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Tensor Categories

av

Benjamin Andersson

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Benjamin Andersson

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Abstract

Tensor categories, also known under the name of ”symmetric monoidal categories”, is an area of import for research in pure mathematics, and with links to computation, logic and physics. In this article, we go through their properties, introducing categorical concepts as needed as we go along. We prove some well-known theorems, such as maclanes coherence theorem, for monoidal categories (and in particular monoidal categories with a symmetry), and we zoom in on symmetric monoidal categories with some more restrictive structure (e.g. rigidity). The work tries to lay the ground for understanding the main theorem in P. Deligne & J.S. Milnes article ”Tannakian Categories”, wherein it is shown that an exact, faithful $\Bbbk$-linear tensor functor yields an equivalence of categories between a more restricted type of tensor category, and the category of linear representations of a group (perphaps affine group scheme) G.

Abstrakt

Tensor kategorier, också kända under namnet “symmetriska monoidala kategorier”, är ett område med betydelse för både forskning inom ren matematik, men också med kopplingar till område som teoretisk datavetenskap, logik, och fysik. I den här artikeln, så kommer vi gå igenom dessa typer av kategoriers egenskaper, där vi introducerar kategori-teoretiska koncept då de behövs. Vi bevisar några välkända teorem, så som Maclanes koherensteorem, både för monoidala men också symmetriskt monoidala kategorier, och vi lägger sedan mer fokus på en symmetriskt monoidala kategorier mer fler restriktioner på sin struktur (exempelvis rigiditet). Arbetet försöker bygga en en bas för att kunna förstå huvudsatsen i P. Deligne & J.S. Milnes artikel “Tannakian Categories”, där i de visar att en exakt, trogen, $\Bbbk$-linjär tensor-funktor ger en ekvivalens av kategorier mellan en mer restriktiv typ av tensor kategori, och kategorin av linjära representationer av en grupp (kanske eg. affint grupp schema) G.

Contents

1 Prelude

The author has made his best to indicate where proofs are not fully developed, or when things are assumed for sake of brevity or to avoid too much complication. I've used GPT-4 to ask questions and to create biblatex references (mainly when they were hard to find by a quick google search).

This work is mainly an expansion on themes encountered in the the first chapter of P. Deligne & J.S. Milnes "Tannakian Categories" ([10]). The reader already familiar with category theory at reasonably high level, might find that some of the things stated in the thesis are too elementary. The author of the thesis did not know much beyond quite elementary category theory at the beginning of writing this thesis, whence a lot of work has been done to expand on assumptions and formal machinery that is in the background of a lot definitions and theorems in [10].

Originally, the intent and goal of the article was to apply the theory of tannakian categories to more elementary algebraic structures; as a lot of work went into understanding the theory behind tannakian categories, over time, this morphed into a thesis that seeks to understand and unpack as much as possible (with due time constraints) of the theorems and concepts at work in chapter 1 of [10]. We have mainly used [13] as a reference for the categorical side of the thesis. The author wants to thank several people for being willing to answer questions, or engage in dialogue about the material in the thesis; among them Rikard Bögvad, David Rydh, Gregory Arone, Kilian Liebe, Victor Groth, and the larger math-community at mathoverflow (MO) and mathstackexchange (MSE).

2 Introduction

Tannakian formalism was perhaps first seriously investigated by Grothendiecks student, Neantro Saavedra Rivano. The findings were published in "Catégories Tannakiennes" ([14]). This was later on taken up by another of Grothendiecks students, Pierre Deligne. What in Deligne & Milne is called "tensor categories", also goes by the name of "symmetric monoidal categories", among others. John Baez has at one point called symmetric monoidal categories a "Rosetta Stone", for their connections to areas such as logic, computation and physics (for example, in certain types of topological quantum field theories; see also the *cobordism hypothesis*; [3]). In this work, we will be happy to cover more basic properties of what we call tensor categories; we will cover certain coherence conditions that are imposed by the axioms, maclanes coherence theorem for monoidal (& symmetric monoidal categories), dual objects, morphisms of tensor functors, and some other results related to these constructions. One of the goals of this article can be seen to be, to lay the groundwork for being able to understand what one of the main theorems in ([10, proposition 2.11]) says.

3 Preliminaries; Categories

We begin by introducing some relevant notations and definitions.

Definition 1.1. (Category). A category $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$ consists of

1. A *class* $\mathrm{ob}(\mathcal{C})$ whose elements are **objects** of $\mathcal{C}$.
2. For every $X, Y \in \mathrm{Ob}(\mathcal{C})$, a *class* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ where elements in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are called **morphisms** in $\mathcal{C}$ *from* X *to* Y.
3. For every $X \in \mathrm{Ob}(\mathcal{C})$, there is a special element $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ called the *identity* on X.
4. For every $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a *function* called *composition*

   $$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

   defined by

   $$(g, f) \longmapsto g \circ f = gf.$$

   The composition satisfies *associativity*:

   $$h(gf) = (hg)f \quad (\textit{when} \text{ both sides make sense}).$$

   - Unitality: If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ then

   $$\begin{aligned} \mathrm{id}_Y \circ f &= f \circ \mathrm{id}_X \\ &= f. \end{aligned}$$

Definition 1.2. A category $\mathcal{C}$ is **small** if $\mathrm{Ob}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}) \in$ **Set**.

Example 1.3. Some examples of categories are

- **Set** with objects as sets and morphisms as functions.
- **Top** with objects as topological spaces and morphisms as continuous functions.
- $\mathbf{Vect}_{\Bbbk}$ over a field $\Bbbk$, who:s objects are vector-spaces over $\Bbbk$ and where the morphisms are $\Bbbk$-linear maps.
- **Grp**, where the objects are groups and the morphisms are group homomorphisms.
- Take a set X with a preorder $\leq$. Then $(X, \leq)$ defines a category where the objects are the elements of X, and $\mathrm{Hom}(x, y) := \begin{cases} \{f_{x,y}\}, \text{if } x \leq y \\ \varnothing, \text{if } x \nleq y \end{cases}$
- **Cat**, where objects are *small* categories (1.2), and where the morphisms are functors.

A special type of category, which will be relevant in the context of the bifunctor $- \otimes -$, are product categories.

Definition 1.4. A **product category** is a category $\mathcal{C} \times \mathcal{D}$ whose objects are pairs (X, Y) where $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$, whose morphisms are pairs (f, g) where $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, and $g \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ for objects $E, F \in \mathrm{Ob}(\mathcal{D})$.

Furthermore, we define composition in the product category as $(f_1 \circ f_2, g_1 \circ g_2) = (f_1, g_1) \circ (f_2, g_2)$ and identities $\mathrm{id}_{(A,B)} = (\mathrm{id}_A, \mathrm{id}_B)$ for objects $(A, B) \in \mathrm{Ob}(\mathcal{C} \times \mathcal{D})$.

Definition 1.5. A category $\mathcal{C}$ is **locally small** if for all objects $A, B \in \mathcal{C}$, one has $\mathrm{Hom}_{\mathcal{C}}(A, B) \in \mathbf{Set}$.

Example 1.6. $\mathbf{Top} \times \mathbf{Top}$ defines a product category. Morphisms are pairs of morphisms (f, g) where $f, g \in \mathrm{Mor}(\mathbf{Top})$.

Crucially connected to categories, are functors.

Definition 1.7. A **functor** $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}, \mathcal{D}$ are *categories*, is a map that fulfills the following properties:

- For each object $X \in \mathrm{Ob}(\mathcal{C})$, $F(X) \in \mathrm{Ob}(\mathcal{D})$
- For each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we get a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, so that the following conditions hold
    1. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$.
    2. $F(g \circ f) = F(g) \circ F(f)$ for all *morphisms* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_C(Y, Z)$.

Remark 1.8. For a functor (1.7) $F : \mathcal{C} \to \mathcal{D}$, we generally call $\mathcal{C}$ the **source category** of F, and $\mathcal{D}$ its **target category**.

Definition 1.9. A **covariant functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ such that if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ then $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$.

Definition 1.10. A **bifunctor** is a functor F whose domain is a *product category* (1.4). That is, $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ for categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$.

We will mostly talk about the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. In this case, we will take as a given that the following conditions hold (as in [9, chapter 2]):

- For every pair of objects $X, Y \in \mathcal{C}$ gives us an object $X \otimes Y \in \mathcal{C}$.
- For pairs of morphisms $f : X \to X'$ and $g : Y \to Y'$, we have a morphism $f \otimes g : X \otimes Y \to X' \otimes Y'$.
- For identity morphisms id_X and id_Y, we have $\mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X \otimes Y} : X \otimes Y \to X \otimes Y$.
- For defined composites $f' \circ f$ and $g' \circ g$ in $\mathcal{C}$, it holds that $(f' \circ f) \otimes (g' \circ g) = f' \otimes g' \circ f \otimes g$.

We will take these as *facts* in the following text; so we will use these properties freely without mentioning them each time.

Definition 1.11. A forgetful functor $F : \mathcal{C} \to \mathbf{Set}$ is a functor that "forgets" some of the underlying structure in the source category (1.8) $\mathcal{C}$. There is, as of yet, no precise formal definition of exactly what a "forgetful" functor is. Hence this serves more as a *heuristic* for when to call a functor "forgetful".

Example 1.12. We have a forgetful functor $F : \mathrm{Rep}_{\Bbbk}(G) \to \mathbf{Vect}_{\Bbbk}$ from the category of representations of a group G (with fixed field $\Bbbk$), to the category of vector spaces over $\Bbbk$. It takes a representation (V, ρ) to the vector space V, and takes G-equivariant maps to the linear transformation corresponding to them in $\mathbf{Vect}_{\Bbbk}$.

If $H \leq G$ is a subgroup, then one has a forgetful (restrictive) functor $\mathrm{Res}^G_H : \mathbf{Rep}_{\Bbbk}(G) \to \mathbf{Rep}_{\Bbbk}(H)$.

If $H \leq G$ with $\phi : H \hookrightarrow G$ the inclusion homomorphism, then $\rightsquigarrow \varphi^* : \mathbf{Rep}_{\Bbbk}(G) \to \mathbf{Rep}_{\Bbbk}(H)$ "pulls back" a representation (V, ρ) from $\mathbf{Rep}_{\Bbbk}(G)$ to $\mathbf{Rep}_{\Bbbk}(H)$ by $\varphi^*(\rho) := \rho \circ \varphi : H \to \mathrm{GL}(V)$, as illustrated below.

$$\begin{tikzcd} H \arrow[r, "\varphi"] \arrow[dr, "\varphi^* \rho"'] & G \arrow[d, "\rho"] \\ & \mathrm{GL}(V) \end{tikzcd}$$

Remark 1.13. Note that ρ in the above example is always $\Bbbk$-linear.

Example 1.14. If $\mathcal{C} = \mathbf{Grp}$, the category of groups, then we have a forgetful functor $F : \mathbf{Grp} \to \mathbf{Set}$ that takes objects $G \in \mathbf{Grp}$ to their set in $\mathbf{Set}$ (without any relations imposed), and *group homomorphisms* $f : G \to H$ to *functions* $f : G \to H$ between sets H, G in $\mathbf{Set}$.

We now introduce the following lemma.

Lemma 1.15. *Functors preserve isomorphisms.*

Proof. As in [13]: Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $f : X \to Y$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$ be an isomorphism with inverse $f^{-1} : Y \to X$. By 1.7, we find that

$$\begin{aligned} \mathrm{id}_{F(X)} &= F(\mathrm{id}_X) \\ &= F(f \circ f^{-1}) \\ &= F(f) \circ F(f^{-1}) \end{aligned}$$

so that $F(f^{-1})$ is a right-inverse to $F(f)$. By the same property, we have that

$$\begin{aligned} \mathrm{id}_{F(Y)} &= F(f^{-1} \circ f) \\ &= F(f^{-1}) \circ F(f) \end{aligned}$$

so that $F(f^{-1})$ is a left-inverse to $F(f)$. Hence $F(f)$ is an isomorphism. ☐

A category $\mathcal{C}$ can have substructures that themselves are categories; this motivatives the following definition

Definition 1.16. Let $\mathcal{C}$ be a category. Then we say that $\mathcal{C}'$ is a **subcategory** of $\mathcal{C}$ if

- if $A \in \mathcal{C}' \implies A \in \mathcal{C}$.
- if $f \in \mathrm{Mor}(\mathcal{C}') \implies f \in \mathrm{Mor}(\mathcal{C})$.
- $\forall A \in \mathcal{C}'$, we have that $\mathrm{id}_A \in \mathrm{Mor}(\mathcal{C}')$, inherited from $\mathrm{Mor}(\mathcal{C})$.
- $\forall f \in \mathrm{Mor}(\mathcal{C}')$, where $f : X \to Y$, we have that $X, Y \in \mathcal{C}'$.
- If $\forall f, g \in \mathrm{Mor}(\mathcal{C}')$ with $g : A \to B$ and $f : B \to C$, we have that $f \circ g \in \mathrm{Mor}(\mathcal{C}')$.

  This in turn makes $\mathcal{C}' = (\mathrm{Ob}(\mathcal{C}'), \mathrm{Mor}(\mathcal{C}'))$ a category.

Definition 1.17. Let $\mathcal{C}$ be a category and let $\mathcal{C}'$ be a subcategory. Then we say that $\mathcal{C}'$ is a **full subcategory** if $\forall X, Y \in \mathrm{Ob}(\mathcal{C}')$, one has $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Definition 1.18. Let $\mathcal{C}$ be a category, and let $f \in \operatorname{Hom}_{\mathcal{C}}(B, C)$. Then we say that f is an **monomorphism**, if

$$\begin{aligned} & f \circ g_1 = f \circ g_2 \\ \implies & g_1 = g_2 \end{aligned}$$

for $g_1, g_2 : A \rightrightarrows B$.

Definition 1.19. Let $\mathcal{C}$ be a category, and let $f \in \operatorname{Hom}_{\mathcal{C}}(B, C)$. f is an **epimorphism** if

$$\begin{aligned} & g_1 \circ f = g_2 \circ f \\ \implies & g_1 = g_2 \end{aligned}$$

for morphisms $g_1, g_2 : C \rightrightarrows D$ in $\mathcal{C}$.

More generally, we characterize a *diagram* in a category the following way.

Definition 1.20. Let $\mathcal{C}$ be a category. Then a **diagram** in $\mathcal{C}$ is a functor $F : \mathcal{J} \to \mathcal{C}$ where the *index category* $\mathcal{J}$ is *small* (see 1.2).

Usually, one write the diagram pictorially as its image, and leave the source category implicit.

Lemma 1.21. *Functors preserve commutative diagrams*

Proof. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram in $\mathcal{C}$, with $\mathcal{J}$ small. Let $(f_1 \circ \ldots \circ f_n), (g_1 \circ \ldots \circ g_m) \in \operatorname{Mor}(\mathcal{J})$ with $f_1 \circ \ldots \circ f_n = g_1 \circ \ldots \circ g_m$ and $m, n \in \mathbb{Z}_{\geq 0}$. Then we find that $F(g_1 \circ \ldots \circ g_m) = F(g_1) \circ \ldots \circ F(g_m)$ and $F(f_1 \circ \ldots \circ f_n) = F(f_1) \circ \ldots \circ F(f_n)$ by functoriality of F (use definition 1.7 repeatedly).

But since F by assumption is well-defined, we get

$$\begin{aligned} & F(f_n \circ \ldots \circ f_1) = F(g_m \circ \ldots \circ g_1) \\ \implies & F(f_n) \circ \ldots \circ F(f_1) = F(g_m) \circ \ldots \circ F(g_1). \end{aligned}$$

□

1.1 Some special objects of interest, in categories

We want to introduce some objects in a given category $\mathcal{C}$, with certain properties of interest.

Definition 1.22. Let $\mathcal{C}$ be a category, and let $\mathcal{I} \in \mathcal{C}$. If for each object $X \in \operatorname{Ob}(\mathcal{C})$ there *exists a uniqe morphism* $f : \mathcal{I} \to X$, then we call $\mathcal{I}$ an **initial object** of $\mathcal{C}$.

Definition 1.23. Let $\mathcal{C}$ be a category, and let $\mathcal{T} \in \mathcal{C}$ such that for all objects $Y \in \mathcal{C}$ there is a *unique morphism* $t : Y \to \mathcal{T}$. Then we say that $\mathcal{T}$ is a **terminal object** of $\mathcal{C}$.

Definition 1.24. Let $\mathcal{C}$ be a category, and let 0 be an object in $\mathcal{C}$ that is both an initial and terminal object. Then we call 0 a **zero object**.

1.2 Natural transformations

We want to introduce a definition that is crucial to understanding the relationship between and within categories

We will use the following definition

Definition 1.25. . Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be functors from a category $\mathcal{C}$ to $\mathcal{D}$. We then say that α is a **natural transformation** from $\mathcal{C}$ to $\mathcal{D}$ if the following two conditions hold

1. For every object $X \in \mathrm{Ob}(\mathcal{C})$, there is a morphism, $\alpha_X : F(X) \to G(X)$, called the **component** of α at X.

2. For all components, and for every morphism $f : X \to Y$ where $X, Y \in \mathrm{Ob}(\mathcal{C})$, the following diagram commutes

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"] & G(Y)
\end{tikzcd} \tag{1.1}$$

Furthermore, if for *all* $X \in \mathrm{Ob}(C)$, the morphism α_X is an isomorphism in $\mathcal{D}$, then we say that α is a **natural isomorphism**.

2 Tensor categories

Before defining *tensor categories* we need some axioms.

In the following, let $\mathcal{C}$ be a category and let $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a bifunctor, where $\mathcal{C} \times \mathcal{C}$ is a **product category** (1.4).

Definition 2.1. (Associativity constraint and pentagon-axiom). We define the *natural isomorphism* (1.25) ϕ with components $\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ for $X, Y, Z \in \mathcal{C}$. We call ϕ an **associativity constraint** for $(\mathcal{C}, \otimes)$, as in [10], if the following diagram commutes

$$\begin{tikzcd}
 & X \otimes (Y \otimes (Z \otimes T)) \arrow[dl, "\mathrm{id}_X \otimes \phi_{Y,Z,T}"'] \arrow[dr, "\phi_{X,Y,Z\otimes T}"] & \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[d, "\phi_{X,Y\otimes Z,T}"'] & & (X \otimes Y) \otimes (Z \otimes T) \arrow[d, "\phi_{X\otimes Y,Z,T}"] \\
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\phi_{X,Y,Z} \otimes \mathrm{id}_T"] & & ((X \otimes Y) \otimes Z) \otimes T
\end{tikzcd} \tag{2.1}$$

We call this the **pentagon axiom**.

Definition 2.2. (Commutativity constraint and hexagon-axiom). As in [10], and similarly to definition 2.2, we also define a *natural isomorphism* ψ with components

$$\psi_{X,Y} : X \otimes Y \to Y \otimes X$$

for $X, Y \in \mathcal{C}$, which we call a **commutativity constraint**, where we have that

$$\psi_{Y,X} \circ \ \psi_{X,Y} = \mathrm{id}_{X \otimes Y} : X \otimes Y \to X \otimes Y.$$

We say that ϕ and ψ are **compatible** if for all objects $X, Y, Z \in \mathcal{C}$, the following diagram commutes

$$\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[r, "\phi_{X,Y,Z}"] \arrow[dl, "\mathrm{id}_X \otimes \psi_{Z,Y}"'] & (X \otimes Y) \otimes Z \arrow[dr, "\psi_{X \otimes Y, Z}"] & \\
X \otimes (Z \otimes Y) \arrow[dr, "\phi_{X,Z,Y}"'] & & & Z \otimes (X \otimes Y) \arrow[dl, "\phi_{Z,X,Y}"] \\
& (X \otimes Z) \otimes Y \arrow[r, "\psi_{X,Z} \otimes \mathrm{id}_Y"] & (Z \otimes X) \otimes Y &
\end{tikzcd} \tag{2.2}$$

We call this the **hexagon axiom**.

Before proceeding, we cover some more definitions.

Definition 2.3. (Equivalence of categories, see [13]) . We say that we have an equivalence of categories $\mathcal{C}, \mathcal{D}$ if we are given functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : F'$ such that $\eta : \mathrm{id}_{\mathcal{C}} \simeq F \circ F'$ and $\rho : \mathrm{id}_{\mathcal{D}} \simeq F' \circ F$. We write $\mathcal{C} \simeq \mathcal{D}$ for this **equivalence of categories**.

Remark 2.4. Given a functor $F : \mathcal{C} \to \mathcal{D}$, we say that it *yields an equivalence of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that the conditions in 2.3 are fulfilled.

Definition 2.5. . A pair (U, u) where $U \in \mathcal{C}$ together with an isomorphism $u : U \to U \otimes U$ is called an **identity object** of $(C, \otimes)$, if $U \otimes - : \mathcal{C} \to \mathcal{C}$ defined by taking $X \in \mathcal{C}$ to $U \otimes X \in \mathcal{C}$, yields an *equivalence* of categories (2.3, 2.4).

We are now ready to define a **tensor category**.

Definition 2.6. A **tensor category** consists of a 4-tuple $(\mathcal{C}, \otimes, \phi, \psi)$ where ϕ and ψ are *compatible* (see 2.2), and there exists an identity object (2.5). One could also call these categories *symmetric monoidal categories*, as [10] points out.

Remark 2.7. Henceforth, we shall denote a tensor category in a more succinct way as $(\mathcal{C}, \otimes)$, or if we are talking about objects X in $(\mathcal{C}, \otimes)$, we will denote this as $X \in \mathcal{C}$, and morphisms $f : X \to Y$ in $\mathrm{Mor}((\mathcal{C}, \otimes))$ as $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ or $f \in \mathrm{Mor}(\mathcal{C})$.

2.1 Representations of a group G

Definition 2.8. We call a *category* G a **groupoid** if every morphism $f : g \to g'$ for $g, g' \in G$ is *invertible.*

Example 2.9. Every group $(G, \bullet)$ forms a groupoid with one object G and morphisms $g : G \to G$ for each element $g \in G$. Explicitly, we define $g(g') := g \bullet g'$.

Definition 2.10. (G-representation & G-Equivariance).

- Let $(\bullet, G)$ be the category consisting of one object, $\bullet$, with elements $g \in G$ as morphisms. Let $F : (\bullet, G) \to \mathcal{C}$ be a functor from $(\bullet, G)$ into another category $\mathcal{C}$. The functor F specifies objects $F(\bullet)$ in $\mathcal{C}$ together with an endomorphisms $g_* : F(\bullet) \to F(\bullet)$ that has the following properties

    1. $h_* g_* = (hg)_*$ for all $h, g \in G$.

2. $e_* = \mathrm{Id}_{F(\bullet)}$.

