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Dynamics of Quadratic Polynomials
–a Real Approach to Chaos–

av

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Dynamics of Quadratic Polynomials
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Abstract

In this thesis we present theory regarding dynamical systems; by which we refer to the process of iterating a function. In particular, the focus is on the real-valued affine family of functions and the real-valued quadratic family of functions. We present concepts such as orbits and fixed points, as well as methods to graphically examine these for given functions. Further, we examine how the character of a fixed point affects the behavior of neighbouring points. Lastly, the Period-3 Theorem is presented and used as a stepping stone into the theory of chaos, which we conclude with a brief overview using symbolic dynamics.

Sammanfattning

I denna uppsats presenterar vi teori för dynamiska system; med vilket vi menar processen av att iterera en funktion. I synnerhet kommer fokus vara på den reellvärda affina familjen av funktioner och den reellvärda kvadratiska familjen av funktioner. Vi presenterar koncept såsom banor och fixpunkter, samt metoder för att grafiskt analysera dessa för givna funktioner. Vidare undersöker vi hur en fixpunkts karaktär påverkar beteendet hos närliggande punkter. Avslutningsvis presenterar vi Period-3 Satsen och använder denna som startpunkt för kaosteori, vilket vi ger en kort överblick av med hjälp av symbolisk dynamik.

Acknowledgements

First and foremost I would like to thank my supervisor Alan Sola for introducing me to the incredibly interesting subject that is dynamical systems, and for the support during the work of this thesis. I would also like to thank my friends who always listen when I need a "rubber duck" and who have supported me during all twist and turns of this thesis.

Contents

1 Introduction

Dynamical systems concern processes in motion. Our mathematical approach to dynamical systems will be through *iteration* of functions. The process of iteration involves evaluating a function over and over, using the output of the prior application as input in the next; in other words, composing a function with itself repeatedly. In particular, we are interested in the long-term behavior of a point under a given function. More specifically, we wish to predict if a point tends to a particular point, jumps between multiple points, or tends to infinity.

Intuitively, dynamical systems in the form of iteration of functions is about predicting "the next step" given current knowledge. Iteration can, for instance, be used to understand how the size of a population changes over time. Let us denote the population size at generation n by P_n. How does P_{n+1} relate to P_n? Suppose (in a rather naïve manner) this relation is proportional by a factor r, i.e., that $P_{n+1} = rP_n$. Then for a given initial population of size P_0 we could determine the population size of all succeeding generations. In particular, we have $P_n = r^n P_0$. If $r > 1$ the population size would tend to infinity, while the population would go extinct if $r < 1$, and remain the same if $r = 1$.

This thesis to large extent follows the exposition in the book *A First course in Chaotic Dynamical Systems - Theory and Experiment* by Robert L. Devaney [2]. However, a selection of the content has been made and the topics have been chosen to be presented in a different order. Apart from what is presented in Devaney additional examples and details have been added. For instance, an entire section has been included to thoroughly examine the affine family of functions. Additionally, every image is generated by the author: For python code for graphical analysis and orbit diagrams see Appendix B and C respectively.

Notation and assumptions

Unless otherwise stated we will let f denote a real-valued continuously differentiable function and x a real variable. For a function f we will write the n:th iterate of f applied to x,

$$\underbrace{f \circ \cdots \circ f}_{n \text{ times}}(x),$$

as $f^n(x)$. This is the *n-fold* composition of f with itself.

2 Orbits and Fixed Points

To examine the iteration of a function for particular points we begin with the following definition:

Definition 2.1. We define the *orbit* of a point x_0 under a function f as the sequence of points $\{x_n\}_{n\in\mathbb{N}}$, where $x_j = f^j(x_0)$. The initial point x_0 is called the *seed* of the orbit.

Definition 2.2. The most elementary type of orbit is the *fixed point*, where the seed x_0 satisfies

$$f(x_0) = x_0. \tag{1}$$

Considering the condition (1) as a first iteration, we may perform the second iteration as follows: $f^2(x_0) = f(f(x_0)) = f(x_0) = x_0$. Continuing this reasoning we obtain $f^n(x_0) = x_0$ for all $n \in \mathbb{N}$, such that the constant sequence $\{x_0\}$ satisfies Definition 2.1.

Example 2.3. The real-valued identity function $\mathrm{Id}(x) = x$ fixes all points, while $-\mathrm{Id}(x) = -x$ only fixes 0.

Definition 2.4. A more general type of orbit is the *periodic orbit* or *cycle.* Here the seed x_0 satisfies $f^m(x_0) = x_0$ for some $m > 0$, where the least such m is called the *prime period* of the orbit.

So for a periodic orbit the seed does not need to "return" after one iteration. However, if the prime period is $m = 1$ then it is a fixed point. Fixing the prime period m we have

$$\begin{aligned}\{x_n\}_{n\in\mathbb{N}} &= \{x_0, f(x_0), f^2(x_0), \dots, f^{m-1}(x_0), f^m(x_0), f^{m+1}(x_0), \dots\} \\ &= \{x_0, f(x_0), f^2(x_0), \dots, f^{m-1}(x_0), x_0, f(x_0), \dots\}.\end{aligned}$$

from which we note that if the seed has period m then the orbit consists of exactly m elements, all with the common period of m.

If the seed x_0 itself is not fixed, but a point on its orbit is fixed (periodic) then we say that x_0 is *eventually fixed* (*eventually periodic).*

Prior to examining different kinds of fixed points we present the following theorem about existence from Devaney [2, p. 45]:

Theorem 2.5. *[Fixed Point Theorem] Suppose $f : [\alpha, \beta] \to [\alpha, \beta]$ is continuous. Then there is a fixed point for f in $[\alpha, \beta]$.*

Proof. Let $h(x) = f(x) - x$. Then h is a continuous function which satisfies

$$h(\alpha) = f(\alpha) - \alpha \geq 0,$$
$$h(\beta) = f(\beta) - \beta \leq 0,$$

since f maps the interval $[\alpha, \beta]$ onto itself. Hence, by the *Intermediate Value Theorem*, there is a γ in $[\alpha, \beta]$ such that $h(\gamma) = 0$. By our definition of h we have that $f(\gamma) = \gamma$ so that γ is a fixed point of f. □

Note that the theorem asserts existence of at least one fixed point on the interval, but that there may be more.

Example 2.6. Consider the real-valued function $f(x) = x^3$ on the interval $[-1, 1]$, then $f : [-1, 1] \to [-1, 1]$. The equation $f(x) = x$ has three solutions on the interval $[-1, 1]$, namely -1, 0, and 1. Thus, by Definition 2.2, f has three fixed points in the given interval.

Now, having convinced ourselves that there exist fixed points, apart from the ones found in example 2.3, we categorize fixed points as follows:

Definition 2.7. Suppose x_0 is a fixed point for $f \in \mathcal{C}^1$. Then x_0 is said to be an *attracting fixed point* if $|f'(x_0)| < 1$. The point x_0 is said to be a *repelling fixed point* if $|f'(x_0)| > 1$. Finally, if $|f'(x_0)| = 1$, the fixed point is called *neutral* or *indifferent*.

Example 2.8. For all real x, i.e. all fixed points, of the identity function we have $\mathrm{Id}'(x) = 1$, meaning they are all neutral. Also, $|-\mathrm{Id}'(x)| = 1$ for its only fixed point $x = 0$. However, we may note that all other points are periodic with prime period 2. In particular for every $x_0 \neq 0$ under $-\mathrm{Id}(x)$ we have the orbit

$$\{(-1)^n x_0\}_{n \in \mathbb{N}} = \{x_0, -x_0, x_0, -x_0, \dots\}.$$

The above definition of attracting/repelling fixed points tells us something about neighbouring points, which Devaney [2, p.51-53] presents in the following two theorems:

Theorem 2.9 (Attracting Fixed Point Theorem)**.** *Suppose x_0 is an attracting fixed point of $f \in \mathcal{C}^1$. Then there is an open interval I that contains x_0 in its interior and in which the following condition is satisfied: if $x \in I$, then $f^n(x) \in I$ for all n and, moreover, $f^n(x) \to x_0$ as $n \to \infty$.*

Proof. Since $|f'(x_0)| < 1$, there is number $\lambda > 0$ such that $|f'(x_0)| < \lambda < 1$. By continuity, we may therefore choose a number $\delta > 0$ so that $|f'(x)| < \lambda$ provided x belongs to the interval $I = (x_0 - \delta, x_0 + \delta)$. Thus by letting p be an arbitrary point in I, and x a point between x_0 and p (in particular in I), by the *Mean Value Theorem*

$$|f'(x)| = \frac{|f(p) - f(x_0)|}{|p - x_0|} < \lambda$$

so that

$$|f(p) - f(x_0)| < \lambda |p - x_0|.$$

Since x_0 is a fixed point, this is equivalent to

$$|f(p) - x_0| < \lambda |p - x_0|.$$

Given that $0 < \lambda < 1$, the above inequality means that the distance between $f(p)$ and x_0 is strictly smaller than the distance between p and x_0. In particular, $f(p)$ also lies in the interval I, making it possible to apply the same argument to $f(p)$ and $f(x_0)$, yielding

$$\begin{aligned} |f^2(p) - x_0| &= |f^2(p) - f^2(x_0)| \\ &< \lambda |f(p) - f(x_0)| \\ &< \lambda^2 |p - x_0|. \end{aligned}$$

Again, using the properties of λ, we have that $\lambda^2 < \lambda$, so that the points $f^2(p)$ and x_0 are even closer together. Continuing this argument we find that, for any $n > 0$ and any p in I,

$$|f^n(p) - x_0| < \lambda^n |p - x_0|.$$

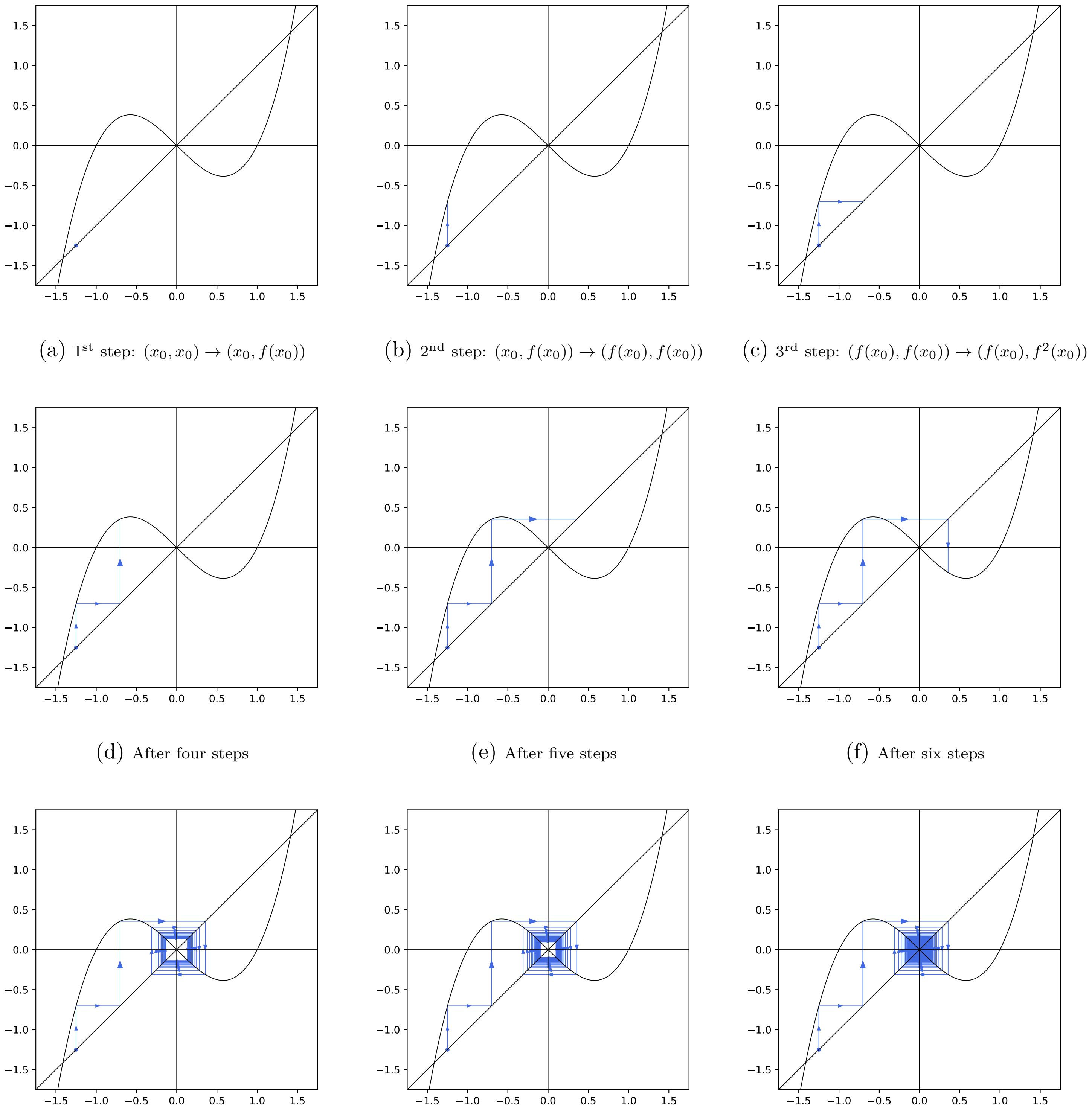
Noting that $\lambda^n \to 0$ as $n \to \infty$, we deduce that $f^n(p) \to x_0$ as $n \to \infty$, as desired. □

The second theorem contains a corresponding result for repelling fixed points, and can be proven analogously, but here we omit the proof:

Theorem 2.10 (Repelling Fixed Point Theorem)**.** *Suppose x_0 is a repelling fixed point for $f \in \mathcal{C}^1$. Then there is an open interval I that contains x_0 in its interior and in which the following condition is satisfied: if $x \in I$ and $x \neq x_0$, then there is an integer $n > 0$ such that $f^n(x) \notin I$.*

3 Graphical Analysis

To obtain intuition about the behavior of points under iteration we introduce the procedure of *graphical analysis*: We again follow Devaney [2, Sec. 4]. Let f be a given function and x_0 a point we want to examine under f. To begin, we impose the line $y = x$, and position ourselves at the point (x_0, x_0) on it; that is, directly above the seed on the x-axis. Next, we draw a vertical line to the graph of f reaching the point $(x_0, f(x_0))$; followed by a horizontal line back to the diagonal, reaching the point $(f(x_0), f(x_0))$. This point is the next on the orbit of x_0, with its x-coordinate being $f(x_0)$. The procedure is then continued in the same manner, alternating between vertical lines from the diagonal to the graph of f, and horizontal lines from the graph back to the diagonal. See Figure 1 for an example illustrating this method. Python code for the graphical analysis performed for the remainder of this thesis can be found in Appendix B.



