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The Brown Representability Theorem and Representing Reduced Cohomology Theories

av

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Abstract

The goal of this paper is to state and prove the Brown representability theorem, named after Edgar Brown, which gives sufficient conditions for when a functor from the homotopy category of connected based CW complexes to the category of pointed sets is representable. The theorem will then be used to prove a bijective correspondence between reduced cohomology theories on CW complexes and Ω-spectra.

The proof presented is based on the proof by Macerato and Slaoui in their article "The Brown Representability theorem, old and new" [6] with general lemmas that can be generalized to the ∞-categorical version of the theorem which we will not touch on. However, we correct an oversight in the paper of Macerato and Slaoui as they neglect to mention the crucial assumption of restricting to connected CW complexes.

Finding the correspondence between reduced cohomology theories and Ω-spectra is what Edgar Brown himself did in his paper "Cohomology theories" [1] with the first proof of his theorem.

Sammanfattning

Målet med denna artikel är att formulera och bevisa Browns representerbarhetssats, först bevisad av Edgar Brown. Teoremet ger tillräckliga krav för att en funktor från homotopi kategorin av sammanhängande baserade CW-komplex till kategorin av baserade mängder ska vara representerbar. Denna sats kommer sedan ge oss en bijektiv korrespondens mellan reducerade kohomologi teorier på CW-komplex och Ω-spektra.

Beviset av satsen är baserad på beviset av Macerato och Slaoui i deras artikel "The Brown Representability theorem, old and new" [6] som innehåller generella lemman som kan generaliseras till den ∞-kategoriska versionen av satsen som vi inte kommer ge oss in på. Dock så tar vi upp varför hypotesen om att begränsa till de sammanhängande komplexen behövs och var i beviset detta används vilket saknas i deras artikel.

Korrespondensen mellan kohomologi teorier och Ω-spektra var det Edgar Brown själv tog upp som följdsats i hans artikel "Cohomology theories" [1] med det första beviset av satsen.

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Contents

1 Introduction and Preliminaries

We start by considering the categories which shall be of interest

Definition 1.1. The category of pointed sets $\mathbf{Set}_*$ is the category whose objects are sets with a distinguished element, also called basepoints, and whose morphisms are functions that preserves the distinguished element.

We will usually denote sets with capital letters such as A, B and the corresponding basepoint as a_0, b_0. It is also common to write it as a pair (A, a_0) so that the morphism $f : (A, a_0) \to (B, b_0)$ means a function from A to B so that $f(a_0) = b_0$.

Every category with objects that can be given the structure of sets with a special element and morphisms preserving those elements has a forgetful functor to $\mathbf{Set}_*$. For example the category of groups $\mathbf{Grp}$ where we view the identity element of each group as the distinguished element. Since group homomorphisms map identities to identities, $\mathbf{Grp}$ has a forgetful functor to $\mathbf{Set}_*$. Similarly the category of pointed topological spaces $\mathbf{Top}_*$ has a forgetful functor to $\mathbf{Set}_*$ where the basepoint of the space is seen as the distinguished element of the underlying set.

Definition 1.2. The homotopy category of pointed CW complexes, denoted by $\mathrm{h}\mathbf{CW}_*$ is the category whose objects consists of pointed CW complexes and whose morphisms are based homotopy classes of based continuous maps which we denote by $[X, Y]_* := \hom_{\mathrm{h}\mathbf{CW}_*}(X, Y)$. The subcategory of connected CW complexes will be denoted $\mathrm{h}\mathbf{CW}_*^c$.

The category $\mathrm{h}\mathbf{CW}_*$ is locally small so we may consider the covariant functors $\hom_{\mathrm{h}\mathbf{CW}_*}(X, -) = [X, -]_*$ and contravariant $[-, X]_*$ from $\mathrm{h}\mathbf{CW}_*$ to $\mathbf{Set}_*$ for any pointed CW complex X. The distinguished element in $[X, Y]_*$ being the homotopy class of the constant map $c(x) = y_0$ where y_0 is the basepoint of Y. The functors are defined on morphisms as follows. If $f : Y \to Z$ is in some homotopy class of maps $[f]$ then f_* is the image of the morphism under $[X, -]_*$ defined by $f_*[\alpha] = [f \circ \alpha]$ where $[\alpha] \in [X, Y]_*$ so that $f \circ \alpha : X \to Z$. Similarly f^* is the notation of the image of f under $[-, X]_*$ and defined by $f^*[\alpha] = [\alpha \circ f]$. This is well defined as homotopic maps stay homotopic under composition. The functors $[X, -]_*$ are sometimes denoted π_X and the pointed set $[X, Y]_*$ is thus $\pi_X(Y)$. While $[-, X]_*$ is sometimes denoted by π^X. A special case is the notation for the homotopy groups, which is most commonly written $\pi_n(Y) := \pi_{S^n}(Y) = [S^n, Y]_*$.

A map between CW complexes is called *cellular* if the image of each n-skeleton lies in the codomain's n-skeleton. A famous result we will use is the "cellular approximation theorem" which states that every map between CW complexes is homotopic to a cellular map. Furthermore, if the map is cellular for some subcomplex we may take the homotopy to be stationary on the subcomplex. For a proof of this theorem, see Hatcher page 349 [4]. One important consequence of this theorem is that for every homotopy class of maps of based maps $[f] \in [X, Y]_*$ we may represent it with a cellular map, using that the basepoint is a subcomplex.

We also remark that if A is a subcomplex of a CW complex X then the pair (X, A) has the so called homotopy extension property with respect to every space Y. Alternatively one says that the inclusion $i : A \hookrightarrow X$ is a cofibration. This means that if two maps $f, g : A \to Y$ are homotopic via some homotopy $H : A \times I \to Y$ and $F : X \to Y$ satisfies $F \circ i = F_{|A} = f$ then H extends to a homotopy $H' : X \times I \to Y$ such that $H'(a, t) = H(a, t)$ for all $a \in A$. If we identify $H'(x, 1) : X \times \{1\} \to Y$ with a map $G : X \to Y$ we see that we get $G \simeq F$ and $G_{|A} = g$. In other words we have extended the homotopy to all of X. For a proof of this fact see theorem 7.2 in Lundell & Weingram [5].

Lastly we consider the general definition of a representable functor, in our case we shall be working with contravariant ones.

Definition 1.3. A contravariant functor $h : \mathcal{C} \to \mathbf{Set}$, where $\mathcal{C}$ is locally small, is said to be representable if there is a natural isomorphism between h and a functor of the form $\hom_{\mathcal{C}}(-, y)$ for some $y \in \mathrm{ob}(\mathcal{C})$.

If the set $\hom_{\mathcal{C}}(x, y)$ has some naturally distinguished element then we get a similar definition of representable contravariant functors into $\mathbf{Set}_*$. The exact condition is that for every $x, y \in \mathrm{ob}(\mathcal{C})$ there exists a morphism $c_{x,y} \in \hom_{\mathcal{C}}(x, y)$ such that for any $f : w \to x$ and $g : y \to z$ we have $c_{x,y} f = c_{w,y}$ and $g c_{y,x} = c_{y,z}$. So that $c_{x,y}$ acts like a distinguished element in $\hom_{\mathcal{C}}(x, y)$.

We consider some standard constructions on topological spaces. These constructions are used a lot in the proof of the theorem and may not be familiar to the reader. We state all of them in their reduced form since these are most natural when working with based spaces.

Definition 1.4. Given two based maps $f : X \to Y$ and $g : X \to Z$, the reduced double mapping cylinder $M(f, g)$ is defined as

$$X \times I \sqcup Y \sqcup Z / \sim$$

with the identification $(x_0, t) \sim (x_0, s)$ for all $t, s \in I$, $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$, since the maps are based the identification makes sense.

When X, Y, Z are CW complexes and f, g are cellular the reduced double mapping cylinder can be given CW structure, this fact follows from theorem 5.11 in Lundell & Weingram [5]. In addition, if Y and Z are connected (X need not be) then $M(f, g)$ will be a connected space.

Here are some commonly used special cases worthy of their own notation.

Definition 1.5. Let X, Y be based space with basepoints x_0 and y_0 and let $\star$ be the one point space. Let $f : X \to Y$ be any based map, $c : X \to \star$ be the unique constant map.

1. The reduced cone on X, denoted CX, is $M(c, \mathrm{Id}_X)$ i.e the reduced double mapping cylinder of the constant map and the identity on X.

2. The reduced mapping cone on X with respect to the based map $f : X \to Y$, denoted Cf, is the space $M(c, f)$.

3. The reduced suspension of X, denoted by ΣX, is the space $M(c, c)$ i.e taking the constant map on both ends of $X \times I$.

Since $\star$ has a CW structure consisting of one 0-cell and the constant map being cellular the reduced mapping cone and the reduced suspension can be given CW structure when X is one. The reduced mapping cone of a based map $f : X \to Y$ can also be given CW structure if Y is a CW complex and f is cellular. Notice also that CX and ΣX are connected for all X and Cf is connected if Y is.

There are alternative ways to view the later two. For the reduced mapping cone one can take the adjunction space $CX \cup_{\tilde{f}} Y$ where $\tilde{f} : X \times 0 \to Y$ by $\tilde{f}(x, 0) = f(x) \in Y$. One then verifies that this is exactly the identification of the reduced mapping cone. The reduced suspension ΣX can be seen as taking two reduced cones and adjoining them via just the identity map on $X \times 1$.

We also note here that CX is contractible by the homotopy $H : CX \times I \to CX$ defined by $H((y, t), s) = (y, (1 - s)t)$.

Lastly we also define a construction similar to the reduced double mapping cylinder but for a whole sequence of maps.

Definition 1.6. Given a sequence of based maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots,$$

we create the reduced mapping telescope for this sequence. This is the space

$$MT = \coprod_{i=0}^{\infty} X_i \times I/\sim$$

where $X_i \times I \ni (x_i, 1) \sim (f(x_i), 0) \in X_{i+1} \times I$ for all i and $(\star_i, t) \sim (\star_j, s)$ for all i, j and $t, s \in I$ where $\star_i$ is the basepoint of X_i.

Similarly as for the previous constructions this can be given CW structure when X_i are CW complexes and all maps f_i are cellular. It is obviously connected if all X_i are connected.

The following definition is central in homotopy theory.

