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p-adic numbers, Modular Forms and Representations

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Contents

Introduction

This thesis is the culmination of an study of the paper[3] titled “Formes modulaires et representations de $\mathsf{GL}(2)$” (in English: modular forms and representations of $\mathsf{GL}(2)$) authored by P. Deligne, which aims to complement Robert’s report[10] on Jacquet-Langlands’ [5]. Given the advanced nature of the original works and the author’s limitations, this thesis will primarily concentrate on comprehending the foundational aspects of the subject matter.

The initial step involves introducing the concept of adelic numbers. Adelic is a fundamental concept in the study of automorphic forms. In the context of Adelic numbers, an essential component is the consideration of p-adic numbers denoted as $\mathbb{Q}_p$. These p-adic numbers are completions of the rational numbers $\mathbb{Q}$ with respect to the p-adic valuation. The real number $\mathbb{R}$ can also be regarded as a special case of p-adic numbers, when we take $p = \infty$. Ostrowski’s theorem states that p-adic is the only non-trivial absolute value on $\mathbb{Q}$ when regarding the real absolute value as $|\cdot|_\infty$, And the ring of adelic numbers provide us the theory to study all the valuation at the same time.

From the classical point of view, modular forms are a class of complex analytic functions that possess specific symmetries and growth conditions. Later on, E. Hecke made a significant advancement by associating L-functions with modular forms. He demonstrated[4] that if a modular form satisfies certain conditions and is an eigenfunction of a specific operator, then its associated L-function possesses an Euler product expansion. This motivation prompts a study of modular forms from the point of representation theory and within the framework of adeles[9]. Jacquet and Langlands [5] introduced the significant notion of an admissible representation.

Outline

The first section provides essential background information that serves as a foundation for the subsequent parts. At the beginning, we introduce the concept of topological groups and illustrate it by using $\mathsf{GL}_n(F)$ as a prominent example. Moreover, some important subgroups, including discrete subgroups, totally disconnected groups are studied. In addition, we present some pertinent propositions, accompanied by either proofs or direct statements. These propositions serve as valuable tools for subsequent analysis or aid in enhancing understanding of the topic. The subsequent three sections primarily focus on modular forms. In particular, the utilization of linear fractional transformations enables us to examine modular forms from two distinct perspectives: the upper half-plane and homogeneous coordinates. The classification of these transformations subsequently leads us to the definition and exploration of cusps. Then we define the cusps and show an example in section 1.3. Next, we define the fundamental domain, which serves as a fundamental concept for modular forms. Concluding this section, we discuss Haar measure, which is proven to exist on locally compact groups. Haar measure enables us to define integrals and measure sets on such groups.

In the second section, we study p-adic numbers. We will depart from Deligne's approach to define p-adic numbers, which employs the language of projective limits. Instead, we start with the concept of valuation and consider $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ with respect to the p-adic absolute value. Then we introduce the concept of the restricted product, which preserves the locally compact nature of the individual components within the product. This ultimately leads us to the establishment of the ring of adelic numbers. Alongside the definitions, we also present several topological properties associated with these structures.

In the Last two sections, we define continous representations and study a certein class of it, which is called admissible representations. Furthermore, we introduce modular forms defnined on different groups and discuss admmissible representations on it.

1 Topological Groups

1.1 Definitions and Results

Let $(G, \mathcal{T})$ be a Hausdorff topological space, we call it a *topological group* if the group operations i.e. the multiplication $G \times G \to G : (x, y) \mapsto xy$ and the inverse map $G \to G : g \mapsto g^{-1}$ are continuous.

Let X be a Hausdorff topological space and G acts on X. The action is *transitive* if there is only one orbit. Denote the action by $a : G \times X \to X$, it is said to be *continuous* if a is a continuous map viewed as a map between topological spaces. Under a continuous action, we define the quotient space $G\backslash X$ to be the collection of all the orbits, with the quotient map $\pi : X \to G\backslash X$. The subset $U \subset G\backslash X$ is open iff $\pi^{-1}(U) \subset X$ is open.

Example 1.1. If F is a field such that $(F, +)$ and $(F^{\times}, \cdot)$ are topological groups. Consider $\mathsf{GL}_n(F) \subset M_n(F) \cong F^{n^n}$. Since the determinant map, regarded as a polynomial of entries, is continuous, the inverse map $A \mapsto A^{-1} = \mathrm{adj}A/\det A, A \in \mathsf{GL}_n(F)$ is thus continuous. The continuity of multiplication map follows from our assumption of F and by subgroup topology. Therefore $\mathsf{GL}_n(F)$ is a topological group.

Subgroups.

Proposition 1.2. *Let $H \subset G$ be a subgroup, then*

1. *the quotient map $\pi : G \to G/H$ is open;*
2. *if G is locally compact, then G/H is locally compact; and*
3. *H is closed iff G/H is Hausdorff.*

Proposition 1.3. *Let G be a locally compact group admits a countable basis and acts transitively and continuously on locally compact space X, then the orbit map*

$$\begin{aligned}\mathrm{orb}_x : G/\mathrm{Stab}_G(x) &\to X \\ g &\mapsto gx\end{aligned}$$

is a homeomorphism.

Discrete subgroups. Now we discuss a special kind of closed subgroups of G. Let G be a topological group and its subgroup Γ a discrete subset of G, then Γ is called a *discrete subgroup* of G. By subset topology it can be easily checked that

Lemma 1.4. *Subgroup $\Gamma \subset G$ is discrete iff there is an open subset $U \subset G$ such that $U \cap \gamma = \{1\}$.*

It is essential to familiarize ourselves with the next proposition.

Proposition 1.5. *Any discrete subgroup Γ of topological group G is closed.*

Proof. Pick open $U \subset G$ such that $1 \in U$ and $U \cap \Gamma = \{1\}$. For any fixed $g \notin \Gamma$, by continuity of multiplication, there exists open neighborhood $V \ni g$ such that $VV^{-1} \subset U$. For $x, y \in V$, $xy^{-1} \in \Gamma$ iff. $x = y$. This shows that $|V \cap \Gamma| = 0$ or 1. The first case is trivial. Otherwise, take open W such that $g \in W$ and $x \notin W$. Such W exists by Hausdorff. Then $W \cap V \ni g$ is an open neighborhood of g which does not intersect with Γ.

□

Definition 1.6 (lattice)**.** A *lattice* in $\mathbb{C}$ is a subgroup Λ of $(\mathbb{C}, +)$ such that

$$\Lambda = \mathbb{Z}a \oplus \mathbb{Z}b,$$

where $a, b \in \mathbb{C}$ are $\mathbb{R}$-linearly independent.

