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Clark measures of bivariate inner functions

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Abstract

To each inner function on the unit disc, one can associate a corresponding family of Clark measures, which in turn can be linked to a family of unitary operators. Hence Clark measures form a link between inner functions, singular measures and operator theory. While Clark theory in one variable has been thoroughly studied and well-developed following D. N. Clark's 1972 paper, it is only recently that progress has been made in extending this theory to the multivariate setting. Our goal is to provide an overview of recent research as well as investigate Clark measures for some new examples of bivariate inner functions on the unit bidisc. In particular, we characterize the Clark measures for certain kinds of multiplicative embeddings in inner functions as well as products of inner functions.

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Contents

1 Introduction

Clark theory has been of great interest since the original paper [9] by D. N. Clark from 1972. He initiated his study by introducing a family of unitary operators with a corresponding family of singular measures on the unit disc, which we now call Clark measures. Furthermore, to each analytic function on the unit disc with modulus one almost everywhere on the boundary (a so called *inner function*), one can associate a family of Clark measures. In this way, Clark measures form an unexpected link between analytic functions, singular measures and operator theory, three significant areas of mathematical analysis. We will mainly focus on the first two topics in this work.

In one variable, there are clear characterizations of Clark measures and their behavior, and the theory is well-developed enough to be explored in introductory textbooks such as [15]. Recently, progress has been made in extending this theory to Clark measures of inner functions in several variables. There are two natural settings for multivariate Clark theory; one could either study Clark measures on the unit polydisc $\mathbb{D}^d$ or the unit d-ball. Since these spaces are not biholomorphically equivalent for $d \geq 2$, the Clark theory will vary based on what space one chooses to work in. Clark theory on the unit d-ball has been explored in detail in e.g. [2]. We restrict ourselves to $\mathbb{D}^d$ in this work, and mainly dimension $d = 2$.

The case of rational inner functions in higher dimensions has been systematically studied in e.g. [3], but less is known about other classes of inner functions. In this work, we aim to provide an overview of recent research as well as investigate Clark measures for some new examples of bivariate inner functions. For instance, we will study constructions of non-rational inner functions from inner functions with more or less known structure. In particular, we investigate the relationship between Clark measures and multiplication; first by introducing compositions of inner functions and multiplicative embeddings, and then by multiplying inner functions with each other. The idea is to study how these operations affect the corresponding Clark measures.

2 Preliminaries

We begin by introducing some central concepts of multivariate complex analysis. Let

$$\mathbb{D}^d := \{(z_1, z_2, \dots, z_d) \in \mathbb{C}^d : |z_j| < 1, \quad j = 1, 2, \dots, d\}$$

denote the unit polydisc in d variables, and

$$\mathbb{T}^d := \{(\zeta_1, \zeta_2, \dots, \zeta_d) \in \mathbb{C}^d : |\zeta_j| = 1, \quad j = 1, 2, \dots, d\}$$

its so called *distinguished boundary*. Note that this is only a subset of the boundary $\partial\mathbb{D}^d$. For $d = 2$, the set $\mathbb{T}^2$ defines a two-dimensional torus.

Much of our discussion will be centered around holomorphic and harmonic functions in several (but mainly two) variables. Let $U \subset \mathbb{C}^d$ be an open set. The function $f : U \to \mathbb{C}$ is said to be holomorphic or analytic if it is locally bounded (i.e., for every $p \in U$, there is a neighborhood N of p such that $f|_N$ is bounded) and complex-differentiable in each variable separately; so the limit

$$\lim_{\mathbb{C} \ni \xi \to 0} \frac{f(z_1, \dots, z_{j-1}, z_j + \xi, z_{j+1}, \dots, z_d) - f(z)}{\xi}$$

exists for all $z \in U$ and all $j = 1, 2, \dots, d$.

Just as in one variable, we may characterize holomorphic functions using power series expansions. Let $\rho = (\rho_1, \ldots, \rho_d) \in \mathbb{R}_+^d$ and $a = (a_1, \ldots, a_d) \in \mathbb{C}^d$ be the polyradius and center of a polydisc

$$\Delta := \{z \in \mathbb{C}^d : |z_j - a_j| < \rho_j, \quad j = 1, \ldots, d\}.$$

Let $f : \overline{\Delta} \to \mathbb{C}$ be a continuous function, holomorphic in Δ. Then, on Δ, f is given by a power series that converges uniformly absolutely on compact subsets of Δ:

$$f(z) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha (z-a)^\alpha.$$

Here we have used multinomial notation, where $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ for $\alpha_j \in \mathbb{N}$. Conversely, if we define $f : \Delta \to \mathbb{C}$ by a power series as above (converging uniformly absolutely on compact subsets of Δ), then f is holomorphic on the same polydisc.

We say that a function $f : U \to \mathbb{C}$ is *pluriharmonic* if it is harmonic along each complex line. Formally, for every $a, b \in \mathbb{C}^d$, we require $\xi \mapsto f(a + b\xi)$ to be harmonic for each $\xi \in \mathbb{C}$ such that $a + b\xi \in U$. Note that the pluriharmonic functions define a subclass of the harmonic functions on $\mathbb{C}^d$. Moreover, $f : U \to \mathbb{C}$ is pluriharmonic if and only if it is locally the real or imaginary part of a holomorphic function (see pp. 68-69 in [19]). For an extensive introduction to multivariate complex analysis, see [19].

Let us recall some properties of *Blaschke products*, which we define as

$$B(z) := e^{ia} z^K \prod_{k \geq 1} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z}_k z},$$

where $a \in \mathbb{R}, K \in \mathbb{N}$ and $\{z_k\}_{k=1}^\infty \subset \mathbb{D}$ is a sequence satisfying the *Blaschke condition*

$$\sum_{k \geq 1} (1 - |z_k|) < \infty.$$

This condition ensures that $B(z)$ converges uniformly on compact subsets of $\mathbb{D}$, and is thus analytic on $\mathbb{D}$. For any Blaschke product $B(z)$, the radial limit $\lim_{r \to 1-} B(r\zeta)$ exists and has modulus one for Lebesgue-almost every $\zeta \in \mathbb{T}$. A *finite* Blaschke product is defined as

$$B(z) := e^{ia} z^K \prod_{k=1}^{n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z}_k z} \tag{1}$$

for $z_k \in \mathbb{D}$ and $k = 1, 2, \ldots, n$. As opposed to their infinite counterparts, the radial limits of finite Blaschke products exist everywhere on $\mathbb{T}$, and they are analytic on an open set containing the closed unit disc.

Finite Blaschke products have many notable properties — for one, they map the unit disc $\mathbb{D}$ to itself, and the unit circle $\mathbb{T}$ to itself. For $K = 0$, the function (1) will be a finite Blaschke product of degree n, and $B(z) = 0$ will have exactly n solutions in $\mathbb{D}$. Moreover, for any $w \in \mathbb{T}$, the equation $B(z) = w$ will have n distinct solutions on $\mathbb{T}$. Another useful property is that the derivative of a finite Blaschke product is non-zero on the unit circle. Finally, recall that by Fatou's theorem, if a function f is analytic on $\mathbb{D}$ with $|f(z)| \to 1$ as $z \to 1$, then f is a finite Blaschke product. For more on finite Blaschke products, see [14].

Next, we review some basic measure-theoretic concepts. Throughout this text, we will let m and m_d denote the normalized Lebesgue measure on $\mathbb{T}$ and $\mathbb{T}^d$ respectively. Whenever we use the words “measurable” or ”almost everywhere” without specifying a measure, we will be referring to the Lebesgue measure. For a given μ on some measure space X, we define its *support* $\text{supp}(\mu)$ as the closure of the set of points $x \in X$ such that every open neighborhood N of x satisfies $\mu(N) > 0$. Moreover, we say that two measures μ and σ on X are *singular* if there exist measurable sets A and B with $X = A \cup B$ and $A \cap B = \emptyset$, such that $\mu(A) = \sigma(B) = 0$.

To be able to define Clark measures in particular, we must first introduce the Poisson kernel on the unit polydisc. For $z \in \mathbb{D}^d$ and $\zeta \in \mathbb{T}^d$, we define the Poisson kernel on $\mathbb{D}^d$ as a product of one-variable Poisson kernels:

$$P_z(\zeta) = P(z, \zeta) := \prod_{j=1}^{d} P_{z_j}(\zeta_j) \quad \text{where} \quad P_{z_j}(\zeta_j) := \frac{1 - |z_j|^2}{|\zeta_j - z_j|^2}.$$

Given a complex Borel measure μ on $\mathbb{T}^d$, we define its Poisson integral as

$$P[d\mu](z) := \int_{\mathbb{T}^d} P(z, \zeta) d\mu(\zeta).$$

The Poisson integral will be of great importance to this work — in fact, by the multivariate version of Herglotz’ theorem (see e.g. Theorem 2.1.3 in [21]), each positive pluriharmonic function on $\mathbb{D}^d$ can be written as the Poisson integral of some unique Borel measure. We will use this to define Clark measures in what comes next.

Furthermore, note that $P[dm_d](z) = 1$ for all $z \in \mathbb{D}^d$. As in the one-variable case, we may express the Poisson integral as a series: for $z_j = r_j e^{i\theta_j}$,

$$P[d\mu](z) = \sum_{k \in \mathbb{Z}^d} \hat{\mu}(k) r^{|k|} e^{ik \bullet \theta}, \tag{2}$$

where we have used multinomial notation again, so $r^{|k|} = r_1^{|k_1|} \cdots r_d^{|k_d|}$ and $k \bullet \theta = k_1\theta_1 + \ldots + k_d\theta_d$. We call

$$\hat{\mu}(k) := \int_{\mathbb{T}^d} \overline{\zeta}^k d\mu(\zeta)$$

the *Fourier coefficients* of μ, where $\overline{\zeta}^k = \overline{\zeta_1}^{k_1} \cdots \overline{\zeta_d}^{k_d}$. The series (2) converges uniformly for points $z \in \mathbb{D}^d$.

Let $\alpha \in \mathbb{T}$; we say that α is a *unimodular* constant. If $\phi : \mathbb{D}^d \to \mathbb{D}$ is a bounded holomorphic function, then

$$\Re\left(\frac{\alpha + \phi(z)}{\alpha - \phi(z)}\right) = \frac{1 - |\phi(z)|^2}{|\alpha - \phi(z)|^2}$$

is positive and pluriharmonic on $\mathbb{D}^d$, as $(\alpha + \phi(z))/(\alpha - \phi(z))$ is holomorphic on the unit polydisc. Hence, by Herglotz’ theorem, there exists a unique positive Borel measure σ_α on $\mathbb{T}^d$ such that

$$\frac{1 - |\phi(z)|^2}{|\alpha - \phi(z)|^2} = P[d\sigma_\alpha](z) = \int_{\mathbb{T}^d} P(z, \zeta) d\sigma_\alpha(\zeta).$$

We call these measures $\{\sigma_\alpha\}_{\alpha\in\mathbb{T}}$ the *Aleksandrov-Clark measures* associated to ϕ. As

$$\int_{\mathbb{T}^d} d\sigma_\alpha = \int_{\mathbb{T}^d} P(0,\zeta)d\sigma_\alpha = \frac{1-|\phi(0)|^2}{|\alpha-\phi(0)|^2} < \infty,$$

we see that σ_α is a finite measure for each $\alpha \in \mathbb{T}$.

Recall that for a function $f : \mathbb{D} \to \mathbb{C}$ and some point $\zeta \in \mathbb{T}$, we say that $f(z)$ approaches $L \in \mathbb{C}$ *non-tangentially*, denoted

$$L := \angle \lim_{z\to\zeta} f(z),$$

if $f(z) \to L$ whenever $z \to \zeta$ in every fixed *Stolz domain*

$$\Gamma_\alpha(\zeta) := \{z \in \mathbb{D} : |z-\zeta| < \alpha(1-|z|)\}, \quad \alpha > 1.$$

This notion extends to multivariate functions: let $f : \mathbb{D}^d \to \mathbb{C}$ and $\zeta \in \mathbb{T}^d$. We say that f has a non-tangential limit L at ζ if $f(z) \to L$ as $z \to \zeta$, where each $z_j \to \zeta_j$ non-tangentially in the one-variable sense. For any bounded holomorphic function $\phi : \mathbb{D}^d \to \mathbb{C}$, Fatou's theorem for polydiscs (see Chapter XVII, Theorem 4.8 in [23]) ensures that the non-tangential limits

$$\phi^*(\zeta) = \angle \lim_{\mathbb{D}^d \ni z\to\zeta} \phi(z)$$

exist for m_d-almost every $\zeta \in \mathbb{T}^d$. Moreover, in one variable, we know that $\phi(r\zeta)$ converges to $\phi^*(\zeta)$ as $r \to 1-$ in $L^p(\mathbb{T})$ (Theorem 11.16, [22]).

We say that $\phi : \mathbb{D}^d \to \mathbb{D}$ is an *inner* function if it is bounded, holomorphic and $|\phi^*(\zeta)| = 1$ for m_d-almost every $\zeta \in \mathbb{T}^d$. For example, any finite Blaschke product is an inner function in $\mathbb{D}$. If ϕ is inner, we call $\{\sigma_\alpha\}_{\alpha\in\mathbb{T}}$ the *Clark measures* of ϕ.

Note that ϕ being inner implies that

$$P[d\sigma_\alpha](z) = \frac{1-|\phi(z)|^2}{|\alpha-\phi(z)|^2} = 0 \qquad m_d\text{-almost everywhere on } \mathbb{T}^d.$$

Clearly, the numerator goes to zero m_d-almost everywhere. As $\phi-\alpha$ is bounded, it lies in the Hardy space $H^2(\mathbb{T}^d)$ defined as the space of functions f analytic on $\mathbb{D}^d$ which satisfy

$$\sup_{0\le r<1} \left(\int_{\mathbb{T}^d} |f(r\zeta)|^2 dm_d(\zeta) \right)^{1/2} < \infty.$$

This is a subspace of the so called Nevanlinna space $N(\mathbb{T}^d)$; hence, Theorem 3.3.5 in [21] states that $\log(\phi^*-\alpha)$ lies in $L^1(\mathbb{T}^d)$. This in turn implies that $\phi^*-\alpha$ must be non-zero m_d-almost everywhere on $\mathbb{T}^d$. Hence, $P[d\sigma_\alpha](z) = 0$ m_d-almost everywhere on $\mathbb{T}^d$, as asserted.

A notable consequence of this result is that if ϕ is an inner function, then its Clark measures $\{\sigma_\alpha\}_{\alpha\in\mathbb{T}}$ must be singular with respect to the Lebesgue measure. To see this, we decompose σ_α into an absolutely continuous measure τ_α^1 and a m_d-singular measure τ_α^2 (see Theorem 6.10, [22]). Then Theorem 2.3.1 in [21] states that the function

$$u(z) := P[d\sigma_\alpha](z) = P[d\tau_\alpha^1 + d\tau_\alpha^2](z)$$

satisfies $u^*(\zeta) = \tau_\alpha^1(\zeta)$ for m_d-almost every $\zeta \in \mathbb{T}^d$. However, we saw already that $P[d\sigma_\alpha] = 0$ m_d-almost everywhere on $\mathbb{T}^d$; hence $\tau_\alpha^1 = 0$ m_d-almost everywhere on $\mathbb{T}^d$. We can thus conclude that σ_α is a m_d-singular measure for each $\alpha \in \mathbb{T}$. Moreover, as asserted in [11], two Clark measures σ_α and σ_β associated to an inner function ϕ are mutually singular whenever $\alpha \neq \beta$.

For an inner function ϕ and a constant $\alpha \in \mathbb{T}$, we define the so called *unimodular level set*

$$\mathcal{C}_\alpha(\phi) := \mathrm{Clos}\Big\{\zeta \in \mathbb{T}^d : \lim_{r\to 1-} \phi(r\zeta) = \alpha\Big\},$$

where the closure is taken with respect to $\mathbb{T}^d$. The following proposition is a generalization of Lemma 2.1 in [3], and is likely known to experts.

Proposition 2.1. *Let $\phi : \mathbb{D}^d \to \mathbb{C}$ be an inner function, and let α be a unimodular constant. Then* $\mathrm{supp}(\sigma_\alpha) \subset \mathcal{C}_\alpha(\phi)$.

With the exception of some tweaks, the proof uses the same arguments as in [3]. We include the details for the interested reader.