Definition 1.7. A continuous function $f : X \to Y$ is called:

1. A 0-equivalence if f induces a surjection on the set of path components.

2. An n-equivalence, for a positive integer n, if f induces an isomorphism on the set of path components and for any choice of basepoint x_0 of X the induced homomorphism on the k-th homotopy group $f_* : \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is an isomorphism for $k < n$ and surjective for $k = n$.

3. A weak (homotopy) equivalence if it induces an isomorphism on the set of path components, and for any choice of basepoint x_0 of X the induced map $f_* : \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is an isomorphism for all k.

Given a path from x_0 to x_1 there exists a change of basepoint isomorphism from $\pi_k(X, x_0)$ to $\pi_k(X, x_1)$ for all $k > 0$. Thus, f is a weak equivalence given that f induces a bijection between the set of path components $\pi_0(X)$ and $\pi_0(Y)$ and the induced maps $f_* : \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ are isomorphisms for some x_0 in each path component of X, and similarly for n-equivalence. In particular for path connected spaces we only need to check the condition for one point. For example any connected CW complex is path connected and thus if it has a basepoint one only needs to check equivalence on that point.

A function $f : X \to Y$ being a weak equivalence is unsurprisingly a weaker condition than f being a homotopy equivalence between X and Y. But, for CW complexes we have Whitehead's theorem which says that a weak equivalence of CW complexes X and Y is a homotopy equivalence. We shall use this in the proof of the Brown representability theorem. For a proof see Hatcher page 346 [4]. Furthermore, our definition of h$\mathbf{CW}_*$ is equivalent to the category h$\mathbf{Top}_*$ formed from the category of pointed topological spaces by inverting all the weak equivalences, see remark 2 page 1.16 chapter 1 in Quillen [7]. The reason why we can choose CW complexes as the representatives of these weak equivalences of all topological spaces is that for any space X there exists a CW complex that is weakly equivalent to X. This is called CW approximation and can be deduced from cellular approximation, see page 352 in Hatcher [4].

One important example of an n-equivalence is the following. Given a CW complex X that is obtained from $\tilde{X}$ by attaching $(n+1)$-cells, a finite or infinite amount, the inclusion $i : \tilde{X} \hookrightarrow X$ is an n-equivalence. This is since the pair $(X, \tilde{X})$ is n-connected by corollary 4.12 in Hatcher [4] as $X - \tilde{X}$ only contains $(n+1)$-cells. Thus one can conclude from the long exact sequence in relative homotopy that the induced map $i_* : \pi_k(\tilde{X}, x_0) \to \pi_k(X, x_0)$ is an isomorphism for $k < n$ and surjective for $k = n$. For more details see Hatcher [4] pages 344 to 351.

2 Proof of necessity and statement of theorem

We start by defining the conditions we require of our functors.

Definition 2.1. A contravariant functor $h : \mathrm{h}\mathbf{CW}_* \to \mathbf{Set}_*$ from the homotopy category of pointed CW complexes to the category of pointed sets is called a Brown functor if it satisfies the following 2 axioms:

1. (Wedge axiom) for any wedge sum of pointed CW complexes $X = \bigvee_{\alpha \in I} X_\alpha$ the based inclusion maps $\iota_\alpha : X_\alpha \hookrightarrow X$ induces bijection of pointed sets
$$(h(\iota_\alpha))_{\alpha \in I} : h(X) \to \prod_{\alpha \in I} h(X_\alpha).$$

2. (Mayer-Vietoris axiom) Given some based CW complex X with subcomplexes $A, B \subset X$ such that the basepoint of X lies in $A \cap B$ and $A \cup B = X$ with the commutative square
$$\begin{tikzcd} X & A \arrow[l, hook', "i"'] \\ B \arrow[u, hook, "j"] & A \cap B \arrow[l, hook', "l"] \arrow[u, hook, "k"'] \end{tikzcd}$$
where i, j, k, l are based inclusion maps. The induced commutative square after applying h
$$\begin{tikzcd} h(X) \arrow[r, "h(i)"] \arrow[d, "h(j)"'] & h(A) \arrow[d, "h(k)"] \\ h(B) \arrow[r, "h(l)"'] & h(A \cap B) \end{tikzcd}$$
is a so called weak pullback square, which means that given $x \in h(A)$ and $y \in h(B)$ such that $h(k)(x) = h(l)(y)$ there exists a $z \in h(X)$ such that $h(i)(z) = x$ and $h(j)(z) = y$. In other words, the canonical map
$$h(X) \to h(A) \times_{h(A \cap B)} h(B)$$
is surjective.

We can extend the definition of a Brown functor to functors defined on the full category of based CW complexes $h : \mathbf{CW}_* \to \mathbf{Set}_*$ by adding the condition that if

$f \simeq g : X \to Y$ relative to the basepoint then $h(f) = h(g)$, but this is equivalent to our functor factoring through the homotopy category

$$\begin{tikzcd} \mathbf{CW}_* \arrow[d, "q"'] \arrow[dr, "h"] & \\ \mathrm{h}\mathbf{CW}_* \arrow[r, "h'"'] & \mathbf{Set}_*. \end{tikzcd}$$

We shall therefore stick to the homotopy category for convenience, but if a functor from the ordinary category $\mathbf{CW}_*$ is said to be a Brown functor we mean that it satisfies this additional axiom.

The Mayer-Vietoris axiom is named this way due to its connection with the long exact Mayer-Vietoris sequence that arises in any reduced cohomology theory, which we will encounter in Section 4.

One immediate consequence of the Wedge axiom is that for a one-point CW complex $\star$ we have $h(\star) = \{a\}$ a singleton. This follows from the fact that if X is any space then $X \vee \star = X$ so that $h(X) = h(X \vee \star) \to h(X) \times h(\star)$ is a bijection for every X implying that $h(\star)$ is a singleton. This in turn means that any contractible space X also has $h(X) = \{a\}$ since h takes homotopy equivalences to bijections.

We prove that the two conditions in 2.1 are necessary for a contravariant functor to be representable.

Proposition 2.1. *Any representable contravariant functor from* $\mathrm{h}\mathbf{CW}_*$ *to* $\mathbf{Set}_*$ *is a Brown functor.*

Proof. Let $h : \mathrm{h}\mathbf{CW}_* \to \mathbf{Set}_*$ be a representable contravariant functor with natural isomorphism $\varphi : h(-) \xrightarrow{\cong} [-, K]_*$ for some based CW complex K.

Let $X = \bigvee_{\alpha \in I} X_\alpha$ with inclusions $\iota_\alpha : X_\alpha \to X$ then

$$\vartheta : [X, K]_* \to \prod_{\alpha \in I} [X_\alpha, K]_*$$

is a bijection induced by the inclusions ι_α by the characteristic property of wedge sums, namely that to define a based map with domain wedge sum is the same as defining a based map from each of the summands independently. Thus we have the commutative diagram

$$\begin{tikzcd} h(X) \arrow[d, "\varphi_X"] & \prod_{\alpha \in I} h(X_\alpha) \arrow[d, "(\varphi_{X_\alpha})_{\alpha \in I}"'] \\ {[X, K]_*} \arrow[r, "\vartheta"] & \prod_{\alpha \in I} [X_\alpha, K]_* \end{tikzcd}$$

giving us the bijection $(\varphi_{X_\alpha}^{-1})_\alpha \circ \vartheta \circ \varphi_X : h(X) \to \prod_\alpha h(X_\alpha)$.

Now given a based CW complex X with subcomplexes A, B as in the Mayer-Vietoris axiom with the diagram of inclusions

$$\begin{tikzcd} X & A \arrow[l, "i"'] \\ B \arrow[u, "j"] & A \cap B, \arrow[l, "l"] \arrow[u, "k"'] \end{tikzcd}$$

applying $[-, K]_*$ and h and connecting via φ, we get

$$\begin{tikzcd}[row sep=small, column sep=small] & {[X,K]_*} \arrow[rr, "i^*"] \arrow[dd, "j^*"] & & {[A,K]_*} \arrow[dd, "k^*"] \\ h(X) \arrow[ur, "\varphi_X", "\cong"'] \arrow[rr, "h(i)"] \arrow[dd, "h(j)"'] & & h(A) \arrow[ur, "\varphi_A", "\cong"'] \arrow[dd, "h(k)"' near start] & \\ & {[B,K]_*} \arrow[rr, "l^*"] & & {[A\cap B,K]_*} \\ h(B) \arrow[ur, "\varphi_B", "\cong"'] \arrow[rr, "h(l)"] & & h(A\cap B). \arrow[ur, "\varphi_{A\cap B}", "\cong"'] & \end{tikzcd}$$

where we wish to verify that the front square is a weak pullback square.

Let $x \in h(A)$ and $y \in h(B)$ be such that $h(k)(x) = h(l)(y)$ and let $\varphi_A(x) \in [A, K]_*$ be represented by f and $\varphi_B(y)$ be represented by g. Then $k^*[f] = [f \circ k] = [g \circ l] = l^*[g]$ by commutativity, meaning that $f \circ k = f_{|A\cap B} \simeq g_{|A\cap B} = g \circ l$. We have that subcomplexes have the homotopy extension property with respect to any space so we get that the homotopy $H : A \cap B \times I \to K$ from $f_{|A\cap B}$ to $g_{|A\cap B}$ extends to a homotopy $H' : A \times I \to K$ such that $H'(x, 0) = f$ and $H'(x, t) = H(x, t)$ for $x \in A \cap B$. In particular, we get a map $H'(x, 1) : A \to K$ that agrees with g on $A \cap B$ and is homotopic to f. Combining $H'(x, 1)$ with g by the gluing lemma we get a map $F : X \to K$ such that $i^*[F] = [F_{|A}] = [H'(x, 1)] = [f]$ and $j^*[F] = [F_{|B}] = [g]$. Letting $z = \varphi_X^{-1}[F]$ we get our required element $z \in h(X)$ since

$$h(i)(z) = \varphi_A^{-1} i^* \varphi_X(z) = \varphi_A^{-1} i^*[F] = \varphi_A^{-1}[F_{|A}] = \varphi_A^{-1}[f] = x$$

and similarly for $h(j)(z) = y$. □

What we did to prove the Mayer Vietoris axiom is basically just proving that the back square of the commutative box above is a weak pullback square and then by naturality of the isomorphism φ concluding that the front one is. This argument shall be used again later.