Take the canonical basis $\{e_1, e_2\}$ of $\mathbb{Z}^2$. Define the set

$$L := \left\{ g \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}) : g(\mathbb{Z}^2) \subset \mathbb{C} \text{ is a lattice} \right\}.$$

For $g \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C})$, it can be extends to $\mathbb{R}^2$ and by viewing $\mathbb{R}^2 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R}$. This induces a map

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}) &\to \mathrm{Hom}(\mathbb{R}^2, \mathbb{C}) \\ g &\mapsto g_{\mathbb{R}}. \end{aligned}$$

In addition, we can identity $L \subset \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C})$ as $\mathrm{Isom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C})$.

Proof. For $g \in L$, the image $g(z^2)$ is a lattice if and only if $g(\mathbb{Z}^2)$ contains a basis of $\mathbb{C}$ over $\mathbb{R}$. On the other hand, by our construction of $g_{\mathbb{R}}$, We thus have that $\{g(e_1), g(e_2)\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$, which is equivalent to say that $g_1 := g(e_1)$ and

$g_2 \coloneqq g(e_2)$ are linearly independent. it is surjective, and the fact that $\dim_{\mathbb{R}} \mathbb{R}^2 = \dim_{\mathbb{R}} \mathbb{C} = 2$ implies that $g_{\mathbb{R}} \in \mathrm{Isom}(\mathbb{R}^2, \mathbb{C})$.

Therefore we have a map

$$\begin{aligned} \Phi : L &\to \mathrm{Isom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C}) \\ g &\mapsto g_{\mathbb{R}}. \end{aligned}$$

For $h_{1,\mathbb{R}}, h_{2,\mathbb{R}} \in \mathrm{Isom}(\mathbb{R}^2, \mathbb{C})$, if $h_{1,\mathbb{R}} = h_{2,\mathbb{R}}$, then for all $a, b \in \mathbb{R}$, the equation $h_1(ae_1 + be_2) = h_2(ae_2 + be_2)$ holds. But since $\mathbb{Z} \subset \mathbb{R}$, this implies that $h_1(ae_1 + be_2) = h_2(ae_2 + be_2)$ also holds for all $a, b \in \mathbb{Z}$. Therefore $h_1 = h_2$. This shows the injectivity. Now we check the surjectivity. For $T \in \mathrm{Isom}(\mathbb{R}^2, \mathbb{C})$, it is uniquely determined by $T(e_1)$ and $T(e_2)$ and we have $T(ae_1 + be_2) = aT(e_1) + bT(e_2)$ for any $a, b \in \mathbb{R}$. Define $g^T(e_i) = T(e_i)$ for $i \in \{1, 2\}$. Then $\Phi(g^T) = g^T_{\mathbb{R}} = T$. It remains to check that $g^T \in T$, this is true if and only if $g^T(\mathbb{Z}^2)$ is a lattice, which is ensured by the choice of T. □

The above isomorphism is one of the four interpretations of the set L, showed in 1.1.1 of [3]. Furthermore, we can identify G to $\mathsf{GL}_2(\mathbb{R})$ by $\mathsf{GL}_2(\mathbb{R}) \ni h \mapsto (i, 1)h \in L$. We are now able to define lattice functions based on set L.

Proper actions

Definition 1.7 (proper actions)**.** Let X is a locally compact Hausdorff space and a discrete group Γ acts on X continuously. If for any compact subsets $K, K' \subset X$ the set

$$\{\gamma \in \Gamma : \gamma K \cap K' \neq \varnothing\}$$

is finite, then the action is called *proper*.

Proper actions has a nice property:

Proposition 1.8. *If a group Γ acts properly on a Hausdorff space X, then the quotient $\Gamma \backslash X$ is also a Hausdorff space.*

Proof. For $x, y \in X$ such that their image under the quotient map $[x] \neq [y]$. Since we assume that X is locally compact, there is $A \ni x$ and $B \ni y$ such that their closure $\bar{A}$ and $\bar{B}$ are compact. Assume $|\{\gamma : \gamma\bar{A} \cap \bar{B} \neq \varnothing\}| = n$, define open sets

$$x \in U = A \cap \bigcap_{i=1}^{n} \gamma_i^{-1} U_i,$$
$$y \in V = B \cap \bigcap_{i=1}^{n} V_i.$$

Then $[x]$ and $[y]$ are seperated by open U and V such that $U \cap V = \varnothing$. □

Totally disconnected groups.

Definition 1.9 (totally disconnected groups and groups of td-type)**.** A Hausdorff topological group G is *totally disconnected* if every neighborhood of its unit element 1 contains a compact open subgroup.

A locally compact totally disconnected group is also called a *group of td-type* or a *t.d. group.*

Example 1.10. Let F be a local field and $n \geqslant 1$ be an integer, then $\mathsf{GL}_n(F)$ is a group of td-type.

The next proposition from 1.2 of [2] provides us some tools to construct new t.d. groups from old ones.

Proposition 1.11.
1. *Every open subgroup or closed subgroup of a group of td-type is again of td-type.*
2. *Finite direct product of groups of td-type if a t.d. group.*
3. *Let $(G_i)_{i\in I}$ be an infinite family of t.d. groups, and $K_i \subset G_i$ be a compact open subgroup. Then the restricted product of G_i w.r.t K_i is a group of td-type.*

1.2 Linear Fractional Transformations

Let $\widehat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$. The group $\mathsf{GL}_2(\mathbb{C})$ acts on $\widehat{\mathbb{C}}$ by

$$\begin{aligned} \mathsf{GL}_2(\mathbb{C}) \times \widehat{\mathbb{C}} &\to \widehat{\mathbb{C}} \\ (\alpha, \tau) &\mapsto \frac{a\tau + b}{c\tau + d} \end{aligned}$$

where $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. In this case, when $\tau = \infty$ the result of the right hand side becomes $\frac{a}{c}$; when $c\tau + d = 0$, the result of the right hand side becomes ∞.

On the other hand, we can view $\widehat{\mathbb{C}}$ as projective space $\mathbb{P}^1(\mathbb{C})$ via

$$\begin{aligned} \mathbb{P}^1(\mathbb{C}) &\to \widehat{\mathbb{C}} \\ [x:y] &\mapsto \frac{x}{y} \end{aligned}$$

then the action is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} .(x:y) = (ax+by : cx+dy) = \begin{cases} \left(\frac{ax+by}{cx+dy}, 1\right), & cx+dy \neq 0; \\ \infty, & cx+dy = 0. \end{cases}$$

For every $\tau \in \mathbb{C}$,

$$\begin{bmatrix} \tau & \tau - 1 \\ 1 & 1 \end{bmatrix} .\infty = \tau, \tag{1}$$

therefore $\mathsf{GL}_2(\mathbb{C})$ acts transitively on $\widehat{\mathbb{C}}$. Note that

$$\det\left(\begin{bmatrix} * & * - 1 \\ 1 & 1 \end{bmatrix}\right) = 1,$$

the equation (1) shows that $\mathsf{SL}_2(\mathbb{C})$ also acts transitively on $\widehat{\mathbb{C}}$.