Thus, F specifies a G-action of the group G on the object $F(\bullet) \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$ of all vector spaces over the field $\Bbbk$, we call $F(\bullet)$ a **G-representation**. In the case of G-representations $F : (\bullet, G) \to \mathrm{Vect}_{\Bbbk}$ where $F(\bullet) = V$ for some vector space V over $\Bbbk$, we have that $g_* : V \to V$ is an *automorphism* of V, so that $g_* \in \mathrm{Aut}(V)$ for all $g \in G$. This follows from lemma 1.15, since *F preserves isomorphisms.*

Given functors $F, F' : (\bullet, G) \rightrightarrows \mathcal{C}$, such that $F(\bullet), F'(\bullet) \in \mathrm{Ob}(\mathcal{C})$. Let $\eta \in \mathrm{Hom}_{\mathcal{C}}(F(\bullet), F'(\bullet))$, and $g_* \in \mathrm{Aut}(F(\bullet))$ as well as $g_* \in \mathrm{Aut}(F'(\bullet))$. Then we say that η is **G-equivariant** if the following diagram commutes

$$\begin{tikzcd}
F(\bullet) \arrow[r, "\eta"] \arrow[d, "g_*"'] & F'(\bullet) \arrow[d, "g_*"] \\
F(\bullet) \arrow[r, "\eta"'] & F'(\bullet)
\end{tikzcd} \tag{2.3}$$

that is, $\eta \circ g_* = g_* \circ \eta$, where

$$\begin{tikzcd}
\bullet \arrow[d, "g"] \\
\bullet
\end{tikzcd}$$

and $g \in \mathrm{Mor}((\bullet, G))$.

Example 2.11. Let G be a group and let $\Bbbk$ be a field. Then the category $\mathbf{Rep}_{\Bbbk}(G)$ of representations of G over $\Bbbk$ has

- As **objects** in said category, $\mathrm{Ob}(\mathbf{Rep}_{\Bbbk}(G))$, are pairs (V, ρ) of vector spaces V over $\Bbbk$ and *representations* ρ of G on V.

- For (V, ρ_V), (W, ρ_W) we have that

  $$\mathrm{Hom}_{\mathbf{Rep}_{\Bbbk}(G)}((V, \rho_V), (W, \rho_W))$$

  consists of G-*equivariant* morphisms (2.10), where $\rho_V : G \to \mathrm{GL}(V)$ is a G-equivariant group homomorphism into the *automorphism group* $\mathrm{GL}(V)$ of V. Note here that if $V = W$, but $\rho_V \neq \rho_W$, then (V, ρ_V) and (W, ρ_W) are *different* objects in $\mathbf{Rep}_{\Bbbk}(G)$.

- **Composition** consists of composition of G-equivariant maps.

- As **identity object** (2.5), we have the *identity representation* $\mathbf{1} : G \to \mathrm{GL}(1, \Bbbk) \cong \Bbbk^{\times}$ defined by $\mathbf{1}(g) \equiv g$, for all $g \in G$, and $V = \Bbbk$, i.e. $(\Bbbk, \mathbf{1})$ is the identity object (see definition 2.4). We have canonical isomorphisms

  $$\begin{aligned} V \otimes_{\Bbbk} \Bbbk &\cong \Bbbk \otimes_{\Bbbk} V \\ &\cong \Bbbk \end{aligned}$$

  defined by

  $$\begin{aligned} v \otimes k &\mapsto k \otimes v \\ &\mapsto kv \end{aligned}$$

(recall that scalar multiplication of $v \in V$ by $k \in \Bbbk$ is well-defined, since V is a vector space over $\Bbbk$).

- With $\otimes$ as bifunctor, $\mathbf{Rep}_{\Bbbk}(G)$ forms a *monoidal* category. One defines

$$\rho_{V \otimes W}(g) := \rho_V(g) \otimes \rho_W(g).$$

2.2 Tensor categories, again

We continue our investigation into tensor categories.

Proposition 2.12. *Let (U, u) be an identity object (2.5) of the tensor category $(\mathcal{C}, \otimes)$.*

(a) We get a natural isomorphism l with components $l_X : X \to U \otimes X$ for each object X in $\mathcal{C}$, so that $l_U : U \to U \otimes U$ is the same map as u in 2.5, that is, $u = l_U$.

Furthermore, the following diagrams commute

$$\begin{tikzcd}
X \otimes Y \arrow[r, "l_{X \otimes Y}"] \arrow[d, equal] & U \otimes (X \otimes Y) \arrow[d, "\phi_{U,X,Y}"] \\
X \otimes Y \arrow[r, "l_X \otimes \mathrm{id}_Y"] & (U \otimes X) \otimes Y
\end{tikzcd} \tag{2.4}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "l_X \otimes \mathrm{id}_Y"] \arrow[d, "\mathrm{id}_X \otimes l_Y"] & (U \otimes X) \otimes Y \arrow[d, "\psi_{U,X} \otimes \mathrm{id}_Y"] \\
X \otimes (U \otimes Y) \arrow[r, "\phi_{X,U,Y}"] & (X \otimes U) \otimes Y
\end{tikzcd} \tag{2.5}$$

Remark 2.13. Note that there is a typo in the proof of (b) in [10], in that l_U in

$$a : U \xrightarrow{l_U} U' \otimes U \xrightarrow{\psi_{U',U}} U \otimes U' \xrightarrow{l_{U'}^{-1}} U'$$

should be l'_U.

Proof. (2.4): Let's note the following; since we have an equivalence of categories $X \rightsquigarrow U \otimes X$, we have functors $F : \mathcal{C} \leftrightarrows \mathcal{C} : G$ so that $GF(X) \cong X$ and $FG(U \otimes X) \cong U \otimes X$ for all objects $X \in \mathcal{C}$. Furthermore, note that $F(X) \cong U \otimes X$ so that

$$\begin{aligned} GF(X) &\cong G(U \otimes X) \\ &\cong X. \end{aligned}$$

Assuming l_X exists, then we have that $l_X : X \cong U \otimes X$ and $l_{U \otimes X} : U \otimes X \cong U \otimes (U \otimes X)$. It follows that l_X is completely determined by the map $\mathrm{id}_U \otimes l_X : U \otimes X \to U \otimes (U \otimes X)$.

We find that $\mathrm{id}_U \otimes l_X : U \otimes X \to U \otimes (U \otimes X)$ can be defined as the composition

$$(U \otimes X) \overset{u \otimes \mathrm{id}_X}{\longrightarrow} ((U \otimes U) \otimes X) \overset{\phi^{-1}_{U,U,X}}{\longrightarrow} (U \otimes (U \otimes X)) \tag{2.6}$$

where we have used that $\phi_{U,U,X}$ is an isomorphism, hence has an inverse $\phi^{-1}_{U,U,X}$.

The following part is inspired by [1].

We start with 2.4, then tensor the diagram with $U \otimes -$, which gives us the first diagram below. The second diagram (the one after the downward arrow) is gotten by the *naturality* of ϕ.

$$\begin{tikzcd}
U \otimes (X \otimes Y) \arrow[r, "\mathrm{id}_U \otimes l_{X \otimes Y}"] \arrow[d, equal] & U \otimes (U \otimes (X \otimes Y)) \arrow[d, "\mathrm{id}_U \otimes \phi_{U,X,Y}"] \\
U \otimes (X \otimes Y) \arrow[r, "\mathrm{id}_U \otimes (l_X \otimes \mathrm{id}_Y)"] & U \otimes ((U \otimes X) \otimes Y)
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
U \otimes (X \otimes Y) \arrow[r, "\mathrm{id}_U \otimes l_{X \otimes Y}"] \arrow[d, equal] & U \otimes (U \otimes (X \otimes Y)) \arrow[d, "\mathrm{id}_U \otimes \phi_{U,X,Y}"] \\
U \otimes (X \otimes Y) \arrow[r, "\mathrm{id}_U \otimes (l_X \otimes \mathrm{id}_Y)"'] \arrow[d, "\phi_{U,X,Y}"] & U \otimes ((U \otimes X) \otimes Y) \arrow[d, "\phi_{U,U \otimes X,Y}"] \\
(U \otimes X) \otimes Y \arrow[r, "(\mathrm{id}_U \otimes l_X) \otimes \mathrm{id}_Y"] & (U \otimes (U \otimes X)) \otimes Y
\end{tikzcd}$$

From here, we use 2.6 together with the fact that by naturality (of ϕ) together with ϕ being an *isomorphism at each component*, we have

$$\begin{aligned}
&\phi_{U,U \otimes X,Y} \circ \mathrm{id}_U \otimes (l_X \otimes \mathrm{id}_Y) = (\mathrm{id}_U \otimes l_X) \otimes \mathrm{id}_Y \circ \phi_{U,X,Y} \\
\iff\ &\mathrm{id}_U \otimes (l_X \otimes \mathrm{id}_Y) = \phi^{-1}_{U,U \otimes X,Y} \circ (\mathrm{id}_U \otimes l_X) \otimes \mathrm{id}_Y \circ \phi_{U,X,Y}
\end{aligned}$$

$$\begin{tikzcd}
U \otimes (X \otimes Y) \arrow[r, "u\otimes \mathrm{id}_{X\otimes Y}"] \arrow[dd, "\phi_{U,X,Y}"'] & (U \otimes U) \otimes (X \otimes Y) \arrow[r, "\phi^{-1}_{U,U,X\otimes Y}"] & U \otimes (U \otimes (X \otimes Y)) \arrow[dr, "\mathrm{id}_U \otimes \phi_{U,X,Y}"] & \\
 & & & U \otimes ((U \otimes X) \otimes Y) \\
(U \otimes X) \otimes Y \arrow[r, "(u\otimes \mathrm{id}_X)\otimes \mathrm{id}_Y"'] & ((U \otimes U) \otimes X) \otimes Y \arrow[r, "\phi^{-1}_{U,U,X}\otimes \mathrm{id}_Y"'] \arrow[d, Rightarrow] & (U \otimes (U \otimes X)) \otimes Y \arrow[ur, "\phi^{-1}_{U,U\otimes X,Y}"'] & \\
U \otimes (X \otimes Y) \arrow[r, "u\otimes \mathrm{id}_{X\otimes Y}"] \arrow[dd, "\phi_{U,X,Y}"'] & (U \otimes U) \otimes (X \otimes Y) \arrow[r, "\phi^{-1}_{U,U,X\otimes Y}"] \arrow[dd, dashed, "\phi_{U\otimes U,X,Y}"] & U \otimes (U \otimes (X \otimes Y)) \arrow[dr, "\mathrm{id}_U \otimes \phi_{U,X,Y}"] & \\
 & & & U \otimes ((U \otimes X) \otimes Y) \\
(U \otimes X) \otimes Y \arrow[r, "(u\otimes \mathrm{id}_X)\otimes \mathrm{id}_Y"'] & ((U \otimes U) \otimes X) \otimes Y \arrow[r, "\phi^{-1}_{U,U,X}\otimes \mathrm{id}_Y"'] & (U \otimes (U \otimes X)) \otimes Y \arrow[ur, "\phi^{-1}_{U,U\otimes X,Y}"'] &
\end{tikzcd} \tag{2.7}$$

where we have added $\phi_{U\otimes U,X,Y}$ to the second square. We want to show that

$$\mathrm{id}_U \otimes \phi_{U,X,Y} \circ \phi^{-1}_{U,U,X\otimes Y} = \phi^{-1}_{U,U\otimes X,Y} \circ \phi^{-1}_{U,U,X} \otimes \mathrm{id}_Y \circ \phi_{U\otimes U,X,Y} \tag{2.8}$$

In (2.1), we set

$$\begin{cases} X := U \\ Y := U \\ Z := X \\ T := Y \end{cases} \tag{2.9}$$

which by assumption gives us the following commutative diagram

$$\begin{tikzcd}
 & U \otimes (U \otimes (X \otimes Y)) \arrow[dl, "\mathrm{id}_U \otimes \phi_{U,X,Y}"'] \arrow[dr, "\phi_{U,X,X\otimes Y}"] & \\
U \otimes ((U \otimes X) \otimes Y) \arrow[d, "\phi_{U,U\otimes X,Y}"] & & (U \otimes U) \otimes (X \otimes Y) \arrow[d, "\phi_{U\otimes U,X,Y}"] \\
(U \otimes (U \otimes X)) \otimes Y \arrow[rr, "\phi_{U,U,X}\otimes \mathrm{id}_Y"] & & ((U \otimes U) \otimes X) \otimes Y
\end{tikzcd}$$

If we start at $(U \otimes U) \otimes (X \otimes Y)$ move towards $U \otimes ((U \otimes X) \otimes Y)$ along the two paths available (noting that we need to invert some maps), we find that

$$\begin{aligned} & \mathrm{id}_U \otimes \phi_{U,X,Y} \circ \phi^{-1}_{U,X,X\otimes Y} = \phi^{-1}_{U,U\otimes X,Y} \circ (\phi_{U,U,X} \otimes \mathrm{id}_Y)^{-1} \circ \phi_{U\otimes U,X,Y} \\ \iff \; & \mathrm{id}_U \otimes \phi_{U,X,Y} \circ \phi^{-1}_{U,X,X\otimes Y} = \phi^{-1}_{U,U\otimes X,Y} \circ \phi^{-1}_{U,U,X} \otimes \mathrm{id}_Y \circ \phi_{U\otimes U,X,Y} \end{aligned}$$

where we have used that

$$\begin{aligned}(\phi_{U,U,X} \otimes \mathrm{id}_Y) \circ (\phi^{-1}_{U,U,X} \otimes \mathrm{id}_Y) &= \mathrm{id}_{(U\otimes U)\otimes X} \otimes \mathrm{id}_Y \\ (\phi^{-1}_{U,U,X} \otimes \mathrm{id}_Y) \circ (\phi_{U,U,X} \otimes \mathrm{id}_Y) &= \mathrm{id}_{U\otimes (U\otimes X)} \otimes \mathrm{id}_Y \\ \implies (\phi_{U,X,X} \otimes \mathrm{id}_Y)^{-1} &= \phi^{-1}_{U,U,X} \otimes \mathrm{id}_Y \,.\end{aligned}$$

This is exactly what we wanted to show (cf. (2.8)). Thus the rightmost subsquare in the square the vertical arrow is pointing toward, in (2.7), commutes.

To elaborate on the first square: We note that $\phi_{(-),(-),(-)}$ is a *natural* (1.25) with respect to all arguments. We rewrite the first square as

$$\begin{tikzcd} U\otimes(X\otimes Y) \arrow[r, "\phi_{U,X,Y}"] \arrow[d, "u\otimes(\mathrm{id}_X\otimes\mathrm{id}_Y)"] & (U\otimes X)\otimes Y \arrow[d, "(u\otimes\mathrm{id}_X)\otimes\mathrm{id}_Y"] \\ (U\otimes U)\otimes(X\otimes Y) \arrow[r, "\phi_{U\otimes U,X,Y}"] & ((U\otimes U)\otimes X)\otimes Y \end{tikzcd}$$

We see that this square commutes since ϕ is natural with respect to functors $F := (-) \otimes ((-) \otimes (-))$ and $G := ((-) \otimes (-)) \otimes (-)$. so that the leftmost square in the lowermost diagram in (2.7) commutes. Hence the lowermost diagram in (2.7) commutes. One verifies that the larger diagram then commutes.

Lemma 2.14. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor which yields an equivalence of categories, then F* reflects *commutative diagrams.*

Proof. As we will show later (4.7), if F yields an equivalence of categories, then F is full, faithful and essentially surjective. But then it follows that

$$\begin{aligned}F(g_1) \circ \cdots \circ F(g_n) &= F(f_1) \circ \cdots \circ F(f_n) \\ \implies g_1 \circ \cdots \circ g_n &= f_1 \circ \cdots \circ f_n.\end{aligned}$$

□

Using 2.14 with $F = U \otimes -$, we find that (2.4) commutes.

(2.5): Let's define $\mathrm{id}_X \otimes l_Y$ explicitly, as the composition

$$X\otimes Y \xrightarrow{l_{X\otimes Y}} U\otimes(X\otimes Y) \xrightarrow{\psi_{U,X\otimes Y}} (X\otimes Y)\otimes U \xrightarrow{\phi_{X,Y,U}} X\otimes(Y\otimes U) \xrightarrow{\mathrm{id}_X\otimes\psi_{Y,U}} X\otimes(U\otimes Y).$$

We use (2.4) to see that $l_X \otimes \mathrm{id}_Y = \phi_{U,X,Y} \circ l_{X\otimes Y}$.

We can then rewrite (2.5) as

$$\begin{tikzcd}
X \otimes Y \arrow[r, "l_{X\otimes U}"] \arrow[dddr, "\mathrm{id}_X \otimes l_Y"'] & U \otimes (X \otimes Y) \arrow[r, "\phi_{U,X,Y}"] \arrow[d, "\psi_{U,X\otimes Y}"] & (U \otimes X) \otimes Y \arrow[ddd, "\psi_{U,X} \otimes \mathrm{id}_Y"] \\
& (X \otimes Y) \otimes U \arrow[d, "\phi_{X,Y,U}"] & \\
& X \otimes (Y \otimes U) \arrow[d, "\mathrm{id}_X \otimes \psi_{Y,U}"] & \\
& X \otimes (U \otimes Y) \arrow[r, "\phi_{X,U,Y}"] & (X \otimes U) \otimes Y
\end{tikzcd}$$

The inverted "right triangle" to the left commutes by definition of $\mathrm{id}_X \otimes l_Y$, and we find that the second square is really the hexagon in (2.2), hence commutes.

That is

$$\begin{aligned}
\phi_{X,U,Y} \ \circ \ (\mathrm{id}_X \otimes l_Y) &= \underbrace{(\phi_{X,U,Y} \circ (\mathrm{id}_X \otimes \psi_{Y,U}) \circ \phi_{X,Y,U} \circ \psi_{U,X\otimes Y})}_{= \ (\psi_{U,X} \otimes \mathrm{id}_Y) \circ \phi_{U,X,Y}} \circ l_{X\otimes Y} \\
&= ((\psi_{U,X} \otimes \mathrm{id}_Y) \circ (\phi_{U,X,Y})) \ \circ \ l_{X\otimes Y}.
\end{aligned}$$

□

Remark 2.15. As in [10] we will henceforth denote the (up to isomorphism) unique identity object in $(\mathcal{C}, \otimes)$ as $\mathbf{1}$.

Before proving our next proposition, we need to introduce some definitions, and lemmas. The structure of our lemmas and proofs follows [6]

Definition 2.16. With respect to *any* identity object $(U, u = l_U)$, we define $r_X := \psi_{U,X} \circ \ l_X : X \to X \otimes U$, which is a composition of isomorphisms, hence an isomorphism. Since this is well-defined for all objects $X \in \mathcal{C}$, we can assemble r_X for all X into a map between functors; since r is then defined as the composition of *natural* transformations ψ, l, we get a *natural isomorphism* r with *components* r_X (see 4.4 for a proof that *a composition of natural transformations, is a natural transformation*).

Lemma 2.17. *Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be functors, and let $\lambda : F \Rightarrow G$ be a natural isomorphism. Then $\lambda^{-1} : G \Rightarrow F$ is a natural isomorphism.*

Proof. By naturality of λ, we have that, for arbitrary morphism $f : X \to Y$ in $\mathcal{C}$, we have

$$\begin{aligned}
& \lambda_Y \circ F(f) = G(f) \circ \lambda_X \\
\iff & \lambda_Y \circ G(f) = F(f) \circ \lambda_X,
\end{aligned}$$

or in diagram form,

$$\begin{tikzcd}
G(X) \arrow[r, "\lambda_X^{-1}"] \arrow[d, "G(f)"'] & F(X) \arrow[d, "F(f)"] \\
G(Y) \arrow[r, "\lambda_Y^{-1}"] & F(Y)
\end{tikzcd}$$

So that $\lambda^{-1} : G \Rightarrow F$ is a natural transformation (1.25). Since λ_X^{-1} is an isomorphism at each $X \in \mathcal{C}$, λ^{-1} is indeed a *natural isomorphism.* ☐

Proposition 2.18. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category (2.6), and let* $X \in \mathcal{C}$ *be arbitrary, and let* $\mathbf{1}$ *be an identity object in* $\mathcal{C}$*. Then it holds that*

$$l_{\mathbf{1}\otimes X}^{-1} = \mathrm{id}_{\mathbf{1}} \otimes l_X^{-1}$$

and

$$r_{X\otimes\mathbf{1}}^{-1} = r_X^{-1} \otimes \mathrm{id}_{\mathbf{1}} .$$

Proof. We note that from 2.17 it follows that l^{-1} and r^{-1} are *natural isomorphisms*. For l_X^{-1}, the following diagram commutes by naturality of l^{-1}, with respect to the functors $\mathbf{1} \otimes -$ and the *identity functor* $\mathrm{id}_{\mathcal{C}}$ (that takes objects to themselves, and morphisms to themselves).

$$\begin{tikzcd}
\mathbf{1} \otimes (\mathbf{1} \otimes X) \arrow[r, "l_{\mathbf{1}\otimes X}^{-1}"] \arrow[d, "\mathrm{id}_{\mathbf{1}} \otimes l_X^{-1}"] & \mathbf{1} \otimes X \arrow[d, "l_X^{-1}"] \\
\mathbf{1} \otimes X \arrow[r, "l_X^{-1}"] & X
\end{tikzcd} . \tag{2.10}$$

So, by commutativity of (2.10), we have

$$\begin{aligned}
& l_X^{-1} \circ \mathrm{id}_{\mathbf{1}} \otimes l_X^{-1} = l_X^{-1} \circ l_{\mathbf{1}\otimes X}^{-1} \\
\iff & \mathrm{id}_{\mathbf{1}} \otimes l_X^{-1} = l_{\mathbf{1}\otimes X}^{-1} \\
\iff & l_{\mathbf{1}\otimes X} = \mathrm{id}_{\mathbf{1}} \otimes l_X
\end{aligned}$$

Similarly, we have that by naturality of r^{-1}, the following diagram commutes

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes \mathbf{1} \arrow[r, "r_{X\otimes\mathbf{1}}^{-1}"] \arrow[d, "r_X^{-1} \otimes \mathrm{id}_{\mathbf{1}}"'] & X \otimes \mathbf{1} \arrow[d, "r_X^{-1}"] \\
X \otimes \mathbf{1} \arrow[r, "r_X^{-1}"] & X
\end{tikzcd} . \tag{2.11}$$

So (2.11) gives us that

$$
\begin{aligned}
&r_X^{-1} \circ r_X^{-1} \otimes \mathrm{id}_{\mathbf{1}} = r_X^{-1} \circ r_{X\otimes\mathbf{1}}^{-1} \\
\iff &r_X^{-1} \otimes \mathrm{id}_{\mathbf{1}} = r_{X\otimes\mathbf{1}}^{-1}.
\end{aligned}
$$

□

Proposition 2.19. *Let $(\mathcal{C}, \otimes)$ be a tensor category. Let $\mathbf{1}$ be an identity object in $\mathcal{C}$, and let $X, Y \in \mathcal{C}$ be arbitrary. Then the following diagram commutes*

