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A Homotopy Theoretic Approach to Quillen's Conjecture

av

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Abstract

In 1978 the American mathematician Daniel Quillen was studying the topological properties of a certain poset associated to a group. He provided necessary and sufficient conditions for a solvable group to contain a normal p-subgroup and conjectured that it will also hold for any finite group. In this paper we will develop some of the techniques one will need to understand, restate and perhaps attack the conjecture. We will start by developing some of the theory of finite topological spaces, simplicial complexes and study their homotopy types. Then we turn our attention to groups, in particular we will look at the equivariant properties of the poset $S_p(G)$. We finish with a theorem that allows one to attack Quillen's conjecture from a lot of different angles and a neat result by Brown on the Euler characteristic of $S_p(G)$.

Abstrakt

År 1978 studerade den amerikanske matematikern Daniel Quillen de topologiska egenskaperna hos en viss poset associerad med en grupp. Han gav nödvändiga och tillräckliga villkor för att en lösbar grupp skulle innehålla en normal p-delgrupp och förmodade att den också skulle gälla för vilken ändlig grupp som helst. I denna artikel kommer vi att utveckla några av de tekniker man behöver för att förstå, omformulera och kanske attackera förmodan. Vi kommer att börja med att utveckla en del av teorin om ändliga topologiska rum, simpliciska komplex och studera deras homotopityper. Sedan riktar vi vår uppmärksamhet mot grupper, i synnerhet kommer vi att titta på de ekvivarianta egenskaperna hos poset:en $S_p(G)$. Vi avslutar med ett sats som låter en attackera Quillens förmodan från många olika vinklar och ett snyggt resultat av Brown på Euler-karaktäristiken av $S_p(G)$.

Contents

Introduction

Quillen's conjecture is the converse of a theorem proved by Quillen. Together with the theorem, Quillen's conjecture gives a necessary and sufficient condition for a group to contain a nontrivial normal subgroup. The conjecture says that given a prime p, a group G has a nontrivial normal p-subgroup if the geometric realization of a poset associated to the group is contractible.

There are a lot of different approaches to attack Quillen's conjecture. The goal of this paper is to build up the preliminaries for studying Quillen's conjecture from the perspective of homotopy theory and finite topology. In Chapter 1 we will develop the connection between posets and finite topological spaces and then focus on finite spaces and their topological and combinatorial properties. Then in Chapter 2 we will move on to simplicial complexes, where we describe how one can associate a simplicial complex to a finite topological space and also associate a finite topological space to a simplicial complex. In Chapter 3, the homotopy theory of finite spaces and simplicial complexes is developed. In particular we will look at the weak, simple and strong homotopy types of finite spaces and how that relates to the associated complex defined in Chapter 2. Finally, in Chapter 4 we apply what we have developed in the previous chapters in the setting of G-spaces (topological spaces on which a group G acts on) where we restrict ourselves to finite groups. Here we develop the equivariant homotopy theory of finite spaces and their associated simplicies complex. Once the necessary results and definitions are given, we apply them to $S_p(G)$, the poset of all nontrivial and proper p-subgroups of G under inclusion, and state Quillen's conjecture. We finish this chapter with a theorem by Brown on the Euler characteristic of $S_p(G)$ and the main theorem of this paper which gives 8 different statements that are all equivalent to Quillen's conjecture.

1 Preorders and Finite Topological Spaces

1.1 Preorders and Posets

Definition 1.1.1. *Let X and Y be arbitrary sets and $R \subseteq X \times Y$ be a relation. R is called a* ***preorder*** *if it is reflexive and transitive. If R is a preorder and it is also antisymmetric then it is called a* ***partial order****. A* ***preordered*** *X is a set with a relation $R \subseteq X \times X$ that is a preorder and a* ***partially ordered set*** *(a.k.a a* ***poset****) is a preordered set X with a relation that is a partial order.*

Given any set X and a relation R on X, we write xRy if x is related to y under R, that is if $(x, y) \in R$. If R happens to be a preorder we usually write $x \leq y$, with $\leq$ replacing R, however sometimes $x < y$ is written to indicate that x and y are distinct.

If X is a poset, an element $x \in X$ is called **maximal** if $x \leq y$ implies $y = x$ and it is called a **maximum** if $y \leq x$ for all $y \in X$. Similarly, we define the notion of a **minimal** and **minimum** element. Note that a finite poset has a maximum (minimum) if and only if there is a unique maximal (minimal) element.

Let X be a preordered set and $x, y \in X$. We say that y **covers** x if $x < y$ and there exists no $z \in X$ such that $x < z < y$. In this case we also say x is **covered by** y. We write $x \prec y$ if y covers x.

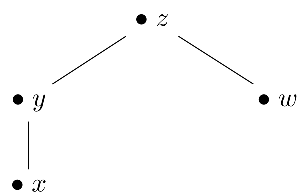
A point $x \in X$ is called an **up beat point** if x is covered by exactly one point of X and it is called a **down beat point** if it covers exactly one point of X. A point is a **beat point** if it is either an up beat point or a down beat point. Stong was the first mathematician to define and study beat points. He called up beat points **linear points** and down beat points **colinear points**, see [9]. The term beat point was introduced in [5] by J.P. May. We will, for the most part, stick with May's terminology.

Given a subset A of a preorder X, it is called a **down-set** if for any $x \in A$ and $y \in X$, $y \leq x$ implies $y \in A$ and it is called an **up-set** if for any $x \in A$ and $y \in X$, $x \leq y$ implies $y \in A$.

A **fence** in a preordered set X is a sequence of points $x_1, \ldots, x_n$ such that any two consecutive points are comparable. We say that a finite preordered set X is **order-connected** if for any two points $x, y \in X$, there exists a fence starting at x and ending at y.

There is a neat and compact way to fully describe a given finite poset. The **Hasse diagram** of a finite poset X is a diagram with the elements of X as points and a line is drawn upward from x to y if y covers x. We illustrate this with an example.

Example 1.1.2. *The Hasse diagram*



(1)

describes the poset $S = \{x, y, z, w\}$ with $x \prec y$, $y \prec z$, $w \prec z$ and $x \leq z$. As one can see the level we are placing the elements matters, z is placed at the top level because it is a maximal element (it also happens to be a maximum here).

Just because there is a "path" from x to w in Example 1.1.2, doesn't mean they are comparable. They are however comparable when the path is "monotone" in the sense that it is only going up (or down). This corresponds to the transitivity of the poset. Note that if we think of the Hasse diagram as a directed graph then the existence of a maximum (minimum) is characterized by having a sink (source) in the Hasse diagram. That is, there is an element where all the lines terminate at (initiate from). Maximal (minimal) elements are those at the top (bottom) level of the diagram. Hasse diagrams will be our main way of providing an example or counter-example for the upcoming results.

1.2 Finite Topological Spaces

The idea of a finite topological space might sound unpromising at first, because the only Hausdorff topology on a finite set is the discrete topology. However, it turns out that the homotopy theory of non-Hausdorff finite spaces is very intriguing.

When studying finite spaces, we lose a lot of the geometric intuition that motivated topology in the first place. But we will see below that by restricting our attention to finite topological spaces, we will gain a new type of intuition, one of a more combinatorial nature.

We start with a given finite preordered set X. For each element $x \in X$, define the down-set $U_x = \{y \in X \mid y \leq x\}$. The collection $\{U_x\}_{x \in X}$ forms a basis for a topology on X; it clearly covers X and if $U_x \cap U_y$ is non empty, then by the transitive property, $U_z \subseteq U_x \cap U_y$ for any $z \in U_x \cap U_y$. On the other hand, if we start with a finite topological space X and define, for each point $x \in X$, the **minimal open set** U_x to be the intersection of all the open sets that contain x, we get a preorder on X by $x \leq y$ if $x \in U_x \cap U_y$. Indeed $x \leq x$ because $x \in U_x$ and if $x \leq y$ and $y \leq z$ then $x \in U_y$ and $y \in U_z$ but U_y is the minimal open subset containing y so $x \in U_y \subseteq U_z$ implying $x \leq z$.

If x belongs to two finite spaces X and Y simultaneously, we denote the minimal sets of x in X and Y as U_x^X and U_x^Y respectively.

This shows that if X is a finite set, there is a one-to-one correspondence between the topologies and the preorders on X. The open sets are generated by the minimal sets which are precisely the down-sets. We also define, for each $x \in X$, the **closure** of the set $\{x\}$ to be the maximal sets $F_x = \{y \in X \mid x \leq y\}$.

Recall that if X is a finite topological space, **opposite** of X, denoted X^{op} is the space whose underlying set is X and whose open sets are the closed sets of X. The order on X^{op} is the inverse order of X. If $x \in X$ then $U_x^{X^{op}} = F_x^X$.

Finally, for any $x \in X$, we define the **punctured minimal set** $\hat{U}_x = U_x \setminus \{x\}$ and similarly, we define the **punctured maximal set** $\hat{F}_x = F_x \setminus \{x\}$.

Remark 1.2.1. *Note that x is an up beat point if and only if $\hat{F}_x$ has a minimum and it is a down beat point if and only if $\hat{U}_x$ has a maximum.*

Example 1.2.2. *Consider the poset S described in the Hasse diagram (1). The minimal basis is*

$$\{\{x\}, \{w\}, \{x, y\}, \{x, y, w, z\}\},$$

so the open sets are

$$\{x\}, \{w\}, \{x, y\}, \{x, w\}, \{x, y, w\}, \{x, y, w, z\}.$$

Note that since the underlying set is finite, an arbitrary intersection of open sets is open, this means that the minimal open set U_x of a point $x \in X$ is indeed open. However the property that an arbitrary intersection of open sets is open is not limited to finite spaces. A topological space is called an **Alexandorff space**[1] if arbitrary intersections of open sets are open. It follows that the correspondence between the topologies and the preorders on a finite set extends to a correspondence between Alexandorff spaces and preordered sets. Note that any set is an Alexandorff space with the discrete topology, so there are Alexandorff spaces that are not finite. From now on we may think of a finite topological space X as a preordered set and a preordered set X as a topological space as described above. It turns out that continuity of maps between finite spaces has an interesting but rather expected characterization.

Proposition 1.2.3. *A function between finite topological spaces is continuous if and only if it is order preserving, that is, $x \leq y$ implies $f(x) \leq f(y)$.*

Proof. Suppose $f : X \to Y$ is continuous and $x \leq y$ for some $x, y \in X$. It follows that $f^{-1}(U_{f(y)})$ is open, and since $y \in f^{-1}(U_{f(y)})$, by minimality $U_y \subseteq f^{-1}(U_{f(y)})$. By assumption $x \in U_y$ so $x \in f^{-1}(U_{f(y)})$ implying $f(x) \in U_{f(y)}$ which means $f(x) \leq f(y)$.

Conversely, if f is order preserving, we will show that for any $y \in X$, $f^{-1}(U_y)$ is open by showing it is a down-set. This is enough because recall that the set $\{U_x\}_{x \in X}$ defines a basis of X. Let $y \in Y$ be arbitrary. Now take any $x \in f^{-1}(U_y)$ and let x' be such that $x' \leq x$. Now the fact that $x \in f^{-1}(U_y)$ means precisely that $f(x) \in U_y$, which implies $f(x) \leq y$. Since f is order preserving, $f(x') \leq f(x) \leq y$, but then $x' \in f^{-1}(U_y)$ proving that $f^{-1}(U_y)$ is a down-set and hence open. □

Note that if X if a finite space and $f : X \to X$ is a continuous and injective, then f is a homeomorphism.

Lemma 1.2.4. *Let X be a finite space and $x, y \in X$ be two comparable points. Then there exists a path from x to y in X.*

Proof. Assume without loss of generality that $x \leq y$ and define the path $\alpha : I \to X$ by $\alpha(t) = x$ if $0 \leq t < 1$ and $\alpha(1) = y$. If U is a neighborhood of y, then $\alpha^{-1}(U) = I$. On the other hand if U is a neighborhood of x and $y \notin U$, then $\alpha^{-1}(U) = [0, 1)$, and if $x \notin U$ then $\alpha^{-1} = \emptyset$. In any case, $\alpha^{-1}(U)$ is open. □

[1]Alexandorff was the first mathematician to study finite topological spaces in the 1930s.

Proposition 1.2.5. *Let X be a finite topological space. The following are equivalent:*

(1) X is order-connected as a preorder.

(2) X is path-connected.

(3) X is connected.

Proof. If X is order-connected, by the lemma above, it is path connected. If it is path-connected then of course it is connected. If it is connected then let $x \in X$ be arbitrary and let $A = \{y \in X \mid \text{there is a fence from } y \text{ to } x\}$. If $y \in A$ and $z \leq y$ then $z \in A$ and similarly if $y \in A$ and $z \geq y$ then $z \in A$. So A is both a down-set and an up-set. It is therefore open and closed. Note that A is nonempty by reflexivity. Therefore by connectedness, $A = X$. □

By a **subbasis** $\mathcal{B}$ for a topological space $(X, \mathcal{T})$ we mean a subcollection of $\mathcal{T}$ such that any open subset of X can be written as a union of finite intersections of elements of $\mathcal{B}$. If X and Y are topological spaces, we denote by Y^X the set of continuous maps $f : X \to Y$.

Let $\mathcal{T}$ be the topology on Y^X whose subbasis is given by the sets $S(K, W) = \{f \in Y^X \mid f(K) \subseteq W\}$ where $K \subseteq X$ is compact and $W \in Y$ is open. This is called the **compact-open topology** on Y^X. When X and Y are finite, Y^X is also finite and we endow Y^X with the pointwise order: $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Proposition 1.2.6. *The pointwise order on Y^X induces the compact-open topology.*

Proof. Let $K \subseteq X$ be compact and $W \subseteq Y$ be open, then the set $S(K, W) = \{f \in Y^X \mid f(K) \subseteq W\}$ is in the subbase of the compact-open topology. Let $f \in S(K, W)$ and suppose $g \leq f$, this means in particular that $g(x) \leq f(x) \in W$ for every $x \in K$ and so $g \in S(K, W)$ showing that $S(K, W)$ is a down set and hence an open set. On the other hand if $f \in Y^X$, then $\{g \in Y^X \mid g \leq f\} = \bigcap_{x \in X} S(\{x\}, U_{f(x)})$. □

Given a finite space X and a space Y (not necessarily finite) there is a natural correspondence between the class of homotopies $\{H : X \times I \to Y\}$ and the class of paths $\{\alpha : I \to Y^X\}$. This follows from the exponential law for sets, which says that given three sets X, Y and Z, there is a natural bijection ϕ between the class of functions $Z \to F(X, Y)$, where $F(X, Y)$ denotes the class of all functions from X to Y, and the class of functions $f : X \times Z \to Y$ given by $\phi(f)(z)(x) = f(x, z)$. This unfortunately doesn't apply if X, Y and Z are any topological spaces with Y^X endowed with the compact-open topology. One can find a discontinuous function f such that $\phi(f)$ is continuous. However, the correspondence holds if for every point $x \in X$ and every neighborhood U of x, there exists a compact neighborhood of x contained in U. But this condition is trivially satisfied when X is finite because U itself will be compact (see [7] Theorem 46.11).