Now we restrict the the action to the subgroup $\mathsf{GL}_2(\mathbb{R})$, by directly computation we conclude

Lemma 1.12. *For $\alpha \in \mathsf{GL}_2(\mathbb{R})$ and $z \in \widehat{\mathbb{C}}$ we have*

$$\mathrm{Im}(\alpha z) = |cz+d|^{-2} \det(\alpha) \mathrm{Im}(z)$$

where

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Observe that

$$\mathsf{GL}_2^+(\mathbb{R}) = \mathbb{R}_{>0}^\times \cdot \mathsf{SL}_2(\mathbb{R}) \tag{2}$$

and $\mathbb{R}_{>0}^\times$ acts trivially on $\mathbb{H}$. This inspires us to study the subgroup $\mathsf{SL}_2(\mathbb{R})$. By definition of $\mathsf{SL}_2(\mathbb{R})$, all its element have determinant 1. Then the previous lemma indicates that $\mathbb{H}, -\mathbb{H}$ and $\widehat{\mathbb{R}} = \mathbb{R} \sqcup \{\infty\}$ are invariant under the action of $\mathsf{SL}_2(\mathbb{R})$. In particular, they are three orbits. The equation (1) also applies to $\widehat{\mathbb{R}}$.

Now pick $z = x + iy \in \mathbb{H}$, define the matrix

$$\alpha = \begin{bmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ & \frac{1}{\sqrt{y}} \end{bmatrix}.$$

Then $z = \alpha.i$ implies that $\mathsf{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$. The proof for $-\mathbb{H}$ is similar.

Furthermore, according to 1.3

Theorem 1.13. *We can identify*

$$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2(\mathbb{R}) \cong \mathbb{H}$$

via the orbit homeomorphism orb_i.

We denote by $\mathrm{Aut}(\mathbb{H})$ the group of all holomorphic automorphisms of $\mathbb{H}$. The next theorem tells us that holomorphic automorphisms of $\mathbb{H}$ can be determined by $\mathsf{GL}_2(\mathbb{R})$ or $\mathsf{SL}_2(\mathbb{R})$.

Theorem 1.14. *We have the following identities:*

$$\mathsf{GL}_2^+(\mathbb{R})/\mathbb{R}^\times \cong \mathsf{SL}_2(\mathbb{R})/\{\pm 1\} \cong \mathrm{Aut}(\mathbb{H}).$$

Proof. The first isomorphism can be deduced from equation (2).

The second isomorphism can be given by

$$\begin{aligned} \mathsf{GL}_2^+(\mathbb{R}) &\to \mathrm{Aut}(\mathbb{H}) \\ g &\mapsto (z \mapsto g.z) \end{aligned}$$

□

Now we classify the non-scalar elements of $\mathsf{GL}_2^+(\mathbb{R})$.

Following §1.3 of [8], we define.

Definition 1.15 (elliptic, parabolic and hyperbolic element)**.** Let $\alpha \in \mathsf{GL}_2^+(\mathbb{R})$ be a non-scalar element. Then α is

- *elliptic* if $\mathrm{tr}(\alpha)^2 < 4\det(\alpha)$;
- *parabolic* if $\mathrm{tr}(\alpha)^2 = 4\det(\alpha)$;

- *hyperbolic* if $\operatorname{tr}(\alpha)^2 > 4\det(\alpha)$.

The geometric interpretation of this definition related to the fixed points of the elements of $\mathsf{GL}_2^+(\mathbb{R})$. Therefore we have

Proposition 1.16. *Let $\alpha \in \mathsf{GL}_2^+(\mathbb{R})$ be a non-scalar element. Then α is*

- elliptic *iff α has the fixed points z_0 and $\bar{z}_0$ with $z \in \mathbb{H}$;*
- parabolic *iff α has a unique fixed point on $\widehat{\mathbb{R}}$;*
- hyperbolic *iff α has two distinct fixed points on $\widehat{\mathbb{R}}$.*

1.3 Congruence Subgroups

Definition 1.17 (modular group and its congruence subgroups)**.** The group $\Gamma_0 := \mathsf{SL}_2(\mathbb{Z})$ is called the *modular group*. For a positive integer $N \in \mathbb{Z}_{>0}$, we define the *principal congruence subgroup* $\Gamma(N)$ of Γ_0 to be

$$\Gamma(N) := \left\{ \gamma \in \mathsf{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \pmod N \right\}.$$

A subgroup $\Gamma \subset \mathsf{SL}_2(\mathbb{Z})$ is said to be a *congruence subgroup* if it contains a principal congruence subgroup. In particular, $\Gamma(1) = \Gamma_0$.

By subgroup topology, it is obvious that congruence subgroups are discrete subgroups of $\mathsf{SL}_2(\mathbb{R})$. Note that for a fixed $N \in \mathbb{Z}_{>0}$, the map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/N\mathbb{Z}$ is surjective.

A discrete subgroup of $\mathsf{SL}_2(\mathbb{R})$ is called a *Fuchsian group*.

Definition 1.18 (cusps)**.** Let $\Gamma \subset \mathsf{SL}_2(\mathbb{R})$ be a Fuchsian group. Consider the set

$$P_\Gamma = \left\{ t \in \widehat{\mathbb{R}} : \overline{\Gamma}_t \text{ contains parabolic element} \right\}.$$

We call elements of $\Gamma \backslash P_\Gamma$ the *cusps* of Γ.

Example 1.19. Consider the modular group $\Gamma_0 = \mathsf{SL}_2(\mathbb{Z})$. It has one cusp ∞. By computation we have $\overline{\Gamma}_0 = \begin{bmatrix} 1 & \mathbb{Z} \\ & 1 \end{bmatrix}$, thus $\infty \in P_{\Gamma_0}$. For the similar argument as the equation (1), we can show $\Gamma_0.\infty = \widehat{\mathbb{Q}} \subset P_{\Gamma_0}$. Note that entries of elements in $\mathsf{SL}_2(\mathbb{Z})$ are integers, the discriminant of the eigenequation must have integral coefficients, we conclude that $\widehat{\mathbb{Q}} \supset P_{\Gamma_0}$. Therefore $\widehat{\mathbb{Q}} = P_{\Gamma_0}$.

