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Multi-Avoidance of Permutation Patterns

av

Patryk Gozdur

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Patryk Gozdur

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Handledare: Per Alexandersson

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1 Abstract

1.1 English

Permutation patterns is an interesting and niche branch of mathematics that give rise to many fascinating sequences ranging from simple combinatorial operations all the way to the Catalan and Fibonacci numbers. In this thesis we study the basics of permutation patterns and prove some of the more well-known results that have been discovered during the last century. We look at permutation patterns from a more combinatorial approach rather than an algorithmical approach that is more common as seen in the works of Donald Knuth. We cover avoidance of permutation patterns and explore multi-avoidance and the kind of sequences that occur as a result of patterns avoiding more than two 3-patterns.

1.2 Svenska

Permutationmönster är ett intressant och nisch gren av matematik som ger upphov till många fascinerande sekvenser, allt från enkla kombinatoriska operationer till de Katalanska och Fibonacci-talen. I det här arbetet studerar vi grunderna i permutationmönster och bevisar några av de mer välkända resultaten som har upptäcks under det senaste århundradet. Vi tittar på permutation-mönster från ett mer kombinatoriskt perspektiv snarare än ett algoritmiskt synsätt vilket är något som är mer vanligt i verken av Donald Knuth. Vi förklarar och ger exempel av undvikande av permutationmönster och går igenom mång-undvikande samt vilka sorter av sekvenser som uppstår som ett resultat av mång-undvikande av fler än två 3-mönster.

Contents

2 Introduction and Background

The history of permutation patterns is relatively brief and has been ongoing for barely over a century. Percy MacMahon was one of the first mathematicians who delved into the topic and showed in 1915 that 123-avoiding permutations are counted by the Catalan numbers in Section 5 of his book Combinatory Analysis although back then he called them "lattice permutations"[7]. During the latter half of the 19th century Donald Knuth came along and showed that stack-sortable permutations are enumerated by the Catalan numbers[6]. It was not until later when Simion and Schmidt constructed the first useful bijection between 123- and 231-avoiding permutations[11]. The topic of avoiding permutations of size 3 has been thoroughly studied since and their relation to the Catalan numbers is well-understood.

In this thesis we introduce permutation patterns and introduce the necessary definitions, we prove important theorems and show examples to provide the reader with a good understanding of the subject even if the reader does not have proficiency of the topic. Furthermore in Section 4 we expand on the previous concepts and examine multi-avoidance. We show interesting results and sequences that arise when avoiding multiple permutations simultaneously and prove the formula for containing all permutation patterns of length 3. In section 5 we cover Wilf-classes and some interesting topics related to permutation patterns, we also touch upon recent discoveries within the field in order to inspire the reader to delve deeper into the topic of permutation patterns.

3 Basics

3.1 Permutation Patterns

Definition 3.1 (Permutation)**.** Define a *permutation* π of length n as a bijection from $[n]$ onto itself where $[n]$ denotes the first n positive integers. The length of π can be denoted as $\text{size}(\pi) = n$ and π can be denoted as an ordered list:

$$\pi = [\pi(1), \pi(2), \dots, \pi(n)]. \tag{1}$$

We write the empty permutation as $\emptyset = [\,]$.

Example 3.1 (Permutation examples)**.** Let $\pi_1 = [4, 1, 6, 2, 3, 5]$ be a permutation of length 6 and let $\pi_2 = [3, 1, 4, 2]$ be a permutation of length 4. We can verify that both π_1 and π_2 are valid permutations as all the elements are unique integers less than or equal to the length given but we can explicitly confirm by inspecting the bijections π_1 and π_2:

$$\pi_1 = [\pi(1), \pi(2), \pi(3), \pi(4), \pi(5), \pi(6)] = [4, 1, 6, 2, 5, 3], \tag{2}$$

$$\pi_2 = [\pi(1), \pi(2), \pi(3), \pi(4)] = [3, 1, 4, 2]. \tag{3}$$

Definition 3.2 (Permutation matrix)**.** Define a *permutation matrix* $M(\pi)$ of a permutation π of length n as a $n \times n$ matrix where $\pi(i) = j \implies M_{j,i}(\pi) = 1$ and $\pi(i) \neq j \implies M_{j,i}(\pi) = 0$. In this text we will refer to non-zero entries of the matrix as *elements* for the sake of simplicity.

Example 3.2 (Permutation matrix example)**.** Inspect the permutation from earlier $\pi_2 = [3, 1, 4, 2]$. We get the following values from the definition:

$$\begin{aligned} \pi(1) = 3 &\implies M_{3,1}(\pi_2) = 1, \\ \pi(2) = 1 &\implies M_{1,2}(\pi_2) = 1, \\ \pi(3) = 4 &\implies M_{4,3}(\pi_2) = 1, \\ \pi(4) = 2 &\implies M_{2,4}(\pi_2) = 1, \end{aligned} \tag{4}$$

which gives us the following 4×4 matrix:

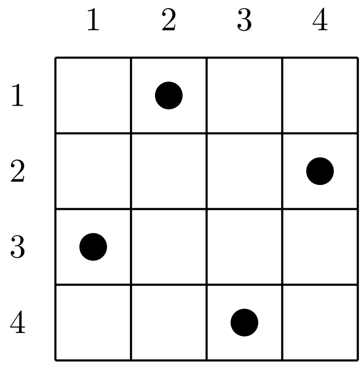
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Definition 3.3 (Permutation group)**.** Define the *permutation group* S_n to be the group of all possible permutations of length n. This group has $n!$ elements.

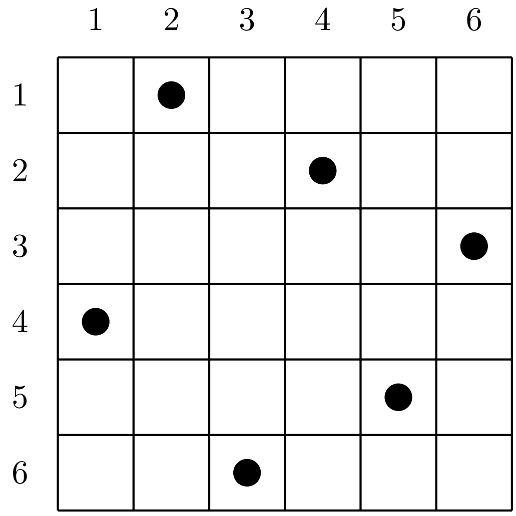
Example 3.3 (Permutation group example)**.** $S_3 = \{[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]\}$. Note that creating a new group of all the permutation matrices of the permutations in S_3 is a subgroup of $\text{GL}_3(\mathbf{R})$. This is true in general regardless of what n we choose.

Definition 3.4 (Permutation grid)**.** A more common way to visualize permutation matrices is with the help of a *permutation grid*, the usefulness of this becomes apparent as we move onto the subject of permutation patterns. Define a permutation grid of a permutation π as its permutation matrix where each 1 is transformed into a dot and where each 0 is omitted.

Example 3.4. Consider the permutation $\pi_2 = [3, 1, 4, 2]$ from earlier, we get the following permutation grid. Notice the similarity to its permutation matrix.



Performing the same process for $\pi_1 = [4, 1, 6, 2, 5, 3]$ of length 6 from earlier we get the following permutation grid:



Note that different sources use different conventions for the orientation of the permutation grids. Some of them use the bottom left corner as origin while some place the elements directly on the gridlines.

Definition 3.5 (Identity permutation)**.** Define the *identity permutation* e_n of length n as the permutation that maps every element onto itself.

Example 3.5. The identity permutation of length 4 is $e_4 = [1, 2, 3, 4]$. Note that the permutation matrix of every identity permutation is the identity matrix. In this case the permutation matrix of e_4 is I_4 where I_n denotes the $n \times n$ identity matrix.

Definition 3.6 (Permutation inverse)**.** Define the *permutation inverse* π^{-1} as the unique permutation that fulfills $\pi \cdot \pi^{-1} = e_n$.

Example 3.6. Let $\pi = [1, 6, 2, 5, 3, 4]$, we get that $\pi^{-1} = [1, 3, 5, 6, 4, 2]$ because

$$\begin{aligned} e_6 &= [\pi(\pi^{-1}(1)), \pi(\pi^{-1}(2)), \pi(\pi^{-1}(3)), \pi(\pi^{-1}(4)), \pi(\pi^{-1}(5)), \pi(\pi^{-1}(6))] \\ &= [\pi(1), \pi(3), \pi(5), \pi(6), \pi(4), \pi(2)] \\ &= [1, 2, 3, 4, 5, 6]. \end{aligned} \tag{5}$$

Notice that the matrix of a permutation inverse is the inverse of the original permutation matrix.

$$M(\pi) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, M(\pi^{-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{6}$$

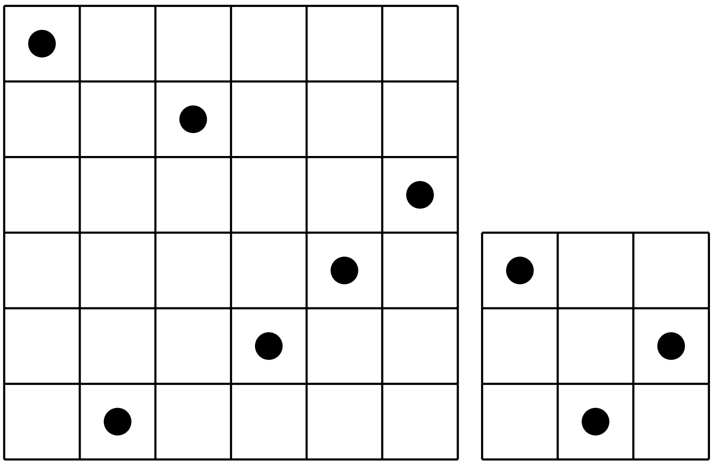
We can indeed verify that $(M(\pi))^{-1} = M(\pi^{-1}) = M^{\mathbf{T}}(\pi)$, this is true because the matrix is orthogonal. The latter part is true because of Lemma 1.4.