When two maps $f, g : X \to Y$ are homotopic we will write $f \simeq g$. If they are homotopic relative to a subset $A \subseteq X$ we write $f \simeq_A g$ or $f \simeq g$ rel A when the expression for the set A is too large to fit as a subscript.

Coming up next is a corollary that will lay the foundation for a lot of the proofs in the later sections.

Corollary 1.2.7. *Let $f, g : X \to Y$. Then $f \simeq g$ iff there is a fence $f = f_0 \leq f_1 \geq f_2 \leq \ldots f_n = g$. Moreover, if $A \subseteq X$ then $f \simeq_A g$ iff there is a fence $f = f_0 \leq f_1 \geq f_2 \leq \ldots f_n = g$ such that $f_i|_A = f|_A$ for all $0 \leq i \leq n$.*

Proof. By the correspondence of homotopies $X \times I \to Y$ and paths $I \to Y^X$, there is a homotopy $H : f \simeq_A g$ if and only if there is a path $\alpha : I \to Y^X$ from f to g such that $\alpha(t)|_A = f|_A$ for all $t \in I$. This is equivalent to saying that there is a path $\alpha : I \to M$ from f to g where M is the subspace of Y^X of maps which coincide with f in A. By Proposition 1.2.5, there is a fence from f to g in M. But since the order of M is induced by the order of Y^X, the proof is complete. □

The following corollary will be used extensively.

Corollary 1.2.8. *If X is a poset with either a maximum or a minimum then it is contractible as a topological space.*

Proof. Suppose we have a maximum $M \in X$. Let $m : X \to X$ be the constant map $m(x) = M$. It follows that the identity is homotopic to m since $id_X \leq m$. The proof for a minimum is similar. □

1.3 F-spaces

Recall that a topological space X is said to be a T_0**-space** if for any distinct pair $x, y \in X$ we can find an open subset containing either x or y but not the other and it is a T_1**-space** if for any two distinct pair $x, y \in X$ can find two open sets U and V such that $x \in U$ and $y \in V$ but $x \notin V$ and $y \notin U$. Being a T_1-space is equivalent to saying that the singletons are closed. For this reason, T_1-spaces (let alone Hausdorff spaces) are of no interest to us if we are considering finite spaces. This is because the closeness of the singletons makes every subset closed and we end up with the discrete topology. T_0-spaces on the other hand are of special interest to us due to the following proposition.

Proposition 1.3.1. *A finite topological space X is a T_0-space if and only if the corresponding preorder is a partial order.*

Proof. Suppose X is T_0. Let $x, y \in X$ such that $x \leq y$ and $y \leq x$. We will show that $x = y$. We argue by contradiction. Suppose that $x \neq y$, then since X is T_0, we may assume that there exists an open set U such that $x \in U$ and $y \notin U$. Now $x \in U$ implies that $U_x \subseteq U$ by the minimality of U_x. But since $y \leq x$, $y \in U_x$ and so $y \in U$, a contradiction.

Conversely, suppose $x \leq y$ and $y \leq x$ imply $x = y$ for all $x, y \in X$. Let $x, y \in X$ be distinct points. If x and y are not comparable, then the open set U_x separates x and y and we are done. Suppose therefore that they are. By the hypothesis, either $x < y$ or $y < x$, exclusively. Suppose that $x < y$, then $y \notin U_x$ but of course $x \in U_x$. The proof is identical if $y < x$. □

We will follow [5] and use the abbreviation F**-space** for finite T_0-spaces and A**-spaces** for Alexandorff T_0-spaces. From this point on we shall only focus on the former, but a lot of the proofs apply to general finite spaces and Alexandorff spaces. Of course arguments done by repeating a process until it terminates are among those that do not hold for general Alexandorff spaces.

The following proposition ensures that we will not miss out on any thing, at least as far as homotopy is concerned, if we restrict ourselves to F-spaces.

Proposition 1.3.2. *Let X be a finite space and define an equivalence relation $\sim$ by $x \sim y$ if $x \leq y$ and $y \leq x$. Then $X/\sim$ is T_0 and the quotient map $q : X \to X/\sim$ is a homotopy equivalence. In particular, every finite space is homotopy equivalent to a an F-space.*

Proof. Being a quotient map, q is obviously surjective and so a section (a right inverse) exists as a map of sets. Let $i : X/\sim \to X$ be any section. The map i will take a class $[a]$ and map it to any representative. It is well-defined because of the definition of $\sim$. It follows that i is continuous since if $[a] \leq [b]$, $i([a]) = a \leq b = i([b])$. Moreover, the composition $i \circ q$ is order preserving and since $iq \leq 1_X$, i is a homotopy inverse for q.

To show that the quotient is T_0, we show that the preorder on the quotient set is antisymmetric. Since q is surjective, let $x, y \in X$ such that $q(x) \leq q(y)$. It follows that $x \leq (i \circ q)(x) \leq (i \circ q)(y) \leq y$. Suppose also that $q(y) \leq q(x)$ and $y \leq x$, then by the definition of q, $q(x) = q(y)$ and hence the preorder is antisymmetric. □

So any finite space is homotopy equivalent to an F-space.

Proposition 1.3.3. *Let X be a finite space and let $x \in X$ be a beat point, then $X \setminus \{x\}$ is a strong deformation retract of X.*

Proof. Let x be a down beat point (the proof for an up beat point is analogous), and denote by y the maximum of $\hat{U}_x$ which exists by Remark 1.2.1. Define the map $r : X \to X \setminus \{x\}$ by declaring it to be the identity on $X \setminus \{x\}$ and $r(x) = y$. Now r is clearly order preserving and so it is continuous. Let $i : X \setminus \{x\} \hookrightarrow X$ be the inclusion. Then $I \circ r \leq 1_X$. By Corollary 1.2.7, $i \circ r \simeq 1_X$ relative to $X \setminus \{x\}$. □

In particular, removing a beat point from our space doesn't effect its homotopy type. A finite space that has no beat points is called **minimal**. A core of a finite space X is a minimal strong deformation retract of X.

Theorem 1.3.4. *Let X be a minimal finite space. A map $f : X \to X$ is homotopic to the identity if and only if $f = 1_X$.*

Proof. Suppose f is homotopic to the identity. By Corollary 1.2.7, we may assume that $f \leq 1_X$ or $f \geq 1_X$. Suppose that $f \leq 1_X$. The case of $f \geq 1_X$ is treated similarly. We proceed by induction of the elements of X. Suppose that $f|_{\hat{U}_x} = 1_{\hat{U}_x}$. If $f(x) \neq x$, then $f(x) \in \hat{U}_x$ and if $y < x$, we have that $y \in \hat{U}_x$ and so $y = f(y) \leq f(x)$ implying that $f(x)$ is the maximum of $\hat{U}_x$. But this means that x is a beat point which is a contradiction, so indeed $f(x) = x$.

The converse is trivial. □

Proposition 1.3.5. *Let X be an F-space, $x \in X$ a point and $f : X \to X$ be an automorphism. If x and $f(x)$ are comparable, then $x = f(x)$.*

Proof. Suppose that $x \leq f(x)$. Since f is an automorphism, it is in particular continuous and so order preserving, therefore $f(x) \leq f^2(x)$ where $f^2(x)$ means $f(f(x))$. We inductively get $f^n(x) \leq f^{n+1}(x)$ for any positive integer n. But X is finite, which implies that $f^m(x) = f^{m+1}(x)$ for some positive integer m. Suppose we found such an m, then since f is a homeomorphism, f^{-1} exists and so f^{-m} also exists. But then applying f^{-m} to $f^m(x) = f^{m+1}(x)$ we get $x = f(x)$. The proof where $f(x) \leq x$ is similar. □

The following corollary is immediate.

Corollary 1.3.6. *Let X be an F-space and let $f_1, f_2 : X \to X$ be two automorphisms and $x \in X$. If $f_1(x)$ and $f_2(x)$ are comparable, then $f_1(x) = f_2(x)$.*

Proof. Apply the proposition to the automorphism $f_2^{-1}f_1$. □

Given two F-spaces X and Y, define the **ordinal sum** $X \circledast Y$ to be the disjoint union $X \sqcup Y$ with the same ordering within X and Y but we let $x \leq y$ for every $x \in X$ and $y \in Y$. The proof of the following proposition can be found in [1] page 30.

Proposition 1.3.7. *Let X and Y be F-spaces. Then $X \circledast Y$ is contractible if and only if either X is contractible or Y is contractible.*

We now introduce the first notion of a collapse. Collapses will be studied in different settings through out the upcoming sections.

Definition 1.3.8. *Let X be an F-space. If $x \in X$ is a beat point, we say that there is an **elementary strong collapse** from X to $X \setminus \{x\}$ and write $X \searrow\!\!\searrow^e X \setminus \{x\}$. If Y is a subspace, we say that there is a **strong collapse** $X \searrow\!\!\searrow Y$ (or a strong **expansion** $Y \nearrow\!\!\nearrow X$) if there is a sequence of elementary strong collapses starting at X and ending at Y.*

Definition 1.3.9. *A pair (X, A) of F-spaces is called a **minimal pair** if all the beat points of X are in A.*

Proposition 1.3.10. *Let (X, A) be a minimal pair and let $f : X \to X$ be continuous such that $f \simeq_A 1_X$. Then $f = 1_X$.*

Proof. Suppose that $f \simeq_A 1_X$, meaning in particular $f|_A = 1_X|_A = 1_A$. Consider the case $f \leq 1_X$. Let $x \in X$. If x is minimal, then $f(x) = x$, for otherwise if $f(x) = y \neq x$ then $y = f(x) \leq 1_X(x) = x$, but then $y = x$ by the minimality of x. Suppose we have showed that $f|_{\hat{U}_x} = 1_{\hat{U}_x}$. It follows that $f(x) = x$. On the other hand, if $x \notin A$, then since (X, A) is a minimal pair, x is not a beat and in particular it is not a down beat point of X. Now since $f|_{\hat{U}_x} = 1_{\hat{U}_x}$, if $y < x$, then $y = f(y) \leq f(x) \leq x$ and so we must have $f(x) = x$. The case $f \geq 1_X$ is treated similarly and the general case follows from Corollary 1.2.7. □

From this we get the following corollaries.

Corollary 1.3.11. *Let (X, A) and (Y, B) be minimal pairs and let $f : X \to Y$ and $g : Y \to X$ be continuous maps such that $g \circ f \simeq_A 1_X$ and $f \circ g \simeq_B 1_Y$. Then f and g are homeomorphisms.*

Proof. Apply the proposition to the maps $g \circ f$ and $f \circ g$. □

Corollary 1.3.12. *Let X be an F-space and let $A \subseteq X$. Then $X \searrow\!\!\searrow A$ if and only if A is a strong deformation retract of X.*

Proof. Suppose that $A \subseteq X$ is a strong deformation retract. We can preform arbitrary elementary strong collapses by removing beat points in which are not in A to obtain a strong collapse $X \searrow\!\!\searrow Y$. Now $A \subseteq Y$ and every beat point of Y is in A by construction of Y. It follows that (Y, A) is a minimal pair and since both Y and A are strong deformation retracts of X, we apply Corollary 1.3.11 to (A, A) and (Y, A) and get that $Y \approx A$ and so $X \searrow\!\!\searrow Y = X \searrow\!\!\searrow A$. The converse follows directly from Proposition 1.3.3. □

Now that we have covered some of the theory of F-spaces we will move on to simplicial complexes and define the necessary notions we will need to arrive at a beautiful theory that connects F-spaces with finite simplicial complexes.

We finish this section by defining the Euler characteristic of an F-space. We will need this when we state and prove Brown's theorem from [3].

Unlike general topological spaces, the Euler characteristic of a finite space can be found without knowing the homology groups[2] or without having to look for a polygonal presentation. We simply only need to know how the set of nonempty chains looks like.

Definition 1.3.13. *Let X be an F-space, the* ***Euler characteristic*** *of X is the number*

$$\chi(X) = \sum_{C \in \mathcal{C}(X)} (-1)^{\#C+1},$$

where $\mathcal{C}(X)$ is the set of nonempty chains of X and $\#C$ is the cardinality of C.

If Y is a subspace, we define the ***relative Euler characteristic*** *as*

$$\chi(X, Y) = \sum_{C \in \mathcal{C}(X \setminus Y)} (-1)^{\#C+1}.$$

Remark 1.3.14. *If X is an F-space and Y is any subspace, then using the definitions above one can easily see that*

$$\chi(X) = \chi(X, Y) + \chi(Y).$$

The following theorem will also be needed. However, the proof requires familiarity with homology, which is beyond the scope of this paper. We will however give a sketch of the proof for the readers who are familiar with some homology theory.

Theorem 1.3.15. *If X and Y are two topological spaces with the same homotopy type, then $\chi(X) = \chi(Y)$.*

Sketch of proof. By Theorem 2.44 in [4], the Euler characteristic is only dependent on the homology groups. However, by Corollary 2.11 in [4], homotopy equivalent spaces have isomorphic homology groups. It follows that the Euler characteristics of homotopy equivalent spaces coincide. □

[2]If the homology groups with coefficients in $\mathbb{Z}$ of a space X are finitely generated then the Euler characteristic is given by $\chi(X) = \sum_{n \geq 0} (-1)^n rank(H_n(X))$

2 Simplicial Complexes

In this section we cover the basics of the theory for simplicial complexes. We will start with some basic definitions and later see how we can associate a simplicial complex to a finite poset (or equivalently an F-space). We will also look at the homotopy and weak homotopy types of the geometric realization of the associated complex and compare it with that of the poset (viewed as a topological space).

2.1 Abstract Simplicial Complexes

As usual, we start with some definitions.

Definition 2.1.1. *An **abstract simplicial complex** $\mathcal{K}$ consists of a set V_K called the set of vertices and a set S_K of finite subsets of V_K called the set of simplicies satisfying that any subset of S_K of cardinality one is in V_K and any subset of a set in S_K is in S_K. It is customary to write $v \in K$ if $v \in V_K$ and $\sigma \in K$ if $\sigma \in S_K$. We identify a simplicial complex with its set of simplices.*

Note that the word abstract is there to remind us that we are dealing with an abstract construction here. A **geometric simplicial complex** is obtained by taking a collection geometric simplices that satisfy the abstract simplicial complex axioms. A **geometric simplex** σ is a subset of $\mathbb{R}^n$, for some positive integer n, such that all the points in σ are affinely independent. Note that this implies in particular that the number of points in σ is $\leq n$. From this point on whenever we speak of a simplicial complex K we always mean an abstract simplicial complex, but for the convince of the reader we will show the geometric simplicial complex in figures that will provide some intuition for the notion in question because it is more intuitive to do so.