1.4 Fundamental Domain

Definition 1.20 (fundamental domain)**.** Let X be a locally compact Hausdorff space and Γ be a discrete group. Assume Γ acts properly on X, a *fundamental domain* of X is a subset $\mathcal{F} \subset X$ such that

1. $\mathcal{F}$ is the closure of its interior $\mathcal{F}^\circ$;

2. For different $\gamma, \gamma' \in \Gamma$, $(\gamma\mathcal{F})^\circ \cap (\gamma'\mathcal{F})^\circ = \varnothing$; and

3. $X = \bigcup_{\gamma \in \Gamma} \gamma\mathcal{F}$, and this covering is locally finite i.e. for any $x \in X$, there exists open $U \ni x$ such that for all but finitely many $\gamma \in \Gamma$, $U \cap \gamma\mathcal{F} = \varnothing$.

1.5 Haar Measure

In the following section, we will assume that G is a fixed locally compact group with topology $\mathcal{T}$ (i.e. G is a locally compact Hausdorff topological group). Then $\mathcal{T}$ generates a σ-algebra $\mathcal{B}$ which is called the *Borel algebra*.

Definition 1.21 (Radon measure)**.** A measure μ defined on $\mathcal{B}$ is called a *Radon measure* if it has the following properties:

1. μ is finite on compact subsets;

2. For all $A \in \mathcal{B}$, $\mu(A) = \sup\{\mu(K) : A \supset K, K \subset G \text{ compact}\}$; and

3. For all $A \in \mathcal{B}$, $\mu(A) = \inf\{\mu(K) : A \subset U, U \subset G \text{ open}\}$.

An example of a Radon measure is the Lebesgue measure on the real line $\mathbb{R}$. The subsequent theorem demonstrates the existence and uniqueness, up to scaling, of a specific type of Radon measure. We refer to chapter D of [12] for the proof of this theorem.

Theorem 1.22 (Haar)**.** *If G is a locally compact group, then there exists a non-zero Radon measure μ on the Borel algebra such that $\mu(xA) = \mu(A)$ for every $x \in G$ and every measurable set $A \subset G$. This property is called* left-invariant*. Furthermore, μ is unique up to scaling by positive numbers.*

Definition 1.23 (Haar measure)**.** The Borel measure stated in the theorem 1.22 is called a *Haar* measure of G.

Starting from this point within this section, we consider G as a locally compact group with a Haar measure μ. Now we can discuss the integration.

Definition 1.24 (convolution)**.** The *convolution* of two functions $f, g \in L^1(G)$ is defined as

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy.$$

Proposition 1.25. *Convolution exists and the space $L^1(G)$ becomes an algebra over $\mathbb{C}$. Moreover $L^2(G)$ is a Hilbert space.*

2 p-adic numbers and Adeles

The field $\mathbb{R}$ of real numbers is complete. More specifically, $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the standard absolute value, denoted as $|\cdot|_\infty$. According to Ostrowski's theorem, any absolute value on $\mathbb{Q}$ is equivalent to either the standard absolute value $|\cdot|_\infty$, or the p-adic absolute value, represented as $|\cdot|_p$, where p is a prime number. The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

Recall that $\mathbb{R}$ is locally compact, we will later that $\mathbb{Q}_p$ is also locally compact. However, the direct product $\prod_p \mathbb{Q}_p$ is not locally compact. To study all the completion of $\mathbb{Q}$ at the same time, we take the restricted product of all $\mathbb{Q}_p$, which preserves the local compactness. This forms the ring $\mathbb{A}_f$ of finite adeles. The ring $\mathbb{A}$ of adeles is the direct product $\mathbb{R} \times \mathbb{A}_f$.

2.1 Construction of p-adic numbers

All along this section, k denotes a field and p is a prime integer.

Definition 2.1 (valuation)**.** A *valuation* on k is a map

$$v : k \to \mathbb{R} \cup \{\infty\}$$

satisfying:

1. $v(0) = \infty$,
2. $v(xy) = v(x) + v(y)$ for $x, y \in k$, and
3. $v(x + y) \geqslant \min\{v(x), v(y)\}$.

An example of a valuation is the p-adic valuation, where p is a fixed prime. First of all, consider the map

$$\begin{aligned} v_p : \mathbb{Z} &\to \mathbb{R} \cup \{\infty\} \\ 0 &\mapsto \infty \\ 0 \neq a &\mapsto \sup\{n \in \mathbb{Z} : a \in p^n\mathbb{Z}\} \end{aligned}$$

defined on the ring of integers. We can extend it to the field of rational numbers $\mathbb{Q}$ by letting $v_p(x) = v_p(a) - v_p(b)$ for $x = a/b \in \mathbb{Q}$.

Proposition 2.2. *p-adic valuation is a valuation.*

Proof. First we check that the value of v_p does not depend on the representative of the rational number as a quotient of two integers. If $x \in \mathbb{Q}$ such that $x = a/b = c/d$ for $a, b, c, d \in \mathbb{Z}$, then $ad = bc$. By unique prime factorization, we can write $a = p^{n_a}a'$ and $d = p^{n_d}d'$ for some $a', d' \in \mathbb{Z}$ such that p does not divide a' or d', thus $ad = p^{n_a+n_d}a'd'$. By definition of v_p, we have

$$v_p(ad) = v_p(a) + v_p(d). \tag{3}$$

Similarly

$$v_p(bc) = v_p(b) + v_p(c).$$

Therefore

$$v_p(a) + v_p(d) = v_p(ad) = v_p(bc) = v_p(b) + v_p(c)$$

implies that

$$v_p(a/b) = v_p(a) - v_p(b) = v_p(c) - v_p(d) = v_p(c/d).$$

Note that equation (3) also verifies condition 2 for integers. Without loss of generality, assume $n_a \leqslant n_d$, then

$$a + d = p^{n_a}(a' + p^{n_b - n_a}b')$$

shows that condition 3 also holds for integers.

Now let $y = r/s$, then

$$\begin{aligned} v_p(xy) &= v_p(ar/bs) \\ &= v_p(ar) - v_p(bs) \\ &= v_p(a) + v_p(r) - v_p(b) - v_p(s) \\ &= v_p(a) - v_p(b) + v_p(r) - v_p(s) \\ &= v_p(x) + v_p(y). \end{aligned}$$

And

$$\begin{aligned} v_p(x+y) &= v_p\left(\frac{as+rb}{bs}\right) \\ &= v_p(as+rb) - v_p(bs) \\ &\geqslant \min\{v_p(as), v_p(rb)\} - v_p(bs) \\ &= \min\{v_p(as) - v_p(bs), v_p(rb) - v_p(bs)\} \\ &= \min\{v_p(x), v_p(y)\}. \end{aligned}$$

The above equations prove the rational cases for condition 2 and 3 respectively.