Proof. Let $B \subset \mathbb{T}^d$ be an open ball such that $\lim_{r\to 1-} \phi(r\zeta) \neq \alpha$ for all $\zeta \in B$. Our goal is to show that $\sigma_\alpha(B) = 0$. Recall that the Poisson kernel is positive (easily checked from its definition); hence,

$$\int_B P(r\zeta,\eta)d\sigma_\alpha(\eta) \leq \int_{\mathbb{T}^d} P(r\zeta,\eta)d\sigma_\alpha(\eta) = \frac{1-|\phi(r\zeta)|^2}{|\alpha - \phi(r\zeta)|^2}$$

for all $\zeta \in B$ and every $0 \leq r < 1$. We make two observations now: first of all, we note that since ϕ is inner, the right-hand side tends to zero for m_d-almost every $\zeta \in B$ as $r \to 1-$. So

$$\lim_{r\to 1-} \int_B P(r\zeta,\eta)d\sigma_\alpha(\eta) = 0 \quad m_d\text{-almost everywhere in } B.$$

Secondly, since ϕ is bounded on the unit polydisc and $\phi(r\zeta) \not\to \alpha$ on B, we have that

$$\limsup_{r\to 1-} \int_B P(r\zeta,\eta)d\sigma_\alpha(\eta) \leq \limsup_{r\to 1-} \frac{1-|\phi(r\zeta)|^2}{|\alpha - \phi(r\zeta)|^2} < \infty \tag{3}$$

for *all* $\zeta \in B$. Here, we take the limit superior instead of the limit, as the limit of the right-hand side need not exist for every point in B.

Now define

$$D_r(\zeta) := \{\eta : |r\zeta_j - \eta_j| \leq 2(1-r) : \quad j = 1, \dots, d\ \}.$$

For every η in this set, we have $|r\zeta_j - \eta_j|^2 \leq 4(1-r)^2$, which implies

$$\frac{1-r^2}{4(1-r)^2} = \frac{1+r}{4(1-r)} \leq \frac{1-r^2}{|r\zeta_j - \eta_j|^2}.$$

By the definition of the Poisson kernel, we find that

$$\left(\frac{1+r}{4(1-r)}\right)^d \leq P(r\zeta,\eta).$$

Then

$$\left(\frac{1+r}{4(1-r)}\right)^d \sigma_\alpha(B \cap D_r(\zeta)) \leq \int_{B\cap D_r(\zeta)} P(r\zeta,\eta)d\sigma_\alpha(\eta) \leq \int_B P(r\zeta,\eta)d\sigma_\alpha(\eta),$$

which in turn implies that

$$\lim_{r\to 1-} \frac{\sigma_\alpha(B\cap D_r(\zeta))}{(1-r)^d} = 0 \quad m_d\text{-almost everywhere in } B \tag{4}$$

and, moreover, that the limit superior of this quotient is finite for all $\zeta \in B$ by (3).

Since $\zeta_j, \eta_j \in \mathbb{T}$, we can express

$$|r\zeta_j - \eta_j| = |r - \eta_j\overline{\zeta_j}| < 2(1-r)$$

in polar coordinates as

$$\begin{aligned}
&2(1-r) > |r - e^{i\theta_j}| \\
&\iff 4(1-2r+r^2) > 1 + r^2 - 2r\cos(\theta_j) \\
&\iff 3r^2 - 6r + 3 = 3(1-r)^2 > 2r - 2r\cos(\theta_j) \\
&\iff \cos(\theta_j) > 1 - \frac{3(1-r)^2}{2r}.
\end{aligned}$$

This means that we can express $D_r(\zeta)$ as

$$D_r(\zeta) = \left\{ \zeta e^{i\theta} : |\theta_j| < \cos^{-1}\left(1 - \frac{3(1-r)^2}{2r}\right), \quad j = 1,\dots,d \right\}.$$

We observe that this is, as a subset of $\mathbb{T}^d$, a product of d copies of the same interval. Hence, as $r \to 1-$, we may estimate the Lebesgue measure of this set as

$$|D_r(\zeta)| = 2^d \cos^{-1}\left(1 - \frac{3(1-r)^2}{2r}\right)^d \geq c(d)\sqrt{\frac{3(1-r)^2}{2r}}^{\,d} \geq c'(d)(1-r)^d$$

for constants $c(d), c'(d)$ dependent on d. Together with (4), this shows that

$$\lim_{r\to 1-} \frac{\sigma_\alpha(B\cap D_r(\zeta))}{|D_r(\zeta)|} = 0 \quad m_d\text{-almost everywhere in } B,$$

and that the limit superior must be finite for all $\zeta \in B$.

Note that per definition, $D_r(\zeta)$ is a d-dimensional cube with volume tending to zero as $r \to 1-$ for every $\zeta \in B$. We now claim that

$$\limsup_{r\to 1-} \frac{\sigma_\alpha(B\cap D_r(\zeta))}{|D_r(\zeta)|} = 0 \quad \text{for every } \zeta \in B. \tag{5}$$

To prove this, suppose there exists some $z \in B$ such that the limit superior in (5) is nonzero. Since σ_α is a finite measure, we have that $\sigma_\alpha(B \cap D_r(z)) < \infty$. Together with the fact that the denominator tends to zero, this would imply that the limit, and hence limit superior, is infinite for $z \in B$, which is a contradiction by our previous arguments. Hence (5) holds.

Since $|D_r(\zeta)| \to 0$ as $r \to 1-$, the limit in (5) implies that the n-dimensional upper density of the restriction measure $(\sigma_\alpha)_{|B}$, defined as $(\sigma_\alpha)_{|B}(A) := \sigma_\alpha(B \cap A)$, is zero for every point in $\mathbb{T}^d$ (see e.g. Proposition 2.2.2 in [18]). Thus, $(\sigma_\alpha)_{|B}$ is equal to zero, which in turn implies that $\sigma_\alpha(B) = 0$. □

Example 2.2. Let $\phi(z_1, z_2) := z_1$. This is clearly an inner function, and for each given unimodular constant α, its Clark measure σ_α satisfies

$$\frac{1-|z_1|^2}{|\alpha - z_1|^2} = \int_{\mathbb{T}^2} P(z,\zeta) d\sigma_\alpha(\zeta) = \int_{\mathbb{T}^2} P_{z_1}(\zeta_1) P_{z_2}(\zeta_2) d\sigma_\alpha(\zeta).$$

We may see this as

$$\frac{1-|z_1|^2}{|\alpha - z_1|^2} \cdot 1 = \int_{\mathbb{T}^2} P_{z_1}(\zeta_1) P_{z_2}(\zeta_2) d\sigma_\alpha(\zeta).$$

In the variable z_1, the measure will be supported on a single point $z_1 = \alpha$. In the variable z_2, it can be seen as the measure μ for which the Poisson integral evaluates to one. But this implies that μ is just the usual normalized Lebesgue measure. Hence, we may conclude that

$$\sigma_\alpha(z) = \delta_\alpha(z_1) \otimes m(z_2).$$

This measure is supported on the line $\{(\alpha, \zeta_2) : \zeta_2 \in \mathbb{T}\}$ in $\mathbb{T}^2$. By Theorem 4 in [4], measures of this kind cannot be supported on sets of Hausdorff dimension less than one, and cannot possess any point masses. Hence, this is essentially the simplest support we can find for Clark measures in two variables, in the sense that it is very easy to characterize.

Next, we establish a result which will be used in several proofs down the line:

Lemma 2.3. *The linear span of Poisson kernels $\mathcal{M} := span\{P_z : z \in \mathbb{D}^2\}$ is dense in $C(\mathbb{T}^2)$.*

The following is a generalization of the proof of Proposition 1.17 in [15] and uses some Hilbert space theory we hope the reader will find familiar:

Proof. It is a well-known fact that the dual space of $C(\mathbb{T}^2)$ is the space $M(\mathbb{T}^2)$ of all so called *Radon* measures on $\mathbb{T}^2$. As $\mathbb{T}^2$ is compact, $M(\mathbb{T}^2)$ is just the space of complex, finite Borel measures on $\mathbb{T}^2$ (see Theorem 7.8 in [13]).

By Riesz' representation theorem (Theorem 6.19, [22]), if ℓ is a bounded linear functional on $C(\mathbb{T}^2)$, then there is a unique $\mu \in M(\mathbb{T}^2)$ such that

$$\ell(f) = \int_{\mathbb{T}^2} f d\mu.$$

As a consequence, $\mathcal{M}$ is dense in $C(\mathbb{T}^2)$ if the only measure $\mu \in M(\mathbb{T}^2)$ such that

$$P[d\mu](z) = \int_{\mathbb{T}^2} P_z(\zeta) d\mu(\zeta) = 0 \quad \text{for every } z \in \mathbb{D}^2$$

is the zero measure (see e.g. Theorem 1.27 in [15]). In this case, we say that the only *annihilator* of $\mathcal{M}$ is the zero measure. Using the series expansion of the Poisson integral, we see that if μ is an annihilator, then for all $n \in \mathbb{Z}^2$,

$$0 = \int_{\mathbb{T}^2} \overline{\zeta}^n P[d\mu](r\zeta) dm_2(\zeta) = \int_{\mathbb{T}^2} \overline{\zeta}^n \left(\sum_{k\in\mathbb{Z}^2} \hat{\mu}(k) r^{|k|} \zeta^k \right) dm_2(\zeta) = \int_{\mathbb{T}^2} \left(\sum_{k\in\mathbb{Z}^2} \hat{\mu}(k) r^{|k|} \zeta^{k-n} \right) dm_2(\zeta), \tag{6}$$

where $0 < r < 1$ and $\zeta \in \mathbb{T}^2$. Under these conditions, the last sum converges uniformly, so we may interchange summation and integration.

Note now that for $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{T}^2$ and $(k_1, k_2) \in \mathbb{Z}^2$,

$$\int_{\mathbb{T}^2} e^{i\theta_1 k_1 + i\theta_2 k_2} dm_2(\theta) = \int_{\mathbb{T}} e^{i\theta_1 k_1} dm(\theta_1) \int_{\mathbb{T}} e^{i\theta_2 k_2} dm(\theta_2) = \begin{cases} 1 & \text{if } (k_1, k_2) = (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

By the above, (6) reduces to

$$0 = \hat{\mu}(n) r^{|n|} + \sum_{\substack{k \neq n \\ k \in \mathbb{Z}^2}} \hat{\mu}(k) r^{|k|} \int_{\mathbb{T}^2} \zeta^{k-n} dm_2(\zeta) = \hat{\mu}(n) r^{|n|}.$$

Thus all the Fourier coefficients of μ are zero, which implies that μ is the zero measure — see e.g. Proposition 1.16 in [15] (all the arguments can be directly generalized to several variables). □

3 Clark measures in one variable

Before getting into Clark measures of inner functions in two variables, we give a brief overview of Clark theory in one variable. In this section, we summarize the results we will need when extending our scope to the bivariate case.

To formulate our main result, we must first introduce the concept of *angular derivatives*.

Theorem 3.1. *For an analytic function f on $\mathbb{D}$ and $\zeta_0 \in \mathbb{T}$, the following are equivalent:*

(i) The non-tangential limits

$$f(\zeta_0) = \angle \lim_{z\to\zeta_0} f(z) \quad \textit{and} \quad \angle \lim_{z\to\zeta_0} \frac{f(z) - f(\zeta_0)}{z - \zeta_0}$$

exist;

(ii) The derivative function f' has a non-tangential limit at ζ_0.

Under the equivalent conditions above,

$$\angle \lim_{z\to\zeta_0} \frac{f(z) - f(\zeta_0)}{z - \zeta_0} = \angle \lim_{z\to\zeta_0} f'(z).$$

Proof. See Theorem 2.19 in [15]. □

Definition 3.2. *Assuming the conditions of the theorem, we call*

$$f'(\zeta_0) := \angle \lim_{z \to \zeta_0} \frac{f(z) - f(\zeta_0)}{z - \zeta_0} = \angle \lim_{z \to \zeta_0} f'(z)$$

the angular derivative of f at ζ_0. Furthermore, if f maps $\mathbb{D}$ to itself, we say that f has an angular derivative in the sense of Carathéodory at $\zeta_0 \in \mathbb{T}$ if f has an angular derivative at ζ_0 and $f(\zeta_0) \in \mathbb{T}$.

We now have the machinery needed to state the following proposition, which will be extremely useful to us in later sections:

Proposition 3.3. *Let ϕ be an inner function in one variable and $\alpha \in \mathbb{T}$. Then the associated Clark measure σ_α has a point mass at $\zeta \in \mathbb{T}$ if and only if*

$$\phi^*(\zeta) = \lim_{r \to 1-} \phi(r\zeta) = \alpha$$

and ϕ has a finite angular derivative in the sense of Carathéodory at ζ. In this case,

$$\sigma_\alpha(\{\zeta\}) = \frac{1}{|\phi'(\zeta)|} < \infty \quad \textit{and} \quad \phi'(\zeta) = \frac{\alpha\bar{\zeta}}{\sigma_\alpha(\{\zeta\})}.$$

Proof. See Proposition 11.2 in [15]. □

This result is a powerful tool for Clark theory in one variable, and establishes a natural connection between derivatives of inner functions and their associated Clark measures. We will see that while Clark measures in two variables do not possess point masses (see [4]), the proposition will still be useful in determining the weights of these measures along curves in the unimodular level sets.

Example 3.4. The function

$$\phi(z) := \exp\left(-\frac{1+z}{1-z}\right), \quad z \in \mathbb{D},$$

is inner, and $\phi^*(\zeta)$ exists everywhere on $\mathbb{T}$; this because

$$\left|\exp\left(-\frac{1+z}{1-z}\right)\right| = \exp\left(\Re\left(-\frac{1+z}{1-z}\right)\right) = \exp\left(-\frac{1-|z|^2}{|1-z|^2}\right),$$

from which we can see that $\phi^*(1) = 0$. Observe that every point $\zeta \neq 1$ on the unit circle solves the equation $\phi^* = \alpha$ for some $\alpha \in \mathbb{T}$, so $\phi^*(\mathbb{T} \setminus \{1\}) = \mathbb{T}$. Moreover, these points accumulate in the limit point $\zeta = 1$ for every α-value. Since the unimodular level sets are closed per definition, this implies that $1 \in \mathcal{C}_\alpha(\phi)$ for all $\alpha \in \mathbb{T}$.

Now let $\alpha = 1$. As seen in Example 11.3(ii) in [15], the solutions to $\phi^*(\zeta) = 1$ are given by

$$\eta_k = \frac{2\pi k - i}{2\pi k + i}, \quad k \in \mathbb{Z},$$

and

$$\frac{1}{|\phi'(\eta_k)|} = \frac{8}{1+4\pi^2k^2}.$$

By Proposition 3.3, the Clark measure of ϕ associated to $\alpha = 1$ may thus be expressed as

$$\sigma_1 = \sum_{k\in\mathbb{Z}} \frac{8}{1+4\pi^2k^2}\delta_{\eta_k}.$$

We will revisit variations of this example in later sections.

4 Clark measures of monomials

We begin our study of bivariate inner functions and their Clark measures with a very simple class of functions: monomials. The general technique applied here to calculate the supports and density of their Clark measures will prove useful even for more complicated inner functions, and we will revisit some of these ideas further down the line.

Let

$$\phi(z_1, z_2) = z_1^N z_2^M$$

for non-negative integers M, N. Clearly, ϕ is continuous and analytic everywhere on the closed unit bidisc, so we need not worry about non-tangential limits at all. In this case, the unimodular level sets are simply given by

$$\mathcal{C}_\alpha(\phi) = \{\zeta \in \mathbb{T}^2 : \phi(\zeta) = \alpha\}.$$

We note that $\phi(z_1, z_2) = \alpha$ can be expressed as $z_2^M = \alpha/z_1^N$. As $z \in \mathbb{T}$, this has precisely M solutions; let $z_1 = e^{i\theta}$ and $\alpha = e^{i\nu}$. Then $z_2^M = e^{i(\nu - N\theta)}$ has solutions

$$\eta_k := \exp\left(i\left(\frac{(\nu - N\theta)}{M} + \frac{2\pi k}{M}\right)\right), \quad k = 0, 1, \ldots, M-1.$$

We see that η_k depends on θ and thus on z_1, so this defines a parameterization of $\mathcal{C}_\alpha(\phi)$ by graphs

$$\{(\zeta, \eta_k(\zeta)) : \zeta \in \mathbb{T}, k = 0, \ldots, M-1\}.$$

In other words, for each k, we get a curve of the form

$$\left\{\left(e^{i\theta}, \exp\left(i\left(\frac{(\nu - N\theta)}{M} + \frac{2\pi k}{M}\right)\right)\right) : 0 \leq \theta \leq 2\pi\right\}.$$

Note that the second coordinate can be rewritten as $\exp\left(i\left(\frac{(\nu-N\theta)}{M}+\frac{2\pi k}{M}\right)\right) = (e^{i\nu}e^{2\pi k i})^{1/M}(e^{-i\theta})^{N/M}$, where $e^{-i\theta}$ is the conjugate of $e^{i\theta}$. This implies that $\eta_k(\zeta)$ is of the form $c_k\bar{\zeta}^{N/M}$ for a unimodular constant c_k. Then $\{(\zeta, \eta_k(\zeta)) : \zeta \in \mathbb{T}\}$ defines an antidiagonal in $\mathbb{T}^2$, and $\mathcal{C}_\alpha(\phi)$ consists of $M-1$ such antidiagonals. To summarize, for $\alpha = e^{i\nu}$, the associated Clark measure of ϕ is supported on

$$\mathcal{C}_\alpha(\phi) = \bigcup_{k=0}^{M-1} \left\{\left(\zeta, c_k\bar{\zeta}^{N/M}\right) : \zeta \in \mathbb{T}, c_k = e^{i(\nu+2\pi k)/M}\right\}.$$

Figure 1 shows the level curves for the inner function $\phi = z_1^3 z_2^2$ and $\alpha = 1$. More precisely, we have plotted the argument of the curves $\eta_k(e^{i\theta})$ for $-\pi \le \theta \le \pi$ and $\alpha = 1$, $M = 2$ and $N = 3$.