If σ and τ are simplices such that $\tau \subseteq \sigma$, we say that τ is a **face** of σ, and a **proper face** if $\tau \subsetneq \sigma$.

Given a simplex $\sigma \in S_K$, we say σ is an **n-simplex**, or **has dimension** n, if it contains $n+1$ vertices. Note that this makes each vertex a 0-simplex. As a convention, the empty simplex has dimension -1. We also define the dimension of the simplicial complex K to be the least upper bound of the dimension of all its simplices. Note that the least upper bound can be infinite if K contains simplices of arbitrary dimension, in which case, we say that K is infinite dimensional. If K is finite dimensional and $\sigma \in S_K$ is a simplex that is maximal (meaning it is not a face of any other simplex but itself) then we call σ a **facet**. We say that a finite dimensional simplicial complex K is **homogeneous** if all facets have the same dimension. A face is called a **free face** if it is the face of only one facet in the simplex. If σ is a free face of the facet τ we will sometimes abbreviate this by saying that the pair $\sigma \subsetneq \tau$ is a **collapsible pair**.

As with most mathematical objects we define the notion of a subcomplex of a simplicial complex. A subcomplex L of a simplicial complex K is a simplicial complex such that $V_L \subseteq V_K$ and $S_L \subseteq S_K$. The subcomplex L is **full** if for any simplex σ in K such that its vertices are in L, it is also in a simplex in L. If this is the case we say that L is the subcomplex of K spanned by the vertices in V_L. Furthermore, if we have two simplicial complexes K and L, we define the **simplicial join** (or just the **join**) of K and L, denoted $K * L$ to be the simplicial complex whose vertex set is the disjoint union of V_K and V_L and whose simplices are the simplices of K and L and all disjoint unions of simplices in

K and simplices in L.

Simplicial complexes have a rather combinatorial nature, however one can add some topological flavor by introduction the **geometric realization** $|K|$ of a simplicial complex K to be the set of formal convex combinations $\sum_{v\in K}\alpha_v v$ such that the set $\{v \mid \alpha_v > 0\}$ is a simplex of K. Given an n-simplex $\sigma = \{v_0, v_1, \ldots, v_n\}$, we define the **closed simplex** $\overline{\sigma}$ to be the set of formal convex combination $\sum_{i=0}^{n}\alpha_i v_i$ where each α_i is a nonnegative real number and $\sum_i \alpha_i = 1$. We can make $\overline{\sigma}$ into a metric space by defining the metric

$$d(\sum_{i=0}^{n}\alpha_i v_i, \sum_{i=0}^{n}\beta_i v_i) = \sqrt{\sum_{i=0}^{n}(\alpha_i - \beta_i)^2}.$$

Note that if K a finite simplicial complex, that is V_K is a finite set, the topology of $|K|$ coincides with the metric topology on the unions of the closed simplices.

We can now make $|K|$ into a topological space by declaring a subset $U \subseteq |K|$ to be open if $U \cap \overline{\sigma}$ is open in the metric space $\overline{\sigma}$ for every $\sigma \in K$.

Given a point $p = \sum_{v\in K}\alpha_v v \in |K|$, we define the **support** of p to be the set $support(p) = \{v_j \mid \alpha_j > 0\}$. If $\sigma \in K$ is a simplex, we define the **open simplex** $\overset{\circ}{\sigma}$ to be the subset of $\overline{\sigma}$ such that the support of any point in $\overset{\circ}{\sigma}$ is σ. Note that the open simplices are disjoint and need not be open subsets despite what the name suggests.

Given two simplicial complexes K and L, a **simplicial map** $\varphi : K \to L$ is a map from V_K into V_L that maps simplices of K into simplices of L. If $\varphi : K \to L$ is a simplicial map it induces a well-defined continuous map $|\varphi| : |K| \to |L|$ defined by $|\varphi|(\sum_{v\in K}\alpha_v v) = \sum_{v\in K}\alpha_v\varphi(v)$.

Definition 2.1.2. *Two simplicial maps $\varphi, \psi : K \to L$ are **contiguous** if for every simplex $\sigma \in K$, $\phi(\sigma) \cup \psi(\sigma)$ is a simplex of L. Two simplicial maps lie in the same **contiguity** class if there is a sequence of pairwise contiguous maps $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_n = \psi$. We denote this by $\varphi \sim \psi$.*

It is not hard to see that if $\varphi_1, \varphi_2 : K \to L$ and $\psi_1, \psi_2 : L \to M$ are simplicial maps with $\varphi_1 \sim \varphi_2$ and $\psi_1 \sim \psi_2$, then $\psi_1\varphi_1 \sim \psi_2\varphi_2$.

Lemma 2.1.3. *Let K be a simplicial complex and $F \subseteq |K|$ be compact. There exists a finite subcomplex L of K such that $F \subseteq L$.*

Proof. Let $F \subseteq |K|$ be compact. For each simplex $\sigma \in K$ such that $F \cap \overset{\circ}{\sigma} \neq \emptyset$, choose a point $x_\sigma \in F \cap \overset{\circ}{\sigma}$ and let D be the set of all these x_σ. Note that each point $x_{\sigma'} \in D$, $support(x_{\sigma'}) = \sigma'$ by the definition of the open simplex. Let $A \subseteq D$. We claim that the intersection of A with any other closed simplex is finite. To see why, suppose $\sigma = \{\sigma_1 \ldots, \sigma_n\}$ is a simplex in K. Now $A \cap \overline{\sigma} = \{x_\sigma \in |K| \mid x_\sigma = \sum_{i=1}^{n}\alpha_i\sigma_i, \alpha_i > 0\}$. By construction, for each simplex τ, D contains at most one point with support equal to τ. It follows that the cardinality of $D \cap \overline{\sigma}$ is at most the cardinality of the power set of τ and since $A \cap \overline{\sigma} \subseteq D \cap \overline{\sigma}$, we conclude that $A \cap \overline{\sigma}$ is finite and thus closed. It follows that A is closed in $|K|$, but since A was an arbitrary subset, this implies that D is discrete and closed (since its a subset of a compact set) and thus finite. It follows that F intersects only finitely many open simplices. Now let L be the complex generated by the simplices σ for which the intersection $F \cap \overset{\circ}{\sigma}$ is nonempty. It follows that L has the desired property. $\square$

Remark 2.1.4. *If $L \subseteq K$ is a subcomplex, then $|L|$ is a closed subspace of $|K|$. The proof of this is somewhat similar to the result in analysis which states that a finite dimensional subspace of a normed space is closed.*

Proposition 2.1.5. *Let K and L be two simplicial complexes and $f, g : |K| \to |L|$ be two continuous maps such that for every $x \in |K|$ there exists a simplex $\sigma \in L$ such that $f(x), g(x) \in \overline{\sigma}$. Then f and g are homotopic.*

Proof. Let $H : |K| \times I \to |L|$ be the straight line homotopy, that is, $H(x,t) = tg(x) + (1-t)f(x)$. Since f and g map each x to the same closed simplex it follows that H is well-defined. Moreover, to show continuity, it suffices to show that H is continuous on $\overline{\sigma} \times I$ for every simplex $\sigma \in K$. Let $\sigma \in K$. It follows that $\overline{\sigma}$ is compact and since f and g are continuous, $f(\sigma)$ and $g(\sigma)$ are also compact. By the lemma above, $f(\overline{\sigma}) \subseteq |L_1|$ and $g(\overline{\sigma}) \subseteq |L_2|$ where L_1 and L_2 are finite subcomplexes of L. If we denote by M the full subcomplex of L generated by the vertices of L_1 and L_2, we get that $H(\overline{\sigma} \times I)$ is contained in $|M|$. And since $\overline{\sigma}$ and $|M|$ have the same metric topology, we get

$$\begin{aligned} d(H(x,t), H(y,s)) &= d(tg(x) + (1-t)f(x), sg(y) + (1-s)f(y)) \\ &\leq d(tg(x) + (1-t)f(x), sg(x) + (1-s)f(x)) \\ &\quad + d(sg(x) + (1-s)f(x), sg(y) + (1-s)f(y)) \\ &\leq 2|t-s| + d(f(x), f(y)) + d(g(x), g(y)). \end{aligned}$$

But f and g are continuous so the proof is complete. ☐

In the case of contiguous maps we get the following useful corollary.

Corollary 2.1.6. *If φ and ψ are contiguous maps then the induced maps $|\varphi|$ and $|\psi|$ are homotopic.*

Proof. This is immediate because if $\sigma \in K$, then for every $x \in \overline{\sigma}$ we have $|\varphi|(x), |\psi|(x) \in \overline{\varphi(\sigma) \cup \psi(\sigma)}$. ☐

A simplicial complex K is called a **simplicial cone with apex** v if v is a vertex in K such that for any simplex $\sigma \in K$, $\sigma \cup \{v\}$ is also a simplex. If K is any simplicial complex, we can define the cone aK to be the join of K and the complex with just one vertex $\{a\}$ (note that $a \notin K$ by the definition of the join).

Corollary 2.1.7. *If K is a simplicial cone, then $|K|$ is contractible.*

Proof. Suppose v is an apex of K. Note that the constant vertex map $w \mapsto v$ is contiguous to the identity by the definition of a simplicial cone. By the previous corollary, the identity of $|K|$ is homotopic to a constant map and hence $|K|$ is contractible. ☐

A **chain of simplices** of a simplicial complex K is a set $\{\sigma_0, \sigma_1, \dots, \sigma_m\}$ of simplices of K such that $\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_m$.

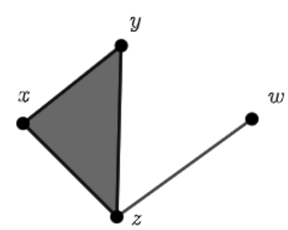
Definition 2.1.8. *Given a simplicial complex K, we define the* ***barycentric subdivision*** *of K, denoted by K', to be the simplicial complex whose vertices are the simplices of K and simplices are chains of K. The* ***barycenter*** *of a simplex $\sigma \in K$ is the point in $|K|$ given by $b(\sigma) = \sum_{v \in \sigma} \frac{1}{\#\sigma} v$ where $\#\sigma$ denotes the cardinality of σ. The map $s_K : |K'| \to |K|$ given by $s_K(\sigma) = b(\sigma)$ is a homeomorphism and it is called the* ***linear map****.*

2.2 The Order Complex of a Poset and the Face Poset of a Complex

In this section we will associate a simplicial complex to a given finite poset P called the **order complex of** P and we will define for a finite simplicial complex K a poset (or equivalently a T_0-space) called the **face poset of** K. We will then study the connection between these two notions and in the next chapter we will look closely at their homotopy types and how they relate to each other. We start with the definitions.

Definition 2.2.1. *Given a poset X or equivalently, an F-space, we assign to it a simplicial complex $\mathcal{K}(X)$ called the **order complex** whose vertex set is X and a simplex in K is a totally ordered subset of X. In other words, $\mathcal{K}(X) = \{U \subseteq X \mid U \text{ is a total order}\}$ where we consider the singletons as total orders.*

Example 2.2.2. *The order complex of the poset S from Example 1.1.2 has the following form*

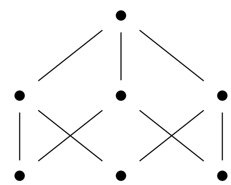


The order complex will allow us to study Quillen's conjecture, we will associate a poset to a given finite group and then look at the order complex (or rather, its geometric realization) and see what (topological) properties it has and how they relate to those of the group. In many sources when one is referring to the topology of a poset they usually mean the topology of the geometric realization of the order complex assigned to the poset as above. However we will not follow this convention here, instead when we refer to a poset as a topological space we will mean the topology induced by the partial order. If we wish to refer to the topology of the order complex we will explicitly say that.

Note that if $f : X \to Y$ is a map between F-spaces, then there is an induced simplicial map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ given by $\mathcal{K}(f)(x) = f(x)$. We say that $\mathcal{K}(f)$ is the **associated simplicial map** of f. By the continuity of f we can deduce that the vertex map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is simplicial because f is order-preserving and maps chains to chains.

Definition 2.2.3. *Let K be a finite simplicial complex. The **face poset of** K is the F-space $\mathcal{X}(K)$ of simplices of K ordered by inclusion. If $\varphi : K \to L$ is a simplicial map and K and L are finite simplicial complexes, there is a continuous map $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$ given by $\varphi(\sigma) = \mathcal{X}(\varphi)(\sigma)$ for every simplex $\sigma \in K$.*

Example 2.2.4. *Let K be the 2-simplex, then $\mathcal{X}(K)$ is the following space*



The maximum is K itself, the points on the level below are the 1*-simplices of K and the ones below them are the vertices.*

Proposition 2.2.5. *Let $K = aL$ be a finite simplicial cone. Then $\mathcal{X}(K)$ is contractible.*

Proof. The map $f : \mathcal{X}(K) \to \mathcal{X}(K)$ given by $f(\sigma) = \sigma \cup \{a\}$ is clearly order preserving and hence continuous. Define $g : \mathcal{X}(K) \to \mathcal{X}(K)$ to be the constant map $g(\sigma) = \{a\}$ for all $\sigma \in \mathcal{X}(K)$. But then we get a fence $1_{\mathcal{X}(K)} \leq f \geq g$ showing that the identity is homotopic to a constant map. □

We can also define the Euler characteristic in the setting of finite simplicial complexes as follows. If K is an n-simplicial complex, denote by r_k number of k-simplices of K, then the Euler characteristic of K is given by

$$\chi(|K|) = \sum_{i=0}^{n} (-1)^i r_j.$$

Note that the Euler characteristic of K is actually the Euler characteristic of $|K|$.

Remark 2.2.6. *By comparing the formulas for the Euler characteristic of a simplicial complex and an F-space and recalling that simplicies in the order complex are chains of the underlying poset, it is fairly easy to see that if X is an F-space then $\chi(X) = \chi(|\mathcal{K}(X)|)$.*

2.3 Homotopy and Contiguity

Lemma 2.3.1. *If $f, g : X \to Y$ are two homotopic maps between F-spaces, then there exists a sequence $f = f_0, f_1, \ldots, f_n = g$ such that for every $0 \leq i < n$, there is a point $x_i \in X$ such that:*

(1) $f_i(x) = f_{i+1}(x)$ for all $x \in X \setminus \{x_i\}$.

(2) $f_i(x_i) \prec f_{i+1}(x_i)$ or $f_{i+1}(x_i) \prec f_i(x_i)$.