We state the main theorem

Theorem 2.1 (Brown Representability theorem). *Every Brown functor restricted to connected CW complexes is representable. More specifically, given a contravariant functor $h : \mathrm{h}\mathbf{CW} \to \mathbf{Set}_*$ satisfying the wedge axiom and Mayer-Vietoris axiom there exists a connected CW complex K, unique up to homotopy, and an element $u \in h(K)$ such that $T_u : [-, K]_* \to h(-)$ defined by $[f] \mapsto h(f)(u)$ is a natural isomorphism for all connected CW complexes X.*

The restriction to connected CW complexes can not be removed. A counter-example is given in the paper "Splitting homotopy idempotents II" by Peter Freyd & Alex Heller [3].

Proving this theorem will take some work and we will therefore divide it up into small lemmas which build upon each other. The first two concern algebraic properties in the form of exactness of a certain sequence and when group structure can be given to $h(X)$. Using these lemmas we prove that we can build a space K with an element $u \in h(K)$ with properties we are looking for. The last lemma will then give us the last tool needed to prove that $T_u(X) : [X, K]_* \to h(X)$ is an isomorphism for all X.

Before we start we shall note that the uniqueness of K is purely categorial and follows from the Yoneda lemma. Spelling it out clearly, if $T_u : [-, K]_* \to h(-)$ and $T_v : [-, L]_* \to h(-)$ are both natural isomorphisms, composing T_u with the inverse of T_v gives a natural isomorphism $N : [-, K]_* \to [-, L]_*$. In particular we get two bijections $N_K : [K, K]_* \to [L, K]_*$ and $N_L : [L, K]_* \to [L, L]_*$. Let $[f] = N_K[\mathrm{Id}_K] \in [K, L]_*$ and let $[g] = N_L^{-1}[\mathrm{Id}_L] \in [L, K]_*$. We get by naturality of N the following diagram

$$\begin{tikzcd} {[K,K]_*} \arrow[r, "N_K"'] \arrow[d, "g^*"] & {[K,L]_*} \arrow[d, "g^*"'] \\ {[L,K]_*} \arrow[r, "N_L"] & {[L,L]_*} \end{tikzcd} \qquad \begin{tikzcd} {[\mathrm{Id}_K]} \arrow[r, mapsto] \arrow[d, mapsto] & {[f]} \arrow[d, mapsto] \\ {[g]} \arrow[r, mapsto] & {[\mathrm{Id}_L] = [fg]} \end{tikzcd}$$

so by commutativity we get $\mathrm{Id}_L \simeq fg$. Similarly

$$\begin{tikzcd} {[L,L]_*} \arrow[r, "N_L^{-1}"'] \arrow[d, "f^*"] & {[L,K]_*} \arrow[d, "f^*"'] \\ {[K,L]_*} \arrow[r, "N_K^{-1}"] & {[K,K]_*} \end{tikzcd} \qquad \begin{tikzcd} {[\mathrm{Id}_L]} \arrow[r, mapsto] \arrow[d, mapsto] & {[g]} \arrow[d, mapsto] \\ {[f]} \arrow[r, mapsto] & {[\mathrm{Id}_K] = [gf]} \end{tikzcd}$$

so therefore $gf \simeq \mathrm{Id}_K$ and thus $K \simeq L$ are homotopy equivalent spaces.

Naturality of T_u is also immediate for any CW complex K and element $u \in h(K)$.

If $f : X \to Y$ is any based map we have a commutative square

$$\begin{tikzcd}
{[Y,K]_*} \arrow[r, "T_u(Y)"'] \arrow[d, "f^*"] & h(Y) \arrow[d, "h(f)"'] \\
{[Y,K]_*} \arrow[r, "T_u(X)"] & h(X)
\end{tikzcd}$$

since if $[\alpha] \in [Y,K]_*$ we have that

$$h(f)T_u(Y)[\alpha] = h(f)h(\alpha)(u) = h(\alpha f)(u) = T_u(X)[\alpha \circ f] = T_u(X)f^*[\alpha].$$

3 Proof of the Brown representability theorem

Throughout this section let h be a fixed Brown functor.

Given a based cellular map $f : X \to Y$ of based CW complexes with basepoints x_0 and y_0 we can build its associated cofiber sequence $X \xrightarrow{f} Y \xrightarrow{i} Cf$ where Cf is the reduced mapping cone of f and i is the inclusion of Y into the mapping cone. Then i is a cofibration since Cf can be given a CW structure such that Y is a subcomplex. In a cofiber sequence such as this the composition $i \circ f$ is nullhomotopic via the homotopy $H : X \times I \to Cf$ by $H(x,t) = (x, 1-t)$ using that $(x,1) \sim f(x)$ in Cf.

Lemma 3.1. *Given $X \xrightarrow{f} Y \xrightarrow{i} Cf$ as above the induced sequence*

$$h(Cf) \xrightarrow{h(i)} h(Y) \xrightarrow{h(f)} h(X)$$

is exact. Under the notion that for functions of pointed sets $g : A \to B$ the kernel is $\ker(g) = \{a \in A \mid g(a) = b_0\}$ *where b_0 is the distinguished element of B.*

Proof. We have the following diagram

$$\begin{tikzcd}
Cf & Y \arrow[l, "i"'] \\
CX \arrow[u, "k"] & X \arrow[l, "j"] \arrow[u, "f"']
\end{tikzcd}$$

where i, j and k are inclusion of subcomplexes by giving Cf and CX the appropriate CW structure. Applying our functor h and noting that since CX is contractible we get

$$\begin{array}{ccc} h(Cf) & \xrightarrow{h(i)} & h(Y) \\ {\scriptstyle h(k)}\downarrow & & \downarrow{\scriptstyle h(f)} \\ \{a\} & \xrightarrow[h(j)]{} & h(X). \end{array}$$

It follows that $\mathrm{im}(h(i)) \subset \ker(h(f))$ since $h(f)h(i)(z) = h(j)h(k)(z) = h(j)(a) = x_0$ the base point of $h(X)$, the other inclusion takes some more thought.

Given the right CW structure we can decompose the cone Cf into the subcomplexes $A = X \times [0, \frac{1}{2}]/\sim$ and $B = Y \cup_f X \times [\frac{1}{2}, 1]/\sim$, so that $A \cap B = X \times \left\{\frac{1}{2}\right\}$. Now A is contractable via the homotopy $H((x,t),s) = (x, t(1-s))$ from Id_A to the constant map from A to the basepoint $(x,0) \sim (x_0, 0)$. Similarly B is homotopy equivalent to Y seen as a subset of Cf via

$$H(z,s) = \begin{cases} z & \text{if } z \in Y \\ (x, t + (1-t)s) & \text{if } z = (x,t) \in CX. \end{cases}$$

For every s these agree on the subset $X \times 1 \sim f(X) \subset Y$ under the identification so H is continuous by the gluing lemma. Lastly, we identify $A \cap B = X \times \frac{1}{2}$ with just X. We summarize as $A \simeq \star, B \simeq Y, A \cap B \simeq X$ and note that the homotopy equivalences can be made compatible with inclusions so that the diagram

$$\begin{array}{ccccc} \star & \xleftarrow{j} & X & \xrightarrow{f} & Y \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ A & \hookleftarrow & A \cap B & \hookrightarrow & B \end{array}$$

commutes up to homotopy given the right CW structure on Cf. A similar diagram can be put up to show it compatibility for the inclusions of A and B into Cf. Now using the Mayer Vietoris axiom we get the weak pullback square

$$\begin{array}{ccc} h(Cf) & \xrightarrow{h(\iota_3)} & h(B) \\ {\scriptstyle h(\iota_4)}\downarrow & & \downarrow{\scriptstyle h(\iota_1)} \\ h(A) & \xrightarrow[h(\iota_2)]{} & h(A \cap B). \end{array}$$

Because of the homotopy equivalences, this diagram is in bijective correspondence to the one above so we get a cube

$$
\begin{tikzcd}
 & h(Cf) \arrow[rr] \arrow[dd] & & h(A) \arrow[dd] \\
h(Cf) \arrow[ur, "\cong"] \arrow[rr, "h(k)"] \arrow[dd, "h(i)"'] & & \{a\} \arrow[ur, "\cong"] \arrow[dd, "h(j)"] & \\
 & h(B) \arrow[rr] & & h(A \cap B) \\
h(Y) \arrow[ur, "\cong"] \arrow[rr, "h(f)"'] & & h(X). \arrow[ur, "\cong"] &
\end{tikzcd}
$$

with commutative sides from the compatibility of the homotopies with inclusions. Now just like in the proof of Proposition 2.1 we have two isomorphic squares where the back one is a weak pullback square. We thus conclude by the same type of diagram chasing as before that the front one also is. Thus if $y \in \ker h(f)$ then $h(f)(y) = h(j)(a) = x_0$ the distinguished element in $h(X)$ then there exists $z \in h(Cf)$ such that $h(i)(z) = y$ proving that $\ker h(f) \subset \operatorname{im} h(i)$. $\square$

Although we are just assuming that h takes values in the category of pointed sets $\mathbf{Set}_*$ there are spaces X where $h(X)$ has a natural group structure, namely when X is a suspension $X = \Sigma Y$. The connection here comes from that ΣY is a so called *H-cogroup* for any based space Y. When a space is a H-cogroup the covariant functor $[\Sigma Y, -]_*$ takes values in the category of groups, see section 1.6 in Spanier [8].

Lemma 3.2. *For any based CW complex Y the pointed set $h(\Sigma Y)$ has a natural group structure. With this structure, for any other CW complex K and element $u \in h(K)$ the based set map $T_u(\Sigma Y) : [\Sigma Y, K]_* \to h(\Sigma Y)$ is a group homomorphism.*

Proof. Consider the map $p : \Sigma Y \to \Sigma Y \vee \Sigma Y$ that "pinches" the center $Y \times \frac{1}{2} \subset \Sigma Y$ to the base point, defined explicitly as

$$
p(y,t) = \begin{cases} (y, 2t)_1 & \text{if } t \in [0, \frac{1}{2}] \\ (y, 2t-1)_2 & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}
$$

where the index denotes which copy of ΣY in $\Sigma Y \vee \Sigma Y$ we are in. We get the induced map $h(p) : h(\Sigma Y \vee \Sigma Y) \to h(\Sigma Y)$ and by the wedge axiom we can precompose it with the isomorphism $h(\Sigma Y \vee \Sigma Y) \cong h(\Sigma Y) \times h(\Sigma Y)$ to get a potential multiplication map $m : h(\Sigma Y) \times h(\Sigma Y) \to h(\Sigma Y)$, we just have to check the axioms.

