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# Introduction to Representation Theory of Finite Groups

av

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# Introduction to Representation Theory of Finite Groups

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## Abstract

This text serves as an introduction to representation theory of finite groups, beginning with a background in group theory and linear algebra before formally defining representations, some representations are shown to be irreducible, while others are composed as a direct sum of the irreducibles (Maschke's Theorem 3.24), culminating in the proof of Schur's Lemma 3.25, which ensure the uniqueness of these compositions. Finally, character theory is introduced, simplifying representation theory by focusing on the trace of matrices, providing systematic methods to identify and decompose any given representation. Examples to demonstrate the these theoretical concepts are some cyclic and symmetric groups of small order. In particular, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$, $\mathcal{S}_3$, and $\mathcal{S}_4$ are examined and completely decomposed and the results are presented in Tables 3, 4, 5, 13 and 14.

## Sammanfattning

Den här texten ger en introduktion till ändliga gruppers representationsteori. Utifrån en inledning med ett par bakgrundsämnen inom gruppteori och linjär algebra definieras representationer formellt. Vissa representationer visar sig vara oreducerbara, medan andra visar sig vara den direkta summan av oreducerbara representationer (Maschkes Sats 3.24), vilket kulminerar i beviset av Schurs Lemma 3.25 som visar att dessa sammansättningar är unika. Slutligen introduceras karaktärsteorin som förenklar arbetet med att analysera representationer genom att undersöka matrisers spår. Karaktärsteorin ger systematiska metoder för att identifiera och sönderdela varje given representation. För att demonstrera dessa teoretiska koncept ges några cykliska och symmetriska grupper som exempel, särskilt så undersöks och sönderdelas $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$, $\mathcal{S}_3$ och $\mathcal{S}_4$ helt och resultaten presenteras i Tabellerna 3, 4, 5, 13 och 14.

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# 1 Introduction

The main purpose of this text is to study linear representations of groups, or in other words replacing abstract groups with sets of matrices behaving the same way as the elements of the group and replacing the group operation with matrix multiplication. This is the intuition behind a homomorphism. The reader is assumed to be familiar with first-year linear algebra and the definitions and basic concepts of vector spaces and groups.

Only representations of finite groups in finite dimensional vector spaces will be treated. As [Ser77, Sect.1.1.] notes, we are usually interested in a finite number of elements of a vector space, and then we could span a finite dimensional subspace with those elements. The restriction to finite groups is more severe however, as some results in this text does not apply on infinite groups, or even compact infinite groups (see [FH04, Ser77]).

Calculations performed in this text are my own and the examples are also my own, unless otherwise noted. As practical examples we study the abelian cyclic groups and the symmetric groups of small order extensively.

Section 2 introduces some background topics from group theory and linear algebra required before introducing representation theory. Especially the symmetric group, recollections of linear algebra and a brief introduction to tensor operations on vector spaces are presented.

The idea of a representation is introduced in section 3 as a group homomorphism (a map respecting the group operation) bringing the elements of a group into the set of invertible transformations of some vector space along with a few basic consequences of the definition. Some typical representations follow with examples applied on some small cyclic and symmetric groups. The notion of subrepresentation, analogous to vector subspaces, is introduced along with tensor operations on representations, allowing us to construct new representations out of already known ones. Some representations are indivisible however and are called irreducible, inviting the main theorem of this text that any representation of a finite group in a finite dimensional vector space is composed of a finite number of constituent subrepresentations (Maschke's Theorem 3.24), which along with Schur's Lemma (Theorem 3.25) allows us to claim that these decompositions are unique.

The last section is on character theory which strips back the "big" concepts of linear algebra and matrix calculations into the trace of those matrices, a metric related to the eigenvalues of a matrix, releasing us from the ambiguities of choices of bases. The representations found earlier are reevaluated and we see that the results allows us to find every irreducible representation of a group by providing a limit for the number of irreducibles, and also completely decompose any arbitrary representation into irreducibles with some “character calculus”. The cyclic groups are quickly studied, and finally the symmetric groups of degree 3 and 4 are closely studied.

# 2 Preliminary topics

## 2.1 Groups

The action of a group $G$ on a set $X$ is a map

$$\sigma : G \times X \to X$$

satisfying for the identity element $e \in G$

$$\sigma(e, x) = x, \text{ for all } x \in X$$

and for all $g, h \in G$,

$$\sigma(g, \sigma(h, x)) = \sigma(gh, x), \text{ for all } x \in X.$$

It is easy to see that, for every fixed $g \in G$ the map $\sigma(g, x) : X \to X$ is a permutation, that is bijective. To simplify notation, we write

$$g \cdot x := \sigma(g, x).$$

Recall ([DF04, 1.3.], [Sag01, 1.1]) that for a positive integer $n$, we denote by $\mathcal{S}_n$ the set of permutations of the set $\{1, 2, \ldots, n\}$, which is a group under composition of permutations, called the *Symmetric group of order* $n$. The number of elements in $\mathcal{S}_n$ is $n!$.

The elements may be represented in several ways or notations, one of which is *cycle decomposition.* A *cycle* is a string of integers

$$(a_1, a_2, \ldots, a_m)$$

which represent which elements of $\{1, 2, \ldots, n\}$ it permutes. The cycle above will permute $a_1$ to $a_2$, $a_2$ to $a_3$ and $a_i$ to $a_{i+1}$ for $1 \leq i \leq m-1$. Lastly it will permute $a_m$ back to $a_1$, completing the *cycle.* This cycle is of length $m$, hence it is an $m$-cycle. Usually 1-cycles are omitted. 2-cycles are called transpositions. Two cycles are *disjoint* if they have no integers in common. A cycle representation of an element of $\mathcal{S}_n$ is not unique, however it can be uniquely expressed as a composition of *disjoint* cycles.

*Example* 2.1. The elements of $\mathcal{S}_3$, expressed in cycle decomposition are

$$(1), \quad (1,2), \quad (1,3), \quad (2,3), \quad (1,2,3), \quad (1,3,2).$$

There are $3! = 6$ elements: the identity permutation (denoted $(1)$), three transpositions and two 3-cycles. For example, the element $(1,2)$ is the permutation that maps $1 \mapsto 2$, $2 \mapsto 1$ and $3 \mapsto 3$. 3 is called a *fixed point of* $(1,2)$.

*Note.* The group $\mathcal{S}_n$ is generated by all sequent transpositions $(1,2),(2,3),\ldots,(n-1,n)$, that is any element of $\mathcal{S}_n$ can be expressed as the composition of transpositions. In $\mathcal{S}_3$ we for example have $(1,2,3) = (1,2)(2,3)$, $(1,3,2) = (2,3)(1,2)$ and $(1,3) = (1,2)(2,3)(1,2)$.

**Definition 2.2** (Sign of a permutation)**.** Let $\sigma \in \mathcal{S}_n$, and let $s$ be the number of transpositions required to compose $\sigma$. Then the function $\operatorname{sgn} : \mathcal{S}_n \to \{\pm 1\}$ is defined as

$$\operatorname{sgn} : \sigma \mapsto (-1)^s.$$

If $s$ is an even integer, then $\sigma$ is called an even permutation, and vice versa for an odd $s$.

Note that if $\sigma$ is composed of $s$ transpositions and $\tau$ is composed of $t$ transposition, then their composition $\sigma\tau$ can be composed of $s+t$ transpositions, in some cycle decomposition of $\sigma\tau$. The sign function is still well-defined, that is, a permutation can not be expressed both as a composition of even number and of a odd number of transpositions [Big04, Thm.12.6.1.].

The *cycle type* of a permutation $\sigma \in \mathcal{S}_n$ is an $n$-tuple $(k^{m_k})_{k=1}^n$ where $m_k$ is the number of $k$-cycles in $\sigma$. For example, the cycle type of $(1,2)(3,4,5) \in \mathcal{S}_6$ is $(1^1,2^1,3^1,4^0,5^0,6^0) = (1^1,2^1,3^1)$, where in the last step the cycles of zero multiplicity are omitted.

Another way to classify a permutation in $\mathcal{S}_n$ is to compare it to an *integer partition of* $n$. An integer partition of $n$ is a sum $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$, where $l \leq n$ and $\lambda_i \geq \lambda_{i+1}$. The element $(1,2)(3,4,5) \in \mathcal{S}_6$ corresponds to the partition $3+2+1$ since it contains one 3-cycle, one transposition and one fixed point [Sag01, Sect.1.1.].

*Example* 2.3 ($\mathcal{S}_3$). The elements of $\mathcal{S}_3$ and their signs, cycle types and corresponding integer partitions are presented in Table 1.

| $\mathcal{S}_3$ | $(1)$ | $(1,2)$ | $(1,3)$ | $(2,3)$ | $(1,2,3)$ | $(1,3,2)$ |
|---|---|---|---|---|---|---|
| Type | $(1^3)$ | $(1^1,2^1)$ | $(1^1,2^1)$ | $(1^1,2^1)$ | $(3^1)$ | $(3^1)$ |
| Part. | 1+1+1 | 2+1 | 2+1 | 2+1 | 3 | 3 |
| Sign | $+1$ | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ |

Table 1: Elements of $\mathcal{S}_3$.

Since the number of elements of $\mathcal{S}_n$ is $n!$, for larger $n$ it becomes increasingly cumbersome to describe every element of $\mathcal{S}_n$, however as we will see, we can instead study the conjugacy classes of $\mathcal{S}_n$.

Recall that two elements $g, g' \in G$ are said to be *conjugate* if there exists an element $h \in G$ such that $g' = hgh^{-1}$. "Being conjugate in a group" is an equivalence relation and the equivalence classes, called *conjugacy classes*, partition $G$ into disjoint subsets [Sag01, Sect.1.1.]. Denote by $[g]$ the conjugacy class in $G$ containing $g$. If another element $g'$ is conjugate to $g$, then they share the same conjugacy class, ie. $[g] = [g']$, and both $g$ and $g'$ are said to be *representatives* of their class. The size of the conjugacy class can be calculated with the *centralizer of $g$ in $G$*, defined by

$$\mathrm{Cent}(g) = \left\{ h \in G \;\middle|\; hgh^{-1} = g \right\},$$

and by the orbit-stabilizer theorem [Big04, Thm.21.3], the relationship between $\mathrm{Cent}(g)$ and the elements of $[g]$ is

$$|[g]| = \frac{|G|}{|\mathrm{Cent}(g)|},$$

where $|\cdot|$ denotes the size of a set.

Returning to the symmetric group, two permutations $\sigma$ and $\tau$ share conjugacy class if and only if they are of the same cycle type [Sag01, Sect.1.1.], [Big04, Thm.12.5.]. Since the cycle type was linked to an integer partition of the degree of the symmetric group, there are as many conjugacy classes in $\mathcal{S}_n$ as there are integer partitions of $n$. For example, there are three conjugacy classes in $\mathcal{S}_3$, five in $\mathcal{S}_4$ and seven in $\mathcal{S}_5$. Also, if the cycle type of a $\sigma \in \mathcal{S}_n$ is $(k^{m_k})_{k=1}^n$, then the size of its centralizer is

$$\prod_{k=1}^{n} k^{m_k} m_k!,$$

([Sag01, Prop.1.1.1.] has a nice combinatorical proof).

*Example* 2.4 ($\mathcal{S}_4$). The conjugacy classes of $\mathcal{S}_4$, along with their sizes, cycle types and signs are presented in Table 2.

| $\mathcal{S}_4$ | $(1)$ | $(1,2)$ | $(1,2)(3,4)$ | $(1,2,3,4)$ | $(1,2,3)$ |
|---|---|---|---|---|---|
| $\lvert\text{Cent}(\sigma)\rvert$ | 24 | 4 | 8 | 4 | 3 |
| $\lvert[\sigma]\rvert$ | 1 | 6 | 3 | 6 | 8 |
| Type | $(1^4)$ | $(1^2,2^1)$ | $(2^2)$ | $(4^1)$ | $(1^1,2^1)$ |
| Part. | 1+1+1+1 | 2+1+1 | 2+2 | 4 | 3+1 |
| Sign | $+1$ | $-1$ | $+1$ | $-1$ | $+1$ |

Table 2: Classes of $\mathcal{S}_4$.