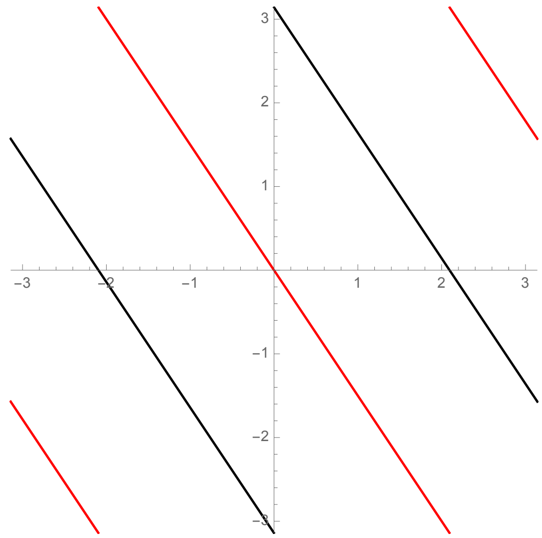


Figure 1: Level curves η_k for $k = 0$ (red) and $k = 1$ (black) when $\alpha = 1$, $M = 2$ and $N = 3$.

A natural next step is to investigate whether the Clark measures have densities along these lines. To this end, fix $\zeta \in \mathbb{T}$ and define $\Phi_\zeta(z_2) := \phi(\zeta, z_2)$. Given $\alpha = e^{i\nu} \in \mathbb{T}$, the solutions to $\Phi_\zeta(z_2) = \alpha$ on the unit circle are $\eta_k = \eta_k(\zeta) = c_k \bar{\zeta}^{N/M}$ as defined above for $k = 0, 1, \ldots, M-1$. Then, by Proposition 3.3, the Clark measure of Φ_ζ is given by

$$\sum_{k=0}^{M-1} \frac{1}{|\Phi_\zeta'(\eta_k)|}\delta_{\eta_k} = \sum_{k=0}^{M-1} \frac{1}{|M\eta_k^{M-1}\zeta^N|}\delta_{\eta_k} = \sum_{k=0}^{M-1} \frac{1}{M}\delta_{\eta_k},$$

where we have used that $\eta_k(\zeta)^{M-1}\zeta^N$ is unimodular for each $\zeta \in \mathbb{T}$. Hence, per definition of this Clark measure,

$$\begin{aligned}\frac{1 - |\Phi_\zeta(z_2)|^2}{|\alpha - \Phi_\zeta(z^2)|^2} = \frac{1 - |\phi(\zeta, z_2)|^2}{|\alpha - \phi(\zeta, z^2)|^2} &= \sum_{k=0}^{M-1} \frac{1}{M} \int_{\mathbb{T}} P_{z_2}(\zeta)\delta_{\eta_k}(\zeta) \\ &= \sum_{k=0}^{M-1} \frac{1}{M} P_{z_2}(\eta_k). \end{aligned} \tag{7}$$

Let σ_α be the associated Clark measure of ϕ for $\alpha = e^{i\nu}$. Now fix $z_2 \in \mathbb{D}$ and define

$$u_{z_2}(z_1) := \frac{1 - |\phi(z_1, z_2)|^2}{|\alpha - \phi(z_1, z_2)|^2} = \int_{\mathbb{T}^2} P_{z_1}(\xi_1)P_{z_2}(\xi_2)d\sigma_\alpha(\xi), \quad z_1 \in \mathbb{D}. \tag{8}$$

Since ϕ is continuous on the closed unit bidisc and $|z_1^N z_2^M| < 1$ for all $z_1 \in \overline{\mathbb{D}}$, the denominator of u_{z_2} is always non-zero. Hence, u_{z_2} is continuous on $\overline{\mathbb{D}}$, and harmonic on $\mathbb{D}$ as the middle expression is pluriharmonic per definition. Thus, we may apply the Poisson integral formula, which states that

$$u_{z_2}(z_1) = \int_{\mathbb{T}} \frac{1 - |\phi(\zeta, z_2)|^2}{|\alpha - \phi(\zeta, z_2)|^2} P_{z_1}(\zeta) dm(\zeta).$$

Combining (7) and (8) with the above equation yields

$$\begin{aligned}\int_{\mathbb{T}^2} P_{z_1}(\xi_1) P_{z_2}(\xi_2) d\sigma_\alpha(\xi) &= \int_{\mathbb{T}} \frac{1 - |\phi(\zeta, z_2)|^2}{|\alpha - \phi(\zeta, z_2)|^2} P_{z_1}(\zeta) dm(\zeta) \\ &= \sum_{k=0}^{M-1} \frac{1}{M} \int_{\mathbb{T}} P_{z_1}(\zeta) P_{z_2}(\eta_k(\zeta)) dm(\zeta).\end{aligned}$$

This shows that

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_\alpha(\xi) = \sum_{k=0}^{M-1} \frac{1}{M} \int_{\mathbb{T}} f(\zeta, \eta_k(\zeta)) dm(\zeta)$$

for $f = P_{z_1} P_{z_2}$. We can now use the fact that linear combinations of Poisson kernels are dense in $C(\mathbb{T}^2)$ by Lemma 2.3 — this proves the following result:

Theorem 4.1. *Let $\phi(z_1, z_2) = z_1^N z_2^M$ for positive integers M, N and $\alpha = e^{i\nu} \in \mathbb{T}$. Then the associated Clark measure σ_α of ϕ satisfies*

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_\alpha(\xi) = \sum_{k=0}^{M-1} \frac{1}{M} \int_{\mathbb{T}} f(\zeta, c_k \bar{\zeta}^{N/M}) dm(\zeta)$$

for all $f \in C(\mathbb{T}^2)$, where $c_k = e^{i(\nu + 2\pi k)/M}$.

Intuitively, this shows that the Clark measures of monomials are very simple; they live on lines in $\mathbb{T}^2$ with negative slope, and the density takes the same constant value along each line. In what comes next, we will study more general classes of functions and investigate whether their Clark measures behave as nicely.

Remark 4.2. One could of course choose to parameterize z_1 as a function of z_2 instead; the same argument as above then yields the formula

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_\alpha(\xi) = \sum_{k=0}^{N-1} \frac{1}{N} \int_{\mathbb{T}} f(d_k \bar{\zeta}^{M/N}, \zeta) dm(\zeta)$$

for $f \in C(\mathbb{T}^2)$ and $d_k = e^{i(\nu + 2\pi k)/N}$. Let us explicitly show that this is indeed the same integral as the one obtained in Theorem 4.1 for a simple toy example. Define $\phi(z) = z_1^2 z_2$ and fix $\alpha = e^{i\nu}$; by Theorem 4.1, the associated Clark measure satisfies

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_\alpha(\xi) = \int_{\mathbb{T}} f(\zeta, c_0 \bar{\zeta}^2) dm(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}, e^{-2it} e^{i\nu}) dm(t). \tag{9}$$

Applying Theorem 4.1 with parameterization in z_2 instead yields formula

$$\begin{aligned}\int_{\mathbb{T}^2} f(\xi)d\sigma_\alpha(\xi) &= \frac{1}{2}\int_{\mathbb{T}} f(d_0\overline{\zeta}^{1/2}, \zeta)dm(\zeta) + \frac{1}{2}\int_{\mathbb{T}} f(d_1\overline{\zeta}^{1/2}, \zeta)dm(\zeta) \\ &= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi} f(e^{-i\theta/2}e^{i\nu/2}, e^{i\theta})dm(\theta) + \int_{-\pi}^{\pi} f(e^{-i\theta/2}e^{i(\nu/2+\pi)}, e^{i\theta})dm(\theta)\right).\end{aligned}$$

We can now change integration variable via $\theta = -2\eta$, obtaining

$$\frac{1}{2\pi}\left(\int_{-\pi/2}^{\pi/2} f(e^{i\eta}e^{i\nu/2}, e^{-2i\eta})dm(\eta) + \int_{-\pi/2}^{\pi/2} f(e^{i\eta}e^{i(\nu/2+\pi)}, e^{-2i\eta})dm(\eta)\right).$$

Now we switch variable to $\eta + \pi$ in the second integral, which yields

$$\begin{aligned}&\frac{1}{2\pi}\left(\int_{-\pi/2}^{\pi/2} f(e^{i\eta}e^{i\nu/2}, e^{-2i\eta})dm(\eta) + \int_{\pi/2}^{3\pi/2} f(e^{i\eta}e^{i\nu/2}, e^{-2i(\eta-\pi)})dm(\eta)\right) \\ &\qquad = \frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} f(e^{i\eta}e^{i\nu/2}, e^{-2i\eta})dm(\eta).\end{aligned}$$

Finally, let $t = \eta + \nu/2$. Then

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} f(e^{i\eta}e^{i\nu/2}, e^{-2i\eta})dm(\eta) &= \frac{1}{2\pi}\int_{-\pi/2+\nu/2}^{3\pi/2+\nu/2} f(e^{it}, e^{-2i(t-\nu/2)})dm(\eta) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it}, e^{-2it}e^{i\nu})dm(t),\end{aligned}$$

where the periodicity of the integrand allows us to change integration limits in the last step. Hence we arrive at the same formula as (9). A generalization of this argument can be used to show that we can switch variables in Theorem 4.1 without any trouble.

5 Rational inner functions

We now move on to a more general — and more complicated — class of functions. In this section, we review some recent results concerning the Clark measures of rational inner functions in two variables. This situation will require extra care, as we now have to deal with potential singularities of these functions. However, we will see that the support of any associated Clark measure is actually a finite union of graphs, and that we can explicitly write out its weights along these — much like in the previous section. We will mainly present results and proofs from [3], potentially with some added detail.

We will first need some terminology specific to rational inner functions. We say that a polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$ is *stable* if it has no zeros in $\mathbb{D}^d$, and that it has *polydegree* $(n_1, \dots, n_d) \in \mathbb{N}^d$ if p has degree n_j when viewed as a polynomial in z_j. By Theorem 5.2.5 in [21], any rational inner function (abbreviated RIF) in $\mathbb{D}^d$ can be written as

$$\phi(z) = e^{ia}\prod_{j=1}^{d} z_j^{k_j}\frac{\tilde{p}(z)}{p(z)}$$

where $a \in \mathbb{R}$, $k_1, \ldots, k_d \in \mathbb{N}$, p is a stable polynomial of polydegree $(n_1, \ldots, n_d)$, and

$$\tilde{p}(z) := z_1^{n_1} \cdots z_d^{n_d}\ \overline{p}\left(\frac{1}{\overline{z_1}}, \ldots, \frac{1}{\overline{z_2}}\right)$$

is its *reflection*. Note that any zero of p will be a zero of $\tilde{p}$ and vice versa, and that p and $\tilde{p}$ have the same polydegree. For simplicity, we will always assume that $\phi(z) = \frac{\tilde{p}}{p}$, where p and $\tilde{p}$ are so called *atoral* — a concept explored in e.g. [1]. In the context of this project, atoral simply means that p and $\tilde{p}$ share no common factors, and that in two dimensions in particular, p and $\tilde{p}$ have finitely many common zeros on $\mathbb{T}^2$. Hence, a rational inner function ϕ in two variables will have finitely many singularities on $\mathbb{T}^2$.

Moreover, we define the polydegree of a rational function $\phi = q/p$ as $(n_1, \ldots, n_d)$, where p and q have no common factors, and n_j is the maximum of the degrees of p and q when viewed as polynomials in variable z_j. Thus, the polydegree of $\phi = \tilde{p}/p$ as defined above agrees with the polydegrees of both its numerator and denominator.

It is known that for any rational inner function ϕ, the non-tangential limit $\phi^*(\zeta)$ exists and is unimodular for every $\zeta \in \mathbb{T}^d$ (see Theorem C, [16]). Moreover, note that for any index j, given some fixed values $\zeta_1, \ldots, \zeta_{j-1}, \zeta_{j+1}, \ldots, \zeta_d \in \mathbb{T}$, the function $z_j \mapsto \phi(\zeta_1, \ldots, z_j, \ldots, \zeta_d)$ is a univariate rational function in $\mathbb{D}$. Then, by [16], this function has unimodular non-tangential limits at every point on $\mathbb{T}$. By Fatou's lemma, this implies that it must be a finite Blaschke product of degree at most n_j. The following lemma refines this result:

Lemma 5.1. *Let $\phi = \tilde{p}/p$ be a RIF in $\mathbb{D}^d$ of polydegree $(n_1, \ldots, n_d)$. Given any $\zeta = (\zeta_1, \ldots, \zeta_{d-1}, \zeta_d) \in \mathbb{T}^d$, set $\zeta' := (\zeta_1, \ldots, \zeta_{d-1}) \in \mathbb{T}^{d-1}$. For a fixed ζ', we define*

$$\phi_{\zeta'}(z_d) := \phi(\zeta_1, \ldots, \zeta_{d-1}, z_d).$$

If ϕ has no singularities at any points of the form $(\zeta', w) \in \mathbb{T}^d$ where $w \in \mathbb{T}$, then $\phi_{\zeta'}$ is Blaschke product of precisely degree n_d.

The proof uses elementary properties of Blaschke products and the structure of RIFs, and is omitted here (see Lemma 2.3 in [3]).

As a first step in our analysis, we would like to characterize the supports of the associated Clark measures of RIFs. By Proposition 2.1, we may do this via the unimodular level sets. The following result gives us a straight-forward expression for the level sets of RIFs specifically.

Theorem 5.2. *For fixed $\alpha \in \mathbb{T}$, let*

$$\mathcal{L}_\alpha(\phi) := \{\zeta \in \mathbb{T}^d : \tilde{p}(\zeta) - \alpha p(\zeta) = 0\}.$$

Then $\mathcal{C}_\alpha(\phi) = \mathcal{L}_\alpha(\phi)$.

Proof. See Theorem 2.6 in [8]. □

Note that for any zero of p, the equation $\tilde{p} \ - \alpha p = 0$ is trivially satisfied. This implies that all singularities of ϕ on $\mathbb{T}^d$ are contained in $\mathcal{C}_\alpha(\phi)$. Moreover, recall that $\mathcal{C}_\alpha(\phi)$ is defined as the closure of $\{\zeta \in \mathbb{T}^d : \lim_{r\to 1-} \phi(r\zeta) \to \alpha\}$. This is in general not a closed set, since ϕ is not necessarily continuous on $\overline{\mathbb{D}}^d$. However, by the theorem above, we may characterize $\mathcal{C}_\alpha(\phi)$ as the zeros of a polynomial when ϕ is a RIF.

When we restrict ourselves to $d = 2$, we have an even nicer characterization of the unimodular level sets:

Lemma 5.3. *Let $\phi = \frac{\tilde{p}}{p}$ be a RIF of bidegree (m, n), and fix $\alpha \in \mathbb{T}$. For any choice of $\tau_0 \in \mathbb{T}$, there exists a finite number of functions $g_1^\alpha, \ldots, g_n^\alpha$ defined on $\mathbb{T}$ and analytic on $\mathbb{T} \setminus \{\tau_0\}$ such that $\mathcal{C}_\alpha(\phi)$ can be written as a union of curves*

$$\{(\zeta, g_j^\alpha(\zeta)) : \zeta \in \mathbb{T}\}, \quad j = 1, \ldots, n,$$

potentially together with a finite number of vertical lines $\zeta_1 = \tau_1, \ldots, \zeta_1 = \tau_k$, where each $\tau_j \in \mathbb{T}$.