Proof. Suppose that $f \simeq g$. By Corollary 1.2.7, we may assume without loss of generality that $f \leq g$. Let $A = \{x \in X \mid f(x) \neq g(x)\}$. If A is empty, there is nothing to prove, so suppose A is nonempty. We will construct functions f_i with the desired properties. Let $f_0 = f$ and let $x_0 \in A$ be a maximal point of A. Let $y \in Y$ such that $f(x_0) \prec y \leq g(x_0)$. Define $f_1 : X \to Y$ by $f_1|_{X \setminus \{x\}} = f|_{X \setminus \{x\}}$ and $f_1 = y$. If $x_0 < x'$ then $x' \notin A$ and

$$f_1(x') = f(x') = g(x') \geq g(x_0) \geq y = f(x_0),$$

so f_1 is order preserving and hence continuous. We iterate this construct to obtains $f_2, f_3, \ldots, f_n$. We know the process must end because X and Y are finite spaces. □

We can prove the following. (cf. Corollary 2.1.6)

Proposition 2.3.2. *If $f, g : X \to Y$ are two homotopic maps between F-spaces. Then the associated simplicial maps $\mathcal{K}(f), \mathcal{K}(g) : \mathcal{K}(X) \to \mathcal{K}(Y)$ lie in the same contiguity class.*

Proof. By the lemma above we can assume that there is an $x \in X$ such that $f(y) = g(y)$ for all $y \neq x$ and $f(x) \prec g(x)$. If follows that if C is a chain in X, then $f(C) \cup g(C)$ is a chain in Y. Since simplexes in $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are nothing but chains in X and Y respectively, it follows that if $\sigma \in \mathcal{K}(X)$ is a simplex, then $f(\sigma) \cup g(\sigma)$ is in $\mathcal{K}(Y)$. □

The next proposition shows that for two simplicial maps, the induced maps under $\mathcal{X}(\cdot)$ are homotopic if the maps are in the same contiguity class.

Proposition 2.3.3. *Let $\varphi, \psi : K \to L$ be two simplicial maps in the same contiguity class. Then $\mathcal{X}(\varphi) \simeq \mathcal{X}(\psi)$*

Proof. The map $f : \mathcal{X}(K) \to \mathcal{X}(L)$ given by $f(\sigma) = \varphi(\sigma) \cup \psi(\sigma)$ is continuous. It is also well-defined by the contiguity of φ and ψ. Note that $\mathcal{X}(\varphi) \leq f \geq \mathcal{X}(\psi)$ and therefore $\mathcal{X}(\varphi) \simeq \mathcal{X}(\psi)$. □

3 Homotopy Types

In this section we study the weak, simple and strong homotopy types of simplicial complexes and F-spaces and find a connection between the three notions.

3.1 Weak Homotopy Types

Definition 3.1.1. *Let X and Y be topological spaces. A continuous map $f : X \to Y$ is a* ***weak homotopy equivalence*** *if the induced maps $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ are isomorphisms for all $n \geq 1$ and $x_0 \in X$ and $f_* : \pi_0(X, x_0) \to \pi_0(Y, f(x_0))$ is a bijection. Two spaces are* ***weakly homotopy equivalent*** *or have the same* ***weak homotopy type*** *if there is a sequence of spaces $X = X_0, X_1, \ldots, X_n = Y$ such that there are weak homotopy equivalences $X_i \to X_{i+1}$ or $X_{i+1} \to X_i$ for every $0 \leq i < n$. In this case, we write $X \overset{we}{\simeq} Y$. A topological space is said to be* ***weakly contractible*** *if it is weak homotopy equivalent to a point.*

Note that homotopy equivalences are weak homotopy equivalences. It is obvious enough to see that being weak homotopy equivalent is an equivalence relation. Weak homotopy equivalences satisfy the so-called 2-out-of-3: If f and g are composable maps and two out of the three maps f, g, $g \circ f$ are weak homotopy equivalences, then so is the third map.

The Whitehead theorem, Theorem 4.5 in [4], says that a weak homotopy equivalence between CW-complexes is a homotopy equivalence. Since simplices complexes are CW-complexes, we get the following equivalence.

Proposition 3.1.2. *If K and L are simplicial complexes, then $f : |K| \to |L|$ is a weak homotopy equivalence if and only if f is a homotopy equivalence.*

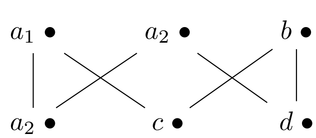
Definition 3.1.3. *Let X be a topological space. An open cover $\mathcal{U}$ of X is called a* ***basis like cover*** *if $\mathcal{U}$ is a basis for some topology of X.*

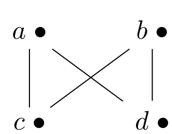
Note that the topology that is generated by $\mathcal{U}$ may be different from the given topology on X. The reason for this somewhat strange definition is the following theorem, known as McCords theorem. We shall use this theorem in the coming sections to prove that a map is a weak homotopy equivalence. See [6] Theorem 6 for the proof.

Theorem 3.1.4 (McCord)**.** *Let X and Y be topological spaces and $f : X \to Y$ be a continuous map. Suppose that $\mathcal{U}$ is a basis like open cover of Y and for every $U \in \mathcal{U}$, the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a weak homotopy equivalence. Then f is a weak homotopy equivalence.*

Luckily for us, the minimal basis is a basis like cover so we will not need to look for a basis like cover every time we want to apply McCord's theorem.

Example 3.1.5. *To illustrate the theorem, consider the following two spaces*





and suppose we had a continuous (order preserving) map f given by $f(a_i) = a$ for $i = 1, 2, 3$, $f(b) = b$, $f(c) = c$ and $f(d) = d$. The preimage of each minimal open set U_x is contractible and since U_x is contractible, it follows that $f|_{U_x} : f^{-1}(U_x) \to U_x$ is a weak homotopy equivalence. It follows by McCord's theorem that f is a weak homotopy equivalence. On the other hand, since the spaces are minimal and not homeomorphic, f cannot be a homotopy equivalence.

In some sense, McCords theorem says that if a continuous map is locally a weak homotopy equivalence, then it is globally a weak homotopy equivalence. This theorem, as one might guess, plays an important role in the homotopy theory of finite spaces.

We will now study the order complex $\mathcal{K}(X)$ of an F-space X through the lens of weak homotopy equivalences. We start with a definition.

Definition 3.1.6. *Let X be an F-space. Define the **$\mathcal{K}$-McCord map** $\mu_X : |\mathcal{K}(X)| \to X$ to be the map $\mu_X(\alpha) = \min(support(\alpha))$.*

With this we may state the following theorem.

Theorem 3.1.7. *The $\mathcal{K}$-McCord map $\mu_X : |\mathcal{K}(X)| \to X$ is a weak homotopy equivalence for any F-space.*

Proof. We will show that $\mu_X^{-1}(U_x)$ is open and contractible for all $x \in X$. This will imply that μ_X is continuous and a weak homotopy equivalence. The latter follows from McCord's theorem because if $\mu_X^{-1}(U_x)$ is contractible, then $\mu_X|_{(U_x)} : \mu_X^{-1}(U_x) \to U_x$ is a weak homotopy equivalence because both $\mu_X^{-1}(U_x)$ and U_x are contractible. Let us start with continuity.

Pick an arbitrary point $x \in X$ and define $L = \mathcal{K}(X \setminus U_x)$, the full subcomplex of $\mathcal{K}(X)$ that is spanned by the vertices which are not in U_x. Since $|L|$ is closed, it suffices to show that

$$\mu_X^{-1}(U_x) = |\mathcal{K}(X)| \setminus |L|.$$

If $\alpha \in |\mathcal{K}(X)| \setminus |L|$ then $\alpha \notin |L|$. But then there exists a $y \in support(\alpha)$ such that $y \in U_x$. But by definition of the $\mathcal{K}$-McCord map, $\mu_X(\alpha) = \min(support(\alpha)) \leq y \leq x$, so $\mu_X(\alpha) \in U_x$. Conversely, if $\alpha \in \mu_X^{-1}(U_x)$, then $\min(support(\alpha)) \in U_x$ so by definition of $|L|$, we have that $\alpha \notin |L|$ so $\alpha \in |\mathcal{K}(X)| \setminus |L|$. Therefore $\mu_X^{-1}(U_x) = |\mathcal{K}(X)| \setminus |L|$.

Now we show that $\mu_X^{-1}(U_x)$ is contractible. Note that every element of U_x is related to x, which means that $\mathcal{K}(U_x)$ is a simplicial cone with apex x. By Corollary 2.1.7, $|\mathcal{K}(U_x)|$ is contractible. The proof is finished if we show that $|\mathcal{K}(U_x)|$ is homotopy equivalent to $|\mathcal{K}(X)| \setminus |L| = \mu_X^{-1}(U_x)$. We show that $|\mathcal{K}(U_x)|$ is a strong deformation retract of $|\mathcal{K}(X)| \setminus |L|$. Let $i : |\mathcal{K}(U_x)| \hookrightarrow |\mathcal{K}(X)| \setminus |L|$ be the inclusion. If $\alpha \in |\mathcal{K}(X)| \setminus |L|$, we can find two (unique) points $\beta \in |\mathcal{K}(U_x)|$ and $\gamma \in |L|$ such that $\alpha = t\beta + (1-t)\gamma$ for some $0 < t \leq 1$. Define $r : |\mathcal{K}(X)| \setminus |L| \to |\mathcal{K}(U_x)|$ by $r(\alpha) = \beta$. Since every complex is finite, the topology on every simplicial complex mentioned here is the metric topology. It follows that r is continuous and the straight line homotopy $H : (|\mathcal{K}(X)| \setminus |L|) \times I \to |\mathcal{K}(X)| \setminus |L|$ from $1_{|\mathcal{K}(X)| \setminus |L|}$ to $r \circ i$ is well defined and continuous. It follows that $\mu_X(U_x)$ is contractible. □

Note that if $f : X \to Y$ is a continuous map between finite T_0-spaces then for $\alpha \in |\mathcal{K}(X)|$, we have

$$(f \circ \mu_X)(\alpha) = f(\min(support(\alpha))) = \min(f(support(\alpha)))$$

$$= \min(support(|\mathcal{K}|(\alpha))) = (\mu_Y \circ |\mathcal{K}(f)|)(\alpha).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} |\mathcal{K}(X)| & \xrightarrow{|\mathcal{K}(f)|} & |\mathcal{K}(Y)| \\ \downarrow{\mu_X} & & \downarrow{\mu_Y} \\ X & \xrightarrow{f} & Y. \end{array} \tag{2}$$

We are actually interested in the following corollary of the theorem.

Corollary 3.1.8. *Let $f : X \to Y$ be a map between finite T_0-spaces. Then f is a weak homotopy equivalence if and only if $|\mathcal{K}(f)| : |\mathcal{K}(X)| \to |\mathcal{K}(Y)|$ is a homotopy equivalence.*

Proof. By Proposition 3.1.2, $|\mathcal{K}(f)|$ is a homotopy equivalence if and only if it is a weak homotopy equivalence. Since the McCord map μ_Y is a weak homotopy equivalence, by the 2-out-of-3 property, $|\mathcal{K}(f)|$ is a weak homotopy equivalence if and only if $\mu_Y \circ |\mathcal{K}(f)| = f \circ \mu_X$ is a weak homotopy equivalence. But we know that μ_X is a weak homotopy equivalence and so this is equivalent to saying that f is a weak homotopy equivalence. □

Corollary 3.1.9. *If X and Y are weak homotopic F-spaces, then $\chi(X) = \chi(Y)$.*

Proof. By Remark 2.2.6, it suffices to show that $\chi(|\mathcal{K}(X)|) = \chi(|\mathcal{K}(Y)|)$. Since $X \overset{we}{\simeq} Y$, by Corollary 3.1.8, $|\mathcal{K}(X)| \simeq |\mathcal{K}(Y)|$ and the result follows from Theorem 1.3.15. □

If K is a finite simplicial complex, it is not too difficult to show that $\mathcal{K}(\mathcal{X}(K))$ is the barycentric subdivision of K. Recall that for each K, there is a canonical homeomorphism $s_K : |K'| \to |K|$ called the linear map, which is given by $s_K(\sigma) = b(\sigma)$, where $b(\sigma)$ is the barycenter of σ.

Consider the $\mathcal{K}$-McCord map $\mu_{\mathcal{X}(K)} : |\mathcal{K}(\mathcal{X}(K))| = K' \to \mathcal{X}(K)$ for $\mathcal{X}(K)$. We define the $\mathcal{X}$-McCord map $\mu_K = \mu_{\mathcal{X}(K)} s_K^{-1} : |K| \to \mathcal{X}(K)$. Since s'_K is a homeomorphism and $\mu_{\mathcal{X}(K)}$ is a weak homotopy equivalence, it follows that μ_K is a weak homotopy equivalence.

Theorem 3.1.10. *For every finite simplicial complex K, the $\mathcal{X}$-McCord map μ_K is a weak homotopy equivalence.*

If $\varphi : K \to L$ is a simplicial map, we denote by φ' the induced map $\mathcal{K}(\mathcal{X}(\varphi)) : K' \to L'$.

Proposition 3.1.11. *If K and L are finite simplicial complexes and $\varphi : K \to L$ is a simplicial map, then the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} |K| & \xrightarrow{|\varphi|} & |L| \\ \downarrow{\mu_K} & & \downarrow{\mu_L} \\ \mathcal{X}(K) & \xrightarrow{\mathcal{X}(\varphi)} & \mathcal{X}(L). \end{array}$$

Proof. Note that there is a composition of squares

$$
\begin{array}{ccc}
|K| & \xrightarrow{\quad|\varphi|\quad} & |L| \\
\Big\downarrow{\scriptstyle s_K^{-1}} & & \Big\downarrow{\scriptstyle s_L^{-1}} \\
|\mathcal{K}(\mathcal{X}(K))| & \xrightarrow{|\varphi'|} & |\mathcal{K}(\mathcal{X}(L))| \\
\Big\downarrow{\scriptstyle \mu_{\mathcal{X}(K)}} & & \Big\downarrow{\scriptstyle \mu_{\mathcal{X}(L)}} \\
\mathcal{X}(K) & \xrightarrow{\quad\mathcal{X}(\varphi)\quad} & \mathcal{X}(L).
\end{array}
$$

We claim that $|\varphi|s_K \simeq s_L|\varphi'|$. Let $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a simplex of K', the barycentric subdivision of K. Note that $\sigma_1 \subsetneq \sigma_2 \subsetneq \cdots \subsetneq \sigma_n$ by the definition of K'. Let $\alpha \in \overline{\sigma}$. Then $s_K(\alpha) \in \overline{\sigma}_n$ and so $|\varphi|s_K(\alpha) \in \overline{\varphi(\sigma_n)} \subseteq |L|$. But on the other hand we have $|\varphi'|(\alpha) \in \overline{\{\varphi(\sigma_1), \varphi(\sigma_2), \ldots, \varphi(\sigma_n)\}}$ and so $s_L|\varphi'|(\alpha) \in \overline{\varphi(\sigma_n)}$. It follows that the straight line homotopy $H : |K'| \times I \to |L|$ from $|\varphi|s_K(\alpha)$ to $s_L|\varphi'|(\alpha)$ is well defined and continuous, so indeed, $|\varphi|s_K \simeq s_L|\varphi'|$. Now the rest follows from diagram (2), since then

$$\mu_L|\varphi| = \mu_{\mathcal{X}(L)}s_L^{-1}|\varphi| \simeq \mu_{\mathcal{X}(L)}|\varphi'|s_K^{-1} = \mathcal{X}(\varphi)\mu_{\mathcal{X}(K)}s_K^{-1} = \mathcal{X}(\varphi)\mu_K.$$

□

A map homotopic to a weak homotopy equivalence is also a weak homotopy equivalence because they will induce the same maps on the homotopy groups. With this in mind and the 2-out-of-3 property we deduce the following corollary.