## 2.2 Linear algebra

We recall from linear algebra the concept of the trace of a square matrix $(a_{ij})_{n\times n}$. It is the sum of the elements along the diagonal,

$$\text{Tr }(a_{ij}) = \sum_{i=1}^{n} a_{ii} = a_{11} + \cdots + a_{nn},$$

and has the following properties:

**Proposition 2.5.** i) It is the sum of the eigenvalues of a matrix [Nic18, Cor.8.6.1.].

ii) Two similar matrices have the same trace [Nic18, Thm.5.5.1.], consequently the trace is independent of the basis chosen.

iii) It is constant under conjugation (Equivalent to ii) and also for two square matrices $A, B$ we have $\text{Tr } AB = \text{Tr } BA$, see [Nic18, Ex.2.3.30.]).

Also of importance is the kernel and image of a vector space map, a linear map between two vector spaces.

**Definition 2.6** (Kernel and image of a linear map)**.** Let $V$ and $W$ be two vector spaces and let $\varphi : V \to W$ be a linear map. Then the kernel and the image of the map are defined thusly:

$$\ker\varphi = \{\mathbf{v} \in V \mid \varphi(\mathbf{v}) = \mathbf{0}\},$$
$$\text{im }\varphi = \{\mathbf{w} \in W \mid \exists \mathbf{v} \in V \text{ s.t. } \varphi(\mathbf{v}) = \mathbf{w}\}.$$

*Remark* 2.7. The kernel and the image of a linear map are subspaces of the domain and codomain of the map respectively, that is $\ker \varphi$ is a subspace of $V$ and $\operatorname{im} \varphi$ is a subspace of $W$ [HU15, Sect.5.4.].

**Theorem 2.8.** [HU15, Thm.12.16] Let $V$ be a vector space and $W$ be a vector subspace of $V$. Then there exists a complementary vector subspace $W'$ in $V$ such that $W \cap W' = \emptyset$ and $W \cup W' = V$. This is equivalent to saying that $V$ is *the direct sum of* $W$ *and* $W'$, denoted as $V = W \oplus W'$.

*Example* 2.9. Letting $V$ be a vector space with a subspace $W$, a linear map $\pi : V \to W$ is called a projection of $V$ onto $W$ if we have that

i) $\pi^2 = \pi$.

ii) For any $\mathbf{v} \in W$ we have that $\pi(\mathbf{v}) = \mathbf{v}$.

iii) If $\mathbf{v} \notin W$ we have that $\pi(\mathbf{v}) = \mathbf{0}$.

Hence $\operatorname{im} \pi = W$ and $\ker \pi$ is a subspace of $V$ complementary to $W$, that is $V = W \oplus \ker \pi$.

## 2.3 Tensor operations

For this section, let $V$ and $W$ be vector spaces of dimensions $m$ and $n$ and bases $(\hat{\mathbf{e}}_i)_{i=1}^m$ and $(\hat{\mathbf{f}}_j)_{j=1}^n$. These definitions are from [Jee15] and [Yok92].

In Theorem 2.8 the **direct sum of vector spaces** was introduced. More generally, if $V$ and $W$ are any vector spaces, $V \oplus W$ is the set of pairs $(\mathbf{v}, \mathbf{w}) := \mathbf{v} \oplus \mathbf{w}$ where $\mathbf{v} \in V$ and $\mathbf{w} \in W$. The basis of $V \oplus W$ is constructed from the pairs $\hat{\mathbf{e}}_i \oplus \hat{\mathbf{f}}_j$, hence $\dim V \oplus W = \dim V + \dim W$. In the sense of linear operators, if $F \in \mathrm{GL}(V)$ and $G \in \mathrm{GL}(W)$, then the direct sum of $F$ and $G$ is in some basis the block matrix

$$F \oplus G = \begin{pmatrix} F & \mathbf{0} \\ \mathbf{0} & G \end{pmatrix}$$

and it will act on the elements of $V \oplus W$ bilinearly, that is

$$\begin{aligned}(F \oplus G) \cdot (\mathbf{v} \oplus \mathbf{w}) &= \begin{pmatrix} F & \mathbf{0} \\ \mathbf{0} & G \end{pmatrix} \cdot (\mathbf{v} \oplus \mathbf{w}) \\ &= (F\mathbf{v}) \oplus (G\mathbf{w}).\end{aligned}$$

The direct sum of an arbitrary number of vector spaces is defined similarly [Ser77, Sect.1.3.], [Sag01, Sect.1.5.].

*Notation.* The direct sum of $n$ copies of a vector space $V$ is here denoted $nV$. An element of $nV$ is an $n$-tuple $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ where every $\mathbf{v}_i \in V$.

*Example* 2.10. The direct sum of $n$ copies of a field $\mathbb{K}$, $n\mathbb{K}$, is usually denoted $\mathbb{K}^n$, for example the real space $\mathbb{R}^3$ is the direct sum of three pair-wise orthogonal lines.

A vector space equipped with a map

$$(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \otimes \mathbf{w}$$

for any $\mathbf{v} \in V$ and $\mathbf{w} \in W$ is called a **tensor product of $V$ and $W$** and denoted by $V \otimes W$ if it is linear in each constituent space[1], and that the set $(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{f}}_j)$ is a basis for $V \otimes W$. The dimension of $V \otimes W$ is $\dim V \cdot \dim W$. An element of $V \otimes W$ is a linear combination of tensor products of elements from $V$ and $W$. For the linear maps $F = (f_{ij}) \in \mathrm{GL}(V)$ and $G = (g_{ij}) \in \mathrm{GL}(W)$ we have that their tensor product is the $mn \times mn$ block matrix

$$F \otimes G = (f_{ij}G) = \begin{pmatrix} f_{11}G & \cdots & f_{1m}G \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mm}G \end{pmatrix}$$

and its action on an element $\mathbf{v} \otimes \mathbf{w} \in V \otimes W$ is again bilinear:

$$\begin{aligned} (F \otimes G) \cdot (\mathbf{v} \otimes \mathbf{w}) &= (f_{ij}G) \cdot (\mathbf{v} \otimes \mathbf{w}) \\ &= (F\mathbf{v}) \otimes (G\mathbf{w}). \end{aligned}$$

The tensor product of an arbitrary number of vector spaces is similarly defined [Ser77, Sect.1.5.], [Sag01, Sect.1.7.].

*Notation.* We denote by $V^{\otimes n}$ the tensor product of $n$ copies of $V$. This vector space is called the $n$th tensor power of $V$.

*Example* 2.11. The **tensor square of $V$** is denoted by $V \otimes V$. An element of $V \otimes V$ is a linear combination of pairs of elements from $V$, that is they look like

$$\sum_{i,j} a_{ij} \mathbf{v}_i \otimes \mathbf{v}'_j.$$

[1] This means that $(a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{w}) = a(\mathbf{v}_1, \mathbf{w}) + b(\mathbf{v}_2, \mathbf{w})$ as well as $(\mathbf{v}, a\mathbf{w}_1 + b\mathbf{w}_2) = a(\mathbf{v}, \mathbf{w}_1) + b(\mathbf{v}, \mathbf{w}_2)$ for some $a, b \in \mathbb{C}$.

The **dual space of** $V$, denoted $V^*$ is the set of linear maps $V \to \mathbb{C}$. An element of $V^*$ is a linear map $\varphi$ that takes a vector from $V$ and returns a complex number.

More generally, the **set of linear maps** $V \to W$ are denoted by $\mathrm{Hom}(V, W)$. An element of $\mathrm{Hom}(V, W)$ is a linear map that takes a vector from $V$ as input and returns a vector from $W$, hence it is identified with the tensor product $V^* \otimes W$ [FH04, Sect.1.1.].

# 3 Representation Theory

The layout of this section is based on [FH04] and [Ser77].

Let $G$ be a finite group written multiplicatively. We denote its size by $|G|$ and the action of its elements with $\cdot$, for example by $g \cdot x$, sometimes abbreviated to $gx$.

Let $V$ be a finite-dimensional vector space over the field of complex numbers $\mathbb{C}$. The group of invertible linear transformations of $V$ is denoted by $\mathrm{GL}(V)$. If $V$ is provided with a basis, usually denoted $(\hat{\mathbf{e}}_i)_{i=1}^n$ where $n = \dim V$, then $\mathrm{GL}(V)$ is identified with the set of invertible $n \times n$ matrices, denoted $\mathrm{GL}_n(\mathbb{C})$[DF04, 18.1].

**Definition 3.1** (Representation)**.** [Ser77] A representation of a group $G$ in the vector space $V$ is a homomorphism

$$\rho : G \to \mathrm{GL}(V). \tag{3.1}$$

The dimension of $V$, denoted $\dim V$, is referred to as the *degree* of the representation.

For every $g$ in $G$, there is a linear map $\rho_g$ in $\mathrm{GL}(V)$ that performs the group action on $V$, that is

$$g \cdot \mathbf{v} = \rho_g(\mathbf{v}).$$

The linearity of the map means that for any $\mathbf{v}, \mathbf{w} \in V$,

$$g \cdot (a\mathbf{v} + b\mathbf{w}) = ag \cdot \mathbf{v} + bg \cdot \mathbf{w},$$

with $a, b \in \mathbb{C}$ and also since it is a homomorphism, for any $g, h \in G$, we have

$$\rho_{gh} = \rho_g \rho_h. \tag{3.2}$$

From the homomorphism of $\rho$, two consequences follow.

**Proposition 3.2.** The homomorphism preserves identity and inverses. For the identity element $e$ of $G$ and an arbitrary element $g$ with inverse $g^{-1}$ in $G$ we have,

$$\rho_e = \mathrm{id} \quad \text{and} \quad \rho_g^{-1} = \rho_{g^{-1}}, \tag{3.3}$$

where id is the identity transformation and $\rho_{g^{-1}}$ is the map associated with the inverse of $g$.

*Proof.* Take $g$ as an arbitrary element of $G$. The first identity follows from taking $h = e$ in Equation 3.2:

$$\rho_{eg} = \rho_e \rho_g.$$

Since $eg = g$ for any $g \in G$ we must have that $\rho_g = \rho_e \rho_g$ which is true if and only if $\rho_e = \text{id}$. Now instead taking $h = g^{-1}$ we have:

$$\rho_{gg^{-1}} = \rho_g \rho_{g^{-1}},$$

but $gg^{-1} = e$ and $\rho_e = \text{id}$, implying $\rho_{g^{-1}} = \rho_g^{-1}$. □

*Notation.* A vector space $V$ provided with such a homomorphism discussed above is said to be a *representation space* of $G$

*Note.* The homomorphism $\rho$, the set of maps $\{\rho_g\}_{g \in G}$ and the representation space $V$ may interchangeably and abusively be called the *representation of G*.

If a basis $(\hat{\mathbf{e}}_i)_{i=1}^n$ is provided for a representation space, then a *matrix representation* can be provided. In this case, $\rho_g$ is the matrix in $\text{GL}_n(\mathbb{C})$ associated with the action of $g$ expressed in the basis provided. A matrix representation is not canonical and is dependent on the basis chosen.

## 3.1 Tensor operations on representations

If we have one or more representations, we may construct additional ones using tensor operations. Given two representations $\rho^V : G \to \text{GL}(V)$ and $\rho^W : G \to \text{GL}(W)$, the following are also representations.

