $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ has a model structure as well as a symmetric monoidal structure which behaves well the model structure.

2.2.1 Algebra objects

We are now ready to introduce algebra objects.

Definition 2.41. Let Com and Com^+ denote the nonunital commutative operad and the unital commutative operad respectively. Let $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ denote the category of nonunital commutative algebra objects in $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$, that is algebras in $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ over the operad Com. We require that our commutative algebras satisfy the Koszul sign rule. Let $\mathsf{Alg}_+(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ denote the category of unital commutative algebra objects and $\mathsf{Alg}_+^{\mathsf{aug}}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ denote the category of augmented unital commutative algebra objects in $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$.

Recall that a functor $\mathcal{A} \colon \mathbf{I} \to \mathbf{Ch}_{\mathbb{Q}}$ has the structure of a commutative algebra object if and only if it is a lax symmetric monoidal functor. That is there exists a chain map

$$m_{\alpha,\beta} \colon \mathcal{A}(\alpha) \otimes \mathcal{A}(\beta) \to \mathcal{A}(\alpha \square \beta)$$

for all $\alpha, \beta \in \mathbf{I}$. At first, this may seem complicated however we give an example which may be instructive.

Example 2.42. Let $\mathbf{N}$ denote the category obtained from $\mathbb{Q}$-linearising the category obtained from the partially ordered set $(\mathbb{N}, \leq)$. So the objects are natural numbers and for all $x, y \in \mathbf{N}$ we have that $\mathbf{N}(x, y) = \mathbb{Q}\{x \to y\}$ if $x \leq y$ and $(x, y) = \mathbf{0}$ if $y < x$. With the usual addition $+$ of natural numbers and 0 we get that $(\mathbf{N}, +, 0)$ is a symmetric monoidal category. We wish to define a functor $\mathcal{W} \colon \mathbf{N} \to \mathbf{Ch}_{\mathbb{Q}}$. For each $n \in \mathbf{N}$ let $\mathcal{W}(n) = (A_*(n), d_*)$ where $A_i(n) = \Lambda^i(\mathbb{Q}^n)$ and $d_k \colon A_k(n) \to A_{k-1}(n)$

$$d_k(e_{i_1}^n \wedge e_{i_2}^n \wedge ... \wedge e_{i_k}^n) = \sum_{i=1}^{k} (-1)^j (e_{i_1}^n \wedge ... \wedge \hat{e}_{i_j}^n \wedge ... \wedge e_{i_k}^n)$$

where $\{e_1^n, ..., e_n^n\}$ is the standard basis of $\mathbb{Q}^n$ and the hat means we are omitting e_j. Notice that $\mathcal{W}(0)$ is the chain complex with a copy of $\mathbb{Q}$ in the 0-degree and zero differential. To describe what $\mathcal{W}$ does on morphisms let $x, y \in \mathbf{N}$. If $y < x$ then $(x, y) = \mathbf{0}$ and we define $\mathcal{W}(\mathbf{0}) \colon A_*(x) \to A_*(y)$ to be the zero map. If $x \leq y$ and $f \colon x \to y$ is the generator of $\mathbf{N}(x, y)$ then we define $\mathcal{W}(f) \colon (A_*(x), d_*) \to (A_*(y), d_*)$ to be the map that is

$$\mathcal{W}(f)_k \colon e_{i_1}^x \wedge ... \wedge e_{i_k}^x \mapsto e_{i_1}^y \wedge ... \wedge e_{i_k}^y.$$

We claim that $\mathcal{W}$ is an algebra object.

To see this for all $x, y \in \mathbf{N}$ we define a map $m_{x,y} \colon \mathcal{W}(x) \otimes \mathcal{W}(y) \to \mathcal{W}(x + y)$ by

$$m_{x,y}(e_{i_1}^x \wedge ... \wedge e_{i_k}^x \otimes e_{j_1}^y \wedge ... \wedge e_{j_m}^y) = e_{i_1}^{x+y} \wedge ... \wedge e_{i_k}^{x+y} \wedge e_{j_1}^{x+y} \wedge ... \wedge e_{j_m}^{x+y}$$

for all $e_{i_1}^x \wedge ... \wedge e_{i_k}^x \in \mathcal{W}(x)_k = A_k(x)$ and $e_{j_1}^y \wedge ... \wedge e_{j_m}^y \in \mathcal{W}(y)_m = A_m(y)$. We will often just denote $m_{x,y}(\alpha \otimes \beta)$ by $\alpha \cdot \beta$. To see that this satisfies the koszul sign rule $\alpha \cdot \beta = (-1)^{|\alpha||\beta|}\beta \cdot \alpha$ notice that if $e_{i_1}^x \wedge ... \wedge e_{i_k}^x \in \mathcal{W}(x)_k$ and $e_{j_1}^y \wedge ... \wedge e_{j_m}^y \in \mathcal{W}(y)_m$ then we have that $|e_{i_1}^x \wedge ... \wedge e_{i_k}^x| = k$ and $|e_{j_1}^y \wedge ... \wedge e_{j_m}^y| = m$ so we get that

$$
\begin{aligned}
(e_{i_1}^x \wedge ... \wedge e_{i_k}^x) \cdot (e_{j_1}^y \wedge ... \wedge e_{j_m}^y) &= e_{i_1}^{x+y} \wedge ... \wedge e_{i_k}^{x+y} \wedge e_{j_1}^{x+y} \wedge ... \wedge e_{j_m}^{x+y} \\
&= (-1)^k e_{j_1}^{x+y} \wedge e_{i_1}^{x+y} \wedge ... \wedge e_{i_k}^{x+y} \wedge e_{j_2}^{x+y} \wedge ... \wedge e_{j_m}^{x+y} \\
&= ... \\
&= (-1)^{k \cdot m} e_{j_1}^{x+y} \wedge ... \wedge e_{j_m}^{x+y} \wedge e_{i_1}^{x+y} \wedge ... \wedge e_{i_k}^{x+y} \\
&= (-1)^{k \cdot m} (e_{j_1}^y \wedge ... \wedge e_{j_m}^y) \cdot (e_{i_1}^x \wedge ... \wedge e_{i_k}^x)
\end{aligned}
$$

This proves that we get the koszul sign rule. Also notice that for all $x \in \mathbf{N}$ and $i \in \{1, .., x\}$ we have that $e_i^x \cdot e_i^x = -e_i^x \cdot e_i^x = 0$.

There are two functors of note. There is an equivalence of categories.

Proposition 2.43. *There is an equivalence of categories*

$$\mathsf{Alg}_+^{\mathsf{aug}}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}) \xrightarrow{\cong} \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$$

given by taking the augmentation ideal and the inverse is given by unitalisation.

This is a well-known result of commutative algebra a proof can be found in [43]. By the completeness of $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ and $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ we get the following result.

Proposition 2.44. *The forgetful functor $U \colon \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}) \to \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ has a left adjoint $\Lambda(-) \colon \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}} \to \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$. Similarly, there is a free functor $\Lambda^+(-) \colon \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}} \to \mathsf{Alg}_+^{\mathsf{aug}}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$.*

This result is due to the completeness of $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ and $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$. We use this to induce a model structure on $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ using Lemma 2.25.

Theorem 2.45. *The category* $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ *has a cofibrantly generated model structure with weak equivalences and fibrations given objectwise. Moreover, we may endow* $\mathsf{Alg}_+(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ *and* $\mathsf{Alg}_+^{\mathsf{aug}}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ *with the same structure.*

Recall the definition of a quasi-free object.

Definition 2.46. An algebra $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ is said to be *quasi-free* if after we forget differentials we have an isomorphism $\mathcal{A} \cong \Lambda(\mathcal{F})$ where $\mathcal{F} \in \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$.

It is well-known from the work of Quillen [58] and Sullivan [63] that quasi-free algebras are cofibrant.

Proposition 2.47. *Every quasi-free algebra object in* $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ *is cofibrant.*

A proof can also be found in [43]. A similar definition and statement holds for augmented algebra objects as well. We will need one more thing before we can continue.

2.2.2 Regular sequences

We will need to tool of regular sequences later so we introduce them more generally for Com objects in $\mathbf{Ch}_{\mathbb{Q}}^{I}$ and use them later. To introduce some notation and the notion of a regular sequence.

Definition 2.48. For $\alpha \in \mathbf{I}$, define $ev_{\alpha}^{!} \colon \mathbf{Ch}_{\mathbb{Q}} \to \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ to be the left adjoint of the evaluation map ev_{α}. Let $\Lambda_{\alpha}(z_k)$ denote $\Lambda(ev_{\alpha}^{!} S(k))$.

The left adjoint of the evaluation functor is given by taking left Kan extensions. We will compute this later for a specific $\mathbf{I}$.

Given $\mathcal{A} \in \mathsf{Alg}_+^{\mathsf{aug}}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ and a cycle $z \in \mathcal{A}(*)$ there is a map $S(k) \to \mathcal{A}(*)$ defined by $z_k \mapsto z$. By adjunction, there is a map $ev_{\alpha}^{!} \colon S(k) \to \mathcal{A}$. By the free and forget adjunction there is a map $\Lambda_*(z_k) \to \mathcal{A}$.

Definition 2.49. Let $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ and let $z \in \mathcal{A}(*)$ be a cycle. We define $\mathcal{A}/(z)$ to be the following pushout

$$
\begin{array}{ccc}
\Lambda_*(z_k) & \longrightarrow & \mathcal{A} \\
\downarrow & & \downarrow \\
\Lambda(\mathbf{0}) & \longrightarrow & \mathcal{A}/(z)
\end{array}
$$

Where $\Lambda(\mathbf{0})$ is the functor defined which is the zero complex for all $\alpha \in \mathbf{I}$ such that $\alpha \neq *$ and $\Lambda(\mathbf{0})(*)$ is the chain complex with $\Lambda(\mathbf{0})(*)_0 = \mathbb{Q}$ and zero elsewhere.

Definition 2.50. Let $\mathcal{A} \in \mathsf{Alg}^{\mathsf{aug}}_+(\mathbf{Ch}^{\mathbf{I}}_{\mathbb{Q}})$ and $z \in \overline{\mathcal{A}}(*)$ be a cycle of degree $k \geq 0$. We say that z is a *zero-divisor* if there exists an $\alpha \in \mathbf{I}$ such that $z \cdot (-) \colon \mathcal{A}(\alpha) \to \mathcal{A}(\alpha)$ is not injective.

Note that if $z \in \overline{\mathcal{A}}(*)$ is a non-zero divisor of degree k then k is even by the sign rule $z^2 = (-1)^k z^2$.

Lemma 2.51. *If $\mathcal{A} \in \mathsf{Alg}^{\mathsf{aug}}_+(\mathbf{Ch}^{\mathbf{I}}_{\mathbb{Q}})$ and $z \in \overline{\mathcal{A}}(*)$ is not a zero-divisor then the diagram*

$$
\begin{array}{ccc}
\Lambda_*(z) & \longrightarrow & \mathcal{A} \\
\downarrow & & \downarrow \\
\Lambda(\mathbf{0}) & \longrightarrow & \mathcal{A}/(z)
\end{array}
$$

is a homotopy pushout.

Proof. It follows from $S(k) \hookrightarrow D(k+1)$ being a generating cofibration that map $\Lambda_*(z_k) \to \Lambda(\mathbf{0})$ can be decomposed into

$$\Lambda_*(z_k) \to \Lambda(ev^!_* D(k)) \to \Lambda(\mathbf{0})$$

where $\Lambda_*(z_k) \to \Lambda(ev^!_* D(k))$ is a cofibration. If $\mathcal{X} \to \mathcal{A}$ is a cofibrant replacement of $\mathcal{A}$ which factorizes the map $\Lambda_*(z) \to \mathcal{A}$ then the pushout of $\Lambda(ev^!_* D(k)) \leftarrow \Lambda_*(z) \to \mathcal{A}$ will be $\mathcal{X} \otimes_{\Lambda_*(z_k)} \Lambda(ev^!_* D(k))$. This is isomorphic to $\mathcal{X} \otimes ev^!_* S(k+1)$ with the differential is given by $d(x \otimes 1) = d_{\mathcal{X}}(x) \otimes 1$ and $d(1 \otimes z) = z' \otimes 1$ where $z' \in \mathcal{X}$ is a lift of $z \in \mathcal{A}$. There is morphism

$$\mathcal{X} \otimes \Lambda(ev^!_* S(k+1)) \to \mathcal{A} \otimes \Lambda(ev^!_* S(k+1))$$

induced by $\mathcal{X} \to \mathcal{A}$. Since $id \colon \Lambda(ev^!_* S(k+1) \to \Lambda(ev^!_* S(k+1)$ and $\mathcal{X} \to \mathcal{A}$ are weak equivalences and $\otimes$ preserves weak equivalences, see 2.40, it follows that

$$\mathcal{X} \otimes \Lambda(ev^!_* S(k+1)) \to \mathcal{A} \otimes \Lambda(ev^!_* S(k+1))$$

is a weak equivalence. Notice that there is an isomorphism $\Lambda(ev^!_* S(k+1)) \cong ev^!_* \Lambda(z)$. It then follows of this isomorphism and $k+1$ being odd that $\mathcal{A} \otimes \Lambda_*(z_k)$ is the mapping cone of $z \cdot (-) \colon \mathcal{A} \to \mathcal{A}$. Since $z \cdot (-)$ is injective for all $\alpha \in \mathbf{I}$ we have that the cokernal and the mapping cone are quasi-isomorphic, hence proving the lemma. $\qquad \square$

2.2.3 Homology of algebras in functor categories

In this subsection, we will define homology for our objects in $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ and $\mathsf{Alg}_+(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$. We will start by defining André-Quillen homology of our algebra objects. André-Quillen homology was introduced by André [1] and Quillen [57]. They generalised resolutions of algebras and modules using cofibrant replacements. One main example is that they used the category of simplicial algebras to define a complex called the cotangent complex. We will use a similar idea and use cofibrant replacements to define a homology for our algebra objects. This homology is somewhat hard to work with computationally so we also introduce Harrison homology. We refer the reader to Loday [42] or to Weibel [64] for more details on André-Quillen homology in the classical setting. There is also a good paper by Iyengar [28] which gives a nice introduction to the cotangent complex.

Definition 2.52. For all $\mathcal{A}$ be an object in $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$. We define $Q(\mathcal{A})$ to be the coequalizer of

$$\Lambda(\mathcal{A}) \underset{\epsilon}{\overset{act}{\rightrightarrows}} \mathcal{A} \longrightarrow Q(\mathcal{A})$$

where $\epsilon \colon \Lambda \to \mathsf{Id}$ is defined by sending $a \mapsto a$ and $a \otimes b \mapsto 0$ for all $a, b \in \mathcal{A}$. We call $Q(\mathcal{A})$ the indecomposibles of $\mathcal{A}$.

The indecomposibles has a right adjoint $Z \colon \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}} \to \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ which is the trivial algebra functor. That is for all $\mathcal{B} \in \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ the multiplication on $Z(\mathcal{B})$ is equal to zero.

Lemma 2.53. *The functors*

$$\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}) \underset{Z}{\overset{Q}{\rightleftarrows}} \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$$

is a Quillen adjunction where Z is trivial algebra functor.

A proof of this can be found in [43]. Given an augmented algebra $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$, we denote the indecomposibles of the augmentation ideal of $\mathcal{A}$ by $Q^+(\mathcal{A})$ and its right adjoint by $Z^+(\mathcal{B})$.

Definition 2.54. Define André-Quillen homology of $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ to be

$$AQ_k(\mathcal{A}) = H_k(LQ(\mathcal{A})) \colon \mathbf{I} \to \mathbf{Vec}_{\mathbb{Q}}$$

where LQ is the (total) left derived functors of Q. There is an analogous version for algebras in $\mathsf{Alg}_+(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ which we shall denote by H^+.

The proof of lemma 2.51 then implies the following proposition.

Proposition 2.55. *If $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ and $z \in \mathcal{A}(*)$ is not a zero-divisor then there is a homotopy cofibre sequence*

$$ev_*^!(\mathbb{Q}\{z\}[k]) \to LQ^+(\mathcal{A}) \to LQ^+(\mathcal{A}/(z)).$$

We will now define Harrison homology for our algebra objects and explain it's relation to the André-Quillen homology. In order to do this we will need to use some theory of operads. In particular, we will explain a bit about the co-Lie operad. All of our results can be found in Loday and Vallette [43].

Definition 2.56. Let Lie and coLie denote the Lie and co-Lie operads respectively. Let Ass be the associative operad. Recall that $\mathsf{Ass}(k) = \mathbb{Q}\mathfrak{S}_k$ for all $k \geq 0$.

Recall that if $\mathcal{O}$ is an operad over chain complexes the free $\mathcal{O}$-algebra on $A \in \mathbf{Ch}_{\mathbb{Q}}$ is given by

$$\mathcal{O}[A] = \coprod_k \mathcal{O}(k) \otimes_{\mathfrak{S}_k} A^{\otimes k}.$$

Since $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ a $\mathbf{Ch}_{\mathbb{Q}}$-cosmos we can define the free $\mathcal{O}$-algebra on $\mathcal{A} \in \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ in a similar way. In particular, we get that

$$\mathsf{coLie}(\mathcal{A}) = \coprod_k \mathsf{coLie}(k) \otimes_{\mathfrak{S}_k} \mathcal{A}^{\otimes k}.$$

which inherits the differential from $\mathcal{A}$. We can give a more explicit definition of $\mathsf{coLie}(\mathcal{A})$.

Proposition 2.57. *There is an inclusion $\mathsf{Lie}(k)\otimes \hookrightarrow \mathsf{Ass}(k) = \mathbb{Q}\mathfrak{S}_k$ for all $k \geq 0$. Dualising gives a surjection $\mathbb{Q}\mathfrak{S}_k \to \mathsf{coLie}(k)$ with kernel being the span of non-trivial shuffle-permutations. We denote this vector space by Sh_k, for all $0 \leq k$.*

A proof of this can be found in [43]. We can then give an explicit form of the free coLie-algebra.

Theorem 2.58. *For all $\mathcal{A} \in \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}}$ we have that*

$$\mathsf{coLie}(\mathcal{A}) = \coprod_k \mathbb{Q}\mathfrak{S}_k \otimes_{\mathfrak{S}_k} \mathcal{A}^{\otimes k}/(Sh_k \otimes_{\mathfrak{S}_k} \mathcal{A}^{\otimes k})$$

with a differential inherited from $\mathcal{A}$.

We can now define Harrison complex and Harrison homology.

Definition 2.59. For all $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$, we define $\mathrm{Harr}(\mathcal{A}) = \mathsf{coLie}(\mathcal{A}[1])[-1]$ with the differential inherited from $\mathcal{A}$.

The next result is a consequence of the Koszul duality between $\mathsf{Com\text{-}Lie}$.

Proposition 2.60. *There exists a weak equivalence*

$$\Lambda(\mathsf{coLie}(\mathcal{A}[1])[-1]) \xrightarrow{\simeq} \mathcal{A}$$

for all $\mathcal{A} \in \mathsf{Alg}_+(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$. The differential on $\Lambda(\mathsf{coLie}(\mathcal{A}[1])[-1])$ is induced by the differential on $\mathcal{A}$.

A proof of this can be found in [43]. We can then deduce that.

Lemma 2.61. *There is a weak equivalence $LQ(\mathcal{A}) \simeq \mathrm{Harr}(\mathcal{A})$ for all $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$.*

Proof. Let $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ and $\mathcal{X} \xrightarrow{\simeq} \mathcal{A}$ be a cofibrant replacement. Then $LQ(\mathcal{A}) = Q(\mathcal{X})$. Naturally, we have that $Q(\Lambda(\mathrm{Harr}(\mathcal{A})) = \mathrm{Harr}(\mathcal{A})$ since the indecomposibles of the free algebra object is just the generating functor. Each $\mathcal{X}^{\otimes n}$ can be written as an iterated pushout this implies that since $\mathcal{X}$ is a cofibrant object so is $\mathrm{Harr}(\mathcal{X})$. We showed that weak equivalences are preserved by tensors so we have that $\mathrm{Harr}(-)$ preserves weak equivalences. This implies then that $\mathrm{Harr}(\mathcal{X}) \xrightarrow{\simeq} \mathrm{Harr}(\mathcal{A})$. From our observation above we also have a weak equivalence $\Lambda(\mathrm{Harr}(\mathcal{X})) \xrightarrow{\simeq} \mathcal{X}$. As Q is a left Quillen functor we have that $Q(\Lambda(\mathrm{Harr}(\mathcal{X})) \xrightarrow{\simeq} Q(\mathcal{X})$. Summarising this in a commutative diagram gives us

$$
\begin{array}{ccc}
Q(\mathcal{X}) \xleftarrow{\;\simeq\;} Q(\Lambda(\mathrm{Harr}(\mathcal{X})) & = & \mathrm{Harr}(\mathcal{X}) \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
Q(\Lambda(\mathrm{Harr}(\mathcal{A})) & = & \mathrm{Harr}(\mathcal{A})
\end{array}
$$

which makes it clear that $Q(\mathcal{X}) = LQ(\mathcal{A}) \simeq \mathrm{Harr}(\mathcal{A})$. $\qquad\square$

Given this, we defined the homology of our objects in $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$.

Definition 2.62. For all $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ we define the p-th homology of $\mathcal{A}$ to be

$$H_p(\mathcal{A}) = AQ_p(\mathcal{A}) \cong H_p(\mathrm{Harr}(\mathcal{A}))$$

If $\mathcal{A}$ has values in $\mathbf{GrVec}_{\mathbb{Q}}$ then $\operatorname{Harr}\mathcal{A}$ inherits a second grading. This induces gradings on $AQ_p(\mathcal{A})$ and $H_p(\operatorname{Harr}(\mathcal{A}))$ to keep track of this grading we define the homology.

Definition 2.63. For all $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ with values in $\mathbf{GrVec}_{\mathbb{Q}}$ we define the q-th graded piece of the p-th homology of $\mathcal{A}$ by

$$H_p(\mathcal{A})_q = AQ_{p+q-1}(\mathcal{A})_q \cong H_{p+q}(\operatorname{Harr}(\mathcal{A}))_q$$

We call p the Harrison degree while q is called the internal degree. The total degree is then $p + q - 1$. We may also consider algebra objects $\mathcal{A} \in \mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ which not only have values in $\mathbf{GrVec}_{\mathbb{Q}}$ but also have an additional grading called the weight grading. Such objects will induce a weight grading on the homology $H_p(\mathcal{A})_q$ which we shall denote this like so

$$H_p(\mathcal{A})_{q,w} = AQ_{p+q-1}(\mathcal{A})$$

for each piece w of the weight grading. With this, we are ready to define Koszulness.

Definition 2.64. Let $\mathcal{A} \in \mathsf{Alg}(\mathbf{GrVec}_{\mathbb{Q}}^{I})$ have an extra weight grading. We say that $\mathcal{A}$ is *Koszul in weight* $\leq W$ if $H_p(\mathcal{A})_{q,w} = 0$ when $p \neq w$ and $w \leq W$. If $W = \infty$ we say that $\mathcal{A}$ is *Koszul*.

Lemma 2.65. *If $\mathcal{A}$ is as above and for each $\alpha \in I$ we have that $\mathcal{A}(\alpha)$ is supported in weight ≥ 1 then $H_p(\mathcal{A})_{q,w} = 0$ when $p > w$.*

Proof. Let $\mathcal{A} \in \mathsf{Alg}(\mathbf{GrVec}_{\mathbb{Q}}^{I})$ have an extra grading. Assume that for each $\alpha \in I$ we have that $\mathcal{A}(\alpha)$ is supported in weight ≥ 1. That is the weight of $\mathcal{A}$ is ≥ 1. Neglecting the differential, the Harrison complex may be written as

$$\operatorname{Harr}(\mathcal{A}) = \bigoplus_{k \geq 0} \mathsf{coLie}(k) \otimes_{\mathfrak{S}_k} (\mathcal{A}[1])^{\otimes k}$$

The weight of $\mathcal{A}[1]$ is ≥ 1 which implies that $(\mathcal{A}[1])^{\otimes k}$ has weight $\geq p$. It then follows for $H_p(\mathcal{A})$ being a sub-algebra of a quotient of $\mathsf{coLie}(p) \otimes_{\mathfrak{S}_p} (\mathcal{A}[1])^{\otimes p}$ that $H_p(\mathcal{A})$ is supported in weight $\geq p$. That is $H_p(\mathcal{A})_{q,w} = 0$ when $p > w$. $\qquad\square$

The previous lemma simplifies the criteria of Koszulness (weighted or not) since it ensures that we only need to check that $H_p(\mathcal{A})_{q,w} = 0$ for $p < w$. Before concluding this section we will quickly define relative homology.

Definition 2.66. Given a morphism $f \colon \mathcal{A} \to \mathcal{B}$ in $\mathsf{Alg}(\mathbf{Ch}^{\mathbf{I}}_{\mathbb{Q}})$. We define the relative homology of the pair $(\mathcal{B}, \mathcal{A})$ to be

$$H_p(\mathcal{B}, \mathcal{A}) = H_p(\mathrm{cone}(LQf))$$

where LQf is the induced map $LQf \colon LQ(\mathcal{A}) \to LQ(\mathcal{B})$.

Then as in the classical case, we still obtain a long exact sequence in homology.

Proposition 2.67. *Given a morphism $f \colon \colon \mathcal{A} \to \mathcal{B}$ in $\mathsf{Alg}(\mathbf{Ch}^{\mathbf{I}}_{\mathbb{Q}})$ we get a long exact sequence*

$$... \to H_p(\mathcal{A}) \to H_p(\mathcal{B}) \to H_p(\mathcal{B}, \mathcal{A}) \to H_{p-1}(\mathcal{A}) \to H_{p-1}(\mathcal{B}) \to H_{p-1}(\mathcal{B}, \mathcal{A}) \to ...$$

in homology.

Proof. Recall that $\mathcal{A}[-1]$ can be expressed as the pushout of $0 \leftarrow \mathcal{A} \to 0$. Then we can use the pasting law of pushout to get a diagram

This gives us a sequence

$$... \to LQ(\mathcal{A}) \to LQ(\mathcal{B}) \to \mathrm{cone}(LQf) \to LQ(\mathcal{A})[-1] \to LQ(\mathcal{B})[-1] \to \mathrm{cone}(LQf)[-1] \to ...$$

Using the isomorphism $H_p(\mathcal{A}[-1]) \cong H_{p-1}(\mathcal{A})$ and applying the homology functor we get the long exact sequence

$$... \to H_p(\mathcal{A}) \to H_p(\mathcal{B}) \to H_p(\mathcal{B}, \mathcal{A}) \to H_{p-1}(\mathcal{A}) \to H_{p-1}(\mathcal{B}) \to H_{p-1}(\mathcal{B}, \mathcal{A}) \to ...$$

of homology functors. $\qquad\square$

2.3 Example of homology and the Downward Brauer Catgeory

We will now apply the theory we have developed to the so call downward Brauer category. Just as with the abstract category $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{I}})$ the downward Brauer category should be thought of as a very large algebra. We will define the Brauer category and introduce two examples of objects and a computation of their relative homology.