3.2 Pattern Avoidance

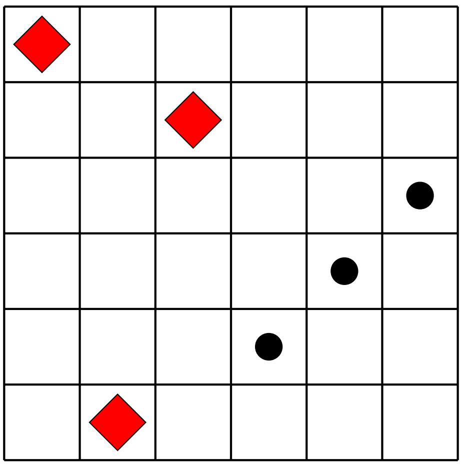
Definition 3.7 (Permutation pattern)**.** Define a *permutation pattern* σ as a "sub-pattern" of a bigger permutation π if σ *occurs* in π. If it does occur then we say that π *contains* σ otherwise we say that π *avoids* σ. Formally we can say that σ occurs in π if there are integers $i_1 < i_2 < \cdots < i_k$ such that:

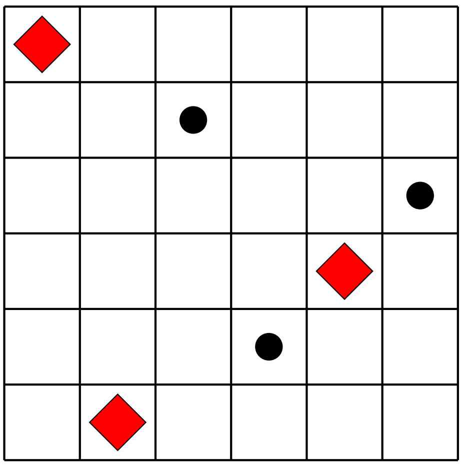
$$\pi(i_a) < \pi(i_b) \iff \sigma(a) < \sigma(b).$$

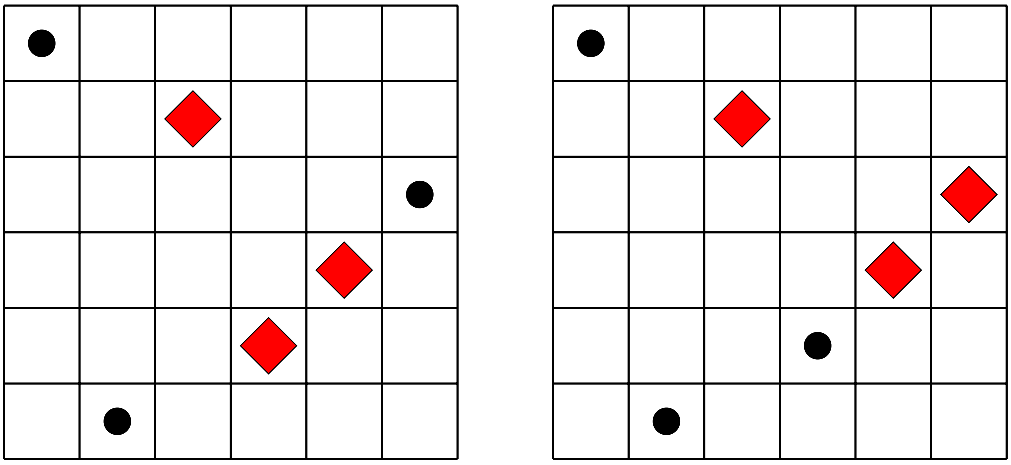
Example 3.7 (Permutation pattern example)**.** Let $\pi = [1, 6, 2, 5, 4, 3]$ and $\sigma = [3, 1, 2]$. Let us investigate if σ occurs in π. Visually one can inspect the permutation grids of π and σ:



Note that the pattern σ does not have to occur exactly as it is shown for it to qualify as occuring in π, it just has to occur in the same relative order. We see that the red diamonds below all fulfill the criteria for σ to occur in π. There are more possible occurences, but we only illustrate four examples here.







Definition 3.8 (Pattern avoidance)**.** Let $\mathrm{AV}_\pi(n)$ be the number of permutations in S_n that are being avoided by π.

Example 3.8 (Pattern avoidance example)**.** Consider the permutation pattern $\sigma = [1, 2, 3]$ and let us investigate $\mathrm{AV}_\sigma(4)$. There is a total of 24 patterns that exist of length 4 but note that σ is avoided by 14 of them, therefore $\mathrm{AV}_\sigma(4) = 14$. One can verify this by checking all of the 24 permutations and seeing that σ does not occur in the following:

$$\begin{aligned}&[1,4,3,2],[2,1,4,3],[2,4,1,3],[2,4,3,1],[3,1,4,2],[3,2,1,4],[3,2,4,1],\\&[3,4,1,2],[3,4,2,1],[4,1,3,2],[4,2,1,3],[4,2,3,1],[4,3,1,2],[4,3,2,1].\end{aligned} \tag{7}$$

Meanwhile σ is contained by the remaining 10 permutations:

$$\begin{aligned}&[\mathbf{1},\mathbf{2},\mathbf{3},4],[\mathbf{1},\mathbf{2},\mathbf{4},3],[\mathbf{1},\mathbf{3},2,\mathbf{4}],[\mathbf{1},\mathbf{3},\mathbf{4},2],[\mathbf{1},4,\mathbf{2},\mathbf{3}],\\&[\mathbf{2},1,\mathbf{3},\mathbf{4}],[\mathbf{2},\mathbf{3},1,\mathbf{4}],[\mathbf{2},\mathbf{3},\mathbf{4},1],[3,\mathbf{1},\mathbf{2},\mathbf{4}],[4,\mathbf{1},\mathbf{2},\mathbf{3}].\end{aligned} \tag{8}$$

Note that some of these have more than one way for σ to occur such as for $[1, 2, 3, 4]$ which has four ways it can occur as shown below:

$$[\mathbf{1},\mathbf{2},\mathbf{3},4],[\mathbf{1},\mathbf{2},3,\mathbf{4}],[\mathbf{1},2,\mathbf{3},\mathbf{4}],[1,\mathbf{2},\mathbf{3},\mathbf{4}]. \tag{9}$$

Definition 3.9 (Reverse permutation)**.** Given a permutation π of length n we define the *reverse permutation* as $\mathrm{rev}(\pi) = [\pi_n, \pi_{n-1}, \ldots, \pi_1]$. One can think of a reverse permutation as flipping the permutation matrix across the vertical axis.

Example 3.9 (Reverse permutation example)**.** If $\pi = [1, 4, 3, 2]$ then $\mathrm{rev}(\pi) = [2, 3, 4, 1]$.

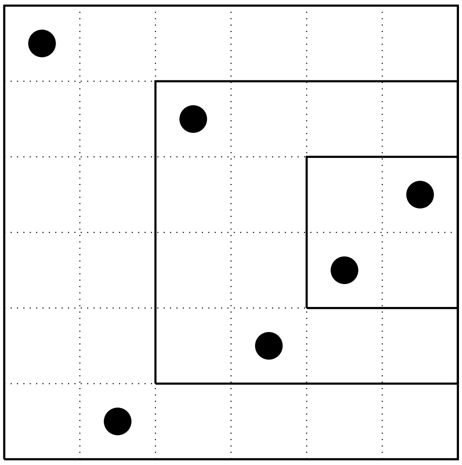
Definition 3.10. Given a permutation π of length n we define the *flip permutation* as $\mathrm{flip}(\pi) = [(n + 1) - \pi_1, (n + 1) - \pi_2, \ldots, (n + 1) - \pi_n]$. One can think of a flip permutation as flipping the permutation matrix across the horizontal axis.

Example 3.10 (Flip permutation example)**.** If $\pi = [1, 4, 3, 2]$ then $\mathrm{rev}(\pi) = [4, 1, 2, 3]$.

Lemma 3.1 (Transitive property)**.** *If π contains σ and σ contains τ then π also contains τ.*

Proof. If π contains σ then from the definition there must exist integers $i_1 < i_2 < \cdots < i_m$ such that $\pi(i_a) < \pi(i_b) \iff \sigma(a) < \sigma(b)$. If σ contains τ then from the definition there must exist integers $j_1 < j_2 < \cdots < j_n$ such that $\sigma(j_a) < \sigma(j_b) \iff \tau(a) < \tau(b)$. We know that $i_1 \leq j_1$ and $j_n \leq i_m$ since $\mathrm{size}(\pi) \geq \mathrm{size}(\sigma) \geq \mathrm{size}(\tau)$. That means for all j we can find an i such that $i_a = j_b$ for some a, b. Therefore we can for all j find a k such that $k_x = i_y$ for some x and $y(x)$ such that it fulfills the following: There exist integers $k_1 = i_{y(1)} < k_2 = i_{y(2)} < \cdots < k_n = i_{y(n)}$ such that $\pi(k_a) < \pi(k_b) \iff \tau(a) < \tau(b)$ Therefore we have proven from the definition that if π contains σ and σ contains τ then π also contains τ. ∎

We illustrate this more clearly by the use of a visual example:



Here we have $\pi = [1, 6, 2, 5, 4, 3]$, $\sigma = [1, 4, 3, 2]$ and $\tau = [2, 1]$. We intuitively see that the transitive property holds and it is not difficult to see that it is true in general.

Lemma 3.2 (Reverse avoidance)**.** *We have that* $\mathrm{AV}_\pi(n) = \mathrm{AV}_{\mathrm{rev}(\pi)}(n)$*.*

Proof. If π contains σ, then $\mathrm{rev}(\pi)$ contains $\mathrm{rev}(\sigma)$. This is clear by considering the permutation matrices. Moreover, $\{\mathrm{rev}(\pi) \mid \pi \in S_n\} = S_n$ because of symmetry. Therefore we have concluded that $\mathrm{AV}_\pi(n) = \mathrm{AV}_{\mathrm{rev}(\pi)}(n)$. □

Lemma 3.3 (Flip avoidance)**.** *We have that* $\mathrm{AV}_\pi(n) = \mathrm{AV}_{\mathrm{flip}(\pi)}(n)$*.*

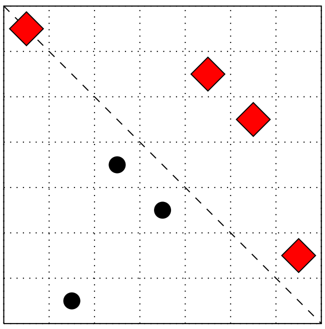
Proof. This follows from a similar argument as above. □

Lemma 3.4 (Inverse equals transpose)**.** *We have that* $\pi^{-1} = \mathrm{rev}(\mathrm{flip}(\pi))$*.*

Proof. Flipping the permutation matrix across the vertical axis and the horizontal axis results in a flip across the diagonal which gives the transpose of the matrix. □

Definition 3.11 (Upper-triangular part of a permutation)**.** We define the *upper-triangular part* $\mathrm{UTP}(\pi)$ of a permutation π as the sub-sequence of elements in π such that $\pi(i) \leq i$. In a similar fashion we define the *lower-triangular part* $\mathrm{LTP}(\pi)$ of a permutation π as the sequence of elements such that $\pi(i) > i$.

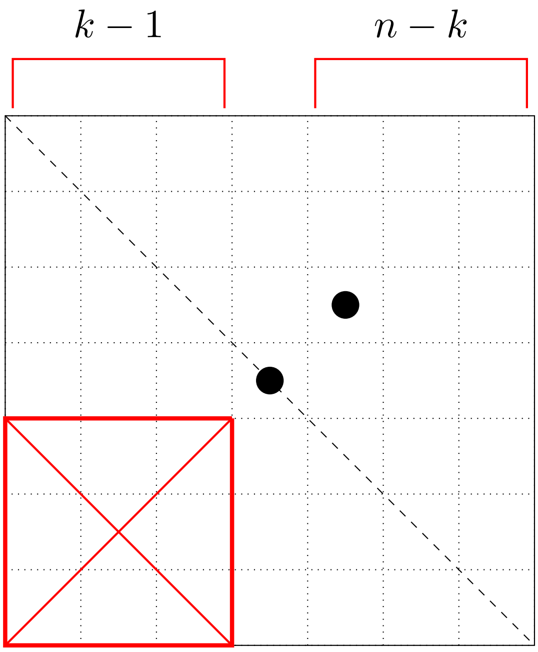
Example 3.11 (Upper-triangular part example)**.** Given $\pi = [1, 7, 4, 5, 2, 3, 6]$ we get $\mathrm{UTP}(\pi) = [1, 2, 3, 6]$ and $\mathrm{LTP}(\pi) = [7, 4, 5]$ as shown below:



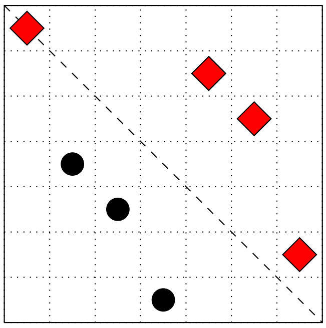
One way to think about $\mathrm{UTP}(\pi)$ is to consider all the elements on or above the diagonal line of the permutation matrix. In a similar fashion one can think of $\mathrm{LTP}(\pi)$ as considering all the elements below the diagonal line.