The following diagram commutes up to homotopy

$$\begin{tikzcd}
\Sigma Y \arrow[r, "p"] \arrow[d, "p"'] & \Sigma Y \vee \Sigma Y \arrow[d, "1\vee p"] \\
\Sigma Y \vee \Sigma Y \arrow[r, "p\vee 1"'] & \Sigma Y \vee \Sigma Y \vee \Sigma Y,
\end{tikzcd}$$

via some tedious homotopies of the upper and lower compositions to the map

$$f(y,t) = \begin{cases} (y,3t)_1 & \text{if } t \in [0,\frac{1}{3}] \\ (y,3t-1)_2 & \text{if } t \in [\frac{1}{3},\frac{2}{3}] \\ (y,3t-2)_3 & \text{if } t \in [\frac{2}{3},1] \end{cases}$$

and using transitivity of homotopy. Now applying our functor h and simplifying using the wedge axiom we get the commutative diagram

$$\begin{tikzcd}
h(\Sigma Y) & h(\Sigma Y) \times h(\Sigma Y) \arrow[l, "m"'] \\
h(\Sigma Y) \times h(\Sigma Y) \arrow[u, "m"] & h(\Sigma Y) \times h(\Sigma Y) \times h(\Sigma Y) \arrow[l, "{(m,1)}"] \arrow[u, "{(1,m)}"']
\end{tikzcd}$$

proving that m is associative.

If $c : \Sigma Y \to \star$ is the unique map to the one point space then the pointed function $h(c) : h(\star) = \{a\} \to h(\Sigma Y)$ must send a to the distinguished element of $h(\Sigma Y)$. We claim that the distinguished element $e = h(c)(a) \in h(\Sigma Y)$ acts like the identity. The sequence of maps

$$\Sigma Y \xrightarrow{p} \Sigma Y \vee \Sigma Y \xrightarrow{1\vee c} \Sigma Y \vee \star = \Sigma Y$$

can be seen to be homotopic to the identity on ΣY via the homotopy defined by

$$H((y,t),s) = \begin{cases} (y,(1+s)t) & \text{if } t \in [0,\frac{1}{1+s}] \\ \star & \text{if } t \in [\frac{1}{1+s},1]. \end{cases}$$

Thus applying h yields the diagram

$$\begin{tikzcd}
h(\Sigma Y) & h(\Sigma Y \vee \Sigma Y) \arrow[l, "h(p)"'] & h(\Sigma Y) \arrow[l, "h(1\vee c)"'] \\
& h(\Sigma Y) \times h(\Sigma Y) \arrow[ul, "m"] \arrow[u, "\cong"] & h(\Sigma Y) \times h(\star) \arrow[l, "{(1,h(c))}"'] \arrow[u, "\cong"]
\end{tikzcd}$$

which commutes by the wedge axiom. We see that since the upper row is equal to the identity on $h(\Sigma Y)$ we get that for any $y \in h(\Sigma Y)$, $m(y, h(c)(a)) = m(y,e) = y$. Thus the distinguished element $e \in h(\Sigma Y)$ acts as the identity.

For inverses let $r : \Sigma Y \to \Sigma Y$ be the map that reverses the direction of the suspension, i.e $(y,t) \mapsto (y, 1-t)$. We have the sequence

$$\Sigma Y \xrightarrow{p} \Sigma Y \vee \Sigma Y \xrightarrow{1 \vee r} \Sigma Y \vee \Sigma Y \xrightarrow{f} \Sigma Y$$

where f just identifies the two copies. We get the following

$$(y,t) \overset{p}{\mapsto} \begin{cases} (y,2t)_1 & t \in [0, \frac{1}{2}] \\ (y, 2t-1)_2 & t \in [\frac{1}{2}, 1] \end{cases} \overset{1\vee r}{\longmapsto} \begin{cases} (y,2t)_1 \\ (y, 2-2t)_2 \end{cases} \overset{f}{\mapsto} \begin{cases} (y,2t) \\ (y,2-2t) \end{cases}$$

where the indexes show whether it is in the first or second copy of ΣY in the wedge $\Sigma Y \vee \Sigma Y$. Now we claim that this composition, call it F, is nullhomotopic. We have a homotopy $H : \Sigma Y \times I \to \Sigma Y$ defined by

$$H((y,t),s) = \begin{cases} (y, 2st) & t \in [0, \frac{1}{2}] \\ (y, 2s(1-t)) & t \in [\frac{1}{2}, 1], \end{cases}$$

the two definitions agrees at $t = \frac{1}{2}$ so it is continuous and it is stationary at the basepoint due to the identification. We see that $H((y,t),0) = (y,0) \sim y_0$ the constant map while $H((y,t),1) = F(y,t)$ showing that the composition F is thus nullhomotopic. The intuition here is to identify ΣY as $Y \wedge S^1$ smash product and then to see that the degree of the map on S^1 is 0. Applying h we get

$$\begin{tikzcd}
h(\Sigma Y) & h(\Sigma Y \vee \Sigma Y) \arrow[l, "h(p)"'] \arrow[d, "\cong"'] & h(\Sigma Y \vee \Sigma Y) \arrow[l, "h(1\vee r)"'] \arrow[d, "\cong"'] & h(\Sigma Y) \arrow[l, "h(f)"'] \arrow[dl, "(1,1)"] \\
& h(\Sigma Y) \times h(\Sigma Y) \arrow[ul, "m"] & h(\Sigma Y) \times h(\Sigma Y) \arrow[l, "(1,h(r))"'] &
\end{tikzcd}$$

and thus going from right to left we see that $m(x, h(r)(x)) = h(F)(x)$ but since F was nulhomotopic it is equal to $h(c)(a) = e$ which is the identity by above. Thus multiplying x by $h(r)(x)$ on the right we get the identity. In exactly the same way one shows that this also holds on the left and hence we have inverses. This proves that $h(\Sigma Y)$ is a group. To prove that $T_u(\Sigma Y)$ is a group homomorphism we have

$$\begin{aligned} T_u(\Sigma Y)([f]\cdot[g]) = T_u(\Sigma Y)[(f \vee g)\circ p] = h(p)h(f\vee g)(u) = \\ = m(h(f)(u), h(g)(u)) = m(T_u(\Sigma Y)[f], T_u(\Sigma Y)[g]) \end{aligned}$$

by definition of multiplication of $[f]$ and $[g]$ in $[\Sigma Y, K]_*$ as the composition with the pinch map. $\square$

By this lemma we have in particular, since $S^n \cong \Sigma S^{n-1}$, that $h(S^n)$ has group structure when $n \geq 1$.

Definition 3.1. For a given CW complex K and a Brown functor h we call an object $u \in h(K)$ n-universal for $n \geq 1$ if $T_u(S^k) : [S^k, K]_* \to h(S^k)$ is an isomorphism for $1 \leq k < n$ and surjective for $k = n$. We call the element universal if it is n-universal for all n i.e that $T_u(S^k)$ is an isomorphism for all $k \geq 1$.

The goal of this next part is to construct a CW complex that has a universal element for our fixed Brown functor h. We shall do this by induction, starting by construction a space with a 1-universal element. Then given a n-universal element we shall construct a new CW complex by adding $(n+1)$-cells which has a $(n+1)$ universal object using the following lemma. The proof uses Lemma 3.2 to prove injectivity since for group homomorphisms injectivity is equivalent to having trivial kernel.

Lemma 3.3. *Given a CW complex K_n and a n-universal for object u_n there exists a CW complex K_{n+1} with a $(n+1)$-universal object u_{n+1} that has K_n as a subcomplex. Furthermore if $i : K_n \to K_{n+1}$ is inclusion then $h(i)(u_{n+1}) = u_n$.*

Proof. Since u_n is n-universal, the map

$$T_{u_n}(S^n) : [S^n, K_n]_* \to h(S^n)$$

is surjective. The kernel $\ker(T_{u_n}(S^n))$ consists of based homotopy classes of maps. Let M be a set of representatives for the kernel so that each $f \in M$ is in exactly one of the homotopy classes of maps in the kernel. By the cellular approximation theorem we may let each $f \in M$ be cellular. Now let $X = \bigvee_{f\in M} S^n$ and $Y = K_n \vee \bigvee_{a\in h(S^{n+1})} S^{n+1}$ i.e X is a wedge of n-spheres, one for each $f \in M$ and Y is K_n wedged with a wedge of $(n+1)$-spheres, one for each $a \in h(S^{n+1})$. We have maps

$$X \xrightarrow{\bigvee_{f\in M} f} K_n \xrightarrow{j} Y$$

where $\bigvee_{f\in M} f$ maps each S^n in X via the associated $f \in M$ to K_n and j is the inclusion of K_n into Y. If we denote the composition with $g : X \to Y$ and consider the cofiber sequence $X \xrightarrow{g} Y \xrightarrow{l} Cg$ where Cg is the reduced mapping cone of $g : X \to Y$ we have by Lemma 3.1 that the sequence

$$h(Cg) \xrightarrow{h(l)} h(Y) \xrightarrow{h(g)} h(X) \tag{1}$$

is exact since a wedge of cellular maps is cellular. The map j can be made cellular by giving Y a CW structure such that K_n is a subcomplex. Now by construction and the wedge axiom we get two based bijections

$$h(Y) \xrightarrow{\varphi} h(K_n) \times \prod_{a\in h(S^{n+1})} h(S^{n+1}) \text{ and } h(X) \xrightarrow{\psi} \prod_{f\in M} h(S^n).$$

Summarizing this we have the following diagram

$$\begin{tikzcd}
h(X) \arrow[d, "\psi"'] & h(Y) \arrow[l, "h(g)"'] \arrow[r, "\varphi"] \arrow[d, "h(j)"] & h(K_n) \times \prod_{a\in h(S^{n+1})} h(S^{n+1}) \arrow[dl, "\pi_1"'] \\
\prod_{f\in M} h(S^n) & h(K_n) \arrow[l, "(h(f))_{f\in M}"] &
\end{tikzcd}$$

where the triangle commutes since the inclusion j is one of the inclusion which induces the bijection φ and the left square commutes since $h(g) = h(j \circ \bigvee_{f\in M} f)$.