2.3.1 Downward Brauer category

First, recall that a partition π of S is a collection of disjoint subsets of S whose union is S. We call the disjoint sets of a partition *blocks* and we say a *block is of size n* if it has cardinality n.

Definition 2.68. Let S be a finite set a *matching* is a partition of S into disjoint sets of cardinality 2. An *ordered matching* of S is a partition of S into disjoint ordered pairs of elements.

An example of a matching of $S = \{1, 2, 3, 4\}$ is $\{\{1, 3\}, \{2, 4\}\}$ and an ordered matching of $\{(1, 3), (2, 4)\}$. Note that the ordered matchings $\{(1, 3), (2, 4)\}, \{(1, 3), (4, 2)\}, \{(3, 1), (4, 2)\}$ and $\{(3, 1), (2, 4)\}$ of S are all different and are not equal.

Definition 2.69. Let dBr be the $\mathbf{Vect}_{\mathbb{Q}}$-enriched category with objects being finite sets. The vector space $dBr(S, T)$ will be the finite-dimensional space spanned by pairs (f, m_S) where $f \colon T \to S$ is an injective function and m_S is a matching of $S \setminus f(T)$. This category is called the *downward Brauer category*.

We give an example.

Example 2.70. We will give an example of two morphisms in dBr^+ and their composition. Let $S = \{1, 2, ..., 7\}$, $T = \{a, b, c, d, e\}$ and $U = \{a, b, c\}$ be sets. Define the injections $f \colon U \to T$ and $g \colon T \to S$ by

$$f(a) = a, \quad f(b) = d, \quad f(c) = c$$

$$g(a) = 3, \quad g(b) = 1, \quad g(c) = 5, \quad g(d) = 2, \quad g(e) = 7.$$

We then have the matchings $\{\{b, e\}\}$ and $\{\{4, 6\}\}$ of the sets $T \setminus f(U)$ and $S \setminus f(T)$. These give us morphisms the morphisms $(f, \{\{b, e\}\}) \in dBr^+(T, U)$

and $(g, \{\{4, 6\}\}) \in dBr^+(S, T)$. In figure 2.1 we have drawn pictures of $(f, \{\{b, e\}\})$ and $(g, \{\{4, 6\}\})$. We can form the composition of $(f, \{\{b, e\}\})$ and $(g, \{\{4, 6\}\})$ which will be the morphism $(g \circ f, \{\{4, 6\}, \{1, 7\}\}) \in dBr^+(S, U)$. The morphism has been drawn in figure 2.2.
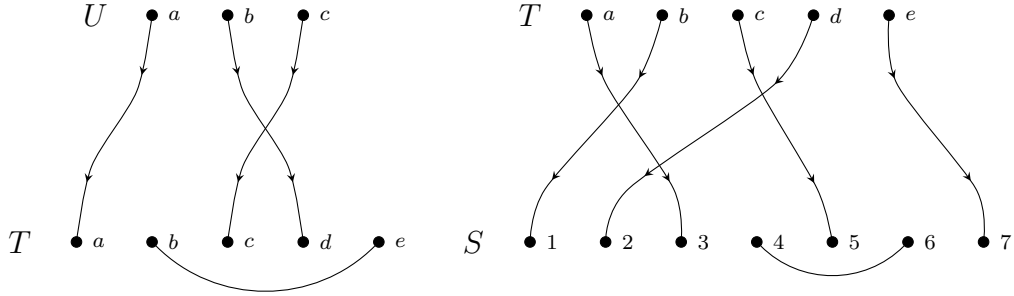
Figure 2.1: The morphisms $(g, \{\{b, e\}\}) \colon U \to T$ and $(f, \{\{4, 6\}\}) \colon S \to T$.
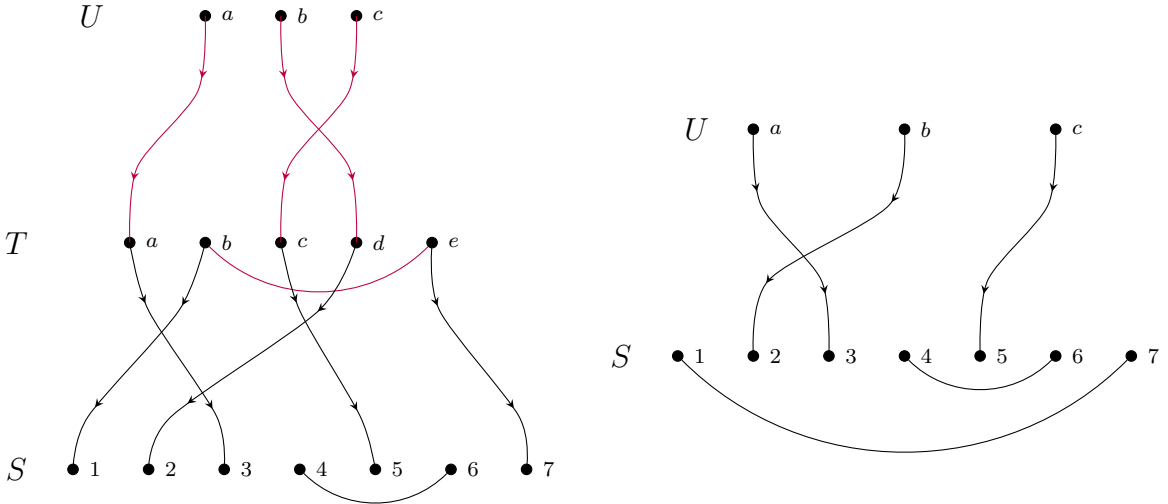
Figure 2.2: The composition morphism $(g, \{\{b, e\}\}) \circ (f, \{\{4, 6\}\})$.

Definition 2.71. Let sdBr be the $\mathbf{Vect}_{\mathbb{Q}}$-enriched category with objects being finite sets. The vector space $\mathrm{sdBr}(S, T)$ will be the finite dimensional space $\mathrm{dBr}(S, T)$ quotiented by the span of the elements $(f, m_S) - (-1)^k(f, m'_S)$ where $f \colon T \to S$ is an injective function and m_S, m'_S are ordered matchings of $S \setminus f(T)$ such that m'_S may be obtained from m_S by

swapping the order of k pairs. This category is called the *signed downward Brauer category*.

Let dB denote the category which is equal to dBr if n is even and sdBr when n is odd, where the use of n will become clearer later. We will also write morphisms of dB as if they were morphisms of sdBr. It should be noted that the morphisms in dBr and sdBr can decompose to a series of compositions of bijections and inclusions, although not uniquely. Below, in figure 2.3, we give an example of a decomposition of the morphism $(g, \{(e, b)\}) \in \text{sdBr}(T, U)$.
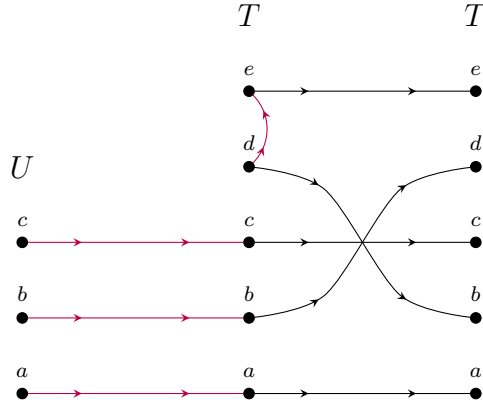
Figure 2.3: A decomposition of $(g, \{(e, b)\})$ into an inclusion of U and a permutation of T.

Both the signed and unsigned downward Brauer categories have a symmetric monoidal structure given by taking disjoint union and the empty set. This implies that that the category $\mathbf{Ch}_{\mathbb{Q}}^{\text{dB}}$ has a model structure where a morphism $\mathcal{X} \to \mathcal{Y}$ in $\mathbf{Ch}_{\mathbb{Q}}^{\text{dB}}$ is a

i) weak equivalences if $\mathcal{X}(S) \to \mathcal{Y}(S)$ is a weak equivalence in $\mathbf{Ch}_{\mathbb{Q}}$ for all sets $S \in \text{dB}$,

ii) or a fibrations if $\mathcal{X}(S) \to \mathcal{Y}(S)$ is a fibration in $\mathbf{Ch}_{\mathbb{Q}}$ for all sets $S \in \text{dB}$.

See Theorem 2.29. We also have that $\mathbf{Ch}_{\mathbb{Q}}^{\text{dB}}$ has a monoidal structure on it given by Day convolution. Kupers and Randal-Williams [40] describe the Day convolution on $\mathbf{Ch}_{\mathbb{Q}}^{\text{dB}}$ explicitly.

Definition 2.72. For all $S', S'', T \in \mathrm{dB}$, let $\mathrm{Pair}(S' \sqcup S'', T)$ denote the subspace of $\mathrm{dB}(S' \sqcup S'', T)$ spanned by the morphisms $(f, m_{S' \sqcup S''})$ where every pair in $m_{S' \sqcup S''}$ has one element from S' and one element from S''.

Theorem 2.73. *For all $\mathcal{F}, \mathcal{G} \in \mathbf{Ch}_{\mathbb{Q}}^{\mathrm{dB}}$ and $T \in \mathrm{dB}$ here is an isomorphism of functors*

$$(\mathcal{F} \otimes \mathcal{G})(T) \cong \bigoplus_{(S', S'')} \mathrm{Pair}(S' \sqcup S'', T) \otimes_{\mathfrak{S}_{S'} \times \mathfrak{S}_{S''}} F(S') \otimes G(S'')$$

where the sum is indexed by isomorphism classes of pairs of finite sets.

A proof can be found in [40]. We can also use the properties of Day convolution to calculate explicitly the functor $\Lambda_{[1]}^{+}(z_n)$. Recall that for sets S, T the set S^T is the set of functions from T to S.

Lemma 2.74. *There is an isomorphism $\Lambda_{[1]}^{+}(z_n)(T) \cong \det(\mathbb{Q}^T)^{\otimes n+1}[|T|(n+1)]$ for all $T \in \mathrm{dB}$ and for each morphism $(f, m) \in \mathrm{dB}$ with $m \neq \varnothing$ we have that $\Lambda_{[1]}^{+}(z_n)(f, m)$ is the zero morphism.*

Proof. Consider the Day convolution $(ev_{[1]}^{!} S(n+1))^{\otimes 2}$. Using proposition 2.15 and the properties of adjoints we can show that

$$\mathbf{Ch}_{\mathbb{Q}}^{\mathrm{dB}}(ev_{[1]}^{!} S(n+1) \otimes ev_{[1]}^{!} S(n+1), \mathcal{A}) \cong \mathbf{Ch}_{\mathbb{Q}}(S(n+1) \otimes S(n+1), \mathcal{A}([2])).$$

Now consider the functor $S(n+1)^{\otimes 2} \otimes \mathrm{dB}([2], -) \colon \mathrm{dB} \to \mathbf{Ch}_{\mathbb{Q}}$. Using the distinguished element $id_{[2]}$ we can get an isomorphism

$$\mathbf{Ch}_{\mathbb{Q}}^{\mathrm{dB}}(S(n+1)^{\otimes 2} \otimes \mathrm{dB}([2], -), \mathcal{A}) \cong \mathbf{Ch}_{\mathbb{Q}}(S(n+1) \otimes S(n+1), \mathcal{A}([2])).$$

We can extend this argument to get

$$ev_{[1]}^{!} S(n+1)^{\otimes p}(T) \cong S(n+1)^{\otimes p} \otimes \mathrm{dB}([p], T)$$

However $\mathfrak{S}_p$ acts diagonally on the $S(n+1)^{\otimes p}$ part and $\mathfrak{S}_p$ acts on $\mathrm{dB}([p], T)$ in the following way. Let $(f, m) \in \mathrm{dB}([p], T)$ and $\sigma \in \mathfrak{S}_p$ then we can define a new map $f \circ \sigma$. The matching of $[p] \setminus f \circ \sigma$ is then defined by

$$m_{\sigma} = \{(\sigma(x), \sigma(y)) \mid (x, y) \in m\}$$

the element $\sigma \cdot (f, m) = (f \circ \sigma, m_\sigma)$. This action implies that

$$\Lambda^+(z_{n+1})(T) = \bigoplus_{p=0}^{\infty} S(n+1)^{\otimes p} \otimes_{\mathfrak{S}_p} \mathrm{dB}([p], T)$$

Now let $z_{n+1} \in S(n+1)$ be the basis element. Let $(f, m) \in \mathrm{dB}([p], T)$ with $(x, y) \in m$. Then $x, y \in [p]$ and so $(xy) \in \mathfrak{S}_p$. By the action which has just been mentioned, we have

$$\begin{aligned}
(z_{n+1} \otimes ... \otimes z_{n+1}) \otimes_{\mathfrak{S}_p} (f, m) &= (xy) \cdot (z_{n+1} \otimes ... \otimes z_{n+1}) \otimes_{\mathfrak{S}_p} (xy) \cdot (f, m) \\
&= (-1)^{n+1}(z_{n+1} \otimes ... \otimes z_{n+1}) \otimes_{\mathfrak{S}_p} (-1)^n(f, m) \\
&= -(z_{n+1} \otimes ... \otimes z_{n+1}) \otimes_{\mathfrak{S}_p} (f, m)
\end{aligned}$$

where the $(xy) \cdot (f, m) = (-1)^n(f, m)$ comes from the definition of dB. Hence $(e_{n+1} \otimes ... \otimes e_{n+1}) \otimes_{\mathfrak{S}_p} (f, m) = 0$. This implies that we only have a non-zero term when $p = |T|$. It then follows

$$\Lambda^+([1]_* S(n+1))(T) = \bigoplus_{p=0}^{\infty} S(n+1)^{\otimes p} \otimes_{\mathfrak{S}_p} \mathrm{dB}([p], T) = S(n+1)^{\otimes |T|} \otimes_{\mathfrak{S}_{|T|}} \mathrm{dB}([|T|], T).$$

But it is easy to see that $\mathrm{dB}([|T|], T) \cong \mathrm{dB}(T, T) \cong \mathbb{Q}S_T$. Therefore by the definition of the tensor over the symmetric group we have that

$$\Lambda^+([1]_* S(n+1))(T) \cong S(n+1)^{\otimes |T|} \otimes_{S_T} \mathbb{Q}S_T \cong S(n+1)^{\otimes |T|}$$

Which is a one-dimensional vector space generated in degree $|T|(n+1)$ so it is isomorphic to $\det(\mathbb{Q}^T)^{\otimes n+1}[|T|(n+1)]$. $\square$

2.3.2 The functors $\mathcal{E}_n$ and $\mathcal{Z}_n$

We will define two functors $\mathcal{E}_n$ and $\mathcal{Z}_n$ in $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathrm{dB}})$

Definition 2.75. Let V be a vector space and define $\det(V) = \Lambda^{\dim V} V$.

Note that $\Lambda^{\dim V} V$ is a one-dimensional vector space since it is generated by the element $e_1 \wedge e_2 \wedge ... \wedge e_n$.

Definition 2.76. Let S be a finite set. A partition of S is a finite collection $\{S_i\}_{i=1}^k$ of, possibly empty, subsets of S that have union equal to S and are pairwise disjoint. We call the subsets in a partition blocks. A partition is admissible if each block is of size 3 or larger. We shall often use the notation $\{S_\alpha\}_{\alpha \in I}$ for a partition. The I is this notation is an indexing set.

We now consider two functors in $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathrm{dB}})$. These will play an important role as they will become the tool we use to prove the Kozulness of the associated graded of the Torelli Lie algebra.

For $n \in \mathbb{N}_{>0}$ let $\mathscr{Z}_n \colon \mathrm{dB} \to \mathbf{GrVec}_{\mathbb{Q}}$ be defined by

$$\mathscr{Z}_n(S) = \mathbb{Q}\{\text{admissible partitions } \{S_\alpha\}_{\alpha \in I} \text{ of } S\} \otimes \det(\mathbb{Q}^S)^{\otimes n}$$

We define the degree of an admissible partition P to be the sum $n \sum_{\alpha \in I}(|S_\alpha| - 2)$. Notice that when we have a map $(f, m_S) \colon S \to T$ in dB it can be broken up a composition of bijections $(g, \varnothing)$ and maps of the form $(i, \{(x,y)\}) \colon S \to S\backslash\{x,y\}$ where $i \colon S\backslash\{x,y\} \to S$ is the inclusion and $\{(x,y)\}$ is the matching. Thus it is enough to define what Z_n does to these morphisms. First consider a bijection $(f, \varnothing) \colon S \to T$. The linear map induced by $(f, \varnothing) \colon S \to T$ will be given by relabelling elements of blocks in partitions and by the induced map on the determinants. The linear map given by $(i, \{(x,y)\}) \colon S \to S \setminus \{x,y\}$ will be defined as follows. Consider $X = [\{S_\alpha\}_{\alpha \in I}] \otimes (x \wedge y \wedge s_3 \wedge ... s_{|S|})^{\otimes n} \in \mathscr{Z}_n(S)$. If $\{x,y\} \subset S_\alpha$ for some $\alpha \in I$ then $\mathscr{Z}_n(f)(X) = 0$. If it is not the case then $x \in S_\beta$ and $y \in S_\gamma$ for some $S_\gamma, S_\beta \in \{S_\alpha\}_\alpha$. We will glue these two blocks of the partition together and remove x and y. Let $\{S'_k\}_{k \in J}$ be the partition of $S \setminus \{x,y\}$ defined by

$$\{S'_k\}_{k \in J} = \{S_\alpha\}_{\alpha \in I \setminus \{\gamma, \beta\}} \cup \{S_\beta \setminus \{x\} \cup S_\gamma \setminus \{y\}\}$$

and then we define

$$\mathscr{Z}_n(f)(X) = [\{S'_k\}_{k \in J}] \otimes (s_3 \wedge s_4 \wedge ... \wedge s_{|S|})^{\otimes n}$$

We then get more general maps by composition.

Definition 2.77. A *weighted partition* $\{(S_\alpha, g_\alpha)\}_{\alpha \in I}$ is a partition $\{S_\alpha\}_{\alpha \in I}$ together with a weight $g_\alpha \in \{0, 1, 2, ...\}$ for each block. A weighted partition is *admissible* if

i) each block of size 0 has weight greater than or equal to 2

ii) each block of size 1 or 2 has weight greater than or equal to 1

Now define the second functor $\mathscr{E}_n \colon \mathrm{dB} \to \mathbf{GrVec}_{\mathbb{Q}}$ which will be of interest to us.

Definition 2.78. For $n \in \mathbb{N}_{>0}$ let $\mathcal{E}_n$ be defined by

$$\mathcal{E}_n(S) = \mathbb{Q}\{\text{admissible weighted partitions } \{(S_\alpha, g_\alpha)\}_{\alpha \in I} \text{ of } S\} \otimes \det(\mathbb{Q}^S)^{\otimes n}$$

We define the degree of an admissible weighted partition P to be the sum $n \sum_{\alpha \in I}(2g_\alpha + |S_\alpha| - 2)$. Again we define what E_n does to morphims by describing does to bijections and maps of the form $(i, \{((x, y), g)\})\colon S \to S \backslash \{x, y\}$. The linear map induced by a bijection $(f, \varnothing)\colon S \to T$ will be given by relabelling elements of blocks in partitions and by the induced map on the determinants. The linear map given by $(i, \{(x, y)\})\colon S \to S \setminus \{x, y\}$ will be defined as follows. Consider $Y = [\{(S_\alpha, g_\alpha\}_{\alpha \in I}] \otimes (x \wedge y \wedge s_3 \wedge ... s_{|S|})^{\otimes n} \in \mathcal{E}_n(S)$. If $\{x, y\} \subset S_\beta$ for some $\beta \in I$ then

$$\mathcal{E}_n(f)(Y) = [\{(S_\alpha, g_\alpha)\}_{\alpha \neq \beta} \cup \{(S_\beta \setminus \{x, y\}, g_\beta + 1)\}] \otimes (s_3 \wedge ... \wedge s_{|S|})^{\otimes n}$$

If it is not the case then $x \in S_\beta$ and $y \in S_\gamma$ for some $(S_\gamma, g_\gamma), (S_\beta, g_\beta) \in \{S_\alpha\}_\alpha$. Then we define

$$\mathcal{E}_n(f)(Y) = [\{(S_\alpha, g_\alpha)\}_{\alpha \in I \setminus \{\gamma, \beta\}} \cup \{(S_\beta \setminus \{x\} \cup S_\gamma \setminus \{y\}, g_\beta + g_\gamma)\}] \otimes (s_3 \wedge s_4 \wedge ... \wedge s_{|S|})^{\otimes n}$$

We use disjoint union to make the functors $\mathcal{E}_n$ and $\mathcal{Z}_n$ into unital commutative algebra objects. This is because they are lax symmetric monoidal functors. Since the homological degree is concentrated at multiples of n we give $\mathcal{E}_n$ and $\mathcal{Z}_n$ additional weight gradings by giving it the homological degree divided by n. Both $\mathcal{E}_n$ and $\mathcal{Z}_n$ have unique augmentations given by sending partitions of non-empty sets to 0 and the empty partition of the empty set to 1.

Example 2.79. Consider the sets $[10] = \{1, 2, ..., 10\}$ and $A = \{a, b, c, d, e\}$. We can form the partitions $\pi_{10} = \{\{2, 3, 10\}, \{1, 4, 9\}, \{5, 6, 7, 8\}\}$ and $\pi_A = \{A\}$. Let $\vec{10} = 1 \wedge 2 \wedge ... \wedge 10$ and $\vec{a} = a \wedge b \wedge ... \wedge e$. Set

$$x_{10,n} = [\pi_{10}] \otimes \vec{10}^{\otimes n}, \quad \& \quad x_{A,n} = [\pi_A] \otimes \vec{a}^{\otimes n}.$$

Then $x_{10,n} \in \mathcal{Z}_n([10])$ and $x_{A,n} \in \mathcal{Z}_n(A)$. The degrees of $x_{10,n}$ and $x_{A,n}$ are

$$\deg(x_{10,n}) = n(1 + 1 + 2) = 4n, \quad \& \quad \deg(x_{A,n}) = n(3) = 3n$$

by the definition. We can multiply $x_{10,n}$ and $x_{A,n}$ together to get

$$x_{10,n} \cdot x_{A,n} = [\pi_{10} \sqcup \pi_A] \otimes (\vec{10} \wedge \vec{a})^{\otimes n} = [\pi_{10} \sqcup \pi_A] \otimes (1 \wedge 2 \wedge ... \wedge 10 \wedge a \wedge ... \wedge e)^{\otimes n}$$

an element of $\mathcal{Z}_n([10] \sqcup A)$. The degree of $x_{10,n} \cdot x_{A,n}$ calculate using the definition is

$$\deg(x_{10,n} \cdot x_{A,n}) = n(1 + 1 + 2 + 3) = 7n$$

as we expect. Also notice that $\vec{10} \wedge \vec{a} = (-1)^{(10 \cdot 5)} \vec{a} \wedge \vec{10} = \vec{a} \wedge \vec{10}$. This implies that we have $x_{10,n} \cdot x_{A,n} = (-1)^{(3n)(4n)} x_{A,n} \cdot x_{10,n} = x_{A,n} \cdot x_{10,n}$ as we may expect.

2.3.3 Relative Homology of $\mathcal{E}_n$ and $\mathcal{Z}_n$

We will now compute the relative homology of $(\mathcal{Z}_n, \mathcal{E}_n)$. To do this we first define a homomorphism. We call the sum of the weights in a weighted partition the total weight.

Definition 2.80. Define the epimorphism $\varphi \colon \mathcal{E}_n \to \mathcal{Z}_n$ by

$$\varphi([\{(S_\alpha, g_\alpha)\}_{\alpha \in I}] \otimes (s_1 \wedge ... \wedge s_k)^{\otimes n}) = \begin{cases} [\{S_\alpha\}_{\alpha \in I}] \otimes (s_1 \wedge ... \wedge s_k)^{\otimes n}, & \sum_{\alpha \in I} g_\alpha = 0 \\ 0 & \sum_{\alpha \in I} g_\alpha > 0 \end{cases}$$

for all finite sets S and admissible weighted partitions $\{(S_\alpha, g_\alpha)\}$ of S.

Note that φ is compatible with the augmentations and preserves the gradings on $\mathcal{E}_n$ and $\mathcal{Z}_n$. To compute the relative homology we will define another functor $\mathcal{E}'_n \colon dB \to \mathbf{Ch}_\mathbb{Q}$ and factor φ through it. Let $[k] = \{1, 2, ..., k\}$. Notice that there is a map

$$S(n) \to \mathcal{E}_n([1])$$

defined by $z_n \mapsto \{([1], 1)\}$. By adjunction we get a map $ev^!_{[1]} S(n) \to \mathcal{E}_n$, where $ev^!_{[1]}$ denotes the left adjoint of the evaluation functor $ev_{[1]}$. This extends to a map $\phi \colon \Lambda^+_{[1]}(z_n) \to \mathcal{E}_n$. We use this to define $\mathcal{E}'_n$.

Definition 2.81. Define $\mathcal{E}'_n \colon dB \to \mathbf{Ch}_\mathbb{Q}$ to be the pushout

$$\begin{array}{ccc} \Lambda_{[1]}(S(n)) & \longrightarrow & \mathcal{E}_n \\ \downarrow & & \downarrow \\ \Lambda(ev^!_{[1]} D(n+1)) & \longrightarrow & \mathcal{E}'_n \end{array}$$

Forgetting differentials gives us the following result.

Proposition 2.82. *If we neglect differentials we get that*

$$\mathcal{E}'_n(S) \cong \bigoplus_{T,T'} \mathrm{Pair}(T \sqcup T', S) \otimes_{S_T \times S_{T'}} \mathcal{E}_n(T) \otimes S(n+1)^{\otimes |T|} \qquad (2.1)$$

for all finite sets $S \in \mathrm{dB}$.