Lemma 3.5 (Structure of 321-avoiding patterns)**.** *A permutation* π *avoids* $[3, 2, 1]$ *if and only if* $\mathrm{UTP}(\pi)$ *and* $\mathrm{LTP}(\pi)$ *are both increasing. Furthermore* $\mathrm{UTP}(\pi)$ *uniquely determines* π*.*

Proof. First we consider the trivial case of $\pi = I_n$, this gives $\text{UTP}(\pi) = I_n$ and $\text{LTP}(\pi) = \emptyset$. We clearly see that I_n avoids $[3, 2, 1]$. Suppose that π is 321-avoiding but $\text{UTP}(\pi)$ is not ascending. This means that there is at least one $k < m$ such that $\pi(k) > \pi(m)$ and because of the definition of $\text{UTP}(\pi)$ we know that $\pi(k) \leq k$ as shown by the figure below:



Since π is 321-avoiding we must have that $\pi(i) < \pi(k)$ for all $i = 1, 2, \ldots, k-1$ otherwise we would get an occurrence of $[3, 2, 1]$. We also note that $\pi(i) \neq \pi(m)$. But this means that $\{\pi(1), \pi(2), \ldots, \pi(k-1)\}$ are only allowed to take different values in $\{\pi(1), \pi(2), \ldots, \pi(k-1)\} \setminus \{\pi(m)\}$ which by the pigeon-hole principle is impossible since the first set has a cardinality of $k-1$ while the latter set has a cardinality of $k-2$. Because of this we get that if π is 321-avoiding then $\text{UTP}(\pi)$ has to be ascending. One can use a similar argument for $\text{LTP}(\pi)$. Now let us prove that if $\text{UTP}(\pi)$ and $\text{LTP}(\pi)$ are ascending then π is 321-avoiding. Suppose that $\pi(a) > \pi(b) > \pi(c)$ where $a < b < c$. This is impossible since either a, b or a, c or b, c must both be in $\text{UTP}(\pi)$ or $\text{LTP}(\pi)$, which contradicts the ascending property. As an example, consider the figure below. Any descent must involve one element from $\text{UTP}(\pi)$ and one from $\text{LTP}(\pi)$.



We now prove the latter part of the lemma, namely that assuming if π is 321-avoiding then $\text{UTP}(\pi)$ determines π. It is clear that the elements in $\text{LTP}(\pi)$ are $\{1, 2, 3, \ldots, n\} \setminus \text{UTP}(\pi)$, furthermore we know that $\text{LTP}(\pi)$ is ascending hence we can easily find $\text{LTP}(\pi)$ if given $\text{UTP}(\pi)$. Therefore we have proven the lemma. □

3.3 Connection to the Catalan Numbers

Definition 3.12 (Catalan numbers)**.** The *Catalan numbers* are a sequence of non-negative integers that frequently occur in combinatorics, especially those that involve recursively defined operations.

One way to calculate the nth Catalan number is by the following formula:

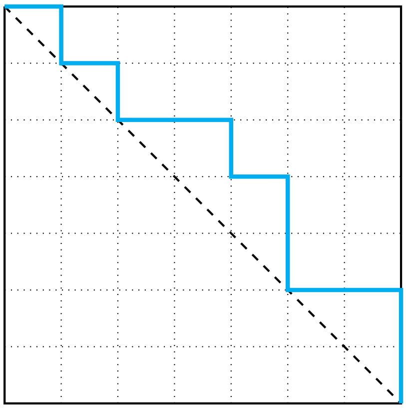
$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

However do note that there are many formulas and recurrence relations one can use to acquire the values of C_n. The recurrence formula that will come of use for us later is $C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}$ with $C_0 = 1$. The first Catalan numbers from $n = 0$ to $n = 9$ are as follows:

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862.$$

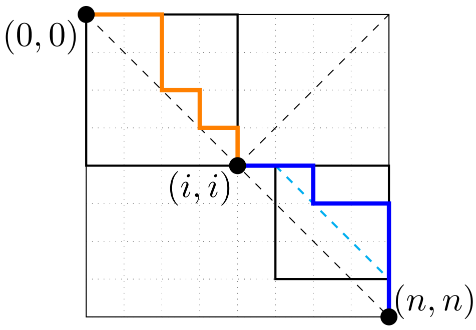
Definition 3.13 (Lattice paths)**.** We define a *lattice path* as a path going from $(0, 0)$ to (a, b) that only allows for downwards and rightwards movement. In this work we only consider lattice paths going from $(0, 0)$ to (n, n) and such that they never cross the diagonal. We say that such a lattice path has length n.

Example 3.12 (Lattice paths example)**.** The figure below shows an example of a lattice path going from $(0, 0)$ to $(7, 7)$. Note that it never crosses the diagonal line.



Lemma 3.6 (Number of lattice paths)**.** *The number of possible lattice paths for a (n, n) grid is equal to the Catalan numbers.*

Proof. Let W_n be the number of possible lattice paths of length n. Let (i, i) be the *last* time that this path reaches the diagonal line except for (n, n). We split the lattice path into two smaller lattice paths of length i and $n - i - 1$ respectively. W_i and W_{n-i-1} denotes the number of such possible paths. The reason for $n - i - 1$ is simply because the latter path has one less option due to being unable to reach the diagonal as is illustrated in the figure below:



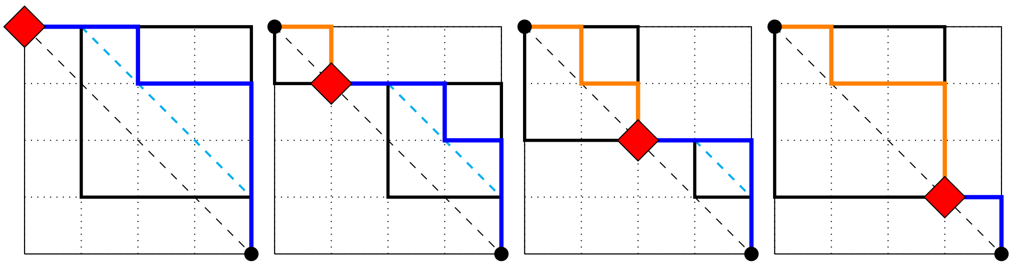
Enumerating from $i = 0$ to $i = n-1$ and assuming $W_0 = 1$ gives us the following sum:

$$
\begin{aligned}
W_n &= W_{n-1} + W_1 W_{n-2} + \cdots + W_{n-2} W_1 + W_{n-1} \\
W_n &= W_0 W_{n-1} + W_1 W_{n-2} + \cdots + W_{n-2} W_1 + W_{n-1} W_0 \\
W_n &= \sum_{i=0}^{n-1} W_i W_{n-i-1} \\
W_n &= \sum_{i=1}^{n} W_{i-1} W_{n-i}.
\end{aligned}
$$

This sum is exactly the recursive formula for the Catalan numbers. Therefore we have that $W_n = C_n$ and we have proven that the number of possible lattice paths of length n is equal to C_n. ☐

Combinatorial Proof. There is another way to prove that the number of possible lattice paths gives rise to the Catalan numbers, the idea is fairly straight forward and makes use of binomial coefficients. The total number of paths from $(0,0)$ to (n,n) without restricting the path to the upper-triangular half is given by $\binom{2n}{n}$. One can find that the number of paths that are below the upper-triangular half is given by $\binom{2n}{n+1}$. Simple algebra gives us $\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$ which aligns with the formula of the Catalan numbers as given in Definition 3.12. ☐

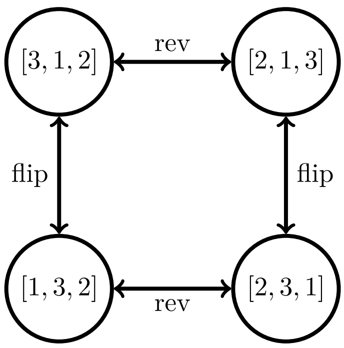
Example 3.13 (Lattice paths of length 4)**.** In the figure below one can see the process of enumerating the sum for the case of $n = 4$.



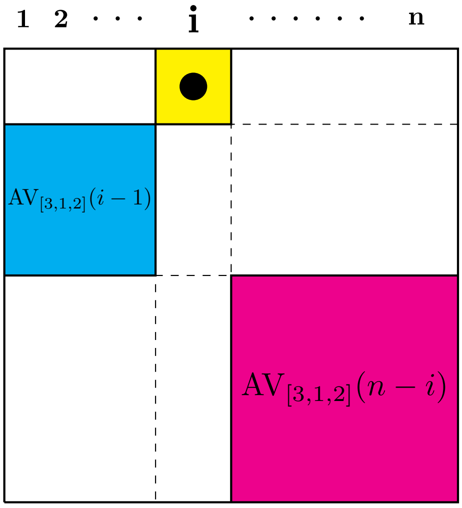
We get the sum $C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 = C_4$. Note that the dashed line is the diagonal line of the latter lattice path.

Theorem 3.7. *Given $\pi \in S_3$ we have that* $\mathrm{AV}_\pi(n) = C_n$.

Proof. We have that $\mathrm{rev}([1,2,3]) = [3,2,1]$ and we also have bijections between the remaining 4 permutations in S_3 by performing flip and reverse operations on them as seen in the figure below:



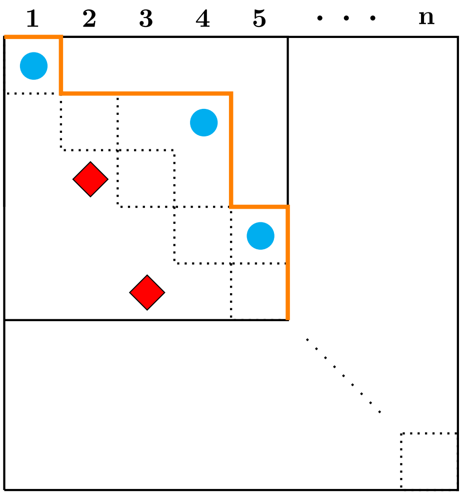
We need to show that $\mathrm{AV}_\pi(n) = C_n$ holds for at least one element in both $P_1 = \{[1,2,3],[3,2,1]\}$ and in $P_2 = \{[3,1,2],[2,1,3],[1,3,2],[2,3,1]\}$. This is because of Lemma 3.2 and 3.3. We start by proving $\mathrm{AV}_{[3,1,2]}(n) = C_n$ by using a combinatorial argument.