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "\phi^{-1}_{X,\mathbf{1},Y}"] \arrow[dr, "r_X^{-1}\otimes \mathrm{id}_Y"] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y^{-1}"] \\
& X \otimes Y &
\end{tikzcd}$$

or equivalently

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y & & X \otimes (\mathbf{1} \otimes Y) \arrow[ll, "\phi_{X,\mathbf{1},Y}"'] \\
& X \otimes Y \arrow[ul, "r_X\otimes \mathrm{id}_Y"] \arrow[ur, "\mathrm{id}_X \otimes l_Y"'] &
\end{tikzcd} \tag{2.12}$$

Proof. By 2.12, we know that (where $U = \mathbf{1}$)

$$
\begin{aligned}
&\phi_{X,\mathbf{1},Y} \circ \mathrm{id}_X \otimes l_Y = \psi_{\mathbf{1},X} \otimes \mathrm{id}_Y \circ l_X \otimes \mathrm{id}_Y \\
\iff &l_X^{-1} \otimes \mathrm{id}_Y \circ \psi_{X,\mathbf{1}} \otimes \mathrm{id}_Y = \mathrm{id}_X \otimes l_Y^{-1} \circ \phi^{-1}_{X,\mathbf{1},Y}
\end{aligned}
$$

but

$$
\begin{aligned}
l_X^{-1} \otimes \mathrm{id}_Y \circ \psi_{X,\mathbf{1}} \otimes \mathrm{id}_Y &= (l_X^{-1} \circ \psi_{X,\mathbf{1}} \otimes \mathrm{id}_Y) \otimes \mathrm{id}_Y \\
&= r_X^{-1} \otimes \mathrm{id}_Y \\
\implies r_X^{-1} \otimes \mathrm{id}_Y &= \mathrm{id}_X \otimes l_Y^{-1} \circ \phi^{-1}_{X,\mathbf{1},Y} \\
\iff r_X \otimes \mathrm{id}_Y &= \phi_{X,\mathbf{1},Y} \circ \mathrm{id}_X \otimes l_Y.
\end{aligned}
$$

□

Proposition 2.20. *Let $(\mathcal{C}, \otimes)$ be a tensor category with identity object $\mathbf{1}$, and let $X, Y \in \mathcal{C}$ be arbitrary. Then the following diagram commutes*

$$\begin{tikzcd}
(X \otimes Y) \otimes \mathbf{1} \arrow[rr, "\phi^{-1}_{X,Y,\mathbf{1}}"] \arrow[dr, "r_{X\otimes Y}^{-1}"'] & & X \otimes (Y \otimes \mathbf{1}) \arrow[dl, "\mathrm{id}_X \otimes r_Y^{-1}"] \\
& X \otimes Y &
\end{tikzcd} .$$

or equivalently, that the following diagram commutes

$$\begin{tikzcd}
X \otimes (Y \otimes \mathbf{1}) \arrow[rr, "\phi_{X,Y,\mathbf{1}}"] & & (X \otimes Y) \otimes \mathbf{1} \\
& X \otimes Y \arrow[ul, "\mathrm{id}_X \otimes r_Y"] \arrow[ur, "r_{X \otimes Y}"'] &
\end{tikzcd} \tag{2.13}$$

Proof. □

Lemma 2.21. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category with identity object* $\mathbf{1}$*, and let* X, Y *be an arbitrary objects in* $\mathcal{C}$*. Then* $r_{\mathbf{1}} = l_{\mathbf{1}}$*.*

Proof. Letting

$$\begin{aligned} X &= Y \\ &= \mathbf{1} \end{aligned}$$

in 2.4, and using 2.18, we see that

$$\begin{aligned} l_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} &= \phi_{\mathbf{1},\mathbf{1},\mathbf{1}} \circ l_{\mathbf{1} \otimes \mathbf{1}} \\ &= \phi_{\mathbf{1},\mathbf{1},\mathbf{1}} \circ \mathrm{id}_{\mathbf{1}} \otimes l_{\mathbf{1}}. \end{aligned} \tag{2.14}$$

□

If we again let

$$\begin{aligned} X &= Y \\ &= \mathbf{1} \end{aligned}$$

in (2.12), we get that

$$r_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} = \phi_{\mathbf{1},\mathbf{1},\mathbf{1}} \circ \mathrm{id}_{\mathbf{1}} \otimes l_{\mathbf{1}}. \tag{2.15}$$

(2.14) and (2.15) together imply that $r_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} = l_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}}$.

Lemma 2.22. *The functor* $- \otimes \mathbf{1} : \mathcal{C} \to \mathcal{C}$ *yields an equivalence of categories.*

Proof. Let $F = - \otimes \mathbf{1}$ and let $G = \mathrm{id}_{\mathcal{C}}$ be the identity functor.

Then we see $r^{-1} : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $r : \mathrm{id}_C \Rightarrow GF$ are natural isomorphisms. □

Then, by 4.7, we have that $F = - \otimes \mathbf{1}$ is faithful, hence $r_1 = l_1$.

Proposition 2.23. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category (2.6). If* $(\mathbf{1}, l_{\mathbf{1}})$ *is an identity object (2.5), and* $(\mathbf{1}', l'_{\mathbf{1}})$ *is another identity object in* $(\mathcal{C}, \otimes)$*, then there is a unique isomorphism* $a : \mathbf{1} \to \mathbf{1}'$ *such that the following diagram commutes*

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "l_{\mathbf{1}}"] \arrow[d, "a"'] & \mathbf{1}\otimes\mathbf{1} \arrow[d, "a\otimes a"] \\
\mathbf{1}' \arrow[r, "l'_{\mathbf{1}'}"'] & \mathbf{1}'\otimes\mathbf{1}'
\end{tikzcd} \tag{2.16}$$

Proof. <u>Existence</u>: We define a as the map

$$a : U \xrightarrow{l'_{\mathbf{1}}} \mathbf{1}'\otimes\mathbf{1} \xrightarrow{\psi_{\mathbf{1}',\mathbf{1}}} \mathbf{1}\otimes\mathbf{1}' \xrightarrow{l^{-1}_{\mathbf{1}'}} \mathbf{1}'$$

Lemma 2.24. *The following diagrams commute.*

$$\begin{tikzcd}
 & \mathbf{1}\otimes X \arrow[dd, "a\otimes\mathrm{id}_X"] & & & X\otimes\mathbf{1} \arrow[dd, "\mathrm{id}_X\otimes a"] \\
X \arrow[ur, "l_X"] \arrow[dr, "l'_X"'] & & & X \arrow[ur, "r_X"] \arrow[dr, "r'_X"'] & \\
 & \mathbf{1}'\otimes X & & & X\otimes\mathbf{1}'
\end{tikzcd} \tag{2.17}$$

Proof. We draw the following diagram (see p. 34 in [14])

$$\begin{tikzcd}
 & X\otimes\mathbf{1} \arrow[d, "\mathrm{id}_X\otimes l'_{\mathbf{1}}"] \arrow[dr, "r'_X\otimes\mathrm{id}_{\mathbf{1}}"] & \\
X \arrow[ur, "r_X"] \arrow[dr, "r'_X"'] & X\otimes(\mathbf{1}'\otimes\mathbf{1}) \arrow[r, "\phi_{X,\mathbf{1}',\mathbf{1}}"] & (X\otimes\mathbf{1}')\otimes\mathbf{1} \\
 & X\otimes\mathbf{1}' \arrow[u, "\mathrm{id}_X\otimes r_{\mathbf{1}'}"] \arrow[ur, "r_{X\otimes\mathbf{1}'}"'] &
\end{tikzcd}$$

If we let $Y=\mathbf{1}$ in (2.5) (also identifying $U=\mathbf{1}'$), the upper right triangle commutes. The lower right triangle commutes by 2.20.

The outer perimeter commutes by naturality of r.

We have

$$\mathrm{id}_X\otimes l'_{\mathbf{1}} = \phi^{-1}_{X,\mathbf{1}',\mathbf{1}}\circ r'_X\otimes\mathrm{id}_{\mathbf{1}} \tag{2.18}$$

and

$$\mathrm{id}_X\otimes r^{-1}_{\mathbf{1}'} = r^{-1}_{X\otimes\mathbf{1}'}\circ\phi_{X,\mathbf{1}',\mathbf{1}} \tag{2.19}$$

so that

$$\begin{aligned}
\mathrm{id}_X \otimes a \circ r_X &= r_X \circ (\mathrm{id}_X \otimes (r_{\mathbf{1}'})^{-1}) \circ (\mathrm{id}_X \otimes l'_{\mathbf{1}}) \circ r_X \\
&= r^{-1}_{X\otimes\mathbf{1}} \circ \phi_{X,\mathbf{1}',\mathbf{1}} \circ \phi^{-1}_{X,\mathbf{1}',\mathbf{1}} \circ r'_X \otimes \mathrm{id}_{\mathbf{1}} \circ r_X \\
&= r^{-1}_{X\otimes\mathbf{1}} \circ \underbrace{r'_X \otimes \mathrm{id}_{\mathbf{1}} \circ r_X}_{r_{X\otimes\mathbf{1}'}\circ r'_X} \\
&= r^{-1}_{X\otimes\mathbf{1}'} \circ r_{X\otimes\mathbf{1}'} \circ r'_X \\
&= r'_X.
\end{aligned}$$

This shows the commutativity of the rightmost diagram in (2.17).

For the leftmost diagram in (2.17), we draw the following diagram.

$$\begin{tikzcd}
 & \mathbf{1}\otimes X \arrow[d, "l'_{\mathbf{1}}\otimes\mathrm{id}_X"] \arrow[rd, "\mathrm{id}_{\mathbf{1}}\otimes l'_X"] & \\
X \arrow[ru, "l_X"] \arrow[rd, "l'_X"'] & (\mathbf{1}'\otimes\mathbf{1})\otimes X \arrow[r, "\phi^{-1}_{\mathbf{1},\mathbf{1}',X}\circ\psi_{\mathbf{1}',\mathbf{1}}\otimes\mathrm{id}_X"] \arrow[d, "r^{-1}_{\mathbf{1}'}\otimes\mathrm{id}_X"] & \mathbf{1}\otimes(\mathbf{1}'\otimes X) \\
 & \mathbf{1}'\otimes X \arrow[ru, "l_{\mathbf{1}'\otimes X}"] &
\end{tikzcd} \tag{2.20}$$

The upper triangle to the right in (2.20) commutes by (2.5), by setting $X = \mathbf{1}$ and with respect to l'. The outer perimeter commutes by *naturality* of l.

For the lower right triangle: From (2.4), we have that $l_{\mathbf{1}'\otimes X} = \phi^{-1}_{\mathbf{1},\mathbf{1}',X} \circ l_{\mathbf{1}'} \otimes \mathrm{id}_X$. Recalling that $r_{\mathbf{1}'} = \psi_{\mathbf{1},\mathbf{1}'} \circ l_{\mathbf{1}'}$, we find that

$$\begin{aligned}
(\phi^{-1}_{\mathbf{1},\mathbf{1}',X} \circ \psi_{\mathbf{1}',\mathbf{1}} \otimes \mathrm{id}_X) \circ (r_{\mathbf{1}'} \otimes \mathrm{id}_X) &= (\phi^{-1}_{\mathbf{1},\mathbf{1}',X} \circ \psi_{\mathbf{1}',\mathbf{1}} \otimes \mathrm{id}_X) \circ ((\psi_{\mathbf{1},\mathbf{1}'} \circ l_{\mathbf{1}'}) \otimes \mathrm{id}_X) \\
&= \phi^{-1}_{\mathbf{1},\mathbf{1}',X} \circ l_{\mathbf{1}'} \otimes \mathrm{id}_X \\
&= l_{\mathbf{1}'\otimes X}
\end{aligned}$$

where we have used that $\psi_{\mathbf{1}',\mathbf{1}} \circ \psi_{\mathbf{1},\mathbf{1}'} = \mathrm{id}_{\mathbf{1}\otimes\mathbf{1}'}$. So the lower right triangle in (2.20) commutes.

Finally, we see that

$$\begin{aligned}
a \otimes \mathrm{id}_X \circ l_X &= ((r^{-1}_{\mathbf{1}'} \circ l'_{\mathbf{1}}) \otimes \mathrm{id}_X) \circ l_X \\
&= (r^{-1}_{\mathbf{1}'} \otimes \mathrm{id}_X) \circ (l_{\mathbf{1}'} \otimes \mathrm{id}_X) \circ l_X \\
&= (r^{-1}_{\mathbf{1}'} \otimes \mathrm{id}_X) \circ (\psi_{\mathbf{1},\mathbf{1}'} \otimes \mathrm{id}_X) \circ \phi_{\mathbf{1},\mathbf{1}',X} \circ \mathrm{id}_{\mathbf{1}} \otimes l'_X \circ l_X \\
&= (r^{-1}_{\mathbf{1}'} \otimes \mathrm{id}_X) \circ \underbrace{(\psi_{\mathbf{1},\mathbf{1}'} \otimes \mathrm{id}_X) \circ \phi_{\mathbf{1},\mathbf{1}',X} \circ l_{\mathbf{1}'\otimes X}}_{r_{\mathbf{1}'}\otimes\mathrm{id}_X} \circ l'_X \\
&= l'_X.
\end{aligned}$$

We conclude that the leftmost diagram in (2.20) commutes. □

Using 2.24 and that $r_{\mathbf{1}}^{-1} = l_{\mathbf{1}}^{-1}$ (2.21), we see that

$$\begin{aligned} a \otimes a &= \underbrace{(a \otimes \mathrm{id}_{\mathbf{1}'})}_{l'_{\mathbf{1}'} \circ l_{\mathbf{1}'}^{-1}} \circ \underbrace{(\mathrm{id}_{\mathbf{1}} \otimes a)}_{r'_U \circ r_{\mathbf{1}}^{-1}} \\ &= (l'_{\mathbf{1}'} \circ l_{\mathbf{1}'}^{-1}) \circ (r'_{\mathbf{1}} \circ \underbrace{r_{\mathbf{1}}^{-1}}_{= l_{\mathbf{1}}^{-1}}) \\ &= (l'_{\mathbf{1}'} \circ l_{\mathbf{1}'}^{-1}) \circ (\psi_{\mathbf{1}',\mathbf{1}} \circ l'_{\mathbf{1}} \circ l_{\mathbf{1}}^{-1}) \\ &= l'_{\mathbf{1}'} \circ (l_{\mathbf{1}'}^{-1} \circ \psi_{\mathbf{1},\mathbf{1}} \circ l'_{\mathbf{1}}) \circ l_{\mathbf{1}}^{-1} \\ &= l'_{\mathbf{1}'} \circ a \circ l_{\mathbf{1}}^{-1} \end{aligned}$$

which is what we wanted to show. Hence (2.16) commutes.

Uniqueness: Similar to [6], we note that if $c : \mathbf{1} \to \mathbf{1}$ is an arbitrary morphism, then by *naturality* of l, the following diagram commutes

$$\begin{tikzcd} \mathbf{1} \arrow[r, "l_{\mathbf{1}}"] \arrow[d, "c"'] & \mathbf{1} \otimes \mathbf{1} \arrow[d, "\mathrm{id}_{\mathbf{1}} \otimes c"] \\ \mathbf{1} \arrow[r, "l_{\mathbf{1}}"] & \mathbf{1} \otimes \mathbf{1} \end{tikzcd} \tag{2.21}$$

Note that for any morphisms a, b such that 2.16 commutes, we have that the leftmost diagram commutes, so that we get the rightmost commutative diagram.

$$\begin{tikzcd} \mathbf{1} \arrow[r, "l_{\mathbf{1}}"] \arrow[d, "a"] & \mathbf{1} \otimes \mathbf{1} \arrow[d, "a \otimes a"] \\ \mathbf{1}' \arrow[r, "l_{\mathbf{1}'}"] \arrow[d, "b^{-1}"'] & \mathbf{1}' \otimes \mathbf{1}' \arrow[d, "b^{-1} \otimes b^{-1}"] \\ \mathbf{1} \arrow[r, "l_{\mathbf{1}}"] & \mathbf{1} \otimes \mathbf{1} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathbf{1} \arrow[r, "l_{\mathbf{1}}"] \arrow[d, "\tau"] & \mathbf{1} \otimes \mathbf{1} \arrow[d, "\tau \otimes \tau"] \\ \mathbf{1} \arrow[r, "l_{\mathbf{1}}"] & \mathbf{1} \otimes \mathbf{1} \end{tikzcd}$$

where $\tau = b^{-1} \circ a$. By (2.21) we know that $(\mathrm{id}_{\mathbf{1}} \otimes \tau) \circ l_{\mathbf{1}} = l_{\mathbf{1}} \circ \tau$. By the rightmost diagram above, we know that

$$\begin{aligned} & l_{\mathbf{1}} \circ \tau = \tau \otimes \tau \circ l_{\mathbf{1}} \\ \implies & (\mathrm{id}_{\mathbf{1}} \otimes \tau) \circ l_{\mathbf{1}} = \tau \otimes \tau \circ l_{\mathbf{1}} \\ \implies & \mathrm{id}_{\mathbf{1}} \otimes \tau = \tau \otimes \tau \\ \iff & \tau^{-1} \otimes \mathrm{id}_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}}\,. \end{aligned}$$

By 2.22 (assuming *choice*), we see that

$$\begin{aligned} \tau^{-1} &= \mathrm{id}_{\mathbf{1}} \\ \iff a^{-1} \circ b &= \mathrm{id}_{\mathbf{1}} \\ \iff a &= b. \end{aligned}$$

□

3 Iterates, Extensions

Let ϕ be a commutativity constraint for a tensor category $(\mathcal{C}, \otimes)$. Then we can as in [10], given $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, create new functors defined by repeated application of $\otimes$.

Definition 3.1. Any functor $F : \mathcal{C}^n \to \mathcal{C}$ constructed by repeated application of $\otimes$ is called an **iterate** of $\otimes$.

Definition 3.2. A **directed graph** $G = (V, E)$ consists of a set of vertices V, and a set of edges E, and two functions $s, t : E \rightrightarrows V$ that takes an *directed* edge $e \in E$ to its source, respectively its target.

To set up the proof of proposition 3.9, we need some definitions:

Definition 3.3. A **preorder** is a binary relation $\mathcal{R} \subset X \times X$, which we will denote as $\leq$, on a set X, such that the following conditions hold:

1. $x \leq x \qquad (\forall x \in X)$.
2. If $x \leq y$ and $y \leq z$, then $x \leq z \qquad (\forall x, y, z \in X)$.

Definition 3.4. If we add *antisymmetric* as a condition to 3.3 then we get a **partial order** on X. That is, for each $x, y \in X$, if $x \leq y$ and $y \leq x$ then $x = y$.

Example 3.5. Take the set $\mathbb{Q}$ of rational numbers with the binary relation $\leq$. Then the set $\mathbb{Q}$ together with $\leq$ forms a preorder $(\mathbb{Q}, \leq)$. As in example 1.2, $(\mathbb{Q}, \leq)$ also gives us a category.

Definition 3.6. A category $\mathcal{C}$ is a **thin category** if there is *at most* one morphism between each pair of objects $X, X' \in \mathcal{C}$.

Example 3.7. A category X, obtained from a partial order (3.4) $(X, \leq)$, is a **thin category**. Here, the objects are elements $x, y \in X$, and there is *at most* one morphism $f : x \to y$. More precisely; when $x \leq y$, then $\exists! f : x \to y, f \in \mathrm{Hom}_X(x, y)$, and if $\neg(x \leq y) \implies \mathrm{Hom}_X(x, y) = \varnothing$.

We will later construct a *thin* category $\mathcal{W}$, consisting of "words" v, w such that if v, w is of the same *length*, then there will be precisely one arrow $v \to w$.

3.1 Coherence in tensor categories

If we let $(C, \otimes)$ be a tensor category (2.6), we see that we can form n-fold tensor-products by repeated application of $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, e.g. if $X, Y, Z \in \mathcal{C}$ then

$$\begin{aligned} \otimes((X, \otimes(Y, Z)) &= \otimes((X, Y \otimes Z)) \\ &= X \otimes (Y \otimes Z). \end{aligned} \tag{3.1}$$

Lemma 3.8. *An n-fold composition of functors $F_n \circ \cdots \circ F_1 : \mathcal{C}_1 \to \mathcal{C}_n$ is a functor (where $n \in \omega = \mathbb{N}$).*

Proof. That composition is preserved is obvious since every F_i preserves composition. The same holds for identity preservation.

Proof sketch by induction for identity preservation: The base case $n = 1$ is obvious (F_1 being a functor). Assume it holds for the composition $F_n \circ \cdots \circ F_1 : \mathcal{C}_1 \to \mathcal{C}_n$ that it is a functor, then we see that

$$\begin{aligned}(F_{n+1} \circ F_n \circ \cdots \circ F_1)(\mathrm{id}_A) &= F_{n+1}(\mathrm{id}_{F_n \circ \cdots \circ F_1(A)}) \\ &= \mathrm{id}_{F_{n+1} \circ \cdots F_1(A)}\end{aligned}$$

where we used induction and that F_{n+1} is a functor. ☐

It follows from 3.8 that we can form n-fold tensor-functors $\otimes : \mathcal{C}^n \to \mathcal{C}$ where $n \in \omega$ is finite, by repeated application of the tensor-product, e.g. as in (3.1). We call such an n-fold tensor functor an **iterate**. If we have two different *iterates* $F, F' : \mathcal{C}^n \rightrightarrows \mathcal{C}$, then one can construct a natural isomorphism of F and F' by (possibly repeated) applications of the *associator* ϕ and its inverse ϕ^{-1}. We can denote this natural isomorphism by $\tau : F \Rightarrow F'$. [8, VII, chapter II] gives a proof of this fact. One should keep in mind that the proof Maclane ([8]) gives is for *monoidal* categories; we don't neccessarily have a commutator $\psi : X \otimes Y \to Y \otimes X$. But there is a way to extend the so called "coherence theorem for monoidal categories" to *tensor categories.*

Proposition 3.9. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category. The tensor structure on* $(\mathcal{C}, \otimes)$ *then admits an extension as follows: for each finite set* I *there is a functor* $\bigotimes_{i \in I} : \mathcal{C}^I \to \mathcal{C}$ *and for each map* $\alpha : I \to J$ *between finite sets* I, J*, there is a natural isomorphism* $\chi(\alpha) : \bigotimes_{i \in I} X_i \to \bigotimes_{j \in J} \left(\bigotimes_{i \mapsto j} X_j \right)$*, such that the following conditions hold:*

a) If I *only has one element, then* $\bigotimes_{i \in I}$ *is the identity functor* id_C*; if* α *is a map between sets with only one element, then* $\chi(\alpha)$ *is the identity automorphism of the identity functor (i.e., the identity of* $\mathrm{id}_{\mathcal{C}}$*, in the functor category* $[\mathcal{C}, \mathcal{C}]$*).*

b) If we have a composition of morphisms $I \xmapsto{\alpha} J \xmapsto{\beta} K$*, they induce natural isomorphisms such that the following diagram commutes.*

$$\begin{tikzcd}
\bigotimes_{i \in I} X_i \arrow[r, "\chi(\alpha)"] \arrow[d, "\chi(\beta\alpha)"] & \bigotimes_{j \in J} \left(\bigotimes_{i \mapsto j} X_i \right) \arrow[d, "\chi(\beta)"] \\
\bigotimes_{k \in K} \left(\bigotimes_{i \mapsto k} X_i \right) \arrow[r, "\otimes(\chi(\alpha \mid I_k))"] & \bigotimes_{k \in K} \left(\bigotimes_{j \mapsto k} \left(\bigotimes_{i \mapsto j} X_i \right) \right)
\end{tikzcd} \tag{3.2}$$

Proof. We claim that this is a direct consequence of maclanes coherence theorem for *symmetric monoidal categories* (see e.g. [8, chapter XI, 2]). This follows from the fact that *any* possible concatenation of maps in the diagram 3.2 can be composed as a "path" of maps only involving ϕ, ψ, r, l, their inverses, and n-fold products, which commutes by the coherence for symmetric monoidal categories. We will prove the coherence for symmetric monoidal categories (i.e. tensor categories) below. ☐

Remark 3.10. In 3.9, we have that $I_k = \beta\alpha^{-1}(k)$.