Proof. If we neglect differentials we have that pushouts of commutative algebra objects are given by the relative tensor product. This gives us

$$
\begin{aligned}
\mathcal{E}'_n &= \mathcal{E}_n \otimes_{\Lambda^+_{[1]}(z_n)} \Lambda^+(D(n+1)) \\
&= \mathcal{E}_n \otimes_{\Lambda^+_{[1]}(z_n)} \Lambda^+ \left(ev^!_{[1]} S(n) \oplus ev^!_{[1]} S(n+1) \right) \\
&= \mathcal{E}_n \otimes_{\Lambda^+_{[1]}(z_n)} \Lambda^+_{[1]}(z_n) \otimes \Lambda^+_{[1]}(z_{n+1}) \\
&= \mathcal{E}_n \otimes \Lambda^+_{[1]}(z_{n+1})
\end{aligned}
$$

The result the follows from Lemma 2.74. $\qquad\square$

Since $\{([1], 1)\}$ is mapped to zero in $\mathcal{Z}_n([1])$ we get a factorisation

$$\mathcal{E}_n \to \mathcal{E}'_n \to \mathcal{Z}_n.$$

However notice that by the construction of $\mathcal{E}'_n$ that the cofibre of $LQ^+(\mathcal{E}_n) \to LQ^+(\mathcal{E}'_n)$ is $ev^!_{[1]} S(n+1)$. Our goal then is to prove that $\mathcal{E}'_n \to \mathcal{Z}_n$ is a weak equivalence as this will simplify the computation of $H_*(\mathcal{Z}_n, \mathcal{E}_n)$. As it stands now we do not know what the differential on $\mathcal{E}'_n$ is. We will define yet another functor $\mathcal{E}''_n \colon \mathrm{dB} \to \mathbf{Ch}_{\mathbb{Q}}$ and prove that it is isomorphic to $\mathcal{E}'_n$. This will allow us to prove that $\mathcal{E}'_n \to \mathcal{Z}_n$ is a weak equivalence.

Definition 2.83. A biweighted partition $\{(A_\alpha, g_\alpha, h_\alpha) \mid \alpha \in I\}$, denoted by $(A_\alpha, g_\alpha, h_\alpha)_{\alpha \in I}$, of a set S is a partition $\{A_\alpha\}_{\alpha \in I}$ of a subset $A \subset S$ together with two weights $g_\alpha \in \{0, 1, 2, ...\}$ and $h_\alpha \in \{0, 1\}$. Here I denotes an indexing set. A biweighted partition is admissible if

i) when $|S_\alpha| = 0$ then $g_\alpha + h_\alpha \geq 2$,

ii) when $|S_\alpha| = 1, 2$ then $g_\alpha + h_\alpha \geq 1$

Definition 2.84. For each finite set S define $\mathcal{E}''_n(S)$ by

$$\mathcal{E}''_n(S) = \bigoplus_{\substack{\text{admissible} \\ \zeta_S = (A_\alpha, g_\alpha, h_\alpha)_{\alpha \in I} \text{ of } S}} \det(\mathbb{Q}^{J(\zeta_S)}) \otimes \det(\mathbb{Q}^{B(\zeta_S)}) \otimes \det(\mathbb{Q}^S)^{\otimes n}$$

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where $J(\zeta_S) = \{\alpha \in I \mid h_\alpha = 1\}$, $A(\zeta_S) = \bigcup_{\alpha \in I} A_\alpha$, and $B(\zeta_s) = S \setminus A(\zeta_S)$. This is made into a graded vector space by defining a biweighted partition to have degree $|B|(n+1) + n \sum_{\alpha \in I}(2g_\alpha + |S_\alpha| - 2) + (2n+1)h_\alpha$.

We also define a differential on $\mathcal{E}''_n$. For short hand let us denote

$$\tilde{j}_\zeta = j_1 \wedge ... \wedge j_{|J(\zeta)|}, \qquad \tilde{b}_\zeta = b_1 \wedge ... \wedge b_{|B(\zeta)|},$$

$$\tilde{S} = a_1 \wedge ... \wedge a_{|A(\zeta)|} \wedge b_1 \wedge ... \wedge b_{|B(\zeta)|}$$

for a biweighted partition ζ of S. We define the differential on $\mathcal{E}''_n(S)$ by sending the element $[(A_\alpha, g_\alpha, h_\alpha)_{\alpha \in I}] \otimes \tilde{j} \otimes \tilde{b} \otimes \tilde{S}^{\otimes n}$ to the element

$$\sum_{\beta \in J(\zeta)} [(A_\alpha, g_\alpha + \delta_{\alpha,\beta}, h_\alpha - \delta_{\alpha,\beta})_{\alpha \in I}] \otimes (j_1 \wedge ... \wedge \hat{\beta} \wedge ... \wedge j_{|J|}) \otimes \tilde{b}_\zeta \otimes \tilde{S}^{\otimes n}$$

$$+ \sum_{b \in \mathcal{B}(\zeta)} [\{(A_\alpha, g_\alpha, h_\alpha)_{\alpha \in I \cup \{b\}}] \otimes \tilde{j} \otimes (b_1 \wedge ... \wedge \hat{b} \wedge ... \wedge b_{|B|}) \otimes \tilde{S}^{\otimes n}$$

where $(A_b, g_b, h_b) = (\{b\}, 1, 0)$.

Lemma 2.85. *There is an isomorphism $\chi_S \colon \mathcal{E}''_n(S) \to \mathcal{E}'_n(S)$ of graded vector spaces for each finite set S.*

Proof. We start by defining $\chi_S \colon \mathcal{E}''_n(S) \to \mathcal{E}'_n(S)$. We define χ_S on the elements which span the $\mathcal{E}''_n(S)$. Let $x = [(A_\alpha, g_\alpha, h_\alpha)_{\alpha \in I}] \otimes \tilde{j} \otimes \tilde{b} \otimes \tilde{S}^{\otimes n} \in \mathcal{E}''_n(S)$. Let $S' = A \cup J$ and $S'' = B \cup J$. Define $\phi = (f, m) \colon S' \sqcup S'' \to S \in \mathrm{dB}(S' \sqcup S'', S)$ to be the injection $f \colon S \to S' \sqcup S''$ defined by $f(s) = s$. Notice that $s \in S'$ if $s \in A$ and $s \in S''$ if $s \in B$. This makes it clear that f is well-defined. As for the matching m, we define m by pair each elements of $J \subset S' = A \sqcup J$ with the same element in $J \subset S'' = B \sqcup J$ placing elements of S' first. This defines an element $\phi \in \mathrm{Pair}(S' \sqcup S'', S)$. Now we define an element in $\mathcal{E}_n(S')$. For each $\alpha \in I$, let

$$S'_\alpha = \begin{cases} S_\alpha & h_\alpha = 0, \\ S_\alpha \cup \{\alpha\} & h_\alpha = 1 \end{cases}$$

which is a subset of S'. We define $\chi_S(x)$ to be

$$\chi_S(x) = \phi \otimes [(S'_\alpha, g_\alpha)_{\alpha \in I}] \otimes (a_1 \wedge ... \wedge a_{|A|} \wedge j_1 \wedge ... \wedge j_{|J|})^{\otimes n} \otimes (b_1 \wedge ... \wedge b_{|B|} \wedge j_1 \wedge ... \wedge j_{|J|})^{\otimes n+1} \in \mathcal{E}'_n(S)$$

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where $\phi \in \text{Pair}(S' \sqcup S'', S)$, $[(S_\alpha, g_\alpha)_{\alpha \in I}] \otimes (a_1 \wedge ... \wedge a_{|A|} \wedge j_1 \wedge ... \wedge j_{|J|})^{\otimes n} \in \mathcal{E}_n(S')$ and $(b_1 \wedge ... \wedge b_{|B|} \wedge j_1 \wedge ... \wedge j_{|J|})^{\otimes n+1} \in \det(\mathbb{Q}^{S''})^{\otimes n+1}[(n+1)|T|]$. This defines a element of $\mathcal{E}'_n(S)$ as in (2.1). Notice that permuting j_k's, a_k's or b_k's act via a (-1), $(-1)^n$ or $(-1)^{n+1}$ respectively, so this shows that χ_S is well-defined.

Having defined χ_S we now wish to show that it is an isomorphism by producing an inverse. Before we do this consider an element in $\mathcal{E}'_n(S)$ represented by a morphism $\phi = (f, m) \colon S' \sqcup S'' \to S$, an element $[(S'_\alpha, g_\alpha)_{\alpha \in I}] \otimes (s'_1 \wedge ... \wedge s'_{|S'|})^{\otimes n}$ of $\mathcal{E}'_n(S')$ and an element $(s''_1 \wedge ... \wedge s''_{|S''|})^{\otimes n+1}$ of $\det(\mathbb{Q}^{S''})^{\otimes n+1}[|S''|(n+1)]$. Assume that there are ordered pairs $(s'_i, s''_i), (s'_j, s''_j) \in m$ with $i \neq j$ and $s'_i, s'_j \in S'_\alpha$ for some $\alpha \in I$. The permutation $(s'_i s'_j)(s''_i s''_j)$ defines a morphism $\psi = ((s'_i s'_j)(s''_i s''_j), \varnothing) \colon S' \sqcup S'' \to S' \sqcup S''$ in dB. Since $s'_i, s'_j, s''_i, s''_j \notin f(S)$ we have that $(s'_i s'_j)(s''_i s''_j) \circ f = f$ and thus $\phi \circ \psi = \phi$. It then follows that

$$\phi \otimes [(S'_\alpha, g_\alpha)_{\alpha \in I}] \otimes (s'_1 \wedge ... \wedge s_{|S'|})^{\otimes n} \otimes (s''_1 \wedge ... \wedge s''_{|S''|})^{\otimes n+1}$$
$$= \phi \otimes (s'_i s'_j) \cdot [(S'_\alpha, g_\alpha)_{\alpha \in I}] \otimes (s'_i s'_j) \cdot (s'_1 \wedge ... \wedge s_{|S'|})^{\otimes n} \otimes (s''_1 \wedge ... \wedge s''_{|S''|})^{\otimes n+1}$$
$$= \phi \otimes (s'_i s'_j) \cdot [(S'_\alpha, g_\alpha)_{\alpha \in I}] \otimes (-1)^n (s'_1 \wedge ... \wedge s_{|S'|})^{\otimes n} \otimes (-1)^{n+1}(s''_1 \wedge ... \wedge s''_{|S''|})^{\otimes n+1}$$
$$= -\phi \otimes [(S'_\alpha, g_\alpha)_{\alpha \in I}] \otimes (s'_1 \wedge ... \wedge s_{|S'|})^{\otimes n} \otimes (s''_1 \wedge ... \wedge s''_{|S''|})^{\otimes n+1}$$
$$= 0$$

Hence we may assume that elements have at most one pair $(s', s'') \in m$ such that $s' \in S'_\alpha$. We may the define the inverse of χ_S to be the morphism that sends an element of $\mathcal{E}'_n(S)$ represented by $\phi = (f, m) \colon S' \sqcup S'' \to S$, $[(S'_\alpha, g_\alpha)_{\alpha \in I}] \otimes (s'_1 \wedge ... \wedge s'_{|S'|})^{\otimes n} \mathcal{E}'_n(S')$ and $(s''_1 \wedge ... \wedge s''_{|S''|})^{\otimes n+1}$ to the element $[(S_\alpha, g_\alpha, h_\alpha)_{\alpha \in I}] \otimes \tilde{j} \otimes \tilde{b} \otimes \tilde{S}^{\otimes n} \in \mathcal{E}''_n(S)$ where $S_\alpha = S'_\alpha \setminus \{s'\}$ with $h_\alpha = 1$ if $(s', s'') \in m$ and $s' \in S'_\alpha$ or $S_\alpha = S'_\alpha$ with $h_\alpha = 0$ otherwise. $\square$

Using this isomorphism $\mathcal{E}'_n(S)$ inherits the differential from $\mathcal{E}''_n(S)$. This gives us the following lemma.

Lemma 2.86. $\mathcal{E}'_n \to \mathcal{Z}_n$ *is a weak equivalence.*

Proof. We need to show that the homology is isomorphic. Recall that the differential on $\mathcal{E}''_n$ is defined by send the element $[(A_\alpha, g_\alpha, h_\alpha)_{\alpha \in I}] \otimes \tilde{j}_\zeta \otimes \tilde{b}_\zeta \otimes \tilde{S}^{\otimes n}$ to the element

$$\sum_{\beta \in J(\zeta)} [(A_\alpha, g_\alpha + \delta_{\alpha,\beta}, h_\alpha - \delta_{\alpha,\beta})_{\alpha \in I}] \otimes (j_1 \wedge ... \wedge \hat{\beta} \wedge ... \wedge j_{|J|}) \otimes \tilde{b}_\zeta \otimes \tilde{S}^{\otimes n}$$
$$+ \sum_{b \in \mathcal{B}(\zeta)} [(A_\alpha, g_\alpha, h_\alpha)_{\alpha \in I \cup \{b\}}] \otimes \tilde{j}_\zeta \otimes (b_1 \wedge ... \wedge \hat{b} \wedge ... \wedge b_{|B|}) \otimes \tilde{S}^{\otimes n}$$

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where $(A_b, g_b, h_b) = (\{b\}, 1, 0)$. With this differential we can view $\mathcal{E}_n''(S)$ as a direct sum of chain complexes, one chain complex for each partition of S. Moreover, the partition $S = (\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{b \in B} \{b\})$ is preserved by the differential and the chain complex associated to it may be identified with the tensor product of chain complexes, one for each block in this partition.

Call the complex associated to the part A_α by A_*^α. This complex is given by

$$\bigoplus_{g_\alpha + 1 \geq r_\alpha} [(A_\alpha, g_\alpha, 1)] \otimes \alpha \otimes 1 \otimes \tilde{A}_\alpha \xrightarrow{d} \bigoplus_{g_\alpha \geq r_\alpha} [(A_\alpha, g_\alpha, 1)] \otimes 1 \otimes 1 \otimes \tilde{A}_\alpha$$

where is given by

$$r_\alpha = \begin{cases} 2, & |A_\alpha| = 0 \\ 1, & |A_\alpha| = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

The homology of this complex is then

$$H_*(A_*^\alpha) = \begin{cases} \text{span}([(A_\alpha, 0, 0)] \otimes 1 \otimes 1 \otimes \tilde{A}_\alpha), & |A_\alpha| > 2 \\ 0, & \text{otherwise} \end{cases}$$

by the requirements on g_α.

The complex corresponding to $\{b\}$ with $b \in B$ is given by

$$[(\varnothing, -, -)] \otimes 1 \otimes s \otimes b \xrightarrow{d} [(\{b\}, 1, 0)] \otimes 1 \otimes 1 \otimes b \xrightarrow{d} 0$$

is an acyclic chain complex (i.e. homology is 0 for all entries).

Thus, the homology of $\mathcal{E}_n'(S)$ is supported on those partitions with $B = \varnothing$, $|A_\alpha| > 2$ and $g_\alpha = h_\alpha = 0$. This is isomorphic to $H_*(\mathcal{Z}_n(S)) \cong \mathcal{Z}_n(S)$. It then follows that $\mathcal{E}_n' \to \mathcal{Z}_n$ is a weak equivalence. $\square$

This allows us to compute the relative homology in the following way.

Theorem 2.87. *We have that the relative homology of $(\mathcal{Z}_n, \mathcal{E}_n)$ is given by*

$$H_p(\mathcal{Z}_n, \mathcal{E}_n)_{q,w}(S) \cong \begin{cases} \mathbb{Q}, & |S| = 1 \text{ and } (p, q, w) = (2, n, 1) \\ 0, & \text{otherwise.} \end{cases}$$

for finite sets $S \in \mathrm{dB}$.

Proof. Recall that the relative homology is defined by taking the homotopy cofibre of $LQ^+(\mathcal{E}_n) \to LQ^+(\mathcal{Z}_n)$, the map induced by $\mathcal{E}_n \to \mathcal{Z}_n$. As we have a weak equivalence $\mathcal{E}'_n \to \mathcal{Z}_n$ it is enough to take the cofibre of the map $LQ^+(\mathcal{E}_n) \to LQ^+(\mathcal{E}'_n)$. However the cofibre of the map $LQ^+(\mathcal{E}_n) \to LQ^+(\mathcal{E}'_n)$ is $ev^!_{[1]}(D(n+1)/S(n)) = ev^!_{[1]}S(n+1)$. As we defined $ev^!_{[1]}S(n+1)$ to have internal degree n and weight 1 the statement follows. $\qquad\square$

Using this we get that.

Corollary 2.88. $\mathcal{E}_n$ *is Koszul if and only if* $\mathcal{Z}_n$ *is Koszul.*

Proof. By our computation of $H_p(\mathcal{Z}_n, \mathcal{E}_n)_{q,w}$ for we have that the long exact sequence is given by

$$ \ldots \to 0 \to H_p(\mathcal{E}_n)_{q,w}(S) \to H_p(\mathcal{Z}_n)_{q,w}(S) \to 0 \to H_{p-1}(\mathcal{E}_n)_{q,w}(S) \to H_{p-1}(\mathcal{Z}_n)_{q,w} \to 0 \to \ldots $$

if $S \not\cong [1]$, $p \neq 2$, $q \neq n$ or $w \neq 1$. This then proves that $H_p(\mathcal{E}_n)_{q,w}(S) \cong H_p(\mathcal{Z}_n)_{q,w}(S)$ in these cases from which the result follows. This leaves us with the case when $|S| = 1$ and $(p, q, w) = (2, n, 1)$. So consider the following section

$$ \ldots \to H_2(\mathcal{E}_n)_{n,1}([1]) \to H_2(\mathcal{Z}_n)_{n,1}([1]) \to \mathbb{Q} \to H_1(\mathcal{E}_n)_{n,1} \to H_1(\mathcal{Z}_n)_{n,1} \to \ldots $$

in the long exact sequence of homology. We wish to show that the connecting map $\phi \colon H_2(\mathcal{Z}_n, \mathcal{E}_n)_{n,1}([1]) \to H_1(\mathcal{E}_n)_{n,1}$ is non-trivial. By the definition of $\mathcal{E}'_n$ and the connecting homomorphism we have that the connecting homomorphism sends the generator of $H_2(\mathcal{Z}_n, \mathcal{E}_n)_{n,1}([1])$ to the image of z_n under the composition of

$$ ev^!_{[1]}S(n) \to \overline{\mathcal{E}_n} \to LQ(\mathcal{E}_n) $$

to a class in $H_1(\mathcal{E}_n)_{n,1}([1]) = AQ_n(\mathcal{E}_n)_{n,1}([1])$. Since 1 is lowest weight for which $\overline{\mathcal{E}_n}$ has non-trivial elements the elements of this weight cannot be decomposable. This implies that the image of z_n does not vanish. Therefore the connecting homomorphism is non-trivial and is injective. It then follows from exactness that we get the isomorphisms

$$ H_2(\mathcal{E}_n)_{n,1}([1]) \cong H_2(\mathcal{Z}_n)_{n,1}([1]) \quad \& \quad H_1(\mathcal{E}_n)_{n,1}([1]) \cong H_1(\mathcal{Z}_n)_{n,1}([1]) $$

concluding the proof. $\qquad\square$

Chapter 3

Representation theory and Cohomology of Torelli groups

The aim of this chapter is to explain the connection between the Lie algebra $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ and the functors $\mathcal{E}_n$. In order to do this we will explain some background on the theory of representations of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Q})$ and the representation theory of categories. We will then explain the relationship between the Torelli group and the representation theory which we shall conclude with a proof connecting $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ and $\mathcal{E}_n$.

3.1 Representation theory

We will begin by explaining some theory of the representations of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Q})$. Similar constructions and theorems exist for the orthogonal group $\mathrm{O}_{g,g}(\mathbb{Q})$. We will be brief, only presenting the theory which we will need. If interested the reader may turn to Fulton and Harris' book [10] for more information, a standard text on the subject. Another great source the encyclopedic text by Goodman and Wallach [18]. We will then introduce some theory of representations of categories. In particular how to realise representations of the downwards Brauer category as $\mathrm{Sp}_{2g}(\mathbb{Q})$-representations. The main sources for categorical representation theory which we shall use are [39] and [61].

3.1.1   Symplectic representation theory

Let V denote a $2d$-dimensional vector space over $\mathbb{Q}$. Let $\langle -, - \rangle \colon V \otimes V \to \mathbb{Q}$ be a non-degenerate skew-symmetric bilinear form on V. Let $G(V)$ denote the automorphism group of $(V, \langle -, - \rangle)$. Since $\langle -, - \rangle$ is skew-symmetric and non-degenerate we have that $G(V) \cong \mathrm{Sp}_{2d}(\mathbb{Q})$.

Definition 3.1. For all $1 \leq k$ we define the action of $\mathfrak{S}_k$ on $V^{\otimes k}$ by

$$(v_1 \otimes v_2 \otimes \ldots \otimes v_k) \cdot \sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(k)}$$

for all $\sigma \in \mathfrak{S}_d$ and $v_1, \ldots, v_k \in V$. We define the action of $G(V)$ on $V^{\otimes k}$ by

$$T \cdot (v_1 \otimes v_2 \otimes \ldots \otimes v_k) = (Tv_1) \otimes (Tv_2) \otimes \ldots \otimes (Tv_k)$$

for all $T \in G(V)$ and $v_1, \ldots, v_k \in V$.

We are interested in the Schur-Weyl Duality theorem which states how these actions and their irreducible representations are related. To discuss such things we need to recall the following definition.

Definition 3.2. A partition of a positive integer k is a tuple $(\lambda_1, \ldots, \lambda_m)$ of non-decreasing positive integers such that $\lambda_1 + \ldots + \lambda_m = k$.

Example 3.3. The partitions of 3 are

$$(3), \qquad (1,1,1), \qquad (1,2)$$

We will abbreviate partitions with repeating entries with a superscript. For example $(1,1,1)$ is denoted by (1^3) and $(1,1,1,2,2)$ is denoted by $(1^3, 2^2)$.

We can now remind the reader of the following result regarding the irreducible representations of $\mathfrak{S}_k$.

Theorem 3.4. *The is a natural actions of $\mathfrak{S}_k$ and $\mathbb{Q}\mathfrak{S}_k$ on the group algebra $\mathbb{Q}\mathfrak{S}_k$ given by right multiplication. Every irreducible representation of $\mathfrak{S}_k$ is $\mathfrak{S}_k$-isomorphic to $S_\lambda = (\mathbb{Q}\mathfrak{S}_k) \cdot c_\lambda$ where λ is a partition on k and c_λ denotes the* Young symmetrizer *of ρ.*

Recall that the Young symmetrizer is an element of $\mathbb{Q}\mathfrak{S}_k$ which is determined by the action of $\mathbb{Q}\mathfrak{S}_k$ on Young diagrams. The $\mathfrak{S}_k$-modules S_λ are called *Specht modules*. We will not prove this theorem but proof maybe found in [10, Chapter 4].

Definition 3.5. For all $1 \leq i < j \leq k$ define the linear maps $\lambda_{i,j} \colon V^{\otimes k} \to V^{\otimes (k-2)}$ by

$$\lambda_{i,j} \colon v_1 \otimes \ldots \otimes v_k \mapsto \langle v_i, v_j \rangle v_1 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes \hat{v}_j \otimes \ldots \otimes v_n$$

called *contractions*. Define

$$V^{\langle k \rangle} = \bigcap_{i,j} \ker(\lambda_{i,j})$$

$$= \ker(\oplus_{i,j} \lambda \colon V^{\otimes k} \to \bigoplus_{i,j} V^{\otimes (k-2)})$$

Definition 3.6. Let $\{a_1, \ldots, a_{2d}\}$ and $\{b_1, \ldots, b_{2d}\}$ be two bases for V such that is $\langle a_i, b_j \rangle = \delta_{i,j}$. Then define $\omega = \sum_{i=1}^{2d} a_i \otimes b_i$. For all $1 \leq i < j \leq k$ define the linear maps $\omega_{i,j} \colon V^{\otimes k-2} \to V^{\otimes k}$ by

$$\omega_{i,j} \colon v_1 \otimes \ldots \otimes v_k \mapsto \sum_{m=1}^{2d} (v_1 \otimes \ldots \otimes v_{i-1} \otimes a_m \otimes v_i \otimes \ldots \otimes v_{j-1} \otimes b_m \otimes v_j \otimes \ldots \otimes v_k).$$

We define

$$V_{\langle k \rangle} = \bigcap_{i,j} \operatorname{coker}(\omega_{i,j})$$

$$= \operatorname{coker}(\oplus_{i,j} \omega \colon \bigoplus_{i,j} V^{\otimes (k-2)} \to V^{\otimes k}).$$

There is an isomorphism between $V^{\langle k \rangle}$ and $V_{\langle k \rangle}$ given by the composition $V^{\langle k \rangle} \to V^{\otimes k} \to V_{\langle k \rangle}$. The spaces $V^{\langle k \rangle}$ and $V_{\langle k \rangle}$ are also $\mathfrak{S}_k$ and $G(V)$ representations. The group $\mathfrak{S}_k$ acts on $V^{\langle k \rangle}$ and $V_{\langle k \rangle}$ on the right while $G(V)$ acts on the left.

If $c_\lambda \in \mathbb{Q}\mathfrak{S}_k$ denotes the Young symmetrizer associated to the partition λ of k then we have the following result on the image of c_λ.

Proposition 3.7. *For all partitions λ of k there is an isomorphism*

$$V \cdot c_\lambda \cong \operatorname{Hom}_{\mathfrak{S}_k}(S_\lambda, V^{\langle k \rangle}) \cong [S_\lambda \otimes V^{\langle k \rangle}]^{\mathfrak{S}_k}.$$

of $G(V)$-representations.

Definition 3.8. For every partition $\lambda \in \operatorname{Par}(k)$ of k define the vector space V_λ by

$$V_\lambda = [S_\lambda \otimes V^{\langle k \rangle}]^{\mathfrak{S}_k}.$$

Proposition 3.9. *For all partitions λ of k the space V_λ is a zero or irreducible representation of $G(V)$. If $\lambda = (\lambda_1, ..., \lambda_m)$ and $2m \leq 2d = \dim(V)$ then V_λ is non-zero.*

The next result, called the *Schur-Weyl duality*, tells us that we may decompose $V^{\langle k \rangle}$ into irreducible representations.

Theorem 3.10 (Schur-Weyl duality). *For all $k \in \mathbb{N}$ and finite dimensional vector spaces V, there exists the decomposition*

$$V^{\langle k \rangle} \cong \bigoplus_\lambda S_\lambda \otimes V_\lambda$$

of $V^{\langle k \rangle}$ as a $\mathfrak{S}_k \times G(V)$-representation, where the sum runs over partitions of k.

The proof is not given in [10] but it can be found in [18]. The idea is to use the double centralizer theorem a result about $\mathbb{Q}$-algebra representations and commuting sub-algebras.

It will be convenient later to have a version of the vector spaces $V^{\otimes k}$, $V^{\langle k \rangle}$, $V_{\langle k \rangle}$ defined for finite sets. To do this we will use the set V^S set of set functions from $S \to V$. This can be made into a vector space but we neglect the vector space structure for the following construction.