In the figure above we have the grid of a permutation π whose structure avoids $[3, 1, 2]$. Assume that 1 is placed in the ith position. Every element to the left of 1 has to be smaller than every element to the right of 1 otherwise we would end up with an occurrence of $[3, 1, 2]$. This is obvious if one considers the structure of $[3, 1, 2]$. Furthermore we know that both the left and the right sub-matrices have to avoid $[3, 1, 2]$. Assuming that π is of length n there are n possibilities for 1 to be placed. We enumerate from $i = 1$ until $i = n$ where the ith term counts the number of 312-avoiding permutations with 1 at position i. We get the following sum:

$$\begin{aligned}\mathrm{AV}_{[3,1,2]}(n) &= \mathrm{AV}_{[3,1,2]}(0)\mathrm{AV}_{[3,1,2]}(n-1) + \cdots + \mathrm{AV}_{[3,1,2]}(n-1)\mathrm{AV}_{[3,1,2]}(0)\\ &= \sum_{i=1}^{n} \mathrm{AV}_{[3,1,2]}(i-1)\mathrm{AV}_{[3,1,2]}(n-i).\end{aligned}$$

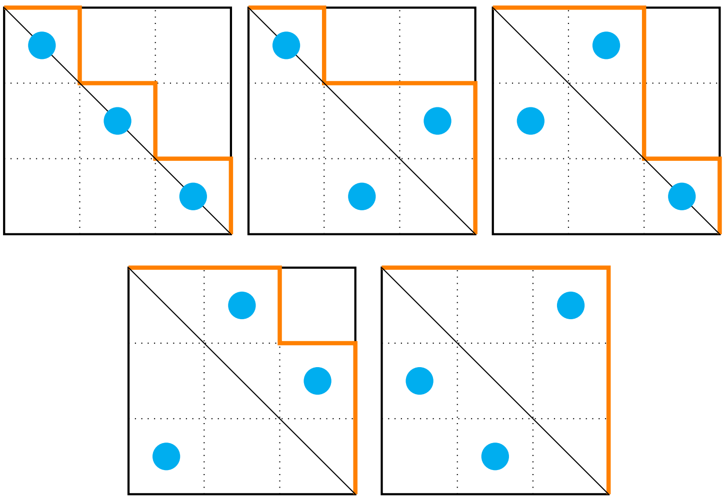
This aligns with the recursive formula for the Catalan numbers C_n given that $\mathrm{AV}_{[3,1,2]}(0) = 1 = C_0$.
We conclude that given $\pi \in P_2$ we know that $\mathrm{AV}_\pi(n) = C_n$.
For $[3, 2, 1]$ and $[1, 2, 3]$ we need a different approach.
We examine $[3, 2, 1]$ to prove that $\pi \in P_1$ satisfy $\mathrm{AV}_\pi(n) = C_n$. By Lemma 3.5 if π avoids $[3, 2, 1]$ then it needs to be uniquely defined by one ascending list of elements $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$.



Consider the figure above of a upper-triangular half of the permutation matrix. We create a path by starting in the top left corner and moving to the rightwards until we reach an element, after that we move downwards until we are at the height of another element or at the diagonal, we repeat this process until we reach (n, n). The path that is formed is indeed a lattice path as every corner of the path corresponds to an element in the upper half of the permutation matrix. We know from

Lemma 3.6 that the number of possible lattice paths of length n is equal to C_n. We know that there is a bijection between the paths caused by this algorithm and the lattice paths because each element in the ascending list gives rise to a "peak" on the path, all of the possible arrangements of these elements gives rise to all the possible lattice paths. This is obvious if one considers the possible placements for the elements and the paths that they produce. We have therefore proven that $\mathrm{AV}_{[3,2,1]}(n) = C_n$ hence proving the case for $[1, 2, 3]$ as well due to Lemma 3.2. Therefore we have proven that given $\pi \in S_3$ we have $\mathrm{AV}_\pi(n) = C_n$. □

Example 3.14 (Bijection example for $C_3 = 5$)**.** Below we show all the cases for the example of $C_3 = 5$:



4 Multi-Avoidance

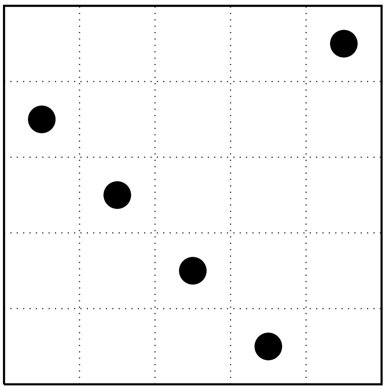
4.1 Introduction

In this section we study multi-avoidance and see how we can extend the concepts shown in the previous section. We study the work of Simion and Schmidt[11] during 1985 and show the most relevant results. We work our way up from double restrictions all the way to restrictions of order 5 and arrive at a very fascinating formula for containing all six permutations of length 3. In order to keep the notation tidy we may omit commas and brackets. For example instead of writing $\mathrm{AV}_{[1,2,3],[1,3,2],[2,1,3]}(n) = \mathrm{AV}_{[2,3,1],[3,1,2],[3,2,1]}(n)$ we may write $\mathrm{AV}_{[123],[132],[213]}(n) = \mathrm{AV}_{[231],[312],[321]}(n)$.

4.2 Multi-avoidance

Definition 4.1 (Multi-avoidance)**.** Given the permutation π we define *multi-avoidance* as π avoiding two or more permutations simultaneously. The number of permutations in S_n that simultaneously avoid the permutations $\pi_1, \pi_2, \ldots, \pi_k$ is denoted by $\mathrm{AV}_{\pi_1,\pi_2,\ldots,\pi_k}(n)$ or for the sake of simplicity as $\mathrm{AV}_{\mathbf{P}}(n)$ where $\mathbf{P} = \{\pi_1, \pi_2, \ldots, \pi_k\}$. In this section we only consider permutations of length 3.

Example 4.1 (Multi-avoidance example)**.** Consider the permutation $\pi = [2,3,4,5,1]$ as shown by the permutation grid below, we note that it is 321- and 213-avoiding.



Avoiding $[1,2,3]$ and $[3,2,1]$ at the same time will be a recurring theme as we investigate multi-avoidance. We refer to the following lemma numerous times throughout the section.

Lemma 4.1 (123- and 321- avoiding permutations)**.** *If* $\{[1,2,3],[3,2,1]\} \subseteq \boldsymbol{P}$ *then* $\mathrm{AV}_{\boldsymbol{P}}(n) = 0$ *for* $n \geq 5$.

Proof. For π to be 321-avoiding we can write it as the union of two ascending sub-sequences. For π to also be 123-avoiding we can write it as the union of two descending sub-sequences. It is obvious that such an arrangement is not possible for $n \geq 5$. ☐

4.3 Double Restrictions

Theorem 4.2 (Bijections when $|\mathbf{P}| = 2$)**.** We proved earlier that by Lemmas 3.2, 3.3 and 3.4 that due to symmetries between permutation patterns we can find a bijection between permutations to reduce the amount of cases that need examining. The same holds true when it comes to multi-avoidance. Simion and Schmidt[11] proved that there are only 6 pairs of permutations that need to be considered as shown below:

1. $\mathrm{AV}_{[1,2,3],[1,3,2]}(n) = \mathrm{AV}_{[1,2,3],[2,1,3]}(n) = \mathrm{AV}_{[2,3,1],[3,2,1]}(n) = \mathrm{AV}_{[3,1,2],[3,2,1]}(n)$.
2. $\mathrm{AV}_{[1,3,2],[2,1,3]}(n) = \mathrm{AV}_{[2,3,1],[3,1,2]}(n)$.
3. $\mathrm{AV}_{[1,3,2],[2,3,1]}(n) = \mathrm{AV}_{[2,1,3],[3,1,2]}(n)$.
4. $\mathrm{AV}_{[1,3,2],[3,1,2]}(n) = \mathrm{AV}_{[2,1,3],[2,1,3]}(n)$.
5. $\mathrm{AV}_{[1,3,2],[3,2,1]}(n) = \mathrm{AV}_{[1,2,3],[2,3,1]}(n) = \mathrm{AV}_{[1,2,3],[3,1,2]}(n) = \mathrm{AV}_{[2,1,3],[3,2,1]}(n)$.
6. $\mathrm{AV}_{[1,2,3],[3,2,1]}(n)$.

Proof. Refer to [11, p. 392]. ☐

The two following lemmas will be useful for us for proving the first 5 cases shown above.

Lemma 4.3. *Given* $f(n) = 1 + \sum_{k=1}^{n-1} f(k)$ *we know that* $f(n) = 2^{n-1}$ *for* $n \geq 1$.

Proof. We use an inductive proof to show that this lemma is true. We first prove the base case $n = 1$. We get that $f(1) = 1 = 2^{1-1}$. Assume that $f(m) = 1 + \sum_{k=1}^{m-1} f(k) = 2^{m-1}$ for some

$m \geq 1$. Now we have that:

$$\begin{aligned} f(m+1) &= 1 + \sum_{k=1}^{m} f(k) \\ &= 1 + \sum_{k=1}^{m-1} f(k) + f(m) \\ &= f(m) + f(m) \\ &= 2 \cdot f(m) \\ &= 2 \cdot 2^{m-1} \\ &= 2^m. \end{aligned}$$

By induction we have shown that $f(n) = 1 + \sum_{k=1}^{n-1} f(k) = 2^{n-1}$. ☐

Lemma 4.4. *Given $f(n) = n - 1 + f(n-1)$ and $f(1) = 1$ we have that $f(n) = \binom{n}{2} + 1$ for $n \geq 2$.*

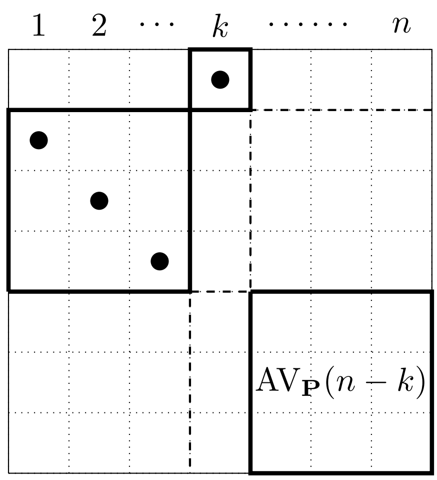
Proof. We use an inductive proof to show that this lemma is true. We first prove the base case $n = 2$. We get that $1 - 1 + f(1) = 2 = \binom{2}{2} + 1$. Assume that $f(m) = \binom{m}{2} + 1$ for some $m \geq 2$. Now we have that:

$$\begin{aligned} f(m+1) &= (m+1) - 1 + f((m+1) - 1) \\ &= m + f(m) \\ &= m + \binom{m}{2} + 1 \\ &= m + \frac{m(m-1)}{2} + 1 \\ &= \frac{m(m+1)}{2} + 1 \\ &= \binom{m+1}{2} + 1. \end{aligned}$$

By induction we have shown that $f(n) = n - 1 + f(n-1) = \binom{n}{2} + 1$. ☐

Lemma 4.5 (Double Restrictions Case 1)**.** *We have that* $\mathrm{AV}_{[3,2,1],[3,1,2]}(n) = 2^{n-1}$.