Now for a word of warning: the curves g_j^α might intersect at singularities of ϕ, in which case analyticity is not obvious. Consider for example the equation $z_1^3 - z_2^2 = 0$; the graphs of the solutions are not analytic at their point of intersection (the origin). We must ensure that the points of intersection in $\mathcal{C}_\alpha(\phi)$ are not of this type, and that we can in some sense "pull apart" any crossed curves and prove that they are each analytic when viewed separately. However, it is shown in [7] that near each singularity of ϕ, the level sets actually consist of smooth curves. Hence, intersections at the singularities in $\mathcal{C}_\alpha(\phi)$ pose no threat. We will use this result in proving Lemma 5.3.

Proof. First, we fix $\tau \in \mathbb{T}$ and seek a parameterization of $\mathcal{C}_\alpha(\phi) \cap (I_\tau \times \mathbb{T})$, where $I_\tau \subset \mathbb{T}$ is a small interval containing τ. The proof is divided into two main steps, depending on whether τ is the z_1-coordinate of a singularity of ϕ or not. Recall that ϕ has finitely many singularities given our assumptions on p and $\tilde{p}$.

Step 1. Suppose first that τ is not the z_1-coordinate of a singularity of ϕ. By Lemma 5.1, $\phi_\tau(z_2) := \phi(\tau, z_2)$ is a non-constant finite Blaschke product of degree n. This implies that ϕ_τ maps $\mathbb{T}$ to itself precisely n times, and so $\phi_\tau(z_2) = \alpha$ has precisely n distinct solutions $\eta_1, \ldots, \eta_n \in \mathbb{T}$. Since ϕ_τ is a non-constant finite Blaschke product, its derivative ϕ_τ' is non-zero on $\mathbb{T}$, and thus

$$\frac{\partial \phi}{\partial z_2}(\tau, \eta_j) = \phi_\tau'(\eta_j) \neq 0, \quad j = 1, \ldots, n.$$

By our choice of τ, the function ϕ is analytic in a neighborhood of each (τ, η_j). The holomorphic implicit function theorem (Theorem 1.4.11 in [17]) then states that we can parameterize $\mathcal{C}_\alpha(\phi) = \mathcal{L}_\alpha(\phi)$ with locally analytic functions in some neighborhood of each such point. Formally, there exist locally analytic functions $g_{1,\tau}^\alpha, \ldots, g_{n,\tau}^\alpha$ and an open interval I_τ containing τ such that $\mathcal{C}_\alpha(\phi)$ is parameterized by

$$\zeta_2 = g_{1,\tau}^\alpha(\zeta_1), \ldots, \zeta_2 = g_{n,\tau}^\alpha(\zeta_1) \tag{10}$$

on $I_\tau \times U$, where U is a union of open arcs containing $\eta_1, \ldots, \eta_n$.

We observe that $\phi(\zeta_1, \zeta_2) = \alpha$ has n distinct solutions for each ζ_1 close to τ. By shrinking I_τ if necessary, we can ensure that there is no singularity lying in its closure; then (10) parameterizes all pieces of $\mathcal{C}_\alpha(\phi)$ contained in the strip $I_\tau \times \mathbb{T}$.

Step 2. Now suppose τ is the z_1-coordinate of a singularity of ϕ. There are two possibilities here: either,

(a) $\tilde{p}(\tau, \zeta) = \alpha p(\tau, \zeta)$ for any $\zeta \in \mathbb{T}$, so the line $\{\zeta \in \mathbb{T}^2 : \zeta_1 = \tau\}$ is contained in $\mathcal{C}_\alpha(\phi)$, or

(b) the intersection $\{\zeta \in \mathbb{T}^2 : \zeta_1 = \tau\} \cap \mathcal{C}_\alpha(\phi)$ consists of points (τ, z_2) where ϕ has a singularity, as well as points (τ, η) for which $\phi(\tau, \eta) = \phi_\tau(\eta) = \alpha$.

Suppose we are in case (a). Let $(\tau, \gamma) \in \mathbb{T}^2$ be the limit of a sequence $\{(\tau_m, \gamma_m)\}_{m=1}^{\infty} \subset \mathcal{C}_\alpha(\phi)$ where $\tau_m \neq \tau$. Our goal is to show that the point of intersection with the line $\zeta_1 = \tau$ must be a singularity of ϕ. To this end, define $\phi_m(z_1) := \phi(z_1, \gamma_m \overline{\tau}_m z_1)$ for each m. Since $\{\zeta \in \mathbb{T}^2 : \zeta_1 = \tau\} \subset \mathcal{C}_\alpha(\phi)$, we must have that $\phi_m(\tau) = \phi(\tau, \gamma_m) = \alpha$. By assumption, $\phi_m(\tau_m) = \phi(\tau_m, \gamma_m) = \alpha$ as well. Note now that ϕ_m is a non-constant finite Blaschke product, so it maps the entire unit circle to itself. This implies that ϕ_m will attain every value on $\mathbb{T}$ on each of the arcs between τ and τ_m. Hence, for any given $\lambda \neq \alpha$ in $\mathbb{T}$, we can find a sequence $\{\rho_m\}_{m=1}^{\infty} \subset \mathbb{T}$ where each ρ_m lies on the smaller of the two arcs of $\mathbb{T}$ between τ and τ_m, with the property that $\phi_m(\rho_m) = \lambda$.

Since $\tau_m \to \tau$, then $\rho_m \to \tau$ as well, and so $(\rho_m, \gamma_m \overline{\tau}_m \rho_m) \to (\tau, \gamma)$ as $m \to \infty$. This implies that ϕ is discontinuous at (τ, γ), and thus has a singularity at this point. We can now apply Theorem 2.9 in [7] at (τ, γ), which says that $\mathcal{C}_\alpha(\phi)$ can be parameterized by analytic functions near each singularity of ϕ. This takes care of case (a).

In case (b), we can again apply Theorem 2.9 in [7] at the singularities, and invoke the implicit function theorem at the other points.

To summarize, for both cases (a) and (b), we get a finite number of analytic functions which — possibly together with a line $\{\zeta \in \mathbb{T}^2 : \zeta_1 = \tau\}$ — parameterize $\mathcal{C}_\alpha$ on a strip $I_\tau \times \mathbb{T}$ for some sufficiently small interval I_τ. Moreover, for any τ that is not one of the singularities from case (b) (of which there are finitely many), there are precisely n distinct points $\eta_1, \ldots, \eta_n \in \mathbb{T}$ such that $\phi(\tau, \eta_j) = \alpha$. This means that in each case, we must get exactly n parameterizing functions, potentially together with a finite number of vertical lines.

It remains to extend this parameterization to $\mathbb{T}^2$. We can form a cover of $\mathbb{T}^2$ from strips $I_\tau \times \mathbb{T}$, where I_τ is as in step 1 or step 2. Since $\mathbb{T}^2$ is compact, we can refine this cover to a finite number of strips such that each of ϕ:s (finitely many) singularities belongs to one of the strips. We have shown that $\mathcal{C}_\alpha(\phi)$ can be analytically parameterized on each such strip, and the parameterizations agree on the overlaps. An issue that may arise is that as we go around the unit circle, one branch might end at the point where another branch began. This implies that we cannot always be sure that $g_j^\alpha(e^{i\theta}) = g_j^\alpha(e^{i\theta+2\pi i})$ for each j, but instead we might find that $g_j^\alpha(e^{i\theta}) = g_k^\alpha(e^{i\theta+2\pi i})$ for $j \neq k$. Hence, we introduce a point $\tau_0 \in \mathbb{T}$ where we allow the branches to jump. This yields functions $g_1^\alpha, \ldots, g_n^\alpha$ globally defined on $\mathbb{T}$ and analytic except at a single point, which, potentially together with a finite number of vertical lines, parameterize $\mathcal{C}_\alpha(\phi)$ as desired. □

The analysis of Clark measures of RIFs must now be divided into two cases; when the unimodular constant α is *generic* versus *exceptional* as defined below.

Definition 5.4. *We say that $\alpha \in \mathbb{T}$ is an exceptional value if $\phi(\tau, \zeta_2) \equiv \alpha$ or $\phi(\zeta_1, \tau) \equiv \alpha$ for some $\tau \in \mathbb{T}$. If α is not exceptional, we say that it is generic.*

The different cases arise from the characterization of $\mathcal{C}_\alpha(\phi)$ in Lemma 5.3; if α is an exceptional value, by the definition above, the level sets will contain lines of the form $\{\zeta_1 = \tau\}$ or $\{\zeta_2 = \tau\}$. If α is generic, $\mathcal{C}_\alpha(\phi)$ can be fully described by the graphs of the functions $g_1^\alpha, \ldots, g_n^\alpha$. We will study these situations separately in what follows.

Generic values

For generic values of α, the situation becomes relatively simple. Given a RIF ϕ, we are able to characterize the density of the Clark measures along the curves in $\mathcal{C}_\alpha(\phi)$ from Lemma 5.3.

Theorem 5.5. *Let $\phi = \frac{\tilde{p}}{p}$ be a RIF of bidegree (m,n) and $\alpha \in \mathbb{T}$ a generic value for ϕ. Then the associated Clark measure σ_α satisfies*

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_\alpha(\xi) = \sum_{j=1}^n \int_{\mathbb{T}} f(\zeta, g_j^\alpha(\zeta)) \frac{dm(\zeta)}{|\frac{\partial \phi}{\partial z_2}(\zeta, g_j^\alpha(\zeta))|}$$

for all $f \in C(\mathbb{T}^2)$, where $g_1^\alpha, \dots, g_n^\alpha$ are the parameterizing functions from Lemma 5.3.

Proof. We first prove this in the case when f is the product of one-variable Poisson kernels. Suppose $\alpha \in \mathbb{T}$ is a generic value for ϕ. Then, by Lemma 5.3,

$$\mathcal{C}_\alpha(\phi) = \bigcup_{j=1}^n \{(\zeta, g_j^\alpha(\zeta)) : \zeta \in \mathbb{T}\}$$

where the functions g_j^α are analytic on $\mathbb{T}$ except at a single point.

For fixed $z_2 \in \mathbb{D}$, define the one-variable function

$$u_{z_2}(z_1) := \frac{1 - |\phi(z_1, z_2)|^2}{|\alpha - \phi(z_1, z_2)|^2} = \int_{\mathbb{T}^2} P_{z_1}(\xi_1) P_{z_2}(\xi_2) d\sigma_\alpha(\xi), \quad z_1 \in \mathbb{D}, \tag{11}$$

where σ_α is the Clark measure of ϕ. By Lemma 10.1 in [16], bivariate RIFs cannot have singularities on $\mathbb{T} \times \mathbb{D}$, and so $\phi(\cdot, z_2)$ is continuous on $\overline{\mathbb{D}}$. We also claim that the denominator is non-zero for all $z_1 \in \overline{\mathbb{D}}$. Recall that for each $\zeta \in \mathbb{T}$, the function $\Phi_\zeta(z_2) := \phi(\zeta, z_2)$ is a finite Blaschke product. By basic properties of Blaschke products, this implies that unless Φ_ζ is constant, it must satisfy $|\Phi_\zeta(z_2)| < 1$ for all $z_2 \in \mathbb{D}$. Thus, if $\Phi_\zeta(z_2) = \alpha$ for some $\zeta \in \mathbb{T}$, it must be a constant function, which in turn implies that α is an exceptional value — a contradiction. Hence, $\phi(\cdot, z_2)$ cannot attain the value α on $\overline{\mathbb{D}}$. This proves that u_{z_2} is continuous on $\overline{\mathbb{D}}$.

Moreover, since the middle expression of (11) is pluriharmonic, u_{z_2} is harmonic in $\mathbb{D}$. Hence, we may apply the Poisson integral formula:

$$u_{z_2}(z_1) = \int_{\mathbb{T}} \frac{1 - |\phi(\zeta, z_2)|^2}{|\alpha - \phi(\zeta, z_2)|^2} P_{z_1}(\zeta) dm(\zeta). \tag{12}$$

We saw in Lemma 5.1 that given $\zeta \in \mathbb{T}$, the function $\Phi_\zeta(z_2) := \phi(\zeta, z_2)$ is a finite Blaschke product of degree n. Let $\{\eta_1, \dots, \eta_n\} \subset \mathbb{T}$ be the solutions to $\Phi_\zeta(z_2) = \alpha$. Then, by Proposition 3.3, the Clark measure of $\Phi_\zeta(z_2)$ will be given by

$$\sum_{j=1}^n \frac{1}{|\Phi_\zeta'(\eta_j)|} \delta_{\eta_j} = \sum_{j=1}^n \frac{1}{|\frac{\partial \phi}{\partial z_2}(\zeta, \eta_j)|} \delta_{\eta_j},$$

where the weights $|\frac{\partial \phi}{\partial z_2}(\zeta, \eta_j)|^{-1}$ are finite.

Using the definition of Clark measures and the parameterization of $\mathcal{C}_\alpha(\phi)$, this implies

$$\begin{aligned}
\frac{1 - |\phi(\zeta, z_2)|^2}{|\alpha - \phi(\zeta, z_2)|^2} &= \sum_{j=1}^n \frac{1}{|\frac{\partial \phi}{\partial z_2}(\zeta, \eta_j)|} \int_{\mathbb{T}} P_{z_2}(\xi) d\delta_{\eta_j}(\xi) \\
&= \sum_{j=1}^n \frac{1}{|\frac{\partial \phi}{\partial z_2}(\zeta, \eta_j)|} P_{z_2}(\eta_j) \\
&= \sum_{j=1}^n \frac{1}{|\frac{\partial \phi}{\partial z_2}(\zeta, g_j^\alpha(\zeta))|} P_{z_2}(g_j^\alpha(\zeta)).
\end{aligned}$$

We may now insert the above in (12) and use (11) to obtain

$$\int_{\mathbb{T}^2} P_{z_1}(\xi_1)P_{z_2}(\xi_2)d\sigma_\alpha(\xi) = \sum_{j=1}^{n}\int_{\mathbb{T}} P_{z_1}(\zeta)P_{z_2}(g_j^\alpha(\zeta))\frac{dm(\zeta)}{|\frac{\partial\phi}{\partial z_2}(\zeta, g_j^\alpha(\zeta))|}.$$

This proves the theorem for $f = P_{z_1}P_{z_2}$. Using that linear combinations of Poisson kernels are dense in $C(\mathbb{T}^2)$ by Lemma 2.3, we obtain the desired result. $\square$

Exceptional values

When α is an exceptional value, the situation becomes more intricate, as we must now take vertical lines in $\mathcal{C}_\alpha(\phi)$ into account. By looking at Theorem 5.5, one might fear that the weights along the lines $\{\zeta \in \mathbb{T} : \zeta_1 = \tau\}$ would grow uncontrollably large, as ϕ has a vanishing partial derivative there. However, it is shown in [3] that this is not the case; the weights along the vertical lines are actually constants. We omit the proof here as it is quite technical, and refer the interested reader to Theorem 3.8 in [3].

Theorem 5.6. *Let $\phi = \frac{\tilde{p}}{p}$ be a RIF of bidegree (m, n) and $\alpha \in \mathbb{T}$ an exceptional value for ϕ. Then, for $f \in C(\mathbb{T}^2)$, the associated Clark measure σ_α satisfies*

$$\int_{\mathbb{T}^2} f(\xi)d\sigma_\alpha(\xi) = \sum_{j=1}^{n}\int_{\mathbb{T}} f(\zeta, g_j^\alpha(\zeta))\frac{dm(\zeta)}{|\frac{\partial\phi}{\partial z_2}(\zeta, g_j^\alpha(\zeta))|} + \sum_{k=1}^{\ell} c_k^\alpha \int_{\mathbb{T}} f(\tau_k, \zeta)dm(\zeta),$$

where $g_1^\alpha, \ldots, g_n^\alpha$ are the parameterizing functions and $\zeta_1 = \tau_1, \ldots, \zeta_1 = \tau_\ell$ the vertical lines in $\mathcal{C}_\alpha(\phi)$ from Lemma 5.3, and $c_k^\alpha := 1/|\frac{\partial\phi}{\partial z_1}(\tau_k, z_2)| > 0$ are constants.