Definition 3.11. Let S be a finite set. Consider the vector space $\mathbb{F}(V^S) = \mathrm{span}(V^S)$ spanned by the set of functions $S \to V$. On $\mathbb{F}(V^S)$ we define the equivalence relation $\sim$ by

    i) if $v, w, w' \in V^S$ such that $v(s) = w(s) + w'(s)$ for some s and for all $t \in S \setminus \{s\}$ we have that $v(t) = w(t) = w'(t)$ then set $v \sim w + w'$,

    ii) if $\alpha \in \mathbb{Q}$ and $v, w \in V^S$ such that $v(s) = \alpha w(s)$ for some $s \in S$ and $v(t) = w(t)$ for all $t \in T$ then $v \sim w$.

We then define $V^{\otimes S}$ to be the space $V^{\otimes S} = \mathbb{F}(V^S)/\sim$.

For functions $v \colon S \to V$ in $\mathbb{F}(V^S)$ and $s \in S$ we will sometimes use the notation v_s to denote $v(s)$. After choosing an ordering $s_1 < ... < s_k$ of S we will sometimes denote $[v] \in V^{\otimes S}$ by $v_{s_1} \otimes v_{s_2} \otimes ... \otimes v_{s_k}$.

Definition 3.12. Let S be a finite set and let $s, t \in S$. We define $\lambda_{s,t} \colon V^{\otimes S} \to V^{\otimes S \setminus \{s,t\}}$ by $[v] \mapsto \langle v_s, v_t \rangle [v|_{S \setminus \{s,t\}}]$. Further more we define the vector space $V^{\langle S \rangle}$ to be the vector space

$$V^{\langle S \rangle} = \bigcap_{s,t} \ker(\lambda_{s,t}).$$

Definition 3.13. Let S be a finite set and let $s, t \in S$. Let $\{a_1, ..., a_{2d}\}$ be a basis of V. Let $\{b_1, ..., b_{2d}\}$ be a basis of V such that $\langle a_i, b_j \rangle = \delta_{i,j}$. We can extend each function $v \colon S \setminus \{s,t\} \to V$ to a function $\tilde{v}_i \colon S \to V$, where $1 \leq i \leq 2d$, by setting

$$\tilde{v}_i(r) \begin{cases} v(r), & r \in S \setminus \{s,t\}; \\ a_i, & r = s; \\ b_i, & r = t \end{cases}$$

for all $r \in S$. We then define $\omega_{s,t} \colon V^{\otimes S \setminus \{s,t\}} \to V^{\otimes S}$ by $\omega_{s,t}[v] = \sum_{i=1}^{2d} [\tilde{v}_i]$ for all $[v] \in V^{S \setminus \{s,t\}}$. Lastly we define the vector space $V_{\langle S \rangle}$ by

$$V_{\langle S \rangle} = \bigcap_{s,t} \operatorname{coker}(\omega_{s,t}).$$

Proposition 3.14. *For all finite sets S and all $s, t \in S$ the maps $\lambda_{s,t}$ and $\omega_{s,t}$ are well-defined and linear.*

Notice that $V^{\otimes \varnothing} \cong \mathbb{Q}$ so we obtain maps $\lambda_{s,t} \colon V^{\otimes \{s,t\}} \to \mathbb{Q}$ and $\omega_{s,t} \colon \mathbb{Q} \to V^{\otimes \{s,t\}}$. Another observation which we will use is that the permutations group $\operatorname{Perm}(S)$ of a finite set S acts on $V^{\otimes S}$ by $[v] \cdot \sigma = [v \circ \sigma^{-1}]$ for all $[v] \in V^{\otimes S}$ and $\sigma \in \operatorname{Perm}(S)$. The group $G(V)$ also acts on $V^{\otimes S}$ by $g \cdot [v] = [g \circ v]$ for all $g \in G(V)$. The next result is an important result in invariant theory. A proof can be found in [18].

Theorem 3.15 (First Fundamental theorem of Invariant theory)**.** *Let S be a finite set with cardinality $|S| = 2k$. Let $\{a_i\}_{i=1}^{2d}$ and $\{b_i\}_{i=1}^{2d}$ be bases of V such that $\langle a_i, b_j \rangle = \delta_{i,j}$. Let $\{(s_i, t_i)\}_{i=1}^{k}$ be an ordered matching of S. If $\omega = \omega_{s_k, t_k} \circ ... \circ \omega_{s_1, t_1}(1)$ then we have that*

$$[V^{\otimes S}]^{G(V)} \cong \operatorname{span}\{\omega \cdot \sigma \mid \sigma \in \operatorname{Perm}(S)\}.$$

Moreover if S is a set of odd cardinality then $[V^{\otimes S}]^{G(V)} = 0$.

Remark 3.16. In this section we have restricted our attention to $\mathrm{Sp}_{2d}(\mathbb{Q})$-representations only however all the same theorems and constructions hold when $\langle \cdot, \cdot \rangle$ is a bilinear form such that $G(V) \cong \mathrm{O}_{d,d}(V)$. We shall use this later in the chapter on high dimensional manifolds.

3.1.2 Realisations of Representations of the downwards Brauer category

In this section we explain how to the representation theory of dB and the represention theory of $\mathrm{Sp}_2\, g(\mathbb{Q})$ and $\mathrm{O}_{g,g}(\mathbb{Q})$ are related. We will need to introduce some categorical representation theory to do this. We follow Kupers and Randal-Williams [39, 40] however it should be noted that this representations of categories approach which Kuper and Randal-Williams use was inspired by Sam and Snowden [61]. The recent paper [62] by Sam and Snowden also provides a good explanation of the theory.

Recall that a representation of a category $\mathbf{C}$ is a functor $\mathcal{F} \colon \mathbf{C} \to \mathbf{A}$, where $\mathbf{A}$ is an abelian category. We will specifically focus on $\mathbf{GrVec}_{\mathbb{Q}}$-representations of dB. Our discussion here can be generalized to $\mathbf{A}$-representations of $\mathbf{C}$, where $\mathbf{A}$ is an abelian category. Some further requirements for $\mathbf{C}$ are needed. The interested reader may turn to [39] for more details.

Fix a $2d$-dimensional vector space V together with a bilinear pairing $\langle \cdot, \cdot \rangle$ such that the automorphism group $G(V)$ of $(V, \langle \cdot, \cdot \rangle)$ is isomorphic to either $\mathrm{Sp}_{2d}(\mathbb{Q})$ or $\mathrm{O}_{d,d}(\mathbb{Q})$. We are now ready explain how to realise functors as representations.

Definition 3.17. Given a $(V, \langle \cdot, \cdot \rangle)$ be a vector space together with a bilinear form such that $G(V) \cong \mathrm{O}_{d,d}(\mathbb{Q})$. We define the functor $\mathcal{K}_V \colon dBr^+ \to \mathbf{GrVec}_{\mathbb{Q}}$ by

$$\mathcal{K}_V(S) = V^{\otimes S}$$

for finite sets $S \in$ dB. For morphisms $(f, m_s) \in \mathrm{dB}(S, T)$ define $\mathcal{K}_V(f) \colon V^{\otimes S} \to V^{\otimes T}$ by applying B to the pairs in m_S and inducing an isomorphism between $V^{\otimes f(T)}$ and $V^{\otimes T}$ by the bijection $f \colon T \to f(T)$.

Similarly if B is a skew symmetric form non-degenerate bilinear form on V then we may define the functor $\mathcal{K}_V \colon$ sdBr $\to \mathbf{GrVec}_{\mathbb{Q}}$ in the same way. Notice though that this is well-define since B is skew symmetric and

swapping the order of an element in a matching adds a sign. We give another example.

Example 3.18. Consider $V = \mathbb{Q}^2$ again but define $\langle \cdot, \cdot \rangle \colon \mathbb{Q}^2 \times \mathbb{Q}^2 \to \mathbb{Q}$ by $\langle \vec{v}, \vec{w} \rangle = v_1 w_2 - v_2 w_1$ for all $\vec{v} = (v_1, v_2), \vec{w} = (w_1, w_2) \in V$. The bilinear form $\langle \cdot, \cdot \rangle$ is skew symmetric and non-degenerate. Just as in the last example set $R = \{x, y\}$ and $T = \{r, s, t, u\}$ in dBr^-. Then we again have $\mathcal{K}_V(R)$ and $\mathcal{K}_V(T)$. Now instead consider the morphism $(f, \{(r, t)\}) \colon T \to R$ where $f \colon R \to T$ is define by $f(x) = s$ and $f(y) = u$. The corresponding morphism $\mathcal{K}_V(f) \colon V^{\otimes T} \to V^{\otimes R}$ is defined by

$$\mathcal{K}_V(f, \{(r, t)\})(v_r \otimes v_s \otimes v_t \otimes v_u) = \langle v_r, v_t \rangle v_x \otimes v_y$$

for all $v_r \otimes v_s \otimes v_t \otimes v_u \in V^{\otimes T}$.

We can dualise $\mathcal{K}_V$ to get a functor $\mathcal{K}_V^\vee \colon (\mathrm{dB})^{\mathsf{op}} \to \mathbf{GrVec}_{\mathbb{Q}}$. That is we define $\mathcal{K}_V^\vee$ by

$$\mathcal{K}_V^\vee(S) = (V^{\otimes S})^\vee = \mathbf{Vec}_{\mathbb{Q}}(V^{\otimes S}, \mathbb{Q})$$

and by pre-composition for morphisms.

Definition 3.19. For all $\mathcal{F} \in \mathbf{GrVec}_{\mathbb{Q}}^{\mathrm{dB}}$ we can define a functor $\mathcal{K}_V^\vee \otimes^{\mathrm{dB}} - \colon \mathbf{GrVec}_{\mathbb{Q}}^{\mathrm{dB}} \to \mathbf{GrVec}_{\mathbb{Q}}$ by

$$\mathcal{K}_V^\vee \otimes^{\mathrm{dB}} \mathcal{F} = \int^{S \in \mathrm{dB}} \mathcal{K}_V(S)^\vee \otimes \mathcal{F}(S)$$

for all $\mathcal{F} \in \mathbf{GrVec}_{\mathbb{Q}}^{\mathrm{dB}}$.

Proposition 3.20. *The functor* $\mathcal{K}_V^\vee \otimes^{\mathrm{dB}} - \colon \mathbf{GrVec}_{\mathbb{Q}}^{\mathrm{dB}} \to \mathbf{GrVec}_{\mathbb{Q}}$ *is strong symmetric monoidal functor.*

Proof. Let $\mathcal{K} = \mathcal{K}_V$. First notice two things. We can apply the same construction to get a functor $(\mathcal{K}^\vee \boxtimes \mathcal{K}^\vee) \otimes^{\mathrm{dB} \times \mathrm{dB}} - \colon \mathbf{GrVec}_{\mathbb{Q}}^{\mathrm{dB} \times \mathrm{dB}} \to \mathbf{GrVec}_{\mathbb{Q}}$. We will use this later. The second thing to notice is that $\mathcal{K}^\vee \circ \sqcup$ is isomorphic to $\mathcal{K}^\vee \boxtimes \mathcal{K}^\vee$. This is because

$$\mathcal{K}^\vee(S \sqcup T) = (V^{\otimes(S \sqcup T)})^\vee \cong (V^{\otimes S} \otimes V^{\otimes T})^\vee \cong (V^{\otimes S})^\vee \otimes (V^{\otimes T})^\vee = \mathcal{K}^\vee \boxtimes \mathcal{K}^\vee(S, T)$$

for sets $S, T \in \mathrm{dB}$. A similar argument holds for morphisms. Using these two facts we get

$$(\mathcal{K}^\vee \otimes^{\mathrm{dB}} \mathcal{F}) \otimes (\mathcal{K}^\vee \otimes^{\mathrm{dB}} \mathcal{G}) = \left(\int^{S \in \mathrm{dB}} \mathcal{K}(S)^\vee \otimes \mathcal{F}(S)\right) \otimes \left(\int^{T \in \mathrm{dB}} \mathcal{K}(T)^\vee \otimes \mathcal{G}(T)\right)$$

$$\cong \int^{(S,T) \in \mathrm{dB} \times \mathrm{dB}} \mathcal{K}(S)^\vee \otimes \mathcal{F}(S) \otimes \mathcal{K}(T)^\vee \otimes \mathcal{G}(T)$$

$$\cong (\mathcal{K}^\vee \boxtimes \mathcal{K}^\vee) \otimes^{\mathrm{dB} \times \mathrm{dB}} (\mathcal{F} \boxtimes \mathcal{G})$$

$$\cong (\mathcal{K}^\vee \circ \sqcup) \otimes^{\mathrm{dB} \times \mathrm{dB}} (\mathcal{F} \boxtimes \mathcal{G})$$

$$\cong \mathcal{K}^\vee \otimes^{\mathrm{dB}} \sqcup^! (\mathcal{F} \boxtimes \mathcal{G})$$

$$= \mathcal{K}^\vee \otimes^{\mathrm{dB}} (\mathcal{F} \otimes \mathcal{G})$$

for $\mathcal{F}, \mathcal{G} \in \mathbf{GrVec}_\mathbb{Q}^{\mathrm{dB}}$. This proves that $\mathcal{K}^\vee \otimes^{\mathrm{dB}} -$ is a strong monoidal functor. The symmetry then follows from the symmetry of $\otimes$ in $\mathbf{GrVec}_\mathbb{Q}$ and the symmetry Day convolution. $\qquad\square$

The previous result implies that if $\mathcal{A} \in \mathbf{GrVec}_\mathbb{Q}^{\mathrm{dB}}$ is a commutative algebra object then we have that $\mathcal{K}_V^\vee \otimes \mathcal{A}$ is a commutative algebra object in $\mathbf{GrVec}_\mathbb{Q}$. Since $G(V)$ is the automorphism group of $(V, \langle \cdot, \cdot \rangle)$ is the group of linear automorphisms which respect $\langle \cdot, \cdot \rangle$ there is a natural action of $G(V)$ on V. This induces an action on $\mathcal{K}_V$ in the obvious way. Using this we may upgrade our functor $\mathcal{K}_V^\vee \otimes^{\mathrm{dB}} -$ to a functor

$$\mathcal{K}_V^\vee \otimes^{\mathrm{dB}} - \colon \mathbf{GrVec}_\mathbb{Q}^{\mathrm{dB}} \to \mathbf{GrRep}_\mathbb{Q}(G),$$

where $\mathbf{GrRep}_\mathbb{Q}(G)$ is the category of graded representations of G.

Definition 3.21. A functor $\mathcal{F} \colon \mathrm{dB} \to \mathbf{Vec}_\mathbb{Q}$ is said to be supported on sets of sizes $\leq N$ if $\mathcal{F}(S) = 0$ whenever $|S| > N$.

Let H denote the rational homology group $H_1(\Sigma_g; \mathbb{Q})$ of the surface Σ_g and let $H_\mathbb{Z}$ denote $H_1(\Sigma_g; \mathbb{Z})$. We will now focus on H. Note that H is a $2g$-dimensional vector space with a skew-symmetric non-degenerate bilinear pairing given by the intersection product. Kupers and Randal-Williams [39, 40] proved the following result.

Lemma 3.22. *There is an natural transformation*

$$\mathrm{sdBr}(T, S) \to [H_{\langle S \rangle} \otimes (H^{\otimes T})^\vee]^{G(H)}$$

of two-variables which is surjective for each $S, T \in \mathrm{sdBr}$ and injective if $|S| + |T| \leq 2g$

We can then use the co-Yoneda lemma to get the next result.

Lemma 3.23. *For all $\mathcal{F} \in \mathbf{Vec}_{\mathbb{Q}}^{\mathrm{sdBr}}$ and finite sets S there is a surjective map*

$$\mathcal{F}(S) \to [H_{\langle S \rangle} \otimes (\mathcal{K}_H^\vee \otimes^{\mathrm{sdBr}} \mathcal{F})]^{G(H)}$$

which is injective when $|S| \leq g$.

Proof. Let $S \in \mathrm{sdBr}$. We have by the co-Yoneda lemma that

$$\mathcal{F}(S) \cong \int^{T \in \mathrm{sdBr}} \mathrm{dB}(T, S) \otimes \mathcal{F}(T).$$

Then there is a surjective map

$$\mathcal{F}(S) \to \int^{T \in \mathrm{sdBr}} [H_{\langle S \rangle} \otimes (H^{\otimes T})^\vee]^{G(H)} \otimes \mathcal{F}(T)$$

which is an isomorphism for $|S| \leq g$. But by definition we have that

$$[H_{\langle S \rangle} \otimes (\mathcal{K}_H^\vee \otimes^{\mathrm{sdBr}} \mathcal{F})]^{G(H)} = \int^{T \in \mathrm{sdBr}} [H_{\langle S \rangle} \otimes (H^{\otimes T})^\vee]^{G(H)} \otimes \mathcal{F}(T).$$

Thus we have that there is a surjective map

$$\mathcal{F}(S) \to [H_{\langle S \rangle} \otimes (\mathcal{K}_H^\vee \otimes^{\mathrm{sdBr}} \mathcal{F})]^{G(H)}$$

which is injective when $|S| \leq g$. $\qquad\square$

Now the functor $\mathcal{K}_H^\vee \otimes^{\mathrm{sdBr}} -$ is a left adjoint which implies that it is right exact. Using the previous proposition we can show that $\mathcal{K}_H^\vee \otimes^{\mathrm{sdBr}} -$ is left exact as well.

Lemma 3.24. *If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence in the category $\mathbf{Vec}_{\mathbb{Q}}^{\mathrm{sdBr}}$ such that $\mathcal{B}$ is supported on sets of size $\leq g$, then*

$$0 \to \mathcal{K}_V^\vee \otimes^{\mathrm{sdBr}} \mathcal{A} \to \mathcal{K}_V^\vee \otimes^{\mathrm{sdBr}} \mathcal{B} \to \mathcal{K}_V^\vee \otimes^{\mathrm{sdBr}} \mathcal{C} \to 0$$

is again exact.

Proof. We need to show that $\mathcal{K}_V^\vee \otimes^{\mathrm{sdBr}} -$ is left exact. Since $\mathcal{B}$ is supported on sets of size $\leq g$ and $\mathcal{A}$ is a subobject of $\mathcal{B}$ we have that $\mathcal{B}$ is also supported on sets of size $\leq g$. From Lemma 3.23 we have a map

$$\mathcal{A}(S) \to [H_{\langle S \rangle} \otimes (\mathcal{K}_H^\vee \otimes^{\mathrm{sdBr}} \mathcal{A})]^{G(H)}$$

which is surjective and an isomorphism for $|S| \leq g$. Since the map is surjective and $\mathcal{A}(S)$ vanishes if $|S| > g$ we must have that $H_{\langle S \rangle} \otimes (\mathcal{K}_H^\vee \otimes^{\mathrm{sdBr}} \mathcal{A})]^{G(H)} = 0$ when $|S| > g$. This implies that this map is an isomorphism. The same arguements applies to $\mathcal{B}$. The induced map

$$\mathcal{K}_H^\vee \otimes^{\mathrm{sdBr}} \mathcal{A} \to \mathcal{K}_H^\vee \otimes^{\mathrm{sdBr}} \mathcal{B}$$

must then be injective. If it we not injective that its kernel is an irreducible $G(H)$-representation. Applying $[H_{\langle S \rangle} \otimes -]^{G(H)}$ would then implies that the map $\mathcal{A}(S) \to \mathcal{B}(S)$ would have a non-trivial kernel which is impossible since $0 \to \mathcal{A} \to \mathcal{B}$ is exact. $\square$

We can now use the exactness of $\mathcal{K}^\vee \otimes^{\mathrm{sdBr}} -$

Lemma 3.25. *If* $\mathcal{A} \in \mathsf{Alg}(\mathbf{GrVec}_{\mathbb{Q}}^{\mathrm{sdBr}})$ *has a weight grading and the weight* w *piece is supported on sets of size* $\leq mw$ *for some* $m \in \mathbb{Z}_{>0}$ *then as long as* $w \leq g/m$ *we have an isomorphism*

$$\mathcal{K}^\vee \otimes^{\mathrm{sdBr}} H_p(\mathcal{A})_{q,w} \cong H_p(\mathcal{K}^\vee \otimes^{\mathrm{sdBr}} \mathcal{A})_{q,w}.$$

Proof. Consider the Harrison complex $LQ(\mathcal{A}) \simeq \mathrm{Harr}(\mathcal{A})_{q,w}$. Since $\mathcal{A}$ is supported on sets of size $\leq mw$ we have that $\mathcal{A}(S)_{q,w} = 0$ if $w < |S|/m$. The Harrison complex inherits this property as well. In particular for $w \leq g/m$ we have that $\mathrm{Harr}_*(\mathcal{A})_{q,w}$ is a chain complex supported on sets of size $\leq g$. By the exactness and strong monoidality of $\mathcal{K}_H^\vee \otimes^{\mathrm{sdBr}} -$ we get that

$$\begin{aligned}
\mathcal{K}^\vee \otimes^{\mathrm{sdBr}} H_p(\mathcal{A})_{q,w} &\cong H_{p+q}(\mathcal{K}^\vee \otimes^{\mathrm{sdBr}} \mathrm{Harr}_*(\mathcal{A}))_{q,w} \\
&\cong H_{p+q}(\mathrm{Harr}_*(\mathcal{K}^\vee \otimes^{\mathrm{sdBr}} \mathcal{A}))_{q,w} \\
&\cong H_p(\mathcal{K}^\vee \otimes^{\mathrm{sdBr}} \mathcal{A})_{q,w}
\end{aligned}$$

as long as $w \leq g/m$. $\square$

3.2 Relation to Torelli group

We will now discuss the relationship between the the cohomology of the Torelli group, the Lie algebra $\mathrm{Gr}^*_{lcs}\mathfrak{t}_g$ and the representation theory of $\mathrm{Sp}_{2g}(\mathbb{Q})$. We will only describe a small part of the very rich theory of mapping class groups. For the interested reader we recommend the lovely survey [53] by Morita.

We will inherit the notation used in the previous section but with an H instead of a V now. From this we can already see that H is a $\mathrm{Sp}_{2g}(\mathbb{Q})$-representation. In fact $H \cong V_{(1)}$ so H is a fundamental representation of $\mathrm{Sp}_{2g}(\mathbb{Q})$. The homology of the Torelli group is also an important representation of $\mathrm{Sp}_{2g}(\mathbb{Q})$. In the early 1980's Johnson [30] constructed a homomorphism and proved the following theorem.

Theorem 3.26. *There exists a surjective homomorphism* $h\colon \mathcal{T}_g \to \Lambda^3 H$. *For* $3 \leq g$ *this induces a* $\mathrm{Sp}(\mathbb{Q})$-*isomorphism*

$$H_1(\mathcal{T}_g;\mathbb{Q}) \cong \Lambda^3 H.$$

The theory of the Johnson homomorphism is rich and there are naturally questions on whether similar results are true for the higher homology groups. There are a few alternative descriptions of the Johnson himself gave one in [29] and Church and Farb [6] found another.

We want to prove introduce a similar result for the Lie algebra homology of $\mathrm{Gr}^*_{lcs}\mathfrak{t}_{g,1}$. Recall we defined in the introduction the Lie homology of Lie algebras.

3.2.1 Construction of the generalised Miller-Morita-Mumford classes

In the papers [49, 52, 54] Miller, Morita and Mumford described characteristic class of surface bundles which came to be known as Miller-Morita-Mumford classes or MMM-classes. These were later generalised by Kawazumi in [31] call the generalised MMM-classes. A generalisation of these to higher dimensions is given in [39]. We will describe the construction of the generalised MMM-classes following [31], [33] and [39].

Recall that group cohomology has a number of different equivalent definition for example we can define the homology groups $H_*(G; M)$ as

$$H_*(G; M) = H_*(BG; M) \quad \text{or as} \quad H_*(G; M) = \text{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}G, M)$$

where BG is the classifying space of G. We will briefly review the group cohomology given by normalised cochain complexes. Kawazumi [31] gives a relative version of these cochain complexes which we will describe and use. More details about the relation between these definitions can be found in [64]. Our goal is to obtain a Gysin homomorphism which will be essential for the definition. We will not provide all the details regarding the normalised cochain complexes as they can be found in [25] and [31].

Definition 3.27. Let A be a group and M a left A-module (that is Abelian group on which A acts on the left). Define $C^m(A; M)$ to be the set of functions $f \colon A^m \to M$ such that $f(a_1, ..., a_m) = 0$ whenever $a_i = 1$ for some $i \in \{1, ..., m\}$. The coboundary map $d \colon C^m(A; M) \to C^{m+1}(A; M)$ is by

$$(df)(a_1, ..., a_{m+1}) = a_1 \cdot f(a_2, ..., a_{m+1}) + \sum_{i=1}^{m} (-1)^i f(a_1, ..., a_i a_{i+1}, ..., a_{m+1}) + (-1)^{m+1} f(a_1, ..., a_m).$$

Definition 3.28. If B is a subgroup of A then the restriction map $res_* \colon C^*(A; M) \to C^*(B; M)$ defines a cochain map. Let $C^m(A, B; M)$ denote the kernel of the restriction map at each $m \in \mathbb{N}$.

We have the following theorem.

Theorem 3.29. *The groups $C^*(A, B; M)$ form a cochain complex. Moreover these relative normalised cochains induce a long exact sequence*

$$... \to H^{m-1}(B; M) \to H^m(A, B; M) \to H^m(A; M) \to H^m(B; M) \to ...$$

of cohomology groups.

If N is a left A-module there is a natural cup product

$$\smile \colon H^m(A; M) \otimes H^k(A, B; N) \to H^{m+k}(A, B; M \otimes N).$$

The details of which can be found in [25]. We can then get a Hochschild-Serre spectral sequence.

Theorem 3.30. *If C is a normal subgroup of A such that $A = BC$ then there exists a convergence spectral sequence*

$$E_2^{p,q} = H^p(A/C; H^q(C, C \cap B; M)) \Rightarrow H^*(A, B; M).$$

The proof is similar to that in [25]. We have not described how A/C acts on $H^q(C, C \cap B; M)$ however this can be found in [31]. As a corollary we can get a Gysin homomorphism for this group cohomology theory.

Corollary 3.31. *If $H^m(C, C \cap B; \mathbb{Z}) = \mathbb{Z}$ for $m = k$ and $H^m(C, C \cap B; \mathbb{Z}) = 0$ for $m > k$ then there exists a homomorphism $\pi_! \colon H^m(A, B; M) \to H^{m-k}(A/C; M)$ called the Gysin homomorphism.*

Now we are ready to construct the characteristic classes. Recall that $\mathcal{M}_{g,1}$ denotes the mapping class group of a surface Σ_g fixing a boundary component. Let $\overline{\mathcal{M}}_{g,1}$ denote the semi-direct product $\pi_1 \Sigma_g \rtimes \mathcal{M}_{g,1}$, where $\pi_1 \Sigma_g$ is the fundamental group of $\Sigma_{g,1}$. We will use this to obtain our characteristic classes. Then there is a split short exact sequence of groups

$$1 \to \pi_1 \Sigma_g \xrightarrow{i} \overline{\mathcal{M}}_{g,1} \xrightarrow{p} \mathcal{M}_{g,1} \to 1 \tag{3.1}$$

where $i(\alpha) = (\alpha, 1)$ and $p(\alpha, \phi) = \phi$ for all $\alpha \in \pi_1 \Sigma_g$ and $\phi \in \mathcal{M}_{g,1}$. The splitting homomorphism $s \colon \mathcal{M}_{g,1} \to \overline{\mathcal{M}}_{g,1}$ is given by $s(\phi) = (1, \phi)$ for all $\phi \in \mathcal{M}_{g,1}$.