(a) 1^{st} step: $(x_0, x_0) \to (x_0, f(x_0))$

(b) 2^{nd} step: $(x_0, f(x_0)) \to (f(x_0), f(x_0))$

(c) 3^{rd} step: $(f(x_0), f(x_0)) \to (f(x_0), f^2(x_0))$

(d) After four steps

(e) After five steps

(f) After six steps

(g) After 50 steps

(h) After 100 steps

(i) After 5000 steps

Figure 1: Graphical analysis step by step for $f(x) = x^3 - x$ and seed $x_0 = -\frac{5}{4}$

4 The Affine Family of Functions

To illustrate the concepts above we will begin by focusing on the most elementary type of polynomials; the affine family of functions.

Definition 4.1. We define the *affine family of functions* as $h(x) = ax + b$, where a and b are real constants and x a real variable.

The general orbit for a point $x_0 \in \mathbb{R}$ under h is, as stated in Definition 2.1, $\{h^n(x_0)\}_{n\in\mathbb{N}}$. Here

$$\begin{aligned}
h^0(x_0) &= x_0 \\
h^1(x_0) &= ax_0 + b \\
h^2(x_0) &= a^2x_0 + ab + b \\
h^3(x_0) &= a^3x_0 + a^2b + ab + b \\
&\vdots \\
h^j(x_0) &= a^jx_0 + b\sum_{k=0}^{j-1} a^k \qquad (2) \\
&\vdots
\end{aligned}$$

Does this orbit constitute a fixed point for some x_0 for every pair a, b? To answer this, we take a step back and consider, with use of (1):

$$\begin{aligned}
h(x) &= x \\
(a-1)x + b &= 0. \qquad (3)
\end{aligned}$$

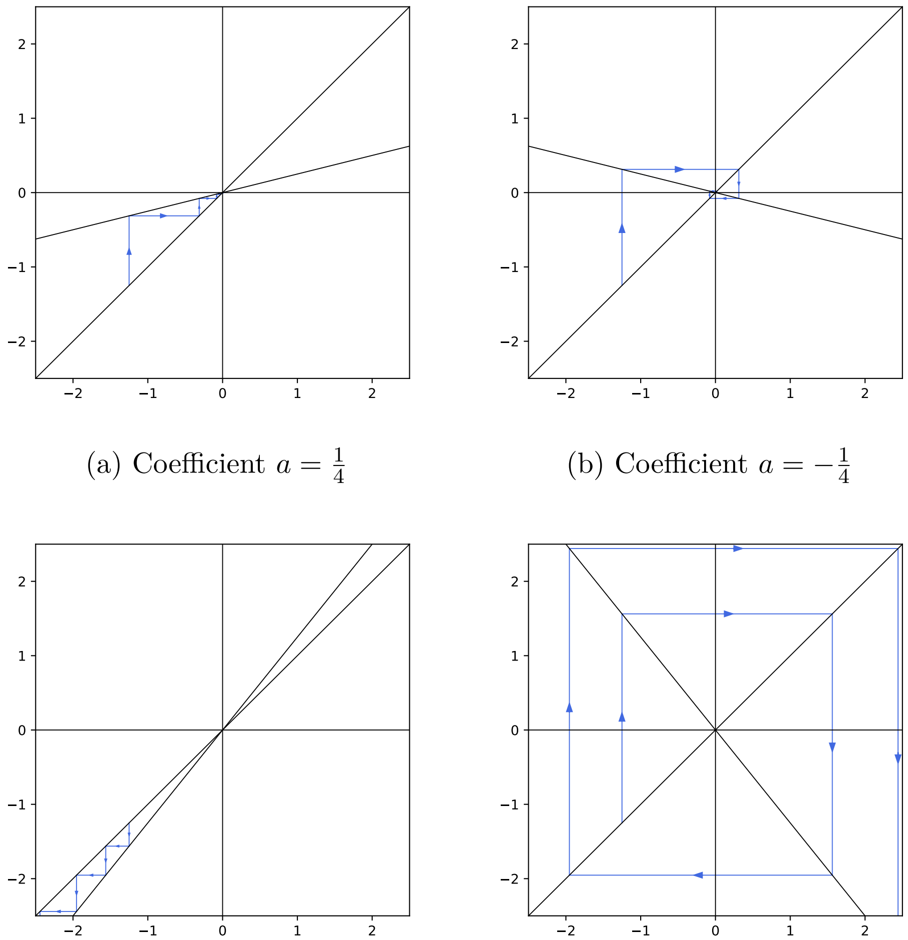
If $a = 1$ then b must be equal to 0 for a fixed point to exist. In fact, if $a = 1$ and $b = 0$, then $h = \text{Id}$ which fixes every point $x_0 \in \mathbb{R}$, as we noted in Example 2.8. If $a = 1$ and $b \neq 0$ then there is no fixed point under h, and every orbit tends to negative or positive infinity. In particular, the orbit of a point x_0 under h in this case is $\{x_0 + bn\}_{n\in\mathbb{N}}$, as can be seen from (2). Moving on, if $a \neq 1$ we may rewrite (2) as

$$h^j(x_0) = a^jx_0 + b\frac{a^j - 1}{a - 1} \qquad (4)$$

and (3) as

$$x = \frac{b}{1-a}.$$

The latter we will denote by p from now on. The character of this fixed point depends on a, since $h'(x) = a$ for all x. According to Definition 2.7, p is attracting when $|a| < 1$, neutral when $|a| = 1$, and repelling when $|a| > 1$. To gain further understanding we begin by fixing $b = 0$, so that the orbit of x_0 under h is $\{a^n x_0\}_{n\in\mathbb{N}}$, which we once again can see from (2). Then the neutral case not yet considered is when $a = -1$ such that $h = -\mathrm{Id}$, which was seen in Example 2.8 as well. We



(a) Coefficient $a = \frac{1}{4}$

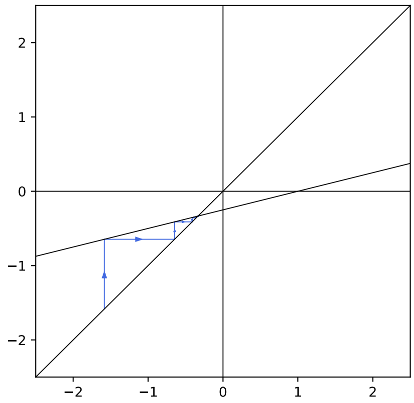
(b) Coefficient $a = -\frac{1}{4}$

(c) Coefficient $a = \frac{5}{4}$

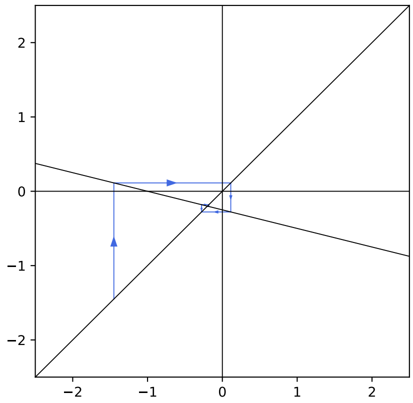
(d) Coefficient $a = -\frac{5}{4}$

Figure 2: Orbits of $x_0 = p - \frac{5}{4}$ under $h(x) = ax$

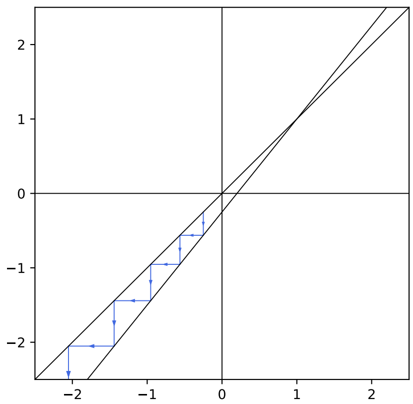
illustrate the remaining cases when $b = 0$, by fixing one value of a in each of the intervals $(0, 1)$, $(-1, 0)$, $(1, \infty)$, and $(-\infty, -1)$, in Figure 2. Letting b be nonzero, we see in Figure 3 that the orbits behave (at least) locally equivalently to the case when $b = 0$, but with the fixed point shifted.



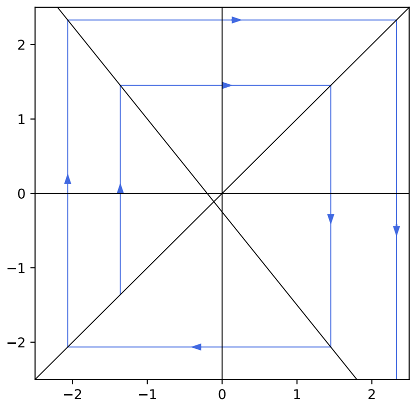
(a) Coefficient $a = \frac{1}{4}$



(b) Coefficient $a = -\frac{1}{4}$



(c) Coefficient $a = \frac{5}{4}$



(d) Coefficient $a = -\frac{5}{4}$

Figure 3: Orbits of $x_0 = p - \frac{5}{4}$ under $h(x) = ax - \frac{1}{4}$

We may prove that this observation is in fact true (globally) by examining a point x_0 shifted by the fixed point p, using (4):

$$\begin{aligned} h^j(x_0 + p) &= a^j \left(x_0 + \frac{b}{1-a} \right) + b\frac{a^j - 1}{a - 1} \\ &= a^j x_0 - b\frac{a^j}{a-1} + b\frac{a^j - 1}{a-1} \\ &= a^j x_0 + \frac{b}{1-a} \\ &= a^j x_0 + p. \end{aligned}$$

So, if $h_a(x) := ax$ and $h_{ab}(x) := ax + b$ we have that

$$\left\{h_a^n(x_0) + \frac{b}{1-a}\right\}_{n\in\mathbb{N}} = \left\{h_{ab}^n\left(x_0 + \frac{b}{1-a}\right)\right\}_{n\in\mathbb{N}}.$$

Now, we understand the affine family of functions for all real constants a and b. Moreover, the orbit of a point x_0 under the function $h_a(x) = ax$ (where $a \neq 1$) is equivalent to that of $x_0 + p$ under $h_{ab}(x) = ax + b$, where p is the fixed point of h_{ab}. Recall that 0 is the fixed point of h_a, while $p = \frac{b}{1-a}$ is the fixed point of h_{ab}. For instance, if x_0 tends to the fixed point 0 under h_a, then $x_0 + p$ will tend to the fixed point p under h_{ab}. The converse is also true.

5 The Quadratic Family of Functions (Part 1)

We leave the rather simple case of the affine family of functions to see what happens as we increase the degree of the polynomial by just one, so we have a quadratic polynomial. As we will see, understanding points under iteration becomes a lot more complicated already increasing the degree by just one, and we will focus on the quadratic family for the remainder of this thesis.

Definition 5.1. We define the *quadratic family of functions* as $g_c(x) = x^2 + c$, with x real. Here the real constant c is called a *parameter*; each c gives rise to a different dynamical system.

We can motivate our use of $g_c(x) = x^2 + c$, instead of the more general form of a second degree polynomial $f(x) = \alpha x^2 + \beta x + \gamma$ (where $\alpha \neq 0$) by conjugation. In particular, we wish to find a conjugacy function φ, such that

$$\varphi \circ f \circ \varphi^{-1} = g_c. \tag{5}$$

As we will see, there is a function $\varphi(x)$ of the form $ax + b$ (with $a \neq 0$) satisfying (5). To determine a and b consider

$$\begin{aligned}
\varphi \circ f \circ \varphi^{-1}(x) &= \varphi \circ f \left(\frac{x-b}{a} \right) \\
&= \varphi \left(\alpha \left(\frac{x-b}{a} \right)^2 + \beta \left(\frac{x-b}{a} \right) + \gamma \right) \\
&= a \left(\alpha \frac{x^2 - 2bx + b^2}{a^2} + \beta \frac{x-b}{a} + \gamma \right) + b \\
&= \frac{\alpha}{a} x^2 + \left(\beta - \frac{2\alpha\beta}{a} \right) x + \frac{\alpha\beta^2}{a} - \beta b + a\gamma + b. \qquad (6)
\end{aligned}$$

Combining (5) and (6) we obtain the following system of equations:

$$\begin{cases}
1 = \frac{\alpha}{a} \\
0 = \beta - \frac{2\alpha\beta}{a} \\
c = \frac{\alpha\beta^2}{a} - \beta b + a\gamma + b.
\end{cases}$$

Thus we have

$$\begin{cases} a = \alpha \\ b = \dfrac{\beta}{2} \end{cases}$$

so that $\varphi(x) = \alpha x + \frac{\beta}{2}$ is a conjugacy function between $f(x) = \alpha x^2 + \beta x + \gamma$ and $g_c = x^2 + c$, where

$$c = \frac{1}{4}(2\beta - \beta^2 + 4\alpha\gamma).$$

So we have found a one-to-one correspondence between any quadratic function of the general form of $f(x) = \alpha x^2 + \beta x + \gamma$ and functions of the same form as $g_c(x) = x^2 + c$. These conjugate functions are equivalent in terms of their dynamics.[1] For instance, suppose x_f is a fixed point for f, i.e. that $f(x_f) = x_f$, then $\varphi(x_f)$ is a fixed point for g_c. To see this, first note that (6) can be rewritten as

$$\varphi \circ f = g_c \circ \varphi. \tag{7}$$

Inserting x_f in (7) gives

$$\begin{aligned} \varphi \circ f(x_f) &= g_c \circ \varphi(x_f) \\ \varphi(x_f) &= g_c(\varphi(x_f)). \end{aligned}$$

So, by Definition 2.2, $\varphi(x_f)$ is indeed a fixed point for g_c. Similar arguments show the correspondence between periodic points of period n between f and g_c.

Now, we find the fixed points of g_c by solving the equation

$$\begin{aligned} g_c(x) &= x \\ x^2 - x + c &= 0 \\ x &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - c}. \end{aligned}$$

We denote the roots by

$$\begin{aligned} p_+ &= \frac{1}{2}(1 + \sqrt{1 - 4c}), \\ p_- &= \frac{1}{2}(1 - \sqrt{1 - 4c}). \end{aligned}$$

[1] In fact φ is a homeomorphism called a topological conjugacy between $f : \mathbb{R} \to \mathbb{R}$ and $g_c : \mathbb{R} \to \mathbb{R}$. For exact definition and further reading we refer to, for instance, Devaney [1, p. 47] and Robinson [5, p. 38-41].