The case of rational inner functions of bidegree $(n, 1)$ specifically has been studied in great detail in [5]. For these RIFs, we obtain a more explicit version of Theorem 5.6. If $\phi = \tilde{p}/p$ has bidegree $(n, 1)$, we may write

$$p(z) = p_1(z_1) + z_2p_2(z_1) \quad \text{and} \quad \tilde{p}(z) = z_2\tilde{p}_1(z_1) + \tilde{p}_2(z_1)$$

for reflections $\tilde{p}_i = z_1^n\overline{p}_i(1/\overline{z}_i)$. In this case, solving $\phi^* = \alpha$ for z_2 yields $z_2 = \frac{1}{B_\alpha(z_1)}$, where

$$B_\alpha(z) := \frac{\tilde{p}_1(z) - \alpha p_2(z)}{\alpha p_1(z) - \tilde{p}_2(z)}.$$

Moreover, define

$$W_\alpha(\zeta) := \frac{|p_1(\zeta)|^2 - |p_2(\zeta)|^2}{|\tilde{p}_1(\zeta) - \alpha p_2(\zeta)|^2}.$$

Then, by Theorem 1.2 in [5], we have

$$\int_{\mathbb{T}^2} f(\xi)d\sigma_\alpha(\xi) = \int_{\mathbb{T}} f(\zeta, \overline{B_\alpha(\zeta)})W_\alpha(\zeta)dm(\zeta) + \sum_{k=1}^{\ell} c_k^\alpha \int_{\mathbb{T}} f(\tau_k, \zeta)dm(\zeta)$$

with $c_k^\alpha = 1/|\frac{\partial\phi}{\partial z_1}(\tau_k, z_2)|$ is non-zero if and only if α is an exceptional value. It is worth noting that for any RIF ϕ of bidegree $(n, 1)$, a value $\alpha \in \mathbb{T}$ is exceptional if and only if it is the non-tangential value of ϕ at some singularity (see Section 3 of [5]).

Example 5.7. For an explicit example, we use Example 5.2 from [5]: let $\phi = \frac{\tilde{p}}{p}$ for

$$p(z) = 4 - z_2 - 3z_1 - z_1 z_2 + z_1^2 \quad \text{and} \quad \tilde{p}(z) = 4z_1^2 z_2 - z_1^2 - 3z_1 z_2 - z_1 + z_2.$$

From here we see that ϕ has only one singularity, which occurs at $(1,1)$. Moreover, for each $\alpha \in \mathbb{T}$, the formulas above yield

$$B_\alpha(z) = \frac{4z_1^2 - 3z_1 + 1 + \alpha + \alpha z_1}{4\alpha - 3z_1\alpha + z_1^2\alpha + z_1^2 + z_1}$$

and

$$W_\alpha(\zeta) = \frac{4|\zeta - 1|^4}{|4\zeta^2 - 3\zeta + 1 + \alpha + \alpha\zeta|^2}.$$

We see that $\alpha = -1$ is an exceptional value, as $\phi = -1$ is solved by $(1, z_2)$ as well as $\left(z_1, \frac{1}{B_{-1}(z_1)}\right) = (z_1, 1/z_1)$. Since ϕ only has one singularity, this point gives rise to the only exceptional value and $\phi^*(1,1) = -1$. Hence, for $\alpha \neq -1$, we have

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_\alpha(\xi) = \int_{\mathbb{T}} f(\zeta, \overline{B_\alpha(\zeta)}) \frac{4|\zeta - 1|^4}{|4\zeta^2 - 3\zeta + 1 + \alpha + \alpha\zeta|^2} dm(\zeta).$$

Moreover, we see that $W_{-1}(\zeta) = \frac{1}{4}|\zeta - 1|^2$ and $\frac{\partial \phi}{\partial z_1}(1, z_2) = -2$, which yields

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_{-1}(\xi) = \frac{1}{4} \int_{\mathbb{T}} f(\zeta, \overline{\zeta}) |\zeta - 1|^2 dm(\zeta) + \frac{1}{2} \int_{\mathbb{T}} f(1, \zeta) dm(\zeta)$$

for $\alpha = -1$.

6 Multiplicative embeddings

The case of Clark measures for inner functions in one variable has been studied extensively with some strong results, e.g. Proposition 3.3 from earlier sections. In this section, we study a certain class of bivariate inner functions constructed from one-variable inner functions, and investigate to what extent the univariate analysis can be applied. We then compare these functions to the rational inner functions from the previous section.

Given an inner function ϕ in one complex variable, we define the multiplicative embedding

$$\psi(z) = \psi(z_1, z_2) := \phi(z_1 z_2), \quad z \in \mathbb{D}^2.$$

The function defined by $(z_1, z_2) \mapsto z_1 z_2$ maps $\mathbb{D}^2$ to $\mathbb{D}$, and so ϕ being an inner function implies that ψ is inner as well. In the following proposition, we characterize the support set of ψ with the help of the original function ϕ. Recall that we define the unimodular level sets of ψ as

$$\mathcal{C}_\alpha(\psi) = \operatorname{Clos}\left\{\zeta \in \mathbb{T}^2 : \lim_{r \to 1-} \psi(r\zeta) = \alpha\right\}.$$

Proposition 6.1. *Let $\phi(z)$ be an inner function in one variable, and $\alpha \in \mathbb{T}$. Define $\psi(z_1, z_2) := \phi(z_1 z_2)$. Then*

$$\mathcal{C}_\alpha(\psi) = \bigcup_{\zeta \in \mathcal{C}_\alpha(\phi)} \{(z, \zeta\overline{z}) : z \in \mathbb{T}\}.$$

Proof. First, for ease of notation, define

$$\mathcal{C}'_\alpha(f) := \left\{\zeta \in \mathbb{T}^d : \lim_{r\to 1-} f(r\zeta) = \alpha\right\}$$

for any inner function f, so that $\mathcal{C}_\alpha(f) = \mathrm{Clos}(\mathcal{C}'_\alpha(f))$.

Let $\zeta \in \mathcal{C}'_\alpha(\phi)$. Then we know that

$$\lim_{r\to 1-} \phi(r\zeta) = \alpha.$$

For every $z \in \mathbb{T}$,

$$\lim_{r\to 1-} \psi(r(z, \zeta\overline{z})) = \lim_{r\to 1-} \phi(r^2\zeta z\overline{z}) = \lim_{r\to 1-} \phi(r\zeta) = \alpha,$$

implying that $(z, \zeta\overline{z}) \in \mathcal{C}_\alpha(\psi)$. Thus

$$\bigcup_{\zeta\in\mathcal{C}'_\alpha(\phi)} \{(z, \zeta\overline{z}) : z \in \mathbb{T}\} \subset \mathcal{C}_\alpha(\psi).$$

To extend this to a union over $\mathcal{C}_\alpha(\phi)$, let $\zeta \in \mathcal{C}_\alpha(\phi)$. Then there exists some sequence $(\zeta_n)_{n\geq 1}$ in $\mathcal{C}'_\alpha(\phi)$ that converges to ζ as n tends to infinity. This also implies that for any $z \in \mathbb{T}$, $(z, \zeta_n\overline{z}) \to (z, \zeta\overline{z}) \in \mathcal{C}_\alpha(\psi)$ as $n \to \infty$. Hence,

$$\bigcup_{\zeta\in\mathcal{C}_\alpha(\phi)} \{(z, \zeta\overline{z}) : z \in \mathbb{T}\} \subset \mathcal{C}_\alpha(\psi).$$

Conversely, let $(z_1, z_2) \in \mathcal{C}'_\alpha(\psi)$, so

$$\lim_{r\to 1-} \psi(r(z_1, z_2)) = \lim_{r\to 1-} \phi(r^2 z_1 z_2) = \alpha.$$

Then $\zeta := z_1z_2 \in \mathcal{C}_\alpha(\phi)$. Since $z_1, z_2 \in \mathbb{T}$, we may write

$$z_2 = \frac{\zeta}{z_1} = \zeta\overline{z_1},$$

so $(z_1, z_2) = (z_1, \zeta\overline{z_1}) \in \{(z, \zeta\overline{z}) : z \in \mathbb{T}\}$. Hence,

$$\mathcal{C}'_\alpha(\psi) \subset \bigcup_{\zeta\in\mathcal{C}_\alpha(\phi)} \{(z, \zeta\overline{z}) : z \in \mathbb{T}\}.$$

Now let $(z_1, z_2) \in \mathcal{C}_\alpha(\psi)$. Then there is some sequence of $(z_{1,n}, z_{2,n})$ in $\mathcal{C}'_\alpha(\psi)$ converging to (z_1, z_2) as $n \to \infty$. But this implies that $z_{1,n}z_{2,n} \to z_1z_2 \in \mathcal{C}_\alpha(\phi)$, and the same argument as above then yields

$$\mathcal{C}_\alpha(\psi) \subset \bigcup_{\zeta\in\mathcal{C}_\alpha(\phi)} \{(z, \zeta\overline{z}) : z \in \mathbb{T}\}.$$

□

Intuitively speaking, this result says that if the Clark measure σ_α of ϕ has point masses at some $\zeta_j \in \mathbb{T}$, the corresponding measure τ_α of ψ will have its support "smeared" across antidiagonals dependent on the points ζ_j. As in the RIF case, the unimodular level sets of this class of functions may be expressed as unions of curves. However, as opposed to in Lemma 5.3, the unions need not be finite — or even countable — here.

Remark 6.2. A skeptic might now point out that so far, we have not seen any Clark measures supported on uncountable sets, except finite unions of curves. To convince ourselves that more complicated examples exist, observe that by Lemma 2.2 in [3], any positive, pluriharmonic, m_d-singular probability measure defines the Clark measure of some inner function. Hence, any such measure with sufficiently intricate support will do as an example.

A natural next step is to investigate whether we can characterize the density of a given Clark measure τ_α on the antidiagonals in $\mathcal{C}_\alpha(\psi)$. To do this, we will first need a formula for integrating $C(\mathbb{T}^2)$-functions with respect to τ_α.

Proposition 6.3. *Let $\phi(z)$ be an inner function in one variable, with Clark measure σ_α for some unimodular constant α. Let τ_α be the corresponding Clark measure of $\psi(z_1, z_2) := \phi(z_1 z_2)$. Then, for any function $f \in C(\mathbb{T}^2)$,*

$$\int_{\mathbb{T}^2} f(\xi) d\tau_\alpha(\xi) = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f(\zeta, x\bar{\zeta}) d\sigma_\alpha(x) \right) dm(\zeta).$$

Proof. We first prove this in the case when f is the product of one-variable Poisson kernels. Fixing $z_2 \in \mathbb{D}$, let

$$u_{z_2}(z_1) := \frac{1 - |\phi(z_1 z_2)|^2}{|\alpha - \phi(z_1 z_2)|^2} = \int_{\mathbb{T}^2} P_{z_1}(\xi_1) P_{z_2}(\xi_2) d\tau_\alpha(\xi), \quad z_1 \in \mathbb{D}.$$

As the middle expression is pluriharmonic, u_{z_2} must be harmonic on $\mathbb{D}$. Since $z_1 z_2 \in \mathbb{D}$ for any $z_1 \in \overline{\mathbb{D}}$, and ϕ is analytic (and hence continuous) on $\mathbb{D}$, we see that $\psi(z_1, z_2) = \phi(z_1 z_2)$ as a function of z_1 is continuous on $\overline{\mathbb{D}}$. Moreover, by the maximum principle, $|\phi| < 1$ on the unit disc, which implies that the denominator will always be non-zero. We conclude that u_{z_2} is continuous $\overline{\mathbb{D}}$, and we may thus apply the Poisson integral formula:

$$u_{z_2}(z_1) = \int_{\mathbb{T}} \frac{1 - |\phi(\zeta z_2)|^2}{|\alpha - \phi(\zeta z_2)|^2} P_{z_1}(\zeta) dm(\zeta). \tag{13}$$

Moreover, for $\zeta \in \mathbb{T}$, we see that

$$\begin{aligned}\int_{\mathbb{T}} P_z(\zeta, x\overline{\zeta}) d\sigma_\alpha(x) &= \int_{\mathbb{T}} P_{z_1}(\zeta) P_{z_2}(x\overline{\zeta}) d\sigma_\alpha(x) \\ &= P_{z_1}(\zeta) \int_{\mathbb{T}} \frac{1-|z_2|^2}{|x\overline{\zeta}-z_2|^2} d\sigma_\alpha(x) \\ &= P_{z_1}(\zeta) \int_{\mathbb{T}} \frac{1-|z_2|^2}{|x\overline{\zeta}-z_2|^2} \frac{|\zeta|^2}{|\zeta|^2} d\sigma_\alpha(x) \\ &= P_{z_1}(\zeta) \int_{\mathbb{T}} \frac{1-|\zeta z_2|^2}{|x-\zeta z_2|^2} d\sigma_\alpha(x) \\ &= P_{z_1}(\zeta) \int_{\mathbb{T}} P_{\zeta z_2}(x) d\sigma_\alpha(x) \\ &= P_{z_1}(\zeta) \frac{1-|\phi(\zeta z_2)|^2}{|\alpha-\phi(\zeta z_2)|^2},\end{aligned}$$

where we used the definition of the Clark measure σ_α in the last step. By integrating the above and applying (13), we get

$$\begin{aligned}\int_{\mathbb{T}} \left(\int_{\mathbb{T}} P_z(\zeta, x\overline{\zeta}) d\sigma_\alpha(x) \right) dm(\zeta) &= \int_{\mathbb{T}} \frac{1-|z_1|^2}{|\zeta-z_1|^2} \frac{1-|\phi(\zeta z_2)|^2}{|\alpha-\phi(\zeta z_2)|^2} dm(\zeta) \\ &= \frac{1-|\phi(z_1 z_2)|^2}{|\alpha-\phi(z_1 z_2)|^2} \\ &= \int_{\mathbb{T}^2} P_{z_1}(\xi_1) P_{z_2}(\xi_2) d\tau_\alpha(\xi).\end{aligned}$$

Lemma 2.3 now yields the desired result. □

Remark 6.4. It is a priori not obvious that $f(\zeta, x\overline{\zeta})$ is integrable with respect to σ_α. Integrability is ensured by the fact that $f(\zeta, x\overline{\zeta})$ is continuous on $\mathbb{T}$, as it is composed by two functions f and $g_x(z) := (z, x\overline{z})$ which are continuous there. Since σ_α is a finite, positive Borel measure on a compact space, all continuous functions on said space are integrable with respect to σ_α.

We now have the machinery we need to prove the following result:

Theorem 6.5. *Let $\phi(z)$ be an inner function in one variable, with Clark measure σ_α for some unimodular constant α. Let τ_α be the corresponding Clark measure of $\psi(z_1, z_2) := \phi(z_1 z_2)$. If σ_α is supported on a countable collection of points $\{\eta_k\}_{k\geq 1} \subset \mathcal{C}_\alpha(\phi)$, then*

$$\int_{\mathbb{T}^2} f(\xi) d\tau_\alpha(\xi) = \sum_{k\geq 1} \int_{\mathbb{T}} f(\zeta, \eta_k \overline{\zeta}) \frac{dm(\zeta)}{|\phi'(\eta_k)|}$$

for all $f \in C(\mathbb{T}^2)$.

Proof. By Proposition 3.3, σ_α having a point mass at some η_k implies that $\sigma_\alpha(\{\eta_k\}) = 1/|\phi'(\eta_k)|$.

Then, following the steps in the proof of Proposition 6.3,

$$\begin{aligned}\int_{\mathbb{T}} P_z(\zeta, x\overline{\zeta})d\sigma_\alpha(x) &= \int_{\mathbb{T}} P_{z_1}(\zeta)P_{z_2}(x\overline{\zeta})d\sigma_\alpha(x)\\ &= P_{z_1}(\zeta)\int_{\mathbb{T}} P_{\zeta z_2}(x)d\sigma_\alpha(x)\\ &= P_{z_1}(\zeta)\int_{\mathbb{T}}\left(\sum_{k\geq 1}\frac{1}{|\phi'(\eta_k)|}P_{\zeta z_2}(x)\right)d\delta_{\eta_k}(x).\end{aligned}$$

This then reduces to

$$P_{z_1}(\zeta)\sum_{k\geq 1}\frac{1}{|\phi'(\eta_k)|}P_{\zeta z_2}(\eta_k) = \sum_{k\geq 1}\frac{1}{|\phi'(\eta_k)|}P_{z_1}(\zeta)P_{z_2}(\eta_k\overline{\zeta})$$

using the same trick as before to go from $P_{\zeta z_2}(\eta_k)$ to $P_{z_2}(\eta_k\overline{\zeta})$. Hence,

$$\int_{\mathbb{T}} P_z(\zeta, x\overline{\zeta})d\sigma_\alpha(x) = \sum_{k\geq 1}\frac{1}{|\phi'(\eta_k)|}P_z(\zeta, \eta_k\overline{\zeta}).$$

Integrating over this and applying Proposition 6.3 then shows that

$$\int_{\mathbb{T}^2} P_z(\xi)d\tau_\alpha(\xi) = \int_{\mathbb{T}}\left(\sum_{k\geq 1}\frac{1}{|\phi'(\eta_k)|}P_z(\zeta, \eta_k\overline{\zeta})\right)dm(\zeta),$$

Since Poisson kernels are positive by definition, the summands in the right-hand side are positive. Hence, by Tonelli's theorem, we may interchange summation and integration:

$$\int_{\mathbb{T}^2} P_z(\xi)d\tau_\alpha(\xi) = \sum_{k\geq 1}\int_{\mathbb{T}} P_z(\zeta, \eta_k\overline{\zeta})\frac{dm(\zeta)}{|\phi'(\eta_k)|}.$$

Lemma 2.3 now yields the desired result. □

It is interesting to compare the above result to the corresponding theorems, Theorem 5.5 and Theorem 5.6, for rational inner functions. In the RIF case, we saw that the weights of Clark measures along the curves in the unimodular level sets were one-variable functions. Theorem 6.5 shows that for the multiplicative embeddings, the weights are simpler than their RIF counterparts — they are constant along each curve in the level sets. This is perhaps unexpected; it implies that given any univariate inner function ϕ, regardless of its complexity, the associated Clark measures of $\phi(z_1z_2)$ will still be very "well-behaved", in the sense that they are supported on straight lines and have constant density along each such line.