Lemma 3.32. *There exists a 1-cocycle $f \in C^1(\pi_1 \Sigma_g; H_\mathbb{Z})$. Moreover we may extend f to a a 1-cocycle $\tilde{f} \in C^1(\overline{\mathcal{M}}_{g,1}, \mathcal{M}_{g,1}; H_\mathbb{Z})$.*

Proof. Let $f \colon \pi_1 \Sigma_g \to H_\mathbb{Z}$ be the quotient map where we view $H_1(\Sigma_g; \mathbb{Z})$ as the abelisation of $\pi_1 \Sigma_g$. The group $\pi_1 \Sigma_g$ acts trivially on $H_\mathbb{Z}$ so for all $a_1, a_2 \in \pi_1 \Sigma_g$ we have that

$$\begin{aligned}
(df)(a_1, a_2) &= [a_2] - [a_1 a_2] + [a_1] \\
&= [a_2] - [a_1] - [a_2] + [a_1] \\
&= 0
\end{aligned}$$

Therefore f is a 1-cocycle in $C^1(\pi_1 \Sigma_g; H_\mathbb{Z})$. Now to extend f notice that there is a $\pi_1 \Sigma_g \rtimes \mathcal{M}_{g,1}$ action on $H_\mathbb{Z}$ given by $(a, \phi) \cdot v = \phi(v)$ for all $(a, \phi) \in \overline{\mathcal{M}}_{g,1}$

and $v \in H_{\mathbb{Z}}$. Now define $\tilde{f} \colon \overline{\mathcal{M}}_{g,1} \to H_{\mathbb{Z}}$ by $\tilde{f}(\alpha, \phi) = f(\alpha)$. Then for all $(a_1, \phi_1), (a_2, \phi_2)$ we get

$$
\begin{aligned}
(df)(a_1, \phi_1, a_2, \phi_2) &= (a_1, \phi_1) \cdot f(a_2, \phi_2) - f(a_1 \phi_1(a_2), \phi_1 \phi_2) + f(a_1, \phi_1) \\
&= \phi_1([a_2]) - [a_1 \phi_1(a_2)] + [a_1] \\
&= \phi_1([a_2]) - [a_1] - \phi_1([a_2]) + [a_1] \\
&= 0
\end{aligned}
$$

which implies that $\tilde{f}$ is a 1-cocycle in $C^1(\overline{\mathcal{M}}_{g,1}; H_{\mathbb{Z}})$. Lastly notice that since $\tilde{f}(1, \phi) = f(1) = 0$ for all $\phi \in \mathcal{M}_{g,1}$ we have that $\tilde{f} \in C^1(\overline{\mathcal{M}}_{g,1}, \mathcal{M}_{g,1}; H_{\mathbb{Z}})$. $\square$

We are now ready to define the generalised MMM-classes.

Proposition 3.33. *There exist non-vanishing cohomology classes*

$$
m_{i,j} = \pi_!(p^*(e)^i \smile [\tilde{f}]^j) \in H^{2i+j-2}(\mathcal{M}_{g,1}; H^{\otimes j})
$$

for all $i, j \geq 0$ such that $i + j \geq 2$.

Proof. Let $e \in H^2(\mathcal{M}_{g,1}; \mathbb{Q})$ denote the Euler class of the universal Σ_g-bundle. We can use $p^* \colon H^2(\mathcal{M}_{g,1}; \mathbb{Q}) \to H^2(\overline{\mathcal{M}}_{g,1}; \mathbb{Q})$ to get a class $p^*(e) \in H^2(\overline{\mathcal{M}}_{g,1}; \mathbb{Q})$. Then we have $(p^*(e))^j \in H^{2j}(\overline{\mathcal{M}}_{g,1}; \mathbb{Q})$. Now consider the class $[\tilde{f}] \in H^1(\overline{\mathcal{M}}_{g,1}, \mathcal{M}_{g,1}; H_{\mathbb{Z}})$. Taking the j-th power of $[\tilde{f}]$ gives a class

$$
[\tilde{f}]^j = \underbrace{[\tilde{f}] \smile \cdots \smile [\tilde{f}]}_{j \text{ times}} \in H^j(\overline{\mathcal{M}}_{g,1}, \mathcal{M}_{g,1}; H_{\mathbb{Z}}^{\otimes j})
$$

Applying the cup product we get $p^*(e)^i \smile [\tilde{f}]^j \in H^{2i+j}(\overline{\mathcal{M}}_{g,1}, \mathcal{M}_{g,1}; H^{\otimes j})$. Finally we may use the Gysin homomorphism to get a class $m_{i,j} = \pi_!(p^*(e)^i \smile [\tilde{f}]^j) \in H^{2i+j-2}(\mathcal{M}_{g,1}; H^{\otimes j})$. $\square$

The classes $m_{i,j}$ are the generalised MMM-classes. The classes $m_{i,0}$ form the standard MMM-classes. The standard MMM-classes are of importance because of the following famous result which was conjectured by Mumford [54] and proved by Madsen and Wiess [46].

Theorem 3.34. *The inclusion*

$$
\Phi \colon \mathbb{Q}[m_{2,0}, m_{3,0}, \ldots] \to \lim_{g \to \infty} H^*(\mathcal{M}_{g,1}; \mathbb{Q})
$$

is an isomorphism.

Kawazumi [32] extended this theorem to calculate the cohomology of $\mathcal{M}_{g,1}$ with $H^{\otimes S}$ coefficients instead. To explain that result consider a finite set S with $|S| = j$ we may extend the definition of the generalised MMM-classes and define the class

$$m_{i,S} = \pi_!(p^*(e)^i \smile [\tilde{f}]^j) \in H^{2i+j-2}(\mathcal{M}_{g,1}; H^{\otimes S})$$

Then given a weighted partition $\{(S_i, w_i)\}_{i=1}^r$ of S with $w_i + |S_i| \geq 2$ for all $i = 1, ..., r$ we define the class

$$\kappa(\{(S_i, w_i)\}_{i=1}^r) = m_{w_1,S_1} \cdot m_{w_2,S_2} \cdot ... \cdot m_{w_r,S_r} \in H^*(\mathcal{M}_{g,1}; H^{\otimes S})$$

with degree $\sum_{i=1}^r (|P_i| - 2) + 2w_i$. This brings us to the following definition.

Definition 3.35. For all finite sets S define $\mathcal{P}(S)$ to be the vector space with the basis being weighted partitions $\{(S_i, w_i)\}_{i \in I}$ such that $|S_i| + w_i \leq 2$ (we allow some S_i to be empty).

Kawazumi [32] then proved the following result, which generalised the Madsen-Wiess theorem.

Theorem 3.36. *Let S be a finite set. The map*

$$\Phi \colon \mathcal{P}(S) \to H^*(\mathcal{M}_{g,1}; H^{\otimes S}), \quad \Phi \colon P \mapsto \kappa(P)$$

is an isomorphism in a range of cohomological degrees tending to infinity with g.

3.2.2 Cohomology of the Torelli group

Now we will construct similar classes as the generalised MMM-classes but for the cohomology of the Torelli group. Recall that the Torelli group is define to be the kernel of the natural map $\mathcal{M}_{g,1} \to \mathrm{Sp}_{2g}(\mathbb{Z})$. This implies we have a short exact sequence

$$1 \to \mathcal{T}_{g,1} \to \mathcal{M}_{g,1} \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to 1.$$

Then for any finite set S the edge homomorphism for the Hichschild-Serre spectral sequence with $H^{\otimes S}$ coefficients gives a map

$$H^*(\mathcal{M}_{g,1}; H^{\otimes S}) \to [H^*(\mathcal{T}_{g,1}; \mathbb{Q}) \otimes H^{\otimes S}]^{\mathrm{Sp}_{2g}(\mathbb{Z})}.$$

It follows from Theorem 3.36 that we have a map $\mathcal{P}(S) \to [H^*(\mathcal{T}_{g,1}; \mathbb{Q}) \otimes H^{\otimes S}]^{\mathrm{Sp}_{2g}(\mathbb{Z})}$. We can view $\mathcal{E}_1(S)$ as a subspace of $\mathcal{P}(S)$ and we can extend this morphism to a natural transformation

$$\mathcal{E}_1 \to [H^*(\mathcal{T}_{g,1}; \mathbb{Q}) \otimes H^{\otimes(-)}]^{\mathrm{Sp}_{2g}(\mathbb{Z})}$$

By adjunction we get a morphism of $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations

$$\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1 \to H^*(\mathcal{T}_{g,1}; \mathbb{Q}).$$

Kupers and Randal-Williams [39] then proved the following.

Theorem 3.37. *In a range of degrees tending to infinity with g the map*

$$\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1/(x_{\mathcal{L}_1}, x_{\mathcal{L}_2}, ...) \to H^*(\mathcal{T}_{g,1}; \mathbb{Q}).$$

is an isomorphism, where $(x_{\mathcal{L}_1}, x_{\mathcal{L}_2}, ...)$ is a regular sequence in $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$.

We will now describe the generators of $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$. Recall that by Poincaré duality we have that

$$H^\vee = H_1(\Sigma_{g,1}; \mathbb{Q})^\vee = H^1(\Sigma_{g,1}; \mathbb{Q}) \xrightarrow{\cong} H_1(\Sigma_{g,1}; \mathbb{Q}) \to H.$$

Definition 3.38. For all $0 \leq k$ we define $\mathfrak{S}_S \times \mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant map μ_k to be the map

$$\mu_k \colon H^{\otimes k} \otimes \mathcal{E}_1([k]) \to \mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1 = \int^{T \in \mathrm{dB}} (H^{\otimes T})^\vee \otimes \mathcal{E}_1(T)$$

given by the universal property of coends.

Recall that $\mathcal{E}_1(S)$ was defined as

$$\mathcal{E}_1(S) = \mathbb{Q}\{\text{admissible weighted partitions } \{(S_\alpha, g_\alpha)\}_{\alpha \in I} \text{ of } S\} \otimes \det(\mathbb{Q}^S)$$

for all finite sets S. For all $k \geq 0$ and $1 \leq i \leq k$ define $b_i \colon [k] \to \mathbb{Q}$ by $i \mapsto 1$ and $j \mapsto 0$ for all $j \in [k] \setminus \{i\}$. Then $\{b_1, ..., b_k\}$ is the usual basis in $\mathbb{Q}^{[k]}$.

Definition 3.39. For $0 \leq k$ and $0 \leq a$, we will call a allowable if $\{(\{1, 2, ..., k\}, a)\}$ is an admissible weight partition of $[k]$ in $\mathcal{E}_1([k])$. For all allowable a define the map $\nu_a \colon H^{\otimes k} \to H^{\otimes k} \otimes \mathcal{E}_1([k])$ by

$$\nu_a(v_1 \otimes ... \otimes v_k) = v_1 \otimes ... \otimes v_k \otimes \{(\{1, ..., k\}, a)\} \otimes (b_1 \wedge ... \wedge b_k)$$

for all $v_1, ..., v_k \in H$.

We can then define the maps which will give us the generators of $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}}$ $\mathcal{E}_1$.

Definition 3.40. For all $0 \leq k$ and $0 \leq a$ such that $\{(\{1, 2, ..., k\}, a)\}$ is an admissible partition of $[k] = \{1, ..., k\}$ we define κ_a to be the composition

$$H^{\otimes k} \xrightarrow{\nu_a} H^{\otimes k} \otimes \mathcal{E}_1([k]) \xrightarrow{\mu_k} \mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1.$$

For all $v_1, ..., v_k$ we have that $\kappa_a(v_1 \otimes ... \otimes v_k)$ has internal degree $2a+k-2$ since $\{(\{1, 2, ..., k\}, a)\}$ has internal degree $2a + k - 2$ while $H^{\otimes k}$ is concentrated in degree 0. The weight of $\kappa_a(v_1 \otimes v_2 \otimes v_3)$ is $2a + k - 2$ for all $v_1, v_2, v_3 \in H$ since the weight grading is equal to the internal grading on $\mathcal{E}_1$ and $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$. Note that we also get maps

$$\kappa_a \colon \mathbb{Q} \to \mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$$

for all $2 \leq a$. We will denote $\kappa_a(1)$ by κ_a.

These maps then have the following property.

Proposition 3.41. *For all* $0 \leq k$, $v_1, ..., v_k \in H$, *allowable* $0 \leq a$ *and permutations* $\sigma \in \mathfrak{S}_k$ *we have that*

$$\kappa_a((v_1 \otimes ... \otimes v_k) \cdot \sigma) = \mathrm{sgn}(\sigma)\kappa_a(v_1 \otimes ... \otimes v_k).$$

Proof. By the definition of ν_a and permuting terms in $b_1 \wedge ... \wedge b_k$ we have that

$$\nu_a((v_1 \otimes ... \otimes v_k) \cdot \sigma) = v_{\sigma^{-1}(1)} \otimes ... \otimes v_{\sigma^{-1}(k)} \otimes \{(\{1, ..., k\}, a)\} \otimes (b_1 \wedge ... \wedge b_k)$$
$$= \mathrm{sgn}(\sigma) \cdot (v_{\sigma^{-1}(1)} \otimes ... \otimes v_{\sigma^{-1}(k)} \otimes \{(\{1, ..., k\}, a)\} \otimes (b_{\sigma^{-1}(1)} \wedge ... \wedge b_{\sigma^{-1}(k)}))$$
$$= \mathrm{sgn}(\sigma)(v_1 \otimes ... \otimes v_k \otimes \{(\{1, ..., k\}, a)\} \otimes (b_1 \wedge ... \wedge b_k)) \cdot \sigma$$

Composing with μ_k, a $\mathfrak{S}_k$-equivariant map, gives us

$$\kappa_a((v_1 \otimes ... \otimes v_k) \cdot \sigma) = \mathrm{sgn}(\sigma)\mu_k((v_1 \otimes ... \otimes v_k \otimes \{(\{1, ..., k\}, a)\} \otimes (b_1 \wedge ... \wedge b_k)) \cdot \sigma)$$
$$= \mathrm{sgn}(\sigma)\mu_k(v_1 \otimes ... \otimes v_k \otimes \{(\{1, ..., k\}, a)\} \otimes (b_1 \wedge ... \wedge b_k))$$
$$= \mathrm{sgn}(\sigma)\kappa_a(v_1 \otimes ... \otimes v_k)$$

since $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$ has trivial $\mathfrak{S}_k$-action. $\qquad \square$

Kupers and Randal-Williams found the relations between these maps. We summarise these properties below.

Proposition 3.42. *Let $\{a_1, ..., a_{2g}\}$ be a basis of H and let $\{b_1, ..., b_{2g}\}$ denote the dual basis characterised by $\langle a_i, b_j \rangle = \delta_{ij}$. If $0 \leq k$ then we have that*

i) κ_a is linear in each v_i for $1 \leq i \leq k$,

ii) $\sum_{i=1}^{2g} \kappa_a(v_1 \otimes ... \otimes v_j \otimes a_i) \cdot \kappa_b(b_i \otimes v_{j+1} \otimes ... \otimes v_r) = \kappa_{a+b}(v_1 \otimes ... \otimes v_r)$,

iii) $\sum_{i=1}^{2g} \kappa_a(v_1 \otimes ... \otimes v_r \otimes a_i \otimes b_i) = \kappa_{a+1}(v_1 \otimes ... \otimes v_r)$,

iv) $\sum_{i=1}^{2g} \kappa_a(a_i) \cdot \kappa_b(b_i) = \kappa_{a+b}$,

v) and $\sum_{j=1}^{2g} \sum_{i=1}^{2g} \kappa_0(v_1 \otimes a_j \otimes a_i) \cdot \kappa_a(b_i) \cdot \kappa_b(b_j) = \kappa_{a+b}(v_1)$

for all allowable $0 \leq a, b$ and all $v_1, ..., v_k \in H$.

We can then combine the above properties, the discussion in [39, Section 8] and Theorem 5.1 of [39] to get the following theorem.

Theorem 3.43. *In a range of degrees tending towards infinity with g the elements $\kappa_0(v_1 \otimes v_2 \otimes v_3)$ generate $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$, for all $v_1, v_2, v_3 \in H$.*

The proof can be found in [39]. Kupers and Randal-Williams computed the earlier regular sequence $(x_{\mathcal{L}_1}, x_{\mathcal{L}_2}, ...)$ in Theorem 3.37 and found that for each $1 \leq i$ we can write $x_{\mathcal{L}_i} = c_i \kappa_{2i}$ for some scalar $c_i \in \mathbb{Q}$. The coefficients c_i for $1 \leq i$ can be computed as the $x_{\mathcal{L}_i}$'s are actually elements which are associated with the Hirzebruch L-polynomials. When working in higher dimensions, as Kupers and Randal-Williams do in [39], one needs to keep track of Pontryagin classes but since we are working with surfaces the Pontryagin classes $p_2, p_3, ...$ are trivial but since $p_1 = e^2$ the Hirzebruch L-polynomials do not completely disappear. Recall that e is the Euler class we used to define the MMM-classes. More information about these Hirzebruch L-polynomials and the $x_{\mathcal{L}_i}$'s can be found in [2] and [39].

Now using the κ_a maps we will give an isomorphism for the commutative algebra $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$ and connect this discussion to the Johnson homomorphism. Remember that we define V_1 and $H \cong V_1$. Now notice it follows from the symmetry of κ_0 that

$$\kappa_0 \colon H^{\otimes 3} \to \mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$$

factors through $\Lambda^3 V_1[1,1]$. That is we can write κ_0 as a composition

$$H^{\otimes 3} \to \Lambda^3 V_1[1,1] \to \mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$$

where the first map is given by $v_1 \otimes v_2 \otimes v_3 \mapsto v_1 \wedge v_2 \wedge v_3$ and the second map is given by $v_1 \wedge v_2 \wedge v_3 \mapsto \kappa_0(v_1 \otimes v_2 \otimes v_3)$ for all $v_1 \otimes v_2 \otimes v_3 \in H^{\otimes 3}$. The degree shift is due to the degree of $\kappa_0(v_1 \otimes v_2 \otimes v_3)$. Notice that shift in the grading on V_1 is due to the degree of κ_0. From the definition $\kappa_0(v_1 \otimes v_2 \otimes v_3$ has internal and weight $2 \cdot 0 + 3 - 2 = 1$ for all $v_1, v_2, v_3 \in H$.

Recall that Johnson constructed a homomorphism $\mathcal{T}_{g,1} \to \Lambda^3 H$ and we know that $(\Lambda^3 H)^\vee \cong \Lambda^3 H$. The discussion in [39, Section 8] proves the following result.

Proposition 3.44. *The composition of*

$$\Lambda^3 V_1[1,1] \to \mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1 \to H^1(\mathcal{T}_{g,1}; \mathbb{Q})$$

is the dual of Johnson homomorphism $H_1(\mathcal{T}_{g,1}; \mathbb{Q}) \to \Lambda^3 V_1[1,1]$.

We can extend the morphism $\Lambda^3 V_1[1,1] \to \mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$ to a morphism

$$\Lambda^* \Lambda^3 V_1[1] \to \mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$$

since $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$ is a commutative algebra and the elements $\kappa_0(v_1 \otimes v_2 \otimes v_3)$ generate $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1$. But from property iii) of Proposition 3.42 and the symmetry relation we have that

$$\sum_{i=1}^{2g} \kappa_0(v_1 \otimes v_2 \otimes a_i) \cdot \kappa_0(b_i \otimes v_3 \otimes v_4) = \sum_{i=1}^{2g} \kappa_0(v_1 \otimes v_3 \otimes a_i) \cdot \kappa_0(b_i \otimes v_4 \otimes v_2)$$

for all $v_1, v_2, v_3 \in H$. This relation is often called the IH relation. This is because we can draw this relation in a way to make it look like an I and an H, an example can be found in [39, Figure 6, Relation (ϵ)].

Definition 3.45. Let $\{a_1, ..., a_{2g}\}$ be a basis of H and let $\{b_1, ..., b_{2g}\}$ be the dual basis characterised by $\langle a_i, b_j \rangle = \delta_{ij}$. We define the maps $f_I, f_H \colon H^{\otimes 4} \to H^{\otimes 3} \otimes H^{\otimes 3}$ by

$$f_I(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = \sum_{i=1}^{2g} (v_1 \otimes v_2 \otimes a_i) \otimes (b_i \otimes v_3 \otimes v_4)$$

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$$f_I(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = \sum_{i=1}^{2g} (v_1 \otimes v_3 \otimes a_i) \otimes (b_i \otimes v_4 \otimes v_2)$$

for all $v_1, v_2, v_3, v_4 \in H$. Let $\phi \colon H^{\otimes 3} \otimes H^{\otimes 3} \to \Lambda^2 \Lambda^3 V_1[1]$ be the map given by

$$\phi(v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5 \otimes v_6) = (v_1 \wedge v_2 \wedge v_3) \wedge (v_4 \wedge v_5 \wedge v_6)$$

for all $v_1, v_2, v_3, v_4, v_5, v_6 \in H$.

Garoufalidis and Nakamura [16] studied the relation IH relation and prove the following result.

Proposition 3.46. *The image of* $(f_I - f_H) \circ \phi \colon H^{\otimes 4} \to \Lambda^2 \Lambda^3 V_1$ *is a* $\mathrm{Sp}_{2g}(\mathbb{Q})$-*representation given by the sum*

$$\mathrm{im}((f_I - f_H) \circ \phi) = V_{2^2} + V_{1^2} + V_0.$$

Using this Kupers and Randal-Williams [39] gave a proof for the next result.

Theorem 3.47. *There is an morphism of graded commutative algebras*

$$\Psi \colon \frac{\Lambda^*(\Lambda^3 V_1[1,1])}{(V_0 + V_{1^2} + V_{2^2})} \to \mathcal{K}_H^\vee \otimes^{dB} \mathcal{E}_1$$

which is an isomorphism in a stable range of degrees. Here $(V_0 + V_{1^2} + V_{2^2})$ *denotes the ideal generated by the direct sum* $V_0 + V_{1^2} + V_{2^2}$ *of irreducible* $\mathrm{Sp}_{2g}(\mathbb{Z})$-*representations.*

Now notice that the regular sequence $(x_{\mathcal{L}_1}, x_{\mathcal{L}_2}, \ldots)$ can be constructed from κ_2 and scalar multiplication. We can write κ_2 as

$$\kappa_2 = -\sum_{i,j,k} \kappa_0(a_i \otimes a_j \otimes a_k) \cdot \kappa_0(b_i \otimes b_j \otimes b_k)$$

using Proposition 3.42. This means that if κ_2 vanishes then so does the regular sequence. We can then derive the following corollary.

Corollary 3.48. *There is an morphism of graded commutative algebras*

$$\frac{\Lambda^*(\Lambda^3 V_1[1,1])}{(2V_0 + V_{1^2} + V_{2^2})} \to \mathcal{K}_H^\vee \otimes^{dB} \mathcal{E}_1/(\kappa_2)$$

which is an isomorphism in a stable range of degrees.

Proof. Let $\{a_1, ..., a_{2g}\}$ be a basis of H and let $\{b_1, ..., b_{2g}\}$ be the dual basis characterised by $\langle a_i, b_j \rangle = \delta_{ij}$. It may be confusing to distinguish the product $\Lambda^* \Lambda^3 V_1$ and an element $v_1 \wedge v_2 \wedge v_3 \in \Lambda^3 V_1$ which generates $\Lambda^* \Lambda^3 V_1$ if we use "$\wedge$" for both. So to avoid this confusion we will denote the product in $\Lambda^* \Lambda^3 V_1$ by "$\cdot$" instead. Let us write k_2 for the vector

$$\sum_{i,j,k} (a_i \wedge a_j \wedge a_k) \cdot (b_i \wedge b_j \wedge b_k)$$

in $\Lambda^* \Lambda^3 V_1$. Then we have that $\Psi(k_2) = \kappa_2$. Notice that the vector space generated by k_2 is one-dimensional. In particular the vector space generated by k_2 is isomorphic to $V_0 = \mathbb{Q}$. We also have that k_2 is not contained in $\mathrm{im}((f_I - f_H) \circ \phi)$. Therefore from Theorem 3.47 we get an induced map

$$\frac{\Lambda^*(\Lambda^3 V_1[1, 1])}{(2V_0 + V_{1^2} + V_{2^2})} \to \mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{E}_1/(\kappa_2)$$

which is an isomorphism in a stable range of degrees. $\qquad\square$

Remark 3.49. By the discussion in [39, Section 9] the range in which Theorem 3.47 holds is in degrees $* \leq g$. So Corollary 3.48 in the same range.

The commutative algebra $\frac{\Lambda^*(\Lambda^3 V_1[1])}{(2V_0 + V_{1^2} + V_{2^2})}$ was proven to be the quadratic dual of $\mathrm{Gr}^*_{lcs} \mathfrak{t}_{g,1}$ by Hain [20]. We give the formulation of this result in the same way as in [40] where a proof can be found.

Theorem 3.50. *There is a homomorphism of tri-graded commutative algebras*

$$\Phi \colon \frac{\Lambda^*[\Lambda^3 V_1[1, 0, 1]]}{(2V_0 + V_{1^2} + V_{2^2})} \to H^*(\mathrm{Gr}^*_{lcs} \mathfrak{t}_{g,1})_{*,*}$$

which is an isomorphism when $4 \leq g$ and $* \leq 2$.

The homomorphism Φ is obtained from the Johnson homomorphism $\Lambda^3 V_1 \xrightarrow{\cong} H^1(\mathcal{T}_{g,1}; \mathbb{Q})$ by factoring through the continuous cohomology of $\mathfrak{t}_{g,1}$, see [20, Section 5] for definition of continuous cohomology. Notice that V_1 is not concentrated in the zero degrees but is instead in the $(1, 0, 1)$-degrees. The first degree shift in the homological degree comes from the identification between the mixed Hogde structure on $\mathfrak{t}_{g,1}$ and the lower central series of $\mathfrak{t}_{g,1}$ given by Hain [20, Theorem 4.10]. The 0-degree is because we set $\mathrm{Gr}^*_{lcs} \mathfrak{t}_{g,1}$ to we concentrated at internal degree 0 by our grading conventions. Finally the last

1-degree comes from giving $\mathrm{Gr}^k_{lcs}\,\mathfrak{t}_{g,1}$ weight k and using Hain [20, Theorem 4.10] again. See the proof in [40] for more details.