Now, noticing that p_+ and p_- are real if and only if $c \leq \frac{1}{4}$, it follows that g_c has no fixed points on the real line whenever $c > \frac{1}{4}$. Moreover, the graph of g_c does not meet the diagonal line, so graphical analysis shows that all orbits tend to infinity, see Figure 4 for instance. In particular, for whichever seed we choose the vertical steps will always be upward and the horizontal to the right, and since the graph of g_c never meets the diagonal this process will continue without restraint ad infinitum.

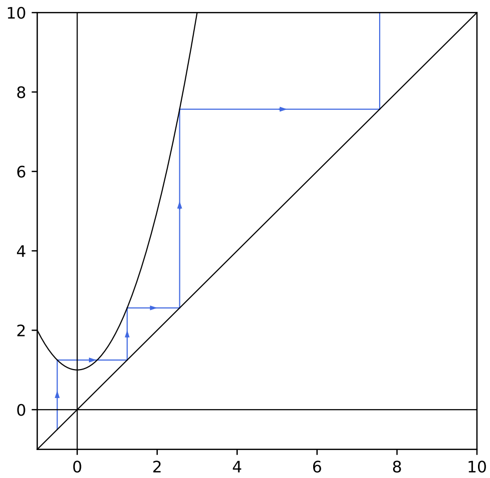
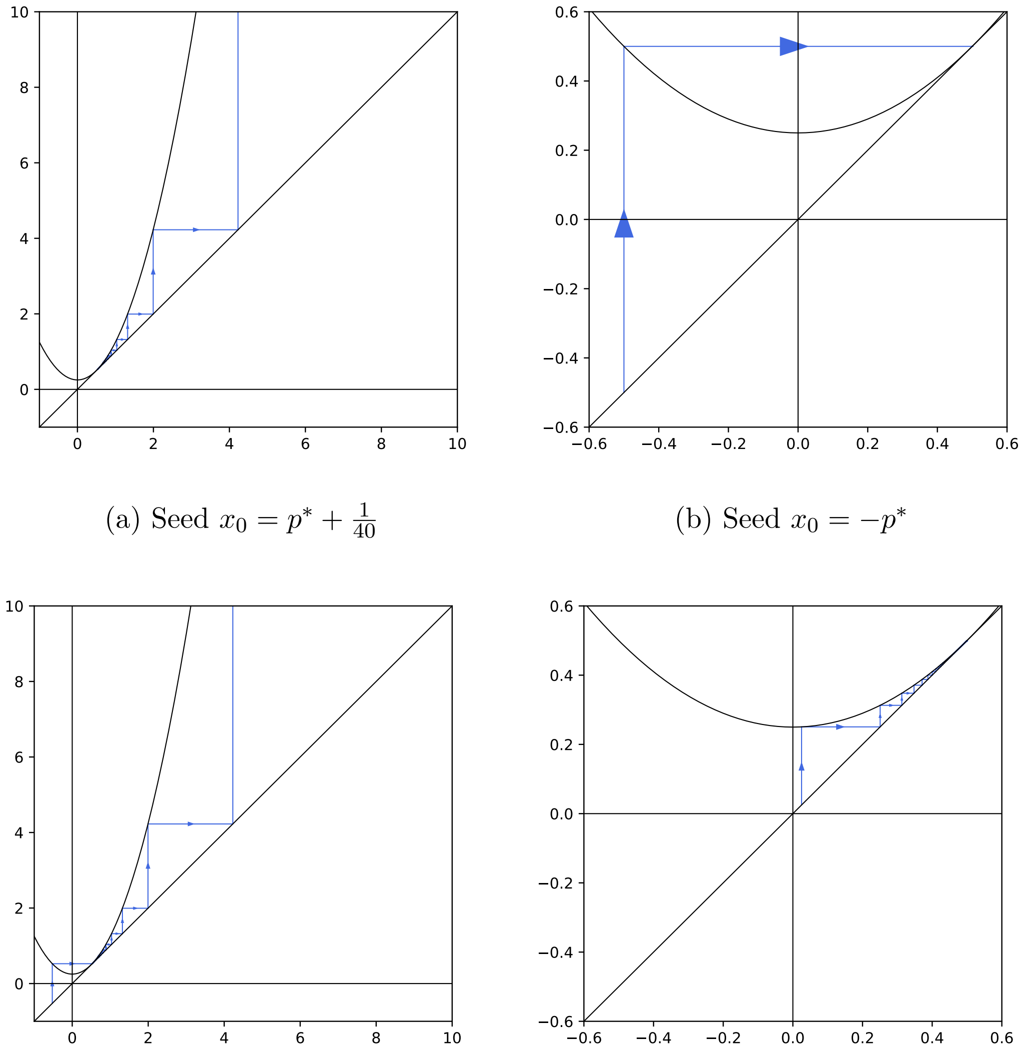


Figure 4: Orbit of $x_0 = -\frac{1}{2}$ under $g_c(x) = x^2 + 1$

Next, consider $c = \frac{1}{4}$. Then $p_+ = p_- = \frac{1}{2}$, so that g_c has exactly one fixed point. Since $g_c'(x) = 2x$ we have $g_c'\left(\frac{1}{2}\right) = 1$, meaning this single fixed point, $p^* := p_+ = p_-$, is neutral. Thus we cannot immediately say anything about the behavior of nearby points. However, through graphical analysis we may examine a selection of the orbits of nearby points, and as we will see by a clever choice of seeds (by partitioning about p_+ and $-p_+$), we will be able to understand the orbit for any real seed x_0.



(a) Seed $x_0 = p^* + \frac{1}{40}$

(b) Seed $x_0 = -p^*$

(c) Seed $x_0 = -p^* - \frac{1}{40}$

(d) Seed $x_0 = \frac{1}{40}$

Figure 5: Orbits of x_0 under $g_c(x) = x^2 + \frac{1}{4}$

First, as seen in Figure 5(a), when $x_0 > p^*$ its orbit tends to infinity just as when $c > \frac{1}{4}$. Secondly, $-p^*$ is an eventually fixed point, since

$$g_c(-p^*) = \left(-\frac{1}{2}\right)^2 + \frac{1}{4} = p^*.$$

In particular, $-p^*$ reach the fixed point p^* after only one iteration, as illustrated in Figure 5(b). Moving below this eventually fixed point, i.e., letting $x_0 < -p^*$, the orbit will once again tend to infinity. This can be seen by performing one iteration on the seed $x_0 = -p^* - \xi$, where $\xi > 0$;

$$\begin{aligned} g_c(x_0) &= (-p^* - \xi)^2 + \frac{1}{4} \\ &= (p^* + \xi)^2 + \frac{1}{4} \\ &= g_c(p^* + \xi), \end{aligned}$$

meaning x_0 shares orbit with $-x_0 = p^* + \xi$, apart from the first element (see Figure 5(c) compared to Figure 5(a)). This observation may be stated more generally:

Proposition 5.2. *Let $g_c(x) = x^2 + c$ be a real quadratic function with $c \le \frac{1}{4}$, and the fixed points p_- and p_+ where $p_- \le p_+$. Then, if $x < -p_+$ or $x > p_+$, the orbit of x under g_c tends to infinity.*

Lastly, we may convince ourselves, based on the orbit in Figure 5(d), that the orbits of x_0, when $-p^* < x_0 < p^*$, tend to the fixed point p^*. (We will go further into why this is the only attracting fixed point in Section 8.)

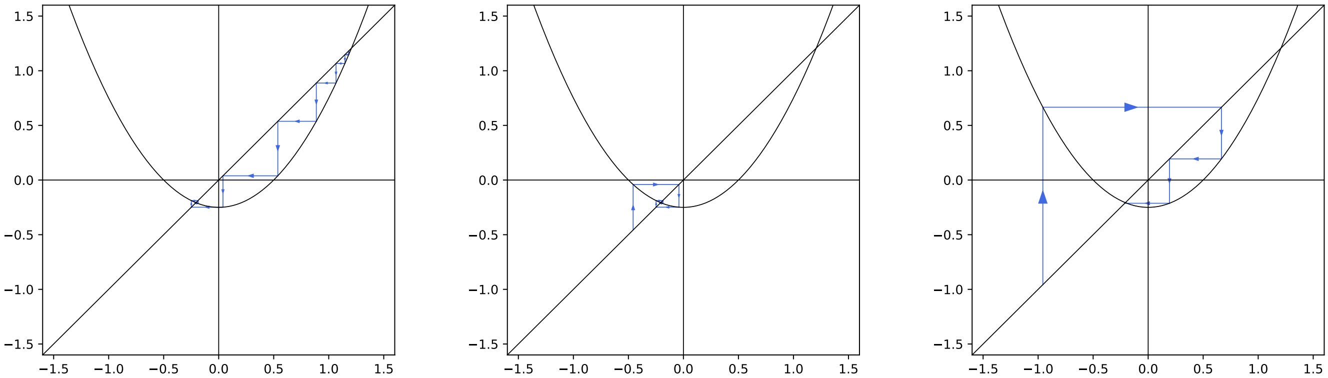
Leaving this single fixed point to consider g_c when $c < \frac{1}{4}$ we begin by noticing that p_+ and p_- now each constitute their own distinct real fixed point. This split into two fixed points is called a *bifurcation*. Recalling that $g_c'(x) = 2x$, we find

$$\begin{aligned} g_c'(p_+) &= 1 + \sqrt{1 - 4c} \\ g_c'(p_-) &= 1 - \sqrt{1 - 4c}. \end{aligned}$$

Thus, when $c < \frac{1}{4}$ the fixed point p_+ is repelling, since $\sqrt{1 - 4c} > 0$. For p_-, on the other hand, to see where $|g_c'(p_-)| < 1$, we compute

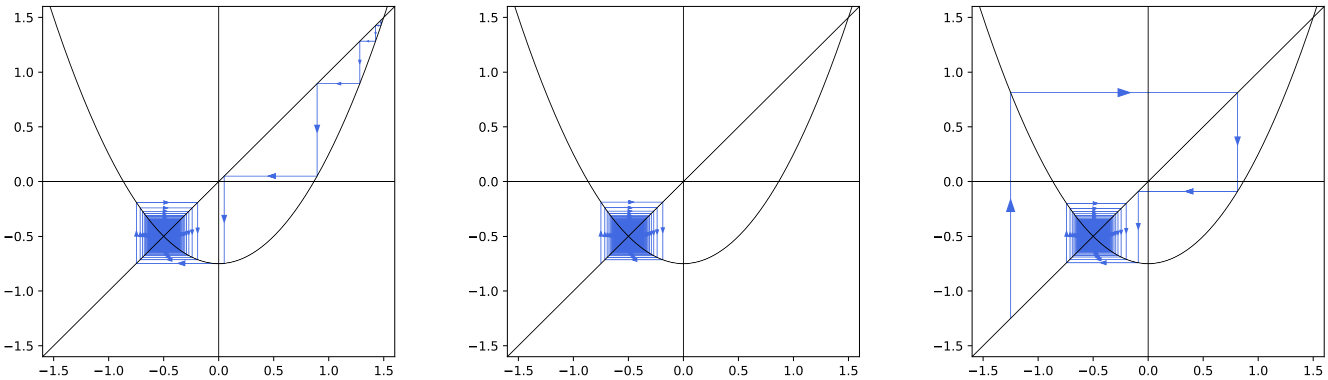
$$\begin{gathered} -1 < 1 - \sqrt{1 - 4c} < 1 \\ 0 < 1 - 4c < 4 \\ -\frac{3}{4} < c < \frac{1}{4}. \end{gathered}$$

Hence p_- is an attracting fixed point when $-\frac{3}{4} < c < \frac{1}{4}$. When $c = -\frac{3}{4}$, we have $g_c'(p_-) = -1$ so that p_- is neutral, and when $c < -\frac{3}{4}$, we have $g_c'(p_-) < -1$ so that p_- is repelling. We show examples of each of these cases in Figures 6, 7 and 8 respectively.



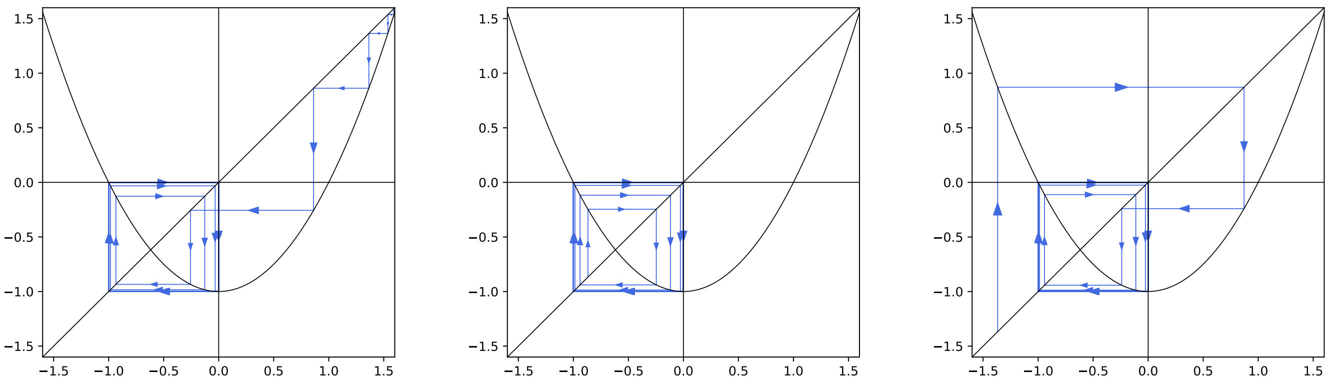
(a) Seed $x_0 = p_+ - \frac{1}{40}$ (b) Seed $x_0 = p_- - \frac{1}{4}$ (c) Seed $x_0 = -p_+ + \frac{1}{4}$

Figure 6: Orbits of x_0 under $g_c(x) = x^2 - \frac{1}{4}$



(a) Seed $x_0 = p_+ - \frac{1}{40}$ (b) Seed $x_0 = p_- - \frac{1}{4}$ (c) Seed $x_0 = -p_+ + \frac{1}{4}$

Figure 7: Orbits of x_0 under $g_c(x) = x^2 - \frac{3}{4}$



(a) Seed $x_0 = p_+ - \frac{1}{40}$ (b) Seed $x_0 = p_- - \frac{1}{4}$ (c) Seed $x_0 = -p_+ + \frac{1}{4}$

Figure 8: Orbits of x_0 under $g_c(x) = x^2 - 1$

As in Devaney [2, p. 63] we summarize these observations in a proposition:

Proposition 5.3 (The First Bifurcation)**.** *For the family $g_c(x) = x^2 + c$:*

1. *All orbits tend to infinity if $c > \frac{1}{4}$.*
2. *When $c = \frac{1}{4}$, g_c has a single fixed point at $p_+ = p_- = \frac{1}{2}$ that is neutral.*
3. *For $c < \frac{1}{4}$, g_c has two fixed points, one at p_+ and one at p_-. The fixed point p_+ is always repelling.*
   a. *If $-\frac{3}{4} < c < \frac{1}{4}$, p_- is attracting.*
   b. *If $c = -\frac{3}{4}$, p_- is neutral.*
   c. *If $c < -\frac{3}{4}$, p_- is repelling.*

In Figure 8 it looks like the orbits of each seed tends to a 2-cycle. To verify whether this is true we need to solve the equation $g_c^2(x) = x$, where $c < -\frac{3}{4}$, i.e.

$$x^4 + 2cx^2 - x + c^2 + c = 0. \tag{8}$$

Note that the fixed points p_+ and p_- are both roots of this fourth-degree polynomial, meaning $(x - p_+)(x - p_-)$ is a factor. Since p_+ and p_- are roots of $g_c(x) - x = 0$ as well, we have that $(x - p_+)(x - p_-) = x^2 - x + c$. Thus we may reduce (8) by this second-degree polynomial:

$$\frac{x^4 + 2cx^2 - x + c^2 + c}{x^2 - x + c} = x^2 + x + c + 1.$$

Now the fixed points of g_c^2 are the solutions of $x^2 + x + c + 1 = 0$, which we find to be

$$q_+ := \frac{1}{2}\left(-1 + \sqrt{-4c - 3}\right),$$
$$q_- := \frac{1}{2}\left(-1 - \sqrt{-4c - 3}\right).$$

Note that q_+ and q_- are real and distinct if and only if $c < -\frac{3}{4}$, meaning no 2-cycle of g_c exists for $c \geq -\frac{3}{4}$. However, there is indeed a cycle of period 2 for g_c when $c < -\frac{3}{4}$. This split when the neutral fixed point $p_- = -\frac{1}{2}$ becomes repelling, as c decreases below $-\frac{3}{4}$, and simultaneously gives rise to 2-cycle is called a *period-doubling bifurcation*.