□

Definition 2.3 (absolute value)**.** An *absolute value* on k is a function

$$|\cdot| : k \to \mathbb{R}_{\geqslant 0}$$

such that

1. $|x| = 0$ iff $x = 0$;
2. $|xy| = |x||y|$ for all $x, y \in k$; and
3. $|x+y| \leqslant |x| + |y|$ for all $x, y \in k$.

An absolute value is called *non-archimedean* if for all $x, y \in k$ it also satisfies

$$|x+y| \leqslant \max\{|x|, |y|\}; \tag{4}$$

otherwise it is *archimedean.*

The next lemma reveals a useful property of non-archimedean absolute values.

Lemma 2.4. *Let* $|\cdot|$ *be a non-archimedean absolute value on* k *and* $x, y \in k$*, if* $|x| \neq |y|$ *then* $|x+y| = \max\{|x|, |y|\}$.

Proof. Assume $|x| > |y|$, then (4) implies that

$$|x+y| \leqslant |x| \tag{5}$$

and

$$|x| = |x+y-y| \leqslant \max\{|x+y|, |y|\}.$$

If $|y|$ from the right hand side is larger than $|x+y|$ i.e. $|x| \leqslant |y|$, then this inequality contradicts to out assumption, therefore $|x| \leqslant |x+y|$. Together with (5), the lemma is proved. □

The p-adic valuation induces an absolute value $|\cdot|_p$ called p-adic absolute value. For $x \in \mathbb{Q}$, we set $|x|_p := p^{-v_p(x)}$ and $|0|_p = p^{-\infty} =: 0$.

Conditions 1 and 2 simply follows from the construction of $|\cdot|_p$, and (4) holds for $|\cot|_p$ since v_p is a valuation. Observe that (4) implies 3. We conclude

Proposition 2.5. *The absolute value $|\cdot|_p$ is a non-archimedean absolute value.*

We employ the concepts about completeness from real analysis. Let $|\cdot|$ be an absolute value on k, a *Cauchy sequence* is a sequence of elements $x_n \in k$ if for every $\varepsilon > 0$ there is a bound M such that $|x_n - x_m| < \varepsilon$ whenever $m, n \geqslant M$. If every Cauchy sequence of elements of k has a limit in k, the field k is called *complete*. A subset $S \subset k$ is *dense* if every open ball around every element of k contains an element of S.

The next proposition is derived from Proposition 2.1 in Chapter XII of Serge Lang's Algebra[7]. We will present it here without providing general proofs. However, the upcoming construction of p-adic numbers demonstrates the proof for this special case.

Proposition-Definition 2.6 (completion)**.** *Up to isomorphism, there exists a unique pair $(\hat{k}, \iota)$ consisting of a field $\hat{k}$, complete under an absolute value $|\cdot|_{\hat{k}}$, and an embedding $\iota : k \to \hat{k}$ such that the absolute value on k is induced by that of $\hat{k}$, and such that $\iota(k) \subset \hat{k}$ is dense in $\hat{k}$.*

The pair $(\hat{k}, \iota)$ or $\hat{k}$ is called the completion *of k.*

Let R be the set of all Cauchy sequence of elements of $\mathbb{Q}$ with respect to $|\cdot|_p$. For sequences $\{x_n\}, \{y_n\} \in R$ we define addition and multiplication as follows:

$$\{x_n\} + \{y_n\} = \{x_n + y_n\}$$

and

$$\{x_n\} \cdot \{y_n\} = \{x_n y_n\}.$$

We denote by $\tilde{x}$ the *constant sequence associated to* $x \in \mathbb{Q}$, that is

$$\tilde{x} := x, x, x, \ldots$$

From analysis the right hand side of equations are Cauchy and thus R is a commutative ring with unity $\tilde{1}$ and zero $\tilde{0}$.

Let $\mathfrak{m} \subset R$ be the set of all Cauchy sequences that converge to 0. We claim that

Lemma 2.7. *Set $\mathfrak{m}$ is a maximal ideal of R.*

Proof. For $\{x_n\} \in R$ and $\{y_n\} \in \mathfrak{m}$, their product $x_n y_n \to 0$ since $y_n \to 0$ and $\{x_n\}$ as a Cauchy sequence is bounded. This shows that $\mathfrak{m}$ is indeed an ideal.

Pick $\{a_n\} \in R - \mathfrak{m} =: \{r \in R : r \notin \mathfrak{m}\}$, since it is Cauchy and does not converge to 0, there exists sufficiently large N such that a_n stays away from 0 for $n > N$. Define

$$a'_n = \begin{cases} 0, & n < N; \\ 1/a_n, & n \geqslant N. \end{cases}$$

This new sequence is again Cauchy i.e. $\{a'_n\} \in R$.

The product

$$\tilde{1}_N := a_n a'_n = \underbrace{0, \ldots, 0}_{N}, 1, \ldots$$

belongs to the ideal generated by $\{a_n\}$. Observe that $\tilde{1} - \tilde{1}_N \in \mathfrak{m}$, we conclude that the unit element $\tilde{1}$ belongs to the ideal generated by $\mathfrak{m}$ and $\{a_n\}$. In other words, any ideal which strictly contains $\mathfrak{m}$ equals to R. Thus $\mathfrak{m}$ is maximal. □

Taking the quotient of R by its maximal ideal $\mathfrak{m}$ we get the *field of p-adic numbers*

$$\mathbb{Q}_p := R/\mathfrak{m}.$$

In addition, we define injection

$$\begin{aligned} \iota : \mathbb{Q} &\hookrightarrow R \\ x &\mapsto \tilde{x}. \end{aligned}$$

For different rational numbers $x \neq y$, $\iota(x) - \iota(y) \notin \mathfrak{m}$, hence the induced map

$$\begin{aligned} \iota : \mathbb{Q} &\hookrightarrow \mathbb{Q}_p \\ x &\mapsto [\tilde{x}] \end{aligned}$$

is also injective. Since $\iota : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$ is an inclusion, we can view $\mathbb{Q}$ as a subset of $\mathbb{Q}_p$ instead of writing $\iota(\mathbb{Q}) \subset \mathbb{Q}_p$.

Now we define an absolute value on $\mathbb{Q}_p$, for $a \in \mathbb{Q}_p$ with representative $\{a_n\}$, set

$$|a|_{\mathbb{Q}_p} \coloneqq \lim_{n\to\infty} |a_n|_p,$$

we claim that this is a well-defined non-archimedean absolute value, which extends $|\cdot|_p$.