Proof. This is Case 1 from Theorem 4.2. Let $\mathbf{P} = \{[3,2,1],[3,1,2]\}$. We know from earlier in Theorem 3.7 that a 312-avoiding permutation must have the following structure:



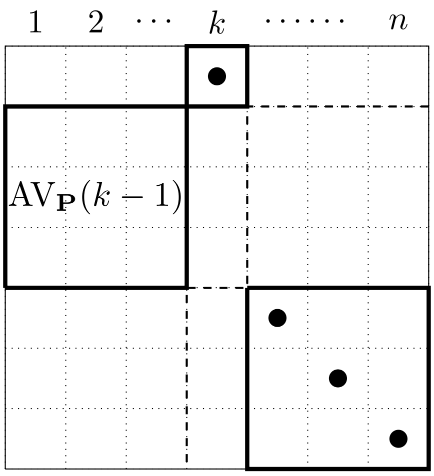
We know from earlier that a 321-avoiding permutation can be written as the union of two ascending sub-sequences which causes a restriction on the elements in the left sub-matrix. We know that $\pi(1) > \pi(2) > \cdots > \pi(k-1)$ has to hold otherwise we would have an occurrence of $[3,2,1]$, this is the only way to avoid $[3,2,1]$ as we saw in the previous section. We know that it is impossible for $[3,2,1]$ occur unless $[3,2,1]$ occurs in the right sub-matrix because all the values to the left of it are smaller than the values in the right sub-matrix. We also know that it is impossible for $[3,1,2]$ to occur unless $[3,1,2]$ occurs in the right sub-matrix because the left sub-matrix already avoids it and all the values to the left of the right sub-matrix are smaller than the values in the right sub-matrix. We conclude that it in order for π to avoid $[3,2,1]$ and $[3,1,2]$ the right sub-matrix needs to also avoid $[3,2,1]$ and $[3,1,2]$. Note that when $k = n$ there is only *one* possibility for the permutation. Enumerating from $k = 1$ to $k = n$ we get the following sum:

$$\begin{aligned}\mathrm{AV}_{\mathbf{P}}(n) &= \mathrm{AV}_{\mathbf{P}}(n-1) + \mathrm{AV}_{\mathbf{P}}(n-2) + \cdots + \mathrm{AV}_{\mathbf{P}}(1) + 1 \\ &= 1 + \sum_{k=1}^{n-1} \mathrm{AV}_{\mathbf{P}}(k).\end{aligned}$$

Because of Lemma 4.3 we know that $\mathrm{AV}_{\mathbf{P}}(n) = 2^{n-1}$ and therefore we have proven the Lemma. □

Lemma 4.6 (Double Restrictions Case 2). *We have that* $\mathrm{AV}_{[2,3,1],[3,1,2]}(n) = 2^{n-1}$.

Proof. This is Case 2 from Theorem 4.2. The proof is similar to the proof of Lemma 4.5 but with a different structure which is as shown below:



We note that the overall structure is 312-avoiding for the same reasons as in 3.7. Furthermore a permutation with this structure is also 231-avoiding as long as the left sub-matrix is also 231-avoiding. Similarly to the previous proof we enumerate from $i = 1$ until $i = n$ and we get $\mathrm{AV}_{\mathbf{P}}(n) = 1 + \mathrm{AV}_{\mathbf{P}}(1) + \cdots + \mathrm{AV}_{\mathbf{P}}(n-2) + \mathrm{AV}_{\mathbf{P}}(n-1) = 1 + \sum_{k=1}^{n-1} \mathrm{AV}_{\mathbf{P}}(k)$ and because of Lemma 4.3 we know that $\mathrm{AV}_{\mathbf{P}}(n) = 2^{n-1}$ and therefore we have proven the lemma. □

Lemma 4.7 (Double Restrictions Case 3). *We have that* $\mathrm{AV}_{[1,3,2],[2,3,1]}(n) = 2^{n-1}$.

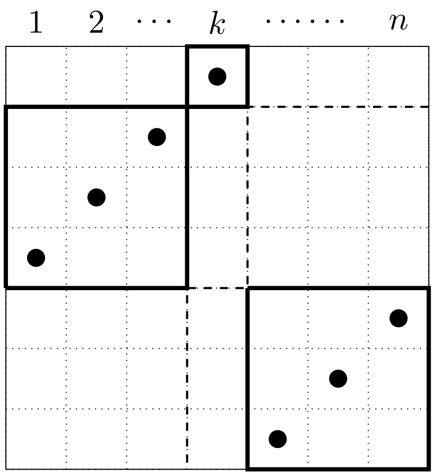
Proof. This is case 3 from Theorem 4.2. The proof is similar to the proof of Lemma 4.5. Refer to [11, p. 393] □

Lemma 4.8 (Double Restrictions Case 4). *We have that* $\mathrm{AV}_{[1,3,2],[3,1,2]}(n) = 2^{n-1}$.

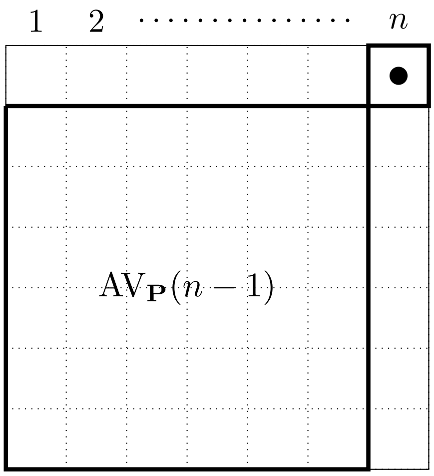
Proof. This is case 4 from Theorem 4.2. The proof is similar to the proof of Lemma 4.5. Refer to [11, p. 393] □

Lemma 4.9 (Double Restrictions Case 5). *We have that* $\mathrm{AV}_{[1,2,3],[3,1,2]}(n) = \binom{n}{2} + 1$.

Proof. Let $\mathbf{P} = \{[1,2,3],[3,1,2]\}$. Once again we consider the structure of a 312-avoiding permutation π. We saw in Theorem 3.7 that a 312-avoiding permutation must have the following structure:



We know that this is true because a 123-avoiding permutation has to be made up of 2 descending lists of elements for the same reason that 321-avoiding permutations have to be made up of 2 ascending lists as we saw in Lemma 3.5. From $k = 1$ to $k = n-1$ there is only one way for π to avoid both $[1,2,3]$ and $[3,1,2]$ for every k. This is obvious if one considers that any other placement would result in an occurrence of $[1,2,3]$. Notice that this structure does not necessarily need to be as shown above in the case of $k = n$:



If $\pi(n) = 1$ then that element does not affect the 123- and 312-avoiding nature of the structure so what we are left with is $\mathrm{AV}_{\mathbf{P}}(n-1)$. Enumerating from $k = 1$ until $k = n$ we get the following sum:

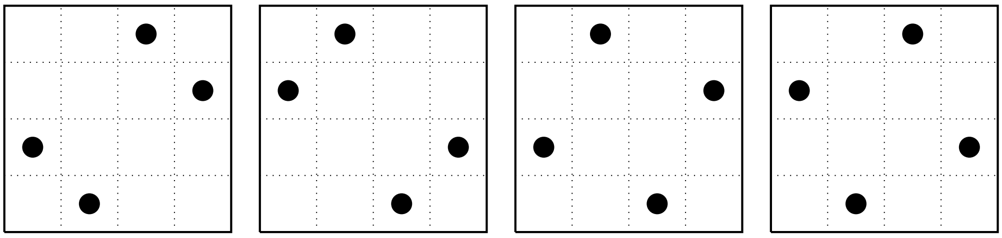
$$\begin{aligned}\mathrm{AV}_{\mathbf{P}}(n) &= \mathrm{AV}_{\mathbf{P}}(n-1) + \sum_{k=1}^{n-1}(1)\\ &= \mathrm{AV}_{\mathbf{P}}(n-1) + (n-1)\\ &= \binom{n}{2} + 1.\end{aligned}$$

This is true because of Lemma 4.4. Therefore we have proven that $\mathrm{AV}_{[1,2,3],[3,1,2]}(n) = \binom{n}{2} + 1$. $\square$

Lemma 4.10 (Double Restrictions Case 6)**.** *The cases of* $\mathrm{AV}_{[1,2,3],[3,2,1]}(n)$ *are as following:*

$$\mathrm{AV}_{[1,2,3],[3,2,1]}(n) = \begin{cases} n & \text{if } n = 1 \text{ or } n = 2 \\ 4 & \text{if } n = 3 \text{ or } n = 4 \\ 0 & \text{if } n \geq 5. \end{cases}$$

Proof. We ignore the trivial cases of $n = 1$ or $n = 2$. For $n = 3$ there are only two cases which do not avoid both $[3, 2, 1]$ and $[1, 2, 3]$. It is indeed $[3, 2, 1]$ and $[1, 2, 3]$ themselves. There is a total of $3! = 6$ permutations which gives us $6 - 2 = 4$ permutations that avoid both $[3, 2, 1]$ and $[1, 2, 3]$. For the case $n = 4$ there are also only four permutations as shown below:



For the case $n \geq 5$ refer to Lemma 4.1, we know that it is not possible to have a permutation avoid both $[1, 2, 3]$ and $[3, 2, 1]$ if $n \geq 5$. □

4.4 Triple Restrictions

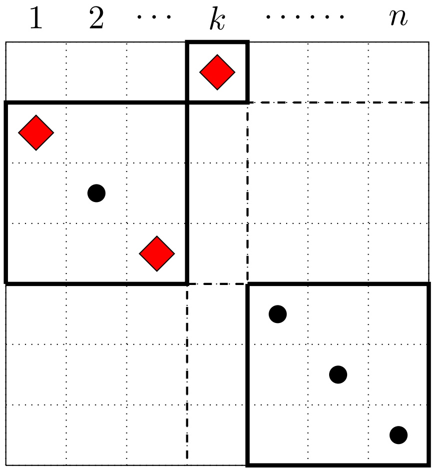
Theorem 4.11 (Bijections when $|\mathbf{P}| = 3$)**.** Similarly to how we reduced the number of cases for double restrictions we can do the same for triple restrictions as shown below:

1. $\mathrm{AV}_{[123],[132],[213]}(n) = \mathrm{AV}_{[231],[312],[321]}(n)$.
2. $\mathrm{AV}_{[123],[132],[231]}(n) = \mathrm{AV}_{[123],[213],[312]}(n) = \mathrm{AV}_{[132],[231],[321]}(n) = \mathrm{AV}_{[213],[312],[321]}(n)$.
3. $\mathrm{AV}_{[132],[213],[312]}(n) = \mathrm{AV}_{[132],[213],[312]}(n) = \mathrm{AV}_{[132],[231],[312]}(n) = \mathrm{AV}_{[213],[231],[312]}(n)$.
4. $\mathrm{AV}_{[123],[132],[312]}(n) = \mathrm{AV}_{[123],[213],[231]}(n) = \mathrm{AV}_{[132],[312],[321]}(n) = \mathrm{AV}_{[213],[231],[321]}(n)$.
5. $\mathrm{AV}_{[123],[231],[312]}(n) = \mathrm{AV}_{[132],[213],[321]}(n)$.
6. $\mathrm{AV}_{[123],[321],[132]}(n) = \mathrm{AV}_{[123],[321],[213]}(n) = \mathrm{AV}_{[123],[321],[231]}(n) = \mathrm{AV}_{[123],[321],[312]}(n)$.