Remark 3.11. Note that by e.g. $i \mapsto j$ in $\bigotimes_{j\in J}\left(\bigotimes_{i\mapsto j} X_i\right)$, we mean $i \in I$ so that $\alpha(i) = j$. So we could just as well have written

$$\bigotimes_{j\in J}\left(\bigotimes_{i\in\alpha^{-1}(j)} X_i\right)$$

.

Furthermore, we define $\mathcal{C}^{\varnothing} = \{*\}$ such that $\bigotimes_{\varnothing} : \mathcal{C}^{\varnothing} = \{*\} \to \mathcal{C}$ is the functor F_e so that $F_e(*) = \mathbf{1}$.

Remark 3.12. In 3.9, when we say that $(\bigotimes, \chi)$ is an *extension* of the tensor structure on $\mathcal{C}$, we mean that e.g. if $\alpha : \{1,2\} \to \{1,2\}$ such that

$$\begin{aligned}\alpha(1) &= 2\\ \alpha(2) &= 1\end{aligned}$$

then

$$\chi(\alpha) = \psi_{X_1,X_2} : X_1 \otimes X_2 \to X_2 \otimes X_1$$

.

Before we prove 3.9, we will show the coherence theorem for tensor categories, without taking ψ into account. The exact same arguments works for *monoidal categories*, that don't have a *commutator* ψ (or even a *braiding* ψ'; the difference is that a *commutator* need to satisfy $\psi_{A,B} \circ \psi_{B,A} = \mathrm{id}_{A\otimes B}$, but this does not hold for a *braiding*).

In the proposition below, we will follow Maclane ([8, chapter 2, VII], but expanded to include the exposition in [18]), but we will restrict our attention to *tensor categories*, which have a bit more structure than just *monoidal categories* (although, formally, tensor categories *are* monoidal categories). We will use $\otimes$ instead of Maclanes $\square$, as $\otimes$ is more apt in this context.

Before proceeding to the proof, we will introduce some definitions. We will call "products" $X \otimes Y$ **binary words**. In relation to binary word, we introduce the *length* of a binary word, recursively, as follows.

Definition 3.13. A **binary word** of **length** 0 is the "empty word" e_0. We can denote this as $\mathscr{L}(e_0) = 0$. A binary word of length 1 is the symbol $(-)$ (i.e. $\mathscr{L}((-)) = 1$).

More generally, we define $\mathscr{L}(X \otimes Y) = \mathscr{L}(X) + \mathscr{L}(Y)$.

Example 3.14. To give a concrete example of how to think about the *length* $\mathscr{L}$ of a binary word, take the binary word $((-) \otimes (-)) \otimes e_0$,

$$\begin{aligned}\rightsquigarrow \mathscr{L}(((-) \otimes (-)) \otimes e_0) &= \mathscr{L}((-) \otimes (-)) + \underbrace{\mathscr{L}(e_0)}_{=0}\\ &= \mathscr{L}((-)) + \mathscr{L}((-))\\ &= 2.\end{aligned}$$

We form a category $\mathcal{W}$ where the objects are the binary words of length $n = 0, 1, 2, \dots$ and where we have exactly one morphism between words if the words are of the same length, so that

$$\mathrm{Hom}_{\mathcal{W}}(X,Y) = \begin{cases}\{\bullet\}, \text{ if } \mathscr{L}(X) = \mathscr{L}(Y).\\ \varnothing, \text{ if } \mathscr{L}(X) \neq \mathscr{L}(Y).\end{cases}$$

It follows easily that every arrow in $\mathcal{W}$ is invertible, since every object (every binary word) is required to have an identity morphism. A consequence of the *uniqueness* of arrows between words of the same length, is that *every diagram in* $\mathcal{W}$ *will commute.* Another consequence of the commutativity of each diagram, is that the coherence conditions for a *monoidal category* will hold in $\mathcal{W}$, with e_0 as the identity object ([8] calls this the "identity object"). Since *functors preserve commutative diagrams* (1.21), it follows that each commutative diagram in $\mathcal{W}$, will yield a commutative diagram in $\mathcal{C}$ under some appropriate functor $\mathcal{W} \to \mathcal{C}$.

Proposition 3.15. *For any tensor category* $(\mathcal{C}, \otimes)$ *and object* $X \in \mathcal{C}$*, there is a* unique *functor* $F : \mathcal{W} \to \mathcal{C}$ *such that* $(-) \mapsto X$*.*

Proof. Our proof will follow [8]. For an excellent survey of the proof, see [18].

As in [8], we can write $w \mapsto w_X$ to mean "substitute X for all blank $(-)$ in the word $w \in \mathcal{W}$". We set

$$\begin{aligned}(e_0)_X &:= \mathbf{1} \\ (-)_X &:= X \\ (v \otimes w)_X &:= v_X \otimes w_X.\end{aligned}$$

Note that this definition is recursive (since all binary words are recursively built up from e_0 and $(-)$ with $\otimes$).

If we fix a length $\mathscr{L} = n$, we can construct a graph $G_{n,X}$, where the *nodes* are the words of length n, *that does not have an instance of* e_0 (i.e. exclusively built up from $(-)$ and $\otimes$), and edges morphisms $v \to w$. We will call these arrows we define **basic arrows**.

Remark 3.16. We identify these arrows with their image $v_X \to w_X$ in $\mathcal{C}$, but will suppress the subscript $_X$ going forward.

Definition 3.17. We will label words of length n without instances of e_0, as **pure binary words** (as in [18]).

Definition 3.18. We define the **basic arrows** in $G_{n,X}$ recursively as

- Arrows of the form $\phi_{u,v,w} : u \otimes (v \otimes w) \to (u \otimes v) \otimes w$, and its inverse ϕ^{-1}.
- $\beta \otimes \mathrm{id}_v$ and $\mathrm{id}_v \otimes \beta$, where id_v is the identity such that its image under F is $\mathrm{id}_v : v_X \to v_X$, and where β is a basic arrow.

Remark 3.19. From the definition of basic arrows (3.18), we see that any basic arrow can only involve *one instance* of ϕ or ϕ^{-1}, exclusively; so, for example, $(\phi \otimes \mathrm{id}_v) \otimes \phi^{-1}$ is *not* a basic arrow. To see this, note that if we recursively start by taking ϕ, and then form $\phi \otimes \mathrm{id}_v$, then, we are not allowed to add another instance of ϕ, nor ϕ^{-1}, since it would then take the form e.g. $\phi \otimes \beta$ or $\beta \otimes \phi^{-1}$, where $\beta \neq \mathrm{id}_v$.

Definition 3.20. If a basic arrow (3.18) has an instance of ϕ, we call it **directed**, and if it has an instance of ϕ^{-1}, we call it **antidirected**.

Observe that in $G_{n,X}$, the paths (in the graph-theoretic sense; a joining of two nodes by a sequence of edges that *are all distinct*) between words u, v of length n, are all possible compositions of basic arrows that when composed yield a morphism $u \to v$. We aim to show that *any sequence of edges yielding a path in G_n between arbitrary words u, v of length n, is the* same *morphism under F in $\mathcal{C}$.*

Definition 3.21. Denote as $w^{(n)}$ the *unique* word of length n with all parenthesis in "the front".

We illustrate the definition of $w^{(n)}$ for $n = 1, \dots, 5$ below.

1. $w^{(1)} = (-)$
2. $w^{(2)} = (-) \otimes (-)$
3. $w^{(3)} = ((-) \otimes (-)) \otimes (-)$
4. $w^{(4)} = (((-) \otimes (-)) \otimes (-)) \otimes (-)$
5. $w^{(5)} = (((((-) \otimes (-)) \otimes (-)) \otimes (-)) \otimes (-)$

Lemma 3.22. *There is a directed path (i.e. only using basic directed arrows) from any pure binary word v (3.17) of length n to $w^{(n)}$.*

Before proceeding with the proof below, note that $w^{(n)} = w^{(n-1)} \otimes (-)$.

Proof. See also [18, proposition 3.3.14]. The results hold vacously for the case $n = 1, 2$, since *there is only one pure binary word of length $n = 1, 2$; i.e.* $w^{(1)}, w^{(2)}$.

Suppose it holds true for n, we want to show it holds for words of length $n+1$. So let $u = v \otimes w$ be a pure binary word of length $n+1$. We divide into two cases.

$\underline{\mathscr{L}(w) = 1}$: If $\mathscr{L}(w) = 1$; then $w = (-)$ and $\mathscr{L}(v) = n$. Then we know that there is a directed path from v to $w^{(n)}$. We can then construct an arrow $\beta \otimes \mathrm{id}_w : v \otimes w \to w^{(n)} \otimes w = w^{(n+1)}$, since $w = (-)$. $\beta \otimes \mathrm{id}_w$ is directed and basic, by induction (and using the recursive definition).

$\underline{\mathscr{L}(w) > 1}$: If $\mathscr{L}(w) > 1$, then w must be on the form $w = s \otimes t$, so that $u = v \otimes (s \otimes t)$. But then we can take $\phi_{v,s,t}$ as our basic directed arrow.

We conclude that the results hold, by induction. $\square$

Since all basic directed arrows are *invertible*, from 3.22 we can, for any two words v, w of length n, find a map $v \to w^{(n)} \to w$, where the map $w^{(n)} \to w$ is gotten by inverting the arrows in the directed path $w \to w^{(n)}$. By the fact that $\mathcal{W}$ is a *thin* category (3.6), this composition must be equal to the *unique arrow* from v to w.

Definition 3.23. We define the **rank** of a word ρ by recursion, as in [8]:

- $\rho(e_0) = 0$.
- $\rho((-)) = 0$.
- $\rho((v \otimes w)) = \rho(v) + \rho(w) + \mathscr{L}(w) - 1$.

By [18, prop 3.3.11, prop. 3.3.12] we see that $\rho(v) = 0 \iff v = w^{(n)}$, given that $\mathscr{L}(v) = n$, and that $\rho(v) \geq 0$ for all pure binary words v.

We will show that $G_{n,X}$ commutes.

Lemma 3.24. *Directed basic arrows decrease rank.*

Proof. In the interest of not being to long-winded, we will exclude a full proof; One can give a proof that proceeds by induction on the structure of directed arrows, where ϕ is the base case, then treating the cases $\phi \otimes \mathrm{id}, \mathrm{id} \otimes \phi$, as in [18, proposition 3.13]. ☐

Remark 3.25. In the lemma below, we will identify arrows between pure binary words v, v' with their image in $\mathcal{C}$ under F in 3.15.

Lemma 3.26. *Let v be a pure binary word of length n, and let $\beta : v \to v'$ and $\gamma : v \to v''$ be two basic directed paths from v to pure binary words v', v'' of length n. Then there exists a pure binary word z and arrows $\beta' : v' \to z, \gamma' : v'' \to z$ that are compositions of directed basic arrows, so that the following diagram commutes*

$$\begin{tikzcd}
 & v \arrow[ld, "\beta"'] \arrow[rd, "\gamma"] & \\
v' \arrow[rd, dotted, "\exists\beta'"'] & & v'' \arrow[ld, dotted, "\exists\gamma'"] \\
 & z &
\end{tikzcd}$$

Proof. We proceed by induction on the rank n of a word v. For the base case $n = 0$, we know that $\rho(v) = 0 \iff v = w^{(n)}$. Then the results hold *vacously*, since there are no basic directed paths β, γ with domain $w^{(n)}$, since by 3.24 they must decrease rank, but we know that $\rho(v'), \rho(v'') \geq 0$.

Suppose it is true for every word of length less than n; we want to show that it holds for a word v of rank n.

So let v be a pure binary word of rank n, and assume that β, γ are given. If $\beta = \gamma$, then we just take

$$\begin{aligned} z &= v' \\ &= v''. \end{aligned}$$

If $\beta \neq \gamma$, write $v = u \otimes w$. Then β will have one of the following three forms:

- $\beta' \otimes \mathrm{id}_w$, so that β only "acts" on u.
- $\mathrm{id}_u \otimes \beta'$, so that β only "acts" on w.
- ϕ, in which case $u \otimes w = u \otimes (s \otimes t)$.

The same three cases holds for γ, so in total we have 9 different cases.

If β and γ acts inside the same factor, e.g. u:

$$\begin{tikzcd}
& u \arrow[dl, "\beta'"'] \arrow[dr, "\gamma'"] & & & & v = u \otimes w \arrow[dl, "\beta'\otimes\mathrm{id}_w"'] \arrow[dr, "\gamma'\otimes\mathrm{id}_w"] & \\
v_1' \arrow[dr, dashed, "\exists h"'] & & v_1'' \arrow[dl, dashed, "\exists h'"] & \rightsquigarrow & v' = v_1' \otimes w \arrow[dr, "h\otimes\mathrm{id}_w"'] & & v'' = v_1'' \otimes w \arrow[dl, "h'\otimes\mathrm{id}_w"] \\
& z_1 & & & & z_1 \otimes w &
\end{tikzcd}$$

The diagram to the left commutes by induction on the length n of words (the base case is clear, since there is only one pure binary word of length 1, which is $(-)$; for which it holds vacously). So the right diagram commutes by induction and bifunctoriality of $\otimes$.

If either β or γ is ϕ: If both β and γ equals ϕ, then we can just take $z = (u \otimes s) \otimes t$ as their common codomain. Assume that $\beta = \phi$, and $\beta \neq \gamma$. Then γ acts inside say u. We then get the following diagram

$$\begin{tikzcd}
& v = u \otimes (s \otimes t) \arrow[dl, "\beta=\phi_{u,s,t}"'] \arrow[dr, "\gamma\otimes\mathrm{id}_{s\otimes t}"] & \\
(u \otimes s) \otimes t \arrow[dr, "(\gamma\otimes\mathrm{id}_s)\otimes\mathrm{id}_t"'] & & u' \otimes (s \otimes t) \arrow[dl, "\phi_{u',s,t}"] \\
& z = (u' \otimes s) \otimes t &
\end{tikzcd}$$

which commutes by *naturality* of ϕ (also note that by byfunctoriality of $\otimes$, $(\gamma \otimes \mathrm{id}_s) \otimes \mathrm{id}_t = \gamma \otimes \mathrm{id}_{s \otimes t}$).

If γ acts inside w, specifically inside s or t; let γ for example act inside t. Then we get a diagram

$$\begin{tikzcd}
& v = u \otimes (s \otimes t) \arrow[dl, "\beta=\phi_{u,s,t}"'] \arrow[dr, "\mathrm{id}_u \otimes(\mathrm{id}_s \otimes\gamma')"] & \\
(u \otimes s) \otimes t \arrow[dr, "(\mathrm{id}_u \otimes \mathrm{id}_s)\otimes\gamma'"'] & & u \otimes (s \otimes t') \arrow[dl, "\phi_{u,s,t'}"] \\
& (u \otimes s) \otimes t' &
\end{tikzcd}$$

that again commutes by *naturality* of ϕ.

$\beta = \phi$ and γ acts inside inside w, but not inside s or t:

$$\begin{tikzcd}
 & v = u \otimes (s \otimes (p \otimes q)) \arrow[dl, "\phi_{u,s,p\otimes q}"'] \arrow[dr, "\mathrm{id}_u \otimes \phi_{s,p,q}"] & \\
(u \otimes s) \otimes (p \otimes q) \arrow[dr, "\phi_{u\otimes s,p,q}"'] & & u \otimes ((s \otimes p) \otimes q) \arrow[dl, "(\phi_{u,s,p}\otimes \mathrm{id})\circ\phi_{u,s\otimes p,q}"] \\
 & ((u \otimes s) \otimes p) \otimes q &
\end{tikzcd}$$

which is the pentagon axiom 2.1, hence commutes (note that we have here used the identification remarked on in 3.16 to use the pentagon axiom). ☐

Lemma 3.27. *If $v \in \mathcal{W}$ is a pure binary word, and $\beta_m \circ \cdots \circ \beta_1, \gamma_\ell \circ \cdots \circ \gamma_1 : w \rightrightarrows w^{(n)}$ are compositions of basic directed arrows in $\mathcal{W}$, then their image under $F : \mathcal{W} \to \mathcal{C}$ defined in 3.15, is equal.*

Proof. We proceed by induction on the rank ρ of a pure binary word v. If $\rho(v) = 0 \iff v = w^{(n)}$ and this is trivial (the compositions must both be the identity in $\mathcal{W}$, and by functoriality they both are the identitiy in the target category).

Assume it holds for rank $\rho < n$; we want to show it holds for $\rho = n$.

We want to show that the following diagram commutes

$$\begin{tikzcd}
 & w \arrow[dl, "\beta_1"'] \arrow[dr, "\gamma_1"] & \\
u_1 \arrow[dr, "\beta_m \circ \cdots \circ \beta_2"'] & & v_1 \arrow[dl, "\gamma_\ell \circ \cdots \circ \gamma_2"] \\
 & w^{(n)} &
\end{tikzcd}$$

By 3.26 we know that there exists a pure binary word z and $\beta' : u_1 \to z$ and $\gamma' : v_1 \to z$ such that the following diagram commutes

$$\begin{tikzcd}
 & w \arrow[dl, "\beta_1"'] \arrow[dr, "\gamma_1"] & \\
u_1 \arrow[dr, "\beta'"'] & & v_1 \arrow[dl, "\gamma'"] \\
 & z &
\end{tikzcd}$$

By 3.22 there is a directed basic arrow $\Gamma^n : z \to w^{(n)} \rightsquigarrow$ we have the following diagram

$$\begin{tikzcd}
 & & w \arrow[dll, "\beta_1"'] \arrow[drr, "\gamma_1"] & & \\
u_1 \arrow[drr, "\beta'"] \arrow[ddrr, "\beta_m\circ\cdots\circ\beta_2"'] & & & & v_1 \arrow[dll, "\gamma'"'] \arrow[ddll, "\gamma_\ell\circ\cdots\circ\gamma_2"] \\
 & & z \arrow[d, "\Gamma^n"] & & \\
 & & w^{(n)} & &
\end{tikzcd}$$

Consider the fact that β_1 and γ_1 decrease rank (3.24), from which it follows that $\rho(u_1) < \rho(w)$ and $\rho(v_1) < \rho(w)$. By induction, both the lower left and lower right triangles commute. It follows that

$$\begin{aligned}
\beta_m \circ \cdots \circ \beta_1 &= \Gamma^n \circ \beta' \circ \beta_1 \\
&= \Gamma^n \circ \gamma' \circ \gamma_1 \\
&= \gamma_\ell \circ \cdots \circ \gamma_1.
\end{aligned}$$

□

Lemma 3.28. *If $\beta_m \circ \cdots \circ \beta_1, \gamma_\ell \circ \cdots \circ \gamma_1 : v \rightrightarrows w$ are compositions of basic arrows from a pure binary word v to a pure binary word w, then their identifications in $\mathcal{C}$ are equal.*

Proof. Let

$$\begin{aligned}
\beta_i &: u_i \to u_{i+1} \\
\gamma_i &: s_i \to s_{i+1}
\end{aligned}$$

where we let

$$\begin{aligned}
u_{i+1} &= s_{i+1} \\
&= w
\end{aligned}$$

and

$$\begin{aligned}
u_1 &= s_1 \\
&= v.
\end{aligned}$$

which leads to the following diagram

$$\begin{tikzcd}
 & u_2 \arrow[r, "\beta_2"] & \cdots \arrow[r, "\beta_{m-1}"] & u_m \arrow[dr, "\beta_m"] & \\
v \arrow[ur, "\beta_1"] \arrow[dr, "\gamma_1"] & & & & w \\
 & s_1 \arrow[r, "\gamma_2"] & \cdots \arrow[r, "\gamma_{\ell-1}"] & s_\ell \arrow[ur, "\gamma_\ell"] &
\end{tikzcd}$$

(3.3)

By 3.22 we know that for each pure binary word u_i and s_i, there is a directed path (composition of basic directed arrows to $w^{(n)}$). Let's denote the corresponding paths as $\Gamma_{s_i} : s_i \to w^{(n)}$ and $\Gamma_{u_i} : u_i \to w^{(n)}$ with

$$\begin{aligned}\Gamma_{u_1} &= \Gamma_{s_1} \\ &= \Gamma_v : v \to w^{(n)} \\ \Gamma_{u_{m+1}} &= \Gamma_{s_{\ell+1}} \\ &= \Gamma_w : w \to w^{(n)}.\end{aligned}$$

Then we get the following (schematic) diagram

$$\begin{tikzcd}
 & & w^{(n)} & & \\
 & u_2 \arrow[ur, bend left, "\Gamma_{u_2}"] \arrow[r, "\beta_2"] & \cdots \arrow[u, bend right] \arrow[r, "\beta_{m-1}"] & u_m \arrow[ul, bend right, "\Gamma_{u_m}"'] \arrow[dr, "\beta_m"] & \\
v \arrow[uurr, bend left, "\Gamma_v"] \arrow[ur, "\beta_1"] \arrow[dr, "\gamma_1"'] & & & & w \arrow[uull, bend right, "\Gamma_w"'] \\
 & s_1 \arrow[uuur, "\Gamma_{s_1}"] \arrow[r, "\gamma_2"] & \cdots \arrow[uuu] \arrow[r, "\gamma_{\ell-1}"] & s_\ell \arrow[uuul, "\Gamma_{s_\ell}"'] \arrow[ur, "\gamma_\ell"] &
\end{tikzcd} \tag{3.4}$$

Note that we have triangles in 3.4 of the form

$$\begin{tikzcd}
 & w^{(n)} & \\
u_i \arrow[ur, "\Gamma_{u_i}"] \arrow[rr, "\beta_i"] & & u_{i+1} \arrow[ul, "\Gamma_{u_{i+1}}"']
\end{tikzcd}$$

where β_i is a *basic directed arrow*. By 3.27, this diagram commutes, for each $i = 1, \ldots, m$. By the same lemma, triangles of the same form involving words s_i, s_{i+1} and $w^{(n)}$ and arrows $\Gamma_{s_i}, \Gamma_{s_{i+1}}$ and γ_i, commutes.