As with fixed points we examine the character of the cycle through the derivative, this time of g_c^2. Since q_+ and q_- together constitute the cycle, their derivatives under

g_c^2 will coincide. Generally, by the Chain Rule we have, given a differentiable function f,

$$
\begin{aligned}
(f^2)'(x_0) &= f'(f(x_0)) \cdot f'(x_0) \\
&= f'(x_1) \cdot f'(x_0)
\end{aligned}
$$

and

$$
\begin{aligned}
(f^3)'(x_0) &= f'(f^2(x_0)) \cdot (f^2)'(x_0) \\
&= f'(x_2) \cdot f'(x_1) \cdot f'(x_0).
\end{aligned}
$$

Continuing in this manner we find:

Proposition 5.4 (Chain Rule Along a Cycle)**.** *Suppose $x_0, x_1, \ldots, x_{n-1}$ lie on a cycle of period n for a function $f \in \mathcal{C}^1$ with $x_j = f^j(x_0)$. Then*

$$(f^n)'(x_0) = f'(x_{n-1}) \cdots f'(x_1) \cdot f'(x_0).$$

Furthermore, since it is arbitrary where we start in the cycle we also have:

Corollary 5.5. *Suppose $x_0, x_1, \ldots, x_{n-1}$ lie on a cycle of period n for a function $f \in \mathcal{C}^1$ with $x_j = f^j(x_0)$. Then*

$$(f^n)'(x_0) = (f^n)'(x_1) = \cdots = (f^n)'(x_{n-1})$$

Returning to q_+ and q_- under g_c^2, we get

$$(g_c^2(q_+))' = (g_c^2(q_-))' = 4c + 4,$$

and

$$
\begin{aligned}
|(g_c^2(q_\pm))'| &= 1 \\
4c + 4 &= -1 \\
c &= -\frac{5}{4}.
\end{aligned}
$$

(Recall, for the second step, that we consider $c < -\frac{3}{4}$ so that we cannot have $4c + 4 = 1$.) So the 2-cycle of g_c is attracting when $-\frac{5}{4} < c < -\frac{3}{4}$, neutral when $c = -\frac{5}{4}$, and repelling when $c < -\frac{5}{4}$.

Once again we summarize our observations as in Devaney [2, p.65]:

Proposition 5.6 (The Second Bifurcation)**.** *For the family $g_c(x) = x^2 + c$:*

1. *For $-\frac{3}{4} < c < \frac{1}{4}$, g_c has an attracting fixed point at p_- and no 2-cycles.*
2. *For $c = -\frac{3}{4}$, g_c has a neutral fixed point at $p_- = q_+ = q_-$ and no 2-cycles.*
3. *For $-\frac{5}{4} < c < -\frac{3}{4}$, g_c has repelling fixed points at p_+ and p_- and an attracting 2-cycle at q_+ and q_-.*

6 Orbit Diagrams

So far we have examined g_c for fixed values of c and the orbits of all possible points $x \in \mathbb{R}$. We have seen g_c go from no fixed point to one fixed point; through a bifurcation to two fixed points of different character; one of these fixed points then giving rise to a 2-cycle through another bifurcation. Do cycles of higher order keep appearing as c decreases further? To answer this question we will shift our approach examining g_c under iteration and let c vary while fixing a seed. In particular we will use *orbit diagrams*, which is another illustrative method used to examine the behavior of g_c. Here, the parameter c is plotted on the horizontal axis while the *asymptotic orbit* of a chosen seed is plotted on the vertical axis. By asymptotic orbit we mean the orbit after a few iterations so it reaches its eventual behavior. See Figure 9 for an example: Here we use the quadratic function $g_c(x)$, with seed 0, on the interval $c \in \left[-\frac{4}{3}, \frac{1}{4}\right]$, i.e., from the single fixed point $p^* = \frac{1}{2}$ just below our last examined value of c; when $c = -\frac{5}{4}$. Python code for the orbit diagrams given for the remainder of this thesis can be found in Appendix C.

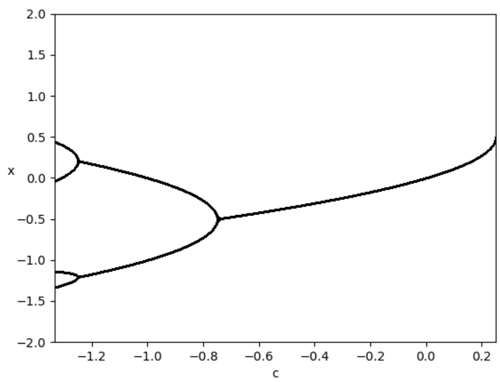


Figure 9: Orbit diagram for $g_c(x) = x^2$ with seed $x_0 = 0$

7 Critical Point

In this section we will go into critical points and explore their significance with the use of the Schwarzian derivative. This will permit us to choose a single point to examine under g_c using orbit diagrams, but still understand a substantial part of its dynamics. We begin with two definitions:

Definition 7.1. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f \in \mathcal{C}^1$. A point x_0 is a *critical point* of f if $f'(x_0) = 0$.

Definition 7.2. The *Schwarzian derivative* of a function $f \in \mathcal{C}^3$ is

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Connected to the Schwarzian derivative is the *pre-Schwarzian derivative* Pf of f. In particular, Pf is the logarithmic derivative of f', i.e., $Pf = \frac{f''}{f'}$. Moreover, we have $Sf = (Pf)' - \frac{1}{2}(Pf)^2$. Both the Schwarzian- and the pre-Schwarzian derivative are important tools in the study of geometric properties for complex-valued functions. For further insight we refer, for instance, to the article *Pre-Schwarzian and Schwarzian Derivatives of Harmonic Mappings* by R. Hernández and M.J. Martín [3].

As we will see, we are not interested in the particular value for the Schwarzian derivative of a function at a given point, but rather its sign for all points in a given domain: namely when the Schwarzian derivative of a function is negative. If for a function f, $Sf(x) < 0$ for all x we write $Sf < 0$. This property, regarding the sign of the Schwarzian derivative, is in fact preserved by composition of functions, and thus by iteration. In particular:

Proposition 7.3 (Chain Rule for Schwarzian Derivatives)**.** *Suppose f_1 and f_2 are real-valued functions and that $f_1, f_2 \in \mathcal{C}^3$. Then*

$$S(f_1 \circ f_2)(x) = Sf_1(f_2(x)) \cdot (f_2'(x))^2 + Sf_2(x).$$

Proof. We differentiate the three times using the Chain Rule for ordinary derivatives:

$$\begin{aligned}(f_1 \circ f_2)'(x) &= f_1'(f_2(x)) \cdot f_2'(x) \\ (f_1 \circ f_2)''(x) &= f_1''(f_2(x)) \cdot (f_2'(x))^2 + f_1'(f_2(x)) \cdot f_2''(x) \\ (f_1 \circ f_2)'''(x) &= f_1'''(f_2(x)) \cdot (f_2'(x))^3 + 3f_1''(f_2(x)) \cdot f_2''(x) \cdot f_2'(x) + f_1'(f_2(x)) \cdot f_2'''(x).\end{aligned}$$

Inserting this into the Schwarzian derivative and performing some tedious calculations (see Appendix A) gives the desired formula. □

This has the immediate consequence:

Corollary 7.4. *Suppose $Sf_1 < 0$ and $Sf_2 < 0$. Then $S(f_1 \circ f_2) < 0$. In particular if $Sf < 0$, then $Sf^n < 0$.*

For a function which has negative Schwarzian derivative we also have the following property for its derivative:

Theorem 7.5 (Schwarzian Min-Max Principle). *[2, p. 161-162] For a function $f \in \mathcal{C}^3$, suppose $Sf < 0$. Then f' cannot have a positive local minimum or a negative local maximum.*

Proof. Suppose x_0 is a critical point of f', so that $f''(x_0) = 0$. Suppose also that $f'(x_0) \neq 0$ (which is done without loss of generalisation since we will consider points where f' is positive or negative, respectively). Then we have

$$\begin{aligned} &Sf(x_0) < 0 \\ \Rightarrow &\frac{f'''(x_0)}{f'(x_0)} < 0.\end{aligned}$$

Now, to examine the local minimum and maximum of f' (not f itself) we examine the second derivative.

If f' has a positive local minimum at x_0, then $f'(x_0) > 0$ and its second derivative f''' must be non-negative, i.e., $f'''(x_0) \geq 0$. Thus we have

$$\frac{f'''(x_0)}{f'(x_0)} \geq 0.$$

This contradicts $Sf < 0$. Similarly, if f' has a negative local maximum at x_0, then $f'(x_0) < 0$ and $f'''(x_0) \leq 0$. Once again yielding a contradiction. □

To further understand the significance of critical points we introduce yet another definition.

Definition 7.6. Suppose x_0 is an attracting fixed point for the function f. The *basin of attraction* of x_0 is the set of all points whose orbits under f tend to x_0. The *immediate basin of attraction* of x_0 is the largest interval containing x_0 that belongs to the basin of attraction.

Similarly, basins of attraction for attracting cycles of period n are defined using f^n instead of f. Note that the Attracting Fixed Point Theorem 2.9 guarantees the existence of immediate basins of attraction for attracting fixed points and cycles.

Now we can formulate the connection between attracting periodic points, fixed points and the Schwarzian derivative:

Theorem 7.7. *For a function $f \in \mathcal{C}^3$, suppose $Sf < 0$. If x_0 is an attracting periodic point for the function f, then either the immediate basin of attraction of x_0 extends to $+\infty$ or $-\infty$, or else, there is a critical point of f whose orbit is attracted to the orbit of x_0.*

Proof. We will prove the theorem for the case of a periodic point having period 1, i.e., being a fixed point. So for an attracting fixed point p we will show that its immediate basin either contains a critical point or extends to infinity.

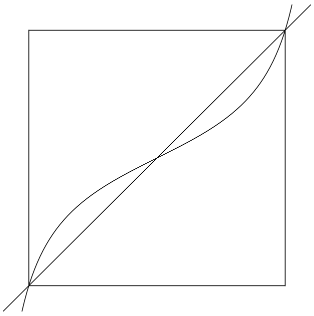
First, we observe that the immediate basin of attraction of p must be an open interval. Otherwise, if the interval were closed, then by the continuity of f, the basin could be extended beyond its endpoints. Hence, let us denote the immediate basin of p by (α, β). If either α or β are infinite we are done, so suppose not, i.e., suppose that both α and β are finite.

Since f maps the interval (α, β) to itself, it must preserve at least one of its endpoints[2]. That is, $f(\alpha)$ and $f(\beta)$ need both be either α or β (not necessarily the same). This yields the four possibilities:

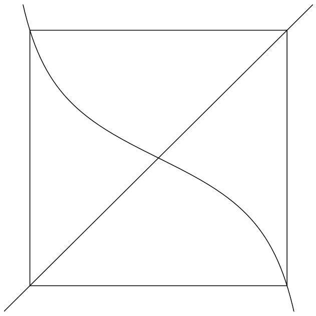
$$\begin{cases} \text{Case 1: } f(\alpha) = \alpha,\ f(\beta) = \beta, \\ \text{Case 2: } f(\alpha) = \beta,\ f(\beta) = \alpha, \\ \text{Case 3: } f(\alpha) = \alpha,\ f(\beta) = \alpha, \\ \text{Case 4: } f(\alpha) = \beta,\ f(\beta) = \beta, \end{cases}$$

[2] If $f(\alpha) \subset (\alpha, \beta)$ then α would be attracted to p, and thus in (α, β) which is a contradiction. On the other hand, α must be mapped to $[\alpha, \beta]$ by continuity of f. The same arguments holds for β.

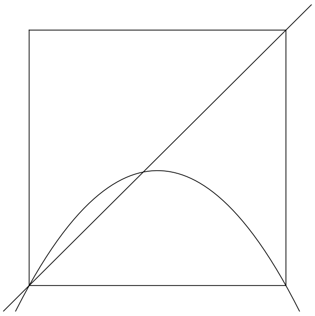
which are illustrated in Figure 10.