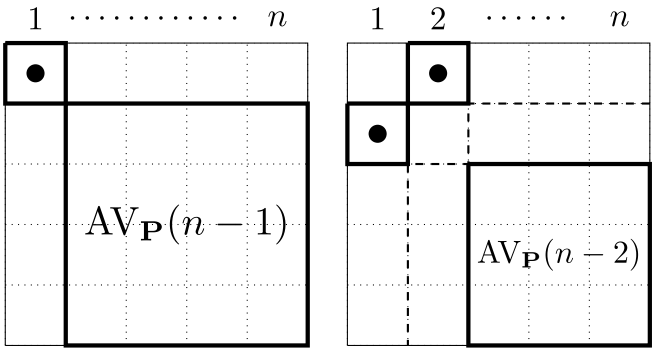
Proof. Refer to [11, p. 396]. □

Lemma 4.12 (Triple Restrictions Case 1)**.** *We have that* $\mathrm{AV}_{[2,3,1],[3,1,2],[3,2,1]}(n) = F_{n+1}$ *where* F_n *is the n:th Fibonacci number.*

Proof. A reminder that F_n is given by the recursive formula $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 1$ and $F_1 = 1$. Let $\mathbf{P} = \{[2, 3, 1], [3, 1, 2], [3, 2, 1]\}$ In order for a permutation π to be 312-avoiding it needs to follow the structure shown in Theorem 3.7. In order for π to be 321-avoiding it needs to follow the structure shown in Lemma 3.5. These two restrictions give us a permutation with a structure as shown below:



We immediately notice that a permutation like this is not 231-avoiding as highlighted by the diamonds above. There are only two placements of $\pi(k) = 1$ that allow for $[2, 3, 1]$ to be avoided, namely when $k = 1$ and $k = 2$.



Note that the case $\pi(1) = 1$ does not have any impact on the occurrence on any of the permutations in $\mathbf{P}$. The same is true for $\pi(2) = 1$. We see that $\mathrm{AV}_{\mathbf{P}}(n) = \mathrm{AV}_{\mathbf{P}}(n-1) + \mathrm{AV}_{\mathbf{P}}(n-2)$. We know that $\mathrm{AV}_{\mathbf{P}}(1) = 1$ and $\mathrm{AV}_{\mathbf{P}}(2) = 2$ which gives us the recursive Fibonacci relation offset by one. Therefore we have shown that $\mathrm{AV}_{\mathbf{P}}(n) = F_{n+1}$. □

Lemma 4.13 (Triple Restrictions Case 2)**.** *We have that* $\mathrm{AV}_{[1,2,3],[1,3,2],[2,3,1]}(n) = n$.

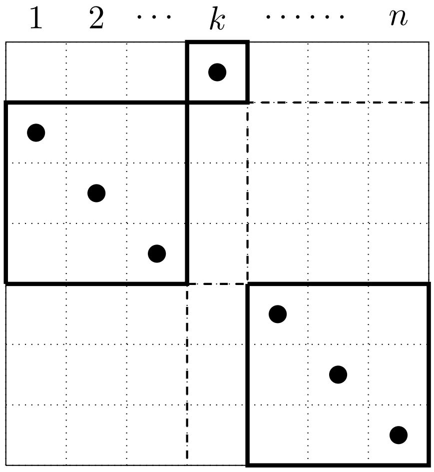
Proof. The proof for this lemma uses a very similar setup to Lemma 4.15 further down. □

Lemma 4.14 (Triple Restrictions Case 3)**.** *We have that* $\mathrm{AV}_{[1,3,2],[2,1,3],[3,1,2]}(n) = n$.

Proof. The proof for this lemma uses a very similar setup to Lemma 4.15 further down. □

Lemma 4.15 (Triple Restrictions Case 4)**.** *We have that* $\mathrm{AV}_{[1,3,2],[3,1,2],[3,2,1]}(n) = n$.

Proof. Let $\mathbf{P} = \{[1, 3, 2], [3, 1, 2], [3, 2, 1]\}$. Similarly to the previous proof we know that in order for a permutation π to be 312- and 321-avoiding is as shown below:



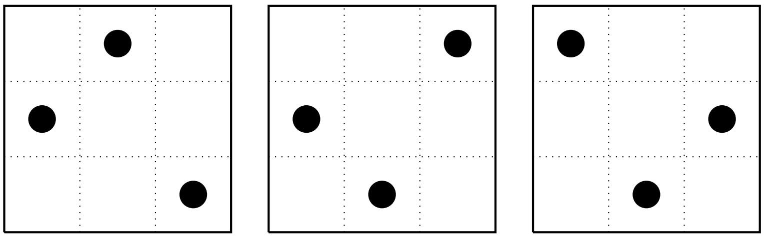
We note that this structure also avoids $[1, 3, 2]$ for every value of $\pi(k) = 1$. There is exactly one way to avoid all permutations in $\mathbf{P}$ for every k. Enumerating from $k = 1$ to $k = n$ gives us the simple formula $\mathrm{AV}_{\mathbf{P}}(n) = \sum_{k=1}^{n} (1) = n$. Therefore we have proven the lemma. ☐

Lemma 4.16 (Triple Restrictions Case 5)**.** *We have that* $\mathrm{AV}_{[1,2,3],[2,3,1],[3,1,2]}(n) = n$.

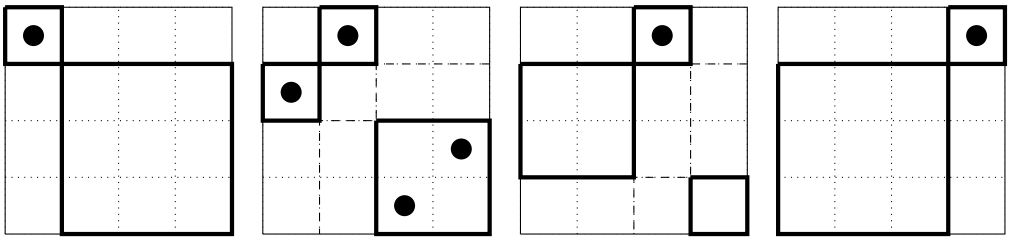
Proof. The proof for this lemma uses a very similar setup to Lemma 4.15. ☐

Lemma 4.17 (Triple Restrictions Case 6)**.** *We have that* $\mathrm{AV}_{[1,2,3],[3,2,1],[3,1,2]}(n) = 0$ *given* $n \geq 5$.

Proof. Let $\mathbf{P} = [1, 2, 3], [3, 2, 1], [3, 1, 2]$. In order to avoid both $[1, 2, 3]$ and $[3, 2, 1]$ it is obvious that if $n \geq 5$ we can no longer construct two ascending and two descending principal lists, this leads us to $\mathrm{AV}_{\mathbf{P}}(n) = 0$. For the trivial cases $n = 1$ and $n = 2$ we know that $\mathrm{AV}_{\mathbf{P}}(1) = 1$ and $\mathrm{AV}_{\mathbf{P}}(2) = 2$. For $n = 3$ we quickly see that there are only 3 permutations in S_3 that avoid all permutations in $\mathbf{P}$ as shown below:



Furthermore, one can easily see that given $|\mathbf{P}| = m$ we get that $\mathrm{AV}_{\mathbf{P}}(3) = 6 - m$ since one must avoid m permutations and there are $3! = 6$ permutations in total in S_3.
For the cases of $n = 4$ we exhaustively examine the possibilities as shown below:



One can easily see that $\pi(1) = 1$, $\pi(3) = 1$ and $\pi(4) = 1$ gives us no way to avoid both $[1, 2, 3]$ and $[3, 2, 1]$. For $\pi(2) = 1$ we get one possibility as shown above, namely $\pi = [2, 1, 4, 3]$. We know from Lemma 4.1 that if $n \geq 5$ then $\mathrm{AV}_{\mathbf{P}}(n) = 0$. Therefore we get the following cases for $\mathrm{AV}_{\mathbf{P}}(n)$:

$$\mathrm{AV}_{\mathbf{P}}(n) = \begin{cases} n & \text{if } 1 \leq n \leq 3, \\ 1 & \text{if } n = 4, \\ 0 & \text{if } n \geq 5. \end{cases}$$

□

4.5 Restrictions of Order 4

Theorem 4.18 (Bijections when $|\mathbf{P}| = 4$). *There are two cases to consider when* $|\boldsymbol{P}| = 4$:

1. $\{[1,2,3],[3,2,1]\} \subset \boldsymbol{P}$.
2. $\{[1,2,3],[3,2,1]\} \not\subset \boldsymbol{P}$.

Proof. We easily see that these are the only two possible cases for $|\mathbf{P}| = 4$. Either both of $[1,2,3]$ or $[3,2,1]$ are in $\mathbf{P}$ or not. □

Lemma 4.19 (Restrictions of Order 4 Case 1). *Given* $\{[1,2,3],[3,2,1]\} \subset \boldsymbol{P}$ *then* $\mathrm{AV}_{\boldsymbol{P}}(n) = 0$ *if* $n \geq 5$.

Proof. We note that the trivial cases are $\mathrm{AV}_{\mathbf{P}}(1) = 1$ and $\mathrm{AV}_{\mathbf{P}}(2) = 2$. As we saw in the proof of Lemma 4.17 there are $6 - |\mathbf{P}|$ ways to avoid all permutations in $\mathbf{P}$ given $n = 3$. Therefore $\mathrm{AV}_{\mathbf{P}}(3) = 2$. For the case of $n = 4$ we refer back to the figure from early on in Theorem 3.7 that showed the bijections between $[3,1,2]$, $[2,1,3]$, $[2,3,1]$ and $[1,3,2]$. The permutations that are not connected by an immediate bijection allow for avoidance of both. For example one immediately notices that if one is to avoid both $[3,1,2]$ and $[2,1,3]$ then $\pi(1)$ or $\pi(4)$ has to be equal to 1 but we know that if $\pi(1) = 1$ or $\pi(4) = 1$ we get I_4 or $\mathrm{Rev}(I_4)$ which results in an occurrence of $[1,2,3]$ or $[3,2,1]$ respectively. There are two cases that avoid this problem. Namely when $\mathbf{P}_1 = \{[1,2,3],[3,2,1],[2,1,3],[1,3,2]\}$ and $\mathbf{P}_2 = \{[1,2,3],[3,2,1],[3,1,2],[2,3,1]\}$. We know that in order to avoid both $[1,2,3]$ and $[3,2,1]$ we have that $\pi(1)$ or $\pi(4)$ can not equal to 1. For $\mathbf{P}_1$ we get one possibility with $[2,1,4,3]$ and for $\mathbf{P}_2$ we get one possibility of $[3,4,1,2]$. If $n \geq 5$ we know from Lemma 4.1 that $\mathrm{AV}_{\mathbf{P}}(n) = 0$. We summarize all cases of n as following:

$$\mathrm{AV}_{\mathbf{P}}(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = 2 \text{ or } n = 3, \\ 1 & \text{if } n = 4 \text{ and } \mathbf{P} \in \{\mathbf{P}_1, \mathbf{P}_2\}, \\ 0 & \text{otherwise.} \end{cases}$$

□

Lemma 4.20 (Restrictions of Order 4 Case 2). *Given* $\{[1,2,3],[3,2,1]\} \not\subset \boldsymbol{P}$ *then* $\mathrm{AV}(n) = 2$.