**The direct sum of** $V \oplus W$, given by

$$\rho_{V \oplus W}(\mathbf{v} \oplus \mathbf{w}) = (\rho_V \oplus \rho_W)(\mathbf{v} \oplus \mathbf{w}) = \rho_V(\mathbf{v}) \oplus \rho_W(\mathbf{w}).$$

By recursion, for a positive integer $n$, the **direct sum of** $n$ **copies of** $V$, denoted

$$nV := \bigoplus_{i=1}^{n} V = \underbrace{V \oplus \cdots \oplus V}_{n \text{ times}}.$$

The **tensor product** $V \otimes W$, given by

$$\rho_{V \otimes W}(\mathbf{v} \otimes \mathbf{w}) = (\rho_V \otimes \rho_W)(\mathbf{v} \otimes \mathbf{w}) = \rho_V(\mathbf{v}) \otimes \rho_W(\mathbf{w}).$$

By recursion, for a positive integer $n$, the **$n$th tensor power of $V$**, denoted

$$V^{\otimes n} := \bigotimes_{i=1}^{n} V = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}.$$

## 3.2 Examples

This section follows [Ser77, 1.2.].

### 3.2.1 Trivial representations

*Example* 3.3 (Trivial representation). For any group $G$ there is a trivial representation of degree 1 defined by mapping each element of $G$ to 1, ie.

$$\rho_g^{\text{Triv}} = 1,$$

for every $g \in G$. It is clearly a homomorphism since for any $g, h \in G$ we have that

$$\rho_{gh}^{\text{Triv}} = 1 = 1 \cdot 1 = \rho_g^{\text{Triv}} \cdot \rho_h^{\text{Triv}}.$$

This could be extended to any vector space by mapping $g$ to the identity transformation of that vector space. For a vector space of dimension $n$, the representation taking every $g$ in a group to the $n \times n$ identity matrix can be described, in the language of Section 3.1, as the direct sum of $n$ copies of the trivial representation.

*Note.* The trivial representation (the mapping is trivial) is not to be confused with the trivial zero space (the vector space containing only the zero vector).

*Example* 3.4 (Alternating representation of $\mathcal{S}_n$). Choosing $G = \mathcal{S}_n$, another degree 1 representation can be found by studying the signs, or parities, of the elements of $\mathcal{S}_n$. By [Big04, Thm.12.6.1.], the sign of a permutation is well-defined, so for any two permutations $\sigma$ and $\tau$ with respective signs $(-1)^s$ and $(-1)^t$, their composition has the sign

$$\operatorname{sgn}(\sigma\tau) = (-1)^{s+t} = (-1)^s \cdot (-1)^t = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau),$$

so clearly the map $\operatorname{sgn} : \mathcal{S}_n \to \{\pm 1\}$ is a homomorphism and thus a representation of degree 1, where even permutations are mapped to $+1$ and odd to $-1$.

### 3.2.2 Degree 1 representations of $\mathcal{C}_n$

Choose $G = \mathcal{C}_n$, and let $g$ be a generator of $\mathcal{C}_n$ such that

$$\mathcal{C}_n = \{e, g, g^2, \dots, g^{n-1}\} \quad \text{and} \quad g^n = e.$$

Consider a map $\rho : \mathcal{C}_n \to \mathbb{C}$ defined as a homomorphism by $\rho_{g^a}\rho_{g^b} = \rho_{g^{a+b}}$ for some integers $a, b$. By Equation 3.3 we have that $\rho_e = 1$, but $g^n = e$ and by induction on Equation 3.2 we have that $\rho_{g^n} = (\rho_g)^n$. Then we must have that $(\rho_g)^n = 1$, that is $\rho_g$ is mapped to a $n$th root of unity. In conclusion, for $\mathcal{C}_n$ we have then found $n$ representations of degree 1, denoted $\rho^0, \rho^1, \dots, \rho^{n-1}$, each mapping $g$ to a $n$th root of unity and the powers of $g$ to the corresponding powers of that root of unity.

*Example* 3.5 ($\mathcal{C}_3$). The three third roots of unity are $1$, $\omega = \frac{-1+i\sqrt{3}}{2}$ and $w^2 = \frac{-1-i\sqrt{3}}{2}$. Three representations of $\mathcal{C}_3$ are presented in table 3.

*Example* 3.6 ($\mathcal{C}_4$). The four fourth roots of unity are $1, i, -1$ and $-i$. Four corresponding representations of $\mathcal{C}_4$ are presented in Table 4.

*Example* 3.7 ($\mathcal{C}_5$). The five fifth roots of unity are $e^{2\pi i m/5}$, $0 \le m \le 4$. Five representations of $\mathcal{C}_5$ are presented in table 5.

| $\mathcal{C}_3$ | $e$ | $g$ | $g^2$ |
|---|---|---|---|
| $\rho_0$ | 1 | 1 | 1 |
| $\rho_1$ | 1 | $\omega$ | $\omega^2$ |
| $\rho_2$ | 1 | $\omega^2$ | $\omega$ |

Table 3: Three reprs. of $\mathcal{C}_3$. $\omega = e^{2\pi i/3}$.

| $\mathcal{C}_4$ | $e$ | $g$ | $g^2$ | $g^3$ |
|---|---|---|---|---|
| $\rho_0$ | 1 | 1 | 1 | 1 |
| $\rho_1$ | 1 | $i$ | $-1$ | $-i$ |
| $\rho_2$ | 1 | $-1$ | 1 | $-1$ |
| $\rho_3$ | 1 | $-i$ | $-1$ | $i$ |

Table 4: Four reprs. of $\mathcal{C}_4$.

| $\mathcal{C}_5$ | $e$ | $g$ | $g^2$ | $g^3$ | $g^4$ |
|---|---|---|---|---|---|
| $\rho_0$ | 1 | 1 | 1 | 1 | 1 |
| $\rho_1$ | 1 | $\omega$ | $\omega^2$ | $\omega^3$ | $\omega^4$ |
| $\rho_2$ | 1 | $\omega^2$ | $\omega^4$ | $\omega$ | $\omega^3$ |
| $\rho_3$ | 1 | $\omega^3$ | $\omega$ | $\omega^4$ | $\omega^2$ |
| $\rho_4$ | 1 | $\omega^4$ | $\omega^3$ | $\omega^2$ | $\omega$ |

Table 5: Five reprs. of $\mathcal{C}_5$. $\omega = e^{2\pi i/5}$.

### 3.2.3 Permutation representation

Given a group $G$, we chose a finite set $X$ that $G$ acts on by permutation. Let $V$ be a vector space spanned by a natural basis $(\hat{\mathbf{e}}_x)_{x\in X}$, then we have a representation $\rho^{\text{Perm}} : G \to \text{GL}(V)$ defined by its action on the basis vectors by, for any $g \in G$,

$$\rho_g^{\text{Perm}} : \hat{\mathbf{e}}_x \mapsto \hat{\mathbf{e}}_{gx},$$

that is $\rho^{\text{Perm}}$ inherited the group action of $G$ on $X$. It is a homomorphism since for any $g, h \in G$ we have

$$\begin{aligned} g \cdot (h \cdot \hat{\mathbf{e}}_x) &= g \cdot \hat{\mathbf{e}}_{hx} \\ &= \hat{\mathbf{e}}_{ghx} \\ &= (gh) \cdot \hat{\mathbf{e}}_x \end{aligned}$$

for any $x \in X$. In the $(\hat{\mathbf{e}}_x)_{x\in X}$-basis, $\rho_g^{\text{Perm}}$ are permutation matrices, which have a one once in every row and column, and the rest of the entries are zero.

Letting $G = \mathcal{S}_n$, it would be appropriate to choose the set $X = \{1, 2, \ldots, n\}$ and to let any $\sigma \in \mathcal{S}_n$ permute any $1 \leq i \leq n$ by $i \mapsto \sigma(i)$. Choosing a basis $(\hat{\mathbf{e}}_i)_{i=1}^n$ to span a vector space $V$, we define a representation $\rho^{\text{Perm}} : \mathcal{S}_n \to \text{GL}(V)$ defined for any $\sigma \in \mathcal{S}_n$ as

$$\rho_\sigma^{\text{Perm}} : \hat{\mathbf{e}}_i \mapsto \hat{\mathbf{e}}_{\sigma(i)}$$

for any $1 \leq i \leq n$. In this basis, the corresponding set of matrix representations are the permutation matrices

$$\rho_\sigma^{\text{Perm}} = (r_{ij})_{n\times n}, \text{ where } r_{ij} = \delta_{j,\sigma(i)} = \begin{cases} 1, \text{ if } j = \sigma(i), \\ 0, \text{ otherwise,} \end{cases}$$

in which the $i$th column has a 1 in the $\sigma(i)$th row, and the rest of the rows have a 0.

Under this action, a vector

$$(a_1, a_2, \ldots, a_n) = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + \cdots + a_n\hat{\mathbf{e}}_n \in V$$

is mapped to

$$(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(n)}) = a_1\hat{\mathbf{e}}_{\sigma(1)} + a_2\hat{\mathbf{e}}_{\sigma(2)} + \cdots + a_n\hat{\mathbf{e}}_{\sigma(n)} \in V.$$

The dimension of $V$ is $|X|$, for example the permutation representation of $\mathcal{S}_n$ is of degree $n$.

*Example* 3.8 (Permutation representation of $\mathcal{S}_2$). The symmetric group of degree 2 has two elements, $\mathcal{S}_2 = \{(1), (1, 2)\}$ and their matrix representations in $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$-space are presented in Table 6.

$$\rho^{\text{Perm}}_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho^{\text{Perm}}_{(1,2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Table 6: Matrix representations of $\mathcal{S}_2$

*Example* 3.9 (Permutation representation of $\mathcal{S}_3$)*.* Likewise, representations of $\mathcal{S}_3$ are presented in Table 7.

$$\rho^{\text{Perm}}_{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho^{\text{Perm}}_{(1,2,3)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho^{\text{Perm}}_{(1,3,2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\rho^{\text{Perm}}_{(1,2)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho^{\text{Perm}}_{(1,3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho^{\text{Perm}}_{(2,3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Table 7: Matrix representations of $\mathcal{S}_3$

*Example* 3.10 (Permutation representation of $\mathcal{S}_4$)*.* Yet again, matrix representations of some elements of $\mathcal{S}_4$ are presented in Table 8. The elements chosen are one representative from each conjugacy class of $\mathcal{S}_4$.

$$\rho^{\text{Perm}}_{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho^{\text{Perm}}_{(1,2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho^{\text{Perm}}_{(1,2)(3,4)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\rho^{\text{Perm}}_{(1,2,3)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho^{\text{Perm}}_{(1,2,3,4)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Table 8: Some matrix representations of $\mathcal{S}_4$

### 3.2.4 Regular representation

Following the same reasoning as in the previous section, but instead we let $G$ act on itself, ie. $X = G$. The corresponding vector space $V$ is spanned by the basis $(\hat{\mathbf{e}}_g)_{g \in G}$ constructed from the elements of $G$. The regular representation of $G$ in $V$ is then a map $\rho^{\text{Reg}} : G \to \text{GL}(V)$ defined by $g \cdot \hat{\mathbf{e}}_h = \hat{\mathbf{e}}_{gh}$ for any $g, h \in G$.

The dimension of $V$ is $|G|$, for example the regular representation of $\mathcal{C}_n$ is of degree $n$ and for $\mathcal{S}_n$ it is of degree $n!$, a number which grows increasingly quick for larger $n$, however the regular representation will be shown to be key in finding every representation of a group.