Finally, combining the previous theorem with Corollary 3.48 and Remark 3.49 together we get the following result.

Theorem 3.51. *If $g \geq 4$ then the realisation $K^\vee \otimes \mathcal{E}_1/(\kappa_{e^2})$ is isomorphic to the quadratic dual of Lie algebra object $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ in degrees $* \leq g$.*

Proof. From Corollary 3.48 and Remark 3.49 there is a map

$$\frac{\Lambda^*(\Lambda^3 V_1[1,1])}{(2V_0 + V_{1^2} + V_{2^2})} \to \mathcal{K}^\vee_H \otimes^{\mathrm{dB}} \mathcal{E}_1/(\kappa_2)$$

which is an isomorphism in degrees $* \leq g$. However by Theorem 3.50 the domain is the quadratic dual of $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ for $g \geq 4$. Therefore for $g \geq 4$ the commutative algebra is $K^\vee \otimes \mathcal{E}_1/(\kappa_{e^2})$ is isomorphic to the quadratic dual of $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$ in degrees $* \leq g$. $\qquad\square$

Our goal is to use this result to prove the Koszulness of $\mathrm{Gr}^*_{lcs}\,\mathfrak{t}_{g,1}$.

Proposition 3.52. *If $\mathcal{E}_1$ is Koszul, then $K^\vee \otimes \mathcal{E}_1/(\kappa_{e^2})$ is Koszul in weight $\leq g/3$.*

Proof. For all $a \geq 2$ let $o_a = \{(\varnothing, a)\} \in \mathcal{E}_1(\varnothing)$. Notice that $\mathcal{E}_1(\varnothing)$ is a polynomial algebra on the elements o_a for $a \geq 2$. Moreover we have that for each $S \in \mathrm{dB}$ we have that $\mathcal{E}_1(S)$ is a free $\mathcal{E}_1(\varnothing)$-module. This implies that $o_2 \in \mathcal{E}_1(\varnothing)_{2,2}$ is not a zero-divisor. From Proposition 2.55 we get a long exact sequence

$$... \to ev^!_\varnothing(\mathbb{Q}\{o_2\}[2]) \to AQ_2(\mathcal{E}_1) \to AQ_2(\mathcal{E}_1/(o_2)) \to ...$$

We have that the image of o_2 under $\mu_\varnothing \colon \mathcal{E}_1(\varnothing) \to \mathcal{K}^\vee_H \otimes^{\mathrm{dB}} \mathcal{E}_1$ is κ_2 and we have that

$$\mathcal{K}^\vee_H \otimes^{dB} (\mathcal{E}_1/(o_2)) \cong \mathcal{K}^\vee_H \otimes^{dB} \mathcal{E}_1/(\kappa_2).$$

This can be checked using the universal property of coends. As κ_2 is decomposable in $\mathcal{K}^\vee_H \otimes^{\mathrm{dB}} \mathcal{E}_1$ we have that o_2 is decomposable. For $(p,q,w) \neq (2,2,2)$ or $S \neq \varnothing$ this gives the short exact sequence

$$0 \to H_p(\mathcal{E}_1)_{q,w}(S) \to H_p(\mathcal{E}_1/o_2)_{q,w}(S) \to 0 \to 0$$

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as well as the short exact sequence

$$0 \to H_2(\mathcal{E}_1)_{2,2}(\varnothing) \to H_2(\mathcal{E}_1/o_2)_{2,2}(\varnothing) \to \mathbb{Q} \to 0.$$

From this we get that $\mathcal{E}_1$ is Koszul if and only if $\mathcal{E}_1/(o_2)$ is Koszul. By the definition of the weight grading on $\mathcal{E}_1$ we have that weight w part of $\mathcal{E}_1/(o_2)$ is supported on sets of size $\leq 3w$ so by Lemma 3.25 we have

$$\mathcal{K}_H^{\vee} \otimes^{\mathrm{dB}} H_p(\mathcal{E}_1/(o_2))_{p,w} \cong H_p(\mathcal{K}_H^{\vee} \otimes^{dB} \mathcal{E}_1/(\kappa_2))_{q,w}$$

for all $w \leq g/3$. Thus if $\mathcal{E}_1$ is Koszul then $H_p(\mathcal{E}_1/(o_2))_{q,w} = o$ for $p \neq q$. It then follows that $H_p(\mathcal{K}_H^{\vee} \otimes^{dB} \mathcal{E}_1/(\kappa_2))_{q,w} = 0$ for all $p \neq w$ and $w \leq g/3$. $\square$

Corollary 3.53. *For all odd $1 \leq n$, if $\mathcal{E}_n$ is Koszul then $\mathrm{Gr}^*_{lcs} \mathfrak{t}_{g,1}$ is Koszul in weight $\leq g/3$.*

Proof. By definition $\mathcal{E}_n$ is isomrphic to $\mathcal{E}_1$ with it's internal grading scaled by n. So we have that $\mathcal{E}_n$ is Koszul if and only if $\mathcal{E}_1$ is Koszul. Now assume that $\mathcal{E}_n$ is Koszul then $\mathcal{E}_1$ is Koszul and $K^{\vee} \otimes \mathcal{E}_1/(\kappa_{e^2})$ is Koszul in weight $\leq g/3$. By Theorem 3.51 we have that $\mathcal{K}_H^{\vee} \otimes^{\mathrm{dB}} \mathcal{E}_1/(\kappa_2)$ is isomorphic to the quadratic dual of $\mathrm{Gr}^*_{lcs} \mathfrak{t}_{g,1}$ in degrees $* \leq g$. By Koszul duality we have that $\mathrm{Gr}^*_{lcs} \mathfrak{t}_{g,1}$ is Koszul in weight $\leq g/3$. $\square$

We will use the next two chapters to prove that $\mathcal{Z}_n$ and $\mathcal{E}_n$ is Koszul for all n. Take note that by the definition of the weight grading on $\mathcal{Z}_n$ and $\mathcal{E}_n$ we have that $(\mathcal{Z}_n)_{q,q/n}$ and $(\mathcal{E}_n)_{q,q/n}$ are the non-zero weighted parts of $\mathcal{Z}_n$ and $\mathcal{E}_n$. So for Koszulness it is enough to show that if $q \neq np$ then $H_p(\mathcal{Z}_n)_{q,w} = 0$ and $H_p(\mathcal{E}_n)_{q,w} = 0$.

Chapter 4

High Dimensional Manifolds

In this chapter, we will describe some recent results from the theory of diffeomorphisms of high-dimensional manifolds. Much of the results come from [38]. We will then these results to compute the homology of the functors $\mathcal{E}_n$ and $\mathscr{Z}_n$. There is a much larger body of research that has been achieved than what we shall include. A good detailed introduction to these results is the survey by Galatius and Randal-Williams [12] as well as their paper [13]. We will assume that all manifolds in this section are smooth (C^∞-) manifolds, possibly with boundary.

4.1 Definitions and Recollections

Here we introduce some of the concepts which will be used throughout the section. We will start by introducing some theory of classifying spaces and function spaces. We then introduce the Thom space and finally turn to some theory of θ-structures. With these, we will be able to provide the necessary results and our main theorem of the chapter. Throughout this subsection set assume that $2n \geq 6$ and $d \geq 3$.

4.1.1 Classifying spaces

Recall that the

Definition 4.1. For a d-manifold W and a manifold of some larger dimension W' let $\mathrm{Emb}(W, W')$ denote the set of embeddings of W into W'. If $\partial W \neq \varnothing$

and $\partial W' \neq \varnothing$ fix an embedding $\partial W \hookrightarrow \partial W'$. Let $\mathrm{Emb}_\partial(W, W')$ denote the set of embeddings which extend the fixed embedding of the boundary.

Definition 4.2. Let $p_1 \colon E_1 \to B_1$ and $p_2 \colon E_2 \to B_2$ be vector bundles. A *bundle map* is a continuous map $\iota \colon E_1 \to E_2$ such that the restriction $\iota|_{F_x}$ to a fibre $F_x = p_1^{-1}(x)$ of p_1 is an isomorphism onto a fibre $F_y = p_2^{-1}(y)$ of p_2 for all $x \in E_1$. Let $\mathrm{Bun}(E_1, E_2)$ denote the set of bundles $E_1 \to E_2$.

The sets $\mathrm{Bun}(E_1, E_2), \mathrm{Diff}(W), \mathrm{Diff}_\partial(W)$ and $\mathrm{Emb}(W, W')$ equipped with the open-compact topology become topological spaces. A definition of the open-compact topology can be found in [23, Chapter 2].

Recall that $\mathbb{R}^\infty$ is the space of sequences of real numbers with only finitely many non-zero terms. We can also view $\mathbb{R}^\infty$ as the inverse limit of $\mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \ldots$ where $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$ sends $(x_1, \ldots, x_k) \in \mathbb{R}^k$ to $(x_1, \ldots, x_k, 0)$.

Lemma 4.3. *The spaces* $\mathrm{Emb}(W, [0, \infty) \times \mathbb{R}^\infty)$ *and* $\mathrm{Emb}_\partial(W, [0, \infty) \times \mathbb{R}^\infty)$ *is contractible for all d dimensional manifolds W.*

Proof. The same proof works for both with and without boundary so we will only show that $\mathrm{Emb}(W, [0, \infty) \times \mathbb{R}^\infty)$ is contractible. By the Whitney embedding theorem there exists an embedding $W \hookrightarrow \mathbb{R}^{2d}$. There also exists a natural embedding of $\mathbb{R}^{2d}$ into $[0, \infty) \times \mathbb{R}^\infty$. Compositing the two gives an embedding $\phi \colon W \hookrightarrow [0, \infty) \times \mathbb{R}^\infty$. The map $h \colon [0, 1] \times \mathrm{Emb}(W, [0, \infty) \times \mathbb{R}^\infty) \to \mathrm{Emb}(W, [0, \infty) \times \mathbb{R}^\infty)$ defined by

$$h(t, \psi) = (1 - t)\psi + t\phi$$

for all $(t, \psi) \in \mathrm{Emb}(W, [0, \infty) \times \mathbb{R}^\infty)$ is a homotopy from the identity to the constant map at ϕ. $\square$

Theorem 4.4. *For any manifold W the group $\mathrm{Diff}(W)$ acts on $\mathrm{Emb}(W, [0, \infty) \times \mathbb{R}^\infty)$ freely and properly discontinuously. Similarly the group $\mathrm{Diff}_\partial(W)$ acts on $\mathrm{Emb}_\partial(W, [0, \infty) \times \mathbb{R}^\infty)$ freely and properly discontinuously.*

The above theorem implies that

$$B\,\mathrm{Diff}(W) = \mathrm{Emb}(W, [0, \infty) \times \mathbb{R}^\infty)/\mathrm{Diff}(W)$$

$$B\,\mathrm{Diff}_\partial(W) = \mathrm{Emb}_\partial(W, [0, \infty) \times \mathbb{R}^\infty)/\mathrm{Diff}_\partial(W)$$

We shall use this later.

4.1.2 θ-Structures

We will now describe some general theory and definitions of θ-structures. We will only restrict ourselves to the frame structure later however we wish to introduce this as θ-structures in greater generality as they have become an essential tool in geometric and algebraic topology. Our main sources for this section are [12, 13, 14].

Recall that $BO(k)$ is the Grassmanian $\mathrm{Gr}_k(\mathbb{R}^\infty)$. Let $\gamma_k \colon U_k \to \mathrm{Gr}_k(\mathbb{R}^\infty)$ denote the universal vector bundle over $\mathrm{Gr}_k(\mathbb{R}^\infty)$. The total space $EO(k)$ is the Stiefel manifolds of k-dimensional normal frames in $\mathbb{R}^\infty$. Let $\nu_{fr} \colon EO(k) \to BO(k)$ denote the associated bundle. Details on these may be found in [51].

Definition 4.5. A *tangential structure* on a topological space B is a map $\theta \colon B \to BO(d)$. This defines a vector bundle over B via the pullback

$$
\begin{array}{ccc}
\theta^* U_d & \longrightarrow & U_d \\
\downarrow & & \downarrow{\scriptstyle \gamma_d} \\
B & \xrightarrow{\ \theta\ } & BO(d)
\end{array}
$$

Definition 4.6. Let $\theta \colon B \to BO(d)$ be a tangential structure on B. Let W be a d-manifold. A θ-*structure* on W is a bundle map $\ell \colon TW \to \theta^* \gamma_d$.

Then the space of θ-structures on W is $\mathrm{Bun}(TW, \theta^* U_k)$. One way to think of θ-structures is to think of them as encoding extra structures on our vector bundles. We have a short example.

Example 4.7. Let G be a topological group. Then a representation of topological groups $\rho \colon G \to \mathrm{GL}_d(\mathbb{R})$ induces a map $B\rho \colon BG \to BO(d)$. If $\theta = B\rho$ is our tangential structure then a θ-structure $\ell \colon TW \to \theta^* U_k$ produces a lift

$$
\begin{array}{ccc}
 & & BG \\
 & \nearrow & \downarrow{\scriptstyle \ell} \\
W & \longrightarrow & BO(d)
\end{array}
$$

where $W \to BO(d)$ classifies the usual tangent vector bundle. The lift $W \to BG$ induces a principle G-bundle with W as the base space.

There is an action of $\text{Diff}(W)$ on TW which is induced by taking the derivative. For each $\phi \in \text{Diff}(W)$ the action the derivative map $D\phi \colon TW \to TW$ is an isomorphism on fibres. We use this action on TW to define an action on $\text{Bun}(TW, \theta^* U_k)$.

Definition 4.8. We define the action of $\text{Diff}(W)$ on $Bun(TW, \theta^* U_k)$ by defining
$$\phi \cdot \ell = \ell \circ (D\phi)^{-1}$$
for all $\phi \in \text{Diff}(W)$ and $\ell \in Bun(TW, \theta^* U_k)$. We define $B^\theta \text{Diff}(W)$ to be the space
$$B^\theta \text{Diff}(W) = \text{Bun}(TW, \theta^* U_k) \times_{\text{Diff}(W)} \text{Emb}(W, [0, \infty) \times \mathbb{R}^\infty).$$

This construction used to define $B^\theta \text{Diff}_\partial(W)$ is called the Borel construction. There is a boundary version of this as well.

Definition 4.9. If W is a manifold with boundary and $\theta \colon B \to BO(d)$ is a tangential structure. Let $\ell \colon T\partial W \to \theta^* U_d$ be a θ-structure on ∂W. We define $\text{Bun}(TW, \theta^* U_d; \ell)$ to be the subset of $\text{Bun}(TW, \theta^* U_d)$ which extends ℓ. Then define $B_\ell^\theta \text{Diff}_\partial(W)$ to be the space
$$B_\ell^\theta \text{Diff}_\partial(W) = \text{Bun}(TW, \theta^* U_k; \ell) \times_{\text{Diff}_\partial(W)} \text{Emb}_\partial(W, [0, \infty) \times \mathbb{R}^\infty).$$

4.2 The manifold $W_{g,1}$ and the homology of $\mathcal{Z}_n$ and $\mathcal{E}_n$

We are now ready to introduce an important collection of manifolds which have been studied in recent years by Randal-Williams, Kupers, Galatius amougst others.

Definition 4.10. For $1 \leq g$ and $6 \leq 2n$ we define manifold $W_{g,1}$ the space
$$W_{g,1} = D^{2n} \# (S^n \times S^n)^{\# g}.$$

Galatius and Randal-Williams [13] generalised the Madsen-Weiss theorem 3.34 for the cohomology of the spaces $\text{BDiff}(W_{g,1})$ and $\text{BDiff}_\partial(W_{g,1})$.

We will now explain some results from [38] which we will use to aid in the computation of the homology of $\mathcal{Z}_n$ and $\mathcal{E}_n$. Consider the homology group $H_n(W_{g,1}; \mathbb{Z})$. The intersection pairing on homology defines a bilinear pairing
$$H_n(W_{g,1}; \mathbb{Z}) \otimes H_n(W_{g,1}; \mathbb{Z}) \to H_{2n}(W_{g,1}; \mathbb{Z}) \cong \mathbb{Z}.$$

Let $G_{W_{g,1}}$ denote the group of automorphisms of $H^n(W_{g,1}; \mathbb{Z})$ which preserve the pairing. Notice that we have the following isomorphism

$$G_{W_{g,1}} \cong \begin{cases} O_{g,g}(\mathbb{Z}) & n \text{ even,} \\ \mathrm{Sp}_{2g}(\mathbb{Z}) & n \text{ odd.} \end{cases}$$

Definition 4.11. The natural action of $\mathrm{Diff}_\partial(W_{g,1})$ on $H^n(W_{g,1}; \mathbb{Z})$ defines a homomorphism

$$\alpha_g \colon \mathrm{Diff}_\partial(W_{g,1}) \to G(W_{g,1}).$$

We denote the image of this map by $G'_{W_{g,1}}$. We define the Torelli group of $W_{g,1}$ to be the group

$$\mathrm{Tor}_\partial(W_{g,1}) = \ker(\alpha_g).$$

Using quadratic refinements of the intersection product Kreck [37] found the following result.

Theorem 4.12. *There is an isomorphism of groups*

$$G'_{W_{g,1}} \cong \begin{cases} O_{g,g}(\mathbb{Z}) & n \text{ even,} \\ \mathrm{Sp}_{2g}(\mathbb{Z}) & n = 1,3,7, \\ \mathrm{Sp}_{2g}^q(\mathbb{Z}) & \text{otherwise,} \end{cases}$$

where $\mathrm{Sp}_{2g}^q(\mathbb{Z})$ is the proper subgroup contain only the symplectic matrices which respect the quadratic refinement $q \colon \mathbb{Z}^{2g} \to \mathbb{Z}/2\mathbb{Z}$ of the bilinear form. In particular $G'_{W_{g,1}}$ is always a finite index subgroup of $G_{W_{g,1}}$.

From the exact sequence

$$\mathrm{Tor}_\partial(W_{g,1}) \to \mathrm{Diff}_\partial(W_{g,1}) \xrightarrow{\alpha_g} G'_{W_{g,1}}$$

we get a Serre fibration

$$B\,\mathrm{Tor}_\partial(W_{g,1}) \to B\,\mathrm{Diff}_\partial(W_{g,1}) \to BG'_{W_{g,1}}.$$

Kupers and Randal-Williams [41, 39, 59] computed the cohomology of $\mathrm{Tor}_\partial(W_{g,1})$ and it's variants in a stable range. They used their results in [41] and some earlier results to compute the rational homotopy groups of even-dimensional disks in [38]. We will use some of the results found in [38]

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to prove the main theorem of this section. To that we require a version of the fibration

$$B \operatorname{Tor}_\partial(W_{g,1}) \to B \operatorname{Diff}_\partial(W_{g,1}) \to BG'_{W_{g,1}}$$

with $B_\ell^\theta \operatorname{Diff}_\partial(W_{g,1})$ instead. To this end let $\nu_{fr} \colon EO(2n) \to BO(2n)$ be the frame bundle associated to the universal vector bundle over $BO(2n)$ and let $B_\ell^{fr} \operatorname{Diff}_\partial(W_{g,1})$ denote $B_\ell^{\nu_{fr}} \operatorname{Diff}_\partial(W_{g,1})$ to simplify notation. Kupers and Randal-Williams [41] proved the following result.

Proposition 4.13. *Up to homotopy there is a unique orientation-preserving boundary condition* $\ell \colon T\partial W_{g,1} \to \nu_{fr}^* U_{2n}$ *on* $\partial W_{g,1}$ *which can be extended to a* ν_{fr}*-structure on all of* $W_{g,1}$*.*

It is not necessarily the case that $B_\ell^{fr} \operatorname{Diff}_\partial(W_{g,1})$ is connected so let $B_\ell^{fr} \operatorname{Diff}_\partial(W_{g,1})_0$ denote connected component of $B_\ell^{fr} \operatorname{Diff}_\partial(W_{g,1})$ which contains the extension of ℓ from the above proposition. The group homomorphism

$$\operatorname{Diff}_\partial(W_{g,1}) \to G_{W_{g,1}}$$

gives us a homomorphism

$$\pi_0(\operatorname{Diff}_\partial(W_{g,1})) \to G_{W_{g,1}}$$

but $\pi_0(\operatorname{Diff}_\partial(W_{g,1})) = \pi_1(B \operatorname{Diff}_\partial(W_{g,1}))$.

Proposition 4.14. *There is a fibre sequence*

$$\operatorname{Diff}_\partial(W_{g,1}) \to \operatorname{Bun}(TW_{g,1}, \nu_{fr}^* U_k; \ell) \times \operatorname{Emb}_\partial(W_{g,1}, [0, \infty) \times \mathbb{R}^\infty) \to B_\ell^{fr} \operatorname{Diff}_\partial(W_{g,1}).$$

Moreover there exists a homomorphism

$$\pi_1(B_\ell^{fr} \operatorname{Diff}_\partial(W_{g,1})_0) \to \pi_0(\operatorname{Diff}_\partial(W_{g,1}))$$

from the long exact sequence of homotopy groups.

There following result was then found in [41].

Lemma 4.15. *The image* $G_{W_{g,1}}^{fr,\ell}$ *of the composition*

$$\pi_1(B_\ell^{fr} \operatorname{Diff}_\partial(W_{g,1})_0) \to \pi_1(B \operatorname{Diff}_\partial(W_{g,1})) \to G'_{W_{g,1}}$$

is a finite index subgroup of $G'_{W_{g,1}}$*.*

We can now define a framed version of the classifying space $B\operatorname{Tor}(W_{g,1})$.

Definition 4.16. We define $B_\ell^{fr}\operatorname{Tor}_\partial(W_{g,1})$ to be the homotopy limit

$$B_\ell^{fr}\operatorname{Tor}_\partial(W_{g,1}) = \operatorname{holim}(B_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1})_0 \to BG_{W_{g,1}}^{fr,\ell}).$$

We are now ready to introduce the main results which are of importance to us. The following result is a special case of [11, Corollary 1.8].

Proposition 4.17. *For $3 \le n$ there is a morphism*

$$\alpha^{fr}\colon B_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1})_0 \to \Omega_0^\infty \mathbb{S}^{-2n}$$

where $\mathbb{S}$ is the sphere spectrum which induces an isomorphism on homology and cohomology. In paticular, this implies that

$$H_1(B_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1})_0; \mathbb{Z}) \xrightarrow{\cong} \pi_{2n+1}(\mathbb{S})$$

and that $H^*(B_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1})_0; \mathbb{Q}) = 0$.

Then Kupers and Randal-Willims [38] found proved the following result.

Proposition 4.18. *There exists a space $\mathbb{M}$ and a homotopy commutative square*

$$
\begin{array}{ccc}
B_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1})_0 & \xrightarrow{\ \alpha^{fr}\ } & \Omega_0^\infty \mathbb{S}^{-2n} \\
\downarrow & & \downarrow \\
BG_{W_{g,1}^{fr,\ell}} & & \\
\downarrow & & \downarrow \\
BG_{W_{g,1}} & \xrightarrow{\hspace{2cm}} & \mathbb{M}
\end{array}
$$

such that the horizontal maps are acyclic in a range of degrees tending towards infinity with g and the right vertical map is a loop map.

Remark 4.19. The space $\mathbb{M}$ is a group completion of the monoid $\bigsqcup_{g \ge 0} BG_{W_{g,1}}$. The idea of the proof of the above proposition is to construct a group completion of a commutative monoid which is of the same homotopy type as $\Omega_0^\infty \mathbb{S}^{-2n}$ and find maps between the spaces in the diagram.

Using Proposition 4.18 we can define the appropriate covering spaces which we will need.

Definition 4.20. Define

- $\overline{B_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1})_0}$ to be the regular $\pi_{2n+1}(\mathbb{S})$-cover of $_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1})_0$,

- $\overline{G}_{W_{g,1}}^{fr,\ell}$ be the image of the composition $\overline{B_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1})_0} \to B_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1}) \to G_{W_{g,1}}$

- $\overline{B_\ell^{fr}\operatorname{Tor}_\partial(W_{g,1})}$ to be the homotopy limit $\operatorname{holim}(\overline{B_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1})_0} \to B\overline{G}_{W_{g,1}}^{fr,\ell})$

- $\overline{\overline{\mathbb{M}}}, \overline{\mathbb{M}}$ and $\overline{\Omega_0^\infty\mathbb{S}^{-2n}}$ to be the universal covers of the corresponding spaces.

From the last proposition Kupers and Randal-Williams [38] then constructed a commutative diagram and computed the homotopy types of the spaces involved.

Lemma 4.21. *There exists a commutative diagram*

$$
\begin{array}{ccccc}
X_1(g) & \longrightarrow & \overline{B_\ell^{fr}\operatorname{Tor}_\partial(W_{g,1})} & \longrightarrow & X_0 \\
\downarrow & & \downarrow & & \downarrow \\
A_1(g) & \longrightarrow & \overline{B_\ell^{fr}\operatorname{Diff}_\partial(W_{g,1})_0} & \longrightarrow & \overline{\Omega_0^\infty\mathbb{S}^{-2n}} \\
\downarrow & & \downarrow & & \downarrow \\
A_2(g) & \longrightarrow & B\overline{G}_g^{fr,\ell} & \longrightarrow & \overline{\overline{\mathbb{M}}}
\end{array}
$$

of spaces such that the rows and columns are fibrations.

Lemma 4.22. *The space* $\overline{B_\ell^{fr}\operatorname{Tor}_\partial(W_{g,1})}$ *is nilpotent and* $X_1(g)$ *is of finite type. Hence spaces in the fibre sequence*

$$X_1(g) \to \overline{B_\ell^{fr}\operatorname{Tor}_\partial(W_{g,1})} \to X_0$$

are nilpotent.

A proof can be found in [41] and in [38].

Theorem 4.23. *For $2n \geq 6$ we have*

i) *the action of $\pi_1(A_1(g))$ on $H^*(X_1(g); \mathbb{Q})$ and $\pi_*(X_1(g)) \otimes \mathbb{Q}$ factors over $\overline{G}_g^{fr,\ell}$,*

ii) *The is a map $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{Z}_n \to H^*(X_1(g); \mathbb{Q})$ of $\overline{G}_g^{fr,\ell}$-representations which is an isomorphism in a stable range.*

Lemma 4.24. *In a stable range, the non-trivial $\overline{G}_g^{fr,\ell}$-representations in $\pi_{*+1}(X_1(g)) \otimes \mathbb{Q}$ vanish in degrees $* \leq \frac{n(n-3)}{2}$, except with $* = r(n-1)$ for some $r \geq 1$.*

Theorem 4.25. *If $q \neq np$ then the functors*

$$H_p(\mathcal{Z}_n)_{q,w} \colon \mathrm{dB} \to \mathbf{Vec}_\mathbb{Q}, \quad \& \quad H_p(\mathcal{E}_n)_{q,w} \colon \mathrm{dB} \to \mathbf{Vec}_\mathbb{Q}$$

vanish when evaluated on non-empty sets.