Corollary 3.1.12. *Let K and L be finite simplicial complexes and $\varphi : K \to L$ a simplicial map. Then $|\varphi| : |K| \to |L|$ is a homotopy equivalence if and only if $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$ is a weak homotopy equivalence.*

Corollary 3.1.13.

(1) Let X and Y be finite T_0-spaces. Then $X \overset{we}{\simeq} Y$ if and only if $|\mathcal{K}(X)| \simeq |\mathcal{K}(Y)|$.

(2) Let K and L be finite simplicial complexes. Then, $|K| \simeq |L|$ if and only if $\mathcal{X}(K) \overset{we}{\simeq} \mathcal{X}(L)$.

3.2 Simple Homotopy Types

We start by generalizing the notion of a beat point which will then generalize the idea of a collapse.

Definition 3.2.1. *A point x in an F-space X is called a* ***down weak point*** *if $\hat{U}_x$ is contractible and it is called an* ***up weak point*** *if $\hat{F}_x$ is contractible. In both cases, we say that x is a* ***weak point****.*

Definition 3.2.2. *Let X be an F-space and $x \in X$ be a point. The* ***star*** *of x is the set $C_x = \{y \in X \mid x \leq y \text{ or } y \leq x\}$. Define also the* ***link*** *of x to be the set $\hat{C}_x = C_x \setminus \{x\}$.*

Note that $C_x = U_x \cup F_x$ and that $\hat{C}_x = \hat{U}_X \circledast \hat{F}_x$.

Remark 3.2.3. *Note that by Proposition 1.3.7, $x \in X$ is a weak point if and only if $\hat{C}_x$ is contractible.*

Proposition 3.2.4. *Let x be a weak point of an F-space. Then the inclusion $i : X \setminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence.*

Proof. Suppose that x is a down weak point. The other case is similar where one considers X^{op} instead of X and notes that $\mathcal{K}(X^{op}) = \mathcal{K}(X)$. We will show that $i^{-1}(U_y) = U_y \setminus \{x\}$ is contractible for all $y \in X$ and then the result will follow from McCord's theorem because the minimal basis is always a basis like cover. If $y = x$ then it is contractible by the definition of a down weak point. So suppose $y \neq x$. But then $U_y \setminus \{x\}$ has a maximum, so $i|_{i^{-1}(U_y)} : i^{-1}(U_y) \to U_y$ is a weak homotopy equivalence. By McCord's theorem, i is a weak homotopy equivalence. □

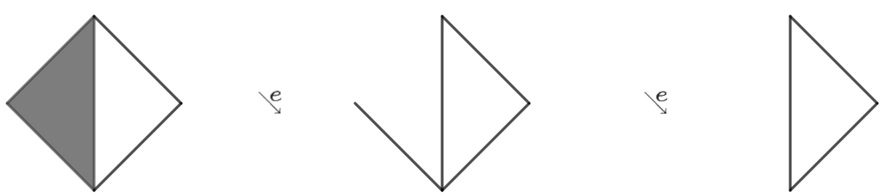
Definition 3.2.5. *Let X be an F-space and $Y \subsetneq X$ be a proper subspace. We say that X* ***collapses to*** *Y* ***by an elementary collapse*** *(or that Y* ***expands to*** *X* ***by an elementary expansion****) if Y is obtained from X by removing a weak point. In this case we write $X \searrow^e Y$ or $Y \nearrow^e X$. If there exists a sequence $X = X_0, X_1, \ldots, X_n = Y$ of F-spaces such that $X_i \searrow^e X_{i+1}$ for every $1 \leq i < n$, we say that X* ***collapses*** *to Y (or that Y* ***expands*** *to X) and write $X \searrow Y$ or $Y \nearrow X$. Two F-spaces X and Y are* ***simple homotopy equivalent*** *if there is a sequence $X = X_0, X_1, \ldots, X_n = Y$ of F-spaces such that $X_i \searrow X_{i+1}$ or $X_i \nearrow X_{i+1}$ for every $1 \leq i < n$. We denote this by $X \diagup\!\!\searrow Y$. We say that an F-space is* ***collapsible*** *if it collapses to a point.*

We can also study simple homotopy types in the setting of finite simplicial complexes.

If K is a simplicial complex and $\sigma \in K$ is a simplex, we define the **boundary** of σ, denoted by $\dot{\sigma}$, to be the union of all proper maximal faces of σ and we denote by σ^c the **complement** of σ which is the subcomplex of K of simplices that do not contain σ.

Definition 3.2.6. *Let K be a finite simplicial complex and L be a subcomplex. We say that there is an* ***elementary simplicial collapse*** *from K to L if there is a simplex σ of K and a vertex v of K that is not in σ such that $K = L \cup v\sigma$ and $L \cap v\sigma = v\dot{\sigma}$. In this case, we write $K \searrow^e L$. Furthermore, we say that K* ***(simplicially) collapses*** *to L if there is a sequence $K = K_1, K_2, \ldots, K_n = L$ of finite simplicial complexes such that $K_i \searrow^e K_{i+1}$ for all i. We denote this by $K \searrow L$. In a similar way we define* ***expansions*** *and write $L \nearrow K$ respectively. A simplicial complex is* ***collapsible*** *if it collapses to one of its vertices.*

In the following diagram, a complex is collapsing to a 2-simplex



Two simplicial complexes K and L are **simple homotopy equivalent** or have the same **simple homotopy type** if there is a sequence $K = K_1, K_2, \ldots, K_n = L$ such that $K_i \searrow K_{i+1}$ or $K_i \nearrow K_{i+1}$ for all $1 \leq i < n$. We denote this also by $K \diagup\!\!\searrow L$.

The proofs of the following two lemmas can be found in [1] section 4.1.

Lemma 3.2.7. *Let aK be a simplicial cone of a finite simplicial complex K. Then K is collapsible if and only if $aK \searrow K$.*

The implication, if K is collapsible then $aK \searrow K$, follows from a more general result due to Whitehead, theorem 4 in [11], namely that if $K \searrow L$, then $\sigma K \searrow \sigma L$, where σ is any stellar subdivision of K.

Lemma 3.2.8. *Let K be a finite simplicial complex and L be a subcomplex of K such that K collapses to L. If M is another finite simplicial complex then $K * M \searrow L * M$.*

The converse of Lemma 3.2.8 is an open problem.

Proposition 3.2.9. *If K and L are subcomplexes of a finite simplicial complex then $K \cup L \searrow K$ if and only if $L \searrow K \cap L$.*

Proof. Suppose there is an elementary collapse from $K \cup L$ to a subcomplex W that contains K. Let $L' = W \cap L$. Then $L' \subseteq L$ and since $W \supseteq K$, we also have $L' \supseteq K \cap L$. This shows that an elementary collapse of $K \cup L$ to a subcomplex containing K, has the form of an elementary collapse to a complex $K \cup L'$ where L' is a subcomplex of L containing $K \cap L$. Now we show that such an elementary collapse is equivalent to (in the sense that it determines and is determined by) an elementary collapse of L to L'.

Now $K \cup L \searrow^e K \cup L'$ is, by definition, equivalent to choosing a simplex $\sigma \in (K \cup L) \setminus (K \cup L')$ and a vertex $v \notin \sigma$ such that $K \cup L = K \cup L' \cup v\sigma$ and $(K \cup L') \cap v\sigma = v\dot{\sigma}$. Therefore $v\sigma \nsubseteq K$ which means $v\sigma \subseteq L$. It follows that $v, \dot{\sigma} \in L$ and so $v, \dot{\sigma} \in K \cup L'$. But since $L \cap (K \cup L') = L'$, we have $v, \dot{\sigma} \in L'$.

This shows that an elementary collapse from $K \cup L$ to $K \cup L'$ with $L' \supseteq K \cap L$, is the same thing as an elementary collapse from L to L'. This extends to collapses, so we have showed that $K \cup L \searrow K$ if and only if $L \searrow K \cap L$. □

If v is a vertex in a simplicial complex K, then the **star** of v in K, denoted $st(v)$, is the subcomplex of all simplicies of K such that $\sigma \cup \{v\}$ is a simplex of K. The **link** of a vertex v, denoted $lk(v)$, is the subcomplex of $st(v)$ such that $v \notin \sigma$ for all $\sigma \in st(v)$. More generally, we define the **star** of a simplex $\sigma \in K$, denoted $st(\sigma)$, to be the subcomplex of K whose simplicies are the simplicies $\tau \in K$ such that $\sigma \cup \tau \in K$, and we define the **link** of a simplex σ, denoted $lk(\sigma)$, as the subcomplex of $st(\sigma)$ whose simplicies are disjoint from σ. With this we may state the following definition.

Lemma 3.2.10. *Let K be a finite simplicial complex and let v be a vertex of K. Then the link $lk(v)$ is collapsible if and only if $K \searrow K \setminus \{v\}$.*

Proof. Note that $st(v) = vlk(v)$ and $lk(v) = st(v) \cap (K \setminus \{v\})$. By Lemma 3.2.7, $lk(v)$ is collapsible if and only if $vlk(v) = st(v) \searrow lk(v) = st(v) \cap (K \setminus \{v\})$. By Proposition 3.2.9, This is equivalent to saying that $st(v) \cup (K \setminus \{v\}) = K \searrow K \setminus \{v\}$. □

We now arrive at the main theorem for this section. We will prove the first part as it is more relevant to our work. The complete proof can be found in [1] Theorem 4.2.11.

Theorem 3.2.11.

(1) *Let X and Y be F-spaces. Then X and Y are simple homotopy equivalent if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. Moreover, if $X \searrow Y$ then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.*

(2) *Let K and L be finite simplicial complexes. Then K and L are simple homotopy equivalent if and only if $\mathcal{X}(K)$ and $\mathcal{L}(L)$ have the same simple homotopy type. Moreover, if $K \searrow L$ then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.*

Proof. Let $x \in X$ be a beat point. Then there exists $x' \in X$ and two subspaces Y and Z of X such that $\hat{C}_x = Y \circledast \{x'\} \circledast Z$. Note that $lk(x) = x'\mathcal{K}(Y \circledast Z)$ is a cone, in particular, it is contractible by Lemma 3.2.7 (since clearly $x'\mathcal{K}(Y \circledast Z) \searrow \mathcal{K}(Y \circledast Z)$. Now by Lemma 3.2.10, $\mathcal{K}(X) \searrow \mathcal{K}(X \setminus \{x\})$, since points in X are precisely the vertices of the order complex. Therefore if X is contractible, $\mathcal{K}(X)$ is collapsible since it is homotopy invariant.

Now let $x \in X$ be a weak point. Then $\hat{C}_x$ is contractible by Remark 3.2.3. Therefore $lk(x) = \mathcal{K}(\hat{C}_x)$ is collapsible. By Lemma 3.2.10, $\mathcal{K}(X) \searrow \mathcal{K}(X \setminus \{x\})$. So $X \searrow Y$ implies $\mathcal{K}(X) \searrow \mathcal{K}(Y)$. In particular $X \diagup\!\!\searrow Y$ implies that $\mathcal{K}(X) \diagup\!\!\searrow \mathcal{K}(Y)$. □

3.3 Strong Homotopy Types

In this section we study another type of collapse.

If K is a simplicial complex and $v \in K$ is a vertex, the **deletion of** v, denoted $K \setminus v$ is the full subcomplex of K spanned by the vertices different from v.

Definition 3.3.1. *Let K be a finite simplicial complex and $v \in K$ be a vertex. We say that there is an* ***elementary strong collapse*** *from K to $K \setminus v$ if the link $lk(v)$ is a simplicial cone. If w is an apex of $lk(v)$, we say that v is* ***dominated*** *by w (or just* ***dominated****) and write $K \searrow\!\!\searrow^e K \setminus v$. If there exists a sequence of elementary strong collapses starting at K and ending at a subcomplex L we say that there is a* ***strong collapse*** *from K to L. In this case, we write $K \searrow\!\!\searrow L$. We define a* ***strong expansion*** *as the inverse of a strong collapse. We say that two finite simplicial complexes K and L have the same* ***strong homotopy type*** *if there is a sequence of strong collapses and strong expansions that starts at K and ends at L.*

Definition 3.3.2. *A simplicial map $\varphi : K \to L$ is a* ***strong equivalence*** *if there exists a map $\psi : L \to K$ such that $\psi\varphi \sim 1_K$ and $\varphi\psi \sim 1_L$. If $\varphi : K \to L$ is a strong equivalence, we write $K \sim L$.*

It is easy to verify that $\sim$ is an equivalence relation.

Definition 3.3.3. *If K is a finite simplicial complex and it has no vertices that are dominated, we say that K is a* ***minimal complex****.*

Proposition 3.3.4. *Let K be a minimal complex and $\varphi : K \to K$ be a simplicial map in the same contiguity class as the identity 1_K. Then $\varphi = 1_K$.*

Proof. If φ is in the same contiguity class as 1_K then there is a sequence of simplicial maps $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_n = 1_K$ that are pairwise contiguous. For this reason we my assume that φ and 1_K are contiguous because the general case follows inductively.

Let $v \in K$ and let $\sigma \in K$ be a maximal simplex with $v \in \sigma$. Since $\varphi \sim 1_K$, $\varphi(\sigma) \cup \sigma$ is a simplex. But since σ is maximal, it follows that $\varphi(v) \in \varphi(\sigma) \cup \sigma = \sigma$. It follows that every maximal simplex that contains v also contains $\varphi(v)$. Since K is minimal, $\varphi(v) = v$. Since v is arbitrary, we conclude that $\varphi = 1_K$. □

An isomorphism between two simplicial complexes K and L is a bijective simplicial map $\psi : K \to L$ such that the inverse ψ^{-1} is also a simplicial map.

The following corollary is immediate.