*Example* 3.11 (Regular representation of $\mathcal{C}_3$). For $G = \mathcal{C}_3$ we calculate the action of $\mathcal{C}_3$ on the $\hat{\mathbf{e}}, \hat{\mathbf{g}}, \hat{\mathbf{g}}^2$-basis thusly:

$$\begin{cases} e \cdot \hat{\mathbf{e}} = \hat{\mathbf{e}}, \\ e \cdot \hat{\mathbf{g}} = \hat{\mathbf{g}}, \\ e \cdot \hat{\mathbf{g}}^2 = \hat{\mathbf{g}}^2, \end{cases} \qquad \begin{cases} g \cdot \hat{\mathbf{e}} = \hat{\mathbf{g}}, \\ g \cdot \hat{\mathbf{g}} = \hat{\mathbf{g}}^2, \\ g \cdot \hat{\mathbf{g}}^2 = \hat{\mathbf{e}}, \end{cases} \qquad \begin{cases} g^2 \cdot \hat{\mathbf{e}} = \hat{\mathbf{g}}^2, \\ g^2 \cdot \hat{\mathbf{g}} = \hat{\mathbf{e}}, \\ g^2 \cdot \hat{\mathbf{g}}^2 = \hat{\mathbf{g}}, \end{cases}$$

hence the permutation matrices in the $\hat{\mathbf{e}}, \hat{\mathbf{g}}, \hat{\mathbf{g}}^2$-basis are

$$\rho_e^{\text{Reg}} = \text{id}, \quad \rho_g^{\text{Reg}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_{g^2}^{\text{Reg}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let's choose a new basis inspired by the third roots of unity 1, $\omega$ and $\omega^2$:

$$\begin{cases} \hat{\mathbf{f}}_1 = \hat{\mathbf{e}} + \hat{\mathbf{g}} + \hat{\mathbf{g}}^2, \\ \hat{\mathbf{f}}_2 = \hat{\mathbf{e}} + \omega^2 \hat{\mathbf{g}} + \omega \hat{\mathbf{g}}^2, \\ \hat{\mathbf{f}}_3 = \hat{\mathbf{e}} + \omega \hat{\mathbf{g}} + \omega^2 \hat{\mathbf{g}}^2, \end{cases}$$

where $\omega = e^{2\pi i/3}$, corresponding to the change-of-basis matrix

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad \text{with inverse} \quad P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

Now, the matrices of $\mathcal{C}_3$ in this new basis are

$$\rho_e^{\text{Reg}} = P^{-1} \cdot \text{id} \cdot P = \text{id},$$

$$\rho_g^{\text{Reg}} = P^{-1} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \text{and}$$

$$\rho_{g^2}^{\mathrm{Reg}} = P^{-1} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

We can clearly see that the regular representation of $\mathcal{C}_3$ is the direct sum of the three degree 1 representations found in Table 3, eg.

$$\rho_g^{\mathrm{Reg}} = (1) \oplus (\omega) \oplus (\omega^2) = \rho_g^0 \oplus \rho_g^1 \oplus \rho_g^2$$

after the change of basis.

## 3.3 Subrepresentations

As laid out in [FH04, Sect.1.2.], we are looking for representations which are said to be "atomic" or "indivisible" and conversely, for any arbitrary representation we wish to find how it is composed of these indecomposable representations. To proceed we need the notion of a vector space map that respects or conserves the group action.

**Definition 3.12** ($G$-linear map)**.** [FH04, Sect.1.1.] Let $V$ and $W$ be two representation spaces of a group $G$. A vector space map $\varphi : V \to W$ is called a $G$-linear map if it commutes with, the group action of $G$, ie. for any $\mathbf{v} \in V$ and $g \in G$ we have

$$\varphi(g \cdot \mathbf{v}) = g \cdot \varphi(\mathbf{v}),$$

or in terms of the maps $\rho_g^V : G \to \mathrm{GL}(V)$ and $\rho_g^W : G \to \mathrm{GL}(W)$,

$$\varphi \circ \rho_g^V(\mathbf{v}) = \rho_g^W \circ \varphi(\mathbf{v}).$$

Equivalently one can say that the diagram in Figure 1 commutes for every $g \in G$.

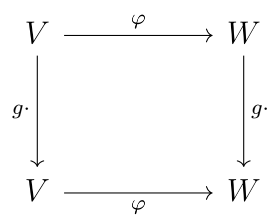


Figure 1: The map $\varphi$ is $G$-linear if the diagram commutes for every $g \in G$.

Consider the case where $W$ is a subspace of $V$ left fixed by $G$, that is $gW \subseteq W$ for all $g \in G$. Such a subspace is called *$G$-invariant*. Let $\rho^V$ be a representation of $G$, then the restriction of $\rho^V$ to $W$, here denoted $\rho^{V|_W}$, is a isomorphism (and then also a homomorphism) of $W$ onto itself since $W$ is $G$-invariant, hence $\rho^{V|_W} : G \to \mathrm{GL}(W)$ is a representation of $G$ in the subspace $W$ of $V$, motivating the following definition:

**Definition 3.13** (Subrepresentation)**.** Let $G$ be a finite group and let $\rho$ be a representation of $G$ in a vector space $V$. A restriction of $\rho$ to a $G$-invariant vector subspace $W$ of $V$ is called a subrepresentation of $\rho$.

*Example* 3.14 (Trivial subspaces)*.* Any representation has itself as well as the zero space as subrepresentations. These are referred to as non-proper or trivial subrepresentations and usually omitted.

### 3.3.1 Proper subrepresentations

Let $G$ be any group and let us study its permutation and regular representations.

*Example* 3.15 (Trivial representation inside the permutation representation)*.* [Sag01, Example 1.4.3.] Let $G$ act on a set $X = \{x_1, x_2, \dots, x_k\}$, where $k = |X|$, and let $V$ be the vector space spanned by the basis $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \dots, \hat{\mathbf{x}}_k)$. Consider the one-dimensional subspace of $V$ spanned by the sum of all basis vectors, ie.

$$W = \mathrm{Span}\{\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2 + \cdots + \hat{\mathbf{x}}_k\}.$$

A vector $\mathbf{w} \in W$ is a scalar multiple of this sum, and for any $g \in G$, the action of $g$ on $\mathbf{w}$ will simply reorder this sum and return the same $\mathbf{w}$, that is for any $g \in G$, we have that $\rho_g^{V|_W} = 1$. Thus $W$ is a $G$-invariant subspace of $V$ and the permutation representation has the trivial representation as a subrepresentation.

*Example* 3.16 (Trivial representation inside the regular representation)*.* [Sag01, Example 1.4.4.] Similarly to the last example, we span a vector space $V$ by a basis $(\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \dots, \hat{\mathbf{g}}_k)$ for every $g_i \in G$, where $k = |G|$, and consider the one-dimensional subspace

$$W = \mathrm{Span}\{\hat{\mathbf{g}}_1 + \hat{\mathbf{g}}_2 + \cdots + \hat{\mathbf{g}}_k\}.$$

Completely analogous to the last example, $W$ is shown to be a $G$-invariant subspace of $V$ and the regular representation also has the trivial representation as a subrepresentation.

*Example* 3.17 (Alternating representation inside the regular representation of $\mathcal{S}_n$). [Sag01, Example 1.4.4.] Let $G = \mathcal{S}_n$, then for the regular representation, $V$ is spanned by a basis vector for every $\sigma \in \mathcal{S}_n$. Let $W$ be the subspace of $V$ spanned by the sum $\sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\sigma)\hat{\mathbf{e}}_\sigma$. The action of a $\tau \in \mathcal{S}_n$ on a $\mathbf{w} \in W$ is

$$\begin{aligned}
\tau(\mathbf{w}) &= \tau \cdot a \sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\sigma)\hat{\mathbf{e}}_\sigma \\
&= a \sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\sigma)\hat{\mathbf{e}}_{\tau\sigma} \\
&= a \frac{1}{\mathrm{sgn}\tau} \sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\tau\sigma)\hat{\mathbf{e}}_{\tau\sigma} \\
&= \{\text{Scalar factor}\} \cdot \mathbf{w} \in \mathrm{Span}(\mathbf{w}),
\end{aligned}$$

where $a \in \mathbb{C}$, hence $W$ is a $G$-invariant subspace of $V$ and the alternating representation is also found inside the regular representation of the symmetric group.

### 3.3.2 Subrepresentations as a kernel

From linear algebra, we are familiar with the kernel and image of a map (see Definition 2.6).

**Proposition 3.18** (Kernel and image of a $G$-linear map)**.** Let $V$ and $W$ be representation spaces of a group $G$ and let $\varphi : V \to W$ be a $G$-linear map. Then the kernel and the image of $\varphi$ are also representations of $G$ and more specifically, $\ker\varphi$ is a subrepresentation of $V$ and $\mathrm{im}\ \varphi$ is a subrepresentation of $W$.

*Proof.*

i) Let $\mathbf{v} \in \ker\varphi$. Then $g \cdot \varphi(\mathbf{v}) = \mathbf{0}$ since by definition $\varphi(\mathbf{v}) = \mathbf{0}$, but then we must also have $\varphi(g \cdot \mathbf{v}) = \mathbf{0}$ since $\varphi$ is a $G$-linear map, implying that $g \cdot \mathbf{v} \in \ker\varphi$. Since this holds for any $g \in G$ and $\mathbf{v} \in \ker\varphi$, the kernel of $\varphi$ is a $G$-invariant subspace of $V$ and a subrepresentation of $V$.

ii) For a $\mathbf{w} \in \mathrm{im}\ \varphi$, by definition there exists a $\mathbf{v} \in V$ such that $\varphi(\mathbf{v}) = \mathbf{w}$. Since $g \cdot \mathbf{v} \in V$, then we have that $\varphi(g \cdot \mathbf{v}) \in W$. This implies that $g \cdot \varphi(\mathbf{v})$ is also in $W$. Since this holds for any $g \in G$ and $\mathbf{v} \in V$, the image of $\varphi$ is a $G$-invariant subspace of $W$ and a subrepresentation of $W$. □

Recall from linear algebra the existence of complementary vector subspaces of a vector space (Theorem 2.8). Is there a similar property of subrepresentations? Consider the case when $W$ is a subrepresentation of $V$ and let $\pi : V \to W$ be the projection of $V$ onto $W$. Recall (Theorem 2.8) that for each subspace $W$ of $V$ there exists a complementary and disjoint subspace $W'$ in $V$ such that $V = W \oplus W'$. This means that every $\mathbf{v} \in V$ is partitioned into $\mathbf{w} \oplus \mathbf{w}'$ where $\mathbf{w} \in W$ and $\mathbf{w}' \in W'$ such that

$$\begin{cases} \pi(\mathbf{v}) = \mathbf{w}, \\ \pi(\mathbf{w}) = \mathbf{w} \text{ and} \\ \pi(\mathbf{w}') = \mathbf{0}. \end{cases}$$

The image of the projection is clearly $W$ and the kernel is $W'$, however it is not clear that $\pi$ a $G$-linear map.

**Proposition 3.19** (Existence of complementary subrepresentations)**.** Let $\rho$ be a representation of a finite group $G$ in a finite-dimensional vector space $V$. Let $W$ be a $G$-invariant subspace of $V$. Then there exists a $G$-invariant subspace $W'$ of $V$ complementary to $W$ such that $V = W \oplus W'$.