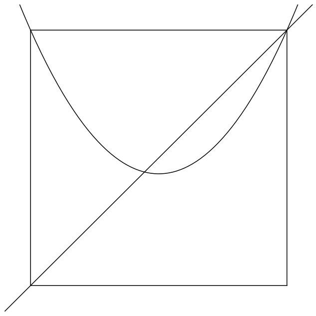
(a) Case 1: $f(\alpha) = \alpha$, $f(\beta) = \beta$



(b) Case 2: $f(\alpha) = \beta$, $f(\beta) = \alpha$



(c) Case 3: $f(\alpha) = \alpha$, $f(\beta) = \alpha$

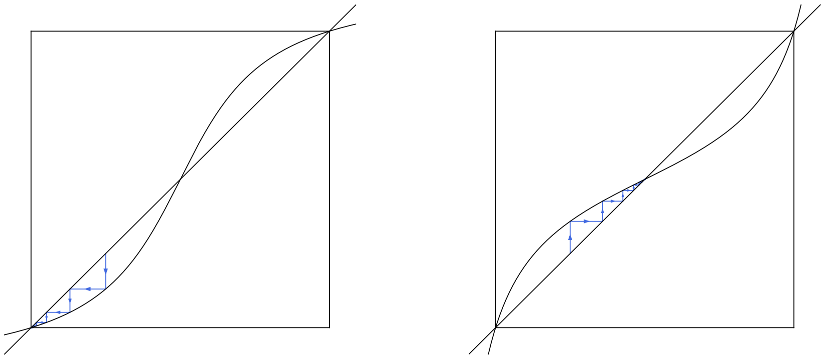


(d) Case 4: $f(\alpha) = \beta$, $f(\beta) = \beta$

Figure 10: The four possible cases for the immediate basin of attraction

In Case 3 and 4, f must have a maximum or minimum in (α, β). This point, x_0, satisfies $f'(x_0) = 0$ and thus constitute the sought critical point which is attracted to p.

Moving on, we will now consider Case 1. We claim that $f(x) > x$ in (α, p), which can be motivated as follows: First, we cannot have $f(x) = x$ in (α, p) since that would mean we have a second fixed point in the immediate basin of attraction of p, which is impossible. Further, if $f(x) < x$ for all x in (α, p), then p would not constitute an attracting fixed point, which can be shown through graphical analysis (see Figure 11). Thus we must have $f(x) > x$ for all x in (α, p). Arguing similarly, we have that $f(x) < x$ for all x in (p, β). We have thereby verified that p is indeed attracting.



(a) $f(x) < x$ in (α, p) (b) $f(x) > x$ in (α, p)

Figure 11: Case 1: graphical analysis for the character of p

Now, by the Mean Value Theorem we have that there is a point γ in (α, p) such that

$$f'(\gamma) = \frac{f(\alpha) - f(p)}{\alpha - p} = \frac{\alpha - p}{\alpha - p} = 1.$$

Since p is an attracting fixed point we also have $f'(p) < 1$, by Definition 2.7, so $\gamma \neq p$. Once again, arguing similarly, we have the existence of a point ϑ in (p, β) such that $f'(\vartheta) = 1$.

So in the interval $[\gamma, \vartheta]$ (where p is an interior point) we have $f'(\gamma) = f'(\vartheta) = 1$ and $f'(p) < 1$. Moreover, by the Schwarzian Min-Max Principle 7.5, f' cannot have a positive local minimum. Hence, f' must become negative in $[\gamma, \vartheta]$. So, by continuity of f', there exists a point τ in (γ, ϑ) such that $f'(\tau) = 0$. We have thus found a critical point in the immediate basin of attraction of p.

To address Case 2, we will consider $\hat{f}(x) := f^2(x)$ instead of $f(x)$. The fixed point p remains attracting for $\hat{f}$, and the immediate basin of attraction of p remains to be (α, β). By the Chain Rule for Schwarzian Derivatives 7.3 we also have that $S\hat{f} < 0$. Under $\hat{f}$ we have $\hat{f}(\alpha) = \alpha$ and $\hat{f}(\beta) = \beta$. Therefore, we have the same setting as in Case 1, so the arguments above show that $\hat{f}$ has a critical point $\hat{x}$ in (α, β). Since $0 = (\hat{f})'(\hat{x}) = f'(f(\hat{x})) \cdot f'(\hat{x})$, either $\hat{x}$ or $f(\hat{x})$ is a critical point of f in (α, β). □

8 The Quadratic Family of Functions (Part 2)

So, what seed should we fix for g_c to examine the behavior under iteration using orbit diagrams? With respect to Section 7 we begin by computing:

$$\begin{aligned} g_c'(x) &= 0 \\ 2x &= 0. \end{aligned}$$

Thus $x_0 := 0$ is the only critical point of g_c. Furthermore, we compute the Schwarzian derivative (see Definition 7.2) of g_c as follows:

$$\begin{aligned} Sg_c(x) &= \frac{g_c'''(x)}{g_c'(x)} - \frac{3}{2}\left(\frac{g_c''(x)}{g_c'(x)}\right)^2 \\ &= -\frac{3}{2x^2}. \end{aligned}$$

So $Sg_c < 0$, and we may apply Theorem 7.7. Since, by Proposition 5.2, we have that if $|x|$ is sufficiently large the orbit of x under g_c tends to infinity, no basin of attraction extends to $+\infty$ or $-\infty$. Hence, since 0 is the only critical point, its orbit must "find" the orbit of every existing attracting cycle. Moreover, this means g_c has at most one attracting cycle.

With the choice of $x_0 = 0$ we will, consequently, be able to see the attracting cycles of g_c as we vary c using orbit diagrams. To begin we consider c in the interval $[-2, \frac{1}{4}]$, so that g_c makes a transition from a type of behavior we already inspected, into something we have yet to understand. The full orbit diagram for this case is shown in Figure 12.

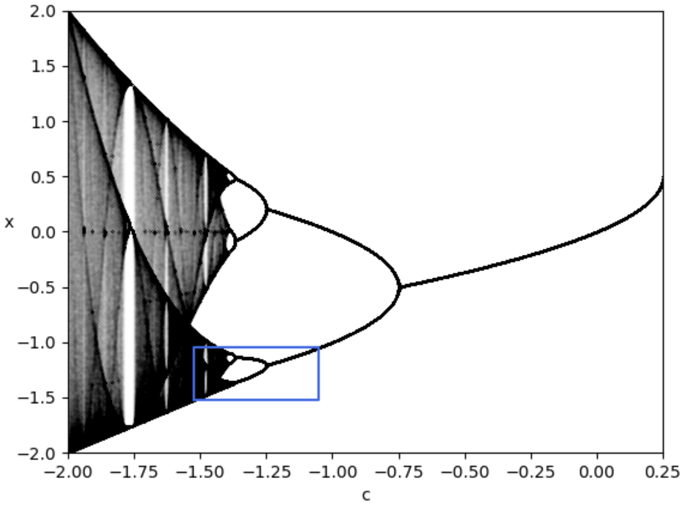
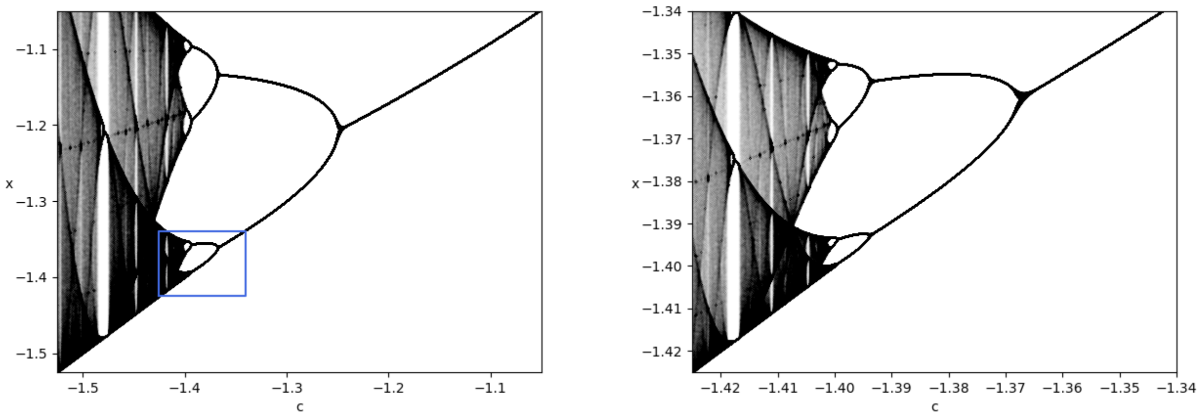


Figure 12: The orbit diagram for $g_c(x) = x^2 + c$ with seed $x_0 = 0$

First, notice that there is exactly one point over c in the interval $[-\frac{3}{4}, \frac{1}{4}]$. This single point is exactly the attracting fixed point the orbit of 0 is attracted to. Analogously the two points over $-\frac{5}{4} < c < -\frac{3}{4}$ corresponds to the attracting 2-cycle born from the bifurcation at $c = -\frac{3}{4}$. All these observations corresponds exactly as expected to what we know from Proposition 5.6.



(a) First magnification

(b) Second magnification

Figure 13: Magnifications of the orbit diagram for $g_c(x) = x^2 + c$ with seed $x_0 = 0$

If we zoom in on the marked square in Figure 12 we arrive at the image in Figure 13(a), and if we in turn zoom in on the square in this image we get Figure 13(b).

These consecutive magnifications suggest that period doubling bifurcations keep occurring as c decreases further, i.e., that periodic points of order $2, 2^2, 2^3, 2^4, \ldots$ appear. On the contrary, slightly below $c = -1.75$ in Figure 12 there is a white area with three crossing blacks regions. Because of its appearance this is called a *period-3 window*.

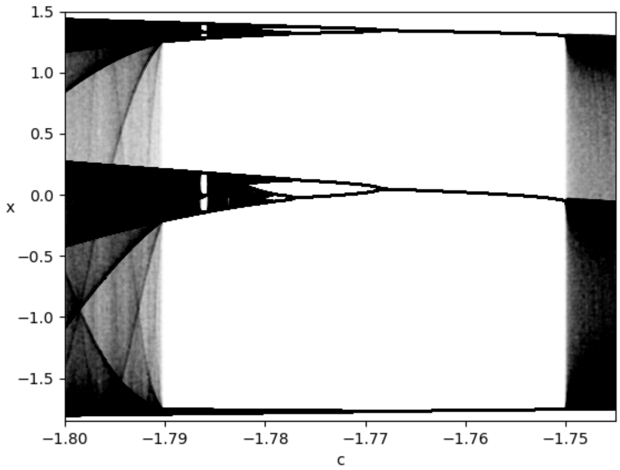
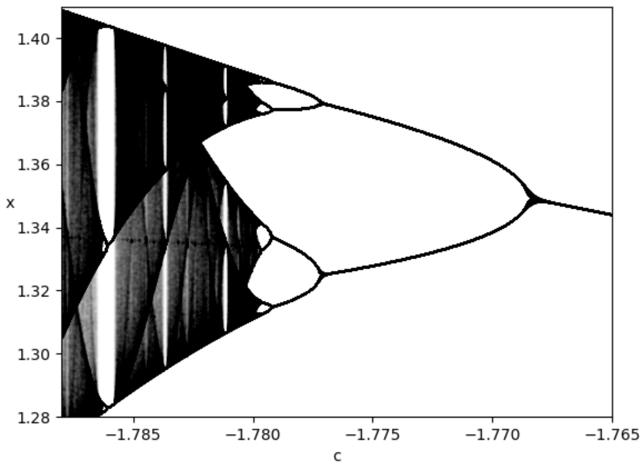


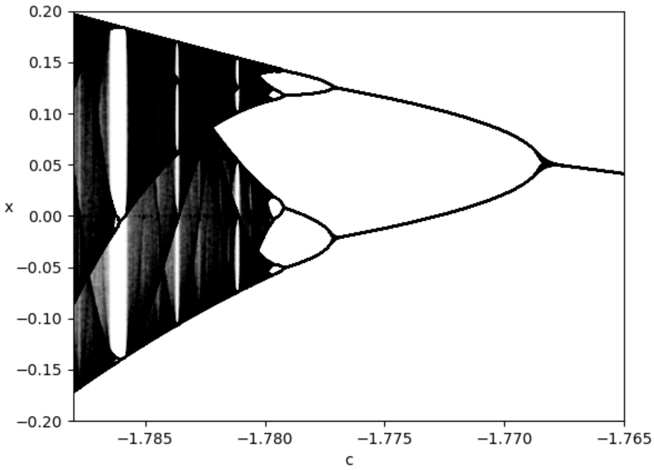
Figure 14: Orbit diagram for $g_c(x) = x^2 + c$ with seed $x_0 = 0$; magnified around period-3 window

Under magnification (see Figure 14) it looks like these three regions in fact begins as an attracting 3-cycle, which then undergo a series of period doubling bifurcations.

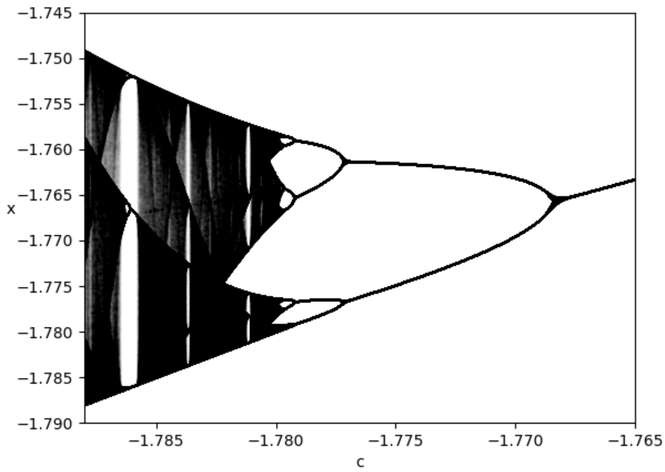
Zooming in on the three points constituting this cycle we indeed see a series of period doubling bifurcations. In particular we note that the magnified images of the 3-cycle in Figure 15 resemble the original orbit diagram in Figure 12. However, since each bifurcation here follows a 3-cycle we now get cycles of order $3 \cdot 2, 3 \cdot 2^2, 3 \cdot 2^3, 3 \cdot 2^4, \ldots$.



(a) Magnification around upper periodic point



(b) Magnification around middle periodic point



(c) Magnification around lower periodic point

Figure 15: Orbit diagram for $g_c(x) = x^2 + c$ with seed $x_0 = 0$; magnified around period-3 points

9 The Period-3 Theorem

This leads to yet another question: Do periodic points of every order exist for the quadratic family of functions? The answer is yes, given that we have found a cycle of period 3. This follows from Sharkovsky's Theorem, which uses the following peculiar ordering of the natural numbers:

$$\begin{aligned}&3, 5, 7, 9, \dots\\&2 \cdot 3, 2 \cdot 5, 2 \cdot 7, 2 \cdot 9, \dots\\&2^2 \cdot 3, 2^2 \cdot 5, 2^2 \cdot 7, 2^2 \cdot 9, \dots\\&2^3 \cdot 3, 2^3 \cdot 5, 2^3 \cdot 7, 2^3 \cdot 9, \dots\\&\qquad\vdots\\&\dots, 2^n, \dots, 2^3, 2^2, 2, 1.\end{aligned}$$

This is called the *Sharkovsky ordering.*

Theorem 9.1. *[Sharkovsky's Theorem] Suppose a function $f : \mathbb{R} \to \mathbb{R}$ is continuous. Suppose that f has a periodic point of prime period n and that n precedes k in the Sharkovsky ordering. Then f also has a periodic point of prime period k.*

As Devaney [2, p. 144] we note that the numbers of the form 2^n form the tail of the Sharkovsky ordering, meaning if a function f only has finitely many periodic point, then they all have periods of a power of 2.