Proof. We note that if $[3,2,1]$ and $[1,2,3]$ are not in $\mathbf{P}$ then there are only two cases that avoid all the permutations in $\mathbf{P}$, namely I_n and $\mathrm{Rev}(I_n)$. If either $[1,2,3]$ or $[3,2,1]$ are in $\mathbf{P}$ then there are also two cases that avoid all the permutations in $\mathbf{P}$. First is either I_n or $\mathrm{Rev}(I_n)$ respectively and the second permutation can be found by placing an element in one of the corners or the permutation matrix and I_n or $\mathrm{Rev}(I_n)$ depending on the last permutation in S_3 that is missing from $\mathbf{P}$. For example given $\mathbf{P} = \{[1,2,3],[3,1,2],[2,1,3],[2,3,1]\}$ we get $[n, n-1, \ldots, 2, 1]$ and $[1, n, n-1, \ldots, 3, 2]$ that avoid all the permutations in $\mathbf{P}$. The same reasoning can be applied for all the other possibilities of $|\mathbf{P}| = 4$. □

4.6 Restrictions of Order 5

Theorem 4.21 (Bijections when $|\mathbf{P}| = 5$). *There are two cases to consider when* $|\boldsymbol{P}| = 5$:

1. *At most one of* $[1,2,3] \in \boldsymbol{P}$ *or* $[3,2,1] \in \boldsymbol{P}$, $\mathrm{AV}_{\boldsymbol{P}}(n) = 1$.
2. $\{[1,2,3],[3,2,1]\} \subset \boldsymbol{P}$, $\mathrm{AV}_{\boldsymbol{P}}(n) = 0$ *if* $n \geq 5$.

Proof. It should be clear to the reader at this point that there is an issue when avoiding both $[1,2,3]$ and $[3,2,1]$ at the same time. Given a permutation π of length n we know from earlier that if $\{[1,2,3],[3,2,1]\} \subseteq \mathbf{P}$ and $n \geq 5$ then $\mathrm{AV}_{\mathbf{P}}(n) = 0$. The other possibility is that either $[1,2,3] \in \mathbf{P}$ or $[3,2,1] \in \mathbf{P}$. In this case we get that either π must equal to $\mathrm{Rev}\,(I_n)$ or to I_n. This is obvious if one considers the proof of Lemma 4.20. □

4.7 Containing all permutations of length 6

In this section we finalize all the results thus far of avoiding different permutations of length 3. We make use of most of the theorems and lemmas that we have proved in this section.

Theorem 4.22 (Number of permutations that contain all permutations of length 3)**.** *Let $X(n)$ be the number of permutations in S_n, given $n \geq 5$, that contain all permutations of length 3 as*

$$X(n) = n! - 6C_n + 5 \cdot 2^n + 4 \cdot \binom{n}{2} - 2F_{n+1} - 14n + 20. \tag{10}$$

This is sequence A124188 in OEIS [9].

Proof. We make use of the inclusion exclusion principle to arrive at the formula above. Note that there are $3! = 6$ permutations of length three. We know from Theorem 3.7 that the number of permutations that avoid π is given by the Catalan numbers C_n. We add all the cases for double restrictions. From Theorem 4.2 and the lemmas following it we know that there is a total of 10 double restrictions that are counted by 2^{n-1}. This gives us $10 \cdot 2^{n-1} = 5 \cdot 2^n$. Case 5 has four distinct possibilities each counted by $\binom{n}{2} + 1$. This gives us

$$n! - (6C_n) + (5 \cdot 2^n + 4 \cdot \binom{n}{2} + 4).$$

Consider the triple restrictions. We know from Theorem 4.11 and the lemmas following it that the total number is $2F_{n+1} + 14n$. The lemma for Case 1 gives us F_{n+1} and there are two such possibilities. Cases 2 through 5 are counted by n and there are 14 possibilities. This gives us

$$n! - (6C_n) + (5 \cdot 2^n + 4 \cdot \binom{n}{2} + 4) - (2F_{n+1} + 14n).$$

Consider the restrictions of order 4. We know from Theorem 4.5 that Case 2 is counted by 2. The number of possibilities is $2 \cdot \binom{5}{4} - 1 = 9$ which gives us $2 \cdot 9 = 18$. This is true because there is only one way to avoid both $[1, 2, 3]$ and $[3, 2, 1]$. There are four ways to include exactly $[1, 2, 3]$ and four ways to include exactly $[3, 2, 1]$ which gives us 9 ways. For restrictions of order 5 we know from Theorem 4.6 that Case 1 is counted by 1. The number of possibilities is $2 \cdot \binom{5}{5} = 2$, either $\mathbf{P} = \{[123], [132], [213], [231], [312]\}$ or $\mathbf{P} = \{[321], [132], [213], [231], [312]\}$. This gives us

$$\begin{aligned} X(n) &= n! - (6C_n) + (5 \cdot 2^n + 4 \cdot \binom{n}{2} + 4) - (2F_{n+1} + 14n) + (18) - (2) \\ &= n! - 6C_n + 5 \cdot 2^n + 4 \cdot \binom{n}{2} - 2F_{n+1} - 14n + 20. \end{aligned}$$

□

5 Literature Study

5.1 Introduction

Permutation patterns have gained popularity over the years. A yearly international conference about permutation patterns has been held since 2003[3]. Every year since then except for 2020 and 2021 due to the COVID pandemic there has been a conference where prestigious and knowledgeable mathematicians have presented research result to other enthusiasts of permutation patterns[2]. It is apparent that permutation patterns are not just a challenge for fun and a branch of mathematics that can be cast aside as a mere puzzle. An example of this is during the 2022 conference where many impactful uses were presented ranging from computer science to combinatorics and graph theory[1]. This section is dedicated to further topics, recent discoveries and unsolved problems within permutation patterns. We cover Wilf-equivalence, avoidance of patterns of length 4, superpermutations, superpatterns and mention other interesting concepts within the topic.

5.2 Avoidance of Length 4

We have studied permutations of length 3 and their avoidance in great depth so far. For permutations of length 4 it becomes a much trickier endeavour. In 1997 Miklós Bóna[4] proved that:

$$\mathrm{AV}_{[1342]}(n) = \frac{7n^2 - 3n - 2}{2} \cdot (-1)^{n-1} + 3\sum_{i=2}^{n} \left[2^{i+1} \cdot \frac{(2i-4)!}{i!\,(i-2)!} \cdot \binom{n-i+2}{2} \cdot (-1)^{n-1} \right].$$

This is sequence A022558 in OEIS [10]. This is a very complicated formula but we can verify that it holds true. The same can not be said for all permutations of length 4 however let alone length 5. For example there is currently no formula for $\mathrm{AV}_{[1324]}(n)$ which remains an open problem. Miklós discussed in his work that an upper bound has been found for [1342], namely $\mathrm{AV}_{[1342]}(n) < 8^n$. Miklós discusses polynomial recursiveness in his work which is a very important tool to help determine not only the upper bound but potentially the formula for the sequence. Miklós further explains that a function $f(n)$ is P-recursive if and only if its ordinary generating function $u(x)$ is differentiably finite (D-finite). If there exist polynomials $p_0(x), p_1(x), \ldots, p_d(x)$ with $p_d(x) \neq 0$ and

$$p_d(x)u^{(d)}(x) + p_{d-1}(x)u^{(d-1)}(x) + \cdots + p_1(x)u^{(1)}(x) + p_0(x)u(x) = 0,$$

where $u^{(i)}(x)$ is the i:th derivative then we say that $u(x)$ is D-finite. Miklós makes use of the fact that all algebraic power series are D-finite and shows that there is a connection between certain permutations of length 4 and rooted bicubic maps whose generating function can be represented by an algebraic power series. Miklós notes that there are only three cases of permutations of length 4 that are relevant and proves that two of them are P-recursive. The third class represented by [1324] remains unknown to this day which raises doubts to whether all permutations are P-recursive and if all permutations have a closed formula. Igor Pak and Scott Garrabrant showed that based on experimental evidence it seems that the sequences $\mathrm{AV}_{[4231],[4123]}$ and $\mathrm{AV}_{[4231],[4123],[4312]}$ are not P-recursive [5]. There are other examples of higher order multi-avoidance that seem to not be P-recursive but it is not yet absolutely certain that the same holds true for avoidance of order 1. Mathematicians suspect that the sequence $\mathrm{AV}_{[1324]}$ is not P-recursive but this remains an open problem.

5.3 Superpermutations

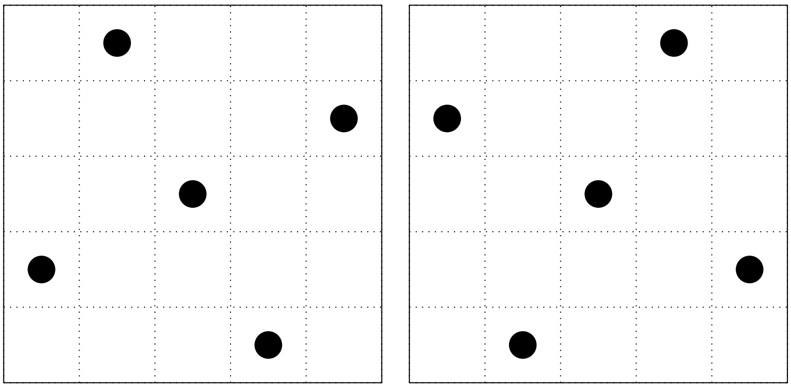
Definition 5.1 (Superpermutation)**.** We define a *superpermutation* $\mathrm{Perm}(n)$ as a sequence that contains all permutations of length n. Note that unlike permutations that were discussed earlier superpermutations allow for elements being repeated. Let $|\mathrm{Perm}(n)|$ be the length of the superpermutation.

Example 5.1 (Superpermutation example Perm(3))**.** We realize that there are $3! = 6$ permutations of length 3. The naive approach would be to put them after eachother in order to achieve 123132213231312321 but one immediately notices that there is a way to do it with significantly fewer elements. In general we want to find the *shortest* superpermutation for each n. In the case of $n = 3$ the shortest superpermutation is actually 123121321. It is certainly shorter and one can verify that it contains all occurrences of all permutations of length 3.

5.4 Superpatterns

Definition 5.2 (Superpattern)**.** We define a *superpattern* $\pi_S(n)$ as a permutation that contains all permutations of length n.