*Proof.* By Proposition 3.18 we know that a $G$-linear map from one representation space $V$ to another $W$ has its kernel as a subrepresentation of $V$. If we can find such a $G$-linear map from $V$ to a $G$-invariant subspace $W$ we are done. The projection $\pi$ may not generally be a $G$-linear map, instead we consider taking the *average* of $\pi$ over $G$,

$$\bar{\pi} := \frac{1}{|G|} \sum_{g \in G} \rho_g \cdot \pi \cdot \rho_g^{-1},$$

and see if it conserves the action of $G$. Note that $\bar{\pi}$ is still a projection of $V$ onto $W$ since $g$ preserves $W$. Now, does $\bar{\pi} \cdot \rho_g = \rho_g \cdot \bar{\pi}$ hold for every $g \in G$? Equivalently, let's consider

$$\begin{aligned} \rho_g \cdot \bar{\pi} \cdot \rho_g^{-1} &= \frac{1}{|G|} \sum_{h \in G} \rho_g \cdot \rho_h \cdot \pi \cdot \rho_h^{-1} \cdot \rho_g^{-1} && \text{(Def. of } \bar{\pi}) \\ &= \frac{1}{|G|} \sum_{h \in G} \rho_{gh} \cdot \pi \cdot \rho_{gh}^{-1} && (\rho \text{ homom., } (gh)^{-1} = h^{-1}g^{-1}) \\ &= \frac{1}{|G|} \sum_{g' \in G} \rho_{g'} \cdot \pi \cdot \rho_{g'}^{-1} && (\text{Let } g' = gh) \\ &= \bar{\pi}. && \text{(Def. of } \bar{\pi}) \end{aligned}$$

So, $\bar{\pi} \cdot \rho_g = \rho_g \cdot \bar{\pi}$ for every $g \in G$ and thus $\bar{\pi} : V \to W$ is a $G$-linear map, which means that its kernel is a $G$-invariant subspace of $V$ complementary to $W$ such that $V = W \oplus \ker \bar{\pi}$. $\square$

### 3.3.3 The standard represention of $\mathcal{S}_n$

In Example 3.15, the trivial representation was found to be a subrepresentation of the permutation representation. Let $G = \mathcal{S}_n$ and let $V$ be the permutation representation and assign it the basis $(\hat{\mathbf{e}}_i)_{i=1}^n$. The one-dimensional subspace $W$ spanned by the sum of all $\hat{\mathbf{e}}_i$ was found to be the trivial representation, then by Proposition 3.19, there exists another subrepresentation $W'$ of $V$ complementary to $W$. The dimension of $W'$ is $n-1$. Let's introduce to $V$ an inner product $(\cdot|\cdot)$ that fulfills the expected properties, in particular:

$$(\hat{\mathbf{e}}_i|\hat{\mathbf{e}}_j) = \delta_{i,j} = \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ if } i \neq j, \end{cases}$$

for the basis $(\hat{\mathbf{e}}_i)_{i=1}^n$. Then the basis $(\hat{\mathbf{f}}_j)_{j=1}^{n-1}$ of $W'$ can be constructed by ensuring that each $\hat{\mathbf{f}}_j$ is orthogonal to the basis vector of $W$, that is

$$\left( \hat{\mathbf{f}}_j \,\middle|\, \sum_{i=1}^n \hat{\mathbf{e}}_i \right) = 0$$

for every $1 \leq j \leq n-1$. One can see that one such basis is found by for example choosing every basis vector $\hat{\mathbf{f}}_j$ to be the difference of two sequent $\hat{\mathbf{e}}_i$, that is choosing the basis $(\hat{\mathbf{e}}_j - \hat{\mathbf{e}}_{j+1})_{j=1}^{n-1}$ for $W'$. This is called the *Standard representation of* $\mathcal{S}_n$ [Sag01, Sect.1.5.].

*Example* 3.20 (Standard representation of $\mathcal{S}_3$). The matrices of the standard representation is found by studying the action inherited from $\mathcal{S}_3$ on the basis

$$\begin{cases} \hat{\mathbf{f}}_1 = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2, \\ \hat{\mathbf{f}}_2 = \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3. \end{cases}$$

For example,

$$(1,2) \cdot \hat{\mathbf{f}}_1 = (1,2) \cdot (\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2) = \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_1 = -\hat{\mathbf{f}}_1, \text{ and}$$
$$(1,2) \cdot \hat{\mathbf{f}}_2 = (1,2) \cdot (\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3) = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_3 = \hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_2,$$

hence $\rho^{\text{Stan}}_{(1,2)} = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The matrices are presented in Table 9.

$$\rho^{\text{Stan}}_{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho^{\text{Stan}}_{(1,2,3)} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \rho^{\text{Stan}}_{(1,3,2)} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\rho^{\text{Stan}}_{(1,2)} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho^{\text{Stan}}_{(1,3)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho^{\text{Stan}}_{(2,3)} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Table 9: Matrices of the standard representations of $\mathcal{S}_3$

If the trivial representation corresponds to the one-dimensional sum of all basis vectors of the permutation representation of $\mathcal{S}_n$ (in $\mathcal{S}_3$ this would be the diagonal through the origin and the point $(1, 1, 1)$) then the standard representation corresponds to the $(n-1)$-dimensional hyperplane perpendicular to that line. Again for $\mathcal{S}_3$, this is the plane intersecting the origin perpendicular to the line through the origin and the point $(1, 1, 1)$.

*Example* 3.21 (Decomposition of the permutation representation of $\mathcal{S}_3$)*.* Letting $V$ be a permutation representation space of $\mathcal{S}_3$, it was found to have two complementary subrepresentations $W$ and $W'$, denoted the trivial and the standard representations respectively. In the $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$-basis, the matrix for $(1, 2)$ is by Example 3.9

$$\rho^{\text{Perm}}_{(1,2)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let's consider a basis change

$$\begin{cases} \hat{\mathbf{f}}_1 = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \\ \hat{\mathbf{f}}_2 = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 \\ \hat{\mathbf{f}}_3 = \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3 \end{cases}$$

with corresponding change-of-basis matrix

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix},$$

in accordance with the bases of the trivial and standard representations. Then the matrix for $(1,2)$ in this new basis is

$$\begin{aligned}\rho^{\text{Perm}}_{(1,2)} = P^{-1} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot P &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \rho^{\text{Triv}}_{(1,2)} \oplus \rho^{\text{Stan}}_{(1,2)}.\end{aligned}$$

The same calculations on the other elements of $\mathcal{S}_3$ confirm that the permutation representation of $\mathcal{S}_3$ decomposes to the direct sum of the trivial and standard representations.

## 3.4 Irreducible representations

In Section 3.3 we found that a given representation can be divided into complementary subrepresentations, now we introduce the notion of an "indivisible" representation.

**Definition 3.22** (Irreducible representation)**.** If there are no proper and non-trivial $G$-invariant subspaces of a representation $V$, it is said to be *irreducible*.

We have already met a few of these.

*Example* 3.23 (Degree 1 representations are irreducible). [Sag01, Example 1.4.2.] A vector space of dimension 1 has no other subspace other than itself and the zero space, thus it is irreducible. Hence the trivial representation of degree 1 of any group and the alternating representation of $\mathcal{S}_n$ discussed in Section 3.2 and the degree 1 representations of $\mathcal{C}_n$ found in Section 3.2.2 are irreducible representations of their respective groups.

The results from Proposition 3.19 invites the notion of *complete reducibility* of an arbitrary representation. This is presented in the following theorem (really a corollary of Proposition 3.19).

**Theorem 3.24** (Maschke's theorem)**.** Let $G$ be a finite group and let $V$ be any representation space of $G$ of finite dimension. Then $V$ is composed as a direct sum of a finite number of subrepresentations $W_i$ of $V$, that is

$$\begin{aligned} V &= W_1 \oplus W_2 \oplus \cdots \oplus W_k \\ &= \bigoplus_{i=1}^{k} W_i. \end{aligned}$$

*Proof.* The theorem will be proved using induction on complementary subrepresentations. If $V$ itself is irreducible, then we are done. If $V$ is not irreducible, then by definition there exists a non-trivial and non-zero subrepresentation $W$ of $V$, and also by Proposition 3.19, there exists a complementary subrepresentation $W'$ of $V$ such that $V = W \oplus W'$. If both $W$ or $W'$ are irreducible, then we are done, if either or both are not, we then apply Proposition 3.19 on them. By the induction hypothesis and by the fact that the dimension of $V$ is finite, $V$ will be decomposed into a direct sum of a finite number of subrepresentations and we are done. □

Maschke's theorem tells us it is possible to decompose any arbitrary representation into the direct sum of irreducible representations, however it does not say anything about the *uniqueness* of a given composition. We need to study which kinds of $G$-linear maps that are permitted between two irreducible representations, a key insight of Schur's Lemma.

**Theorem 3.25** (Schur's Lemma)**.** [FH04, Lemma.1.7.], [Ser77, Prop.2.4.] Let $V$ and $W$ be irreducible representations of a group and let $\varphi : V \to W$ be a $G$-linear map. Then:

i) Either $\varphi$ is an isomorphism, or $\varphi$ is the zero map, $\varphi = \mathbf{0}$.

ii) If $V$ and $W$ are isomorphic, then $\varphi = \lambda \cdot \mathrm{id}$, for some $\lambda \in \mathbb{C}$.

*Proof.* Item i) follows from the $G$-linearity of $\varphi$, since by Proposition 3.18, $\ker \varphi$ is a subrepresentation of $V$, but $V$ is irreducible, hence $\ker \varphi$ must then be all of $V$ ($\varphi = \mathbf{0}$) or $\{\mathbf{0}\}$ ($\varphi$ is injective). Likewise, $\mathrm{im}\ \varphi$ is a subrepresentation of the irreducible $W$, hence $\mathrm{im}\ \varphi$ is either $W$ ($\varphi$ is surjective) or $\{\mathbf{0}\}$ ($\varphi = \mathbf{0}$). Combining the two cases, we must have that either $\varphi$ is an isomorphism, or that $\varphi = \mathbf{0}$.

Consider the case where $V$ and $W$ are isomorphic and let $\lambda \in \mathbb{C}$ be an eigenvalue of $\varphi$. The map $\varphi - \lambda \cdot \mathrm{id}$ is also a $G$-linear map since it is the sum of two such maps. Since $\lambda$ is chosen to be a eigenvalue of $\varphi$, $\ker(\varphi - \lambda \cdot \mathrm{id})$ is non-empty (it contains at least the eigenvector associated with $\lambda$) so by i) we must have that $\varphi - \lambda \cdot \mathrm{id}$ is the zero map, or equivalently $\varphi = \lambda \cdot \mathrm{id}$, and ii) follows. □

In other words, there are no non-trivial maps between two inequivalent irreducible representations.

Schur's Lemma tells us that if $V$ is an arbitrary representation of some group, then it is decomposed into the direct sum of irreducibles $W_i$ by

$$\begin{aligned} V &= m_1 W_1 \oplus m_2 W_2 \oplus \cdots \oplus m_k W_k \\ &= \bigoplus_{i=1}^{k} m_i W_i, \end{aligned}$$

where $m_i$ is the number of occurences of $W_i$ in $V$. This number is called the *multiplicity* of $W_i$ in $V$.

Schur's lemma also allows us to say something about the irreducible representations of an abelian group, that is a group in which every element commutes with every other element (for example $\mathcal{C}_n$).

**Corollary 3.26.** Any irreducible representation of an abelian group is of degree 1.

*Proof.* [FH04, *Mentioned in passing in* Sect.1.3.] Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of an abelian group $G$. Consider the linear map $\rho_h$ of some $h \in G$, then for any $g \in G$ we have

$$\begin{aligned} \rho_h \cdot \rho_g &= \rho_{gh} && (\rho \text{ is a homomorphism}) \\ &= \rho_{hg} && (G \text{ abelian}) \\ &= \rho_h \cdot \rho_g. && (\text{homomorphism again}) \end{aligned}$$

We have found that $\rho_h : V \to V$ is a $G$-linear map, provided $G$ is an abelian group. Then, by Schur's Lemma ii) it is a scalar multiple of the identity function, let's say $\rho_h = \lambda \cdot \mathrm{id}$ for some $\lambda \in \mathbb{C}$. Let $\mathbf{v} \in V$, then we have that

$$\begin{aligned} \varphi(\mathbf{v}) &= \lambda \mathrm{id} \cdot \mathbf{v} \\ &= \lambda \mathbf{v} \in \mathrm{Span}(\mathbf{v}) && (\mathrm{id} \text{ fixes } \mathbf{v}) \end{aligned}$$

Now, this means that the action on $\rho_g$ on an arbitrary $\mathbf{v} \in V$ changes $\mathbf{v}$ by a scalar factor, hence the one-dimensional $\mathrm{Span}(\mathbf{v})$ is a $G$-invariant subspace of $V$. However, $V$ was taken to be an irreducible representation, hence $V$ is identical to this subspace and $\dim V = 1$. □

# 4 Character Theory

This section is based on [Ser77, Ch.2.], as well as [FH04, Sect.2.2.].