Let $v_{n+1} = \varphi^{-1}(u_n, (a)_{a\in h(S^{n+1})}) \in h(Y)$ then we have

$$h(g)(v_{n+1}) = \psi^{-1}(h(f))_{f\in M}(u_n) = x_0$$

the distinguished element of $h(X)$. This follows from that each $f \in M$ satisfies $h(f)(u_n) = T_{u_n}[f] = s_0$ the distinguished element of $h(S^n)$ since f is a representative for a class in $\ker(T_{u_n}(S^n))$. By exactness of (1) we can find an element $u_{n+1} \in h(Cg)$ such that $h(l)(u_{n+1}) = v_{n+1}$ which gives $h(j)h(l)(u_{n+1}) = u_n$ by commutative of the diagram above. We claim that the CW complex $Cg = K_{n+1}$ and the element u_{n+1} is as stated in the lemma. Notice that K_n lives inside Cg since $Y = K_n \vee \bigvee S^{n+1}$ is seen as a subcomplex of Cg. We therefore have the inclusion $l \circ j = i : K_n \to K_{n+1}$, and as seen above $h(i)(u_{n+1}) = h(j)h(l)(u_{n+1}) = u_n$. What we have left to prove is that $u_{n+1} \in h(K_{n+1})$ is a $(n+1)$-universal object.

We look at how we constructed the space K_{n+1}. We began by wedging K_n with one $(n+1)$-sphere for each element in $h(S^{n+1})$ we then created the mapping cone of the map g which was mapping a sum of n-spheres into K_n. Notice that this is equivalent to attaching $n+1$-cells since the reduced cone on S^n is homeomorphic to D^{n+1} and we attach the boundary via the cellular maps $f \in M$. Therefore what we have essentially done is just attached $n+1$-cells with cellular attaching maps $f \in \ker(T_{u_n}(S^n))$ to K_n.

With this in mind we see that $i : K_n \to K_{n+1}$ is a n-equivalence, as defined in the introduction, since we have just attached $n+1$-cells. This is useful since $[S^k, K_n]_* = \pi_k(K_n)$ the k:th homotopy group. We therefore have the commutative diagram

$$\begin{tikzcd}
\pi_k(K_n) = [S^k, K_n]_* \arrow[rr, "i_*"'] \arrow[dr, "T_{u_n}(S^k)"'] & & {[S^k, K_{n+1}]_* = \pi_k(K_{n+1})} \arrow[dl, "T_{u_{n+1}}(S^k)"] \\
& h(S^k) &
\end{tikzcd}$$

where i_* is the induced map of i under the covariant functor $\pi_k(-)$. The diagram is commutative since if $f : S^k \to K_n$ then

$$\begin{aligned} T_{u_n}(S^k)[f] &= h(f)(u_n) = h(f)h(i)(u_{n+1}) = h(if)(u_{n+1}) = \\ &= T_{u_{n+1}}(S^k)[if] = T_{u_{n+1}}(S^k)(i_*[f]). \end{aligned}$$

Now if $k < n$ then $T_{u_n}(S^k)$ is an isomorphism by assumption, and since i is an n-equivalence so is i_*, it follows that $T_{u_{n+1}}(S^k)$ is an isomorphism. For $k = n$, the homomorphism $T_{u_n}(S^k)$ is surjective, thus by commutativity so must $T_{u_{n+1}}(S^k)$ be. What is therefore left to prove is injectivity for $k = n$, and surjectivity for $k = n + 1$. We start with the former.

Let $a : S^n \to K_{n+1}$ be a based map so that $[a] \in \ker(T_{u_{n+1}}(S^n))$, since i_* is surjective for $k = n$ we can find $b : S^n \to K_n$ such that $i_*[b] = [a]$. By commutativity of the diagram it therefore follows that $[b] \in \ker(T_{u_n})$ and is thus represented by some cellular map $b \in M$. We therefore get the following sequence of continuous maps

$$\begin{tikzcd} S^n \arrow[r, hook, "\iota_b"] \arrow[rr, bend right, "b"'] & X \arrow[r, "\bigvee_{f\in M} f"] \arrow[rr, bend left, "g"] & K_n \arrow[r, "j"] \arrow[rr, bend right, "i"'] & Y \arrow[r, "l"] & K_{n+1} \end{tikzcd}$$

where ι_b is inclusion into $X = \bigvee_{f\in M} S^n$ so that $\bigvee_{f\in M} f \circ \iota_b = b$. Also notice here that $l \circ g$ is nullhomotopic since it is the composition in a cofiber sequence as remarked in the beggining of this section, we get that

$$[a] = i_*[b] = [i \circ b] = [l \circ g \circ \iota_b] = [c \circ \iota_b] = [c]$$

where c is the constant map and therefore $[a]$ is equal to 0 in $\pi_n(K_{n+1})$. Having trivial kernel is equivalent to injectivity here since $T_{u_n}(S^n)$ is a group homomorphism by Lemma 3.2 for $n \geq 1$.

For surjectivity in the case $k = n + 1$, let $a \in h(S^{n+1})$. We have the map $\iota_a : S^{n+1} \to Y$ since $Y = K_n \vee \bigvee_{a\in h(S^{n+1})} S^{n+1}$ then composing it with l we get a map $l \circ \iota_a : S^{n+1} \to K_{n+1}$. From this we conclude that

$$\begin{aligned} T_{u_{n+1}}(S^{n+1})[l \circ \iota_a] &= h(l \circ \iota_a)(u_{n+1}) = h(\iota_a)h(l)(u_{n+1}) = h(\iota_a)(v_{n+1}) \\ &= h(\iota_a)(\varphi^{-1}(u_n, (a)_{a\in h(S^{n+1})}) \\ &= \pi_a(u_n, (a)_{a\in h(S^{n+1})}) = a, \end{aligned}$$

where the last line comes from the following commutative diagram induced by the wedge axiom where π_a is the projection of the a-component,

$$\begin{tikzcd} h(S^{n+1}) & \\ h(Y) \arrow[u, "h(\iota_a)"] \arrow[r, "\varphi"] & h(K_n) \times \prod_{a\in h(S^{n+1})} h(S^{n+1}) \arrow[ul, "\pi_a"'] \end{tikzcd}$$

which concludes the proof. $\square$

Note that by construction K_{n+1} is connected if K_n is. We shall now use this to build inductively a space CW complex K that has a universal element $u \in h(K)$.

Lemma 3.4. *For any based CW complex Y and element $v \in h(Y)$ there exists a based CW complex K with universal element $u \in h(K)$ such that Y is a subcomplex of K and if $i : Y \to K$ is inclusion we have $h(i)(u) = v$.*

Proof. We start by creating a space K_1 with a 1-universal element u_1 from Y. Let $K_1 = Y \vee \bigvee_{a \in h(S^1)} S^1$ so that $\varphi : h(K_1) \xrightarrow{\cong} h(Y) \times \prod_{a \in h(S^1)} h(S^1)$ is a bijection of pointed sets by the wedge axiom. Let $u_1 \in h(K_1)$ be $\varphi^{-1}(v, (a)_{a \in h(S^1)})$. Then u_1 is 1-universal as $T_{u_1}(S^1) : [S^1, K_1]_* \to h(S^1)$ is surjective by the same argument as in the last paragraph of the proof of Lemma 3.3. Namely, if $a \in h(S^1)$ we have an inclusion map $\iota_a : S^1 \to K_1$ with $T_{u_1}[\iota_a] = h(\iota_a)(u_1) = \pi_a(v, (a)_{a \in h(S^1)}) = a$. Notice that a similar argument also shows that $h(i_0)(v) = u_1$ if $i_0 : Y \to K_1$ is the inclusion.

Letting $Y = K_0$ and $v = u_0$ for ease of notation we apply Lemma 3.3 inductively starting from the pair K_1, u_1. We get a sequence of spaces K_n, n-universal elements u_n and inclusions $i_n : K_n \hookrightarrow K_{n+1}$ such that $h(i_n)(u_{n+1}) = u_n$ giving the sequence

$$Y = K_0 \overset{i_0}{\hookrightarrow} K_1 \overset{i_1}{\hookrightarrow} K_2 \overset{i_2}{\hookrightarrow} K_3 \overset{i_3}{\hookrightarrow} \cdots . \tag{2}$$

From this sequence we can construct the reduced mapping telescope, denote it by K

$$K = \coprod_{n=0}^{\infty} K_n \times I / \sim .$$

Categorically K is the homotopy colimit of the sequence (2), and we now wish to show that there exists a natural surjection $h(K) \to \varprojlim h(K_n)$ the inverse limit in **Set**$_*$. Define two subspaces

$$A = \coprod_{n=0}^{\infty} K_{2n} \times I / \sim \qquad B = \coprod_{n=0}^{\infty} K_{2n+1} \times I / \sim .$$

visually this would be like coloring the telescope with color A and color B and alternating after every cylinder so that $K = A \cup B$, one may equip K with a CW structure so that these are subcomplexes. Now notice that

$$A \cap B = \coprod_{n=0}^{\infty} K_n \times \{1\} / \sim \; \cong \; \bigvee_{n=0}^{\infty} K_n$$

since $A \supset K_{2n} \times \{1\} \sim i_{2n}(K_{2n}) \times \{0\} \subset K_{2n+1} \times \{0\} \subset B$. Furthermore A and B deformation retracts to

$$A \simeq \coprod_{n=0}^{\infty} K_{2n} \times \{0\}/\sim \;\cong\; \bigvee_{n=0}^{\infty} K_{2n} \tag{3}$$

$$B \simeq \coprod_{n=0}^{\infty} K_{2n+1} \times \{0\}\,/\sim \;\cong\; \bigvee_{n=0}^{\infty} K_{2n+1}. \tag{4}$$