By hypothesis of 3.28, we don't know that β_i is *directed*; so assume that β_i was *anti-directed*.

Then we see that β_i^{-1} is directed, so that, again by 3.28, the following diagram commutes

$$\begin{tikzcd}
 & w^{(n)} & \\
u_i \arrow[ur, "\Gamma_{u_i}"] & & u_{i+1} \arrow[ul, "\Gamma_{u_{i+1}}"'] \arrow[ll, "(\beta_i)^{-1}"']
\end{tikzcd}$$

Finally, note that each $\Gamma_{u_i}, \Gamma_{s_i}$ is an isomorphism in $\mathcal{W}$, and functors preserve isomorphisms, so the identification with the respective morphisms in $\mathcal{C}$ are all isomorphisms, and the same holds for β_i.

We have thus seen that in $\mathcal{C}$, we have either (for β_i)

$$\begin{aligned}\Gamma_{u_{i+1}} \circ \beta_i = \Gamma_i &\iff \beta_i = \Gamma_{u_{i+1}}^{-1} \circ \Gamma_i \\ \Gamma_{u_i} \circ (\beta_i)^{-1} = \Gamma_{u_{i+1}} &\iff \beta_i = \Gamma_{u_{i+1}}^{-1} \circ \Gamma_i.\end{aligned}$$

The same holds for γ_i. Consider the path $\beta_m \circ \cdots \circ \beta_1$. We rewrite this as

$$\begin{aligned}\beta_m \circ \cdots \circ \beta_1 &= (\Gamma_{u_{m+1}}^{-1} \circ \Gamma_{u_m}) \circ (\Gamma_{u_m}^{-1} \circ \Gamma_{u_{m-1}}) \circ \cdots \circ (\Gamma_{u_2}^{-1} \circ \Gamma_{u_1}) \\ &= \Gamma_{u_{m+1}}^{-1} \circ \Gamma_{u_1} \\ &= \Gamma_w^{-1} \circ \Gamma_v.\end{aligned}$$

The path $\gamma_\ell \circ \cdots \circ \gamma_1$ yield the same morphism $\Gamma_w^{-1} \circ \cdots \circ \Gamma_v = \Gamma_w^{-1} \circ \Gamma_v$. Hence 3.3 commutes. □

□

The proof then proceeds to incorporate units **1** in $\mathcal{C}$ and the natural isomorphisms l, r into the results. That is, we want to treat cases where not only we have *pure binary words*, but also words including instances of e_0, the "empty word". Also note that we have only looked at a functor such that it "fills" all instances of $(-)$ with the same object $X \in \mathcal{C}$. But we also want to treat cases where we have multiple objects $X, Y, Z \in \mathcal{C}$ that gives us e.g. tensor-products on the form $X \otimes (Y \otimes Z)$. We will not provide explicit details for every step, but just give a sketch of how to continue the proof.

[8] considers graphs G'_n with vertices all words of a the same length n, where the words are no longer *pure*. One then forms, recursively, directed and anti-derícted *unitor* arrows consisting of r and l, just as we did earlier with ϕ. The graph G'_n then becomes infinite, since by definition of $\mathscr{L}$, the length of an instance of the empty word e_0 is 0. Still, G_n is contained in G'_n. [18] gives the details for proving that G'_n commutes (identifying morphisms with their image in the target category) by a similar proof-strategy to what we have already covered (note that we find that there seems to be some smaller gaps in atleast one proof in [18], but in general, it is a good thesis).

We let $(\mathcal{C}, \otimes)$ be a tensor category. We then define an *iterated functor category*, $\mathrm{It}(\mathcal{C})$, such that:

- Objects in $\mathrm{It}(\mathcal{C})$ are *functors* $F : \mathcal{C}^n \to \mathcal{C}$ for $n \in \mathbb{N}$, where, for $n = 0$, we set $\mathcal{C}^0 = \mathbf{1}$, the *identity object.*
- Morphisms in $\mathrm{It}(\mathcal{C})$ are *natural transformations* $F \Rightarrow G$.
- We give $\mathrm{It}(\mathcal{C})$ a *monoidal structure* as follows:
  - (a) We give $\mathrm{It}(\mathcal{C})$ a product

    $$\odot : \mathrm{It}(\mathcal{C}) \times \mathrm{It}(\mathcal{C}) \to \mathrm{It}(\mathcal{C}),$$

    where we want $\odot$ to have the properties of a *bifunctor*. If $F : \mathcal{C}^n \to \mathcal{C}, G : \mathcal{C}^m \to \mathcal{C}$ are two functors (i.e. objects in $\mathrm{It}(\mathcal{C})$), then we let

    $$F \odot G : \mathcal{C}^{n+m} \to \mathcal{C},$$

    explicitly defined by

    $$F \odot G(X_1, \ldots, X_n, X_{n+1}, \ldots, X_{m+n}) := F(X_1, \ldots, X_n) \otimes G(X_{n+1}, \ldots, X_{m+n})$$

    on objects $X_i \in \mathcal{C}$.

    For *morphisms* (i.e. natural transformations) in $\mathrm{It}(\mathcal{C})$; if $F_1, G_2 : \mathcal{C}^n \rightrightarrows \mathcal{C}$ and $F_2, G_2 : \mathcal{C}^m \rightrightarrows \mathcal{C}$ are objects (functors), and $\eta : F_1 \Rightarrow G_1, \lambda : F_2 \Rightarrow G_2$ are morphisms, then we set

    $$\eta \odot \lambda : F_1 \odot G_1 \Rightarrow F_2 \odot G_2,$$

    such that, for objects $X_i \in \mathcal{C}$, we have

    $$\eta \odot \lambda(X_1, \ldots, X_n, X_{n+1}, \ldots, X_{m+n}) := (\eta)_{X_1, \ldots, X_n} \otimes (\lambda)_{X_{n+1}, \ldots, X_{m+n}}.$$

Remark 3.29. Note that below, we make the weaker assertion that it holds when $(\mathcal{C}, \otimes)$ is a *tensor category*, instead of just a monoidal category. The stronger assertion also holds.

Lemma 3.30. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category. Then* $(\mathrm{It}(\mathcal{C}), \odot, c, \Phi, \mathrm{l}, \mathrm{r})$ *is a monoidal category, where,*

(a) $\odot : \mathrm{It}(\mathcal{C}) \times \mathrm{It}(\mathcal{C}) \to \mathrm{It}(\mathcal{C})$ *is our bifunctor.*

(b) The identity object *is a functor* $c : \{*\} \to \mathcal{C}$ *such that* $c(*) = \mathbf{1}$*.*

(c) For any functors $F_i : \mathcal{C}^{n_i} \to \mathcal{C}$*, for* $i \in \{1, 2, 3\}$*, we define* $\Phi_{F_1,F_2,F_3} : F_1 \odot (F_2 \odot F_3) \Rightarrow (F_1 \odot F_2) \odot F_3$ *as the morphism in* $\mathrm{It}(\mathcal{C})$*, so that for objects* $X_1, \dots, X_{n_1+n_1+n_3} \in \mathcal{C}$*, we have*

$$\Phi_{F_1,F_2,F_3}(X_1, \dots, X_{n_1+n_2+n_3}) = (\Phi)_{F_1(X_1,\dots,X_{n_1}),F_2(X_{n_1+1},\dots,X_{n_2+1}),F_3(X_{n_1+n_2+1},\dots,X_{n_1+n_2+n_3})}.$$

(d) For any functor $F_i : \mathcal{C}^i \to \mathcal{C}$ *in* $\mathrm{It}(\mathcal{C})$*, we define the natural transformation (morphism)* $\mathbf{r}_{F_i} : c \odot F \Rightarrow F$ *such that on objects* $(*, X_1, \dots, X_i)$ *in the product-category* $\{*\} \times \mathcal{C}$*, we have*

$$(\mathrm{r}_{F_i})(*, X_1, \dots, X_i) = (r)_{F(X_1,\dots,X_n)},$$

while $\mathrm{l}_{F_i} : F_i \odot c \Rightarrow F_i$ *is defined similarly.*

We now reach the main result.

Lemma 3.31. *For every tensor category* $(\mathcal{C}, \otimes)$*, there exists a unique functor* $\mathbb{G}_{\mathrm{id}} : \mathcal{W} \to \mathrm{It}(\mathcal{C})$*, where* $\mathbb{G}_{\mathrm{id}}((-)) = \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ *is the identity functor on* $\mathcal{C}$*.*

Proof. By 3.30 we know that $\mathrm{It}(\mathcal{C})$ is a monoidal category. Then from 3.15 we get the existence of a unique functor $\mathbb{G}_{\mathrm{id}}$ such that $\mathbb{G}_{\mathrm{id}}(-) = \mathrm{id}_{\mathcal{C}}$, since $\mathrm{id}_{\mathcal{C}}$ is an object in $\mathrm{It}(\mathcal{C})$. ☐

To illustrate why this gives us coherence in $(\mathcal{C}, \otimes)$ of *formal* diagrams (i.e. diagrams consisting only of ϕ, l, r and their tensored products), we investigate the behavior of $\mathbb{G}_{\mathrm{id}}(-)$.

For a morphism $\phi_{(-),(-),(-)} : (-) \otimes ((-) \otimes (-)) \to ((-) \otimes (-)) \otimes (-)$ in $\mathcal{W}$, this gets mapped to (as in the proof of 3.15)

$$\Phi_{\mathrm{id},\mathrm{id},\mathrm{id}} : \mathrm{id} \odot (\mathrm{id} \odot \mathrm{id}) \to (\mathrm{id} \odot \mathrm{id}) \odot \mathrm{id}\,.$$

We may then give $\Phi_{\mathrm{id},\mathrm{id}\,\mathrm{id}}$ any arguments $X_1, X_2, X_3 \in \mathcal{C}$ to obtain $\phi_{X_1,X_2,X_3} : X_1 \otimes (X_2 \otimes X_3) \overset{\simeq}{\to} (X_1 \otimes X_2) \otimes X_3$ in $\mathcal{C}$.

Recall that $\mathcal{W}$ is a *strict* category, so that all diagrams commute. By 1.21, we see that the image of a diagram in $\mathcal{W}$ is commutative in $\mathrm{It}(\mathcal{C})$. So the image will be a commutative diagram in $\mathrm{It}(\mathcal{C})$ consisting of natural isomorphisms between functors, with the identity functor in each argument, of $\mathrm{It}(\mathcal{C})$, since functors also preserve isomorphisms (1.15). We can then give any arguments $X_1, \dots, X_n$ to the image of a commutative diagram of words of length n (in $\mathcal{W}$), to such a commutative diagram, to see that it commutes in $\mathcal{C}$. This concludes our investigation of coherence in tensor categories (excluding ψ).

Now, we need to fold ψ into this mix, to make a coherence argument for *symmetric monoidal categories*, or, as in [10], *tensor categories*. [8, chapter 2, XI] gives us such an argument. Recall that the *symmetric group* of n letters, S_n, is generated by *adjacent* transpositions $\tau_i = (i\ i+1)$ for $1 \leq i \leq n-1$, subject to the following relations:

1. $\tau_i^2 = 1 \qquad (\forall i \in \{1, \dots, n-1\})$.
2. $(\tau_i \tau_j)^2 = 1 \iff \tau_i \tau_j = \tau_j \tau_i \qquad (\forall |i-j| \geq 2)$.
3. $(\tau_i \tau_{i+1})^3 = 1 \iff \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \qquad (\forall i \text{ such that } 1 \leq i \leq n-2)$.

The relations are easy to check: (a) holds trivially, since the order of any transposition is 2. The second relation hold since disjoint transpositions commutes (disjointness assured by $|i-j| \geq 2$), so that

$$\begin{aligned} (\tau_i \tau_j)^2 &= \tau_i^2 \tau_j^2 \\ &= 1 \end{aligned}$$

where the last equality follows from (a). For (c): note that $\tau_i(\tau_i + 1) = (i \; i+1 \; i+2)$, hence has order 3, and the result follows.

For a given tensor category, what we call a *permuted word* $w\tau$ induces a functor $(w\tau)_{\mathcal{C}} : \mathcal{C}^n \to \mathcal{C}$ defined by permuting the indices of the argument, so that

$$(w\tau)_{\mathcal{C}}(X_1, \dots, X_n) = w(X_{\tau(1)}, \dots, X_{\tau(n)}).$$

For example, if $w = ((-) \otimes (-))$ and $\tau = (1 \; 2)$, then

$$\begin{aligned} (w\tau)_{\mathcal{C}}(X_1, X_2) &= ((-) \otimes (-))(X_{\tau(1)}, X_{\tau(2)}) \\ &= ((-) \otimes (-))(X_2, X_1) \\ &= X_2 \otimes X_1. \end{aligned}$$

One can see that if $w\tau$ and $v\sigma$ in the same # of letters, n (where $\tau, \sigma \in S_n$) there is *atleast one map* between $(w\tau)_{\mathcal{C}}(X_1, \dots, X_n)$ and $(v\sigma)_{\mathcal{C}}(X_1, \dots, X_n)$, gotten from combining instances of directed and anti-directed arrows (as defined earlier for ϕ; see 3.18, 3.20) of ϕ and ψ.

Furthermore, if we suppress ϕ in (2.2) as well as parentheses, we get the following two diagrams:

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "\psi"] \arrow[dr, "\mathrm{id} \otimes \psi"'] & & Z \otimes X \otimes Y \arrow[dl, "\psi \otimes \mathrm{id}"] & X \otimes Y \otimes Z \arrow[rr, "\psi"] \arrow[dr, "\psi \otimes \mathrm{id}"'] & & Y \otimes Z \otimes X \arrow[dl, "\mathrm{id} \otimes \psi"] \\
& X \otimes Z \otimes Y & & & Y \otimes X \otimes Z &
\end{tikzcd} \tag{3.5}$$

where the rightmost one comes from the identification

$$\begin{cases} Z = X \\ X = Y \\ Y = Z \end{cases}$$

at the object $Z \otimes (X \otimes Y)$ in (2.2). From 2.2, with ϕ suppressed, we can also note that any instance of ϕ exchanging blocks

We note that $\psi \circ \psi = \mathrm{id}$, so that ψ mirrors the first relation (1) of transpositions τ_i. For the second relation, by bifunctoriality of $\otimes$, we have

$$\begin{tikzcd}
(X \otimes Y) \otimes (Z \otimes T) \arrow[r, "\mathrm{id} \otimes \psi"] \arrow[d, swap, "\psi' \otimes \mathrm{id}"] & (X \otimes Y) \otimes (T \otimes Z) \arrow[d, "\psi' \otimes \mathrm{id}"] \\
(Y \otimes X) \otimes (Z \otimes T) \arrow[r, swap, "\mathrm{id} \otimes \psi"] & (Y \otimes X) \otimes (T \otimes Z)
\end{tikzcd}$$

which mirrors $\tau_i \tau_j = \tau_j \tau_i$. For the third relation, we consider the following diagram.

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "\psi \otimes \mathrm{id}"] \arrow[d, swap, "\mathrm{id} \otimes \psi"] & & Y \otimes X \otimes Z \arrow[dll, "\psi"] \arrow[dr, "\mathrm{id} \otimes \psi"] & \\
X \otimes Z \otimes Y \arrow[dr, swap, "\psi \otimes \mathrm{id}"] & & & Y \otimes Z \otimes X \arrow[dll, swap, "\psi"] \arrow[d, "\psi \otimes \mathrm{id}"] \\
& Z \otimes X \otimes Y \arrow[rr, swap, "\mathrm{id} \otimes \psi"] & & Z \otimes Y \otimes X
\end{tikzcd}$$

The triangles commute by (3.5), and the square in the middle commutes by naturality of ψ, since naturality in ψ means that for any objects A, A', B, B' and morphisms $f : A \to B$ and $g : A' \to B'$, the following diagram commutes,

$$\begin{tikzcd}
A \otimes B \arrow[r, "\psi_{A,B}"] \arrow[d, swap, "f \otimes g"] & B \otimes A \arrow[d, "g \otimes f"] \\
A' \otimes B' \arrow[r, swap, "\psi_{A',B'}"] & B' \otimes A'
\end{tikzcd}$$

so we take $f = \mathrm{id}$, $g = \psi$ (with appropriate objects) above. If we think of $\psi \otimes \mathrm{id}$ as τ_1 and $\mathrm{id} \otimes \psi$ as τ_2, we see that

$$\tau_1 \tau_2 \tau_1 = \tau_2 \tau_1 \tau_2$$

holds, for fixed arbitrary objects $X, Y, Z \in \mathcal{C}$. This mirrors the third relation of 3.1. Hence ψ fulfills all the relations of transpositions $\tau_i \in S_n$, for fixed n. Since any permutation of a word w of length n corresponds to a composition of directed and anti-directed ψ-arrows (with ϕ suppressed), which fulfills the relations of transpositions, any two paths $f_m \circ \ldots \circ f_1$ and $g_\ell \circ \cdots \circ g_1$ of directed and anti-directed ψ arrows, corresponding to the *same* permutation $\alpha \in S_n$, will be equal, which is forced by the fact that the relations *defining* S_n (i.e. S_n:s group presentation) are fulfilled. From the following diagram(s)

$$\begin{tikzcd}
X \arrow[r, "l_X"'] \arrow[dr, "r_X"] & \mathbf{1} \otimes X \arrow[d, "\psi_{\mathbf{1},X}"] \\
& X \otimes \mathbf{1}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
X \otimes \mathbf{1} \arrow[r, "\psi_{X,\mathbf{1}}"'] \arrow[dr, "r_X^{-1}"] & \mathbf{1} \otimes X \arrow[d, "l_X^{-1}"'] \\
& X
\end{tikzcd}$$

we find that removing an identity object before or after applying ψ are identical operations. This motivates how to fold the natural isomorphisms r, l and the object $\mathbf{1}$ into our coherence results for tensor categories. By this proof (sketch), the coherence theorem for tensor categories is done. We then claim that as a consequence, 3.9 follows.

Given an arbitrary tensor category $(\mathcal{C}, \otimes)$, it is assumed that an extension such as in 3.9 has been given.

4 Invertible objects

We start by giving a weaker version of when a functor has an inverse.

Definition 4.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If there is a functor $G : \mathcal{D} \to \mathcal{C}$ so that $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$, then we call G a **quasi-inverse** to F.

Given a tensor category, we introduce a definition relating to when we should call an object *invertible*.

Definition 4.2. Let $(\mathcal{C}, \otimes)$ be a tensor category. We call an object L in $\mathcal{C}$ **invertible** if

$$L \otimes - : \mathcal{C} \to \mathcal{C}$$

defined by

$$\mathcal{C} \ni X \longmapsto L \otimes X \in \mathcal{C}$$

yields an *equivalence of categories.*

As we shall see in 4.7, it follows that there is an object L' in $\mathcal{C}$ so that $L \otimes L' \cong \mathbf{1}$. On the other hand, if $L \otimes L' \cong \mathbf{1}$ holds, and we denote this isomorphism by δ, then we see that $L \otimes -$ and $- \otimes L'$ acts as *quasi-inverses* of each other: For an arbitrary object X in $\mathcal{C}$, we have

$$(L \otimes -) \circ (- \otimes L')(X) = L \otimes (X \otimes L') \xrightarrow{\mathrm{id}_L \otimes \psi_{X,L'}} L \otimes (L' \otimes X) \xrightarrow{\phi_{L,L',X}} (L \otimes L') \otimes X \xrightarrow{\delta \otimes \mathrm{id}_X} \mathbf{1} \otimes X \xrightarrow{l_X^{-1}} X \tag{4.1}$$

We <u>claim</u> that $\zeta \otimes \mathrm{id}_X$ is a natural isomorphism in X, for an arbitrary isomorphism $\zeta : X \to Y$.

Proof.

$$\begin{tikzcd}
A \arrow[d, "f"'] & X \otimes A \arrow[r, "\zeta \otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_X \otimes f"] & Y \otimes A \arrow[d, "\mathrm{id}_Y \otimes f"] & X \arrow[d, "\zeta"] \\
B & X \otimes B \arrow[r, "\zeta \otimes \mathrm{id}_B"] & Y \otimes B & Y
\end{tikzcd}$$

The square to the right above commutes, for arbitrary morphism $f : A \to B$ in $\mathcal{C}$. ☐

Remark 4.3. One can similarly show that $\mathrm{id}_X \otimes \zeta$ is a natural isomorphism in X. Note that the isomorphism part is not needed to show that for arbitrary morphism ζ, we have that $\zeta \otimes \mathrm{id}_X$ and $\mathrm{id}_X \otimes \zeta$ are natural in X.

Lemma 4.4. *Assume that $\alpha : F \to G$ and $\beta : G \to H$ are natural transformations between parallel functors $F, G, H : \mathcal{C} \to \mathcal{D}$. Then there is a natural transformation $\beta \circ \alpha : F \to H$ so that*

$$(\beta \circ \alpha)_A := \beta_A \circ \alpha_A$$

.

Proof. For $f : A \to B$, we contemplate the following rectangle

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"] & G(A) \arrow[r, "\beta_A"] \arrow[d, "G(f)"] & H(A) \arrow[d, "H(f)"] \\
F(B) \arrow[r, "\alpha_B"] & G(B) \arrow[r, "\beta_B"] & H(B)
\end{tikzcd}$$

Let $x \in F(A)$. Then we have that

$$\begin{aligned}
(H(f) \circ \beta_A \circ \alpha_A)(x) &= (\beta_B \circ G(f) \circ \alpha_A)(x) \\
&= (\beta_B \circ \alpha_B \circ F(f))(x)
\end{aligned}$$

so that the larger square commutes, for an arbitrary morphism $f \in \mathrm{Mor}(\mathcal{C})$. ☐

We see that by 4.4, (4.1) is a natural isomorphism

$$\epsilon : (L \otimes -) \circ (- \otimes L') \xrightarrow{\simeq} \mathrm{id}_{\mathcal{C}}.$$

Similarly, we find that for arbitrary $X \in \mathcal{C}$, we have

$$(- \otimes L') \circ (L \otimes -)(X) = (L \otimes X) \otimes L' \xrightarrow{\psi_{L,X} \otimes \mathrm{id}_{L'}} (X \otimes L) \otimes L' \xrightarrow{\phi^{-1}_{X,L,L'}} X \otimes (L \otimes L') \xrightarrow{\mathrm{id}_X \otimes \delta} X \otimes \mathbf{1} \xrightarrow{r_X^{-1}} X.$$

This is a composition of natural isomorphisms, hence a natural isomorphism. It follows that this defines a natural isomorphism

$$\eta : (- \otimes L') \otimes (L \otimes -) \xrightarrow{\simeq} \mathrm{id}_{\mathcal{C}}\,.$$

This shows that $L \otimes -$ defines an equivalence of categories, with quasi-inverse $(- \otimes L')$.