Let $V$ be a vector space with basis $(\hat{\mathbf{e}}_i)_{i=1}^n$ and let $\rho : G \to \mathrm{GL}(V)$ be a representation. Now, for each $g \in G$, we define a function $\chi : G \to \mathbb{C}$ to be the trace of the matrix of $g$ in $V$. In other words $\chi(g) = \mathrm{Tr}\ (\rho_g)$. This function is called the *character* of a representation. Why is it useful when studying representations?

## 4.1 Basic properties of characters

**Proposition 4.1.** [Ser77, Prop.2.1.] The trace function *characterizes* the representation in some relevant ways:

i) The character of the identity element of $G$ is the degree of the representation, $\chi(e) = \dim V$.

ii) The character of an element in $G$ is the complex conjugate of the character of the inverse element, $\chi(g^{-1}) = \overline{\chi(g)}$.

iii) The character is constant under conjugation, if $g, h \in G$ are conjugate it implies $\chi(g) = \chi(h)$.

*Proof.* i) By Proposition 3.2, $\rho_e$ is the $n \times n$ identity matrix, where $n = \dim V$, hence $\chi(e) = \mathrm{Tr}\ \mathrm{id} = n = \dim V$.

ii) We are free to choose an orthonormal basis, then $\rho_g$ is a unitary matrix with roots of unity $\{\lambda_i\}$ as eigenvalues [Nic18, Exercise.8.6.15.]. The character of $g^{-1}$ is then

$$\begin{aligned}
\chi(g^{-1}) &= \mathrm{Tr}\ \rho_{g^{-1}} && \text{(Def. of character)} \\
&= \mathrm{Tr}\ \rho_g^{-1} && \text{(Prop.3.2.)} \\
&= \sum_i \frac{1}{\lambda_i} && \text{(Eigenvalues of unitary matrix)} \\
&= \sum_i \overline{\lambda_i} && \text{(Reciprocal of root of unity)} \\
&= \mathrm{Tr}\ \overline{\rho_g} && \text{(Trace is sum of eigenvalues)} \\
&= \overline{\chi(g)}. && \text{(Def. of character)}
\end{aligned}$$

iii) It is known that the trace is conserved under conjugation (see Prop.2.5.). □

Now we properly introduce the character.

**Definition 4.2.** Let $V$ be a representation space of a group $G$ of dimension $n$ and let $\rho_g = (r_{ij})$ be the matrix representation of a $g \in G$. Then the *character* of $g$ in $V$ is defined to be the trace of the matrix representation, that is

$$\chi_V(g) := \text{Tr } \rho_g = \sum_{i=1}^{n} r_{ii}.$$

*Notation* (Group character). A "character vector", simply called the (group) character, of $G$ can be defined as the tuple containing the character of every element of $G$, ie. $\chi = (\chi(g))_{g \in G}$. If the group $G$ is partitioned into the conjugacy classes $[k_1], [k_2], \ldots, [k_l]$, then as an ink-saving measure the group character can be abbreviated to contain one representative $k_i$ from every class $[k_i]$, ie. $\chi = (\chi(k_i))_{i=1}^{l}$, since the character is fixed under conjugation.

*Example* 4.3 (Degree 1 representations). The trace of a $1 \times 1$ matrix is of course its only element, hence the character of any representations of degree 1 is the representation itself. For example, the trivial character of any group is $(1, \ldots, 1)$.

*Example* 4.4 (Character of a permutation matrix). The character is the sum along the diagonal of a matrix, which in a permutation matrix (eg. the permutation and regular representations) correseponds to the fixed points of the group action. For example the element $(1, 2) \in \mathcal{S}_3$ has one fixed point (3) and thus has the character 1 in the permutation representation.

*Notation* (Character table). A character table is an array of characters of a group. Every column represents a conjugacy class and every row a representation. The trivial representation is placed in the first row. The sizes of every conjugacy class is placed under each class. An example is shown in Table 10.

*Example* 4.5 ($\mathcal{S}_3$ so far). So far, we have described the trivial, alternating, permutation and standard representations of $\mathcal{S}_3$ (Examples 3.3, 3.4, 3.9, 3.20). These representations are presented in Table 11.

## 4.2 Characters and tensor operations

Addition and multiplication of two characters $\chi$ and $\psi$ are defined component-wise thusly:

$$\chi + \psi = (\chi(g) + \psi(g))_{g \in G}, \text{ and}$$

| $G$ | $\cdots$ | $[g]$ | $\cdots$ |
|---|---|---|---|
| $\|[g]\|$ | $\cdots$ | $\|[g]\|$ | $\cdots$ |
| Triv. | $\cdots$ | 1 | $\cdots$ |
| $\vdots$ | | $\vdots$ | |
| $V$ | $\cdots$ | $\chi_V(g)$ | $\cdots$ |
| $\vdots$ | | $\vdots$ | |

Table 10: Layout of a character table of a group $G$.

| $\mathcal{S}_3$ | $[(1)]$ | $[(1,2)]$ | $[(1,2,3)]$ |
|---|---|---|---|
| $\|[\sigma]\|$ | 1 | 3 | 2 |
| Triv. | 1 | 1 | 1 |
| Alt. | 1 | $-1$ | 1 |
| Perm. | 3 | 1 | 0 |
| Stan. | 2 | 0 | -1 |

Table 11: Character table of $\mathcal{S}_3$.

$$\chi \cdot \psi = (\chi(g) \cdot \psi(g))_{g \in G}.$$

To apply character theory on the tensor toolbox presented in Section 3.1, we propose the following:

**Proposition 4.6.** Let $V$ and $W$ be representation spaces of a group $G$ and let $\chi_V$ and $\chi_W$ be its characters in those representations. Then we propose:

i) The character in $V \oplus W$ is $\chi_V + \chi_W$.

ii) The character in $V \otimes W$ is $\chi_V \cdot \chi_W$.

*Proof.* Statement i) and ii) are consequences of Section 2.3.

i) The trace of $\rho_g^V \oplus \rho_g^W$ is clearly the sum of the traces of $\rho_g^V$ and $\rho_g^W$.

ii) Likewise, the trace of $\rho_g^V \otimes \rho_g^W$ is found to be the product of the traces of $\rho_g^V$ and $\rho_g^W$. □

The character of the direct sum or tensor product of an arbitrary number of representations are defined similarly.

*Example* 4.7 ($\mathcal{S}_3$ again)*.* The permutation representation was earlier found to be the direct sum of the trivial and the standard representations, in fact

$$\chi_{\text{Triv}} + \chi_{\text{Stan}} = (1,1,1) + (2,0,-1) = (3,1,0) = \chi_{\text{Perm}}.$$

*Example* 4.8 ($\mathcal{S}_4$ so far)*.* For $\mathcal{S}_4$, we know the trivial representation and the alternating representation.

Using Example 4.4, we can find the permutation character $(4,2,1,0,0)$ by counting the number of fixed points of every class, equivalently we count the number of ones in the integer partitions of Table 2.

We may also calculate the character of the standard representation by subtracting the trivial character from the permutation character. These representations are presented in Table 12.

| $\mathcal{S}_4$ | $[(1)]$ | $[(1,2)]$ | $[(1,2,3)]$ | $[(1,2,3,4)]$ | $[(1,2)(3,4)]$ |
|---|---|---|---|---|---|
| $\lvert[\sigma]\rvert$ | 1 | 6 | 3 | 6 | 8 |
| Triv. | 1 | 1 | 1 | 1 | 1 |
| Alt. | 1 | -1 | 1 | -1 | 1 |
| Stan. | 3 | 1 | 0 | -1 | -1 |
| Perm. | 4 | 2 | 1 | 0 | 0 |

Table 12: Character table of $\mathcal{S}_4$.

*Example* 4.9 (Tensor power of alternating representation). The character of the alternating representation of a symmetric group is $\chi_{\text{Alt}} = (1, -1, 1, -1, \ldots, 1)$, hence the $n$th tensor power of the alternating representation has the character

$$\begin{aligned}\chi_{\text{Alt}}^n &= (1, (-1)^n), 1, (-1)^n, \ldots, 1)\\ &= \begin{cases}\chi_{\text{Triv}}, \text{ if } n \text{ is even, and}\\ \chi_{\text{Alt}}, \text{ if } n \text{ is odd.}\end{cases}\end{aligned}$$

That is, an even tensor power is isomorphic to the trivial representation and an odd power is isomorphic to the alternating representation.

## 4.3 Orthogonality relations of characters

This section is based on [FH04, Sect.2.2.].

Continuing the discussion of Section 3.4, we want find characters of irreducible representations. We call those irreducible characters. Let $V$ be an arbitrary representation and let $W$ be an irreducible representation of some group $G$.

We denote by $\text{Hom}(V, W)$ the set of all homomorphisms from $V$ to $W$, and by the superscript $(\cdot)^G$ we denote a subset which is fixed by $G$, for example

$$V^G = \{\mathbf{v} \in V \mid g \cdot \mathbf{v} = \mathbf{v},\ \forall g \in G\} \tag{4.1}$$

is the subspace of $V$ in which every element is fixed by all of $G$. Also, $\text{Hom}(V, W)^G$ is the set of all $G$-linear maps $V \to W$, and the set of linearly independent $G$-linear maps for a basis for $\text{Hom}(V, W)^G$.

**Proposition 4.10.** The multiplicity of $W$ in $V$ is $\dim \mathrm{Hom}(W, V)^G$, that is the number of (linearly independent) $G$-linear maps from $W$ to $V$. Conversely, if $W$ is arbitrary and $V$ is irreducible, then $\dim \mathrm{Hom}(V, W)^G$ is the multiplicity of $V$ in $W$. If both $V$ and $W$ are irreducible, then by Schur's Lemma we must have that

$$\dim \mathrm{Hom}(V, W)^G = \begin{cases} 1, & \text{if } V \text{ and } W \text{ are isomorphic, and} \\ 0, & \text{else.} \end{cases} \tag{4.2}$$

Now, let $V^G$ be defined as the fixed points of $V$ under the action of $G$, as defined in Equation 4.1. A map from $V^G$ to itself is called an *endomorphism of* $V^G$ and we set $\mathrm{End}(V^G) := \mathrm{Hom}(V^G, V^G)$. Such a map is not just a representation of $G$, but also a trivial representation by definition, then the number of all such maps, denoted $\dim \mathrm{End}(V^G)$ is the multiplicity of the trivial representation in $V$.

The map associated with any $g$ is not generally a $G$-linear map, however as we have previously seen, we can construct such a map by averaging over $G$, that is we define a map $\varphi : V \to V$ by

$$\varphi = \frac{1}{|G|} \sum_{g \in G} g,$$

which is $G$-linear since for some $h \in G$,

$$\begin{aligned} h \cdot \varphi(h^{-1} \cdot \mathbf{v}) &= \frac{1}{|G|} \sum_{g \in G} (hgh^{-1}) \cdot \mathbf{v} \\ &= \varphi(\mathbf{v}) \end{aligned}$$

for any $\mathbf{v} \in V$, also it is projection of $V$ onto $V^G$ since the image of $\varphi$ is $V^G$ and clearly it then also is an endomorphism of $V^G$.