Example 6.6. In Example 4.2 in [11], the author uses technical properties of Poisson kernels to show that

$$\int_{\mathbb{T}^2} f(\xi)d\tau_\alpha(\xi) = \int_{\mathbb{T}} f(\zeta, \alpha\overline{\zeta})dm(\zeta)$$

for $f \in C(\mathbb{T}^2)$, where τ_α is the Clark measure associated to $\psi(z_1, z_2) := z_1z_2$ and $\alpha \in \mathbb{T}$. By setting $\phi(z) := z$, we can instead apply Theorem 6.5. Note that ϕ^* exists and is equal to ϕ everywhere on $\mathbb{T}$, and $\phi(\zeta) = \alpha$ has only one solution $\zeta = \alpha$. Hence, direct application of the theorem yields the equality above.

Example 6.7. Recall the function

$$\phi(z) := \exp\left(-\frac{1+z}{1-z}\right), \quad z \in \mathbb{D},$$

from Example 3.4. We saw there that the solutions to $\phi^*(\zeta) = 1$ are given by

$$\eta_k = \frac{2\pi k - i}{2\pi k + i}, \quad k \in \mathbb{Z},$$

and

$$\frac{1}{|\phi'(\eta_k)|} = \frac{8}{1 + 4\pi^2 k^2}.$$

Applying Theorem 6.5 for the Clark measure τ_1 of $\psi(z_1, z_2) := \phi(z_1 z_2)$ then results in

$$\int_{\mathbb{T}^2} f(\xi) d\tau_1(\xi) = \sum_{k \in \mathbb{Z}} \frac{8}{1 + 4\pi^2 k^2} \int_{\mathbb{T}} f(\zeta, \eta_k \bar{\zeta}) dm(\zeta)$$

for $f \in C(\mathbb{T}^2)$. This marks our first example of a non-rational bivariate function, for which we can explicitly characterize the Clark measures! Moreover, this is our first example of an inner function whose unimodular level sets consist of infinitely many curves, as opposed to the RIF case. The function ψ is also studied in Example 13.1 in [6].

7 Product functions

After the analysis of multiplicative embeddings from the last section, one might wonder about product functions of the form

$$\Phi(z_1, z_2) := \phi(z_1)\psi(z_2)$$

for one-variable inner functions ϕ and ψ. Note that we have already studied one such class of functions, the monomials from the first section. However, at first sight, general product functions may look elusive. Recall that a key argument in the proofs of Theorem 4.1, Theorem 5.5 and Theorem 6.5 is the Poisson integral formula. To use this for $\Phi(z_1, z_2)$, we require that for fixed $z_2 \in \mathbb{D}$, the function

$$u_{z_2}(z_1) := \frac{1 - |\phi(z_1)\psi(z_2)|^2}{|\alpha - \phi(z_1)\psi(z_2)|^2}$$

is continuous on the closed unit disc. However, for a general inner function ϕ, its non-tangential limits need only exist m-almost everywhere on $\mathbb{T}$. Even if they do exist on the entire unit circle, ϕ^* need not be continuous. For this reason, we introduce the function $\Phi_r(z) := \phi(rz_1)\psi(z_2)$ for $0 < r < 1$. This is not an inner function, as $|\phi(rz_1)| < 1$ on the unit circle. However, since $\Phi_r \to \Phi$ as $r \to 1-$, we can go via Φ_r to investigate the Clark measures of Φ.

Unfortunately, this does not solve all issues for general inner functions ϕ and ψ; we will have to introduce some assumptions on at least one of the functions.

Theorem 7.1. *Let $\Phi(z_1, z_2) := \phi(z_1)\psi(z_2)$ for one-variable inner functions ϕ and ψ, such that*

A. ψ^* is continuous on $\mathbb{T}$ except at a finite set of points,

B. the solutions to $\psi^* = \beta$ for $\beta \in \mathbb{T}$ can be parameterized by functions $\{g_k(\beta)\}_{k\geq 1}$ which are also continuous on $\mathbb{T}$ except at a finite set of points,

C. for every $\beta \in \mathbb{T}$, there are no solutions to $\psi^* = \beta$ with infinite multiplicity, and

D. the Clark measures of ψ are all discrete.

Then the Clark measures of Φ satisfy

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_\alpha(\xi) = \sum_{k\geq 1} \int_{\mathbb{T}} f(\zeta, g_k(\alpha\overline{\phi^*(\zeta)})) \frac{dm(\zeta)}{|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|}$$

for $\in C(\mathbb{T}^2)$.

Remark 7.2. In all our examples so far, if a one-variable inner function ψ had a discrete Clark measure for some α-value, it would hold true for every $\alpha \in \mathbb{T}$. However, there are examples of inner functions where its Clark measure σ_α is discrete for one specific α-value but σ_β is singular continuous for $\beta \in \mathbb{T} \setminus \{\alpha\}$, and vice versa. See Example 1 and 2 in [10].

Before getting into the details of the proof, let us try to locate the difficulties and motivate our assumptions. A potential issue is that the right-hand side may not be integrable; this is remedied by our assumptions A and B on $\{g_k\}_{k\geq 1}$ and ψ^*. This is explored in more detail in the proof.

A more concerning threat is that the sum in the right-hand side might run off to infinity. Per definition, $\zeta = g_k(\beta)$ solves the equation $\psi^*(\zeta) = \beta$ for every $k \geq 1$. By Proposition 3.3 and assumption D, for every fixed $\zeta \in \mathbb{T}$ such that $\phi^*(\zeta)$ is unimodular, the Clark measure of ψ associated to parameter value $\alpha\overline{\phi^*(\zeta)}$ is thus given by

$$\sum_{k\geq 1} \frac{1}{|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|} \delta_{g_k(\alpha\overline{\phi^*(\zeta)})}.$$

As all Clark measures are finite, the sum of weights $\sum_{k\geq 1} |\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|^{-1}$ converges for this fixed ζ, and so the above sum converges as well. However, the equation $\psi^* = \alpha\overline{\phi^*(\zeta)}$ need not have the same number of solutions for each ζ; we might then have points of intersection of the g_k-functions. Since the Clark measures of ψ are mutually singular and in this case discrete, two measures associated to different parameters must be supported on disjoint sets. Hence $g_k(\alpha\overline{\phi^*(\zeta)}) \neq g_j(\alpha\overline{\phi^*(\zeta')})$ for any $\zeta \neq \zeta'$ where $\phi^*(\zeta) \neq \phi^*(\zeta')$. Nevertheless, a situation could arise where the curves have the same limit points; i.e. if $\lim_{k\to\infty} g_k(\alpha\overline{\phi^*(\zeta)}) = \lim_{j\to\infty} g_j(\alpha\overline{\phi^*(\zeta')})$. In this case, infinitely many g_k-functions intersect at these points.

Let us illustrate why this poses a problem. Suppose that there is a point $\zeta \in \mathbb{T}$ such that an infinite subset of $\{g_k\}_{k\geq 1}$ intersect at $\alpha\overline{\phi^*(\zeta)} \in \mathbb{T}$. Call the set of indices for the intersecting functions E. If furthermore $|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))| = c$ is finite for this ζ, then the sum

$$\sum_{k\in E} \frac{1}{|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|} = \sum_{k\in E} \frac{1}{c}$$

diverges. This in turn implies that the integral in Theorem 7.1 is divergent.

A trivial example would be if all g_k intersect at this point. If there is a point $\zeta \in \mathbb{T}$ such that $g_k(\alpha\overline{\phi^*(\zeta)}) = \eta \in \mathbb{T}$ for every $k \geq 1$, then this is the only solution to $\psi^* = \alpha\overline{\phi^*(\zeta)}$. As all Clark measures of ψ are assumed to be discrete, the measure associated to this parameter will just be δ_η. This in turn implies that the weights $|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|^{-1} = |\psi'(\eta)|^{-1} = 1$ for every $k \geq 1$. Hence, the sum $\sum_{k\geq 1} |\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|^{-1}$ diverges for this specific $\zeta \in \mathbb{T}$, and so the integral in Theorem 7.1 diverges as well.

To avoid this situation altogether, by assumption C, we do not allow for solutions of $\psi^* = \beta$ with infinite multiplicity, as any such solution would be a point of intersection for an infinite subset of $\{g_k\}_{k\geq 1}$. Note that this includes discontinuities of ψ^*; even there we do not allow for infinite multiplicity. Furthermore, we have already established that there cannot exist points where finitely many g_k-curves intersect. This implies that all $\{g_k\}_{k\geq 1}$ are distinct, except in potential limit points.

One might then ask if infinite intersections in limit points cause the same problem as above. The answer is no: per definition, a limit point of $g_k(\alpha\overline{\phi^*(\zeta)})$ does not in fact solve the equation $\psi^* = \alpha\overline{\phi^*(\zeta)}$. By Proposition 3.3, the Clark measure of ψ associated to $\alpha\overline{\phi^*(\zeta)}$ has a point mass at this point if and only if it solves $\psi^* = \alpha\overline{\phi^*(\zeta)}$. Hence the weight of the Clark measure at the limit point must be zero, meaning that we avoid the situation where the integral picks up infinitely many positive weights. We will give a detailed example of this situation later on.

Clearly, Theorem 7.1 is not applicable for a general class of product functions; we have a number of restrictions on at least one of the factors. Nevertheless, we will see some interesting examples of product functions which satisfy our assumptions and do not fall under any previous category of bivariate inner functions.

Proof. Define $\Phi_r(z_1, z_2) := \phi(rz_1)\psi(z_2)$ for $0 < r < 1$, and note that

$$\frac{1 - |\Phi_r(z_1, z_2)|^2}{|\alpha - \Phi_r(z_1, z_2)|^2} \to \frac{1 - |\Phi(z_1, z_2)|^2}{|\alpha - \Phi(z_1, z_2)|^2} = \int_{\mathbb{T}^2} P_z(\xi) d\sigma_\alpha(\xi)$$

as $r \to 1-$. Define, for fixed $z_2 \in \mathbb{D}$ and fixed $0 < r < 1$,

$$u^r_{z_2}(z_1) := \frac{1 - |\psi(z_2)\phi(rz_1)|^2}{|\alpha - \psi(z_2)\phi(rz_1)|^2}, \quad z_1 \in \mathbb{D}.$$

As $\phi(rz_1)$ is continuous and satisfies $|\phi(rz_1)| < 1$ on the unit circle, $u^r_{z_2}$ is continuous on $\overline{\mathbb{D}}$. Moreover, even if Φ_r is not an inner function, it holds that

$$\frac{1 - |\Phi_r|^2}{|\alpha - \Phi_r|^2} = \Re\left(\frac{\alpha + \Phi_r}{\alpha - \Phi_r}\right)$$

where $(\alpha + \Phi_r)/(\alpha - \Phi_r)$ is analytic on $\mathbb{D}^2$. Hence, the left-hand side is pluriharmonic in $\mathbb{D}^2$, which in turn implies that $u^r_{z_2}$ is harmonic in $\mathbb{D}$. By the Poisson integral formula,

$$\frac{1 - |\psi(z_2)\phi(z_1)|^2}{|\alpha - \psi(z_2)\phi(z_1)|^2} = \lim_{r\to 1-} u^r_{z_2}(z_1) = \lim_{r\to 1-} \int_{\mathbb{T}} u^r_{z_2}(\zeta) P_{z_1}(\zeta) dm(\zeta).$$

Observe that $\Re((\alpha + \Phi_r(z))/(\alpha - \Phi_r(z)))$ is bounded for every $z \in \overline{\mathbb{D}}$ and every $0 < r < 1$. The dominated convergence theorem then states that we can move the limit into the integral:

$$\frac{1 - |\psi(z_2)\phi(z_1)|^2}{|\alpha - \psi(z_2)\phi(z_1)|^2} = \int_{\mathbb{T}} \lim_{r\to 1-} u^r_{z_2}(\zeta) P_{z_1}(\zeta) dm(\zeta). \tag{14}$$

Moreover,

$$\lim_{r\to 1-} u_{z_2}^r(\zeta) = \lim_{r\to 1-} \frac{1-|\psi(z_2)\phi(r\zeta)|^2}{|\alpha-\psi(z_2)\phi(r\zeta)|^2} = \frac{1-|\psi(z_2)\phi^*(\zeta)|^2}{|\alpha-\psi(z_2)\phi^*(\zeta)|^2}$$

for m-almost every $\zeta \in \mathbb{T}$. Let E denote the set of points ζ such that $|\phi^*(\zeta)| = 1$.

By our assumptions, the solutions to $\psi^* = \beta$ can be parameterized by functions $g_k(\beta)$ continuous on $\mathbb{T}$ except on a finite collection of points. Since we have also assumed that the Clark measures of ψ consist of point masses, by Proposition 3.3, the measure associated to any $\beta \in \mathbb{T}$ is given by $\sum_{k\geq 1} |\psi'(g_k(\beta))|^{-1}\delta_{g_k(\beta)}$ where $|\psi'(g_k(\beta))| > 0$ for each k. For fixed $\zeta \in E$, this holds for $\beta = \alpha\overline{\phi^*(\zeta)}$.

Hence, for $\zeta \in E$, we have that

$$\frac{1-|\psi(z_2)\phi^*(\zeta)|^2}{|\alpha-\psi(z_2)\phi^*(\zeta)|^2} = \sum_{k\geq 1} \frac{1}{|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|} P_{z_2}(g_k(\alpha\overline{\phi^*(\zeta)})).$$

Since the above holds for m-almost every $\zeta \in \mathbb{T}$, the integrals of the left- and right-hand side will coincide. By combining this with (14), we see that

$$\begin{aligned}
\frac{1-|\psi(z_2)\phi(z_1)|^2}{|\alpha-\psi(z_2)\phi(z_1)|^2} &= \int_{\mathbb{T}} \frac{1-|\psi(z_2)\phi^*(\zeta)|^2}{|\alpha-\psi(z_2)\phi^*(\zeta)|^2} P_{z_1}(\zeta)dm(\zeta) \\
&= \int_{\mathbb{T}} \sum_{k\geq 1} \frac{1}{|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|} P_{z_1}(\zeta)P_{z_2}(g_k(\alpha\overline{\phi^*(\zeta)}))dm(\zeta).
\end{aligned}$$

As the summands are all positive, we may apply Tonelli's theorem to interchange summation and integration. Thus,

$$\frac{1-|\Phi(z_1,z_2)|^2}{|\alpha-\Phi(z_1,z_2)|^2} = \sum_{k\geq 1}\int_{\mathbb{T}} P_{z_1}(\zeta)P_{z_2}(g_k(\alpha\overline{\phi^*(\zeta)}))\frac{dm(\zeta)}{|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|},$$

i.e.

$$\int_{\mathbb{T}^2} P_z(\xi)d\sigma_\alpha(\xi) = \sum_{k\geq 1}\int_{\mathbb{T}} P_{z_1}(\zeta)P_{z_2}(g_k(\alpha\overline{\phi^*(\zeta)}))\frac{dm(\zeta)}{|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|}.$$

Lemma 2.3 now yields

$$\int_{\mathbb{T}^2} f(\xi)d\sigma_\alpha(\xi) = \sum_{k\geq 1}\int_{\mathbb{T}} f(\zeta, g_k(\alpha\overline{\phi^*(\zeta)}))\frac{dm(\zeta)}{|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|}$$

for all $f \in C(\mathbb{T}^2)$.