We will not prove the general version of Sharkovsky's Theorem. However, we will prove the following Corollary, which addresses the case when a cycle of period 3 has been found, as we have for g_c.

Corollary 9.2. *[The Period-3 Theorem]* [3] *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Suppose also that f has a periodic point of prime period 3. Then f also has periodic points of all other prime periods.*

To prove this Corollary we first make two observations.

Observation 1. Suppose $I = [\alpha, \beta]$ and $J = [\gamma, \vartheta]$ are closed intervals and that $I \subset J$. If f is a continuous and $f(I) \supset J$, then f has a fixed point in I. (This follows from Theorem 2.5.)

[3] This theorem appeared in the article "Period Three Implies Chaos" by Li and Yorke published 1975 [4], and caused global recognition of Sharkovsky's Theorem, which had been published more than ten years earlier by A.N. Sharkovsky [6]. Both articles can be found online (Sharkovsky's in an English translation [7]) but once again, we choose to follow the exposition given in Devaney [2].

Observation 2. Suppose I and J are two closed intervals and $f(I) \supset J$. Then there is a closed subinterval $I' \subset I$ such that $f(I') = J$.

Proof of corollary. Suppose f has a 3-cycle given by

$$\alpha \mapsto \beta \mapsto \gamma \mapsto \alpha.$$

Assuming α is the leftmost point on the orbit, we have two possibilities; either $\alpha < \beta < \gamma$ or $\alpha < \gamma < \beta$. We assume the former; the latter case may be dealt with similarly.

Let $I_0 = [\alpha, \beta]$ and $I_1 = [\beta, \gamma]$. First, notice that $f(\beta) = \gamma$ and $f(\gamma) = \alpha$. This together with the order of the points; $\alpha < \beta < \gamma$, and the continuity of f gives $f(I_1) \supset I_1$. By Observation 1, f has a fixed point in I_1 so we have found a cycle of period 1. Similarly, $f(I_0) \supset I_1$ and $f(I_1) \supset I_0$, so there is a 2-cycle with one point in each interval.

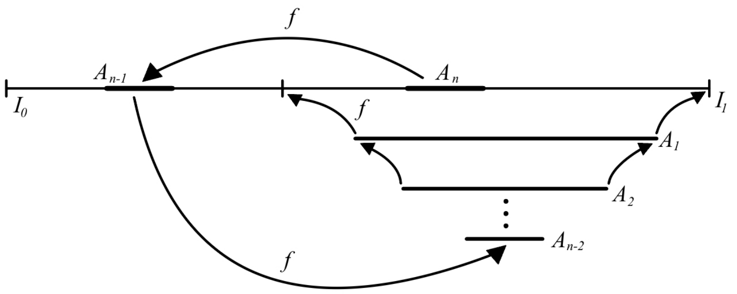


Figure 16: Construction of the subintervals $A_1, \dots, A_n$

Next, we will produce a cycle of period $n > 3$. To start, we choose the closed subinterval $A_1 \subset I_1$, satisfying $f(A_1) = I_1$. Such a subinterval exists, by Observation 2, since $f(I_1) \supset I_1$. Analogously, since $A_1 \subset I_1$ and $f(A_1) = I_1$, so that $f(A_1) \supset A_1$, we find a closed subinterval $A_2 \subset A_1$ satisfying $f(A_2) = A_1$. Through this construction we have that $A_2 \subset A_1 \subset I_1$ and $f^2(A_2) = I_1$. Continuing in this fashion, for a total of $n - 2$ steps, we arrive at a collection of closed subintervals

$$A_{n-2} \subset A_{n-3} \subset \cdots \subset A_2 \subset A_1 \subset I_1,$$

such that $f(A_i) = A_{i-1}$ for $i = 2, \dots, n-2$ and $f(A_1) = I_1$. In particular, $f^{n-2}(A_{n-2}) = I_1$ and $A_{n-2} \subset I_1$. Pairing the latter observation with the fact that $I_1 \subset f(I_0)$, we now find that there is a closed subinterval $A_{n-1} \subset I_0$ such that

$f(A_{n-1}) = A_{n-2}$. Conversely, since $f(I_1) \supset I_0 \supset A_{n-1}$, we obtain our last closed subinterval $A_n \subset I_1$, which satisfies $f(A_n) = A_{n-1}$. Meaning

$$A_n \xrightarrow{f} A_{n-1} \xrightarrow{f} \cdots \xrightarrow{f} A_1 \xrightarrow{f} I_1$$

so that $f^n(A_n) = I_1$ (see Figure 16). Since $A_n \subset I_1$, this means there is a point $x_0 \in A_n$, which is fixed under f^n, by Observation 1. Hence x_0 has period n. To verify that it has prime period n, we note that $f(x_0) \subset I_0$, but $f^i(x_0) \in I_1$ for $i = 2, \dots n$. Thus x_0 has period $\geq n$, i.e., prime period n. □

So, under the surface of the period-3 window (see Figure 14) we have that there exist cycles of every order. To, gain a bit of insight into the behavior of the periodic points constituting these cycles we will fix a c-value in the interval for the attracting 3-cycle and examine which points are attracted to this cycle - and which are not.

10 The Quadratic Family of Functions (Part 3)

We choose to fix the parameter c such that c is nonzero and satisfies

$$\begin{aligned} 0 &= g_c^3(0) \\ 0 &= (c^2 + c)^2 + c. \end{aligned} \tag{9}$$

We denote this c by $c_\triangle$.[4]

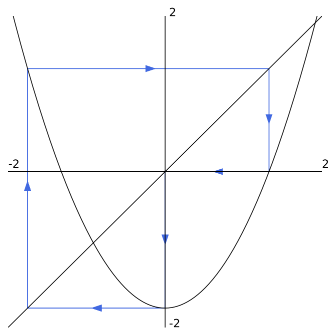


Figure 17: Attracting 3-cycle of $g_{c_\triangle}$

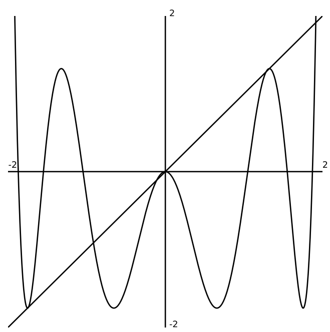


Figure 18: Graph of $g^3_{c_\triangle}$

[4]The exact form of $c_\triangle$ is a complicated expression which we here omit. Approximately we have $c_\triangle \approx -1.7548776662\ldots$

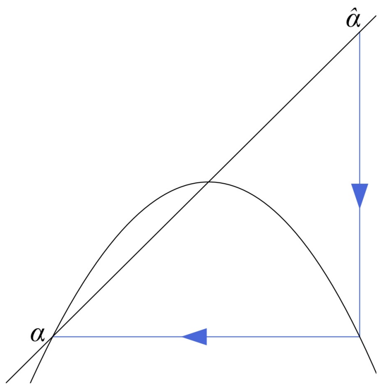
The condition (9) means that 0 lies on the attracting 3-cycle

$$0 \mapsto c_\vartriangle \mapsto c_\vartriangle^2 + c_\vartriangle \mapsto 0 \tag{10}$$

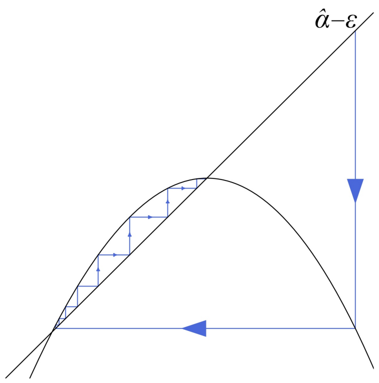
(see Figure 17). Considering the graph of $g_{c_\vartriangle}^3$ in Figure 18, although difficult to see with the naked eye, it can be noted that $g_{c_\vartriangle}^3$ has eight fixed points. Apart from the two fixed points of $g_{c_\vartriangle}$ (p_+ and p_- from Section 5) and the attracting 3-cycle (10), there is a repelling 3-cycle. We denote this cycle by

$$\alpha \mapsto \beta \mapsto \gamma \mapsto \alpha,$$

where $\gamma < \beta < \alpha$.



(a) Orbit of seed $x_0 = \hat{\alpha}$ under $g_{c_\vartriangle}^3$



(b) Orbit of seed $x_0 = \hat{\alpha} - \varepsilon$ under $g_{c_\vartriangle}^3$ (where $\varepsilon > 0$)

Figure 19: Magnification of $g_{c_\vartriangle}^3$ around α and $\hat{\alpha}$

Zooming in to a neighbourhood of the point α (see Figure 19(a)) we note a point $\hat{\alpha}$ satisfying $g_{c_\vartriangle}^3(\hat{\alpha}) = \alpha$. Furthermore, through graphical analysis (see Figure 19(b)[5]) it can be seen that any point in the interval $(\alpha, \hat{\alpha})$ has an orbit that under $g_{c_\vartriangle}$ tends to the attracting fixed point of $g_{c_\vartriangle}^3$, i.e. to the attracting 3-cycle. Similarly, there are points $\hat{\beta}$ and $\hat{\gamma}$ such that $g_{c_\vartriangle}^3(\hat{\beta}) = \beta$ and $g_{c_\vartriangle}^3(\hat{\gamma}) = \gamma$, respectively. Moreover, the orbit of any point in $(\beta, \hat{\beta})$ or $(\hat{\gamma}, \gamma)$ tends to the attracting 3-cycle[6]. In particular, there are no cycles contained in any of the intervals $(\alpha, \hat{\alpha})$, $(\beta, \hat{\beta})$ or $(\hat{\gamma}, \gamma)$.

Since we want to extend our understanding of cycles beyond period 3 we are

[5]Note that ε also need to be small enough so the seed $x_0 = \hat{\alpha} - \varepsilon$ remains in the interval $(\alpha, \hat{\alpha})$.

[6]The reversed order of the interval $(\hat{\gamma}, \gamma)$ is caused by γ lying to the right of the neighbouring periodic point, as opposed to α and β lying to the left of their respective neighbouring periodic points.

therefore interested in what happens outside these intervals. However, before we continue, we examine the images of these intervals under $g_{c_\triangle}$ to understand of how these intervals are mapped each iteration.

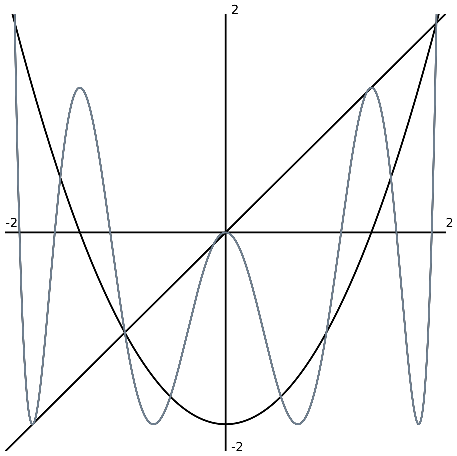


Figure 20: Graph of $g_{c_\triangle}$ and $g^3_{c_\triangle}$

Superimposing the graphs of $g_{c_\triangle}$ and $g^3_{c_\triangle}$ (see Figure 20) we may make the following observations:

$$g_{c_\triangle}(\hat{\alpha}) = \hat{\beta}$$
$$g_{c_\triangle}(\hat{\gamma}) = \hat{\alpha}$$
$$g_{c_\triangle}(\hat{\beta}) = \gamma$$

and furthermore:

$$\begin{aligned} g_{c_\triangle}([\alpha, \hat{\alpha}]) &= [\beta, \hat{\beta}] \\ g_{c_\triangle}([\hat{\gamma}, \gamma]) &= [\alpha, \hat{\alpha}] \\ g_{c_\triangle}([\beta, \hat{\beta}]) &\subset [\hat{\gamma}, \gamma] \end{aligned} \tag{11}$$

where the inclusion follows from the equalities $g_{c_\triangle}(\beta) = g_{c_\triangle}(\hat{\beta}) = \gamma$.

Moving on, let $I_0 = [\gamma, \beta]$ and $I_1 = [\hat{\beta}, \alpha]$. From Figure 20 and the observations in (11), we note on one hand that $g_{c_\triangle}$ maps I_1 in one-to-one fashion onto I_0. On the other hand, $g_{c_\triangle}$ takes I_0 into $I_0 \cup (\beta, \hat{\beta}) \cup I_1$. We illustrate this in Figure 21.

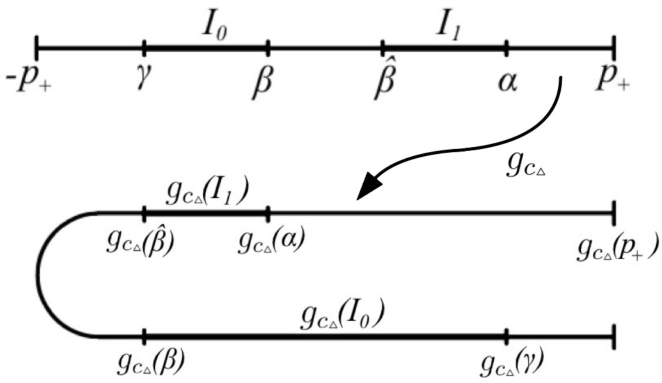


Figure 21: Image of $[-p_+, p_+]$ under $g_{c_\triangle}$

In other words,

$$g_{c_\triangle}(I_1) \supset I_0$$
$$g_{c_\triangle}(I_0) \supset I_0 \cup I_1$$

analogously to the setting in the proof of Corollary 9.2. Thus, we have periodic points of all periods in $I_0 \cup I_1$. Note that the only other possible position for cycles to exist is the intervals $(-p_+, \hat{\gamma})$ and $(\hat{\alpha}, p_+)$. However, graphical analysis (see Figure 22) shows that the orbits of any point in either of the intervals $(-p_+, \hat{\gamma})$ or $(\hat{\alpha}, p_+)$ eventually enters $[\hat{\gamma}, \hat{\alpha}]$. Hence, all cycles lie in $I_0 \cup I_1$. If an orbit ever leaves $I_0 \cup I_1$ then it must enter the interval $(\hat{\beta}, \beta)$ (see Figure 21) and thereby be attracted to the attracting 3-cycle.