Now, the eigenvalues of a projection are 1 for every eigenvector in the image and 0 for every eigenvector in the kernel, hence the trace of the projection $\varphi$ is the dimension of $V^G$, that is

$$\begin{aligned} \dim V^G = \mathrm{Tr}\ \varphi &= \mathrm{Tr} \left( \frac{1}{|G|} \sum_{g \in G} g \right) \\ &= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}\ g \qquad \text{(Trace is linear operator)} \end{aligned}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \qquad \text{(Def. of character)},$$

hence

$$\dim V^G = \frac{1}{|G|} \sum_g \chi_V(g). \tag{4.3}$$

Now, let's apply Equation 4.3 to the set of all maps $V \to W$, that is $V$ becomes $\mathrm{Hom}(V,W)$, where both $V$ and $W$ are irreducible representations of $G$. Earlier in Section 2.3, $\mathrm{Hom}(V,W)$ was identified with $V^* \otimes W$, hence the character of $\mathrm{Hom}(V,W)$ is $\overline{\chi_V(g)} \cdot \chi_W(g)$, then by Equations 4.2 and 4.3 we have:

$$\dim \mathrm{Hom}(V,W)^G = \frac{1}{|G|} \sum_g \overline{\chi_V(g)} \cdot \chi_W(g) = \begin{cases} 1, & \text{if } V \cong W, \text{ and} \\ 0, & \text{if } V \neq W. \end{cases}$$

We have arrived at an expression that looks familiar, interpreting characters as complex-valued vectors in some vector space, we have found an *inner product of characters.*

**Definition 4.11** (Inner product of characters)**.** Let $\varphi$ and $\psi$ be the group characters of a group $G$, then we define the inner product of characters as

$$(\varphi|\psi) := \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)}\psi(g).$$

It is an inner product since it satisfies the expected properties:

i) We have

$$\begin{aligned} \overline{(\varphi|\psi)} &= \frac{1}{|G|} \sum_{g \in G} \overline{\overline{\varphi(g)}\psi(g)} && \text{(Def. inner product)} \\ &= \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\psi(g)} && (\overline{\overline{a}} = a \text{ for all } a \in \mathbb{C}) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\psi(g)}\varphi(g) && \text{(Scalars commute)} \\ &= (\psi|\varphi). && \text{(Def. inner product)} \end{aligned}$$

ii) For $a, b \in \mathbb{C}$ we have

$$
\begin{aligned}
(\varphi | a\psi_1 + b\psi_2) &= \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \left( a\psi_1(g) + b\psi_2(g) \right) && \text{(Def.)} \\
&= a \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi_1(g) + b \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi_2(g) \\
&= a(\varphi|\psi_1) + b(\varphi|\psi_2) && \text{(Def.)}
\end{aligned}
$$

for any characters $\varphi, \psi_1$ and $\psi_2$, hence it is linear in $\psi$.

iii) Lastly, we have

$$
\begin{aligned}
(\varphi|\varphi) &= \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \varphi(g) \\
&= \frac{1}{|G|} \sum_{g \in G} |\varphi(g)|^2 && \text{(Def. modulus)}
\end{aligned}
$$

for any character $\varphi \neq \mathbf{0}$, hence $(\psi|\psi) > 0$ and $(\psi|\psi) \in \mathbb{R}$ since every summand is positive and real.

We have also shown the following property of irreducible representations:

**Theorem 4.12** (Irreducibility criterion)**.** The characters of irreducible representations are orthonormal. In other words, let $\chi \neq \psi$ be characters of irreducible representations of $G$ (called irreducible characters), then we have that

i) $(\chi|\chi) = 1$, and
ii) $(\chi|\psi) = 0$.

In other words, irreducible characters create an orthonormal system, with these irreducibles as a basis.

Now, let $V$ be the direct sum of irreducible representations $W_i$ of $G$ such that

$$V = \bigoplus_i W_i.$$

If $\chi_i$ is the character of $W_i$, then by Proposition 4.6 the character of $V$ is $\varphi = \sum_i \chi_i$. Let $\chi$ be the character of an irreducible representation of $G$, then we have that $(\varphi|\chi) = \sum_i (\chi_i|\chi)$. Since $\chi$ and all of the $\chi_i$ are irreducible characters, all of the inner products $(\chi_i|\chi)$ are either 1 or 0, depending on if $\chi$ and $\chi_i$ are of isomorphic representations. Hence, $(\varphi|\chi)$ will return the multiplicity of $\chi$ in $\varphi$.

*Remark* 4.13. This also means that the composition of a representation $\varphi$ into a direct sum of irreducible subrepresentations $\chi_i$ is unique up to isomorphism, since two isomorphic compositions would have the same decomposition, that is it would be constructed the same way of the same irreducibles.

*Remark* 4.14. The converse is also true, if two representations have the same character, then they are isomorphic since they contain the same irreducible representations.

*Note.* Let $V$ be a representation with the composition

$$V = \bigoplus_{i=1}^{k} m_i W_i,$$

where the $m_i$ are the multiplicities of the irreducible $W_i$. If the character of $W_i$ is denoted by $\chi_i$, then the character of $V$ is

$$\varphi = \sum_{i=1}^{k} m_i \chi_i,$$

where $m_i = (\varphi|\chi_i)$. Taking the inner product of $\varphi$ with itself we have

$$(\varphi|\varphi) = \sum_{i=1}^{k} m_i^2. \tag{4.4}$$

**Theorem 4.15.** The character $\varphi$ is irreducible if and only if $(\varphi|\varphi) = 1$.

*Proof.* If $\varphi$ is irreducible, then by Theorem 4.12 we have that $(\varphi|\varphi) = 1$. For the converse statement, if the sum in Equation 4.4 is equal to one, we must have that only one of the $m_i$ is equal to 1, and the rest are equal to 0. Then $V$ is composed of only *one* irreducible character, hence it is irreducible. □

*Example* 4.16 ($\mathcal{S}_3$). The standard representation of $\mathcal{S}_3$ is irreducible since

$$\begin{aligned}(\chi_{\text{Stan}}|\chi_{\text{Stan}}) &= \frac{1}{6}(2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) \\ &= 1.\end{aligned}$$

hence the standard representation of $\mathcal{S}_3$ is irreducible. The trivial and the standard representations are inequivalent since

$$(\chi_{\text{Triv}}|\chi_{\text{Stan}}) = \frac{1}{6}(2 + 3 \cdot 0 - 2) = 0.$$

We already know that the permutation representation is not irreducible. This is verified by

$$(\chi_{\text{Perm}}|\chi_{\text{Perm}}) = \frac{1}{6}(9 + 3 \cdot 1 + 0) = 2.$$

*Example* 4.17 ($\mathcal{S}_4$). Likewise, the standard character of $\mathcal{S}_4$, $\chi^{\text{Stan}} = (3, 1, 0, -1, -1)$, is also irreducible since

$$\begin{aligned}(\chi_{\text{Stan}}|\chi_{\text{Stan}}) &= \frac{1}{24}(3^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) \\ &= 1.\end{aligned}$$

## 4.4 Decomposition of the regular representation

Earlier, the regular representation of any group where found have the trivial representation as a subrepresentation, likewise, the alternating group was found in the regular representation of $\mathcal{S}_n$. Now, we will completely decompose the regular representation of any group.

For a $g \in G$, it will act on a basis vector $\hat{\mathbf{e}}_h$ of the regular representation space $V$ by $g \cdot \hat{\mathbf{e}}_h = \hat{\mathbf{e}}_{gh}$ and the resulting matrix $\rho_g$ can then be constructed by studying the action of this $g \in G$ on every $\hat{\mathbf{e}}_h$, where $h \in G$. However, the trace of this matrix, ie. the character $\chi_{\text{Reg}}(g)$, only depends on the values on the diagonal, which corresponds to the fixed points under the action of $g$. What this means is that the "$h$th" column will have an 1 in the "$h$th" row if and only if $gh = h$, which holds only if $g = e$ since only $e$ leaves every other element of $G$ fixed, hence $\rho_g$ is a $|G| \times |G|$ permuation matrix with trace

$$\chi_{\text{Reg}}(g) = \begin{cases} |G|, \text{ if } g = e, \\ 0, \text{ otherwise}, \end{cases} \tag{4.5}$$

and the character of the regular representation is then

$$\chi_{\text{Reg}} = (|G|, 0, \ldots, 0)$$

For any group $G$. Now taking the inner product of it with itself we have,

$$\begin{aligned}
(\chi_{\text{Reg}}|\chi_{\text{Reg}}) &= \frac{1}{|G|}\sum_{g\in G}\overline{\chi^{\text{Reg}}(g)}\chi^{\text{Reg}}(g) \\
&= \frac{1}{|G|}\overline{\chi^{\text{Reg}}(e)}\chi^{\text{Reg}}(e) && \text{(All vanish except } g=e.\text{)} \\
&= \frac{1}{|G|}(\dim V)^2 && \text{(Prop. 4.1).} \\
&= \frac{1}{|G|}|G|^2 && \text{(From def of } V.\text{)} \\
&= |G|.
\end{aligned}$$

Applying Equation 4.4, we have that $(\chi^{\text{Reg}}|\chi^{\text{Reg}})$ is the square sum of the multiplicities of every irreducible subrepresentation of the regular representation, hence we have:

$$\sum_i m_i^2 = |G|. \tag{4.6}$$

If $\{W_i\}$ is the family of all irreducible representations of a group with respective characters $\{\chi_i\}$, then the regular representation $V$ is a direct sum of these with (not-necessarily non-zero) multiplicities $\{m_i\}$. Then we have that the multiplicity of some irreducible $W_j$ with character $\chi_j$ in $\chi^{\text{Reg}}$ is

$$\begin{aligned}
m_j &= (\chi_{\text{Reg}}|\chi_j) \\
&= \sum_i m_i(\chi_i|\chi_j) && \text{(Reg. is sum of } m_i\chi_i\text{)} \\
&= \frac{1}{|G|}\sum_i m_i \sum_{g\in G}\overline{\chi_i(g)}\chi_j(g). && \text{(Def. of inner product.)} \\
&= \frac{1}{|G|}\sum_i m_i\overline{\chi_i(e)}\chi_j(e) && (\chi_i \text{ vanish for all } g\neq e.) \\
&= \frac{1}{|G|}\sum_i m_i \dim W_i \dim W_j && \text{(Character of } e \text{ is degree of repr.)} \\
&= \dim W_j \frac{1}{|G|}\sum_i m_i \dim W_i \\
&= \dim W_j \frac{1}{|G|}\dim V && (V=\textstyle\sum_i m_i W_i) \\
&= \dim W_j \frac{1}{|G|}|G| && \text{(From def. of regular repr.)}
\end{aligned}$$

$$= \dim W_j.$$

We have shown the following:

**Theorem 4.18.** Every irreducible representation $W_i$ of a group appears exactly $\dim W_i$ times in the regular representation $V$ of the group. In other words, the regular representation is decomposed as

$$V = \bigoplus_i \dim(W_i) W_i.$$

Returning to Equation 4.6, we have the following:

**Corollary 4.19.** The square sum of the degrees of every irreducible representations of a group is the order of the group.

*Remark* 4.20. By Theorem 4.18 and Corollary 4.19, along with the orthogonality relations and irreducibility criterion, we can use character theory to find **every** irreducible representation of a group.

For example, to ensure we have found an irreducible representations of a group, we take the inner product of its character with itself, expecting 1 if it is irreducible. To ensure we have found every irreducible representation, we calculate the square sum of the degrees of those we found so far. If it does not add up to the order of the group, then we can conjecture possible degrees of the missing irreducibles.

## 4.5 Examples

*Notation.* For this section, a representation space may be denoted by its designation in the subscript, that is a trivial representation space is denoted by $V_{\text{Triv}}$, a permutation by $V_{\text{Perm}}$, etc.

### 4.5.1 Characters of $\mathcal{C}_n$

*Example* 4.21 (Characters of $\mathcal{C}_n$). By Equation 4.5, the character of the regular representation of $\mathcal{C}_n$ is $\chi_{\text{Reg}} = (n, 0, \dots)$. In section 3.2.2 we described $n$ irreducible representations of degree 1 of $\mathcal{C}_n$, and by Theorem 4.18, we have then found every irreducible representation of $\mathcal{C}_n$ since $\sum_{i=1}^{n} 1^2 = n$.

Remember that the trace of a $1 \times 1$ matrix is its only element, therefore the Tables 3, 4 and 5 are the character tables of respectively $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{C}_5$.