It remains to convince ourselves that the right-hand side is indeed integrable and finite. Recall that by Fatou's theorem, $\phi(r\zeta)$ converges to $\phi^*(\zeta)$ as $r \to 1-$ m-almost everywhere on $\mathbb{T}$ and in $L^1(\mathbb{T})$. Moreover, the curves $\{g_k\}_{k\geq 1}$ are assumed to be continuous on the unit circle except on finitely many points. Hence, the composition $f(\zeta, g_k(\alpha\overline{\phi^*(\zeta)}))$ must be measurable for $f \in C(\mathbb{T}^2)$. Similarly, we see that the weights $|\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|$ are measurable, as ψ^* being differentiable almost everywhere on $\mathbb{T}$ implies that its derivative is measurable on $\mathbb{T}$. Since we are integrating over a compact space, this is enough to ensure integrability.

For fixed $\zeta \in E$, the sum $\sum_{k\geq 1} |\psi'(g_k(\alpha\overline{\phi^*(\zeta)}))|^{-1}$ must be finite, since the Clark measure of ψ associated to the parameter value $\alpha\overline{\phi^*(\zeta)}$ exists by assumption. By our previous discussion, as we have excluded the situation where infinitely many of the curves intersect, there is no issue of the weights summing up to infinity — except at limit points, where the weights are zero anyway. $\square$

Roughly speaking, the above result says that as long as we have one sufficiently well-behaved inner function with discrete Clark measures, we may multiply it with any other inner function and obtain a characterization of the Clark measures of the resulting product. Note that the weights of these measures strongly resemble their RIF counterparts from Theorem 5.5 — quite surprisingly, as product functions need not be rational in any variable. Moreover, as in the case of the multiplicative embeddings, Theorem 7.1 allows for infinite collections of parameterizing functions.

Before moving on, let us convince ourselves that there actually exist inner functions ψ that meet the requirements of Theorem 7.1. For example, finite Blaschke products define one such class. Let ψ be a non-constant finite Blaschke product of order n. Then ψ is analytic on $\mathbb{T}$, and $\psi^*(\zeta) = \beta$ has precisely n distinct solutions for each $\beta \in \mathbb{T}$. Moreover, from properties of finite Blaschke products, its derivative is non-zero on $\mathbb{T}$. By the implicit function theorem, we may thus parameterize the solutions with functions $\{g_k(\beta)\}_{k=1}^n$ analytic on the unit circle. Additionally, it follows from Proposition 3.3 that the Clark measures of ψ are discrete for every $\beta \in \mathbb{T}$, with point masses at the distinct solutions to $\psi^* = \beta$. Hence, Theorem 7.1 works for any product function $\Phi(z) = \phi(z_1)\psi(z_2)$ where ψ is a non-constant finite Blaschke product and ϕ is an arbitrary inner function.

In what comes next, we let $g_k^\alpha(\zeta) := g_k(\alpha\overline{\phi^*(\zeta)})$ for ease of notation.

Remark 7.3. In the case where both ϕ and ψ are finite Blaschke products, the theorem reproduces what we know about RIFs. Suppose ϕ is of order m and ψ of order n, and let $\Phi(z_1, z_2) = \phi(z_1)\psi(z_2)$. Note that Φ is a RIF of bidegree (m, n), and since finite Blaschke products are analytic on $\mathbb{T}$, it has no singularities on the torus. Moreover,

$$\psi^*(\zeta_2) = \alpha/\phi^*(\zeta_1) = \alpha\overline{\phi^*(\zeta_1)}$$

has precisely n distinct solutions for each $\zeta_1 \in \mathbb{T}$, which may be parameterized by n analytic functions g_k^α. Then

$$\mathcal{C}_\alpha(\Phi) = \bigcup_{k=1}^{n} \{(\zeta, g_k^\alpha(\zeta)) : \zeta \in \mathbb{T}\}.$$

By Theorem 7.1,

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_\alpha(\xi) = \sum_{k=1}^{n} \int_{\mathbb{T}} f(\zeta, g_k^\alpha(\zeta)) \frac{dm(\zeta)}{|\psi'(g_k^\alpha(\zeta))|} = \sum_{k=1}^{n} \int_{\mathbb{T}} f(\zeta, g_k^\alpha(\zeta)) \frac{dm(\zeta)}{|\frac{\partial \Phi}{\partial z_2}(\zeta, g_k^\alpha(\zeta))|}$$

for $f \in C(\mathbb{T}^2)$, where we have used that $|\phi| = 1$ everywhere on $\mathbb{T}$ to conclude that $|\frac{\partial \Phi}{\partial z_2}(\zeta, g_k^\alpha(\zeta))| = |\psi'(g_k^\alpha(\zeta))|$. Observe that the equality above is precisely the formulation of Theorem 5.5 for RIFs.

Next, we illustrate our results for some examples of non-rational product functions.

Example 7.4. Let

$$\Phi(z_1, z_2) := z_2\phi(z_1) = z_2 \exp\left(-\frac{1+z_1}{1-z_1}\right),$$

where ϕ is as in Example 3.4. Note that the first factor is continuous everywhere, and satisfies the required properties in Theorem 7.1. Since ϕ^* exists everywhere on $\mathbb{T}$, it follows that $\Phi^*(\zeta) = \zeta_2\phi^*(\zeta_1)$. On the line $\{(1, \zeta_2) : \zeta_2 \in \mathbb{T}\}$ in $\mathbb{T}^2$, we have that $\Phi^* = 0$. Elsewhere on the torus, Φ^* is given by the obvious interpretation. Hence, for $\alpha \in \mathbb{T}$, the solutions to $\Phi^* = \alpha$ are given by

$$\zeta_2 = g^\alpha(\zeta_1) := \alpha \exp\left(\frac{1+\zeta_1}{1-\zeta_1}\right).$$

Note that g^α is discontinuous for $\zeta_1 = 1$, and that this point does not solve $\Phi^* = \alpha$. However, since $\mathcal{C}_\alpha(\Phi)$ is a closed set, it must include these singular points. Hence, $\mathcal{C}_\alpha(\Phi)$ can be parameterized by the graph $\{(\zeta, g^\alpha(\zeta)) : \zeta \in \mathbb{T}\}$. In Figure 2, we have plotted the argument of $g^\alpha(e^{i\theta})$ for $-\pi \le \theta \le \pi$. We see that the discontinuity at $\zeta_1 = 1$ creates oscillatory behavior in the level curve.

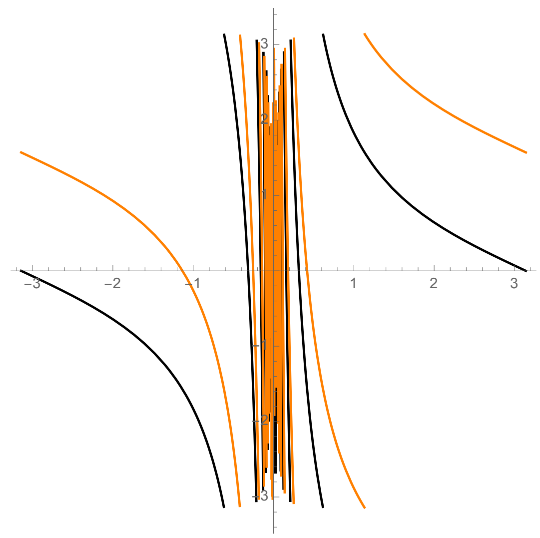


Figure 2: Level curve g^α for $\alpha = 1$ (black) and $\alpha = i$ (orange).

This marks our first example of a two-variable inner function whose associated Clark measures are supported on discontinuous sets. By Theorem 7.1, the Clark measures of Φ satisfy

$$\int_{\mathbb{T}^2} f(\xi)d\sigma_\alpha(\xi) = \int_{\mathbb{T}} f(\zeta, g^\alpha(\zeta))dm(\zeta)$$

for $f \in C(\mathbb{T}^2)$.

Example 7.5. Let

$$\Phi(z_1, z_2) := z_2^M\phi(z_1) = z_2^M \exp\left(-\frac{1+z_1}{1-z_1}\right)$$

for a positive integer M. We see again that $\Phi^*(\zeta_1) = 0$ for $\zeta_1 = 1$. The equation $\Phi^* = \alpha$ can be rewritten as

$$\zeta_2^M = \alpha \exp\left(\frac{1+\zeta_1}{1-\zeta_1}\right).$$

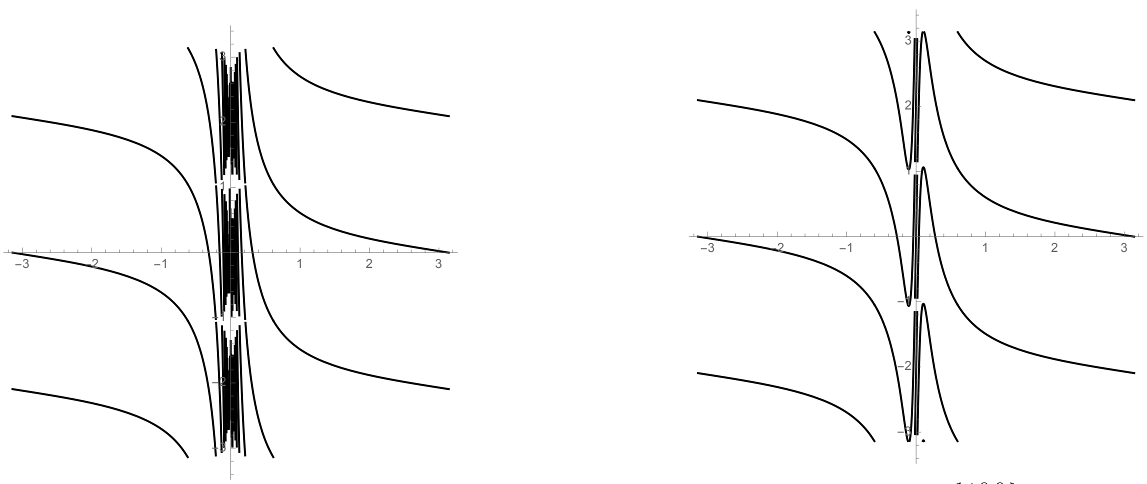
For $\zeta_1 \in \mathbb{T}$, the argument of the exponential function on the right-hand side is precisely the imaginary part

$$\theta := \Im\left(\frac{1+\zeta_1}{1-\zeta_1}\right).$$

Then, for $\alpha = e^{i\nu}$,

$$\zeta_2 = g_k^\alpha(\zeta_1) := e^{i\left(\frac{\theta+\nu}{M} + \frac{2\pi k}{M}\right)}, \quad k = 0, 1, \ldots, M-1.$$

This defines M functions, all of which are discontinuous for $\zeta_1 = 1$. We illustrate these functions in Figure 3. As it is hard to differentiate between the graphs of g_k^α for different k, we have also included a plot of solutions to $\zeta_2^M = \alpha \exp\left(\frac{1+0.9\zeta_1}{1-0.9\zeta_1}\right)$, where it is clear how the components are connected.



(a) Level curves g_k^1 for $M = 3$.

(b) Solutions to $\zeta_2^3 = \exp\left(\frac{1+0.9\zeta_1}{1-0.9\zeta_1}\right)$, to clearly illustrate the different components.

Figure 3: Level curves g_k^α for $M = 3$ and $\alpha = 1$.

Again, as the unimodular level sets $\mathcal{C}_\alpha(\Phi)$ are closed, they must include the line $\zeta_1 = 1$. Hence, the associated Clark measures of Φ are supported on discontinuous graphs for every M. By Theorem 7.1, the Clark measures of Φ satisfy

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_\alpha(\xi) = \sum_{k=0}^{M-1} \frac{1}{M} \int_{\mathbb{T}} f(\zeta, g_k^\alpha(\zeta)) dm(\zeta)$$

for $f \in C(\mathbb{T}^2)$.

Example 7.6. Let

$$\Phi(z_1, z_2) := \psi(z_2)\phi(z_1) = z_2 \frac{\lambda - z_2}{1 - \overline{\lambda} z_2} \exp\left(-\frac{1+z_1}{1-z_1}\right)$$

for some non-zero constant $\lambda \in \mathbb{D}$. Note that $\psi(z_2) = z_2 \frac{\lambda - z_2}{1-\overline{\lambda} z_2}$ is a Blaschke product of order two, which is then continuous on $\mathbb{T}$. As we saw in previous examples, $\Phi^* = 0$ for $\zeta_1 = 1$. The equation $\Phi^* = \alpha$ for $\alpha \in \mathbb{T}$ can be rewritten as

$$\zeta_2 \frac{\lambda - \zeta_2}{1 - \overline{\lambda}\zeta_2} = \alpha \exp\left(\frac{1+\zeta_1}{1-\zeta_1}\right).$$

For $\alpha = e^{i\nu}$, the solutions to this are given by $\zeta_2 = g_k^\alpha(\zeta_1)$, $k = 1, 2$, where

$$g_1^\alpha(\zeta_1) := \frac{1}{2}\left(\lambda + \exp\left(i\nu + \frac{1+\zeta_1}{1-\zeta_1}\right)\overline{\lambda} + \sqrt{-4\exp\left(i\nu + \frac{1+\zeta_1}{1-\zeta_1}\right) + \left(-\lambda - \exp\left(i\nu + \frac{1+\zeta_1}{1-\zeta_1}\right)\overline{\lambda}\right)^2}\right),$$

$$g_2^\alpha(\zeta_1) := \frac{1}{2}\left(\lambda + \exp\left(i\nu + \frac{1+\zeta_1}{1-\zeta_1}\right)\overline{\lambda} - \sqrt{-4\exp\left(i\nu + \frac{1+\zeta_1}{1-\zeta_1}\right) + \left(-\lambda - \exp\left(i\nu + \frac{1+\zeta_1}{1-\zeta_1}\right)\overline{\lambda}\right)^2}\right).$$

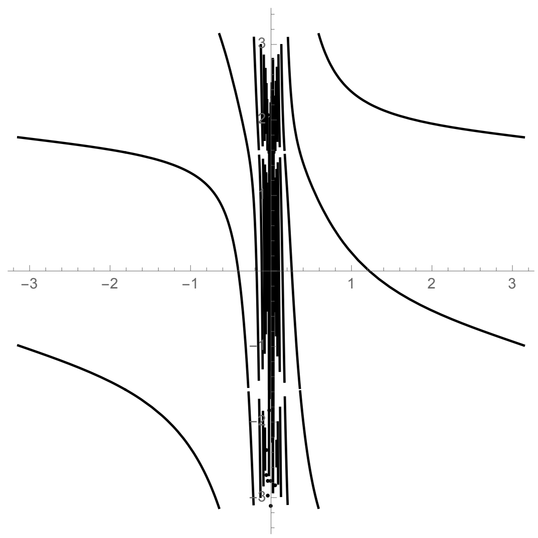


Figure 4: Level curves g_1^α and g_2^α for $\alpha = e^{i\pi/4}$ and $\lambda = i/2$.

Let us calculate the weights of the Clark measures. Observe that

$$\psi'(z_2) = \frac{\lambda - 2z_2 + z_2^2\overline{\lambda}}{(1-\overline{\lambda}z_2)^2}.$$

Hence, by Theorem 7.1,

$$\int_{\mathbb{T}^2} f(\xi) d\sigma_\alpha(\xi) = \sum_{k=1}^{2} \int_{\mathbb{T}} f(\zeta, g_k^\alpha(\zeta)) \frac{|1 - \overline{\lambda} g_k^\alpha(\zeta)|^2}{|\lambda - 2g_k^\alpha(\zeta) + g_k^\alpha(\zeta)^2\overline{\lambda}|} dm(\zeta)$$

for all $f \in C(\mathbb{T}^2)$.

We see in Figure 4 that the graphs are similar to the ones from Figure 3. However, in this case, the weights of the Clark measures are quite complicated one-variable functions, as opposed to the constant weights in the previous example. In the next figure, we have plotted the weights

$$W_k(\zeta) := \frac{|1 - \overline{\lambda} g_k^\alpha(\zeta)|^2}{|\lambda - 2g_k^\alpha(\zeta) + g_k^\alpha(\zeta)^2\overline{\lambda}|}$$

for certain parameter values.