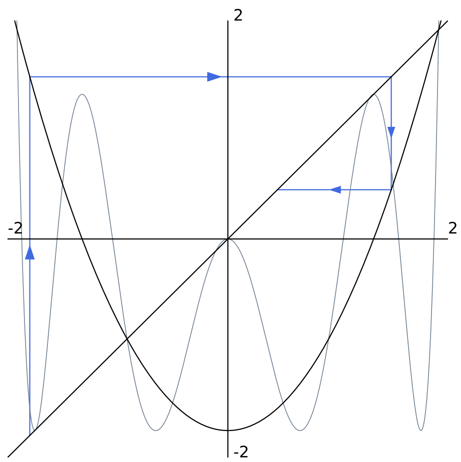


Figure 22: First steps of orbit of seed x_0 in $(-p_+, \hat{\gamma})$ under $g_{c_\triangle}$

11 Symbolic Dynamics

It remains to understand the set

$$\Lambda := \{x \in I_0 \cup I_1 \mid g_{c_\vartriangle}^n(x) \in I_0 \cup I_1,\ \forall n \in \mathbb{N}\}.$$

To do this we will use the concept of symbolic dynamics. To begin, we make the following definition:

Definition 11.1. Suppose $J_0, J_1 \in \mathbb{R}$ are closed intervals, and that Ω is another closed interval such that $\Omega \subset J_0 \cup J_1$. Suppose also that f is a continuous function and let x be a point in Ω. We define the *itinerary* of x as the infinite sequence of 0's and 1's given by

$$\mathcal{S}(x) = (s_0 s_1 s_2 \dots)$$

where $s_j = 0$ if $f^j(x) \in J_0$ and $s_j = 1$ if $f^j(x) \in J_1$.

In our case (with I_k as J_k, Λ as Ω, and $g_{c_\vartriangle}$ as f) we notice that not all sequences of 0's and 1's are possible itineraries: Suppose $x \in \Lambda$ has itinerary $(s_0 s_1 s_2 \dots)$ and that $g_{c_\vartriangle}^j(x) \in I_1$ for some j, so that $s_j = 1$. Since $g_{c_\vartriangle}(I_1) = I_0$, we have that $g_{c_\vartriangle}^{j+1}(x) \in I_0$, and in particular that $s_{j+1} = 0$. So the itineraries for $g_{c_\vartriangle}$ in Λ cannot contain two consecutive 1's. For instance, this means that there is no fixed point in I_1 since $(111\dots)$ is not an allowable itinerary. However, there is a fixed point in I_0 (namely p_-) so $(000\dots)$ is an allowable itinerary. We denote the set of possible itineraries for $g_{c_\vartriangle}$ in Λ as follows

$$\Sigma' := \{(s_0 s_1 s_2 \dots) \mid s_j \in \{0, 1\},\ \text{if } s_j = 1 \text{ then } s_{j+1} = 0\},$$

which is a subset of all possible sequences of 0's and 1's:

Definition 11.2. We define the *sequence space* on the two symbols 0 and 1 as

$$\Sigma = \{(s_0 s_1 s_2 \dots) \mid s_j \in \{0, 1\}\}.$$

Upon this set Σ we define the following function:

Definition 11.3. The *shift map* $\sigma : \Sigma \to \Sigma$ is defined by

$$\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 s_3 \dots).$$

The shift map drops the first entry from a sequence and shifts the remaining entries one position to the left. We may consider the dynamics of the shift map σ. Here, periodic points correspond to repeating sequences. A sequence with period n under σ is a sequence of the form $\mathbf{s} = (s_0 s_1 \dots s_{n-1} s_0 s_1 \dots s_{n-1} s_0 s_1 \dots)$, since

$$\begin{aligned}
\sigma(\mathbf{s}) &= (s_1 s_2 \dots s_{n-1} s_0 s_1 s_2 \dots) \\
\sigma^2(\mathbf{s}) &= (s_2 s_3 \dots s_{n-1} s_0 s_1 s_2 \dots) \\
&\vdots \\
\sigma^{n-1}(\mathbf{s}) &= (s_{n-1} s_0 \dots s_{n-1} s_0 s_1 s_2 \dots) \\
\sigma^n(\mathbf{s}) &= (s_0 s_1 \dots s_{n-1} s_0 s_1 s_2 \dots).
\end{aligned}$$

For instance, the only fixed points for σ is $(000\dots)$ and $(111\dots)$, while the points of period 2 are $(010101\dots)$ and $(101010\dots)$. Eventually periodic points are defined similarly.

The shift map σ may also be defined on Σ'. If $\mathbf{s}$ is a sequence without consecutive 1's, then so is $\sigma(\mathbf{s})$, and all other iterations of σ applied to $\mathbf{s}$. Something even more striking is that we can use the dynamics of the shift map to describe the dynamics of $g_{c_\triangle}$, we have:

Theorem 11.4. *[2, p. 153] The itinerary function $\mathcal{S} : \Lambda \to \Sigma'$ is a conjugacy between $g_{c_\triangle} : \Lambda \to \Lambda$ and $\sigma : \Sigma' \to \Sigma'$.*[7]

For further reading about the conjugacy constituted by $\mathcal{S}$ we refer to Devaney [2, p. 113-117, 151-153] and [1, p. 44-47]. Here we will focus on the fact that this relation means that $g_{c_\triangle}$ and σ are equivalent in terms of their dynamics[8], so that we may use the shift map σ to further understand $g_{c_\triangle}$. In particular, we can prove the following theorem:

Theorem 11.5. *The quadratic map $g_{c_\triangle}$ is chaotic on Λ.*

There are many possible definition of chaos, but we will use the following:

[7] In fact, $\mathcal{S}$ is a conjugacy between g_c and σ for other values of c as well (see Devaney [2, p. 115])

[8] In Section 5 we showcased how a conjugacy relates fixed points between quadratic polynomials of different form.

Definition 11.6. [1, p. 50] Let Ω be a metric space with an infinite number of elements. Then a continuous function $f : \Omega \to \Omega$ is said to be *chaotic* on Ω if:

1. f has sensitive dependence on initial conditions.
2. f has dense orbits.
3. periodic points are dense in Ω.

In Devaney [1] *topological transitivity* is used as the second condition for chaos, but as noted in [1, p. 49] these properties are equivalent for compact subsets of $\mathbb{R}$. In particular, the equivalence holds in our case.[9]

We will go through each of the conditions for chaos one at a time, and prove that $g_{c_\vartriangle}$ satisfies each one on Λ. First, we define sensitive dependence on initial conditions as follows:

Definition 11.7. Let Ω be a set. A continuous function $f : \Omega \to \Omega$ is said to have *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for any $x \in \Omega$ and any $\varepsilon > 0$, there exists a y, different from x with $|x - y| < \varepsilon$, and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

To see that $g_{c_\vartriangle}$ indeed has sensitive dependence on initial conditions on Λ we choose $\delta < |\hat{\beta} - \beta|$, i.e., δ strictly less than the distance between I_0 and I_1. If $x, y \in \Lambda$ are distinct points they must have distinct itineraries, meaning $\mathcal{S}(x) \neq \mathcal{S}(y)$. Say that $\mathcal{S}(x)$ and $\mathcal{S}(y)$ differ at the n:th spot, then $g^n_{c_\vartriangle}(x)$ and $g^n_{c_\vartriangle}(y)$ must lie in different intervals; one in I_0 and the other in I_1, i.e., on different sides of $(\beta, \hat{\beta})$. Thus we have found a δ such that

$$|g^n_{c_\vartriangle}(x) - g^n_{c_\vartriangle}(y)| > \delta.$$

To prove the two density properties we must first define a metric on Σ':

Definition 11.8. [10] Let $\mathbf{s} = (s_0 s_1 s_2 \dots)$ and $\mathbf{t} = (t_0 t_1 t_2 \dots)$ be two sequences in Σ'. We define the distance between them by

$$d[\mathbf{s}, \mathbf{t}] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

[9] Despite the similarities between [1] and [2], both by Devaney, we leave the latter for the moment in favor of the former, since the [1] provide a more "rigours overview" of chaos with intuition better suited for our purpose.

[10] To see that d indeed is a metric see [1, p. 40]

Notice, since $|s_i - t_i|$ is either 0 or 1, this series is bounded by the geometric series

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

and thereby converges.

Moreover, this distance function d implies that two points in Σ' are "close together" if their first few entries agree. More precisely:

Theorem 11.9 (The Proximity Theorem)**.** *Let* $\mathbf{s} = (s_0 s_1 s_2 \dots)$ *and* $\mathbf{t} = (t_0 t_1 t_2 \dots)$ *be two sequences in* Σ'*. Suppose* $s_i = t_i$ *for* $i = 0, 1, \dots n$*. Then* $d[\mathbf{s}, \mathbf{t}] \leq \frac{1}{2^n}$*. Conversely, if* $d[\mathbf{s}, \mathbf{t}] < \frac{1}{2^n}$*, then* $s_i = t_i$ *for* $i \leq n$*.*

Proof. If $s_i = t_i$ for $i \leq n$, then

$$\begin{aligned} d[\mathbf{s}, \mathbf{t}] &= \sum_{i=0}^{n} \frac{|s_i - s_i|}{2^i} + \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \\ &= \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \\ &\leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} \\ &= \frac{1}{2^n}. \end{aligned}$$

Contrarily, if $s_j \neq t_j$ for some $j \leq n$, then

$$d[\mathbf{s}, \mathbf{t}] \geq \frac{1}{2^j} \geq \frac{1}{2^n}.$$

So, if $d[\mathbf{s}, \mathbf{t}] < \frac{1}{2^n}$, then $s_i = t_i$ for $i \leq n$. □

Next, we define density:

Definition 11.10. Let X be a metric space with the metric d_X. A subset $S \subset X$ is said to be dense in X if, for any $\varepsilon > 0$ and $x \in X$, there is some $s \in S$ such that $d_X(x, s) < \varepsilon$.

Or equivalently:

Definition 11.11. Let X be a metric space. A subset $S \subset X$ is dense in X if, for any $x \in X$, there is a sequence $\{s_n\}_{n \in \mathbb{N}} \subset S$ such that

$$\lim_{n \to \infty} s_n = x.$$

Now, we prove that $g_{c_\triangle}$ has dense orbits using σ. We show that there exists a point in Σ' whose orbit under σ comes arbitrarily close to any given sequence in Σ'. This orbit cannot be periodic, since it has to come arbitrarily close to different types of periodic orbits as well as non-periodic orbits. Thus we construct the following sequence:

$$\hat{\mathbf{s}} = (\ \underbrace{0\ 1}_{\text{1 blocks}}\ \ \underbrace{00\ 10\ 01}_{\text{2 blocks}}\ \underbrace{000\ 001\ 010\ 100\ 101}_{\text{3 blocks}}\ \ \underbrace{\ldots}_{\text{4 blocks}}\ )$$

where we list all blocks of Σ' of length $1, 2, 3, \ldots$. Let $\varepsilon > 0$ be given and let $\mathbf{s} \in \Sigma'$ be an arbitrary sequence whose n first entries are $s_0, \ldots, s_{n-1}$, where we choose n such that $\frac{1}{2^n} < \varepsilon$. By construction the n-block constituted by $s_0, \ldots, s_{n-1}$ is contained in $\hat{\mathbf{s}}$. In particular, $\sigma^m(\hat{\mathbf{s}}) = (s_0 \ldots s_{n-1} \ldots)$ for some $m \geq 0$, so, by the Proximity Theorem 11.9, $d[\mathbf{s}, \sigma^m(\hat{\mathbf{s}})] \leq \frac{1}{2^n}$. Since $\frac{1}{2^n} < \varepsilon$, we have found an point $\mathbf{s} \in \Sigma'$ whose orbit is dense in Σ', with respect to Definition 11.10.

Next, to prove that the periodic points of Σ' under σ are dense we use Definition 11.11. So, we wish to find a sequence of periodic points $\{\tau_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \tau_n = \mathbf{s}$$

for an arbitrary point $\mathbf{s} = (s_0 s_1 s_2 \ldots) \in \Sigma'$. If we choose $\tau_n = (s_0 \ldots s_n s_0 \ldots s_n s_0 \ldots)$ so that τ_n agrees with $\mathbf{s}$ up to the n:th entry, then by the Proximity Theorem 11.9, $d[\tau_n, \mathbf{s}] \leq \frac{1}{2^n}$ so that $\tau_n \to \mathbf{s}$ as $n \to \infty$.

Furthermore, we may even find a recursive formula for the number of periodic point of each order. To begin, let $\mathcal{P}_n$ denote the set of sequences in Σ' fixed by σ^n, and let $\#\mathcal{P}_n$ denote the cardinality of $\mathcal{P}_n$. The type of sequences in $\mathcal{P}_n$ can be partitioned as follows:

$$\begin{aligned} A_n &= \{(\overline{s_0 \ldots s_{n-1}}) \in \mathcal{P}_n \mid s_0 = 0, s_{n-1} = 1\} \\ B_n &= \{(\overline{s_0 \ldots s_{n-1}}) \in \mathcal{P}_n \mid s_0 = 1, s_{n-1} = 0\} \\ C_n &= \{(\overline{s_0 \ldots s_{n-1}}) \in \mathcal{P}_n \mid s_0 = s_{n-1} = 0\} \end{aligned}$$

Recall that adjacent 1's not are allowed in Σ' so that we cannot have $s_0 = s_{n-1} = 1$. Since A_n, B_n, and C_n are mutually exclusive we have

$$\#\mathcal{P}_n = \#A_n + \#B_n + \#C_n.$$

Now, we may prove the following:

Theorem 11.12. *[2, p. 153-154] Let $\mathcal{P}_n$ be the set of sequences in Σ' fixed by the n:th iterate of the shift map, σ^n. Then the following recursive formula for the cardinality of $\mathcal{P}_n$ holds:*

$$\#\mathcal{P}_{n+2} = \#\mathcal{P}_{n+1} + \#\mathcal{P}_n$$

for $n > 0$.

Proof. Let $\mathbf{s} = (\overline{s_0 s_1 \dots s_{n+1}}) \in \mathcal{P}_{n+2}$. We will associate a unique sequence in either $\mathcal{P}_{n+1}$ or $\mathcal{P}_n$ to $\mathbf{s}$. First, suppose $s_0 = s_{n+1}$, then both these entries have to be 0. In particular $s_{n+1} = 0$, allowing s_n to be either 0 or 1. Thus, a repeating sequence of length $n+1$ can be determined by $\mathbf{s}$ if $s_0 = s_{n+1} = 0$; namely, $(\overline{0 s_1 \dots s_n})$ which lies in A_{n+1} if $s_n = 1$, and C_{n+1} if $s_n = 0$.