Proof. First note that if $q \neq np$ implies that $H_p(\mathcal{Z}_n)_{q,w}(S) = 0$ for all sets $S \neq \varnothing$ then by our computation of $H_p(\mathcal{Z}_n, \mathcal{E}_n)_{q,w}(S)$ (Theorem 2.87) and the long exact sequence of homology we can conclude that $H_p(\mathcal{E}_n)_{q,w}(S) = 0$. It is thus enough to show that $q \neq np$ implies $H_p(\mathcal{Z}_n)_{q,w}(S) = 0$ for all sets $S \neq \varnothing$.

Our strategy will be to compute the cohomology $H^($X_1(g); \mathbb{Q})$ and compare with the homology of $\mathcal{K}^\vee \otimes^{\mathrm{dB}} \mathcal{Z}_n$ using the map $\mathcal{K}^\vee \otimes^{\mathrm{dB}} \mathcal{Z}_n \to H^*(X_1(g); \mathbb{Q})$ from Theorem 4.23. To this end, using the homology H^{Com} of commutative algebra objects in $\mathbf{Ch}_\mathbb{Q}$, we define a rational Adams spectral sequence by

$$E_{s,t}^2 = H_s^{Com}(H^*(X_1(g); \mathbb{Q}))_t$$

with the differential d^r with bidegree $(-r, -r+1)$. This spectral sequence is unstable and is strongly convergent to $\mathrm{Hom}(\pi_{t-s}(\Omega X_1(g)), \mathbb{Q})$ since $X_1(g)$ is nilpotent and of finite type (Lemma 4.22). It follows from Theorem 4.23 at this is a spectral sequence of $\overline{G}_g^{fr,\ell}$-representations. By the second part of Theorem 4.23, in a stable range, the cohomology of $X_1(g)$ is concentrated in degrees which are multiples of n. This is due to the definitions of the gradings on $\mathcal{Z}_n$. So in a stable range the groups $E_{s,t}^2$ are supported along the lines $t = rn$ where $r \in \mathbb{N}$. However we have by Lemma **??** that $E_{s,rn}^2 = 0$ for $r < s$. Thus $E_{s,t}^2$ is non-trivial only for $1 \leq s \leq r$, contributing to degrees

$$rn - r + 1 \leq t - s + 1 \leq rn$$

where $t = rn$. This implies that as long as $n \geq r - 2$ the spectral sequence collapses n degrees $t - s + 1 \leq rn$ as there are no non-trivial differentials. Thus for $r \leq n + 2$ there is an isomorphism of $\overline{G}_g^{fr,\ell}$-reresentations

$$H_s^{Com}(H^*(X_1(g); \mathbb{Q}))_{rn} \cong \mathrm{Hom}(\pi_{rn-s+1}(X_1(g)), \mathbb{Q})$$

in a stable range.

However, it follows from Lemma 4.24 that $\mathrm{Hom}(\pi_{rn-s+1}(X_1(g), \mathbb{Q}) = 0$ in degrees $rn - s \leq \frac{n(n-3)}{2}$ except when $r = s$. Recall from Lemma 3.25 that

$$\mathcal{K}^\vee \otimes^{\mathrm{dB}} H_p(\mathcal{Z}_n)_q \cong H_p(\mathcal{K}^\vee \otimes^{\mathrm{dB}} \mathcal{Z}_n)_q$$

for $q/n \leq g/3$. For large enough g and $q \leq \frac{n(n-3)}{2}$ the map $\mathcal{K}^\vee \otimes^{\mathrm{dB}} \mathcal{Z}_n \to H^*(X_1(g); \mathbb{Q})$ is an isomorphism and hence there is an isomorphism for homology which implies by the above vanishing result that $\mathcal{K}^\vee \otimes^{\mathrm{dB}} H_p(\mathcal{Z}_n)_q$ is a trivial representation whenever $q \neq pn$. Since $\overline{G}_g^{fr,\ell}$ is a finite index subgroup the vanishing results are true as G_g-representations.

We have just showed that for large enough g and $q \leq \frac{n(n-3)}{2}$ the G_g-representation $\mathcal{K}^\vee \otimes^{\mathrm{dB}} H_p(\mathcal{Z}_n)_q$ is trivial in a stable range. But by Lemma 3.23 if $H_p(\mathcal{Z}_n)_q(S)$ is not zero for a non-empty finite set S then for $|S| \leq g$ the representation $[V^{\langle S \rangle} \otimes (\mathcal{K}^\vee \otimes^{\mathrm{dB}} H_p(\mathcal{Z}_n)_q)]^{G_g}$ is non-zero. The representations $V^{\langle S \rangle}$ are not trivial. We can then conclude that since $\mathcal{K}^\vee \otimes^{\mathrm{dB}} H_p(\mathcal{Z}_n)_q$ is trivial in a stable range by the contrapositive of Lemma 3.23 $H_p(\mathcal{Z}_n)_q(S) = 0$. The bound $q \leq \frac{n(n-3)}{2}$ scales quadratically with n so we can ignore it. Therefore we have that $H_p(\mathcal{Z}_n)_q(S) = 0$ for $S \neq \varnothing$ and $q \neq np$. $\square$

Remark 4.26. While we have used the long exact sequence in homology. It should be noted that in the paper "On the Torelli Lie Algebra" [40] the authors prove the above result for the homology of $\mathcal{E}_n$ in a sightly different way. Instead they define Euler tangential structures. This forces the Pontryagin characteristic classes to vanish leaving only the Euler class. It is then possible, while keeping track of the Euler class, to form another fibre sequence square and similar results as the above and prove Theorem 4.25 using these adapted Euler structure versions.

Chapter 5

Graph Complexes and the end of the proof

In this chapter we will finish the proof of koszulness. It requires us to define some graph complexes and use some of the theory of graph complexes to define models for $LQ^+(\mathcal{E}_n)$ and $LQ^+(\mathcal{Z}_n)$. Then we use vanishing results from the literature to prove koszulness.

5.1 Graph Complexes

We will need to defined graph complexes and give some intuition for them. We will define two functors using graph complexes and show how they give model for $LQ^+(\mathcal{E}_n)$ and $LQ^+(\mathcal{Z}_n)$. We will then end with the proof of the main theorem.

5.1.1 Graphs and Graph Complexes

Graph complexes and graph homology have become an indispensable tool which we do not completely understand yet. Graph complexes were introduced by M. Kontsevich. In his papers [35] and [36] Kontsevich used graph homology to study three infinite dimensional Lie algebras, 3-manifolds, knot invariants, automorphisms of groups and mapping class groups. The idea is simple, we wish to define chain complexes with (combinatorial) graphs as a basis for each of the vector spaces in the complex and the differential of the chain complex is then determined by operations like contraction of edges or

switching of colours in a colouring. We will give two examples of how to go about defining graph complexes as this can be tricky. Later we will then use the ideas to define a graph complex which will model $\mathcal{E}_n$ and $\mathcal{Z}_n$. But to be able speak of such things we will first need a formal definition of a graph.

Definition 5.1. A *graph* Γ is a tuple $\Gamma = (V, H, \iota, \partial)$ where

  i) V is a set of vertices,

 ii) H is a set of half edges,

iii) $\iota \colon H \to H$ is an involution (i.e. $\iota^2 = id_H$),

iv) and $\partial \colon H \to V$ is a function.

We define the edge set E of Γ to be the set of free orbits of ι. We define the legs $L(\Gamma)$ of Γ to be the set of trivial orbits of ι. We denote the data of a graph Γ by $V(\Gamma), E(\Gamma), H(\Gamma), \iota_\Gamma$ and ∂_Γ respectively.

While this definition seems complicated at a first glance it is rather diverse. Using this definition we can easily study multiedged graphs and graphs with loops. This definition can also be use two introduces, what we have called, legs. These can be interpreted in a number of different ways as infinite edges, legs or whiskers. These different types of graphs and edges are encoded using ∂ and ι. The role that ∂ plays is to assign end points edges. The role of ι is to determine edges and half edges. We give an example below.

Example 5.2. Let Γ be the graph with the vertex set $V = V(\Gamma) = \{a, b, c, d\}$ and the half edge set $H = H(\Gamma) = \{h_1, ..., h_{12}\} \cup \{1, 2, 3, 4, 5\}$. The reason for union with $[5] = \{1, 2, 3, 4, 5\}$ is because these will be the labels for our legs. Next we define ∂ and ι. We define $\partial \colon H \to V$ by

$$
\partial(x) = \begin{cases}
b, & x = h_1, ..., h_6; \\
c, & x = h_7, h_8, h_9, h_{10} \text{ or } x = 4, 5; \\
d, & x = h_{11} \text{ or } x = 1, 2, 3; \\
a, & x = h_{12}.
\end{cases}
$$

and we define the involution $\iota \colon H \to H$ by

$$
\iota(h_1) = h_{12}, \quad \iota(h_2) = h_3, \quad \iota(h_4) = h_7,
$$

$$\iota(h_6) = h_9, \quad \iota(h_{10}) = h_{11}, \quad \iota(h_5) = h_8$$

and the elements $1, 2, 3, 4, 5$ are fixed points of ι. We can draw this graph by drawing an edge $\{h_i, h_j\} \in E(\Gamma)$ between two vertices $x, y \in V$ if $\partial(h_i) = x$ and $\partial(h_j) = y$. If $\{h_k, h_n\}$ is an edge of Γ with $\partial(h_k) = \partial(h_n) = x$ then we draw a loop or a reflexive at x. Lastly we draw the legs as edges without an endpoint, these also look like little hairs. The graph Γ is drawn in figure 5.1.
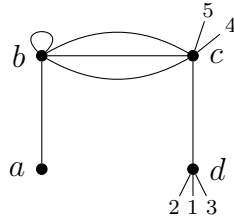

Figure 5.1: The graph Γ with four vertices, six edges and five legs

Now that we know what a graph is we will now define contraction which we will use to define graph complexes and graph homology.

Definition 5.3. Let Γ be a graph. If $e = \{h_1, h_2\} \in E(\Gamma)$ is an edge of Γ, then the graph Γ/e obtained by *contrating the edge e* has the vertex set $V(\Gamma/e) = V(\Gamma)/(\partial_\Gamma(h_1) \sim \partial_\Gamma(h_2))$ and the half edges $H(\Gamma/e) = H(\Gamma) \setminus \{h_1, h_2\}$. We define $\partial_{\Gamma/e}$ to be the composition of the restriction of ∂_Γ to $H(\Gamma/e)$ composed with the quotient map $q \colon V(\Gamma) \to V(\Gamma/e)$. Similarly $\iota_{\Gamma/e} = \iota_\Gamma|_{H(\Gamma/e)}$.

If we do multiple contractions of edges $e_1, e_2, ..., e_n$ of Γ in their respective order then we write $\Gamma/(e_1, e_2, ..., e_n)$.

Example 5.4. Let Γ be the graph from the previous example. Consider the edges $e_1 = \{h_4, h_7\}, e_2 = \{h_5, h_8\}, e_3 = \{h_6, h_9\}$. Then the graph Γ/e_1 We will give an example of edge contractions for Γ. The there graphs drawn in figure 5.2 are the graph Γ and graph Γ/e which is obtain by contracting e_1. This implies that the edges e_2 and e_3 become loops in the drawn graph.

Edge contraction gives one way to define the differential in a graph complex. In order to use this though we will need a notion of isomorphisms of graphs.
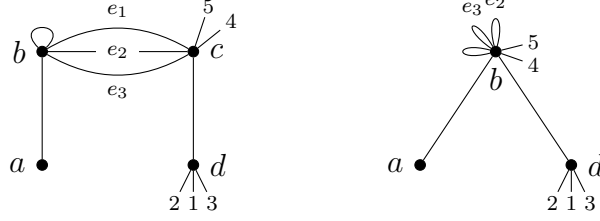
Figure 5.2: Edge contractions

Definition 5.5. We will say that an isomorphism of graphs $(f_V, f_H)\colon \Gamma_1 \to \Gamma_2$ is a pair of bijections $f_V\colon V(\Gamma_1) \to V(\Gamma_2)$ and $f_H\colon H(\Gamma_1) \to H(\Gamma_2)$ such that $f_V \circ \partial_{\Gamma_1} = \partial_{\Gamma_2} \circ f_H$ and $f_H \circ \iota = \iota \circ f_H$.

Now to define a graph complex rigorously we will need to orientate or order our edges. Let $\Gamma = (V, H, \partial, \iota)$ be a graph. For succinctness let use the notation $E = E(\Gamma)$ for the edges and $L = L(\Gamma)$. To order edges consider the vector space $\det(\mathbb{Q}^E) = \Lambda^{|E|}(\mathbb{Q}^E)$. This space is a one dimensional space spanned by $e_1 \wedge e_2 \wedge ... \wedge e_m$ where $E = \{e_1, ..., e_m\}$. Every other such word in $\det(\mathbb{Q}^E)$ can be obtain by a permutation of $e_1 \wedge e_2 \wedge ... \wedge e_m$ up to a sign. Choosing a unit vector in $\det(\mathbb{Q}^E)$ is equivalent to choosing a total ordering of the elements in E. Since $\det(\mathbb{Q}^E)$ is important for us we define $\mathfrak{E}(\Gamma) = \det(\mathbb{Q}^E)$. Similarly we want an ordering of the legs as the legs can be permuted so we define $\mathfrak{L}(\Gamma) = \det(\mathbb{Q}^L)$. As with the edges case the unit vector of $\mathfrak{L}(\Gamma)$ is an ordering of the legs. Assume now we chosen a fixed orderings $\nu \in \mathfrak{E}(\Gamma)$ and $\omega \in \mathfrak{L}(\Gamma)$. Let $e \in E$ be an edge. Contraction of e induces a linear map $-/e\colon \mathfrak{E}(\Gamma) \to \mathfrak{E}(\Gamma/e)$ defined by $-/e\colon e \wedge w \mapsto w$ where w is a word on the letters $E \setminus \{e\}$ in $\mathfrak{E}(\Gamma/e) = \Lambda^{|E|-1}(\mathbb{Q}^{E\setminus\{e\}})$. This is well-defined since $e \wedge w$ spans $\det(\mathbb{Q}^E)$. However the map $\mathfrak{L}(\Gamma) \to \mathfrak{L}(\Gamma/e)$ induced by contraction is an isomorphism. Isomorphism's $\Gamma_1 \cong \Gamma_2$ of graphs induces isomorphism of $\mathfrak{E}(\Gamma_1) \cong \mathfrak{E}(\Gamma_2)$ and $\mathfrak{L}(\Gamma_1) \cong \mathfrak{L}(\Gamma_2)$. We are now ready to define our first graph complex. Let S be a finite set. We can then define the graph complex

$$\mathrm{Grph}(S)_n = \left(\bigoplus_{\substack{\text{graphs } \Gamma, |E(\Gamma)|=n \\ \ell\colon L(\Gamma) \xrightarrow{\cong} S}} \mathfrak{E}(\Gamma) \otimes \mathfrak{L}(\Gamma) \right) \Big/ \cong$$

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We denote elements of $\mathrm{Grph}(S)_*$ by $(\Gamma, \ell, \nu, \omega)$ where Γ is a graph, $\ell \colon L(\Gamma) \to S$ is a bijection, $\nu \in \mathfrak{E}(\Gamma)$ and $\omega \in \mathfrak{L}(\Gamma)$. The differential $d_{con} \colon \mathrm{Grph}(S)_* \to \mathrm{Grph}(S)_{*-1}$ is given by

$$d_{con}(\Gamma, \ell, \nu, \omega) = \sum_{e \in E(\Gamma)} (\Gamma/e, \ell, \nu/e, \omega).$$

There is a variant of the graph complex defined graph complexes by using colouring's.

Definition 5.6. A *black-red colouring* of a graph Γ is a function $c \colon E(\Gamma) \to \{R, B\}$ which colours each edge red or black. A graph Γ along with a black-red colouring $c \colon E(\Gamma) \to \{R, B\}$ is called a *black-red graph* or a *black and red graph*.

Given a black-red graph (Γ, c) we will denote the set of black edges by $E_B(\Gamma)$ and the set of red edges by $E_R(\Gamma)$.

Definition 5.7. Let (Γ, c) be a black-red graph and let $e \in E_B(\Gamma)$. We define the black-red graph $(\Gamma \backslash e, c_e)$ to have the underlining graph $\Gamma \backslash e = \Gamma$ and the colouring $c_e \colon E(\Gamma) \to \{R, B\}$ by $c_e = c$ on $E_B(\Gamma) \setminus \{e\}$ and $c_e(e) = R$.

We then wish to define a graph complex $\mathrm{Grph}^{rb}(S)_*$ using black and red graphs where the differential is given by switching the colours. Again we will need to orientate the edges in order to give a well-defined chain complex. To this end we define $\mathfrak{E}_B(\Gamma) = \det(\mathbb{Q}^{E_B(\Gamma)})$. Note that switching the colour of an edge $e \in E_B(\Gamma)$ then defines a map $-\backslash e \colon \mathfrak{E}_B(\Gamma) \to \mathfrak{E}_B(\Gamma \backslash e)$ defined by $-\backslash e \colon b \wedge w \mapsto w$ where w is a word in $\mathfrak{E}_B(\Gamma \backslash e)$. We can then define another graph complex $\mathrm{Grph}^{rb}(S)_*$ using coloured graphs

$$\mathrm{Grph}^{rb}(S)_n = \left(\bigoplus_{\substack{(\Gamma, c), |E_B(\Gamma)|=n \\ \ell \colon L(\Gamma) \xrightarrow{\cong} S}} \mathfrak{E}(\Gamma) \otimes \mathfrak{L}(\Gamma) \right) \Big/ \cong$$

where the differential $d_{col} \colon \mathrm{Grph}^{rb}(S)_* \to \mathrm{Grph}^{rb}(S)_{*-1}$ is given by

$$d_{col}(\Gamma, \ell, \nu, \omega) = \sum_{e \in E_B(\Gamma)} (\Gamma \backslash e, \ell, \nu \backslash e, \omega).$$

The tools of graph complexes and graph homology are very useful.

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5.1.2 Models for $\mathcal{E}_n$ and $\mathcal{Z}_n$

Now that we have given examples of how to construct graph complexes we will now construct graph complexes which we shall use to model $\mathcal{E}_n$ and $\mathcal{Z}_n$.

Definition 5.8. A *weight* function is a function $w\colon V(\Gamma) \to \mathbb{N}$ which assigns each vertex a weight. A *weighted red and black* graph $(\Gamma, c_\Gamma, w_\Gamma)$ is a graph Γ together with a colouring c_Γ and a weight function w_Γ.

We will not refer to any other types of weights in this chapter so there should be no confusion. We call the edges of a weighted black and red graph $(\Gamma, c_\Gamma, w_\Gamma)$ red edges if they are in $E_R(\Gamma) = c_\Gamma^{-1}(r)$ and we call edges in $E_B(\Gamma) = c_\Gamma^{-1}(b)$ black edges. An isomorphism of weighted black and red graphs is pair of bijections which are an isomorphism of graphs such that they commute with the colouring and weight functions.

Definition 5.9. A weighted black-red graph $(\Gamma, c_\Gamma, w_\Gamma)$ is said to be admissible if for each $v \in V(\Gamma)$ we have that $2w(v) + |\partial_\Gamma^{-1}(\{v\})| \geq 3$, where $|\partial_\Gamma^{-1}(\{v\})|$ is the cardinality of the set $\partial_\Gamma^{-1}(\{v\})$.

When we do contraction of weighted black and red graphs we will also need to keep track of the weights and colourings. This leads to the following definition.

Definition 5.10. Let Γ be a weighed black and red graph. Let $e = \{h_1, h_2\} \in E_B(\Gamma)$. We give the graph Γ/e the induced colouring. We then define the weight function on Γ/e by

   i) if $\partial_\Gamma(h_1) \neq \partial_\Gamma(h_2)$ then the new vertex is given the weight $w(\partial_\Gamma(h_1)) + w(\partial_\Gamma(h_2))$,

   ii) if $\partial_\Gamma(h_1) = \partial_\Gamma(h_2)$ then the new vertex is given the weight $w(\partial_\Gamma(h_1)) + 1$.

   We give an example of what we have just mentioned.

Example 5.11. Let $\Gamma = (V, H, \iota, \partial)$ be the graph as in the previous example. We will define a colouring on Γ.

Definition 5.12. For an admissible weighted red and black graph Γ define the degree of the graph Γ to be

$$\deg(\Gamma) = n \left(\sum_{v \in V(\Gamma)} 2w_\Gamma(v) + \mathsf{val}(v) - 2 \right) + |E_B(\Gamma)|$$

.
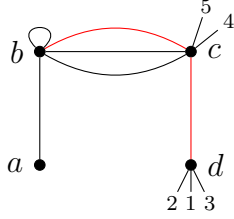
Figure 5.3: A weighted red and black graph

Now we will define the functors $\mathcal{R}_{\mathcal{Z}_n}, \mathcal{R}_{\mathcal{E}_n}\colon \mathrm{dB} \to \mathbf{Ch}_{\mathbb{Q}}$, similar to how we defined the graph complexes $(\mathrm{Grph}(S)_*, d_{con})$ and $(\mathrm{Grph}^{rb}(S)_*, d_{col})$. Starting with $\mathcal{R}_{\mathcal{E}_n}$ for a set $S \in \mathrm{dB}$ let

$$\mathcal{R}_{\mathcal{E}_n}(S)_n = \left(\bigoplus_{\substack{\mathrm{admisible}(\Gamma, w, c), |E_B(\Gamma)| = n \\ L(\Gamma) \cong S}} \mathfrak{E}_B(\Gamma) \otimes \mathfrak{L}(\Gamma)^{\otimes n}[\deg(\Gamma)] \right) \Big/ \cong$$

where we sum over admissible weighted black-red graphs with legs identified with S and the number of black edges being n. We again mod out by the isomorphic vector spaces and graphs. We can denote elements of $\mathcal{R}_{\mathcal{E}_n}(S)$ by $(\Gamma, \ell, \nu \otimes \omega^{\otimes n})$ where Γ is an admissible graph, $\ell\colon L(\Gamma) \to S$ is a bijection, $\nu \in \mathfrak{E}_B(\Gamma)$ and $\omega^{\otimes n} \in \mathfrak{L}(\Gamma)[\deg(\Gamma)]$. We define the maps d_{con}, d_{col} on $\mathcal{R}_{\mathcal{E}_n}(S)$ by

$$d_{con}(\Gamma, \ell, \nu \otimes \omega^{\otimes n}) = \sum_{e \in E_B(\Gamma)} (\Gamma, \ell, (\nu/e) \otimes \omega^{\otimes n})$$

$$d_{col}(\Gamma, \ell, \nu \otimes \omega^{\otimes n}) = - \sum_{e \in E_B(\Gamma)} (\Gamma, \ell, (\nu/e) \otimes \omega^{\otimes n})$$

then we define the differential d on $\mathcal{R}_{\mathcal{E}_n}(S)$ by setting $d = d_{con} + d_{col}$.

We define $\mathcal{R}_{\mathcal{Z}_n}$ to be a quotient of $\mathcal{R}_{\mathcal{E}_n}$ by all weighted red and black graphs with a vertex of weight greater than zero. More explicitly, $\mathcal{R}_{\mathcal{Z}_n}$ is given by weighted red and black graphs have all weights equal to zero. The differential is given by the same formula with the added condition that Γ/e is zero when e is a loop.

We can use the disjoint union of graphs to make $\mathcal{R}_{\mathcal{E}_n}$ and $\mathcal{R}_{\mathcal{Z}_n}$ into commutative algebra objects.

Proposition 5.13. *The algebra objects* $\mathcal{R}_{\mathcal{E}_n}$ *and* $\mathcal{R}_{\mathcal{Z}_n}$ *are quasi-free and hence cofibrant.*

Proof. Let $\mathcal{X} \subseteq \mathcal{R}_{\mathcal{E}_n}$ be sub-object spanned by those weighted red and black graphs whose black graph is connected and nonempty. Neglecting the differentials for a moment we get that the commutative algebra structure on $\mathcal{R}_{\mathcal{E}_n}$ induces a map

$$\Lambda^+(\mathcal{X}) = \bigoplus_{k \geq 0} (\mathcal{X}^{\otimes k})_{\mathfrak{S}_k} \to \mathcal{R}_{\mathcal{E}_n}$$

which we will prove is an isomorphism. By the definition of Day convolution we have that

$$\mathcal{X}^{\otimes k}(S) = \operatorname*{colim}_{S_1 \sqcup \ldots \sqcup S_k \to S \in \mathrm{dB}} \mathcal{X}(S_1) \otimes \ldots \otimes \mathcal{X}(S_k)$$

This means that $\mathcal{X}^{\otimes k}(S)$ is the vector space of weighted red and black graphs with legs S whose black subgraphs have exactly k components which are ordered. This implies that $(\mathcal{X}^{\otimes k})_{\mathfrak{S}_n}(S)$, which is invariant under permutations, is the vector space of weighted red and black graphs with legs S whose black subgraphs have exactly k unordered components. Since we defined $\mathcal{R}_{\mathcal{E}_n}(S)$ to be the complex of all admissible black and red graphs with legs S it is not hard to see how we may decompose $\mathcal{R}_{\mathcal{E}_n}(S)$ by the number of connected components of the black subgraphs. Including the differentials proves that $\mathcal{R}_{\mathcal{E}_n}$ is quasi-free. The same argument applies to the proof for $\mathcal{R}_{\mathcal{Z}_n}^C$. $\qquad\square$

Proposition 5.14. *There are weak equivalences*

$$\mathcal{R}_{\mathcal{Z}_n} \xrightarrow{\sim} \mathcal{Z}_n \quad \& \quad \mathcal{R}_{\mathcal{E}_n} \xrightarrow{\sim} \mathcal{E}_n$$

where $\mathcal{R}_{\mathcal{Z}_n}$ *and* $\mathcal{R}_{\mathcal{E}_n}$ *are cofibrant in* $\mathsf{Alg}(\mathbf{Ch}_{\mathbb{Q}}^{\mathrm{dB}})$.

Proof. We prove the result for $\mathcal{E}_n$. The result for $\mathcal{Z}_n$ is proved using the same arguments. We will give the map for $\mathcal{Z}_n$ at the end. For the $\mathcal{E}_n$ case we define $\phi \colon \mathcal{R}_{\mathcal{E}_n} \to \mathcal{E}_n$ to be the map which sends any graphs with a black edge to zero. For graphs with only red edges ϕ assigns each graph to the partition induced by connected components. That is for each connected component Γ_α with legs L_α we get a block L_α of the partition of $L = L(\Gamma)$ we give this block the weight $(\sum_{v \in V(\Gamma_\alpha)} w(v)) + 1 - \chi(\Gamma_\alpha)$.