Lemma 2.8. *For $x \coloneqq \{x_n\} \in R - \mathfrak{m}$, there exists an integer N such that for $m, n \geqslant N$, $|x_n|_p = |x_m|_p$.*

Proof. Since x is Cauchy and does not tent to zero, there exists sufficiently large N such that $|x_n|_p \geqslant c > 0$ for some positive c whenever $n > N$. Since we are choosing sufficiently large N, for $n, m \geqslant N$, we also have

$$|x_n - x_m|_p < c \leqslant \min\{|x_n|_p, |x_m|_p\}, \tag{6}$$

which implies that $|x_n - x_m|_p \neq |x_m|_p$. Now we apply lemma 2.4 and get

$$|x_n|_p = |x_n - x_m + x_m|_p = \max\{|x_n - x_m|_p, |x_m|_p\} \overset{(6)}{=} |x_m|_p.$$

□

This lemma shows that the limit exits. To show that the absolute value is well-defined, we have to check that it does not depend on the representatives. If two sequences $\{x_n\}$ and $\{y_n\}$ converge to the same limit, then their difference $\{x_n - y_n\}$ tends to zero and thus an element of $\mathfrak{m}$. That $|\cdot|_{\mathbb{Q}_p}$ is non-archimedean and extends $|\cdot|_p$ on $\mathbb{Q}$ is immediate by its definition. Beginning at this point, the notation $|\cdot|_p$ represents both for $|\cdot|_p$ and $|\cdot|_{\mathbb{Q}_p}$.

Proposition 2.9. *The field $\mathbb{Q}$ is dense in $\mathbb{Q}_p$.*

Proof. Pick $x \in \mathbb{Q}_p$ represented by $\{x_n\}$, consider the open ball $B(x, r)$ where $r > 0$ is the radius. There exists $N > 0$ such that for $n, m \geqslant N$, $|x_n - x_m|_p < \frac{r}{2}$. The constant sequence $\tilde{x}_N \in \iota(\mathbb{Q})$ then belongs to the open ball $B(x, r)$. □

Proposition 2.10. *The field $\mathbb{Q}_p$ is complete with respect to $|\cdot|_p$.*

Proof. Let $\{c_n\}$ be a Cauchy sequence in $\mathbb{Q}_p$. By the dense property we can find an element $x_n \in \mathbb{Q}$ such that $|c_n - \tilde{x}_n|_p < 1/n$ for each n. We can verify that $\{x_n\}$ is Cauchy. Let $c \in \mathbb{Q}_p$ be the limit of $\{x_n\}$, it remains to show that $\{c_n\}$ converges to

c. For $\varepsilon > 0$, there exists N such that $|x_m - x_n|_p < \frac{\varepsilon}{2}$ whenever $m, n \geqslant N$. Fix n and we get

$$|c - \tilde{x}_n|_p = \lim_{m\to\infty} |x_m - x_n|_p \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

This means that $(c - \{\tilde{x}_n\})$ converges to $0 \in \mathbb{Q}_p$ i.e. $\{\tilde{x}_n\}$ converges to c. On the other hand, by our choice of $\{x_n\}$, we know that $|c_n - \tilde{x}_n|_p$ converges to 0. Apply the triangle inequality of absolute value, we deduce that $\{c_n\}$ converges to c. □

If there is a pair $(\hat{k}, \iota')$ which is another completion of $\mathbb{Q}$. Take any $\alpha \in \mathbb{Q}_p$, by density of $\mathbb{Q}$, we can find a Cauchy sequence $\{a_n\}$ of $\mathbb{Q}$ whose limit is α. Then $\{\iota'(a_n)\}$ converges to a limit $\alpha' \in \hat{k}$. This algorithm defines a map φ from $\mathbb{Q}_p$ to $\hat{k}$. It is easy to see that φ is an isomorphism that preserves absolute value. This shows the uniqueness of the completion.

By combining the construction and the propositions, we now have

Proposition 2.11. *The pair* $(\mathbb{Q}_p, \iota)$ *is the completion of* $\mathbb{Q}$ *with respect to the* p*-adic absolute value.*

2.2 Properties of p-adic numbers

Absolute values induce metrics, it follows that we can define a topology. For more on metric spaces, we refer to [11]. Since field operations are continuous functions under the metric induced by absolute values, the field of p-adic numbers $\mathbb{Q}_p$ is a topological field.

Next we introduce the ring of p-adic integers.

Proposition-Definition 2.12 (ring of p-adic integers)**.** *The set*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leqslant 1\}$$

is a ring called ring of p-adic integers.

Proof. It is obvious that $0, \pm 1 \in \mathbb{Z}_p$. For any $x, y \in \mathbb{Z}_p$, by non-archimedean inequality (4), $|x + y|_p \leqslant 1$, and by 2 of definition 2.3, $|xy|_p = |x|_p|y|_p \leqslant 1$ and $|-1 \cdot x|_p = |-1|_p|x|_p \leqslant 1$. These show that $\mathbb{Z}_p$ is closed under addition, multiplication and change of sign, thus it is a ring. □

By applying the same argument as to why $\mathbb{Q}_p$ is a topological field, we can conclude that $\mathbb{Z}_p$ is a topological ring.

2.3 Adeles

Definition 2.13 (restricted product)**.** Let I be an index set and $(G_i)_{i\in I}$ be a family of locally compact group, if $K_i \subset G_i$ is an open compact subgroup for each $i \in I$, then we define the *restricted product* as

$$G = \widehat{\prod_{i\in I}}^{K_i} G_i = \left\{ x \in \prod_{i\in I} G_i : x_i \in K_i \text{ for almost all } i \in I \right\}.$$

The basis of open sets are those of the form

$$\prod_{i\in E} U_i \times \prod_{i\notin E} K_i,$$

where $E \subset I$ is a finite subset and $U_i \subset G_i$ is an open set.

Adhere to the notation specified in the definition 2.13, pick $x \in G$ there is a finite subset $E \subset I$ such that $x_i \in K_i$ whenever $i \notin E$. For x_i such that $i \in E$, we choose its compact neighborhood U_i. Then the product of U_i and K_i is a compact neighborhood of x. This shows that the restricted product G is locally compact.