Thus by the wedge axiom we have a bijection $h(A) \cong \prod_{n=0}^{\infty} h(K_{2n})$ via the induced map from inclusions $j_{2n} : K_{2n} \to A$ and similarly for $h(B) \cong \prod_{n=0}^{\infty} h(K_{2n+1})$ and $h(A \cap B) \cong \prod_{n=0}^{\infty} h(K_n)$. In particular, since it is a bijection, we can find elements $a \in h(A)$ such that $h(j_{2n})(a) = u_{2n} \in h(K_{2n})$, the $2n$-universal element, for all $n \geq 0$. Similarly we can find $b \in h(B)$ with $h(j_{2n+1})(b) = u_{2n+1}$ for all $n \geq 0$ and $c \in h(A \cap B)$ such that $h(j_n)(c) = u_n$ for all $n \geq 0$. We have a diagram of inclusions

$$\begin{tikzcd} K & A \arrow[l, "l_3"'] \\ B \arrow[u, "l_4"] & A \cap B \arrow[l, "l_2"] \arrow[u, "l_1"'] \end{tikzcd}$$

which we wish to use the Mayer Vietoris axiom on. We therefore want $h(l_1)(a) = c$ and similarly for b. Since $h(j_{2n})(a) = h(j_{2n})(c) = u_{2n}$ and we have a commutative diagram

$$\begin{tikzcd} h(A) \arrow[r, "h(j_{2n})"] \arrow[d, "h(l_1)"'] & h(K_{2n}) \arrow[d, "h(i_{2n-1})"] \\ h(A \cap B) \arrow[r, "h(j_{2n-1})"'] & h(K_{2n-1}) \end{tikzcd}$$

for $n \geq 1$ so that $h(i_{2n-1})h(j_{2n})(a) = h(i_{2n-1})(u_{2n}) = u_{2n-1} = h(j_{2n-1})(c)$ we can conclude that $h(l_1)(a) = c$ by injectivity of $(h(j_n)) : h(A \cap B) \to \prod_{n=0}^{\infty} h(K_n)$. A similar argument works for B to show that $h(l_2)(b) = c$. By the Mayer-Vietoris axiom we may find $u \in h(K)$ such that $h(l_3)(u) = a$ and $h(l_4)(u) = b$. We have that $h(j_n) : h(K) \to h(K_n)$ gives $h(j_n)(u) = h(j_n)(a \text{ or } b) = u_n$ by slight abuse of notation. In particular $h(j_0)(u) = h(i)(u) = v$ where this i is an in the statement of the lemma.

We are only left with proving that u is universal i.e n-universal for all n. Let $n \geq 1$ be arbitrary and consider the diagram

$$\begin{tikzcd} \pi_k(K_n) = [S^k, K_n]_* \arrow[rr, "(j_n)_*"] \arrow[dr, "T_{u_n}(S^k)"'] & & {[S^k, K]_* = \pi_k(K)} \arrow[dl, "T_u(S^k)"] \\ & h(S^k) & \end{tikzcd}$$

with $1 \leq k < n$. It commutes since if $f : S^k \to K_n$ is a based map then we get $T_{u_n}(S^k)[f] = h(f)(u_n)$ and

$$T_u(S^k)(j_n)_*[f] = T_u(S^k)[j_n f] = h(j_n f)(u) = h(f)h(j_n)(u) = h(f)(u_n).$$

Now the inclusion $j_n : K_n \to K$ is an n-equivalence since K is homotopy equivalent to the mapping telescope starting at K_n and then each K_t with $t > n$ is obtained by attaching cells of dimension $t + 1$ and thus is a t-equivalence. Thus $(j_n)_*$ is an isomorphism when $k < n$. $T_{u_n}(S^k)$ is an isomorphism by assumption and thus it follows that $T_u(S^k)$ is an isomorphism for $k < n$. Since n can be made arbitrarily big we conclude that $T_u(S^k)$ is an isomorphism for all k and thus that u is universal. $\square$

Again just like in Lemma 3.3 we note that by construction this universal space K is connected if Y is. Now is when we need to turn to only considering connected spaces.

Lemma 3.5. *Suppose we are given a based connected CW complex K with universal element $u \in h(K)$, a based connected CW complex X and some element $w \in h(X)$. If there exists a subcomplex A with inclusion $i : A \to X$ and a based cellular map $g : A \to K$ such that $h(g)(u) = h(i)(w)$ then g extends to a map $G : X \to K$ which agrees with g on A and $h(G)(u) = w$.*

Proof. Let Z be the reduced double mapping cone of $g : A \to K$ and $i : A \to X$. Since X and K are connected Z will also be connected and as i and g are cellular Z is a CW complex. The CW structure may be taken so that we can decompose Z into subcomplexes

$$B = (A \times [0, \tfrac{1}{2}] \sqcup K)/\sim \qquad C = (A \times [\tfrac{1}{2}, 1] \sqcup X)/\sim . \tag{5}$$

It is not too hard to see that there are homotopy equivalences $B \simeq K$, $C \simeq X$ and $B \cap C \simeq A$ where the inclusions of $B \cap C$ into B and C are equivalent to g and i so that

$$\begin{tikzcd} B \arrow[d, "\simeq"'] & B \cap C \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\simeq"] & C \arrow[d, "\simeq"] \\ K & A \arrow[l, "g"] \arrow[r, hook, "i"'] & X \end{tikzcd}$$

commutes up to homotopy. Our weak pullback square in the Mayer-Vietoris axiom is thus isomorphic to

$$\begin{tikzcd} h(Z) \arrow[r, "h(l_1)"] \arrow[d, "h(l_2)"'] & h(K) \arrow[d, "h(g)"] \\ h(X) \arrow[r, "h(i)"'] & h(A) \end{tikzcd}$$

since including $g(a) \in K$ into Z is the same as including directly into $(a, 0) \in Z$ by the identification. We conclude that the square above also is a weak pullback square just as in the proof of Lemma 3.1. Thus, we find $z \in h(Z)$ such that $h(l_1)(z) = u$ and $h(l_2)(z) = w$. By Lemma 3.4 we can create a CW complex K' with a universal element $u' \in h(K')$ such that Z is a subcomplex of K' and if $j : Z \to K'$ is inclusion then $h(j)(u') = z$. Since Z is connected K' will be as well. Notice also that K lives inside K' as $K \overset{l_1}{\hookrightarrow} Z \overset{j}{\hookrightarrow} K'$. Let $\varphi = j \circ l_1$ so that $h(\varphi)(u') = h(l_1)h(j)(u') = h(l_1)(z) = u$. We have the diagram

$$\begin{tikzcd} \pi_n(K) \arrow[rr, "\varphi_*"] \arrow[dr, "T_u(S^n)"'] & & \pi_n(K') \arrow[dl, "T_{u'}(S^n)"] \\ & h(S^n) & \end{tikzcd}$$

which commutes since if $f : S^n \to K$ is a map, then

$$T_{u'}(S^n)\varphi_*[f] = h(\varphi f)(u') = h(f)h(\varphi)(u') = h(f)(u)$$

and since u and u' are universal $T_u(S^n)$ and $T_{u'}(S^n)$ are isomorphisms, it thus follows that φ_* is an isomorphism. Since this is true for all $n \geq 1$ and the spaces K and K' consists of just one path components as they are connected we have a weak homotopy equivalence $\varphi : K \to K'$ which by Whitehead theorem is a homotopy equivalence.

Let $H : A \times I/\sim \hookrightarrow Z \hookrightarrow K'$. This defines a pointed homotopy from the map $H_0 : A \times \{0\} \to K'$ to $H_1 : A \times \{1\} \to K'$ and since $A \times \{0\}$ is attached to K via g we see that $H_0 = \varphi g$. Furthermore, if $g' : X \overset{l_2}{\hookrightarrow} Z \overset{j}{\hookrightarrow} K'$ then $H_1 = g'i$. Now by the homotopy extension property of $A \subset X$ we get a based homotopy $H' : X \times I \to K'$ which agrees on $A \subset X$ and $H'_1 = g'$. Now $H'_0 : X \times \{0\} = X \to K'$ has the property that $H'_0 i = H_0 = \varphi g$. Furthermore since φ is a homotopy equivalence say with some homotopy inverse $\psi : K' \to K$ we get a homotopy $\tilde{H} : A \times I \to K$ from $\psi\varphi g$ to g and the map $\psi H'_0 : X \to K$ has the property that $\psi H'_0 i = \psi\varphi g$. Thus, $\tilde{H}$ extends via the homotopy extension property to X which gives a map $G = \tilde{H}'_1 : X \to K$ where $\tilde{H}'$ is the extended homotopy such that $Gi = g$.

All we have left is to prove that this G has the desired property. We have that $h(G)(u) = h(G)h(\varphi)(u') = h(\varphi G)(u')$. Now since $G \simeq \psi H_0'$ we have that

$$\varphi G \simeq \varphi\psi H_0' \simeq H_0' \simeq H_1' = g'.$$

Thus $h(G)(u) = h(g')(u') = h(jl_2)(u') = h(l_2)h(j)(u') = h(l_2)(z) = w$ and we are done. $\square$

We are finally ready to prove the Brown representability theorem.

Proof of Theorem 2.1. Let $\star$ be the one-point CW complex so that $h(\star) = \{a\}$. Using Lemma 3.4 we get a connected CW complex K with universal element $u \in h(K)$. We verify that $T_u(X) : [X, K]_* \to h(X)$ is an isomorphism for all based connected CW complexes X.