Example 4.5. In the category $\mathbf{Rep}^{\mathrm{fd}}_{\Bbbk}(G)$ of representations of G over finite dimensional vector spaces over $\Bbbk$, the invertible objects are precisely the 1-dimensional representations. This should be clear from the fact that if $(\mathbb{L}, \rho_{\mathbb{L}})$ is a 1-dimensional representation, then $\mathbb{L} \cong \Bbbk$. Hence we see that $\mathbb{L} \otimes_{\Bbbk} W \cong \Bbbk \otimes_{\Bbbk} W \cong W$.

To expand on this, we introduce the following definitions

Definition 4.6. Let $\mathcal{C}$ and $\mathcal{D}$ be *locally small* categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Define $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$. We say that F is

- **Faithful** if $F_{X,Y}$ is injective for all objects X, Y in $\mathcal{C}$.
- **Full** if $F_{X,Y}$ is surjective, for all objects X, Y in $\mathcal{C}$.
- **Fully faithful** if it is both faithful and full.
- **Essentially surjective** if for each object $B \in \mathcal{D}, \exists A \in \mathcal{C}$ so that $F(A) \cong B$.

We will prove that any functor that defines an equivalence of categories, possesses these three properties. The structure of our proof will follow [13] (chap. 1.5, page 31).

Proposition 4.7. *A functor $F : \mathcal{C} \to \mathcal{D}$ defining an equivalence of categories, is* full, faithful *and* essentially surjective*.*

Lemma 4.8. *Let $\mathcal{C}$ be a category and let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, together with isomorphisms $A \cong A'$ and $B \cong B'$. Then these morphisms together determines a* unique *morphism $f' : A' \to B'$ so that all of the following four diagrams commute.*

$$\begin{tikzcd} A \arrow[d, "f"] & A' \arrow[l, "\cong"'] \arrow[d, "f'"] \\ B \arrow[r, "\cong"] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B \arrow[r, "\cong"] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "f"] & A' \arrow[l, "\cong"'] \arrow[d, "f'"] \\ B & B' \arrow[l, "\cong"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B & B' \arrow[l, "\cong"'] \end{tikzcd}$$

We start by proving 4.8, going from the leftmost diagram to the rightmost.

Proof. *First diagram*: Let $\delta_A : A \cong A'$ and $\delta_B : B \cong B'$. We let the leftmost diagram define f', so that $f' := \delta_B \circ f \circ \delta_A^{-1}$, where we have used that δ_A, δ_B are isomorphisms, together with the definition of f'.

Second diagram:

$$\begin{aligned} \delta_B \circ f &= \delta_B \circ (\delta_B^{-1} \circ f' \circ \delta_A) \\ &= f' \circ \delta_A. \end{aligned}$$

Third diagram:

$$\begin{aligned} f \circ \delta_A^{-1} &= (\delta_B^{-1} \circ f' \circ \delta_A) \circ \delta_A^{-1} \\ &= \delta_B^{-1} \circ f' \end{aligned}$$

by similar reasoning as in the 2nd diagram.

Fourth diagram:

$$f = \delta_B^{-1} \circ f' \circ \delta_A$$

from the definition of f', and from the fact that δ_A, δ_B are isomorphisms. ☐

We now proceed to prove proposition 4.7.

Proof.

Essentially surjective: We start by supposing that $F, G : \mathcal{C} \rightleftarrows \mathcal{D}$ defines an equivalence of categories, so that $FG \simeq \mathrm{id}_{\mathcal{D}}$ and $GF \simeq \mathrm{id}_{\mathcal{C}}$. Then we see that for objects $d \in \mathcal{D}$ we have $FG(d) \simeq d$, so that F is essentially surjective.

Faithful: Let $f, g : c \rightrightarrows c'$ in $\mathcal{C}$. We let η be a family of morphisms that define the natural isomorphism between GF and $\mathrm{id}_{\mathcal{D}}$. We get the following diagram

$$\begin{tikzcd} GF(c) \arrow[r, "\eta_c^{-1}", "\simeq"'] \arrow[d, "GF(f)=GF(g)"] & c \arrow[d, "f \text{ or } g"] \\ GF(c') \arrow[r, "\eta_{c'}^{-1}", "\simeq"'] & c' \end{tikzcd}$$

that commutes by naturality.

We see that

$$\begin{aligned} f \circ \eta_c^{-1} &= \eta_{c'}^{-1} \circ GF(f) \\ \iff \quad f &= \eta_{c'}^{-1} \circ GF(f) \circ \eta_c \\ &= \eta_{c'}^{-1} \circ GF(g) \circ \eta_c \\ &= g \end{aligned}$$

Remark 4.9. In the third equality above, we used that $GF(f) = GF(g)$.

Then, we see that $f = g$, and that the morphism making the diagram commute is unique (either by definition or by using lemma 4.8). Again, by lemma 4.8, we assert that this morphism $f = g$ is such that diagrams that defines a natural isomorphism $GF \simeq \mathrm{id}_{\mathcal{C}}$ commutes.

It follows that if $F(f) = F(g)$ then $f = g$, so that F is *faithful*. One can appply the same reasoning to $f', g' : d \rightrightarrows d' \in \mathrm{Mor}(\mathcal{D})$ together with $FG \simeq \mathrm{id}_{\mathcal{D}}$ to get a unique morphism $f' = g'$ so that the diagrams associated to $FG \simeq \mathrm{id}_{\mathcal{D}}$ commutes, hence G is also *faithful*, by the same reasoning.

Full: Let $k : F(c) \to F(c')$. From $GF \simeq \mathrm{id}_{\mathcal{C}}$, we get an natural isomorphism η with components $\eta_c, \eta_{c'}$. We use this to define h

$$\begin{tikzcd}
GF(c) \arrow[r, "\eta_c^{-1}", "\simeq"'] \arrow[d, "GF(k)"] & c \arrow[d, dashed, "h"] \\
GF(c') \arrow[r, "\eta_{c'}^{-1}", "\simeq"'] & c'
\end{tikzcd}$$

by lemma 4.8, we see that h is the unique map that makes the diagrams in 4.8 commute. By naturality, we also have

$$\begin{tikzcd}
GF(c) \arrow[r, "\eta_c^{-1}", "\simeq"'] \arrow[d, "GF(h)"'] & c \arrow[d, "h"] \\
GF(c') \arrow[r, "\eta_c^{-1}", "\simeq"'] & c'
\end{tikzcd}$$

we see from this that

$$\begin{aligned} & h = \eta_{c'}^{-1} \circ GF(k) \circ \eta_c \\ \iff & G(k) = \eta_{c'} \circ h \circ \eta_c^{-1} \end{aligned}$$

and

$$\begin{aligned} & h = \eta_{c'}^{-1} \circ GF(h) \circ \eta_c \\ \iff & GF(h) = \eta_{c'} \circ h \circ \eta_c^{-1} \\ \implies & G(k) = GF(h). \end{aligned}$$

Since we already showed that G is *faithful*, we find that $F(h) = k$, so that F is *full*.

$\square$

We will leave the proof of the converse out for now; i.e. that if one assumes the *Axiom of Choice*, then one can prove that any functor F that is faithful, full and essentially surjective defines an equivalence of categories.

Remark 4.10. By definition 4.1, we see that if $L \otimes - : \mathcal{C} \to \mathcal{C}$ defines an equivalence of categories, then by proposition 4.3, $L \otimes -$ is essentially surjective $\rightsquigarrow$ there exists an object L^{-1} in $\mathcal{C}$ so that

$$\zeta : L \otimes L^{-1} \cong \mathbf{1}.$$

The remark above motivates our next definition:

Definition 4.11. Let $(\mathcal{C}, \otimes)$ be a tensor category, and let L be an invertible object. Then we call (L^{-1}, δ) an **inverse** of L, where $\delta : \bigotimes_{i \in \{\pm\}} X_i \cong \mathbf{1}$, with $X_+ := L$ and $X_- := L^{-1}$.

Remark 4.12. Note that in a tensor category $(\mathcal{C}, \otimes)$, one has

$$L \otimes L^{-1} \cong L^{-1} \otimes L \cong \mathbf{1};$$

hence (L, δ) is an inverse of L^{-1}.

Proposition 4.13. *If $(L_1, \delta_1), (L_2, \delta_2)$ are both inverses of an invertible object L, then there exists a* unique *isomorphism $\alpha : L_1 \cong L_2$ so that*

$$\delta_2 \circ (\mathrm{id}_L \otimes \alpha) = \delta_1 : L \otimes L_1 \to L \otimes L_2 \to \mathbf{1}$$

Proof. By assumption, $L \otimes -$ yields an equivalence of categories. Hence, by 4.7 we know that $L \otimes -$ is *fully faithful* $\rightsquigarrow \mathrm{Hom}(L_1, L_2) \to \mathrm{Hom}(L \otimes L_1, L \otimes L_2)$ is bijective mapping, and is explicitly defined by

$$\mathrm{Hom}(L_1, L_2) \ni \gamma \longmapsto \mathrm{id}_L \otimes \gamma \in \mathrm{Hom}(L \otimes L_1, L \otimes L_2). \tag{4.2}$$

Since $\delta_2 : L \otimes L_2 \cong \mathbf{1}$, we get a bijection

$$\mathrm{Hom}(L \otimes L_1, L \otimes L_2) \cong \mathrm{Hom}(L \otimes L_1, \mathbf{1}). \tag{4.3}$$

by post-composing with δ_2, i.e.

$$\mathrm{Hom}(L \otimes L_1, L \otimes L_2) \ni g \longmapsto \delta_{2_*}(g) = \delta_2 \circ g \in \mathrm{Hom}(L \otimes L_1, \mathbf{1}).$$

Composing 4.2 and 4.3, we get a bijection

$$F : \mathrm{Hom}(L_1, L_2) \to \mathrm{Hom}(L \otimes L_1, \mathbf{1}). \tag{4.4}$$

Since $\delta_1 \in \mathrm{Hom}(L \otimes L_1, \mathbf{1})$, it follows that there is a unique $\alpha \in \mathrm{Hom}(L_1, L_2)$ so that

$$F(\alpha) = \delta_2 \circ (\mathrm{id}_L \otimes \alpha) = \delta_1 \iff (\mathrm{id}_L \otimes \alpha) = \delta_2^{-1} \circ \ \delta_1.$$

It remains to show that α is an isomorphism. Similarly, as before, we see that

$$\mathrm{Hom}(L_2, L_1) \to \mathrm{Hom}(L \otimes L_2, L \otimes L_1) \tag{4.5}$$

is a bijection by 4.7, defined by $\mathrm{Hom}(L_2, L_1) \longmapsto \mathrm{id}_L \otimes \gamma$, and that

$$\mathrm{Hom}(L \otimes L_2, L \otimes L_1) \to \mathrm{Hom}(L \otimes L_2, \mathbf{1}) \tag{4.6}$$

is a bijection, defined by $\mathrm{Hom}(L \otimes L_2, L \otimes L_1) \ni f \longmapsto \delta_{1_*}(f) = \delta_1 \circ f \in \mathrm{Hom}(L \otimes L_2, \mathbf{1})$.

By composing 4.5 and 4.6, we have a bijection

$$G : \mathrm{Hom}(L_2, L_1) \to \mathrm{Hom}(L \otimes L_2, \mathbf{1}).$$

Since $\delta_2 \in \mathrm{Hom}(L \otimes L_2, \mathbf{1})$, there is a unique map β so that

$$\begin{aligned} G(\beta) &= \delta_1 \circ (\mathrm{id}_L \otimes \beta) \\ &= \delta_2 \\ \implies (\mathrm{id}_L \otimes \beta) &= \delta_1^{-1} \circ \ \delta_2. \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{id}_{L\otimes L_2} &= (\delta_2^{-1}\delta_1)(\delta_1^{-1}\delta_2) \\ &= (\mathrm{id}_L \otimes \alpha)(\mathrm{id}_L \otimes \beta) \\ &= (\mathrm{id}_L \otimes \alpha\beta) \\ &= \mathrm{id}_L \otimes \mathrm{id}_{L_2} \\ \implies \alpha\beta &= \mathrm{id}_{L_2}\,. \end{aligned}$$

and

$$\begin{aligned} \mathrm{id}_{L\otimes L_1} &= (\delta_1^{-1}\delta_2)(\delta_2^{-1}\delta_1) \\ &= (\mathrm{id}_L \otimes\beta)(\mathrm{id}_L \otimes\alpha) \\ &= (\mathrm{id}_L \otimes\beta\alpha) \\ &= (\mathrm{id}_L \otimes\beta\alpha) \\ &= \mathrm{id}_L \otimes \mathrm{id}_{L_1} \\ \implies \beta\alpha &= \mathrm{id}_{L_1}\,. \end{aligned}$$

Therefore, α is an isomorphism. □

5 Internal Hom

We start by introducing a certain kind of category

Definition 5.1. Given a category $\mathcal{C}$, we call the category $\mathcal{C}^{\mathrm{opp}}$ the **opposite category**, where one has that $\mathrm{Ob}(\mathcal{C}^{\mathrm{opp}}) = \mathrm{Ob}(\mathcal{C})$, but to every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ there is a corresponding morphism $f^{\mathrm{opp}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(B, A)$.

$$\begin{tikzcd}
\mathrm{Ob}(\mathcal{C}^{\mathrm{opp}}) & = & \mathrm{Ob}(\mathcal{C}) & f \in \mathrm{Hom}_{\mathcal{C}}(A,B) & \leftrightsquigarrow & f^{\mathrm{opp}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(B,A) \\
\mathcal{C}^{\mathrm{opp}} \ni A & \leftrightsquigarrow & A \in \mathcal{C} & A \arrow[d, "f"] & & B \arrow[d, "f^{\mathrm{opp}}"] \\
 & & & B & & A
\end{tikzcd}$$

5.1 Hom$(A, -)$ and Hom$(-, A)$

- Let $\mathcal{C}$ be a locally small category, and let $f : X \to Y$ be a morphism. Then we see that for $g \in \mathrm{Hom}(Y, A)$, we can precompose g with f, i.e. $f^*(g) := g \circ f : X \to A$.

  That is, for each object A, and morphism f, we find that $f : X \to Y \rightsquigarrow \mathrm{Hom}(Y, A) \xrightarrow{f^*} \mathrm{Hom}(X, A)$. It follows that we can define an **contravariant functor** $\mathrm{Hom}(-, A) : \mathcal{C} \to$ **Set**, which takes objects X to $\mathrm{Hom}(X, A)$ and morphisms $f : X \to Y$ to $\mathrm{Hom}(f, A) : \mathrm{Hom}(Y, A) \to \mathrm{Hom}(X, A)$.

  Assume that $f : X \to Y$ and $h : Y \to Z$. Then we see that $hf : X \to Z$, and that for

$g \in \mathrm{Hom}(Z, A)$ we have

$$\begin{aligned} \mathrm{Hom}(hf, A)(g) &:= (hf)^*(g) \\ &= g \circ (hf) : X \to A \end{aligned}$$

as well as

$$\begin{aligned} \mathrm{Hom}(hf, Y)(g) &= (\mathrm{Hom}(f, Y) \circ \mathrm{Hom}(h, Y))(g) \\ &= (f^* \circ h^*)(g) \\ &= f^*(g \circ h) \\ &= g \circ h \circ f \end{aligned}$$

where we used that function-composition is associative, . At last, we see that for a morphism $f : X \to Y$ and $\mathrm{Hom}(\mathrm{id}_Y, A) = \mathrm{id}_Y^*$ we have that for $g \in \mathrm{Hom}(Y, A)$ one finds

$$\begin{aligned} \mathrm{id}_Y^*(g) &= g \circ \mathrm{id}_Y \\ &= g. \end{aligned}$$

- On the other hand, let $f : X \to Y$ be a morphism in $\mathcal{C}$, and let $g \in \mathrm{Hom}(A, X)$. Then we can define $f_*(g) := f \circ g : A \to Y$. Thus we claim that $\mathrm{Hom}(A, -)$ is a **covariant functor** that takes objects X in $\mathcal{C}$ to $\mathrm{Hom}(A, X)$ and morphisms $f : X \to Y$ to $\mathrm{Hom}(A, f) = f_*$.

  Let $f : X \to Y$ and let $h : Y \to Z$, and let $g \in \mathrm{Hom}(A, X)$. Then

  $$\begin{aligned} \mathrm{Hom}(A, hf) &= (hf)_*(g) \\ &= hf \circ g : A \to Z \end{aligned}$$

  and

  $$\begin{aligned} (\mathrm{Hom}(A, h) \circ \mathrm{Hom}(A, f))(g) &= h_*(f_*(g)) \\ &= h_*(f \circ g) \\ &= h \circ f \circ g. \end{aligned}$$

  Lastly, we find that if $g \in \mathrm{Hom}(A, X)$, then

  $$\begin{aligned} (\mathrm{id}_X)_*(g) &= \mathrm{id}_X \circ g \\ &= g. \end{aligned}$$

  This shows that $\mathrm{Hom}(A, -)$ is a covariant functor.

5.2 Representable functors, presheafs, and internal-hom adjunctions

Definition 5.2. We call a functor $F : \mathcal{C} \to \mathbf{Set}$ a **representable functor** if it is naturally isomorphic to the hom-functor $\mathrm{Hom}(A, -)$ for some object A of $\mathcal{C}$, and we call (F, ϕ) the **representation** of F, where $\phi : \mathrm{Hom}(A, -) \Rightarrow F$ is the natural isomorphism with components at objects $B \in \mathcal{C}$, i.e. so that the following diagram commutes for each morphism $f : X \to Y$ in $\mathcal{C}$

$$\begin{tikzcd}
f & & \\
X \arrow[d] & \mathrm{Hom}(A,X) \arrow[r, "\phi_X", "\simeq"'] \arrow[d, "\mathrm{Hom}(A,f)"] & F(X) \arrow[d, "F(f)"] \\
Y & \mathrm{Hom}(A,Y) \arrow[r, "\phi_Y", "\simeq"'] & F(Y)
\end{tikzcd} \tag{5.1}$$

and where $\mathrm{Hom}(A,f)$ is defined as $\mathrm{Hom}(A,X) \ni g \longmapsto f \circ g \in \mathrm{Hom}(A,Y)$.

We introduce one more crucial definition, closely related to definition 5.2.

Definition 5.3. Let $\mathcal{C}$ be a category, and let $G : \mathcal{C} \to \mathbf{Set}$ be a *contravariant* functor. Then we call G a **presheaf**. Furthemore, we say that G is **representable** if it is naturally isomorphic to $\mathrm{Hom}(-,A)$ for some object $A \in \mathcal{C}$. We can just as well define a presheaf as a functor $F : \mathcal{C}^{\mathrm{opp}} \to \mathbf{Set}$.

Definition 5.4. Let $\mathcal{C}$ be a category. Then we define $\mathbf{PSh}(\mathcal{C}) := [\mathcal{C}^{\mathrm{opp}}, \mathbf{Set}]$ as the **presheaf category on** $\mathcal{C}$, with objects as functors $F : \mathcal{C}^{\mathrm{opp}} \to \mathbf{Set}$ and morphisms $\alpha \in \mathrm{Mor}(\mathbf{Psh}(\mathcal{C}))$ as natural transformations between functors $F, G : \mathcal{C}^{\mathrm{opp}} \rightrightarrows \mathbf{Set}$.

We want to relate this to tensor-categories.

Definition 5.5. Let $(\mathcal{C}, \otimes)$ be a tensor category. If the contravariant functor $\mathrm{Hom}(-\otimes X, Y) : \mathcal{C} \to \mathbf{Set}$ defined by $\mathcal{C} \ni T \to \mathrm{Hom}(T \otimes X, Y) \in \mathbf{Set}$ is *representable* (5.3) for objects X, Y in $\mathcal{C}$, then we denote the object corresponding to the representation as $\underline{\mathrm{Hom}}(X,Y)$ (given that $\underline{\mathrm{Hom}}(X,Y)$ exists).

Then $\mathrm{Hom}(-\otimes X, Y)$ would be an example of a **representable presheaf**.

To clarify what we mean by $\underline{\mathrm{Hom}}(X,Y)$ being the corresponding object of the representation; we mean that $\mathrm{Hom}(-\otimes X, Y)$ is *naturally isomorphic* to $\mathrm{Hom}(-, \underline{\mathrm{Hom}}(X,Y))$, that is, the following diagram commutes for each morphism $f : B \to A$ in $\mathcal{C}$

$$\begin{tikzcd}
B \arrow[d, "f"] & \mathrm{Hom}(A \otimes X, Y) \arrow[r, "\eta_A", "\simeq"'] \arrow[d, "\mathrm{Hom}(f\otimes X,Y)"] & \mathrm{Hom}(A, \underline{\mathrm{Hom}}(X,Y)) \arrow[d, "\mathrm{Hom}(f,\underline{\mathrm{Hom}}(X,Y))"] \\
A & \mathrm{Hom}(B \otimes X, Y) \arrow[r, "\eta_B", "\simeq"'] & \mathrm{Hom}(B, \underline{\mathrm{Hom}}(X,Y))
\end{tikzcd} \tag{5.2}$$

and we call $\mathrm{ev}_{X,Y} : \mathrm{Hom}(T \otimes X, Y) \to Y$ the morphism corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}$ under the (natural) isomorphism $\eta_{\underline{\mathrm{Hom}}(X,Y)}$ (see diagram below)

$$\begin{tikzcd}
T \arrow[d, "f"] & \mathrm{Hom}(\underline{\mathrm{Hom}}(X,Y)\otimes X, Y) \arrow[r, "\eta_{\underline{\mathrm{Hom}}(X,Y)}", "\simeq"'] \arrow[d, "\mathrm{Hom}(f\otimes X{,}Y)"] & \mathrm{Hom}(\underline{\mathrm{Hom}}(X,Y), \underline{\mathrm{Hom}}(X,Y)) \arrow[d, "\mathrm{Hom}(f{,}\underline{\mathrm{Hom}}(X{,}Y))"] \\
\underline{\mathrm{Hom}}(X,Y) & \mathrm{Hom}(T\otimes X,Y) \arrow[r, "\eta_T", "\simeq"'] & \mathrm{Hom}(T,\underline{\mathrm{Hom}}(X,Y))
\end{tikzcd} \tag{5.3}$$

We see that

$$\mathrm{Hom}(T\otimes X,Y) \overset{\eta_T}{\simeq} \mathrm{Hom}(T,\underline{\mathrm{Hom}}(X,Y)). \tag{5.4}$$

If we assume that $\underline{\mathrm{Hom}}(X,Y)$ exists for every pair of objects X,Y in $\mathcal{C}$, then

$$\mathrm{Hom}(\underline{\mathrm{Hom}}(X,Y)\otimes \mathrm{Hom}(Y,Z)\otimes X, Z) \simeq \mathrm{Hom}(\underline{\mathrm{Hom}}(X,Y)\otimes \underline{\mathrm{Hom}}(Y,Z), \underline{\mathrm{Hom}}(X,Z)) \tag{5.5}$$

where one finds that

$$\underline{\mathrm{Hom}}(X,Y)\otimes\underline{\mathrm{Hom}}(Y,Z)\otimes X \xrightarrow{\phi_{\underline{\mathrm{Hom}}(X,Y),\underline{\mathrm{Hom}}(Y,Z)}\otimes \mathrm{id}_X} \underline{\mathrm{Hom}}(Y,Z)\otimes\underline{\mathrm{Hom}}(X,Y)\otimes X$$
$$\xrightarrow{\mathrm{id}_{\underline{\mathrm{Hom}}(Y,Z)}\otimes \mathrm{ev}_{X,Y}} \underline{\mathrm{Hom}}(Y,Z)\otimes Y \xrightarrow{\mathrm{ev}_{Y,Z}} Z.$$

To prove our next proposition, we need a lemma, known under the name “Yoneda lemma”.