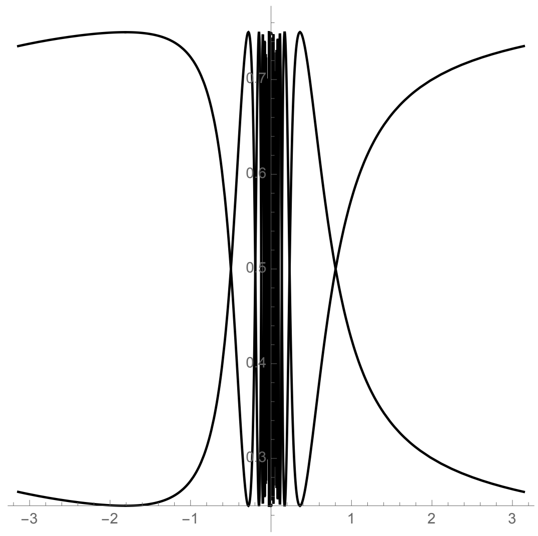


Figure 5: Weight curves W_k for $\alpha = e^{i\pi/4}$ and $\lambda = i/2$.

It is worth noting that in all the examples in this section so far, ψ has been defined as a rational function. In fact, any inner function that is continuous on $\mathbb{T}$ must be a finite Blaschke product, by Corollary 4.2 in [14]. This implies that to cover a wider class of functions, we must allow some discontinuities of ψ^*. Recall that we define the Hardy space $H^p(\mathbb{T})$ as the space of functions f analytic on $\mathbb{D}$ which satisfy

$$\sup_{0\leq r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

It turns out that if ψ' is in $H^1(\mathbb{T})$, then ψ is continuous on $\mathbb{T}$ (Theorem 3.11, [12]) and thus a finite Blaschke product. Hence, to be able to construct varied examples, we need ψ to be a bit more complicated; either through discontinuities or irregularity in the derivative on $\mathbb{T}$. With this in mind, we move on to our final — and perhaps most interesting — example.

A very non-rational example

As we mentioned above, all the examples so far in this section have been rational in (at least) one of two variables. Let us therefore take a look at a function that is not rational in any variable.

Define

$$\Phi(z_1, z_2) := \phi(z_1)\phi(z_2) = \exp\left(-\frac{1+z_1}{1-z_1}\right)\exp\left(-\frac{1+z_2}{1-z_2}\right),$$

where ϕ again is as in Example 3.4. As ϕ^* exists everywhere on $\mathbb{T}$, we have $\Phi^*(\zeta) = \phi^*(\zeta_1)\phi^*(\zeta_2)$. On the lines $\{(1, \zeta_2) : \zeta_2 \in \mathbb{T}\}$ and $\{(\zeta_1, 1) : \zeta_1 \in \mathbb{T}\}$ in $\mathbb{T}^2$, we see that $\Phi^* = 0$. Otherwise, $|\Phi^*| = 1$.

We begin with the following deconstruction of the unimodular level sets of Φ.

Proposition 7.7. *Let $\phi(z) = \exp\left(-\frac{1+z}{1-z}\right)$ and $\alpha \in \mathbb{T}$. Define $\Phi(z_1, z_2) := \phi(z_1)\phi(z_2)$. Then*

$$\mathcal{C}_\alpha(\Phi) = \bigcup_{x\in\mathbb{T}} \{\mathcal{C}_x(\phi) \times \mathcal{C}_{\alpha/x}(\phi)\}.$$

Proof. As in the proof of Proposition 6.1, we write

$$\mathcal{C}'_\alpha(f) := \left\{\zeta \in \mathbb{T}^d : \lim_{r\to 1-} f(r\zeta) = \alpha\right\},$$

so that $\mathcal{C}_\alpha(f) = \mathrm{Clos}(\mathcal{C}'_\alpha(f))$.

Fix $x \in \mathbb{T}$. If $\zeta_1 \in \mathcal{C}'_x(\phi)$ and $\zeta_2 \in \mathcal{C}'_{\alpha/x}(\phi)$, then $\Phi^*(\zeta_1, \zeta_2) = \phi^*(\zeta_1)\phi^*(\zeta_2) = \alpha$ and $(\zeta_1, \zeta_2) \in \mathcal{C}'_\alpha(\Phi) \subset \mathcal{C}_\alpha(\Phi)$. Hence,

$$\bigcup_{x\in\mathbb{T}} \{\mathcal{C}'_x(\phi) \times \mathcal{C}'_{\alpha/x}(\phi)\} \subset \mathcal{C}_\alpha(\Phi).$$

Now let $(\zeta_1, \zeta_2) \in \mathcal{C}_x(\phi) \times \mathcal{C}_{\alpha/x}(\phi)$. Then there exist sequences $(\zeta_{1,n})_{n\geq 1} \subset \mathcal{C}'_x(\phi)$ and $(\zeta_{2,n})_{n\geq 1} \subset \mathcal{C}'_{\alpha/x}(\phi)$ such that $\zeta_{1,n} \to \zeta_1$ and $\zeta_{2,n} \to \zeta_2$ respectively as $n \to \infty$. But this implies that $(\zeta_{1,n}, \zeta_{2,n})$ defines a sequence in $\mathcal{C}'_x(\phi) \times \mathcal{C}'_{\alpha/x}(\phi) \subset \mathcal{C}'_\alpha(\Phi)$ converging to (ζ_1, ζ_2). Thus, (ζ_1, ζ_2) is a limit point of $\mathcal{C}'_\alpha(\Phi)$, and must therefore lie in its closure. This proves that

$$\bigcup_{x\in\mathbb{T}} \{\mathcal{C}_x(\phi) \times \mathcal{C}_{\alpha/x}(\phi)\} \subset \mathcal{C}_\alpha(\Phi).$$

Conversely, let $(\zeta_1, \zeta_2) \in \mathcal{C}'_\alpha(\Phi)$. Then, as ϕ^* exists everywhere on $\mathbb{T}$, we see that $\Phi^*(\zeta_1, \zeta_2) = \phi^*(\zeta_1)\phi^*(\zeta_2) = \alpha$. Set $x := \phi^*(\zeta_1)$. Note that by properties of ϕ, the value x must be unimodular; the only other option is that $x = 0$, but in this case, $\alpha \notin \mathbb{T}$, a contradiction. Clearly $\zeta_1 \in \mathcal{C}_x(\phi)$ and $\zeta_2 \in \mathcal{C}_{\alpha/x}(\phi)$, and so

$$\mathcal{C}'_\alpha(\Phi) \subset \bigcup_{x\in\mathbb{T}} \{\mathcal{C}_x(\phi) \times \mathcal{C}_{\alpha/x}(\phi)\}.$$

Now let $(\zeta_1, \zeta_2) \in \mathcal{C}_\alpha(\Phi)$. Then there must exist a sequence of points $(\zeta_{1,n}, \zeta_{2,n}) \in \mathcal{C}'_\alpha(\Phi)$ such that $(\zeta_{1,n}, \zeta_{2,n}) \to (\zeta_1, \zeta_2)$ as $n \to \infty$. Then $\phi^*(\zeta_{1,n})\phi^*(\zeta_{2,n}) = \alpha$ for each n, and so $\zeta_{1,n} \in \mathcal{C}'_x(\phi)$ and $\zeta_{2,n} \in \mathcal{C}'_{\alpha/x}(\phi)$ for some $x \in \mathbb{T}$. We see that ζ_1 and ζ_2 are limit points of $\mathcal{C}'_x(\phi)$ and $\mathcal{C}'_{\alpha/x}(\phi)$ respectively, so $(\zeta_1, \zeta_2) \in \mathcal{C}_x(\phi) \times \mathcal{C}_{\alpha/x}(\phi)$. This proves that

$$\mathcal{C}_\alpha(\Phi) \subset \bigcup_{x\in\mathbb{T}} \{\mathcal{C}_x(\phi) \times \mathcal{C}_{\alpha/x}(\phi)\},$$

and we are done. $\square$

We saw in Example 3.4 that for $\alpha = 1$, the unimodular level set $\mathcal{C}_\alpha(\phi)$ consists of a countable collection of points. By the above proposition, the level sets of Φ are uncountable unions over cartesian products of countably infinite sets — so it seems that the associated Clark measures of Φ are supported on quite complicated sets. To see if this is true, let us explicitly calculate the solutions to $\Phi^* = \alpha$ for $\alpha = e^{i\nu}$.

Since Φ^* is well-defined and unimodular on $\mathbb{T}^2$ except on the lines $\{\zeta_1 = 1\} \cup \{\zeta_2 = 1\}$ where $\Phi^* = 0$, we need to solve the equation $\Phi = \alpha$. We may see this as

$$\exp\left(-\frac{1+\zeta_1}{1-\zeta_1} - \frac{1+\zeta_2}{1-\zeta_2}\right) = e^{i(\nu+2\pi k)}, \quad k \in \mathbb{Z},$$

i.e.

$$-\frac{1+\zeta_1}{1-\zeta_1} - \frac{1+\zeta_2}{1-\zeta_2} = i(\nu + 2\pi k), \quad k \in \mathbb{Z}.$$

Solving for ζ_2 yields

$$\zeta_2 = g_k^\alpha(\zeta_1) := \frac{\nu(\zeta_1 - 1) + 2\pi k(\zeta_1 - 1) + 2i}{\nu(\zeta_1 - 1) + 2\pi k(\zeta_1 - 1) + 2i\zeta_1}, \quad k \in \mathbb{Z}.$$

Note that functions g_k^α are continuous on the unit circle; their only singularities occur at points $\zeta_1 = \frac{2\pi k+\nu}{\nu+2\pi k+2i}$, which do not have modulus one.

Moreover, all g_k^α pass through the point $(1,1) \in \mathbb{T}^2$, which does not solve $\Phi^* = \alpha$ as $\Phi^*(1,1) = 0$. However, since $\mathcal{C}_\alpha(\Phi)$ is closed, the point $(1,1)$ nevertheless lies in the unimodular level set. Hence,

$$\mathcal{C}_\alpha(\Phi) = \bigcup_{k\in\mathbb{Z}} \{(\zeta, g_k^\alpha(\zeta)) : \zeta \in \mathbb{T}\}$$

where g_k^α is analytic on $\mathbb{T}$ for every k. This proves that the Clark measures of Φ are supported on smooth curves — quite surprisingly, given our previous discussion.

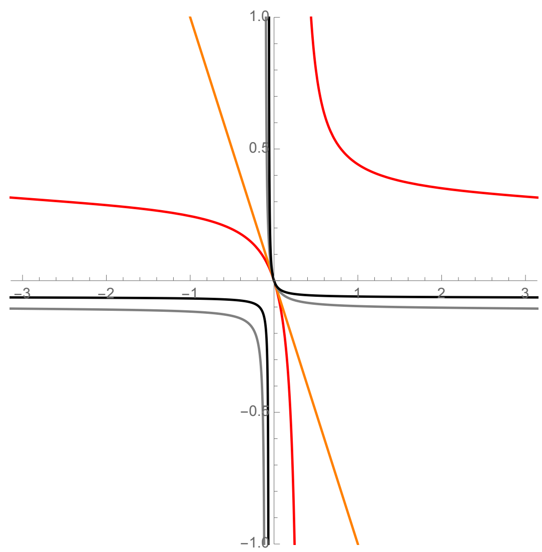


Figure 6: Level curves g_k^1 for $k = -1$ (red), $k = 0$ (orange), $k = 3$ (gray) and $k = 5$ (black).

It is interesting to compare these graphs to the ones in Figure 2, Figure 3 and Figure 4. We see that despite them all having singularities along one axis (both axes in the case of Figure 6), the curves behave very differently close to this line. Recall that by Lemma 5.3, the unimodular level sets of RIFs can be parameterized by graphs that are analytic on $\mathbb{T}^2$ except possibly at a single point. One might then expect that the Clark measures of a product function which is rational in at least one variable would be supported on smoother curves than this Φ. However, we see that in this case, the unimodular level sets are actually parameterized by much more "well-behaved" curves than our previous examples.

At first sight, this function does not seem to meet the requirements of Theorem 7.1; there is a point on $\mathbb{T}^2$ where all g_k intersect, as $g_k(1) = 1$ for all $k \in \mathbb{Z}$. However, as noted above, this value does not in fact solve the equation $\Phi^* = \alpha$ since $\phi^*(1) = 0$. This point would cause a problem if the Clark measure of ϕ had positive weight there. Fortunately, we are saved by Proposition 3.3; the measure associated to α has a point mass at 1 if and only if $\phi^*(1) = \alpha$, and so $|\phi'(1)|^{-1} = 0$.

Let us now calculate the weights of the Clark measures associated to Φ. First note that

$$\phi'(z_2) = -\frac{2\exp\left(-\frac{1+z_2}{1-z_2}\right)}{(1-z_2)^2} = -\frac{2\phi(z_2)}{(1-z_2)^2}.$$

Then

$$\phi'(g_k^\alpha(\zeta_1)) = -\frac{2\alpha}{\phi(\zeta_1)(1-g_k^\alpha(\zeta_1))^2} = \frac{2\alpha}{\phi(\zeta_1)} \frac{(\nu(\zeta_1-1)+2\pi k(\zeta_1-1)+2i\zeta_1)^2}{4(\zeta_1-1)^2}$$

for $\zeta_1 \in \mathbb{T}\setminus\{1\}$. When taking moduli, we find

$$|\phi'(g_k^\alpha(\zeta_1))| = \left|\frac{\nu(\zeta_1-1)+2\pi k(\zeta_1-1)+2i\zeta_1}{2(\zeta_1-1)}\right|^2$$

for $\zeta_1 \in \mathbb{T}\setminus\{1\}$. Hence, Theorem 7.1 yields

$$\int_{\mathbb{T}^2} f(\xi)d\sigma_\alpha(\xi) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} f(\zeta, g_k^\alpha(\zeta)) \left|\frac{2(\zeta-1)}{\nu(\zeta-1)+2\pi k(\zeta-1)+2i\zeta}\right|^2 dm(\zeta) \tag{15}$$

for all $f \in C(\mathbb{T}^2)$, where $\alpha = e^{i\nu}$. Note that the weights reduce to zero for $\zeta = 1$, as expected.

Let us explicitly check that the above integral does indeed converge. Fix $\zeta \in \mathbb{T}$. The sum of weights can be expressed as

$$\sum_{k\in\mathbb{Z}} \left|\frac{2(\zeta-1)}{\nu(\zeta-1)+2\pi k(\zeta-1)+2i\zeta}\right|^2 = \sum_{k\in\mathbb{Z}} \frac{1}{k^2}\left|\frac{2(\zeta-1)}{\nu(\zeta-1)/k+2\pi(\zeta-1)+2i\zeta/k}\right|^2. \tag{16}$$

Note that

$$\left|\frac{2(\zeta-1)}{\nu(\zeta-1)/k+2\pi(\zeta-1)+2i\zeta/k}\right|^2$$

is bounded for every k, and since the series $\sum_{k\in\mathbb{Z}} 1/k^2$ converges, the sum in (16) converges as well. As this holds for every $\zeta \in \mathbb{T}$, we conclude that the integral in (15) is convergent.

Remark 7.8. Throughout this text, we have managed to gain some insight into the Clark measures of certain specific types of bivariate inner functions. However, Clark measures of general inner functions in two variables still remain a mystery. In one variable, any singular probability measure on $\mathbb{T}$ defines the Clark measure of some inner function (pp. 234-235, [15]). In several variables, we need added requirements on a measure for it to be a Clark measure — as discussed in Remark 6.2, any positive, pluriharmonic, singular probability measure defines the Clark measure of some inner function. The distinction arises from the fact that in several variables, it is not as easy to ensure that a given harmonic function is the real part of an analytic function. By Theorem 2.4.1 in [21], the Poisson integral of a real measure μ on $\mathbb{T}^d$ is given by the real part of an analytic function if and only if the Fourier coefficients

$$\hat{\mu}(k) = \int_{\mathbb{T}^d} \overline{\zeta}^k d\mu(\zeta) = \int_{\mathbb{T}^d} \overline{\zeta_1}^{k_1} \cdots \overline{\zeta_d}^{k_d} d\mu(\zeta)$$

are zero for every k outside the set $-\mathbb{Z}_+^d \cup \mathbb{Z}_+^d$. Here $-\mathbb{Z}_+^d$ denotes the set of points $(k_1, \ldots, k_d)$ where every $k_j \leq 0$. We call the measures that satisfy this condition *RP-measures.* Note that Clark measures are included in this class per definition.

Furthermore, we still do not know much about the supports of Clark measures in several variables. The kind of smooth curve-parameterizations that were obtained for the classes of inner functions in this text are certainly not applicable in general. What we do know is that RP-measures cannot be supported on sets of Hausdorff dimension less than one (Theorem 4, [4]). In two dimensions, we have seen examples of Clark measures supported on curves, sets which have Hausdorff dimension exactly one. We did not see any examples of Clark measures whose supports have Hausdorff dimension greater than one here, but in [20], the author constructs an RP-measure whose support has Hausdorff dimension two. For an in-depth discussion about the supports of RP-measures, see [4].

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