For injectivity suppose $h(f_0)(u) = h(f_1)(u)$ where $f_0, f_1 : X \to K$ we shall prove that $f_0 \simeq f_1$ are based homotopic, here f_0 and f_1 represent homotopy classes so we may assume that they are cellular by the cellular approximation theorem. We get the unique cellular map sum $f_0 \vee f_1 : X \vee X \to K$. Consider then $X \wedge I^+$ the smash product with $I^+ = I \sqcup \{x_0\}$ the interval with an extra based point, notice also that $X \vee X$ can be identified as the subcomplex $X \times \{0\} \vee X \times \{1\}$ in $X \wedge I^+$ so we get an inclusion $i : X \vee X \to X \wedge I^+$. We also get the projection $p : X \wedge I^+ \to X$ giving us a diagram which is commutative in the two right triangles, but not the left two

$$\begin{tikzcd}
 & X \arrow[d, "l_0"] \arrow[dr, "f_0"] & \\
X \wedge I^+ \arrow[ur, "p"] \arrow[dr, "p"'] & X \vee X \arrow[l, "i"'] \arrow[r, "f_0 \vee f_1"] & K \\
 & X. \arrow[u, "l_1"'] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

Notice also that $p \circ i \circ l_0 = \mathrm{Id}_X$ while other directions around the two left triangles must not be the identity. Applying h using the wedge axiom on the middle and that $X \wedge I^+$ is homotopy equivalent to X via p we get

$$\begin{tikzcd}
 & h(X) \arrow[dl, "h(p)"'] & \\
h(X \wedge I^+) \arrow[r, "h(i)"] & h(X) \times h(X) \arrow[u, "h(l_0)"'] \arrow[d, "h(l_1)"] & h(K) \arrow[l, "(h(f_0), h(f_1))"] \arrow[ul, bend right, "h(f_0)"'] \arrow[dl, bend left, "h(f_1)"] \\
 & h(X) \arrow[ul, "h(p)"] &
\end{tikzcd}$$

where $h(p)$ is an isomorphism. By the wedge axiom we get that $h(l_0)(x_0, x_1) = x_0$ and similarly for l_1, it then follows $h(i)(x) = (h(p)^{-1}(x), h(p)^{-1}(x))$ for the relation

$$h(p \circ i \circ l_k) = h(l_k)h(i)h(p) = \mathrm{Id}_{h(X)}$$

to be true for both $k = 0$ and $k = 1$. Therefore if $w = h(p)h(f_0)(u) = h(p)h(f_1)(u)$, recall that $h(f_0)(u) = h(f_1)(u)$, then

$$h(i)(w) = (h(p)^{-1}(w), h(p)^{-1}(w)) = (h(f_0)(u), h(f_1)(u))$$

the bijective image of $h(f_0 \vee f_1)(u)$. Now using Lemma 3.5 on $f_0 \vee f_1 : X \vee X \to K$ we get a map $G : X \wedge I^+ \to K$ such that $h(G)(u) = w$ and $G \circ i = f_1 \vee f_2$. This G is then a based homotopy from $G_0 = f_0 : X \to K$ to $G_1 = f_1 : X \to K$ and thus $f_0 \simeq f_1$ so equal in $[X, K]_*$.

For surjectivity we simply note that the base-point $\star$ is included into any based space X. Thus if $w \in h(X)$ is arbitrary we get two maps: $g : \star \to K$ and $i : \star \to X$. As noted earlier $h(\star) = \{a\}$ singleton and thus $h(g)$ and $h(i)$ must map everything to a. So $h(i)(w) = a = h(g)(u)$ and thus by Lemma 3.5 we can extend g to $G : X \to K$ such that $h(G)(u) = w$. Therefore there exists $G \in [X, K]_*$ such that $T_u(X)[G] = h(G)(u) = w$ proving surjectivity and thus we are done. $\square$

4 Representing reduced cohomology theories

Using the Brown representability theorem we wish to understand reduced cohomology theories on $\mathbf{CW}_*$.

Definition 4.1. A reduced cohomology theory on $\mathbf{CW}_*$ is a sequence of functors h^n from $\mathbf{CW}_*$ to $\mathbf{Ab}$ for $n \in \mathbb{Z}$ and natural isomorphisms $h^n(X) \cong h^{n+1}(\Sigma X)$ such that

1. (Homotopy axiom) If $f \simeq g : X \to Y$ with respect to the basepoints of X, then $h^n(f) = h^n(g) : h^n(Y) \to h^n(X)$ for all n.

2. (Exactness axiom) For every inclusion $A \hookrightarrow X$ of subcomplexes and every n the sequence $h^n(X/A) \to h^n(X) \to h^n(A)$ is exact.

3. (Wedge axiom) for any wedge sum of pointed CW complexes $X = \bigvee_{\alpha \in I} X_\alpha$ the based inclusion maps $\iota_\alpha : X_\alpha \hookrightarrow X$ induces a bijection

$$(\iota_\alpha^*)_{\alpha \in I} : h^n(X) \to \prod_{\alpha \in I} h^n(X_\alpha)$$

We would like to classify every reduced cohomology theory on $\mathbf{CW}_*$ as the axioms are fairly general. Since they are homotopy invariant functors to $\mathbf{Ab}$ which has a forgetful functor to $\mathbf{Set}_*$ we can prove the following using Brown Representability theorem.

Theorem 4.1 (Classification of reduced cohomology theories)**.** *There is a bijective correspondence between reduced cohomology theories h^n on CW complexes and Ω-spectra $\{K_n\}$ of CW complexes given by natural isomorphisms $h^n(-) \xrightarrow{\cong} [-, K_n]_*$ for all $n \in \mathbb{Z}$.*

We shall define Ω-spectra when they appear naturally in the proof.

Proof. We start by showing that any reduced cohomology theory on $\mathbf{CW}_*$ defines Brown functors. Axiom 1 makes sure that we can factor each h^n through $\mathrm{h}\mathbf{CW}_*$, composing with the forgetful functor from $\mathbf{Ab}$ to $\mathbf{Set}_*$ we get functors $\tilde{h}^n : \mathrm{h}\mathbf{CW}_* \to \mathbf{Set}_*$. The wedge axiom in Definition 2.1 is exactly the same as the one in 4.1 since the forgetful functor from $\mathbf{Ab}$ to $\mathbf{Set}_*$ respects products. The main difference here will be to translate the second axiom in Definition 4.1 to the Mayer Vietoris axiom.

We have the so called Puppe sequence, see Hatcher [4] page 398,

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \Sigma(X/A) \to \Sigma^2(A) \to ...$$

from thinking of quotients X/A as homotopy equivalent to taking the mapping cone of the inclusion $i : A \hookrightarrow X$ by collapsing the retractable cone. The mapping cone from inclusion can simply be written as $X \cup CA$. We can then look at each adjacent pair of maps in this sequence as inclusion then quotient. The first triplet is obvious, the second one $X \to X/A \simeq X \cup CA \to \Sigma(A) \simeq X \cup CA \cup CX \simeq (X \cup CA)/X$ can be seen as including X into the mapping cone trivially then collapsing all of X to a point giving us ΣA. Thus, by the exactness axiom together with the homotopy axiom applying any of the functors gives an exact sequence

$$\cdots \to h^m(\Sigma(X/A)) \to h^m(\Sigma X) \to h^m(\Sigma A) \to h^m(X/A) \to h^m(X) \to h^m(A).$$

In the case $m = n + 1$ together with the isomorphism $h^{n+1}(\Sigma X) \cong h^n(X)$ we get the sequence

$$\cdots \to h^n(X/A) \to h^n(X) \to h^n(A) \to h^{n+1}(X/A) \to h^{n+1}(X) \to h^{n+1}(A)$$

the reason for considering this is so that the square in the Mayer Vietoris axiom can be extended to the commutative diagram

$$\begin{tikzcd} h^n(X/A) \arrow[r, "\alpha"] \arrow[d, dashed, "\cong"] & h^n(X) \arrow[r, "h^n(i)"] \arrow[d, "h^n(j)"'] & h^n(A) \arrow[r, "\beta"] \arrow[d, "h^n(k)"] & h^{n+1}(X/A) \arrow[d, dashed, "\cong"] \\ h^n(B/A\cap B) \arrow[r, "\alpha'"] & h^n(B) \arrow[r, "h^n(l)"] & h^n(A\cap B) \arrow[r, "\beta'"] & h^{n+1}(B/A\cap B) \end{tikzcd}$$

with exact rows, the isomorphism comes from the fact that $X/A \cong B/(A \cap B)$ assuming $X = A \cup B$. We can from here create a part of the Mayer Vietoris sequence

$$h^n(X) \xrightarrow{\Psi} h^n(A) \oplus h^n(B) \xrightarrow{\Phi} h^n(A \cap B)$$

defined as $\Psi(x) = (h^n(i)(x), -h^n(j)(x))$ and $\Phi(a,b) = h^n(k)(a) + h^n(l)(b)$ remembering here that h^n is a functor into **Ab**. Exactness can now be proved by some classic diagram chasing. First notice that $\Phi(\Psi(x)) = h^n(k)h^n(i)(x) - h^n(l)h^n(j)(x) = 0$ by commutativity of the middle square, which gives $\operatorname{im}\Psi \subset \ker\Phi$. Conversely if $\Phi(a,b) = h^n(k)(a) + h^n(l)(b) = 0$ then $h^n(k)(a) = -h^n(l)(b)$ so $h^n(k)(a) \in \operatorname{im}(h^n(l))$. Thus, $\beta' h^n(k)(a) = 0$ by exactness, so the isomorphism on the right gives $\beta(a) = 0$ and there exists an element $x' \in h^n(X)$ mapping to a by exactness. Then

$$h^n(l)(h^n(j)(x') + b) = h^n(l)h^n(j)(x') + h^n(l)(b) = h^n(k)(a) + h^n(l)(b) = 0$$

so we find some $c \in h^n(B/A \cap B)$ mapping to $h^n(j)(x') + b$ under α'. By commutativity and the left isomorphism we get

$$h^n(j)(x' - \alpha(c)) = h^n(j)(x') - \alpha'(c) = h^n(j)(x') - h^n(j)(x') - b = -b$$

and we see then that $x = x' - \alpha(c)$ will give $h^n(i)(x) = h^n(i)(x') + h^n(i)(\alpha(c)) = a$ and $h^n(j)(x) = -b$. Therefore $\Psi(x) = (h^n(i)(x), -h^n(j)(x)) = (a, b)$.

From this we can finally conclude that the Mayer Vietoris axiom for $\tilde{h}^n$ holds. Since if $\tilde{h}^n(k)(a) = \tilde{h}^n(l)(b)$ seen just as set maps then the pair $(a, -b)$ maps to zero by Φ and hence there exists $x \in h^n(X)$ such that $\Psi(x) = (h^n(i)(x), -h^n(j)(x)) = (a, -b)$ and thus $\tilde{h}^n(i)(x) = a$ and $\tilde{h}^n(j)(x) = b$.

We can now apply the Brown representability theorem to the Brown functors $\tilde{h}^n$ for each n to get representating spaces K_n and universal elements u_n so that $T_{u_n} : \tilde{h}^n(-) \to [-, K_n]_*$ is a natural isomorphism when restricting the functors to $\mathrm{h}\mathbf{CW}^c_*$. Thus for every connected CW complex X the underlying set of the abelian group $h^n(X)$ is in bijective correspondence with $[X, K_n]_*$. We can thus give $[X, K_n]_*$ the same group structure as $h^n(X)$. The next question that arises is what happens when X is a suspension $X = \Sigma Y$. Since then by Lemma 3.2 $\tilde{h}^n(\Sigma Y)$ has group structure and $T_{u_n}(\Sigma Y)$ is a group homomorphism which by the Brown representability theorem is an isomorphism. But $h^n(\Sigma Y)$ has (abelian) group structure by assumption that h^n is a part of a cohomology theory. Thus we wish to know if these coincide.