Corollary 3.3.5. *Any strong equivalence between minimal complexes is an isomorphism.*

Proposition 3.3.6. *Let K be a finite simplicial complex and $v \in K$ a dominated vertex. Then the inclusion $i : K \setminus v \hookrightarrow K$ is a strong equivalence. In particular, if K and L have the same strong homotopy type, then $K \sim L$.*

Proof. Let w be the point that dominates v. Define the vertex map $r : K \to K \setminus v$ by declaring it to be the identity on $K \setminus v$ and $r(v) = w$. Let $\sigma \in K$ be a simplex such that $v \in \sigma$ and let τ be a maximal simplex such that $\sigma \subsetneq \tau$. It follows that $w \in \tau$ and $r(\sigma) = (\sigma \cup \{w\}) \setminus \{v\} \subseteq \tau$ is a simplex of $K \setminus v$. This show that r is a simplicial map. On the other hand $ir(\sigma) \cup \sigma = \sigma \cup \{w\} \subseteq \tau$ shows that $ir(\sigma) \cup \sigma$ is a simplex of K and hence $ir \sim 1_K$. Therefore i is a strong equivalence. □

Definition 3.3.7. *Let K be a finite simplicial complex. A* ***core*** *of K is a minimal subcomplex $K_0 \subset K$ such that $K \searrow\searrow K_0$.*

Theorem 3.3.8. *Let K be a simplicial complex. Then K has a core and it is unique up to isomorphism. Furthermore, two finite simplicial complexes have the same strong homotopy type if and only if their cores are isomorphic.*

Proof. Note that a core of K is obtained by removing dominated points one at a time. This proves existence. If K_1 and K_2 are both cores of K, they have the same strong homotopy type. By Proposition 3.3.6, $K_1 \sim K_2$. But cores are always minimal so it follows by Corollary 3.3.5 that K_1 and K_2 are isomorphic. Therefore a core is unique up to isomorphism.

To prove the second part, let K and L be finite simplicial complexes. Suppose that K and L have the same strong homotopy type. Then so do their cores and by the reasoning above, their cores are isomorphic.

Conversely, if K_0 and L_0 are isomorphic cores of K and L respectively, then they have the same strong homotopy type because isomorphic complexes have the same strong homotopy type. But then so do K and L. □

Corollary 3.3.9. *Two finite simplicial complexes K and L have the same strong homotopy type if and only if $K \sim L$.*

Proof. If $K \sim L$ and K_0 and L_0 denote their cores then K_0 and L_0 are isomorphic. □

Proposition 3.3.10. *A strong equivalence is also a simple homotopy equivalence.*

Proof. Let $\varphi : K \to L$ be a strong equivalence and let K_0 be a core of K and L_0 be a core of L. It follows that the inclusions $i_K : K_0 \hookrightarrow K$ and $i_L : L_0 \hookrightarrow L$ are strong equivalences. Let $r : L \to L_0$ be a homotopy inverse of i_L. Since cores are minimal complexes, it follows that the map $r\varphi i_K : K_0 \to L_0$ is an isomorphism. But then the maps $|i_K|$, $|r|$ and $|r\varphi i_K|$ are simple homotopy equivalences and therefore $|\varphi|$ is a simple homotopy equivalence. □

Definition 3.3.11. *A complex is* ***strong collapsible*** *if it strong collapses to a point.*

The proof of the following proposition can be found in [1] page 77.

Proposition 3.3.12. *If K and L are two finite simplicial complexes, then the join $K * L$ is strong collapsible if and only if K or L is strong collapsible. (cf. Proposition 1.3.7)*

The analogous result of Proposition 3.3.12 for collapsibility (rather than strong collapsibility) does not hold. It is not known whether $K * L$ is collapsible only if K or L is collapsible.

Finally, the following is the main result of this section. (cf. Theorem 3.2.11)

Theorem 3.3.13.

(1) If two F-spaces are homotopy equivalent, then their order complexes have the same strong homotopy type.

(2) If two finite simplicial complexes have the same strong homotopy type, then their face posets are homotopy equivalent.

Proof. Let $f : X \to Y$ be a homotopy equivalence between two F-spaces and let $g.Y \to X$ denote its homotopy inverse. Since $gf \sim 1_X$ and $fg \simeq 1_Y$, by Proposition 2.3.2, $\mathcal{K}(g)\mathcal{K}(f) \sim 1_{\mathcal{K}(X)}$ and $\mathcal{K}(f)\mathcal{K}(g) \sim 1_{\mathcal{K}(Y)}$. Therefore, $\mathcal{K}(X) \sim \mathcal{K}(Y)$.

To prove the second part, let K and L be two complexes with the same strong homotopy type. There exist simplicial maps $\varphi : K \to L$ and $\psi : L \to K$ such that $\psi\varphi \sim 1_K$ and $\varphi\psi \sim 1_L$. But then, by Proposition 2.3.3, $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(Y)$ is a homotopy equivalence with $\mathcal{X}(\psi)$ as its inverse. $\square$

The last result that we will need before we go other to the equivariant homotopy theory in the setting of F-spaces and simplicial complexes is the following theorem. The proof can be found in [1] page 79.

Theorem 3.3.14. *Let K be a finite simplicial complex. Then K is strong collapsible if and only if the barycentric subdivision K' is strong collapsible.*

What we really need is the following corollary.

Corollary 3.3.15. *An F-space X is contractible if and only if $\mathcal{K}(X)$ is strong collapsible.*

3.4 Summary

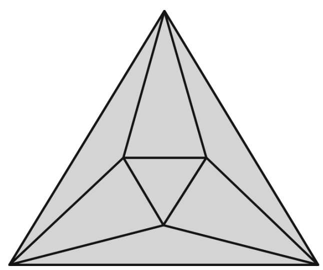
We finish this section with a summary of the important results we have obtained. The following two diagrams illustrate the relations we have found. The first diagram is for F-spaces and their order complexes and the second diagram is for simplicial complexes and their face posets.

$$\begin{array}{ccccccc}
X \simeq Y & \Longrightarrow & X \nearrow\searrow Y & \Longrightarrow & X \overset{we}{\simeq} Y & & \\
\Downarrow & & \Updownarrow & & \Updownarrow & & \\
\mathcal{K}(X) \sim \mathcal{K}(Y) & \Longrightarrow & \mathcal{K}(X) \nearrow\searrow \mathcal{K}(Y) & \Longrightarrow & |\mathcal{K}(X)| \overset{we}{\simeq} |\mathcal{K}(Y)| & \Longleftrightarrow & |\mathcal{K}(X)| \simeq |\mathcal{K}(Y)|
\end{array}$$

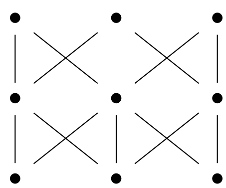
$$\begin{array}{ccccccc}
\mathcal{X}(K) \simeq \mathcal{X}(L) & \Longrightarrow & \mathcal{X}(K) \nearrow\searrow \mathcal{X}(L) & \Longrightarrow & \mathcal{X}(K) \overset{we}{\simeq} \mathcal{X}(Y) & & \\
 & & \Updownarrow & & \Updownarrow & & \\
 & & K \nearrow\searrow L & \Longrightarrow & |K| \overset{we}{\simeq} |L| & \Longleftrightarrow & |K| \simeq |L|
\end{array}$$

We now exhibit some examples to show that some of the implications in the diagrams are strict.

Example 3.4.1. *The simplicial complex below is a homogeneous* 2*-simplex (meaning* 2 *is the common dimension of its facets) is collapsible. But is is also minimal, which means in particular that it cannot have the strong homotopy type of a point, so it is not strong collapsible.*



Example 3.4.2. *The following space is collapsible but not contractible.*



It is collapsible because $\hat{U}_x$ *or* $\hat{F}_x$ *is contractible for any* x *but except for possibly one point, but since the space is minimal (it has no beat points), it is not contractible.*

The analogous example of Example 3.4.2 for simplicial complexes is the-house-with-two-rooms found by Bing in [2] page 170. It is contractible but not collapsible.

4 Equivariant Homotopies of F-spaces

When we let a finite group act on an F-space, we can not only uncover some properties of the underlying space but also some properties of the group. In this section we will let a finite group G act on an F-space.

4.1 G-F-spaces and Their Properties

Let G be a group. A topological space X is said to be a **G-space** if G acts on X and the map $m_g : X \to X$ given by $x \mapsto gx$ is continuous for every $g \in G$. Given a subspace A of a G-space, we say that A is **G-invariant** if $gx \in A$ for all $x \in A$ and all $g \in G$. A map $f : X \to Y$ between G-spaces X and Y is said to be a **G-map** if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. A **G-homotopy** is a homotopy $H : X \times I \to X$ such that $H(gx, t) = gH(x, t)$ for all $g \in G$, $x \in X$ and $t \in I = [0, 1]$. A G-invariant subspace $A \subseteq X$ is an **equivariant strong deformation retract** if there is a retraction $r : X \to A$ that is a G-map such that the homotopy $H : i \circ r \simeq 1_X$ is a G-homotopy which is stationary on A. We shall denote by Gx the orbit of a point x and by X^G the set of points in X fixed by the action. A core that is G-invariant is called a **G-core**. An F-space that is also a G-space will be called a G-F-space. The following lemma by Stong shows that every finite T_0-G-space has a G-core.

Lemma 4.1.1. *Let G be a group and X be a G-F-space. Then there exists a G-core of X which is an equivariant strong deformation retract of X.*

Proof. If X is minimal the statement is trivially satisfied, so suppose that X is not minimal. Then there exists a beat point x. Suppose that it is a down beat point (the proof for an up beat point is analogous), this means that it covers a unique point $y \in X$. We note that the orbits Gx and Gy are disjoint. Indeed, if $gx = hy$ for some $g, h \in G$, then $gx = hy < hx$ so $gx < hx$, but this contradicts Corollary 1.3.6 above because the maps $m_g, m_h : X \to X$ are comparable automorphism. It follows that the retraction $r : X \to X \setminus Gx$ given by $r(gx) = gy$ is a well defined continuous G-map. Let $\alpha : I \to X^X$ be the path defined by $\alpha(t) = i \circ r$, where $i : X \setminus Gx \hookrightarrow X$ is the inclusion, if $0 \leq t < 1$ and $\alpha(1) = 1_X$. The homotopy $X \times I \to X$ corresponding to α is a G-homotopy between $i \circ r$ and 1_X relative to $X \setminus Gx$. It follows that $X \setminus Gx$ is an equivariant strong deformation retract. Now we pick the next beat point and do the same process until we arrive at a core. □

Proposition 4.1.2. *If X is a contractible G-F-space then X^G is nonempty.*

Proof. By Lemma 4.1.1, there exists a G-invariant core. By Theorem 3.3.13, the order complex $\mathcal{K}(X)$ has the same strong homotopy type as a point. But if $\mathcal{K}(X)$ has the same strong homotopy type as a point, by Theorem 3.3.8, its core is unique and isomorphic to a point. But then the G-core of X in question will be homotopy equivalent to the core of $|\mathcal{K}(X)|$, that is, it is nonempty. □

To illustrate the utility of the proposition, we give an alternative proof of a result from group theory.

Example 4.1.3. *Let G be a finite group and let H be a proper subgroup such that if S is a nontrivial subgroup then $S \cap H$ is also nontrivial. Then G is not simple.*

Consider the poset $S(G)$ of nontrivial proper subgroups of G. Note that $S(G)$ is a G-space as G acts on $S(G)$ by conjugation. Let $c_H : S(G) \to S(G)$ be the constant map H and define $f : S(G) \to S(G)$ by $S(S) = S \cap H$. By hypothesis, f is well defined. Furthermore it is clearly order preserving and hence continuous. Now note that we have a fence $1_{S(G)} \geq f \leq c_H$ which means $S(G)$ is contractible. By Proposition 4.1.2, $S(G)$ has a point fixed by the action of G. But since G acted on $S(G)$ conjugation, it follows that $S(G)$ contains a proper and nontrivial normal subgroup of G and hence G cannot be simple.

We now extend the notion of a strong collapse to G-spaces. If X is a G-F-space and $x \in X$ is a beat point, we say that there is an **elementary strong G-collapse** from X to $X \setminus Gx$. Note that if x is a beat (weak) point, then gx is a beat (weak) point for all $g \in G$. It is easily seen that elementary strong G-collapses are strong collapses. A **strong G-collapse** is a sequence of elementary strong G-collapses. If X strong G-collapses to Y we write $X \searrow\!\!\searrow^G Y$. The notion of a strong **G-expansion** is defined dually.

Proposition 4.1.4. *Let X be a G-F-space and let Y be a G-invariant subspace. The following are equivalent:*

(1) $X \searrow\!\!\searrow^G Y$.

(2) Y is an equivariant strong deformation retract of X.

(3) Y is a strong deformation retract of X.

Proof. Suppose we have a strong G-collapse from X to Y. By following the steps in the proof of Lemma 4.1.1 we get that Y is an equivariant strong deformation retract of X.

If Y is an equivariant strong deformation retract, then it is also a strong deformation retract.

Suppose now that Y is a strong deformation retract and let $x \in X \setminus Y$ be a beat point. It follows that $X \searrow\!\!\searrow X \setminus Gx$, so $X \setminus Gx$ is a strong deformation retract of X. Therefore the result follows inductively. ☐

Now we generalize collapses to G-collapses.

Definition 4.1.5. *Let X be a G-F-space. If $x \in X$ is weak point, we say that there is an **elementary G-collapse** from X to $X \setminus Gx$ and write $X \searrow^{Ge} X \setminus Gx$. A **$G$-collapse** is a sequence of G collapses. X is **G-collapsible** if it G-collapses to a point. **G-expansions** are defined dually.*

We say that two G-F-spaces X and Y have the same **equivariant simple homotopy type** if there exists a sequence $X = X_1, X_2, \ldots, X_n = Y$ such that $X_i \searrow^G X_{i+1}$ or $X_i \nearrow^G X_{i+1}$ for $1 \leq i < n$. In this case, we write $X \diagup\!\!\searrow^G Y$.

Unpacking the definition, it is easy to see that strong G-collapses are G-collapses and G-collapses are collapses.

4.2 Group Actions on Simplicial Complexes

A **simplicial G-complex** (or just a **G-complex**) is a simplicial complex with an action of G on the set of vertices V_K such that if $\{v_1, v_2, \ldots, v_n\}$ is a simplex, then $\{gv_1, gv_2, \ldots, gv_n\}$ is also a simplex. In other words, the map $G \times K \to K$ induced by the action is simplicial.