### 4.5.2 Characters of $\mathcal{S}_n$

*Example* 4.22 (Characters of $\mathcal{S}_3$). Returning to Table 11, we have found three irreducible representations of $\mathcal{S}_3$. Their degrees are 1, 1 and 2, and the square sum of the degrees are $1+1+4=6$ which is the order of $\mathcal{S}_3$, hence by Corollary 4.19, we have found every irreducible representation of $\mathcal{S}_3$.

Now we decompose the tensor powers of the standard representation.

**Decomposition of $V_{\mathbf{Stan}} \otimes V_{\mathbf{Stan}}$.** Its character is $(4,0,1)$, which by a quick glance on the character table is seen to be the sum of all irreducibles, hence

$$V_{\text{Stan}} \otimes V_{\text{Stan}} = V_{\text{Triv}} \oplus V_{\text{Alt}} \oplus V_{\text{Stan}}.$$

**Decomposition of $V_{\mathbf{Stan}}^{\otimes n}$.** [FH04, Exercise 2.7.] To find the decomposition of larger tensor powers, we study the character $\chi_{\text{Stan}}^n = (2^n, 0, (-1)^n)$ and take the inner product of it with the irreducibles and find:

$$(\chi_{\text{Triv}}|\chi_{\text{Stan}}^n) = (\chi_{\text{Alt}}|\chi_{\text{Stan}}^n) = \frac{1}{6}\left(2^n + (-1)^n\right), \quad \text{and}$$
$$(\chi_{\text{Stan}}|\chi_{\text{Stan}}^n) = \frac{1}{6}\left(2^{n+1} + (-1)^{n+1}\right),$$

hence

$$V_{\text{Stan}}^{\otimes n} = a_n(V_{\text{Triv}} \oplus V_{\text{Alt}}) \oplus a_{n+1}V_{\text{Stan}}, \quad \text{where} \quad a_n = \frac{1}{6}\left(2^n + (-1)^n\right).$$

We arrive at a "complete" character table for $\mathcal{S}_3$, presented in Table 13.

| $S_3$ | $[(1)]$ | $[(1,2)]$ | $[(1,2,3)]$ |
|---|---|---|---|
| $\lvert[\sigma]\rvert$ | 1 | 3 | 2 |
| $\chi_{\text{Triv}}$ | 1 | 1 | 1 |
| $\chi_{\text{Alt}}$ | 1 | -1 | 1 |
| $\chi_{\text{Stan}}$ | 2 | 0 | -1 |
| $\chi_{\text{Perm}}$ | 3 | 1 | 0 |
| $\chi_{\text{Stan}}^2$ | 4 | 0 | 1 |
| $\chi_{\text{Reg}}$ | 6 | 0 | 0 |

Table 13: Complete character table of $S_3$. The representations above the doublestruck line are irreducibles, and those below are composed.

*Example* 4.23 (Characters of $\mathcal{S}_4$). This section follows the same methods from the previous section on $\mathcal{S}_3$. So far, we have that the trivial, the alternating and the standard representations are irreducibles, see Table 12. Their respective degrees are 1, 1 and 3, with the square sum $1 + 1 + 9 = 11$, which is less than $|\mathcal{S}_4| = 24$, hence by Corollary 4.19 we are expected to find additional irreducible representations of $\mathcal{S}_4$. We will find them while constructing and decomposing new representations.

**Decomposition of $V_{\mathbf{Alt}} \otimes V_{\mathbf{Stan}}$.** Consider an "alternating version of the standard representation", denoted with the space $V'_{\mathrm{Stan}} := V_{\mathrm{Alt}} \otimes V_{\mathrm{Stan}}$ and the character $\chi'_{\mathrm{Stan}} := \chi_{\mathrm{Alt}}\chi_{\mathrm{Stan}} = (3, -1, 0, 1, -1)$. It clearly has the same inner product with itself as $\chi_{\mathrm{Stan}}$, hence it is also irreducible. It increments the square sum of degrees to $11 + 3^2 = 20$.

**Decomposition of $V_{\mathbf{Stan}} \otimes V_{\mathbf{Stan}}$.** To find the next irreducible, we study the tensor square of the standard representation. Its character is $\chi^2_{\mathrm{Stan}} = (9, 1, 0, 1, 1)$ and calculations will show that

$$\begin{aligned}
(\chi_{\mathrm{Triv}}|\chi^2_{\mathrm{Stan}}) &= \frac{1}{24}(9 + 6 + 0 + 6 + 3) = 1,\\
(\chi_{\mathrm{Alt}}|\chi^2_{\mathrm{Stan}}) &= \frac{1}{24}(9 - 6 + 0 - 6 + 2) = 0,\\
(\chi_{\mathrm{Stan}}|\chi^2_{\mathrm{Stan}}) &= \frac{1}{24}(27 + 6 + 0 - 6 - 3) = 1, \quad \text{and}\\
(\chi_{\mathrm{Alt}}\chi_{\mathrm{Stan}}|\chi^2_{\mathrm{Stan}}) &= \frac{1}{24}(27 - 6 + 0 + 6 - 3) = 1.
\end{aligned}$$

However these subrepresentations of non-zero multiplicity are of degrees 1, 3, and 3, and their direct sum is of degree 7, which is less than the dimension of $V_{\mathrm{Stan}} \otimes V_{\mathrm{Stan}}$, hence by Proposition 3.19 there is another (not necessarily irreducible) subrepresentation of degree 2 complementing them. Denoting this representation by $W$, its character is $\chi_W = \chi^2_{\mathrm{Stan}} - \chi_{\mathrm{Triv}} - \chi_{\mathrm{Stan}} - \chi'_{\mathrm{Stan}} = (2, 0, -1, 0, 2)$, which is found to be such that

$$(\chi_W|\chi_W) = \frac{1}{24}(4 + 0 + 8 + 0 + 12) = 1,$$

hence $W$ is irreducible and the tensor square of the standard representation is decomposed to

$$V_{\mathrm{Stan}} \otimes V_{\mathrm{Stan}} = V_{\mathrm{Triv}} \oplus W \oplus V_{\mathrm{Stan}} \oplus V'_{\mathrm{Stan}}.$$

We have now found all five irreducible representations of $\mathcal{S}_4$, since $1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24$, the order of $\mathcal{S}_4$.

**Decomposition of the $n$th tensor power of $W$.** The character of $W^{\otimes n}$ is

$$\chi_W^n = (2^n, 0, (-1)^n, 0, 2^n).$$

Its inner product with the irreducible characters are found to be

$$\begin{aligned}(\chi_{\text{Triv}}|\chi_W^n) = (\chi_{\text{Alt}}|\chi_W^n) &= \frac{1}{6}\left(2^n + 2\cdot(-1)^n\right), \\ (\chi_W|\chi_W^n) &= \frac{1}{6}\left(2^{n+1} + 2\cdot(-1)^{n+1}\right), \text{ and} \\ (\chi_{\text{Stan}}|\chi_W^n) = (\chi'_{\text{Stan}}|\chi_W^n) &= 0,\end{aligned}$$

hence the decomposition is

$$W^{\otimes n} = a_n(T \oplus A) \oplus a_{n+1}W, \quad \text{where} \quad a_n := \frac{1}{6}\left(2^n + 2\cdot(-1)^n\right).$$

**Decomposition of the $n$th tensor power of $V_{\text{Stan}}$.** The character of $V_{\text{Stan}}^{\otimes n}$ is

$$\chi_{\text{Stan}}^n = (3^n, 1, 0, (-1)^n, (-1)^n),$$

and after projecting it on the irreducible characters we find that

$$V_{\text{Stan}}^{\otimes n} = a_n V_{\text{Triv}} \oplus b_n V_{\text{Alt}} \oplus c_n W \oplus a_{n+1} V_{\text{Stan}} \oplus b_{n+1} V'_{\text{Stan}}, \tag{4.7}$$

where

$$\begin{cases} a_n = \frac{1}{24}(3^n + 9\cdot(-1)^n + 6), \\ b_n = \frac{1}{24}(3^n - 3\cdot(-1)^n - 6), \quad \text{and} \\ c_n = \frac{1}{12}(3^n + 3\cdot(-1)^n). \end{cases}$$

**Decomposition of the $n$th tensor powers of $V'_{\text{Stan}}$.** Since $V'_{\text{Stan}} = V_{\text{Alt}} \otimes V_{\text{Stan}}$, the character of $(V'_{\text{Stan}})^{\otimes n}$ is the $n$th power of the character $\chi_{\text{Alt}}\chi_{\text{Stan}}$, which is $\chi_{\text{Alt}}^n \chi_{\text{Stan}}^n$, hence $(V'_{\text{Stan}})^{\otimes n} = V_{\text{Stan}}^{\otimes n} \otimes V_{\text{Alt}}^{\otimes n}$.

By Example 4.9, the $n$th tensor power of $V_{\text{Alt}}$ is $V_{\text{Triv}}$ if $n$ is even and $V_{\text{Alt}}$ if $n$ is odd, so for even $n$, $(V'_{\text{Stan}})^{\otimes n}$ is isomorphic to $V_{\text{Stan}}^{\otimes n}$, and for odd $n$, it is isomorphic to $V_{\text{Stan}}^{\otimes n} \otimes V_{\text{Alt}}$. Tensor multiplying Equation 4.7 with $V_{\text{Alt}}$, we have:

$$\begin{aligned} V_{\text{Stan}}^{\otimes n} \otimes V_{\text{Alt}} &= \left(a_n V_{\text{Triv}} \oplus b_n V_{\text{Alt}} \oplus c_n W \oplus a_{n+1} V_{\text{Stan}} \oplus b_{n+1} V'_{\text{Stan}}\right) \otimes V_{\text{Alt}} \\ &= b_n V_{\text{Triv}} \oplus a_n V_{\text{Alt}} \oplus c_n W \oplus b_{n+1} V_{\text{Stan}} \oplus a_{n+1} V'_{\text{Stan}}, \end{aligned}$$

since

$$\begin{aligned} V_{\text{Triv}} \otimes V_{\text{Alt}} &= V_{\text{Alt}}, \\ V_{\text{Alt}} \otimes V_{\text{Alt}} &= V_{\text{Triv}}, \\ V_{\text{Stan}} \otimes V_{\text{Alt}} &= V'_{\text{Stan}}, \quad \text{and} \\ V'_{\text{Stan}} \otimes V_{\text{Alt}} &= V_{\text{Stan}} \otimes V_{\text{Alt}}^{\otimes 2} = V_{\text{Stan}}. \end{aligned}$$

Then we have found the decomposition of all tensor powers of $V'_{\text{Stan}}$: For even $n$, it is the same as $V_{\text{Stan}}^{\otimes n}$, and lastly for odd $n$, $V_{\text{Triv}}$ switch multiplicities with $V_{\text{Alt}}$, and $V_{\text{Stan}}$ switch with $V'_{\text{Stan}}$.

The findings of the last few paragraphs (except those on larger tensor powers) are presented in table 14.

| $\mathcal{S}_4$ | $[(1)]$ | $[(1,2)]$ | $[(1,2,3)]$ | $[(1,2,3,4)]$ | $[(1,2)(3,4)]$ |
|---|---|---|---|---|---|
| $\lvert[\sigma]\rvert$ | 1 | 6 | 8 | 6 | 3 |
| $\chi_{\text{Triv}}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_{\text{Alt}}$ | 1 | -1 | 1 | -1 | 1 |
| $\chi_W$ | 2 | 0 | -1 | 0 | 2 |
| $\chi_{\text{Stan}}$ | 3 | 1 | 0 | -1 | -1 |
| $\chi'_{\text{Stan}}$ | 3 | -1 | 0 | 1 | -1 |
| $\chi_{\text{Perm}}$ | 4 | 2 | 1 | 0 | 0 |
| $\chi^2_{\text{Stan}}$ | 9 | 1 | 0 | 1 | 1 |

Table 14: Complete character table of $\mathcal{S}_4$. The representations above the doublestruck line are irreducibles, and those below are composed.

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