This defines a map of commutative algebra objects. We also get a map $\phi_S \colon \mathcal{R}_{\mathcal{E}_n}(S) \to \mathcal{E}_n(S)$ of chain complexes. In particular ϕ_S respects the differential. To see this consider a graph $(\Gamma, \ell, \nu, \omega^{\otimes n})$. If $(\Gamma, \ell, \nu, \omega^{\otimes n})$ has at least two black edges then every graph in the sum $d(\Gamma, \ell, \nu, \omega^{\otimes n})$ has at least one black edge so $d(\Gamma, \ell, \nu, \omega^{\otimes n})$ vanishes under ϕ_S. If $(\Gamma, \ell, \nu, \omega^{\otimes n})$ has only one edge then

$$d(\Gamma, \ell, \nu, \omega^{\otimes n}) = (\Gamma/e, \ell, 1, \omega^{\otimes n}) - (\Gamma \setminus e, \ell, 1, \omega^{\otimes n})$$

but the graphs $(\Gamma/e, \ell, 1, \omega^{\otimes n}), (\Gamma \setminus e, \ell, 1, \omega^{\otimes n})$ induce the same partitions so both $(\Gamma, \ell, \nu, \omega^{\otimes n})$ and $d(\Gamma, \ell, \nu, \omega^{\otimes n})$ vanish under ϕ_S.

Now we show that ϕ is a weak equivalence. Let $F^p \mathcal{R}_{\mathcal{E}_n}(S)$ be spanned by those graphs with $\leq p$ edges. This defines a filtration on $\mathcal{R}_{\mathcal{E}_n}(S)$. Since d_{con} reduces the number of edges while d_{col} preserves the number of edges, we have that

$$\mathrm{Gr}(\mathcal{R}_{\mathcal{E}_n}(S), d_{con} + d_{col}) \cong (\mathcal{R}_{\mathcal{E}_n}(S), d_{col})$$

for the associated graded $\mathrm{Gr}(\mathcal{R}_{\mathcal{E}_n}(S), d_{con} + d_{col})$.

This chain complex $\mathrm{Gr}(\mathcal{R}_{\mathcal{E}_n}(S), d_{con} + d_{col})$ splits over isomorphism classes of weighted graphs Γ as a sum of chain complexes $\mathrm{Red}_*(\Gamma)_{\mathrm{Aut}(\Gamma)}$, where $\mathrm{Red}_*(\Gamma)$ has the basis the set of subgraphs of Γ. We consider this basis to be the set of subgraphs whose edges we shall colour red. That means that each subgraph Γ_α of Γ is given the weight $n(\sum_{v \in V(\Gamma)}(2w(v) + \mathsf{val}(v) - 2)) + |E_B(\Gamma)|$. The differential on $\mathrm{Red}_*(\Gamma)$ sums over ways of adding a single red edge to the red subgraph. Up to a degree shift we can identify $\mathrm{Red}_*(\Gamma)$ with the simplicial chain complex on the set of edges of Γ. Thus $\mathrm{Red}_*(\Gamma)$ is acyclic if Γ contains an edge or $\mathrm{Red}_*(\Gamma)$ has homology $\mathbb{Q}[n\sum_{v \in V(\Gamma)}(2w(v) + \mathsf{val}(v) - 2)]$ if Γ has no edges. Since $\mathrm{Red}_*(\Gamma)$ is supported on graphs without edges and the contraction differential d_{con} has no effect. This causes the spectral sequence of this fibration to collapse, this gives us

$$H_*(\mathcal{R}_{\mathcal{E}_n}(S), d) \cong \bigoplus_{\substack{\text{weighted graphs} \\ \text{with no edges}}} \mathbb{Q}[n\sum_{v \in V(\Gamma)}(2w(v) + \mathsf{val}(v) - 2)]$$

which implies that $\phi_{S,*} \colon H_*(\mathcal{R}_{\mathcal{E}_n}(S), d) \to H_*(\mathcal{E}_n(S)) = \mathcal{E}_n(S)_*$ is an isomorphism. This proves that ϕ is a weak equivalence since S was arbitrary. This concludes the proof for the $\mathcal{E}_n$ case for the $\mathcal{Z}_n$ case we do the same. The $\mathcal{Z}_n$

case we can neglect the arguments regarding the weights since $\mathcal{Z}_n$ is not defined with weighted partition and similarly $\mathcal{R}_{\mathcal{Z}_n}$ is not defined with weighted vertices. $\qquad\square$

Let $\mathcal{R}^c_{\mathcal{E}_n}$ be the quotient of $\mathcal{R}_{\mathcal{E}_n}$ by the unit and the graphs which are have black subgraphs which are not connected. Let $\mathcal{R}^c_{\mathcal{Z}_n}$ denote the analogous quotient of $\mathcal{R}_{\mathcal{Z}_n}$.

Corollary 5.15. *We have that $LQ(\mathcal{Z}_n) \cong \mathcal{R}^c_{\mathcal{Z}_n}$ and $LQ(\mathcal{E}_n) \cong \mathcal{R}^c_{\mathcal{E}_n}$.*

This is an application of the definitions.

Definition 5.16. For each finite set S let $G^{\mathcal{E}_n}(S)$ be the quotient of $\mathcal{R}^c_{\mathcal{E}_n}(S)$ by the subcomplex spanned by those weighted red and black graphs having a non-zero amount of red edges. Similarly let $G^{\mathcal{Z}_n}(S)$ denote the analogous quotient of $\mathcal{R}^c_{\mathcal{Z}_n}(S)$ by red and black graphs having a non-zero amount of red edges.

The complexes $G^{\mathcal{E}_n}$ and $G^{\mathcal{Z}_n}$ inherit an internal grading which gives an internal and a total grading to the homology groups. However the reader may notice that we did not define $G^{\mathcal{E}_n}$ and $G^{\mathcal{Z}_n}$ on morphisms. This is because $G^{\mathcal{E}_n}$ and $G^{\mathcal{Z}_n}$ are only functorial on bijections. That is $G^{\mathcal{E}_n}, G^{\mathcal{Z}_n}$ are functors from **FB**, the $\mathbb{Q}$-linearised category of finite sets and bijections to $\mathbf{Ch}_{\mathbb{Q}}$. If we ignore the differential every graph with a connected black subgraph can be obtained by adding a red edge to the connected black subgraph. We can view $G^{\mathcal{E}_n}$ and $G^{\mathcal{Z}_n}$ as complexes of connected graphs with only black edges. From our observation about adding red edges to the connected black graphs we get a decomposition

$$\mathcal{R}^c_{\mathcal{E}_n}(-) \cong \bigoplus_n G^{\mathcal{E}_n}([n]) \otimes_{\mathfrak{S}_n} \mathrm{dB}([n], -)$$

$$\mathcal{R}^c_{\mathcal{Z}_n}(-) \cong \bigoplus_n G^{\mathcal{Z}_n}([n]) \otimes_{\mathfrak{S}_n} \mathrm{dB}([n], -)$$

where the $\mathfrak{S}_n$-tensor by $\mathrm{dB}([n], -)$ can be viewed as adding new red edges (we take the $\mathfrak{S}_n$-tensor as we have not orientated the red edges).

There is an inclusion $\mathcal{I} \colon \mathbf{FB} \to \mathrm{dB}$ which sends finite sets to themselves and bijections $f \colon S \to T$ to morphisms $(f, \varnothing) \colon T \to S$ with the empty matching. Define the functor $\mathcal{J} \colon \mathrm{dB} \to \mathbf{FB}$ to be the identity functor on

sets. On morphisms we defined $\mathcal{J}$ by sending $(f, m_S)\colon S \to T$ to $\mathcal{J}(f, m_S) = f\colon T \to S$ if $|S| = |T|$ and $m_S = \varnothing$. If $m_S \neq \varnothing$ and $|S| \neq |T|$ then $\mathcal{J}(f) = 0 \in \mathrm{dB}(S, T)$. The functor $\mathcal{J}$ is a retraction of $\mathcal{I}$ since for all $S \in \mathbf{FB}$ we have that $\mathcal{J} \circ \mathcal{I}(S) = \mathcal{J}(S) = S$ and $f \in \mathbf{FB}(S, T)$ then $\mathcal{J} \circ \mathcal{I}(f) = \mathcal{J}(f, \varnothing) = f$. Naturally we get functors $\mathcal{J}^*\colon \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{FB}} \to \mathbf{Ch}_{\mathbb{Q}}^{\mathrm{dB}}$ and $\mathcal{I}^*\colon \mathbf{Ch}_{\mathbb{Q}}^{\mathrm{dB}} \to \mathbf{Ch}_{\mathbb{Q}}^{\mathbf{FB}}$. Using left Kan extensions we get left adjoints $\mathcal{J}_*$ and $\mathcal{I}_*$ to the functors $\mathcal{J}^*$ and $\mathcal{I}^*$. From chapter 1 we know that $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{FB}}$ has a model structure. It is not hard to see that $\mathcal{J}_* \dashv \mathcal{J}^*$ is a Quillen adjunction. This means that $\mathcal{J}_*$ admits a left derived functor $L\mathcal{J}_*$.

Lemma 5.17. *The maps*

$$L\mathcal{J}_*(\mathcal{R}_{\mathcal{E}_n}^c) \to \mathcal{J}_*(\mathcal{R}_{\mathcal{E}_n}^c) = G^{\mathcal{E}_n}$$

$$L\mathcal{J}_*(\mathcal{R}_{\mathcal{Z}_n}^c) \to \mathcal{J}_*(\mathcal{R}_{\mathcal{Z}_n}^c) = G^{\mathcal{Z}_n}$$

are weak equivalences in $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{FB}}$.

Proof. We only prove the $\mathcal{E}_n$ case, proof for $\mathcal{Z}_n$ is identical. We can define a filtration on $\mathcal{R}_{\mathcal{E}_n}^c$ by its grading by defining $\mathcal{F}^p \mathcal{R}_{\mathcal{E}_n}^c(S) = \bigoplus_{i=0}^p (\mathcal{R}_{\mathcal{E}_n}^c(S))_p$ for all finite sets S. This induces a filtration on $L\mathcal{J}_*(\mathcal{R}_{\mathcal{E}_n}^c)$ by

$$\mathcal{F}^* L\mathcal{J}_*(\mathcal{R}_{\mathcal{E}_n}^c) = L\mathcal{J}_*(\mathcal{F}^* \mathcal{R}_{\mathcal{E}_n}^c)$$

as well as a filtration on $\mathcal{R}_{\mathcal{E}_n}^c$ where

$$\mathcal{F}^* L\mathcal{J}_*(\mathcal{R}_{\mathcal{E}_n}^c) = L\mathcal{J}_*(\mathcal{F}^* \mathcal{R}_{\mathcal{E}_n}^c).$$

This gives a spectral sequence defined by the filtration. Since $\mathcal{J}_*$ preserves exact sequences the map

$$L\mathcal{J}_*(\mathcal{R}_{\mathcal{E}_n}^c) \to \mathcal{J}_*(\mathcal{R}_{\mathcal{E}_n}^c) = G^{\mathcal{E}_n}$$

is a map of E^1 pages. We need to show that this map is an equivalence. Notice that from the decomposition of $\mathcal{R}_{\mathcal{E}_n}^c$ we have that $\mathrm{Gr}(\mathcal{R}_{\mathcal{E}_n}^c) = \mathcal{I}_*(G^{\mathcal{E}_n})$ where $\mathcal{I}_*(G^{\mathcal{E}_n})$ has trivial differential. As we are working over a field of characteristic zero all objects in $\mathbf{Ch}_{\mathbb{Q}}^{\mathbf{FB}}$ cofibrant so $L\mathcal{I}_*(G^{\mathcal{E}_n}) \overset{\cong}{\to} \mathcal{I}_*(G^{\mathcal{E}_n})$. Therefore using $\mathcal{J}_* \circ \mathcal{I}_* = id_{\mathbf{FB}}$, and similarly $L\mathcal{J}_* \circ L\mathcal{I}_* = id_{\mathbf{FB}}$ we have that the claim follows. $\qquad\square$

5.1.3 End of the proof

We now conclude the proof of the main theorem. To do this we will make use of a spectral sequence.

Proposition 5.18. *For all $\mathcal{A}\colon \mathrm{dB} \to \mathbf{Ch}_{\mathbb{Q}}$ there exists a spectral sequence $E^*_{p,q}$ which converges strongly to $H_{s+t}(L\mathcal{J}_*(\mathcal{A}))$. The E^1-page is given by*

$$E^1_{s,t} = \bigoplus_{S_0,\ldots,S_s \in \mathrm{dB}} H_t(\mathcal{A})(S_s) \otimes \mathrm{dB}(S_{s-1}, S_s) \otimes \ldots \otimes \mathrm{dB}(S_0, S_1) \otimes \mathbf{FB}(S_0, -)$$

and the d^1-differential is given by the alternating sum of the face maps. Furthermore if $\mathcal{A}$ has an extra grading this gets carried along with as well.

Proof. Consider $L\mathcal{J}_*(\mathcal{A})$ where $\mathcal{A}\colon \mathrm{dB} \to \mathbf{Ch}_{\mathbb{Q}}$. Recall that the Bar construction model homotopy colimits. Using this we get a simplicial object with m-simplices

$$\underline{m} \mapsto \bigoplus_{S_0,\ldots,S_m \in \mathrm{dB}} \mathcal{A}(S_m) \otimes \mathrm{dB}(S_{m-1}, S_m) \otimes \ldots \otimes \mathrm{dB}(S_0, S_1) \otimes \mathbf{FB}(\mathcal{J}(S_0), -)$$

which is the geometric realisation of $L\mathcal{J}_*(\mathcal{A})$. We filter the geometric realisation by skeleta to give a spectral sequence

$$E^1_{s,t} = \bigoplus_{S_0,\ldots,S_s \in \mathrm{dB}} H_t(\mathcal{A})(S_s) \otimes \mathrm{dB}(S_{s-1}, S_s) \otimes \ldots \otimes \mathrm{dB}(S_0, S_1) \otimes \mathbf{FB}(S_0, -).$$

The spectral sequence converges strongly to $H_{s+t}(L\mathcal{J}_*(\mathcal{A}))$ due to $\mathcal{A}$ being non-negatively graded. $\qquad\square$

Applying the above proposition to $\mathcal{R}^c_{\mathcal{E}_n}$ gives a spectral sequence with E^1-page

$$E^1_{s,t,q} = \bigoplus_{S_0,\ldots,S_s \in \mathrm{dB}} H_{t-q+1}(\mathcal{E}_n)_q(S_s) \otimes \mathrm{dB}(S_{s-1}, S_s) \otimes \ldots \otimes \mathrm{dB}(S_0, S_1) \otimes \mathbf{FB}(S_0, -).$$

This converges to $H_{s+t}(G^{\mathcal{E}_n})_q$. Applying the proposition to $\mathcal{R}^c_{\mathcal{Z}_n}$ produces a similar spectral sequence. Now we are ready to finish the proof of the main theorem.

Lemma 5.19. *If $n \cdot (p+1) < q$ and $T \neq \varnothing$ then the homology $H_{p+q}(G^{\mathcal{E}_n})_q$ and $H_{p+q}(G^{\mathcal{Z}_n})_q$ vanishes.*

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Proof. Let $T \neq \varnothing$. As we have mentioned we apply Proposition 5.18 to $\mathcal{R}^c_{\mathcal{E}_n}$ for a fixed q and T. To get a spectral sequence which starts from

$$E^1_{s,t,q} = \bigoplus_{S_0,\ldots,S_s \in \mathrm{dB}} H_{t-q+1}(\mathcal{E}_n)_q(S_s) \otimes \mathrm{dB}(S_{s-1}, S_s) \otimes \ldots \otimes \mathrm{dB}(S_0, S_1) \otimes \mathbf{FB}(S_0, T)$$

and converges to $H_{s+t}(G^{\mathcal{E}_n})_q(T)$. By the definition of dB the parts of the tensor

$$\mathrm{dB}(S_{s-1}, S_s) \otimes \ldots \otimes \mathrm{dB}(S_0, S_1) \otimes \mathbf{FB}(S_0, T)$$

are non-zero if we have that

$$|S_s| \geq \ldots \geq |S_1| \geq |S_0| = |T|$$

so $T \neq \varnothing$ implies that $S_s \neq \varnothing$. By Theorem **??** for $H_{t-q+1}(\mathcal{E}_n)_q(S_s)$ to be non-zero we must have $q = n(t - q + 1)$. If $q \neq n(t - q + 1)$ then $E^1_{s,t,q} = 0$. Rearranging this implies that if $t < q(n+1)/n - 1$ then $E^1_{s,t,q} = 0$. Therefore $H_{s+t}(G^{\mathcal{E}_n})_q = 0$ when $s + t < q(n+1)/n - 1$. Setting $s + t = p + q$ and rearranging gives $n(p+1) < q$. This proves the theorem. $\qquad \square$

The next result was proved by Turchin and Willwacher [**empty citation**] for $\mathcal{Z}_n$. It was then proved for $\mathcal{E}_n$ by Chan, Galatius and Payne [**empty citation**]. The proof itself is not hard but it is long so we will not explain it. The interested reader can also consult our main source [40] for a proof.

Lemma 5.20. *There are injections*

$$H_{p+q}(G^{\mathcal{E}_n})_q(\varnothing) \to H_{p+q+n}(G^{\mathcal{E}_n})_{q+n}([1])$$

$$H_{p+q}(G^{\mathcal{Z}_n})_q(\varnothing) \to H_{p+q+n}(G^{\mathcal{Z}_n})_{q+n}([1])$$

We can derive the following

Corollary 5.21. *If $np < q$ then $H_{p+q}(G^{\mathcal{E}_n})_q(\varnothing) = 0$ and $H_{p+q}(G^{\mathcal{Z}_n})_q(\varnothing) = 0$.*

Proof. By Lemma 5.19 since $[1] = \{1\} \neq \varnothing$, if $n(p+1) < q + n$ then $H_{p+q}(G^{\mathcal{E}_n})_{q+n}([1]) = 0$ and $H_{p+q}(G^{\mathcal{Z}_n})_{q+n}([1]) = 0$. By Lemma 5.20 $H_{p+q}(G^{\mathcal{E}_n})_q(\varnothing)$ and $H_{p+q}(G^{\mathcal{Z}_n})_q(\varnothing)$ are subspaces of $H_{p+q}(G^{\mathcal{E}_n})_{q+n}([1])$ and $H_{p+q}(G^{\mathcal{Z}_n})_{q+n}([1])$ respectively. This implies the result. $\qquad \square$

We can use this Corollary together with the spectral sequence to prove the vanishing of the homology of $\mathcal{Z}_n$ and $\mathcal{E}_n$.

Theorem 5.22. *If $np < q$ then $H_p(\mathcal{E}_n)_q(\varnothing)$ and $H_p(\mathcal{Z}_n)_q(\varnothing)$ vanish.*

Proof. We only prove the theorem for $\mathcal{E}_n$ as the $\mathcal{Z}_n$ case is the same. Now we apply the spectral sequence from Proposition 5.18 to a fixed q and $T = \varnothing$ to get the spectral sequence starting with

$$E^1_{s,t,q} = \bigoplus_{S_0,\ldots,S_s \in \mathrm{dB}} H_{t-q+1}(\mathcal{E}_n)_q(S_s) \otimes \mathrm{dB}(S_{s-1}, S_s) \otimes \ldots \otimes \mathrm{dB}(S_0, S_1) \otimes \mathbf{FB}(S_0, \varnothing)$$

and converging strongly to $H_{s+t}(G^{\mathcal{E}_n})_q(\varnothing)$. Set $D_{*,t,q}$ to be the subcomplex of $(E^1_{*,t,q}, d^1)$ with the terms $S_s = \varnothing$. Notice that the only non-zero factors of the direct sum of $D_{*,t,q}$ are those with $S_0 = S_1 = \ldots = S_s$ by the definition of dB. Writing out $D_{*,t,q}$ explicitly gives us

$$D_{s,t,q} = \bigoplus_{S_0,\ldots,S_{s-1} \in \mathrm{dB}} H_{t-q+1}(\mathcal{E}_n)_q(\varnothing) \otimes \mathrm{dB}(S_{s-1}, \varnothing) \otimes \ldots \otimes \mathrm{dB}(S_0, S_1) \otimes \mathbf{FB}(S_0, \varnothing)$$
$$= H_{t-q+1}(\mathcal{E}_n)_q(\varnothing) \otimes \mathbb{Q} \otimes \ldots \otimes \mathbb{Q} \otimes \mathbb{Q}$$
$$= H_{t-q+1}(\mathcal{E}_n)_q(\varnothing)$$

so by the definition of the differentials $D_{*,t,q}$ is the complex

$$H_{t-q+1}(\mathcal{E}_n)_q(\varnothing) \xleftarrow{0} H_{t-q+1}(\mathcal{E}_n)_q(\varnothing) \xleftarrow{id} H_{t-q+1}(\mathcal{E}_n)_q(\varnothing) \xleftarrow{0} H_{t-q+1}(\mathcal{E}_n)_q(\varnothing) \xleftarrow{id} \ldots$$

which has trivial homology in degrees $s > 0$.

Consider the quotient $E^1_{*,t,q}/D_{*,t,q}$. This is the same direct sum as $E^1_{*,t,q}$ except we don't allow for $S_s = \varnothing$. For $H_{t-q+1}(\mathcal{E}_n)_q(S_0)$ to be non-zero we must have that $q = n(t - q + 1)$. This implies that $E^2_{0,t,q} = H_{t-q+1}(\mathcal{E}_n)_q(\varnothing)$ and if $s > 0$ and $q \neq n(t - q + 1)$ then $E^2_{s,t,q} = 0$.

We collect these facts together to reach the conclusion of the proof. We already know, from Lemma 2.65, that if $q < np$ then $H_{p+q}(\mathcal{E}_n)_q(\varnothing)$ and $H_{p+q}(\mathcal{Z}_n)_q(\varnothing)$ vanish. So assume that $q > np$. Notice that there are no non-tirvial differentials leaving $H_{t-q+1}(\mathcal{E}_n)_q(\varnothing) = E^2_{0,p+q-1,q}$. The differentials which go to the $(0, p + q - 1, q)$ position of the sequence come from $E^r_{r,p+q-r,q}$ with $r \geq 2$. For $E^r_{r,p+q-r,q}$ to be non-zero we require that $q = n(p - r + 1) \leq np$ which is impossible due to our assumption that $q > np$. Thus there are no differentials entering $E^2_{0,p+q-1,q}$ and $E^\infty_{0,p+q-1,q} = H_{t-q+1}(\mathcal{E}_n)_q(\varnothing)$. Since $E^\infty_{0,p+q-1,q}$ is a filtered quotient of $H_{p+q-1}(G^{\mathcal{E}_n})_q(\varnothing)$. But from Corollary 5.21 $H_{p+q-1}(G^{\mathcal{E}_n})_q(\varnothing)$ vanishes for $n(p-1) < q$. rearranging gives $np < q+n$ now our assumption was $np < q$ which implies that $np < q < n + q$. Therefore the homology of $H_p(\mathcal{E}_n)_q(\varnothing)$ vanishes when $q \neq np$. $\qquad\square$

The main theorem is now essentially proven but we write out the argument, connecting the main results form each of the chapters.

Main Theorem. *The Lie algebra $\mathrm{Gr}_{lcs}\,\mathfrak{t}_{g,1}$ is Koszul in weight $\leq g/3$.*

Proof. We know from Theorem 3.51 that for $g \geq 4$ and $* \leq g$ the realisation $\mathcal{K}_H^\vee \otimes \mathcal{E}_1/(\kappa_{e^2})$ is the quadratic dual of $\mathrm{Gr}_{lcs}^*\,\mathfrak{t}_{g,1}$. It then follows that it is enough to prove that $\mathcal{E}_1$ is Koszul, Corollary 3.53. Since $\mathcal{E}_1$ and $\mathcal{E}_n$ only differ by multiplying n to the gradings of $\mathcal{E}_n$ we have that $\mathcal{E}_1$ is Koszul if and only if $\mathcal{E}_n$ is Koszul.

In chapter 1 we defined a morphism $\mathcal{E}_n \to \mathcal{Z}_n$ which gave us a long exact sequence in homology. From the long exact sequence in homology we know that $\mathcal{E}_n$ is Koszul if and only if $\mathcal{Z}_n$ is, Corollary 2.88. The realisation $\mathcal{K}_H^\vee \otimes^{\mathrm{dB}} \mathcal{Z}_n$ is isomorphic the $H^*(X_1(g);\mathbb{Q})$ in a stable range. We used this and showed that if $q \neq np$ and S is a non-empty set then $H_p(\mathcal{Z}_n)_{q,w}(S)$ vanishes. Using the long exact sequence we get that if $q \neq np$ and S is a non-empty set then $H_p(\mathcal{E}_n)_{q,w}(S)$ vanishes. Since the weight grading of $\mathcal{Z}_n$ and $\mathcal{E}_n$ has non-zero pieces when $w = q/n$ this implies the vanishing result for $p \neq w$.

All that is left to show is that $H_p(\mathcal{Z}_n)_{q,w}(\varnothing)$ and $H_p(\mathcal{E}_n)_{q,w}(\varnothing)$ vanish when $p \neq w$. We proved this in Theorem 5.22. From Lemma 2.65 we can conclude that $\mathcal{Z}_n$ and $\mathcal{E}_n$ are Koszul proving that $\mathrm{Gr}_{lcs}^*\,\mathfrak{t}_{g,1}$ is Koszul in weight $w \leq g/3$. $\qquad\square$

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Corrections for Masters Thesis:

Chapter 1

- There are a couple of typo's in Definition 1.8. It should state:

  Definition. For every commutative algebra A we define the homology of A by

  $$H_p(A) = H_p(\mathrm{Harr}_*(A))$$

  If A is a graded commutative algebra then the homology inherits the grading from A. We denote the q-th part by

  $$H_p(A)_q$$

  and we shall call this the internal grading. If A is graded and has a weight grading the homology also inherits this weight grading we denote the w-th part by

  $$H_p(A)_{q,w}.$$

  We call this additional grading on $H_p(A)$ the weight grading.

- Definition 1.9 should state "A graded-commutative algebra A with a weight grading is Koszul in weight..." instead.

Chapter 2

- From the comment after Definition 2.56 until Proposition 2.60 the differential on $\mathsf{coLie}(\mathcal{A}[1])$ is not correct. The differential on $\mathsf{coLie}(\mathcal{A}[1])$ is a functorial deformation of the differential on $\mathcal{A}$.

Chapter 5

- Before Lemma 5.20 there are two missing empty citations and reference. It should read "... proved by Turchin and Willwacher [2] for $\mathcal{E}_n$" with the corresponding reference given below. The second should then read "proved for $\mathcal{E}_n$ by Chan, Galatius and Payne [1]."

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