As we will see below, the notion of G-collapses extends to simplicial complexes. Let K be a finite G-complex and $\sigma \in K$ a free face of $\tau \in K$. Then $g\sigma$ is a free face of $g\tau$ for every $g \in G$. Note that the isotropy group[3] of σ, denoted G_σ is included in G_τ, the isotropy group of τ. To see why, suppose $g \in G_\sigma$, that is $g\sigma = \sigma$. Then $g\sigma = \sigma \subsetneq g\tau$. But since σ is a free face, it follows that $g\tau = \tau$ meaning $g \in G_\tau$. The other inclusion does not hold in general (see [1] Example 8.3.4 on page 113), which motivates the following definition.

Definition 4.2.1. *Let K be a finite simplicial G-complex and let $\sigma \in K$ be a free face of $\tau \in K$. Let $L = K \setminus \bigcup_{g\in G}\{g\sigma, g\tau\}$. Note that L is G-invariant. We say that there is an* ***elementary G-collapse*** *$K \searrow^{Ge} L$ from K to L or that $\sigma \subsetneq \tau$ is a* ***G-collapsible pair*** *if $G_\sigma = G_\tau$ (or equivalently, if $G_\tau \subseteq G_\sigma$). If there is a sequence of elementary G-collapses initiating at K and terminating at L we say that there is a* ***G-collapse*** *from K to L and write $K \searrow^G L$. As usual, a* ***G-expansion*** *is defined dually. A simplicial G-complex is* ***G-collapsible*** *if it G-collapses to a vertex.*

We can also define equivariant simple homotopy types in the setting of simplicial complexes. If there is a sequence $K = K_1, K_2, \ldots, K_n = L$ such that $K_i \searrow^G K_{i+1}$ or $K_i \nearrow^G K_{i+1}$ then we say that K and L have the same **equivariant simple homotopy type**. We denote this also by $K \diagup\!\!\searrow^G L$

We now state the following proposition.

Proposition 4.2.2. *Let K be a finite simplicial G-complex and let $\sigma \subsetneq \tau$ be a collapsible pair. Then the following are equivalent:*

(1) $\sigma \subsetneq \tau$ is a G-collapsible pair.

(2) $K \searrow L$, where $L = K \setminus \bigcup_{g\in G}\{g\sigma, g\tau\}$.

Proof. Suppose that $\sigma \subsetneq \tau$ is a collapsible pair. Then for each g, the pair $g\sigma \subsetneq g\tau$ can be collapsed too. By iterating on all elements of g, we get that $K \searrow L$, where L is as in the hypothesis. On the other hand, if $K \searrow L$ and σ is an n-simplex, then the number of n-simplices of the set $\cup_{g\in G}\{g\sigma, g\tau\}$ is equal to the number of $(n+1)$-simplices. In other words, the sets $G \cdot \sigma = \{g\sigma\}_{g\in G}$ and $G \cdot \tau = \{g\tau\}_{g\in G}$ have the same cardinality. It follows that the cardinalities of the isotropy groups G_σ and G_τ are equal since

$$\#G_\sigma = \#G/\#G\cdot\sigma = \#G/\#G\cdot\tau = \#G_\tau,$$

by the orbit-stabilizer theorem. But we know that $G_\sigma \subseteq G_\tau$, and thus $G_\sigma = G_\tau$. □

Lemma 4.2.3. *Let G be a group acting on a finite simplicial cone aK such that the action fixes the vertex a. Then $aK \searrow^G a$.*

Proof. Let $\sigma \in K$ be a maximal simplex. Then $\sigma \subsetneq a\sigma$ is a G-collapsible pair. By the preceding proposition, $aK \searrow^G aK \setminus \bigcup_{g\in G}\{g\sigma, ga\sigma\} = a(K \setminus \bigcup_{g\in G}\{g\sigma\})$. By induction on the maximal simplices, we arrive at our result. □

[3] Strictly speaking, we defined the action on the set of vertices but it is easily extended to an action on the simplices by "linearity", that is, acting on a simplex can be seen as acting on all the vertices which the simplex contains.

If X is a G-F-space, there is a natural induced action of G on $\mathcal{K}(X)$. There is a natural isomorphism $\mathcal{K}(G \times X) = G \times \mathcal{K}(X)$ if G is considered as a discrete topological group on the left and as a discrete simplicial complex on the right. If $\theta : G \times X \to X$ is the action of G on X, then we have an induced action $\mathcal{K}(\theta) : G \times \mathcal{K}(X) = \mathcal{K}(G \times X) \to \mathcal{K}(X)$. We will always assume that the action on $\mathcal{K}(X)$ is the induced one unless we state otherwise.

The proof of the following lemma can be found in [1] page 115.

Lemma 4.2.4. *Let X be a G-F-space and $x \in X$. The stabilizer of x, G_x, acts on $\hat{C}_x = C_x \setminus \{x\}$ and then on $\mathcal{K}(\hat{C}_x)$. If $\mathcal{K}(\hat{C}_x)$ is G_x-collapsible, then $\mathcal{K}(X) \searrow^G \mathcal{K}(X \setminus Gx)$.*

Now we state an important theorem.

Theorem 4.2.5.

(1) Let X be a G-F-space and $Y \subseteq X$ a G-invariant subspace. If $X \searrow^G Y$, then $\mathcal{K}(X) \searrow^G \mathcal{K}(Y)$.

(2) Let K be a finite G-complex and $L \subseteq K$ a G-invariant subcomplex. If $K \searrow^G L$, then $\mathcal{X}(K) \searrow^G \mathcal{X}(L)$.

Proof. Suppose $x \in X$ is a down beat point (what follows will also apply for an up beat point). Then there exists a unique $y \in X$ such that x covers y. If $z \in C_x$, then either $z \in U_x$ or $z \in F_x$. If $z \in U_x$ then $z \leq x$ and since x covers y, $z \leq y$ and in particular, $z \in C_y$. If $z \in F_x$, then by transitivity $z \in F_y$. So $C_x \subseteq C_y$. Now suppose that $g \in G_x$, then $gx = x$ and so $gy \leq gx = x$, by Lemma 1.3.5, $gy = y$. Since $\hat{C}_x$ is a union of a down-set and an up-set, it follows that we may write $\mathcal{K}(\hat{C}_x) = y\mathcal{K}(\hat{C}_x \setminus \{y\})$. Note that the stabilizer G_x of x acts on $\hat{C}_x$ which induces an action on $\mathcal{K}(\hat{C}_x)$. But $G_x \subseteq G_y$ so G_x fixes y. By Lemma 4.2.3, $\mathcal{K}(\hat{C}_x) = y\mathcal{K}(\hat{C}_x \setminus \{y\}) \searrow^{G_x} y$. But this means that $\mathcal{K}(\hat{C}_x)$ is G_x collapsible so by Lemma 4.2.4 $\mathcal{K}(X) \searrow^G \mathcal{K}(X \setminus Gx)$. It follows that if X is contractible, then $\mathcal{K}(X)$ is G-collapsible.

Now suppose that $x \in X$ is a weak point. It follows that the star C_x is contractible and so $\mathcal{K}(C_x)$ is G-collapsible. By Lemma 4.2.4 we obtain that $\mathcal{K}(X) \searrow^G \mathcal{K}(X \setminus Gx)$.

For the second part, suppose that K elementary G-collapses to L. Let $\sigma \subsetneq \tau$ be the G-collapsible pair such that $L = K \setminus \{g\sigma, g\tau\}_{g \in G}$. Then $\sigma \in \mathcal{X}(K)$ is an up beat point covered by τ and therefore $\mathcal{X}(K) \searrow^{Ge} \mathcal{X}(K) \setminus \{g\sigma\}_{g \in G}$. Note that $\tau \in \mathcal{X}(K) \setminus \{g\sigma\}_{g \in G}$ is a down weak point since $\overline{\tau} \setminus \{\sigma, \tau\}$ is a simplicial cone and then by Proposition 2.2.5, $\hat{U}_\tau^{\mathcal{X}(K) \setminus \{g\sigma\}_{g \in G}} = \hat{U}_\tau^{\mathcal{X}(K) \setminus \{\sigma\}} = \mathcal{X}(K)(\{\overline{\tau} \setminus \{\sigma, \tau\}\})$ is contractible. Thus $\mathcal{X}(K) \setminus \{g\sigma\}_{g \in G} \searrow^{Ge} \mathcal{X}(K) \setminus \{g\sigma, g\tau\}_{g \in G} = \mathcal{L}$ and $\mathcal{X}(K) \searrow^G \mathcal{X}(L)$, which completes the proof. □

Corollary 4.2.6. *Let X and Y be G-F-spaces. X and Y have the same equivariant simple homotopy type if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same equivariant simple homotopy types.*

Given a map $f : X \to Y$ between F-spaces, define the **non-Hausdorff mapping cylinder** $B(f)$ to be the F-space whose underlying set is $X \sqcup Y$ and is topologized by keeping the same ordering within X and Y and setting $x \leq y$ in $B(f)$ if $f(x) \leq y$ in Y. We denote by $i : X \hookrightarrow B(f)$ and $j : Y \hookrightarrow B(f)$ the canonical inclusions of X and Y into $B(f)$.

Lemma 4.2.7. *Let $f : X \to Y$ be a map between F-spaces. Then Y is a strong deformation retract of $B(f)$.*

Proof. Let $r : B(f) \to Y$ be the map given by $r(x) = f(x)$ for all $x \in X$ and restrict to the identity on Y. It is order preserving and hence continuous. Moreover, $1_{B(f)} \leq jr$ which means $jr \simeq_Y 1_{B(f)}$. □

A map $f : X \to Y$ between F-spaces is called **distinguished** if $f^{-1}(U_y) \subseteq X$ is contractible for every $y \in Y$.

Lemma 4.2.8. *Let $f : X \to Y$ be a distinguished G-map between G-F-spaces. Then $B(f) \searrow^G X$.*

Proof. If $x \leq y$ for some $x \in X$ and $y \in Y$, then $f(x) \leq y$ and thus $f(gx) = gf(x) \leq gy$ for every $g \in G$. It follows that $gx \leq gy$ and thus the map $X \sqcup Y \ni z \mapsto gz$ is order preserving and hence continuous. Therefore $B(f)$ is a G-space with the action induced by X and Y.

Let $Y_1, \ldots, Y_k$ be the orbits of Y under the action of G. We define a partial order $\leq$ on the set of orbits by declaring $Y_i \leq Y_j$ if there are elements $y_i \in Y_i$ and $y_j \in Y_j$ such that $y_i \leq y_j$. Reflexivity and transitivity are clear to see. To show that this is antisymmetric, suppose we have two orbits Y and Y' such that $Y \leq Y'$ and $Y' \leq Y$. We will show that $Y = Y'$. Since $Y \leq Y'$, there exist a $y_1 \in Y$ and $z_1 \in Y'$ such that $y_1 \leq\leq z_1$ and similarly since $Y' \leq Y$, there exist elements $y_2, \in Y$ and $z_2 \in Y'$ such that $z_2 \leq y_2$. Sine y_1 and y_2 are in the same orbit, there exists a $g \in G$ such that $gy_1 = y_2$. Similarly, there exists an $h \in G$ such that $hz_1 = z_2$. It follows that we have inequalities $y_1 \leq z_1$ and $hz_1 \leq gy_1$ and so $z_1 \leq h^{-1}gy_1$. Let $g' = h^{-1}g$. Then $y_1 \leq z_1 \leq g'y_1$. But since the map $y \mapsto gy$ is order preserving, we get that $z_1 = y_1$ and hence $Y = Y'$.

Now consider an arrangement of the orbits $Y_1, Y_2, \ldots, Y_k$ such that $Y_m \leq Y_n$ implies $m \leq n$. Define $X_r = X \cup Y_{r+1} \cup Y_{r+2} \cup \cdots \cup Y_k \subseteq B(f)$ for all $0 \leq r \leq k$. Then if y_r is a representative of Y_r, the minimal set

$$\hat{U}_{y_r}^{X_{r-1}} = \{x \in X_{r-1} \mid x \leq y_r\} = \{x \in X_{r-1} \mid f(x) \leq y_r\}$$

is homeomorphic to $f^{-1}(U_{y_r}^Y)$ and the latter is contractible by the hypothesis. It follows that y_r is a weak point of X_{r-1} and thus $X_{r-1} \searrow^{Ge} X_r$ for all $1 \leq r \leq k$. Since $B(f) = X_0$ and $X = X_k$, it follows that $B(f) \searrow^G X$.

□

Proposition 4.2.9. *Let X and Y be G-F-spaces and $f : X \to Y$ be a distinguished G-map. Then X and Y have the same equivariant simple homotopy type.*

Proof. By the proof of the preceding lemma, $B(f)$ is a G-space with the action induced by X and Y. If $y \in Y \subseteq B(f)$ (we identity Y with its image under i), then $gy \in Y$ for otherwise $gy \in X$ and $f(gy) = gf(y)$ but f is not defined on y. It follows that Y is G-invariant and so by Proposition 4.1.4 $B(f) \searrow\searrow^G Y$. But by the preceding lemma, $B(f) \searrow^G X$ so X and Y have the same equivariant simple homotopy type. □

Recall that if X is a G-space we can define an equivalence relation $\sim$ on X by $x \sim y$ if $gx = y$ for some $g \in G$. The quotient space corresponding to this relation is called the **orbit space** and is denoted by X/G. If X is a finite G-space, then the order on X/G

is given by $\overline{x} \leq \overline{y}$ if there exists a g such that $gx \leq y$. From topology we know that a quotient q map is open if and only if $q^{-1}(q(U))$ is open for any open subset U. We claim that $q^{-1}(q(U_x)) = \bigcup_{g\in G} U_{gx}$. If $y \in q^{-1}(q(U_x))$ then $q(y) \in q(U_x)$ and so $y \in U_x$, in particular, $y \in \bigcup_{g\in G} U_{gx}$. On the other hand, if $y \in U_{gx}$ for some $g \in G$, then $y \leq gx$ and so $y \in \{z \in X \mid q(z) \in q(U_x)\} = q^{-1}(q(U_x))$. Therefore $q^{-1}(q(U_x)) = \bigcup_{g\in G} U_{gx}$. In particular, $q^{-1}(q(U_x))$ is open for all x. It follows that q is an open map and hence we can deduce that $q(U_x) = U_{\overline{x}}$ for all $x \in X$. Indeed, $U_{\overline{x}} \subseteq q(U_x)$ by minimality of the minimal basis (since $q(U_x)$ is open) and the other inclusion follows from the continuity of q.

Finally, it follows that if X is T_0, then so is X/G. To see why, suppose that $\overline{x} \leq \overline{y}$ and $\overline{y} \leq \overline{x}$. Then there exist $g, h \in G$ such that $y \leq gx$ and $x \leq hy$. Therefore, $y \leq gx \leq ghy$. By Proposition 1.3.5, $y = gx = ghy$. But then $\overline{y} = \overline{x}$.