Example 5.2 (Superpattern example $\pi_S(3)$)**.** Similarly to superpermutations we could take a naive approach and end up with $\pi_S(3) = [1, 2, 3, 4, 6, 5, 8, 7, 9, 14, 15, 13, 18, 16, 17, 21, 20, 19]$. It is obvious however that we can find a superpattern that is smaller. It turns out that the smallest $\pi_S(3)$ has length 5. There are two such patterns and one can see that they are symmetrical:



5.5 Wilf-equivalence

An interesting aspect of avoiding permutations that we have noticed so far is that certain patterns are avoided equally. For example we know from Theorem 3.7 that $\mathrm{AV}_{[1,2,3]}(n) = \mathrm{AV}_{[1,3,2]}(n)$. We also saw many other examples in Section 4. In general when talking about Wilf-equivalence however we usually refer to *classes* of permutations and for that we need a few definitions.

Definition 5.3 (Symmetry class)**.** We define a *symmetry class* of a permutation π as the set of all permutations that can be acquired through the operations Flip and Rev or through a combination of them.

Definition 5.4 (Wilf-equivalence)**.** We say that two distinct symmetry classes $\mathbf{A}$ and $\mathbf{B}$ are *Wilf-equivalent* if and only if for each permutation $\pi \in \mathbf{A}, \tau \in \mathbf{B}$, we have $\mathrm{AV}_\pi(n) = \mathrm{AV}_\tau(n)$ for all n.

Example 5.3 (Symmetry class and Wilf-equivalence example)**.** We saw two examples of symmetry classes in Theorem 3.7. We note that there are two symmetry classes in S_3 namely $\mathbf{A} = \{[123], [321]\}$ and $\mathbf{B} = \{[132], [213], [231], [312]\}$. We clearly see that these are Wilf-equivalent from Theorem 3.7.

Definition 5.5 (Wilf-class)**.** We define a *Wilf-class* as a set that is made up of all relevant symmetry classes that are Wilf-equivalent.

Example 5.4 (Wilf-class examples)**.** We saw earlier that there are two symmetry classes in S_3, it is easy to see that there is only one Wilf-class in S_3 which is just S_3 itself. Throughout Section 4 we saw many different symmetry classes and Wilf-classes. Below we have combined the results for avoiding two and three patterns of length 3.

Table 1: Classes avoiding two patterns of length 3

A	Sequences enumerating $AV_{\mathbf{A}}(n)$	Formula	OEIS
$\{[123],[321]\}$	$1, 2, 4, 4, 0, 0, 0, \ldots$	—	—
$\{[132],[321]\}$	$1, 2, 4, 7, 11, 16, 22, \ldots$	$\binom{n}{2}+1$	A000124
$\{[123],[132]\}$ $\{[132],[213]\}$ $\{[132],[231]\}$ $\{[132],[312]\}$	$1, 2, 4, 8, 16, 32, 64, \ldots$	2^{n-1}	A000079

Table 2: Classes avoiding three patterns of length 3

A	Sequences enumerating $AV_{\mathbf{A}}(n)$	Formula	OEIS
$\{[123],[321],\pi\}$	$1, 2, 3, 1, 0, 0, 0, \ldots$	—	—
$\{[123],[132],[213]\}$	$1, 2, 3, 5, 8, 13, 21, \ldots$	F_{n+1}	A000045
$\{[123],[132],[231]\}$ $\{[132],[213],[312]\}$ $\{[123],[132],[312]\}$ $\{[123],[231],[312]\}$	$1, 2, 3, 4, 5, 6, 7, \ldots$	n	A000027

5.6 Further Reading

We have presented the reader a few interesting topics related to permutation patterns such as superpatterns, superpermutations and Wilf-equivalence during this section. If one wants to pursue the topic of combinatorics further with a more rigorous resource then the book "Handbook of Enumerative Combinatorics" compiled by Miklós Bóna who we mentioned earlier might be of interest[8]. This is a large piece of literary work that is over a thousand pages long and features several contributors one of whom is Vincent Vatter who has made many contributions with regards to the topic of permutation patterns. It does not come as a surprise that his contribution to this book is also about permutation patterns. He defines and explains Wilf-classes and gives examples of them and then delves deeper into the topic of growth rates of principal classes and notions of structure. There are many other topics covered in the book that might be of interest. Trees, graphs, lattice path enumeration and Catalan paths are just a few of the interesting topics in the book.

6 References

[1] 2022 conference booklet. https://www.valpo.edu/mathematics-statistics/files/2022/06/pp2022-program-book.pdf. Accessed on 23th April, 2024.

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[4] Miklós Bóna. Exact enumeration of 1342-avoiding permutations: A close link with labeled trees and planar maps. *Journal of Combinatorial Theory, Series A*, 80(2):257–272, 1997.

[5] Scott Garrabrant and Igor Pak. *Permutation patterns are hard to count*, pages 923–936.

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[9] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A124188.

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A Permutation Avoidance Program

During my work on this thesis I wrote a program in C# that was of great use when checking the avoidance of certain patterns.

A.1 Permutation Handler Class

Below is the code of the permutation handler class:

```
public class PermutationHandler
{
    public static int Factorial(int n)

    {
        if (n == 0)
            return 1;
        return (n * Factorial(n - 1));
    }
    static IEnumerable<IEnumerable<T>> GetPermutations<T>(IEnumerable<T> list, int length)
    {
        if (length == 1) return list.Select(t => new T[] { t });

        return GetPermutations(list, length - 1)
            .SelectMany(t => list.Where(e => !t.Contains(e)),
                (t1, t2) => t1.Concat(new T[] { t2 }));
    }

    public static List<int[]> AllPossiblePatterns(int length)
    {
        // Get all possible permutations of given length
        List<int[]> list = new List<int[]>();
        int[] ints = new int[length];
        for (int i = 0; i < length; i++)
        {
            ints[i] = i + 1;
        }


        var test = GetPermutations(ints, length);
        foreach(IEnumerable<int> t in test.ToList())
        {
            int[] temp = new int[length];
            for (int i = 0; i <  length; i++)
            {
                temp[i] = t.ToArray()[i];
            }

            list.Add(temp);
        }

        return list;
    }

    public static List<int[]> AvoidsPattern(int[] pattern, int n, bool reverse = false)
    {
        List<int[]> avoidsPattern = new List<int[]>();
        List<int[]> allPatterns = AllPossiblePatterns(n);
```

```
    for (int i = 0; i < allPatterns.Count(); i++)
    {
        List<int[]> allPossibleCombos
        = AllPossibleCombinations(pattern.Length, allPatterns[i]);

        List<int[]> validPerms = ValidPermutations(allPossibleCombos, pattern);

        if (!reverse)
        {
            if (validPerms.Count == 0)
                avoidsPattern.Add(allPatterns[i]);
        }
        else
        {
            if (validPerms.Count > 0)
                avoidsPattern.Add(allPatterns[i]);
        }

    }

    return avoidsPattern;
}



/// <summary>
/// Returns a list of all arrays of length (permLength) from permutation
/// </summary>
/// <param name="permLength"></param>
/// <param name="permutationMatrix"></param>
/// <returns></returns>
public static List<int[]> AllPossibleCombinations
(int permLength, int[] permutationMatrix)
{
    List<int[]> indexList = new List<int[]>();
    int[] finalMatrix = new int[permLength];
    int[] currentMatrix = new int[permLength];

    for (int i = 0; i < permLength; i++)
    {
        finalMatrix[i] = permutationMatrix.Length - permLength + i;
    }


    for (int i = 0; i < permLength; i++)
        currentMatrix[i] = i;

    indexList.Add(currentMatrix);

    while (!currentMatrix.SequenceEqual(finalMatrix))
    {
        int[] newMatrix = AddToMatrix(currentMatrix);

        if (!ValidMatrix(newMatrix, permutationMatrix.Length))
            FixMatrix(newMatrix, permutationMatrix.Length);
```

```
        indexList.Add(newMatrix);
        currentMatrix = newMatrix;
    }


    List<int[]> result = new List<int[]>();

    // Get the actual values

    for (int i = 0; i < indexList.Count; i++)
    {
        int[] valueArray = new int[permLength];
        for (int j = 0; j < permLength; j++)
        {
            valueArray[j] = permutationMatrix[(indexList[i])[j]];
        }
        result.Add(valueArray);
    }

    return result;
}

static int[] AddToMatrix(int[] previousMatrix)
{
    int[] newMatrix = new int[previousMatrix.Length];
    for (int i = 0; i < previousMatrix.Length; i++)
    {
        newMatrix[i] = previousMatrix[i];
    }
    newMatrix[newMatrix.Length - 1] += 1;
    return newMatrix;
}

static bool ValidMatrix(int[] matrix, int maxVal)
{
    for (int i = 0; i < matrix.Length; i++)
    {
        if (matrix[i] >= maxVal)
            return false;
    }
    return true;
}

static int[] FixMatrix(int[] matrix, int maxVal)
{
    bool validM = false;

    while (!validM)
    {
        for (int i = matrix.Length - 1; i > 0; i--)
        {
            if (matrix[i] == maxVal)
            {
                matrix[i - 1] += 1;
                for (int j = 0; j < (matrix.Length - i); j++)
```

```
                {
                    matrix[i + j] = matrix[i + j - 1] + 1;
                }
            }
        }
        validM = ValidMatrix(matrix, maxVal);
    }

    return matrix;
}

public static List<int[]> ValidPermutations(List<int[]> allCombinations,
int[] mainPermutation, bool stopEarly = false)
{
    List<int[]> validCombinations = new List<int[]>();

    int[] ordInd = new int[mainPermutation.Length];
    for (int i = 0; i < ordInd.Length; i++)
    {
        ordInd[mainPermutation[i] - 1] = i;
    }


    for (int i = 0; i < allCombinations.Count; i++)
    {
        bool validPerm = true;
        int[] currentTest = allCombinations[i];
        int[] newTest = new int[currentTest.Length];

        for (int j = 0; j < currentTest.Length; j++)
        {
            newTest[j] = currentTest[ordInd[j]];
        }

        for (int j = 0; j < newTest.Length - 1; j++)
        {
            if (validPerm && newTest[j] >= newTest[j + 1])
                validPerm = false;
        }

        if (validPerm)
        {
            if (stopEarly)
                return new List<int[]>();
            validCombinations.Add(currentTest);
        }
    }


    return validCombinations;
}
}
```

A.2 Testing Code

Below is sample code used for testing:

```
int[] testPattern = { 1, 2, 3};
int length = 5;
List<int[]> allAvoids = PermutationHandler.AvoidsPattern(testPattern, length, false);

Console.WriteLine("Testing how many and which of the permutation patterns that are "
+ string.Join("", testPattern) + "-avoiding.");

for (int i = 0; i < allAvoids.Count; i++)
{
    Console.WriteLine(string.Join("", allAvoids[i]));
}

Console.WriteLine("It turns out that " + string.Join("", testPattern) + " avoids "
+ allAvoids.Count() + "/" + PermutationHandler.Factorial(length).ToString()
+ " patterns, as you can see above.");
```

A.3 Testing Code Output

This gives us the following output:

```
Testing how many and which of the permutation patterns that are 123-avoiding.
15432
21543
25143
25413
25431
31542
32154
32514
32541
35142
35214
35241
35412
35421
41532
42153
42513
42531
43152
43215
43251
43512
43521
45132
45213
45231
45312
45321
51432
52143
52413
52431
53142
53214
53241
53412
53421
54132
54213
54231
54312
54321
It turns out that 123 avoids 42/120 patterns, as you can see above.
```