Lemma 5.6. *Let $F : \mathcal{C} \to \mathbf{Set}$ be a any (covariant) functor from a* locally small *category $\mathcal{C}$, and let X be an object of $\mathcal{C}$. Then there is a bijection*

$$\mathbf{Nat}(\mathrm{Hom}_{\mathcal{C}}(X,-),F) \cong F(X).$$

Furthermore, natural transformations $\alpha : \mathrm{Hom}_{\mathcal{C}}(X,-) \to F$ correspond to $\alpha_X(1_X) \in F(X)$, and this correspondence is natural *in X and F.*

Proof. We start by proving the bijective property.

Bijection: Let $\phi : \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(A,-),F) \to F(A)$, where $\mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(A,-),F)$ is the class of (*natural transformations*) from $\mathrm{Hom}_{\mathcal{C}}(A,-)$ to F.

ϕ is defined explicitly by taking a natural transformation $\alpha : \mathrm{Hom}(A,-) \to F$ to $\alpha_A(1_A) \in F(A)$, i.e.

$$\alpha \xmapsto{\phi} \alpha_A(1_A).$$

This map is clearly right unique and left total, hence a function.

We want to find an inverse function $\Psi : F(A) \to \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(A,-),F)$ such that for $x \in F(A)$, we get a natural transformation $\Psi(x) : \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(A,-) \to F$.

It follows that we must define components $\Psi(x)_B : \mathrm{Hom}(\mathrm{Hom}(A,B)) \to F(B)$, so that for $f : A \to B \in \mathrm{Mor}(\mathcal{C})$, we get

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[r, "\Psi(x)_A"] \arrow[d, "\mathrm{Hom}(A,f)=f_*"] & F(A) \arrow[d, "F(f)"] \\
\mathrm{Hom}(A,B) \arrow[r, "\Psi(x)_B"] & F(B)
\end{tikzcd}$$

We find that $1_A \in \mathrm{Hom}(A,A)$ get's taken to $(\Psi(x)_B)(f)$ by going downward left and then right, while 1_A get's taken to $F(f) \circ (\Psi(x)_A)(1_A)$ via the right-downward route.

We see that we need to define $\Psi(x)_A(1_A) = x$, since we want

$$\begin{aligned}\Phi(\Psi(x)) &= \Psi(x)_A(1_A) \\ &= x.\end{aligned}$$

It follows, since we need naturality, that

$$F(f)(x) = \Psi(x)_B(f) \tag{5.6}$$

We conclude by showing that for a generic morphism $g : B \to D \in \mathrm{Mor}(\mathcal{C})$, $\Psi(x)$ is a natural transformation. I.e. showing that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(A,B) \arrow[r, "\Psi(x)_B"] \arrow[d, "\mathrm{Hom}(A,g)=g_*"] & F(B) \arrow[d, "F(g)"] \\
\mathrm{Hom}(A,D) \arrow[r, "\Psi(x)_D"] & F(D)
\end{tikzcd}$$

For that, let $f \in \mathrm{Hom}(A,B)$. Then along the right-downward path we get that f gets taken to

$$\begin{aligned}(F(g) \circ \Psi(x)_B)(f) &= F(g) \circ \Psi(x)_B(f) \\ &= F(g)(F(f)(x)).\end{aligned}$$

On the downward-right path, f gets taken to

$$\begin{aligned}(\Psi(x)_D \circ g)(f) &= \Psi(x)_D(gf) \\ &= F(gf)(x)\end{aligned}$$

using 5.6

Since F is a functor, we have $F(gf) = F(g) \circ F(f)$, so that the paths gives the same element in $F(D)$, i.e. the diagram commutes, for arbitrary morphism $g \in \mathrm{Hom}(B,D)$, where $B, D \in \mathcal{C}$ are arbitrary.

We have already seen that Ψ is a right-inverse to ϕ, and we want to show that Ψ is a left-inverse to ϕ; that is, that

$$\begin{aligned} \Psi\phi(\alpha) &= \Psi\alpha_A(1_A) \\ &= \alpha \end{aligned}$$

for a natural transformation $\alpha : \mathrm{Hom}(A, -) \to F$. From (5.6) (with $x = \alpha_A(1_A) \in F(A)$) have that

$$\Psi(\alpha_A(1_A))_B(f) = Ff(\alpha_A(1_A)) \tag{5.7}$$

Since α is a natural transformation, the following square commutes

$$\begin{tikzcd} A \arrow[d, "f"] & \mathrm{Hom}(A, A) \arrow[r, "\alpha_A"] \arrow[d, "\mathrm{Hom}(A,f)=f_*"] & F(A) \arrow[d, "F(f)"] \\ B & \mathrm{Hom}(A, B) \arrow[r, "\alpha_B"] & F(B) \end{tikzcd} \tag{5.8}$$

from which it follows that

$$\begin{aligned} F(f)(\alpha_A(1_A)) &= \alpha_B(f)(1_A) \\ &= \alpha_B(f) \end{aligned}$$

so that $\Psi(\alpha_A(1_A))_B(f) = \alpha_B(f)$ by (5.7).

This concludes the proof of the bijection $\mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(A, -), F) \cong F(A)$.

We prove *naturality*. Naturality: The assertion about naturality in X and F corresponds to the following claims:

1. Naturality in F means that, given a natural transformation $\beta : F \to G$, the element of $G(A)$ representing the composite natural transformation $\beta\alpha : \mathrm{Hom}(A, -) \to F \to G$ is the image under $\beta_A : F(A) \to G(A)$ of the element of $F(A)$ representing $\alpha : \mathrm{Hom}(A, -) \to F$, that is, the following diagram commutes

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{Hom}(A, -), F) \arrow[r, "\phi_F", "\cong"'] \arrow[d, "\beta_*"] & F(A) \arrow[d, "\beta_A"] \\ \mathrm{Hom}(\mathrm{Hom}(A, -), G) \arrow[r, "\phi_G", "\cong"'] & G(A) \end{tikzcd}$$

where we have $\phi_F : \mathrm{Hom}(A, -) \to F(A)$ and $\phi_G : \mathrm{Hom}(A, -) \to G(A)$ defined explicitly by

$$\phi_F(\alpha) = \alpha_A(1_A)$$

$$\phi_G(\beta \circ \alpha) = (\beta \circ \alpha)_A(1_A).$$

To show that $\beta_A(\phi_F(\alpha)) = \phi_G(\beta \circ \alpha)$ for $\alpha \in \mathrm{Hom}(\mathrm{Hom}(A,-),F)$, we will use

We note that we have parallel functors $\mathrm{Hom}(A,-), F, G : \mathcal{C} \rightrightarrows \mathbf{Set}$, and natural transformations $\alpha : \mathrm{Hom}(A,-) \to F$ and $\beta : F \to G$. This gives us that $(\beta \circ \alpha)_A(1_A) := (\beta_A \circ \alpha_A)(1_A)$ by 4.4.

That is, we have

$$\begin{aligned} \phi_G(\beta \circ \alpha) &= (\beta \circ \alpha)_A(1_A) \\ &= (\beta_A \circ \alpha_A)(1_A) \\ &= \beta_A(\phi_F(\alpha)) \end{aligned}$$

where the last equality follows from $\phi_F(\alpha) := \alpha_A(1_A)$ and associativity of morphisms.

2. "Naturality in X" amounts to the assertion that given $f : A \to B \in \mathrm{Mor}(\mathcal{C})$, the element of $F(B)$ representing the composite natural transformation $\alpha f^* : \mathrm{Hom}(B,-) \to \mathrm{Hom}(A,-) \to F$ is the image under $F(f) : F(A) \to F(B)$ of the element of $F(A)$ representing α, i.e. the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(\mathrm{Hom}(A,-),F) \arrow[r, "\phi_A", "\cong"'] \arrow[d, "(f^*)^*"] & F(A) \arrow[d, "F(f)"] \\
\mathrm{Hom}(\mathrm{Hom}(B,-),F) \arrow[r, "\phi_B", "\cong"'] & F(B)
\end{tikzcd} \tag{5.9}$$

We see that the image of α along the right-downward path is $F(f)\phi_A(\alpha) = F(f)(\alpha_A(1_A))$ and that the image of α along the downward-right path is

$$\begin{aligned} \phi_B((f^*)^* \circ \alpha) &= \phi_B(\alpha \circ f^*) \\ &= (\alpha \circ f^*)_B(1_B). \end{aligned}$$

We use the following lemma

Lemma 5.7. *Let $\mathcal{C}$ be a category, and let $f : A \to B \in \mathrm{Mor}(\mathcal{C})$. Then the pullback $f^* : \mathrm{Hom}(B,-) \to \mathrm{Hom}(A,-)$ is a natural transformation.*

Proof. We contemplate the following diagram

$$\begin{tikzcd}
X \arrow[d, "g"] & \mathrm{Hom}(B,X) \arrow[r, "f_X^*"] \arrow[d, "\mathrm{Hom}(B,g)=g_*"] & \mathrm{Hom}(A,X) \arrow[d, "\mathrm{Hom}(A,g)=g_*"] \\
Y & \mathrm{Hom}(B,Y) \arrow[r, "f_Y^*"] & \mathrm{Hom}(A,Y)
\end{tikzcd} \tag{5.10}$$

We want to show that for arbitrary $h \in \mathrm{Hom}(B,X)$ we have that $(g_* \circ f_X^*)(h) = (f_Y^* \circ g_*)(h)$. This should be clear by definition. □

It follows that αf^* as assumed, is a composition of natural transformations. We have parallel functors $\mathrm{Hom}(A,-), \mathrm{Hom}(B,-), F : \mathcal{C} \to \mathbf{Set}$, and natural transformations $f^* : \mathrm{Hom}(B,-) \to \mathrm{Hom}(A,-)$ and $\alpha : \mathrm{Hom}(A,-) \to F$. By 4.4, we have that

$$\begin{aligned}(\alpha f^*)_B(1_B) &= (\alpha_B \circ f^*_B)(1_B) \\ &= \alpha_B(f).\end{aligned}$$

From diagram (5.8) we then see that $\alpha_B(f) = F(f)(\alpha_A(1_A))$. It follows that

$$\begin{aligned}F(f)\phi_A(\alpha) &= F(f)(\alpha_A(1_A)) \\ &= \alpha_B(f) \\ &= (\alpha f^*)_B(1_B) \\ &= \phi_B((f^*)^* \circ \alpha)\end{aligned}$$

so that diagram (5.9) commutes.

□

There is a dual statement of the “Yoneda Lemma”:

Lemma 5.8. *Let $F : \mathcal{C} \to \mathbf{Set}$ be any (contravariant) functor from a* locally small category *$\mathcal{C}$, and let X be an object of $\mathcal{C}$. Then there is a bijection*

$$\mathbf{Nat}(\mathrm{Hom}_{\mathcal{C}}(-,X), F) \cong F(X).$$

Furthermore, natural transformations $\alpha : \mathrm{Hom}_{\mathcal{C}}(-,X) \to F$ correspond to $\alpha_X(1_X) \in F(X)$, and this correspondence is natural *in X and F.* □

A corollary of 5.6, called **Yoneda embedding**, follows. Before we give the corollary, we want to clarify what we mean by *embedding* (in this context).

Definition 5.9. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F is *faithful*, and *injective* on objects in $\mathcal{C}$, then we call F an **embedding**.

Lemma 5.10. *The functors y and y' below, define full and faithful embeddings*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, hook, "y"] & & \mathbf{Set}^{\mathcal{C}^{opp}} \\
A \arrow[rr, mapsto] \arrow[dd, "f"'] & & \mathrm{Hom}(-,A) \arrow[dd, "f_*"] \\
{} \arrow[rr, mapsto] & & {} \\
B \arrow[rr, mapsto] & & \mathrm{Hom}(-,B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[rr, hook, "y'"] & & \mathbf{Set}^{\mathcal{C}} \\
A \arrow[rr, mapsto] \arrow[dd, "f"'] & & \mathrm{Hom}(A,-) \\
{} \arrow[rr, mapsto] & & {} \\
B \arrow[rr, mapsto] & & \mathrm{Hom}(B,-) \arrow[uu, "f^*"']
\end{tikzcd}$$

Proof. To show that y, y' are *injections*, we use the following lemma:

Lemma 5.11. *Let $f, g : A \rightrightarrows B$ be parallel morphisms in a category $\mathcal{C}$. Then the induced natural transformations*

$$f_*, g_* : \mathrm{Hom}(-, A) \rightrightarrows \mathrm{Hom}(-, B)$$

and

$$f^*, g^* : \mathrm{Hom}(B, -) \rightrightarrows \mathrm{Hom}(A, -)$$

by post-and-pre composition, respectively, are distinct.

Proof. We have that $(f_*)_A(\mathrm{id}_A) := f \circ \mathrm{id}_A$ and

$$\begin{aligned} (g_*)_A(\mathrm{id}_A) &:= g \circ \mathrm{id}_A \\ &= g \end{aligned}$$

but since f, g are distinct

$$\begin{aligned} &\rightsquigarrow (f_*)_A(\mathrm{id}_A) \neq (g_*)_A(\mathrm{id}_A) \\ &\implies f_* \neq g_*. \end{aligned}$$

Similarly, we have that

$$\begin{aligned} &(f^*)_B(\mathrm{id}_B) \neq (g^*)_B(\mathrm{id}_B) \\ &\implies f^* \neq g^*. \end{aligned}$$

□

Using 5.11, we see that y, y' are injective.

For each pair of objects $A, B \in \mathcal{C}$, and natural transformation

$$\alpha_f : \mathrm{Hom}(A, -) \to \mathrm{Hom}(B, -),$$

we want to show that $\exists f \in \mathrm{Mor}(\mathcal{C})$ so that

$$\begin{aligned} y'(f) &= f^* \\ &= \alpha_f. \end{aligned}$$

From 5.6 we know that there is a bijection

$$\mathbf{Nat}(\mathrm{Hom}(A, -), F) \cong F(A).$$

Taking $F = \mathrm{Hom}(B, -)$, we see that there is a bijection

$$\mathbf{Nat}(\mathrm{Hom}(A, -), \mathrm{Hom}(B, -)) \cong \mathrm{Hom}(B, A)$$

Let $f : B \to A$ correspond to α_f. Then again, by 5.6, we know that $(\alpha_f)_A(\mathrm{id}_A) = f$.

We note that

$$f^* : \mathrm{Hom}(A, -) \Rightarrow \mathrm{Hom}(B, -) \in \mathbf{Nat}(\mathrm{Hom}(A, -), \mathrm{Hom}(B, -))$$

is such that

$$\begin{aligned} (f^*)_A(\mathrm{id}_A) &= \mathrm{id}_A \circ f \\ &= f. \end{aligned}$$

By the bijective property of 5.6, we need $\alpha_f = f^*$.

Similarly, for pair of objects $A, B \in C$, and natural transformation

$$\alpha_f : \mathrm{Hom}(-, A) \Rightarrow \mathrm{Hom}(B, -)$$

we want to show that there exists a morphism f in $\mathcal{C}$ so that

$$\begin{aligned} y(f) &= f_* \\ &= \alpha. \end{aligned}$$

By 5.6, and taking $F = \mathrm{Hom}(-, B)$, we have

$$\mathbf{Nat}(\mathrm{Hom}(-, A), \mathrm{Hom}(-, B)) \cong \mathrm{Hom}(A, B).$$

Hence to each morphism $f \in \mathrm{Hom}(A, B)$, there exists a unique natural transformation α_f so that $(\alpha_f)_A(\mathrm{id}_A) = f$.

We find that $f : A \to B$ is such that

$$f_* : \mathrm{Hom}(-, A) \Rightarrow \mathrm{Hom}(-, B) \in \mathbf{Nat}(\mathrm{Hom}(-, A), \mathrm{Hom}(-, B)).$$

Then

$$\begin{aligned} (f_*)_A(\mathrm{id}_A) &= f \circ \mathrm{id}_A \\ &= f. \end{aligned}$$

It follows that $f_* = \alpha_f$.

□

For the next proposition, we will introduce a further notion of a certain type of functor.

Definition 5.12. A functor $F : \mathcal{C} \to \mathcal{D}$ is called **conservative** if $F(f)$ being an isomorphism in $\mathcal{D}$ implies that f itself is an isomorphism in $\mathcal{C}$.

Proposition 5.13. *Assume that* $\underline{\mathrm{Hom}}(X, Y)$ *exists for every pair of objects* X, Y *in a tensor category* $(\mathcal{C}, \otimes)$*. Then* $\underline{\mathrm{Hom}}(Z \otimes X, Y) \cong \underline{\mathrm{Hom}}(Z, \underline{\mathrm{Hom}}(X, Y))$ *for all objects* X, Y, Z *in* $\mathcal{C}$*.*

Proof. We follow the structure of the proof in [11].

Let A, X, Y, Z be arbitrary objects of $\mathcal{C}$. Then we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(A, \underline{\mathrm{Hom}}(X \otimes Y, Z)) &\simeq \mathrm{Hom}_{\mathcal{C}}(A \otimes (X \otimes Y), Z) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}((A \otimes X) \otimes Y, Z) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(A \otimes X, \underline{\mathrm{Hom}}(Y, Z)) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(A, \underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y, Z)) \end{aligned}$$

where we have repeatedly used that $\mathrm{Hom}_{\mathcal{C}}(T \otimes X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(T, \underline{\mathrm{Hom}}(X, Y))$ and the *associator* ϕ for the third isomorphism (recall that ϕ is a natural isomorphism).

Since A was arbitrary, we have that for all objects A in $\mathcal{C}$, the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(A, \underline{\mathrm{Hom}}(X \otimes Y, Z)) \simeq \mathrm{Hom}_{\mathcal{C}}(A, \underline{\mathrm{Hom}}(X\, \underline{\mathrm{Hom}}(Y, Z)))$$

holds. It follows that

$$\mathrm{Hom}_{\mathcal{C}}(-, \underline{\mathrm{Hom}}(X \otimes Y, Z)) \simeq \mathrm{Hom}_{\mathcal{C}}(-, \underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y, Z))) \tag{5.11}$$

in $\mathbf{PSh}(\mathcal{C})$.

Recall that from 5.10 we get that the functor $F : \mathcal{C} \to \mathbf{PSh}(\mathcal{C})$ defined by $\mathcal{C} \ni A \longmapsto \mathrm{Hom}_{\mathcal{C}}(-, A) \in \mathbf{PSh}(\mathcal{C}) = [\mathcal{C}^{\mathrm{opp}}, \mathbf{Set}]$ is *fully faithful*.

Let's denote the natural isomorphism in (5.11) with α. By the fullness of F, we find that there exists an $f : X \to Y \in \mathrm{Mor}(\mathcal{C})$ so that $F(f) = \alpha$.

Lemma 5.14. *Fully faithful functors are conservative.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, and let $\alpha \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ for objects A, B in $\mathcal{C}$, be an isomorphism. Then there exists $\alpha^{-1} \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$. Since F is full, there are $f : A \to B$ and $f^{-1} : B \to A$ in $\mathrm{Mor}(\mathcal{C})$ so that $F(f) = \alpha$ and $F(f^{-1}) = \alpha^{-1}$. By functoriality of F we have

$$\begin{aligned} F(f^{-1} \circ f) &= F(f^{-1}) \circ F(f) \\ &= \alpha^{-1} \circ \alpha \\ &= \mathrm{id}_{F(A)} \end{aligned}$$

and

$$\begin{aligned} F(f \circ f^{-1}) &= F(f) \circ F(f^{-1}) \\ &= \alpha \circ \alpha^{-1} \\ &= \mathrm{id}_{F(B)} \,. \end{aligned}$$

By functoriality of F, we have that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(\mathrm{id}_B) = \mathrm{id}_{F(B)}$. By faithfulness of F, one finds that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$. It follows that f is an isomorphism. ☐

Using lemma 5.14, we find that $\underline{\mathrm{Hom}}(X \otimes Y, Z) \cong \underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y, Z))$. ☐

5.3 Duals

Let $(\mathcal{C}, \otimes)$ be a tensor category. We note that we have an isomorphism

$$\mathrm{Hom}(\mathbf{1}, \underline{\mathrm{Hom}}(X, Y)) \simeq \mathrm{Hom}(\mathbf{1} \otimes X, Y) \simeq \mathrm{Hom}(X, Y) \tag{5.12}$$

using 5.4 for the first natural isomorphism, and $l_X^{-1} : \mathbf{1} \otimes X \simeq X$ in the last isomorphism. We introduce *duals*.

Definition 5.15. Let X be an object in $\mathcal{C}$. We denote the **dual** of X as $X^\vee$, defined explicitly as $X^\vee = \underline{\mathrm{Hom}}(X, \mathbf{1})$.