Proposition 4.2.10. *Let X be a G-F-space that strongly G-collapses to a G-invariant subspace Y. Then X/G strongly collapses to Y/G and X^G strongly collapses to Y^G.*

Proof. Suppose that there is an elementary strong G-collapse from X to Y. The general case follows inductively. It follows that $Y = X \setminus Gx$ for some beat point $x \in X$. Suppose x is a down beat point and let $y \in X$ be the point such that $y \prec x$. Then $\overline{y} < \overline{x}$ in X/G. Our goal is to show that $\overline{x}$ is a beat point in X/G by showing it covers $\overline{y}$ and only $\overline{y}$ because then X/G strongly collapses to $X/G \setminus \{\overline{x}\} = Y/G$. Suppose that $\overline{z} < \overline{x}$. Then there exists a $g \in G$ such that $gz < x$. Since y is the unique element covered by x, $gz \leq y$, but then $\overline{z} \leq \overline{y}$, proving that $\overline{x}$ is indeed a beat point of X/G.

To prove the next part, note that if x is not fixed by G then $Y^G = X^G$ and there is nothing to prove. Suppose therefore that $x \in X^G$. Then if $g \in G$, we have $gy < gx = x$ and therefore $gy \leq y$. It follows that $gy = y$ and so $y \in X^Y$. It follows that x is a beat point of X^G. This implies that $X^G \searrow\!\!\searrow Y^G$. □

In particular, if X is contractible then so are X/G and X^G. Indeed, if X is contractible then X strongly G-collapses to a G-core which is a point, and then both X/G and X^G are contractible.

Proposition 4.2.11. *Let X be a G-F-space which G collapses to Y. Then X^G collapses to Y^G. In particular, if X is G-collapsible, then X^G is collapsible.*

Proof. Again, we suppose that $X \searrow^{Ge} Y$ and the fact that $X \searrow^{G} Y$ follows inductively. Suppose $X \searrow^{Ge} Y = X \setminus Gx$. As in the proof of the preceding proposition, if $x \notin X^G$ then $Y^G = X^G$. Suppose $x \in X^G$, then $\hat{C}_x^X$ is G-invariant and contractible. By the preceding proposition, $\hat{C}_x^{X^G} = (\hat{C}_x^X)^G$ is contractible and so x is a weak point of X^G. This means that $X^G \searrow Y^G$. □

Corollary 4.2.12. *Let X and Y be equivariant simple homotopy equivalent G-F-spaces. Then X^G and Y^G have the same simple homotopy type.*

If K is a G-simplicial complex, we denote by K^G the full subcomplex of K spanned by the vertices fixed by the acts.

We can prove an analogous result of Proposition 4.2.11 for simplicial complexes.

Proposition 4.2.13. *Let K be a finite G-simplicial complex that G-collapses to a subcomplex L. Then K^G collapses to L^G. In particular, if K is G-collapsible, then K^G is collapsible.*

Proof. The proof is very similar to the preceding proposition. Suppose that $K \searrow^{Ge} L = K \setminus \bigcup_{g\in G}\{g\sigma, g\tau\}$, where $\sigma \subsetneq \tau$ is a G-collapsible pair. The general case follows inductively. If $\sigma \notin K^G$, then $L^G = K^G$. So suppose $\sigma \in K^G$. Since σ is a free face of τ, it follows that $\tau \in K^G$. Then $L = K \setminus \{\sigma, \tau\}$ and $L^G = K^G \setminus \{\sigma, \tau\}$. Since $\sigma \subsetneq \tau$ is a collapsible pair in K^G, it follows that $K^G \searrow L^G$. □

Corollary 4.2.14. *Suppose K and L are finite G-simplicial complexes with the same equivariant simple homotopy type. Then K^G and L^G have the same simple homotopy type. In particular, K has a vertex which is fixed by the action of G if and only if L has a vertex fixed by G.*

4.3 Quillen's Conjecture

In this section we state and prove Quillen's theorem and introduce Quillen's conjecture.

Let G be a group and p be a prime. Let $S_p(G)$ denote the poset of nontrivial p-subgroups of G under inclusion. Denote by $A_p(G)$ the subspace of $S_p(G)$ consisting of nontrivial elementary abelian p-subgroups. Recall that an abelian group is elementary if all the nontrivial elements have the same order. Note that the maximal elements of $S_p(G)$ are precisely the Sylow p-subgroups and the minimal elements are the subgroups of order p.

Theorem 4.3.1 (Quillen)**.** *If G has a normal p-subgroup, then the space $S_p(G)$ is contractible.*

Proof. Suppose N is a nontrivial normal p-subgroup of G. Since N is normal, then NH is a subgroup of G for any p-subgroup H of G, and since both N and H are p-subgroups and $|NH| = |N||H|/|N \cap H|$, it follows that the map $f : S_p(G) \to S_p(G)$ given by $H \mapsto NH$ is well defined. Now clearly $f(H) \geq H$ so $f \geq 1_{S_p(G)}$. Furthermore, if c_N denotes the constant map N, we get a fence $c_N \leq f \geq 1_{S_P(G)}$. By Corollary 1.2.7, it follows that the identity is null-homotopic and so $S_p(G)$ is indeed contractible. □

Let $\mathcal{K}(S_p(G))$ be the order complex of S_p and denote by $|\mathcal{K}(S_p(G))|$ its geometric realization. With this notation, the following corollary is immediate by a combination of Proposition 3.1.2 and Corollary 3.1.13.

Corollary 4.3.2. *If G has a nontrivial normal p-subgroup then $|\mathcal{K}(S_p(G))|$ is contractible.*

The converse of this corollary is Quillen's conjecture.

Conjecture 4.3.3 (Quillen)**.** *If $|\mathcal{K}(S_p(G))|$ is contractible, then G has a nontrivial normal p-subgroup.*

The conjecture is known to hold for solvable groups as Quillen showed in [8]. Stong was the first mathematician to apply the theory of finite topological spaces to Quillen's work. Brown and Quillen studied the topological properties of $S_p(G)$ through the geometric realization of its order complex.

In [10], Stong showed that $A_p(G)$ and $S_p(G)$ are not homotopy equivalent as finite spaces in general, but they are weakly homotopy equivalent.

Proposition 4.3.4. *The inclusion $A_p(G) \hookrightarrow S_p(G)$ is a weak homotopy equivalence.*

Proof. Since U_H is contractible for all $H \in S_p(G)$, it suffices to show that $i^{-1}(U_H)$ is contractible, by the same reasoning as in the proof of Theorem 3.1.7. Since H is a nontrivial p-subgroup, by the class equation, its center Z is not trivial. Note that $i^{-1}(U_H) = A_p(H)$. Let $N \subseteq Z$ be the subgroup whose nontrivial elements have order p. Recall that if S, T are subgroups of G, the set $ST = \{st \mid s \in S, t \in T\}$ is a group with with operation inherited from G if either S or T is normal. Let $K \in A_p(H) = i^{-1}(U_H)$. From the formula

$$|ST| = \frac{|S||T|}{|S \cap T|},$$

it follows that $TN \in A_p(H)$. But $T \leq TN \geq N$. So $A_p(H) = i^{-1}(U_H)$ is contractible.

With this in mind, we apply McCords theorem to the inclusion map while considering the minimal basis for $S_p(G)$ to be the basis like cover. □

Proposition 4.3.5. *$A_p(G)$ and $S_p(G)$ have the same equivariant simple homotopy type.*

Proof. From the proof of Proposition 4.3.4, we deduce that the inclusion $A_p(G) \hookrightarrow S_p(G)$ is a distinguished map. The result follows from Proposition 4.2.9. □

Corollary 4.3.6. *If G has a nontrivial normal p-subgroup, then it has a nontrivial normal elementary abelian p-subgroup.*

Proof. Since $S_p(G)$ and $A_p(G)$ have the same equivariant simple homotopy type, by Corollary 4.2.12, $S_p(G)^G$ and $A_p(G)^G$ have the same equivariant simple homotopy type. Therefore, if $S_p(G)^G$ is nonempty, so is $A_p(G)^G$. □

4.4 The Euler Characteristic of $S_p(G)$: Brown's Theorem

Before we finish with a theorem that allows us to attack Quillen's conjecture from different sides, we would like to provide an alternative proof of Browns result in [3] on the Euler characteristic of $S_p(G)$.

Theorem 4.4.1 (Brown)**.** *Let G be a finite group and P be a Sylow p-subgroup. Then $\chi(S_p(G)) \cong 1$ mod$(\#P)$.*

Before we start with the proof we will need some preliminary result.

If H is a subgroup of G, then H acts on $S_p(G)$ by conjugation. We will denote by $S_p(G)^H$ the fixed points of this action.

Proposition 4.4.2. *Let H be a nontrivial p-subgroup of G. Then $S_p(G)^H$ is contractible.*

Proof. Let $T \in S_p(G)^H$. Then $H \subseteq N_G(T)$, where $N_G(T)$ denotes the normalizer of T in G. Then TH is a subgroup. It is in fact a p-subgroup since

$$|TH| = \frac{|T||H|}{|T \cap H|}.$$

It follows that $TH \in S_p(G)^H$. Note that $T \leq TH \geq H$. Since T was arbitrary, it follows that the constant map $c_H : S_p(G)^H \to S_p(G)^H$ is homotopic to the identity, that is $S_p(G)^H$ is contractible. □

If X is an F-space, then the **barycentric subdivision** of X is the F-space $X' := \mathcal{X}(\mathcal{K}(X))$. If X is a G-F-space, then the barycentric subdivision X' is also a G-space with the action given by $g \cdot \{x_1, x_2, \dots x_n\} = \{gx_1, gx_2, \dots, gx_n\}$.

Let P be a Sylow p-subgroup of G. Since P acts on $S_p(G)$ by conjugation, we get an induced action of P on $S_p(G)'$. If $c \in S_p(G)'$, we let $P_c = \{g \in P \mid gc = c\}$ denote the stabilizer of c and we define $Y := \{c \in S_p(G)' \mid P_c \neq 0\}$.

Lemma 4.4.3. $\chi(S_p(G)', Y) \equiv 0 \ mod(\#P)$.

Proof. Let $C = \{c_0 < c_1 < \cdots < c_n\}$ be a chain of $S_p(G)' \setminus Y$. It follows that $c_j \notin Y$ for some $0 \leq j \leq n$. This means that $P_{c_j} = 0$, or that the cardinality of the orbit of c_j under the action of P is equal to $\#P$. Now note that C is an element of $S_p(G)''$, the barycentric subdivision of $S_p(G)'$ (the second barycentric subdivision of $S_p(G)$), and since P acts on $S_p(G)'$, we get an induced action of P on $S_p(G)$. For this reason, it follows that the orbit of C also has $\#P$ elements. This implies that $\#P$ divides $\chi(S_p(G)', Y) = \sum_{i \geq 0}(-1)^i \alpha_i$, where α_i is the number of chains of length $(i+1)$ of $S_p(G)'$ that are not chains of Y. $\square$

Lemma 4.4.4. $\chi(Y) \equiv 1 \ mod(\#P)$.

Proof. We will show that Y is weakly contractible, since then the result follows from Corollary 3.1.9. Let $f : Y \to S_p(P)^{op}$ be defined by $f(c) = P_c$. Note that P_c is nontrivial by the definition of Y, therefore f is well defined. Also, if $c_0 \leq c_1$, then $P_{c_0} \supseteq P_{c_1}$, showing that f is order preserving and thus continuous. Let $H \subseteq P$ be nonempty, then $f^{-1}(U_H) = \{c \in Y \mid H \subseteq P_c\} = (S_p(G)^H)'$. By Proposition 4.4.2, $f^{-1}(U_H)$ is contractible. By McCord's theorem (Theorem 3.1.4), f is a weak homotopy equivalence. But $S_p(P)^{op}$ has a minimum, in particular, it is contractible and thus it follows that Y is weakly contractible. $\square$

Armed with these results, we can now prove Theorem 4.4.1.

Proof of Theorem 4.4.1. Since $S_p(G)$ and $S_p(G)'$ are homeomorphic, they have the same homotopy type, and so by Theorem 1.3.15, $\chi(S_p(G)) = \chi(S_p(G)')$. By Remark 1.3.14 and Lemma 4.4.4,

$$\chi(S_p(G)') = \chi(Y) + \chi(S_p(G)', Y) \equiv 1 \ mod(\#P).$$

$\square$

4.5 The Main Theorem

We are finally ready to state and prove the main theorem of this paper.

Theorem 4.5.1. *For any finite group G and prime p, the following are equivalent:*

(1) G has a nontrivial normal p-subgroup.

(2) $S_p(G)$ is a contractible space.

(3) $S_p(G)$ is G-collapsible.

(4) $S_p(G)$ has the equivariant simple homotopy of a point.

(5) $\mathcal{K}(S_p(G))$ is G-collapsible.

(6) $\mathcal{K}(S_p(G))$ *has the equivariant simple homotopy type of a point.*

(7) $\mathcal{K}(S_p(G))$ *is strong collapsible.*

(8) $A_p(G)$ *has the equivariant simple homotopy type of a point.*

(9) $\mathcal{K}(A_p(G))$ *has the equivariant simple homotopy type of a point.*

Proof. If G has a nontrivial normal p-subgroup then $S_p(G)$ is contractible by Quillen's theorem. If $S_p(G)$ is contractible, its G-core is a point, combining Lemma 4.1.1 with Proposition 4.1.4 we see that there is a strong G-collapse from a G-F-space to its G-core, so it G-collapses to a point and therefore it is G-collapsible. If $S_p(G)$ is G-collapsible then by Theorem 4.2.5, $\mathcal{K}(S_p(G))$ is also G-collapsible.

We have shown that (1) $\implies$ (2) $\implies$ (3) $\implies$ (5). We will show that (5) $\implies$ (6) $\implies$ (4) $\implies$ (8) $\implies$ (9) $\implies$ (1) and complete the proof by showing that (7) is equivalent to (2) (which we showed is equivalent to all others).

If $\mathcal{K}(S_p(G))$ is G-collapsible, then it trivially have the same equivariant simple homotopy type of a point. By Corollary 4.2.6, $S_p(G)$ has the equivariant simple homotopy type of a point. By Proposition 4.3.5, $A_p(G)$ also has the equivariant simple homotopy type of a point, and by Theorem 4.2.5, $\mathcal{K}(A_p(G))$ has the equivariant simple homotopy type of a point. Now we show that if $\mathcal{K}(A_p(G))$ has the equivariant simple homotopy type of a point, then G has a nontrivial normal p-subgroup, but this follows from Corollary 4.2.14, since this implies that $\mathcal{K}(A_p(G))$ has a vertex that is fixed by the action of G. But since the action of G is conjugation then this fixed vertex corresponds to a nontrivial normal p-subgroup of G. Finally, the equivalence of (7) with (2) (and thereby with all the others) follows from Corollary 3.3.15. $\square$

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