Next, if $s_0 \neq s_{n+1}$, we have two cases. On the one hand, if $s_0 = 0$ and $s_{n+1} = 1$, then $s_n = 0$. Moving further, s_{n-1} may be either 0 or 1. Similarly to above, $\mathbf{s}$ determines a repeating sequence, now of length n; $(\overline{0 s_1 \dots s_{n-1}})$, which lies in A_n if $s_{n-1} = 1$, and C_n if $s_{n-1} = 0$. On the other hand, if $s_0 = 1$ and $s_{n+1} = 0$, then s_n may be either 0 or 1. If $s_n = 0$ then $\mathbf{s}$ determines the sequence $(\overline{1 s_1 \dots s_{n-1} 0})$ in B_{n+1}. If $s_n = 1$ then $s_{n-1} = 0$ so $\mathbf{s}$ instead determines the sequence $(\overline{1 s_1 \dots s_{n-2} 0})$ in B_n.

Recalling that $\mathcal{P}_n = A_n \cup B_n \cup C_n$, we may associate a unique sequence in either $\mathcal{P}_n$ or $\mathcal{P}_{n+1}$ to any given sequence in $\mathcal{P}_{n+2}$. The converse may be proven by reversing the above process. □

To be able to use this recursive formula we need two base-cases. There is only one sequence in Σ' which is fixed by σ, namely $(00\overline{0})$. So $\#\mathcal{P}_1 = 1$. Next, $\mathcal{P}_2$ contains $(01\overline{01})$, $(10\overline{10})$, and $(00\overline{0})$, so $\#\mathcal{P}_2 = 3$. Using the recursive relation in Theorem 11.12 yields:

$$\mathcal{P}_3 = 4$$
$$\mathcal{P}_4 = 7$$
$$\mathcal{P}_5 = 11$$
$$\mathcal{P}_6 = 18$$
$$\mathcal{P}_7 = 29$$
$$\vdots$$

We have now proved that $g_{c_\triangle}$ is chaotic on Λ. What do the three properties in Definition 11.6 mean intuitively? First, if a function possesses sensitive dependence on initial conditions it is unpredictable. We can no longer use graphical analysis to examine the orbits of a given point, since the error caused by round-off grows upon iteration. So the illustrated/computed orbit may be far from the actual orbit. Secondly, if a function has dense orbits then its domain cannot be decomposed into disjoint open sets which are invariant under iteration of the function. Lastly, amidst the unpredictability we have periodic points of every order, and even more they are dense in the domain of the function.

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Appendix

A Calculations for Chain Rule for Schwarzian Derivatives

Recall that we want to prove the equality:

$$S(f_1 \circ f_2)(x) = Sf_1(f_2(x)) \cdot (f_2'(x))^2 + Sf_2 \tag{12}$$

for $f_1, f_2 \in \mathcal{C}^3$, where the Schwarzian derivative Sf is given

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \tag{13}$$

The Chain Rule for ordinary derivatives gave us

$$\begin{aligned}
(f_1 \circ f_2)'(x) &= f_1'(f_2(x)) \cdot f_2'(x) \\
(f_1 \circ f_2)''(x) &= f_1''(f_2(x)) \cdot (f_2'(x))^2 + f_1'(f_2(x)) \cdot f_2''(x) \\
(f_1 \circ f_2)'''(x) &= f_1'''(f_2(x)) \cdot (f_2'(x))^3 + 3f_1''(f_2(x)) \cdot f_2''(x) \cdot f_2'(x) + f_1'(f_2(x)) \cdot f_2'''(x).
\end{aligned} \tag{14}$$

First, we expand the right-hand side of (12) using (13) as follows:

$$\begin{aligned}
r.h.s. &= Sf_1(f_2(x)) \cdot (f_2'(x))^2 + Sf_2 \\
&= \left(\frac{f_1'''(f_2(x))}{f_1'(f_2(x))} - \frac{3}{2}\left(\frac{f_1''(f_2(x))}{f_1'(f_2(x))}\right)^2\right) \cdot (f_2'(x))^2 + \frac{f_2'''(x)}{f_2'(x)} - \frac{3}{2}\left(\frac{f_2''(x)}{f_2'(x)}\right)^2.
\end{aligned}$$

For the left-hand side of (12) we begin by simplifying one term of (13) at a time, with the use of (14). The first term becomes:

$$\begin{aligned}
\frac{(f_1 \circ f_2)'''(x)}{(f_1 \circ f_2)'(x)} &= \frac{f_1'''(f_2(x)) \cdot (f_2'(x))^3}{f_1'(f_2(x)) \cdot f_2'(x)} + 3\frac{f_1''(f_2(x)) \cdot f_2''(x) \cdot f_2'(x)}{f_1'(f_2(x)) \cdot f_2'(x)} + \frac{f_1'(f_2(x)) \cdot f_2'''(x)}{f_1'(f_2(x)) \cdot f_2'(x)} \\
&= \frac{f_1'''(f_2(x)) \cdot (f_2'(x))^2}{f_1'(f_2(x))} + 3\frac{f_1''(f_2(x)) \cdot f_2''(x)}{f_1'(f_2(x))} + \frac{f_2'''(x)}{f_2'(x)},
\end{aligned} \tag{15}$$

and the second becomes:

$$-\frac{3}{2}\left(\frac{(f_1\circ f_2)''(x)}{(f_1\circ f_2)'(x)}\right)^2 = -\frac{3}{2}\left(\frac{f_1''(f_2(x))\cdot(f_2'(x))^2}{f_1'(f_2(x))\cdot f_2'(x)} + \frac{f_1'(f_2(x))\cdot f_2''(x)}{f_1'(f_2(x))\cdot f_2'(x)}\right)^2$$

$$= -\frac{3}{2}\left(\frac{f_1''(f_2(x))\cdot f_2'(x)}{f_1'(f_2(x))} + \frac{f_2''(x)}{f_2'(x)}\right)^2$$

$$= -\frac{3}{2}\left(\frac{f_1''(f_2(x))\cdot f_2'(x)}{f_1'(f_2(x))}\right)^2 - 3\frac{f_1''(f_2(x))\cdot f_2''(x)}{f_1'(f_2(x))} - \frac{3}{2}\left(\frac{f_2''(x)}{f_2'(x)}\right)^2. \tag{16}$$

The middle terms of (15) and (16), respectively, cancel each other out, so the left-hand side of (12):

$$l.h.s. = \frac{f_1'''(f_2(x))\cdot(f_2'(x))^2}{f_1'(f_2(x))} + \frac{f_2'''(x)}{f_2'(x)} - \frac{3}{2}\left(\frac{f_1''(f_2(x))\cdot f_2'(x)}{f_1'(f_2(x))}\right)^2 - \frac{3}{2}\left(\frac{f_2''(x)}{f_2'(x)}\right)^2$$

which is the right-hand side rearranged and we are done.

B Code for Graphical Analysis

```
import matplotlib.pyplot as plt
import numpy as np
import decimal as dec

def f_cubic(x,a,_):
    try:
        return dec.Decimal(x**3+a*x)
    except TypeError:
        return x**3+a*x

def h_ab(x,a=0,b=0):
    try:
        return dec.Decimal(a)*dec.Decimal(x)+dec.Decimal(b)
    except TypeError:
        return a*x+b

def g_c(x,c,_):
    try:
        return dec.Decimal(x**2)+dec.Decimal(c)
    except TypeError:
        return x**2+c

def point_generator(func,seed,n_iter,param1=0,param2=0):
    x_val = seed
    y_val = seed
    x_points = [x_val]
    y_points = [y_val]
    for i in range(n_iter):
        try:
            if i%2 == 0:
                y_val = func(y_val,param1,param2)
            else:
                x_val = y_val       # same as x_val = func(x_val,param1,param2)
```

```
            x_points.append(x_val)
            y_points.append(y_val)
        except dec.DecimalException:
            break
    return x_points, y_points

def graphical_analysis(name,func,seed,n_iter,s,param1=0,param2=0):
    x_list, y_list = point_generator(func,seed,n_iter,param1,param2)
    plt.figure(figsize=(5.0,5.0))
    func_x_min = float(min(x_list))
    func_x_max = float(max(x_list))

    for i in range(len(x_list)-2):
        plt.plot(x_list[i:i+2],y_list[i:i+2],color="royalblue",linewidth=0.7)
        if i <= 1000:
            dx = dec.Decimal(x_list[i+1])-dec.Decimal(x_list[i])
            dy = dec.Decimal(y_list[i+1])-dec.Decimal(y_list[i])
            if not (dx.is_zero() and dy.is_zero()):
                if dx.is_zero() and np.abs(float(dy)) > .025:
                    x_cor = float(x_list[i])
                    y_cor = float(dec.Decimal(y_list[i])+dy/dec.Decimal(2))
                    arr_length = float(dy*dec.Decimal(.025))
                    arr_width = float(min(0.07,dec.Decimal(.05)*np.abs(dy)))
                    if x_cor < 10 and y_cor < 10:
                        plt.arrow(x_cor,y_cor,0,arr_length,color="royalblue",
                                  length_includes_head=True,head_width=arr_width)
                elif dy.is_zero() and np.abs(float(dx)) > .025:
                    x_cor = float(dec.Decimal(x_list[i])+dx/dec.Decimal(2))
                    y_cor = float(y_list[i])
                    arr_length = float(dx*dec.Decimal(.025))
                    arr_width = float(min(0.07,dec.Decimal(.05)*np.abs(dx)))
                    if x_cor < 10 and y_cor < 10:
                        plt.arrow(x_cor,y_cor,arr_length,0,color="royalblue",
                                  length_includes_head=True,head_width=arr_width)
```

```python
    if func_x_max > 10 and func_x_min < -10:
        id_x = np.linspace(-10,10,10000)
        plt.xlim(-10,10)
        plt.ylim(-10,10)
    elif func_x_max > 10:
        id_x = np.linspace(-1,10,10000)
        plt.xlim(-1,10)
        plt.ylim(-1,10)
    elif func_x_min < -10:
        id_x = np.linspace(-10,1,10000)
        plt.xlim(-10,1)
        plt.ylim(-10,1)
    else:
        id_x = np.linspace(-s,s,10000)
        plt.xlim(-s,s)
        plt.ylim(-s,s)

    plt.plot(id_x,id_x,color="black",linewidth=0.7)
    plt.plot(id_x,[0]*len(id_x),"black",linewidth=0.7)
    plt.plot([0]*len(id_x),id_x,"black",linewidth=0.7)
    f_y = func(id_x,param1,param2)
    plt.plot(id_x, f_y, color="black",linewidth=0.7)
    plt.savefig(name+".pdf")
    plt.close()
    plt.clf()

def plot_step_by_step():
    # Step by step graphical analysis for f(x)=x^3-x, with seed x_0=-5/4
    n_steps = [1,2,3,4,5,6,50,100,5000]
    for n_step in n_steps:
        graphical_analysis("GAnalysis"+str(n_step),f_cubic,-5/4,n_step,1.75,-1)

def plot_affine_family():
    # Graphical analysis of the Affine Family of Functions: h(x) = ax+b
    a_list = [1/4,-1/4,5/4,-5/4]
```

```python
    b_list = [0,-1/4]
    n = 0
    for b in b_list:
        for a in a_list:
            n += 1
            p = b/(1-a)
            graphical_analysis("AffineFamily"+str(n),h_ab,p-5/4,5000,2.5,a,b)

def plot_quadratic_family():
    # Graphical analysis of the Quadratic Familyof Functions

    # No fixed point
    graphical_analysis("gcNoFixed",g_c,-1/2,7500,5,1)

    # One fixed point
    p_star = 1/2
    x0_list = [p_star+1/40,-p_star,-(p_star+1/40),1/40]
    c = 1/4
    n = 0
    for x0 in x0_list:
        n += 1
        graphical_analysis("gcOneFixed"+str(n),g_c,x0,7500,0.6,c)

    # Two fixed points
    c_list = [-1/4,-3/4,-1]
    n = 0
    n_iter = 10000
    for c in c_list:
        p_plus = (1+np.sqrt(1-4*c))/2
        p_minus = (1-np.sqrt(1-4*c))/2
        graphical_analysis("gcTwoFixed1"+str(n),g_c,p_plus-1/40,n_iter,1.6,c)
        graphical_analysis("gcTwoFixed2"+str(n),g_c,p_minus-1/4,n_iter,1.6,c)
        graphical_analysis("gcTwoFixed3"+str(n),g_c,-p_plus+1/4,n_iter,1.6,c)
        n += 1
```

```python
def main():

    plot_step_by_step()
    plot_affine_family()
    plot_quadratic_family()

main()
```

C Code for Orbit Diagrams

```
import matplotlib.pyplot as plt
import numpy as np
import decimal as dec
from sympy.solvers import solve
from sympy import Symbol

c_var = Symbol('c_var')
c_values = solve((c_var**2+c_var)**2+c_var, c_var)
c = c_values[-1]

def g_c(x,c):
    try:
        y = dec.Decimal(x**2)+dec.Decimal(c)
        return y
    except TypeError:
        return x**2+c

def orbit_diagram(name,func,seed,n_iter,min_c,max_c,min_x,max_x,s1=None,s2=None):
    c_list = []
    x_list = []
    c_range = np.linspace(min_c,max_c,n_iter)

    for c in c_range:
        x = seed
        for i in range(1001):
            x = func(x,c)
            if i > 250:          # To obtain the asymptotic orbit
                c_list.append(c)
                x_list.append(x)
    plt.plot(c_list,x_list,ls='',marker='.',markersize=0.005,color='black')
    plt.plot([s1,s2,s2,s1,s1],
             [s1,s1,s2,s2,s1],color='royalblue')     # Square which to magnify
    plt.xlim(min_c,max_c)
```

```python
    plt.ylim(min_x,max_x)
    plt.xlabel('c')
    plt.ylabel('x',rotation=0)
    plt.savefig(name+'.png')
    plt.close()
    plt.clf()

def main():
    x_0 = 0
    n = 10000

    orbit_diagram('ODEx',g_c,x_0,n,-4/3,1/4,-2,2)

    orbit_diagram('ODOriginal',g_c,x_0,n,-2,1/4,-2,2,-1.525,-1.05)
    orbit_diagram('ODMagn1',g_c,x_0,3*n,-1.525,-1.05,-1.525,-1.05,-1.425,-1.34)
    orbit_diagram('ODMagn2',g_c,x_0,4*n,-1.425,-1.34,-1.425,-1.34)

    orbit_diagram('ODPeriod3',g_c,x_0,2*n,-1.8,-1.745,-1.85,1.5)
    orbit_diagram('ODPer3Magn1',g_c,x_0,6*n,-1.788,-1.765,1.28,1.41)
    orbit_diagram('ODPer3Magn2',g_c,x_0,6*n,-1.788,-1.765,-0.2,0.2)
    orbit_diagram('ODPer3Magn3',g_c,x_0,6*n,-1.788,-1.765,-1.79,-1.745)

main()
```