Recall that $\mathbb{Z}_p$ is open in $\mathbb{Q}_p$, we define the set *finite adeles* as

$$\mathbb{A}_f = \widehat{\prod_{p \text{ prime}}}^{\mathbb{Z}_p} \mathbb{Q}_p$$

and the set of *adeles*

$$\mathbb{A} = \mathbb{A}_f \times \mathbb{R}.$$

Proposition 2.14. *Both $\mathbb{A}_f$ and $\mathbb{A}$ are topological rings.*

Proposition 2.15. *The field $\mathbb{Q}$ is a discrete subset of $\mathbb{A}$ and $\mathbb{A}/\mathbb{Q}$ is compact.*

Proposition 2.16. *The field $\mathbb{Q}$ is dense in $\mathbb{A}_f$.*

2.4 Haar measure

Let G be a locally compact group with topology $\mathcal{T}$ (i.e. G is a locally compact Hausdorff topological group). Then $\mathcal{T}$ generates a σ-algebra which is called the *Borel algebra.*

Definition 2.17 (Radon measure)**.** A measure μ defined on the Borel algebra $\mathcal{B}$ is called a *Radon measure* if it has the following properties:

1. μ is finite on compact subsets.
2. For all $A \in \mathcal{B}$, $\mu(A) = \sup\{\mu(K) : A \supset K, K \subset G \text{ compact}\}$
3. For all $A \in \mathcal{B}$, $\mu(A) = \inf\{\mu(K) : A \subset U, U \subset G \text{ open}\}$

An example of a Radon measure is the Lebesgue measure on the real line.

3 Representations

3.1 Continuous Representations

Definition 3.1 (continuous representations)**.** Let G be a locally compact group and let V be a Banach space. A *continuous representation* of G (on the left) is a pair (π, V) where $\pi : G \to \mathsf{GL}(V)$ is a group homomorphism such that

$$\begin{aligned} G \times V &\to V \\ (g, v) &\mapsto \pi(g)v \end{aligned}$$

is continuous.

Following Catier's definition[2], we introduce smooth representations and admissible representations.

Definition 3.2 (smooth representations)**.** A representation (π, G) of a group of td-type G is *smooth* iff the stabilizer of every vector in V is open. Equivalently, if $V = \bigcup_K V^K$ where K runs over the compact open subgroups of G and V^K is the space of vectors $v \in V$ such that $\pi(k) \cdot v = v$ for any $k \in K$.

Definition 3.3 (admissible representations)**.** Following the notations in definition 3.2, a representation (π, V) of G is said to be *admissible* of it is smooth and the space V^K of vectors invariant under K is finite-dimensional for every compact open subgroup $K \subset G$.

Fix the basis $\{1, i\}$ of $\mathbb{C}$. Consider $\mathsf{GL}_2(\mathbb{R}) \cong \mathsf{GL}_{\mathbb{R}}(\mathbb{C})$ where the $\mathsf{GL}_{\mathbb{R}}(\mathbb{C}) = \mathsf{GL}(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C})$, the notation $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{C})$ means restricting scalars [13] from $\mathbb{C}$ to $\mathbb{R}$. For $a + bi = \lambda \in \mathbb{C}^\times$ such that $a, b \in \mathbb{R}$, $a \neq 0$ or $b \neq 0$. The map $z \mapsto \lambda z$ (for $z \in \mathbb{C}$) gives a matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ representing λ. Therefore we have

$$\mathbb{C}^\times \cong \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R}, a \neq 0 \text{ or } b \neq 0 \right\}.$$

Note that the unitary group

$$U_1 = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\}$$

is contained in $\mathbb{C}^\times$.

Let s be the complex conjugate map i.e. in the basis $\{1, i\}$, it maps 1 to 1 and i to $-i$. Then s is $\mathbb{R}$-linear and the corresponding matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. By computation we have

$$sU_1 = \left\{ \begin{bmatrix} a & -b \\ -b & -a \end{bmatrix} : a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\}.$$

Furthermore, for $g \in U_1$ or sU_1, the product $g^t g = I$. This shows that

$$U_1 \cup sU_1 \subset O(2) \coloneqq \left\{ A \in \mathsf{GL}_2(\mathbb{R}) : A^t A = I \right\}$$

On the other hand, pick $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{GL}_2(\mathbb{R})$, by definition of $O(2)$ we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = I,$$

solve this equation and we conclude that $U_1 \cup sU_1 \cong O(2)$. Similarly, one can verify that

$$U(1) \cong SU(1) \coloneqq \{A \in O(2) : \det A = 1\}.$$

The group $O(2)$ is not connected. More specifically, it has two connected components $O(2) = SO(2) \sqcup sSO(2)$.

Now consider the group

$$GO(2) \coloneqq \left\{ A \in \mathsf{GL}_2(\mathbb{R}) : \text{there exists } \lambda \in \mathbb{R}^\times \text{ s.t. } A^t A = \lambda I \right\}$$

and the multiplier map

$$\begin{aligned} \lambda : GO(2, \mathbb{R}) &\to \mathbb{R}^\times \\ A &\mapsto \lambda : A^t A = \lambda I. \end{aligned}$$

We have $A^t A = \lambda(A) I$. Take determinant of both sides one gets $\det^2 A = \lambda(A)^2$. Hence the 2 connected components are

$$\mathbb{C}^\times = \{\det A = \lambda(A)\}$$

and

$$s\mathbb{C}^\times = \{\det A = -\lambda(A)\}\,.$$

Furthermore, $(\mathbb{C}^\times \cup s\mathbb{C}^\times)/\mathbb{R}^\times$ is a compact, maximal component in $\mathsf{GL}_2(\mathbb{R})/\mathbb{R}^\times$.

Let $K = U_1 \cup sU_1$, we combine the previous argument with 0.2.2 of [3], a complex admissible representation π of $\mathsf{GL}_{\mathbb{R}}(\mathbb{C})$ is a linear representation such that $\pi \mid K : K \to \mathsf{GL}_{\mathbb{C}}(V)$, which is the direct sum of irreducible representings of K, each occurring finitely many times.

4 Modular forms

4.1 Modular forms for SL_2

For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{GL}_2(\mathbb{C})$, define its *automorphy factor* to be

$$j(\gamma, \tau) := c\tau + d, \tau \in \mathbb{H}.$$

We can verify that it satisfies

$$j(\gamma\gamma', \tau) = j(\gamma, \gamma'\tau)j(\gamma', \tau)$$

where $\gamma, \gamma' \in \mathsf{GL}_2^+(\mathbb{R})$.