Proposition 4.1. *For a cohomology theory the group structure of $\tilde{h}^n(\Sigma Y)$ is the same as that of $h^n(\Sigma Y)$. In particular it is abelian.*

Proof. Just like in the proof of Lemma 3.2 we get $m : h^n(\Sigma Y) \times h^n(\Sigma Y) \to h^n(\Sigma Y)$ induced from the pinch map $p : \Sigma Y \to \Sigma Y \vee \Sigma Y$, but unlike before this is a homomorphism since h^n takes values in **Ab**. The identity for m is the element $h^n(c)(0)$ where

$c : \Sigma Y \to \star$ and $h^n(\star) = \{0\}$ from the wedge axiom. But then $h^n(c)(0) = 0$ since it is a homomorphism. Thus $m(x, 0) = m(0, x) = x$ for all $x \in h^n(\Sigma Y)$. This gives that $m(x, y) = m(x, 0) + m(0, y) = x + y$ thus the multiplication map m is exactly the addition (abelian multiplication) of $h^n(\Sigma Y)$. Since this m is exactly what gives $\tilde{h}^n(\Sigma Y)$ its group structure the result follows. □

We sidetrack by introducing a construction dual to that of the suspension.

Definition 4.2. Given a based space X with basepoint x_0, let ΩX be the set of loops $\alpha : I \to X$ based at x_0 (so that $\alpha(0) = \alpha(1) = x_0$). By giving ΩX the compact-open topology it becomes a based space where the base point is the constant loop.

Notice that if X is not path connected then ΩX will just capture the path connected component where the basepoint lies i.e $\Omega X \cong \Omega X_0$ if X_0 is the path component of x_0.

For any based space X the loop space ΩX is a type of space called a *H-group*, see chapter 1.5 in Spanier [8]. What's special about H-groups is that the contravariant functor $[-, \Omega X]_*$ naturally takes it values in the category of groups **Grp**. Just like how the covariant functor $[\Sigma X, -]_*$ also takes it values in **Grp** for any based space X since ΣX is a H-cogroup as remarked in Section 3.

The reason why the suspension and loop space are central is that they define functors $\Sigma(-)$ and $\Omega(-)$ from the category of based spaces to the category of H cogroups and H groups respectively with continuous homomorphisms. The former sends a based map $f : X \to Y$ to $\Sigma f : \Sigma X \to \Sigma Y$ defined by $\Sigma f(x, t) = (f(x), t)$ which is well defined since f is based. The later sends f to $\Omega f : \Omega X \to \Omega Y$ defined by $\Omega f(\alpha) = f \circ \alpha$ for loops $\alpha \in \Omega X$.

In addition these functors are adjoint to eachother, which means that there is a natural isomorphism for every pair of spaces X, Y

$$[\Sigma X, Y]_* \cong [X, \Omega Y]_*.$$

Which means that for every homotopy class of based maps $f : \Sigma X \to Y$ there exists a class of maps $\tilde{f} : X \to \Omega Y$. Given a map $f : \Sigma X \to Y$ then $\tilde{f}$ sends x to the loop α_x given by $\alpha_x(t) = f(x, t)$, and given a function $\tilde{f}$ assigning points of x to loops α_x we get a function $f : \Sigma X \to Y$ given by $f(x, t) = \alpha_x(t)$. One then checks that if $f \simeq g$ based homotopic then $\tilde{f} \simeq \tilde{g}$ and vice versa which gives the correspondence. We can even iterate to get for $n \geq 1$

$$[\Sigma^n X, Y]_* \cong [X, \Omega^n Y]_*.$$

In our case we do not need to worry about the general theory behind this but if the reader is worried about the details section 1.6 in Spanier [8] is a good resource.

We continue where we left off. The bijection of pointed sets $[X, K_n]_* \cong \tilde{h}^n(X)$ for connected X from the Brown representability theorem makes us equip $[X, K_n]_*$ with group structure isomorphic to $h^n(X)$. By assumption we also have that $h^n(X)$ is isomorphic $h^{n+1}(\Sigma X)$ as abelian groups. The isomorphism $\tilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, K_{n+1}]_*$ from $T_{u_{n+1}}(\Sigma X)$ in the Brown representability theorem is a group homomorphism by Lemma 3.2. But now that we have proved that the two group structures are identitical we get that $h^{n+1}(\Sigma X) \cong [\Sigma X, K_{n+1}]_*$ which by the adjoint relation of Σ and Ω is isomorphic to $[X, \Omega K_{n+1}]_*$. Therefore we can conclude that $[X, K_n]_* \cong [X, \Omega K_{n+1}]_*$ so that our arbitrary choice of giving $[X, K_n]_*$ the same structure as $h^n(X)$ was justified with the natural group structure of $[X, \Omega K_{n+1}]_*$ from the fact that ΩK_{n+1} is a H-group.

Another consequence of the isomorphism $[X, K_n]_* \cong [X, \Omega K_{n+1}]$ is that inserting $X = S^k$ gives that $\pi_k(K_n) = [S^k, K_n]_* \cong [S^k, \Omega K_{n+1}]_* = \pi_k(\Omega K_{n+1})$ for all k, thus by Whitehead theorem they are homotopically equivalent $K_n \simeq \Omega K_{n+1}$ (in the construction all the K_n are connected), alternatively one can see that they are homotopically equivalent since representing spaces are unique up to homotopy by the Yoneda lemma discussed in Section 2.

These types of spaces are of interest

Definition 4.3. A collection of pointed spaces $\{K_n\}$ for $n \in \mathbb{Z}$ together with based maps $\sigma_n : \Sigma K_n \to K_{n+1}$ is called a spectrum. If the adjoints $\tilde{\sigma}_n : K_n \to \Omega K_{n+1}$ are weak homotopy equivalences then the spectrum is an Ω-spectrum.

Therefore for any cohomology theory h^n the representing spaces $\{K_n\}$ with the adjoint of the homtopy equivalences $K_n \simeq \Omega K_{n+1}$ define an Ω-spectrum of CW complexes.

All of these have only been for connected X, when X is not connected we cannot make use of the Brown representability theorem directly. But because of the suspension isomorphism together with the fact that ΣX is connected for all X we have natural isomorphisms of abelian groups

$$h^n(X) \cong h^{n+1}(\Sigma X) \cong [\Sigma X, K_{n+1}]_* \cong [X, \Omega K_{n+1}]_* \tag{6}$$

and as $\Omega K_{n+1} \simeq K_n$ the pointed set $[X, K_n]_*$ gets it group structure, but this is isomorphic to the group structure of $h^n(X)$ by the equation above. Therefore we can conclude that $h^n(-) \xrightarrow{\cong} [-, K_n]_*$ also holds for non-connected CW complex.

Conversely we can prove that any Ω-spectrum of based CW complexes defines a reduced cohomology theory.

Proposition 4.2. *Given a Ω-spectrum $\{E_n\}$ of based CW complexes with structure maps $\sigma_n : \Sigma E_n \to E_{n+1}$ the functors $h^n(-) := [-, E_n]_*$ defines a reduced cohomology theory on $\mathbf{CW}_*$.*

Proof. Since $E_n \simeq \Omega E_{n+1}$ and $E_{n+1} \simeq \Omega E_{n+2}$ we can replace E_n by $\Omega^2 E_{n+2}$ so that $[X, E_n]_*$ gets its abelian group structure from $[X, \Omega^2 E_{n+2}]_*$ where the constant map acts as the zero. We also see that

$$h^n(X) = [X, E_n]_* \cong [X, \Omega E_{n+1}]_* \cong [\Sigma X, E_{n+1}]_* = h^{n+1}(\Sigma X)$$

by the adjoint relation. We are therefore just left to check the axioms.

We have clearly that if $f \simeq g : X \to Y$ then $f^* = g^*$ as composition respects homotopy $f^*[\alpha] = [\alpha \circ f] = [\alpha \circ g] = g^*[\alpha]$ for any $\alpha : Y \to E_n$.

If A is a subcomplex of X we consider

$$[X/A, E_n]_* \xrightarrow{j^*} [X, E_n]_* \xrightarrow{i^*} [A, E_n]_* \tag{7}$$

where we identify X/A with the homotopy equivalent space $X \cup CA$ the reduced mapping cone of $i : A \to X$. If $f : X \to E_n$ is in the kernel of i^* then $f \circ i = f_{|A} \simeq c$ the constant map. If H is the homotopy then we get a map $F : X \cup CA \to E_n$ defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \\ H(a,t) & \text{if } (a,t) \in CA \end{cases}$$

since $H(a,0) = f_{|A}$ and all the maps here are based it makes sense with the identification. Also clearly $F \circ j = F_{|X} = f$. Thus $f \in \operatorname{im}(j^*)$. Similarly if $F : X \cup CA \to E_n$ then $F_{|CA}$ defines a homotopy $H : A \times I \to E_n$ with $H(a,0) = F_{|A}$ and $H(a,1) = c$ thus $i^* j^*(F) = F_{|A} \simeq c$ so $j^*(F) \in \ker i^*$. Thus (7) is exact.

The wedge axiom is immediate by the universal property of the wedge sum. We conclude that Ω-spectra defines reduced cohomology theories on $\mathbf{CW}_*$. □

We conclude the bijective correspondance between Ω-spectrum and reduced cohomology theories proving Theorem 4.1. □

For example singular cohomology with coefficients in an arbitrary abelian group G is a cohomology theory by setting $H^n(X; G)$ as $h^n(X)$ for $n \geq 0$ and the 0-functor for $n < 0$. Restricting these to CW complexes we get Ω-spectrum representing the functors. These are the so called *Eilenberg-Maclane spaces* named after Samuel Eilenberg and Saunders Mac Lane and denoted $K(G, n)$. In section 4.3 of Hatcher [4] the isomorphisms $H^n(X; G) \cong [X, K(G, n)]$ are proved without Theorem 4.1 by assuming that these spaces exists. But now using the Brown representability theorem we know that these spaces do exist.

The various cohomology theories of Cobordism are represented via Theorem 4.1 with Ω-spectrum called Thom Spectrum, named after René Thom. The classifying spaces of vector bundles corresponds to the cohomology theory that arises from K-theory. See lectures 38 and 43 in Fomenko and Fuchs [2].

References

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