Now fix $k \in \mathbb{Z}, \gamma \in \mathsf{GL}_2^+(\mathbb{R})$, for any function $f : \mathbb{H} \to \mathbb{C}$, we define

$$f \mid_k \gamma : \tau \mapsto (\det \gamma)^{\frac{k}{2}} j(\gamma, \tau)^{-k} f(\gamma\tau), \tau \in \mathbb{H}.$$

This gives us a right action of $\mathsf{GL}_2^+(\mathbb{R})$ on the space of functions $\mathbb{H} \to \mathbb{C}$ by

$$f \mid_k (\gamma\gamma') = (f \mid_k \gamma) \mid_k \gamma'.$$

Consider the map

$$q_N(\tau) := e^{2\pi i\tau/N}$$

where $N \in \mathbb{Z}_{>0}$. This defines a surjective holomorphic map from $\mathbb{H}$ to $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Since $q_N(\tau) = q_N(\tau') \iff (\tau - \tau') \in N\mathbb{Z}$, if a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is periodic of period N i.e. $f(\tau + N) = f(\tau)$ then it has a Laurent expansion:

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q_N^n, a_n \in \mathbb{C}. \tag{7}$$

Remark 4.1. Let $\tau = x + iy$ such that $y > 0$ and $x \in \mathbb{R}$, we can view equation (7) as the Fourier expansion of a real function $x \mapsto f(x + iy)$ of period N. Thus a_n is also called the *Fourier coefficient* of f.

Definition 4.2. Following the above notations, the function f is said to be *holomorphic* at ∞ if $a_n = 0$ for every $n < 0$; f *vanishes* at ∞ if $a_n = 0$ for every $n \leqslant 0$.

Now we can define modular forms.

Definition 4.3 (modular forms for congruence subgroups)**.** Let Γ be a congruence subgroup and $k \in \mathbb{Z}_0$. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is a *modular form of weight* k (w.r.t. Γ)if

1. f is holomorphic on $\mathbb{H}$;
2. For all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z);$$
and
3. f is holomorphic also at the cusps of H w.r.t. Γ.

A *cusp form* is a modular form which in addition vanishes at ∞.

We denote by $\mathcal{M}_k(\Gamma)$ the $\mathbb{C}$-vector space of all modular forms of weight k and $\mathcal{S}_k(\Gamma)$ the subspace of cusp forms of weight k.

This is the classical definition of a modular form for SL_2.

4.2 Modular forms for GL_2

In this section we adopt Deligne's[3] approach to define modular forms on GL_2. For a function g on $\mathsf{GL}_2(\mathbb{R})$, we define

$$\begin{aligned} \|g\| &= \mathrm{tr}({}^t gg + ({}^t gg)^{-1}) \\ &= (a^2+b^2+c^2+d^2)(1+\det(g)^{-2}). \end{aligned}$$

Definition 4.4 (moderate growth)**.** A function f on $\mathsf{GL}_2(\mathbb{R})$ is said to be C^∞ to *moderate growth* if there exist $A > 0$ and $N > 0$ such that

$$f(g) \leqslant A\|g\|^N,$$

and all its derivatives of f satisfie analogous conditions.

Next we define modular forms of group $\mathsf{GL}_2(\mathbb{Z})$.

Definition 4.5 (modular forms)**.** Let

$$L \coloneqq \left\{ g \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}) : g(\mathbb{Z}^2) \subset \mathbb{C} \text{ is a lattice} \right\}.$$

A holomorphic modular form of weight k of group $\mathsf{GL}_2(\mathbb{Z})$ is a function on L, such that

1. f is holomorphic;
2. $f(\lambda g) = \lambda^{-k} f(g)$ where $g \in L, \lambda \in \mathbb{C}^\times$;
3. $f(g\gamma) = f(g)$ where $g \in L, \gamma \in \mathsf{GL}_2(\mathbb{Z})$; and
4. f is in moderate growth.

If we identify f as a lattice function from $L/\mathsf{GL}_2(\mathbb{Z})$ to $\mathbb{C}$, then the condition 3 requiring the right action to be invariant can be omitted, and the condition 2 can be modified to

2' $f(\lambda R) = \lambda^{-k} f(R)$ where R is an lattice and $\lambda \in \mathbb{C}^\times$.

The other two conditions stay unchanged.

Remark 4.6. Comparing the two definitions of modular forms in definition 4.3 and definition 4.5. The condition of requiring f to be holomorphic at cusps and in moderate growth are highly related. We refer to section [1] and 1.3 of [3], for further details.

4.3 Representations of $\mathsf{GL}_2(\mathbb{R})$ and Holomorphic Modular Forms

Fix $k \geqslant 1$, let D_{k-1} be the irreducible admissible representation of $\mathsf{GL}_\mathbb{R}(\mathbb{C})$ satisfying:

1. D_{k-1} has a base e_n indexed by the integers $n \equiv k \pmod 2$ such that $|n| \geqslant k$.
2. The action of $\mathfrak{gl}_\mathbb{R}(\mathbb{C})$ and of $(\mathbb{C}^\times \cup s\mathbb{C}^\times) \subset \mathsf{GL}_\mathbb{R}(\mathbb{C})$ such that
   (a) $\lambda * e_n = \lambda^k (\lambda\bar{\lambda}^{-1})^{\frac{n-k}{2}} e_n$ where $\lambda \in \mathbb{C}^\times$;
   (b) $s * e_n = e_{-n}$;
   (c) $H * e_n = n.e_n$;

(d) $X * e_n = \frac{k+n}{2} e_{n+2}$; and

(e) $Y * e_n = \frac{k-n}{2} e_{n-2}$,

where we have

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad X = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Y = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Note that $\{H, X, Y\}$ is a standard choice of basis for Lie algebra

$$\mathfrak{sl}_2(\mathbb{C}) = \{A \in M_2(\mathbb{C}) : \mathrm{tr} A = 0\}.$$

Now we discuss this representation with $(\mathfrak{g}, K)$-module, following the symbols from [6].

Definition 4.7. Let k be a field, G be an algebraic group and K be a compact subgroup of $G(k)$, a $(\mathfrak{g}, K)$-module is a vector space V equipped with an action of $\mathfrak{g}_{\mathbb{C}} \coloneqq \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and K such that the V as a K-module is an (infinite) direct sum of finite-dimensional representations of K, and V is also a Lie algebra representation of $\mathfrak{g}$.

Take $\mathfrak{g} = \mathfrak{gl}_{\mathbb{R}}(\mathbb{C})$ and $K = \mathbb{C}^\times \cup s\mathbb{C}^\times \cong O(2)$. Note that K is not connected, we define $K' = SO(2)$, and consider the $(\mathfrak{g}, K')$-module V. First we have

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

where

$$V_n = \{v \in V : z\dot{v} = z^n, \text{ for all } z \in K'\}.$$

The following map diagonalizes the group K' :

$$\left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\} \cong \left\{ \begin{bmatrix} a + bi & 0 \\ 0 & a - bi \end{bmatrix} : a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\}$$
$$A \mapsto \gamma^{-1} A \gamma,$$

where

$$\gamma = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}.$$

By definition of $(\mathfrak{g}, K')$-module, we have that K' acts on $\mathfrak{g}$ by conjugation. Finally, by derivation of, we have that H acts as multiplication by n on V_n. This corresponds to 2c.

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