Definition 5.16. There is an alternative characterization of the dual of an object $X \in \mathcal{C}$, for $(\mathcal{C}, \otimes)$ a tensor category: It is a pair $(Y, Y \otimes X \overset{\mathrm{ev}}{\to} \mathbf{1})$ with a morphism $\epsilon : \mathbf{1} \to X \otimes Y$ such that

$$X \overset{l_X}{\simeq} \mathbf{1} \otimes X \overset{\epsilon \otimes \mathrm{id}_X}{\longrightarrow} (X \otimes Y) \otimes X \overset{\phi^{-1}_{X,Y,X}}{\simeq} X \otimes (Y \otimes X) \overset{\mathrm{id}_X \otimes \mathrm{ev}}{\longrightarrow} X \otimes \mathbf{1} \overset{r_X^{-1}}{\simeq} X \tag{5.13}$$

and

$$Y \overset{r_Y}{\simeq} Y \otimes \mathbf{1} \xrightarrow{\mathrm{id}_Y \otimes \epsilon} Y \otimes (X \otimes Y) \overset{\phi^{-1}_{Y,X,Y}}{\simeq} (Y \otimes X) \otimes Y \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_Y} \mathbf{1} \otimes Y \overset{l_Y^{-1}}{\simeq} Y \tag{5.14}$$

are the identity maps id_X, and id_Y, respectively. (5.13) and (5.14) together are usually called the **zigzag-equations** or the **snake equations**.

We then have a natural map $\mathrm{ev}_X : \underline{\mathrm{Hom}}(X, \mathbf{1}) \otimes X \to \mathbf{1}$ or, more succinctly, $\mathrm{ev}_X : X^\vee \otimes X \to \mathbf{1}$.

From 5.4, we see that $\mathrm{Hom}(T, X^\vee) = \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, \mathbf{1})) \simeq \mathrm{Hom}(T \otimes X, \mathbf{1})$. Since T is an arbitrary object, we have a natural isomorphism

$$\gamma : \mathrm{Hom}(-, X^\vee) \simeq \mathrm{Hom}(- \otimes X, \mathbf{1}) \tag{5.15}$$

where we let F denote the functor $F : \mathrm{Hom}(- \otimes X, \mathbf{1}) : \mathcal{C}^{\mathrm{opp}} \to \mathbf{Set}$ (note that we are implicitly assuming that $\mathcal{C}$ is a *locally small* category here). By 5.6, we find that γ is uniquely determined by $\gamma_{X^\vee}(\mathrm{id}_{X^\vee}) = \mathrm{ev}_X$ (see 5.2).

If we let $\mathcal{C} = \mathcal{C}^{\mathrm{opp}}$ in 5.6, we see that $\mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(X^\vee, -) \cong \mathrm{Hom}_{\mathcal{C}}(-, X^\vee)$. This clarifies the concordance between what we wrote above, and our formulation of 5.6.

For a tensor category $(\mathcal{C}, \otimes)$, we let the functor $\mathrm{Hom}(- \otimes X, Y)$ act on morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(T, B)$ by $\mathrm{Hom}(- \otimes X, Y)(f) = \mathrm{Hom}(f \otimes X, Y) : \mathrm{Hom}(B \otimes X, Y) \to \mathrm{Hom}(T \otimes X, Y)$, defined by

$$\mathrm{Hom}(f \otimes X, Y)(g) = g \circ (f \otimes \mathrm{id}_X) : T \otimes X \to Y$$

for $g \in \mathrm{Hom}(B \otimes X, Y)$.

The following diagram shows us that

$$\begin{aligned} \gamma_T(f) &= F(f)(\mathrm{ev}_{X,Y}) \\ &= \mathrm{ev}_X \circ (f \otimes \mathrm{id}_X) \end{aligned} \tag{5.16}$$

$$\begin{tikzcd}
f & & & & \\
T \arrow[ddd] & \mathrm{Hom}(X^\vee, X^\vee) \arrow[rrr, "\gamma_{X^\vee}", "\cong"'] \arrow[ddd, "\mathrm{Hom}(f, X^\vee)"'] & & & F(X^\vee) \arrow[ddd, "F(f)"] \\
 & & \mathrm{id}_{X^\vee} \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{ev}_X \arrow[d, mapsto] & \\
 & & f \arrow[r, mapsto] & \gamma_T(f) = F(f)(\mathrm{ev}_X) & \\
X^\vee & \mathrm{Hom}(T, X^\vee) \arrow[rrr, "\gamma_T", "\cong"'] & & & F(T)
\end{tikzcd} \tag{5.17}$$

Since γ_T is an isomorphism, it follows that for every $g \in \mathrm{Hom}(T \otimes X, Y)$ we get a *unique* $f : T \to X^\vee$ so that 5.16 holds. We also see that this applies more generally, since instead of $X^\vee = \underline{\mathrm{Hom}}(X, \mathbf{1})$ we could have taken $\underline{\mathrm{Hom}}(X, Y)$ for arbitrary X, Y (atleast assuming we are in a tensor category where $\underline{\mathrm{Hom}}(X, Y)$ exists for all objects X, Y).

The following diagram (cf. diagram 1.6.1 in [10]) illustrates the more general situation:

$$\begin{tikzcd}
T \arrow[d, dotted, "f"] & T \otimes X \arrow[d, dotted, "f\otimes \mathrm{id}_X"] \arrow[dr, "g"] & \\
\underline{\mathrm{Hom}}(X,Y) & \underline{\mathrm{Hom}}(X,Y)\otimes X \arrow[r, "\mathrm{ev}_{X,Y}"] & Y
\end{tikzcd} \tag{5.18}$$

From this, we can make the map $X \longmapsto X^\vee$ into a *functor*:

Let $f : X \to Y \in \mathrm{Mor}(\mathcal{C})$. We note that $\mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f) : Y^\vee \otimes X \to \mathbf{1}$, hence $\mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f) \in F(Y^\vee)$.

Since $\gamma_{Y^\vee} : \mathrm{Hom}(Y^\vee, X^\vee) \cong \mathrm{Hom}(Y^\vee \otimes X, \mathbf{1})$ is an isomorphism, there is a *unique* map ${}^t f : Y^\vee \to X^\vee$ so that $\gamma_{Y^\vee}({}^t f) = \mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f)$. But recall that $\gamma_{Y^\vee}({}^t f) = \mathrm{ev}_X \circ ({}^t f \otimes \mathrm{id}_X)$

$$\rightsquigarrow \mathrm{ev}_X \circ ({}^t f \otimes \mathrm{id}_X) = \mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f). \tag{5.19}$$

Or, in diagrammatic form

$$\begin{tikzcd}
Y^\vee \otimes X \arrow[r, "{}^t f \otimes \mathrm{id}_X"] \arrow[d, "\mathrm{id}_{Y^\vee} \otimes f"'] & X^\vee \otimes X \arrow[d, "\mathrm{ev}_X"] \\
Y^\vee \otimes Y \arrow[r, "\mathrm{ev}_Y"'] & \mathbf{1}
\end{tikzcd} \tag{5.20}$$

Let $f : X \to Y$ be an isomorphism in a tensor category $(\mathcal{C}, \otimes)$ and assume that $X^\vee, Y^\vee$ exists. Then its image under our functor $(-)^\vee$ is an isomorphism, since functors preserve isomorphisms (1.15). Hence it follows that ${}^t f : Y^\vee \to X^\vee$ has an inverse $({}^t f)^{-1} : X^\vee \to Y^\vee$. We denote this inverse by $f^\vee := ({}^t f)^{-1} : X^\vee \to Y^\vee$.

By properties of the tensor product as a bifunctor, we find that

$$\begin{aligned}
\mathrm{ev}_X \circ ({}^t f \otimes \mathrm{id}_X) \circ (f^\vee \otimes \mathrm{id}_X) &= \mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f) \circ (f^\vee \otimes \mathrm{id}_X) \\
\implies \mathrm{ev}_X &= \mathrm{ev}_Y \circ (f^\vee \otimes f).
\end{aligned}$$

Remark 5.17. Note that

$$\begin{aligned}
\mathrm{ev}_X \circ ({}^t f \otimes \mathrm{id}_X) \circ (f^\vee \otimes \mathrm{id}_X) &= \mathrm{ev}_X \circ ({}^t f f^\vee \otimes \mathrm{id}_X) \\
&= \mathrm{ev}_X \circ (\mathrm{id}_{X^\vee} \otimes \mathrm{id}_X) \\
&= \mathrm{ev}_X .
\end{aligned}$$

by 5.15.

Remark 5.18. If $f : X \to Y$, then ${}^t f : Y^\vee \to X^\vee$ is explicitly defined as (see [4, chap. 2.4])

$${}^t f : Y^\vee \overset{\mathrm{id} \otimes \epsilon}{\to} Y^\vee \otimes X \otimes X^\vee \overset{\mathrm{id} \otimes f \otimes \mathrm{id}}{\to} Y^\vee \otimes Y \otimes X^\vee \overset{\mathrm{ev} \otimes \mathrm{id}}{\to} X^\vee \tag{5.21}$$

With the above remark in mind; if $f : X \to Y$ and $g : Y \to Z$, so that $g \circ f : X \to Z$ we have that

$$^t(g \circ f) = (\mathrm{ev}_Z \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (g \circ f) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \epsilon_X) : Z^\vee \to X^\vee,$$

We aim to show that this is equal to $^tf \circ {}^tg$: Consider the following diagram

$$\begin{tikzcd}
Z^\vee \arrow[r, "\mathrm{id}\otimes\epsilon_Y"] \arrow[dr, swap, "\mathrm{id}\otimes\epsilon_X"] & Z^\vee\otimes Y\otimes Y^\vee \arrow[r, "\mathrm{id}\otimes g\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\epsilon_X"] & Z^\vee\otimes Z\otimes Y^\vee \arrow[r, "\mathrm{ev}_Z\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}\otimes\mathrm{id}\otimes\epsilon_X"] & Y^\vee \arrow[r, "\mathrm{id}\otimes\epsilon_X"] & Y^\vee\otimes X\otimes X^\vee \arrow[r, "\mathrm{id}\otimes f\otimes\mathrm{id}"] & Y^\vee\otimes Y\otimes X^\vee \arrow[r, bend left, "\mathrm{ev}_Y\otimes\mathrm{id}"] & X^\vee \\
 & Z^\vee\otimes X\otimes X^\vee \arrow[r, swap, "\mathrm{id}\otimes\epsilon_Y\dots"] \arrow[dr, swap, "\mathrm{id}\otimes f\otimes\mathrm{id}"] & Z^\vee\otimes Y\otimes Y^\vee\otimes X\otimes X^\vee \arrow[r, swap, "\mathrm{id}\otimes g\ \otimes\dots"] \arrow[dr, "\dots f\otimes\mathrm{id}"] & Z^\vee\otimes Z\otimes Y^\vee\otimes X\otimes X^\vee \arrow[r, swap, "\dots f\otimes\mathrm{id}"] \arrow[ur, "\mathrm{ev}_Z\otimes\mathrm{id}\otimes\mathrm{id}"] & Z^\vee\otimes Z\otimes Y^\vee\otimes Y\otimes X^\vee \arrow[r, swap, "\dots\mathrm{ev}_Y\otimes\mathrm{id}"] \arrow[ur, "\mathrm{ev}_Z\otimes\mathrm{id}\dots"] & Z^\vee\otimes Z\otimes X^\vee \arrow[ur, "\mathrm{ev}_Z\otimes\mathrm{id}"] & \\
 & & Z^\vee\otimes Y\otimes X^\vee \arrow[r, "\mathrm{id}\otimes\epsilon_Y\otimes\mathrm{id}"] \arrow[rr, bend right, "\mathrm{id}"] & Z^\vee\otimes Y\otimes Y^\vee\otimes Y\otimes X^\vee \arrow[r, "\dots\mathrm{ev}_Y\otimes\mathrm{id}"] \arrow[ur, "\mathrm{id}\otimes g\dots"] & Z^\vee\otimes Y\otimes X^\vee \arrow[ur, "\mathrm{id}\otimes g\otimes\mathrm{id}"] & &
\end{tikzcd}$$

(5.22)

In the diagram above, we have *suppressed all canonical isomorphisms.* All subdiagrams except the lowermost commutes by bifunctoriality of $\otimes$, and the lowermost diagram is the a snake identity (5.16). To clarify what we mean by bifunctoriality, we can exhibit an expanded diagram for the uppermost, leftmost diagram as

$$\begin{tikzcd}
Z^\vee \arrow[r, "\simeq"] \arrow[d, "\simeq"] & Z^\vee\otimes\mathbf{1}\otimes\mathbf{1} \arrow[r, "\mathrm{id}\otimes\epsilon_Y\otimes\mathrm{id}"] & Z^\vee\otimes Y\otimes Y^\vee\otimes\mathbf{1} \arrow[r, "\mathrm{id}\otimes\mathrm{id}\otimes\epsilon_X"] & Z^\vee\otimes Y\otimes Y^\vee\otimes X\otimes X^\vee \arrow[d, equal] \\
Z^\vee\otimes\mathbf{1}\otimes\mathbf{1} \arrow[r, swap, "\mathrm{id}\otimes\mathrm{id}\otimes\epsilon_X"] & Z^\vee\otimes\mathbf{1}\otimes X\otimes X \arrow[rr, swap, "\mathrm{id}\otimes\epsilon_Y\otimes\mathrm{id}"] & & Z^\vee\otimes Y\otimes Y^\vee\otimes X\otimes X^\vee
\end{tikzcd}$$

which commutes by bifunctoriality.

By following the outermost perimeter in (5.22) , and using the snake identity, we see that

$$(\mathrm{ev}_Y\otimes\mathrm{id})\circ(\mathrm{id}\otimes f\otimes\mathrm{id})\circ(\mathrm{id}\otimes\epsilon_X)\circ(\mathrm{ev}_Z\otimes\mathrm{id})\circ(\mathrm{id}\otimes g\otimes\mathrm{id})\circ(\mathrm{id}\otimes\epsilon_Y) = (\mathrm{ev}_Z\otimes\mathrm{id})\circ(\mathrm{id}\otimes g\otimes\mathrm{id})\circ(\mathrm{id}\otimes f\otimes\mathrm{id})\circ(\mathrm{id}\otimes\epsilon_X)$$

$$\iff {}^tf\circ{}^tg = {}^t(g\circ f).$$

For arbitrary $X \in \mathcal{C}$, we have

$$\begin{aligned}
^t(\mathrm{id}_X) &= (\mathrm{ev}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\epsilon) \\
&= (\mathrm{ev}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\epsilon) \\
&= \mathrm{id}_{X^\vee}
\end{aligned}$$

again, by a snake identity (5.16).

We conclude that $(-)^\vee$ is indeed a *functor.*

We see that $\mathrm{ev}_X \circ \psi_{X,X^\vee} : X \otimes X^\vee \to \mathbf{1}$, where we remember that $\psi_{X,X^\vee}$ is the *commutator* with respect to $X, X^\vee$. We let $X^{\vee\vee} := (X^\vee)^\vee$. By (5.15) we then have a natural isomorphism

$$\gamma : \mathrm{Hom}(-, X^{\vee\vee}) \simeq \mathrm{Hom}(-\otimes X^\vee, \mathbf{1}) \tag{5.23}$$

which affords us with a bijection

$$\gamma_X : \mathrm{Hom}(X, X^{\vee\vee}) \cong \mathrm{Hom}(X \otimes X^{\vee}, \mathbf{1}). \tag{5.24}$$

Hence we have a *unique* map $i_X : X \to X^{\vee\vee}$ so that $\gamma_X(i_X) = \mathrm{ev}_X \circ \psi_{X,X^{\vee}}$.

We introduce a definition in relation to i_X, and a crucial definition related to *representable functors*.

Definition 5.19. Let $(\mathcal{C}, \otimes)$ be a tensor category. If $i_X : X \to X^{\vee\vee}$ (the map we induced from the canonical isomorphism (5.23)) is an *isomorphism*, then we call X **reflexive**.

Definition 5.20. Let $\mathcal{C}$ be a category and let A be an object in $\mathcal{C}$. A **universal property** of A is a *representable* functor F together with a **universal element** $X \in F(A)$ that together define a *natural isomorphism* $\mathrm{Hom}_{\mathcal{C}}(A, -) \simeq F$ or $\mathrm{Hom}_{\mathcal{C}}(-, A) \simeq F$, by 5.6.

Remark 5.21. F in 5.20 is *covariant* in the first case, and *contravariant* in the second.

Definition 5.22. Let $F : \mathcal{C} \to \mathcal{D}$ be a (covariant) functor, and let X be an object of $\mathcal{D}$, and A, A' be objects of $\mathcal{C}$. A **universal morphism from** X **to** F is a *unique* pair $(A, \mu : X \to F(A))$ that has the following **universal property** (5.20): For each morphism $f : X \to F(A')$, there exists a unique map $h : A \to A'$ in $\mathcal{C}$, so that the following (leftmost) diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[dr, "\forall f"'] & F(A) \arrow[d, dotted, "F(h)"'] & A \arrow[d, dotted, "\exists! h"] \\
 & F(A') & A'
\end{tikzcd} \tag{5.25}$$

Dually, a **universal morphism from** F **to** X is a *unique* pair $(A, \mu : F(A) \to X)$ that satisfies the following **universal property**: For each morphism $f' : F(A') \to X$, there exists a *unique* morphism $h' : A' \to A$ so that the following diagram commutes

$$\begin{tikzcd}
X & F(A) \arrow[l, "\mu"'] & A \\
 & F(A') \arrow[ul, "\forall f'"] \arrow[u, dotted, "F(h')"] & A' \arrow[u, dotted, "\exists! h'"']
\end{tikzcd} \tag{5.26}$$

Lemma 5.23. *Let* $(A, \mu : F(A) \to X)$ *and* $(A', \mu' : F(A') \to X)$ *both be universal morphism from* F *to* X*. Then there is a unique isomorphism* $k : A \to A'$ *so that* $\mu' \circ F(k) = \mu$*.*

We contemplate the following diagrams

$$\begin{tikzcd}
X & F(A) \arrow[l, "\mu"'] & A & X & F(A') \arrow[l, "\mu'"'] & A' \\
 & F(A') \arrow[ul, "\mu'"] \arrow[u, dotted, "F(k')"'] & A' \arrow[u, dotted, "\exists! k'"'] & & F(A) \arrow[ul, "\mu"] \arrow[u, dotted, "F(k)"'] & A \arrow[u, dotted, "\exists! k"']
\end{tikzcd}$$

We find that there are unique k', k so that

$$\mu \circ F(k') = \mu' \tag{5.27}$$

$$\mu' \circ F(k) = \mu. \tag{5.28}$$

It follows that

$$\begin{aligned} \mu \circ F(k') &= \mu' \\ \iff (\mu' \circ F(k)) \circ F(k') &= \mu \\ \iff \mu' \circ F(kk') &= \mu'. \end{aligned}$$

and

$$\begin{aligned} \mu' \circ F(k) &= \mu \\ \iff (\mu \circ F(k')) \circ F(k) &= \mu \\ \iff \mu \circ F(k'k) &= \mu. \end{aligned}$$

Again, by the universal property, we have

$$\begin{tikzcd}
X & F(A') \arrow[l, "\mu'"'] & A & X & F(A) \arrow[l, "\mu"'] & A' \\
& F(A') \arrow[ul, "\mu'"] \arrow[u, dotted, "F(h')"'] & A' \arrow[u, dotted, "\exists! h'"'] & & F(A) \arrow[ul, "\mu"] \arrow[u, dotted, "F(h)"'] & A \arrow[u, dotted, "\exists! h"']
\end{tikzcd}$$

Clearly $h' = \mathrm{id}'_A$ and $h = \mathrm{id}_A$. By uniqueness of h', h together with (5.27) and (5.28), we see that $k \circ k' = \mathrm{id}'_A$ and $k' \circ k = \mathrm{id}_A$ so that k is the unique isomorphism such that $\mu' \circ F(k) = \mu$.

Proposition 5.24. *Let $(\mathcal{C}, \otimes)$ be a tensor category, and let X be an object in $\mathcal{C}$ that has an inverse $(X^{-1}, \delta : X^{-1} \otimes X \overset{\cong}{\to} \mathbf{1})$. Then X is* reflexive, *and the map*

$$f : X^{-1} \to X^\vee$$

determined by δ, is an isomorphism.

Proof. By 5.8, we have an isomorphism

$$\mathbf{Nat}(\mathrm{Hom}(-, X^{-1}), \mathrm{Hom}(- \otimes X, \mathbf{1})) \cong \mathrm{Hom}(X^{-1} \otimes X, \mathbf{1}).$$

Since $\delta \in \mathrm{Hom}(X^{-1} \otimes X, \mathbf{1})$, there is a *unique* corresponding natural transformation

$$\alpha \in \mathbf{Nat}(\mathrm{Hom}(-, X^{-1}), \mathrm{Hom}(- \otimes X, \mathbf{1}))$$

.

Again, by 5.8, for each object $Y \in \mathcal{C}$, with $f : Y \to X^{-1}$, we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(X^{-1}, X^{-1}) \arrow[rrr, "\alpha_{X^{-1}}"] \arrow[ddd, "\mathrm{Hom}(f, X^{-1})"'] & & & \mathrm{Hom}(X^{-1} \otimes X, \mathbf{1}) \arrow[ddd, "\mathrm{Hom}(f \otimes X, \mathbf{1})"] \\
& \mathrm{id}_{X^{-1}} \arrow[r, mapsto] \arrow[d, mapsto] & \delta \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & \alpha_Y(f) = F(f)(\delta) & \\
\mathrm{Hom}(Y, X^{-1}) \arrow[rrr, "\alpha_Y"] & & & \mathrm{Hom}(Y \otimes X, \mathbf{1})
\end{tikzcd}$$

Similar to the proof of 4.13, we get a bijection

$$\mathrm{Hom}(Y, X^{-1}) \to \mathrm{Hom}(X \otimes X^{-1}, \mathbf{1})$$

defined by

$$\mathrm{Hom}(Y, X^{-1}) \ni f \longmapsto f \otimes \mathrm{id}_X \longmapsto \delta \circ (f \otimes \mathrm{id}_X) \in \mathrm{Hom}(X \otimes X^{-1}, \mathbf{1}).$$

It follows that α_Y is isomorphism, for each $Y \in \mathcal{C}$, so that α is a natural isomorphism

$$\mathrm{Hom}(-, X^{-1}) \simeq \mathrm{Hom}(- \otimes X, \mathbf{1}). \tag{5.29}$$

By (5.15), we also have

$$\mathrm{Hom}(-, X^{\vee}) \simeq \mathrm{Hom}(- \otimes X, \mathbf{1}).$$

Lemma 5.25. *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, and let $X, Y \in \mathcal{C}$. Then we have that*

$$F(X) \cong F(Y) \implies X \cong Y.$$

Proof. Let $f : F(X) \overset{\cong}{\to} F(Y)$ in $\mathcal{D}$. Therefore, since F is *full*, there exist a morphism $g : X \to Y$ so that $F(g) = f$. By 5.14, g itself must be an isomorphism. ☐

From this we see that both (X^{-1}, δ) and $(X^{\vee}, \mathrm{ev}_X)$ satisfy the same universal property (5.22). To be more precise, let $F = \mathrm{Hom}(- \otimes X, \mathbf{1})$ and let $A = X^{-1}, \mu' = \delta$ and let $X = \mathbf{1}$ in (5.26). Similarly, we can let $A = X^{\vee}$, $\mu' = \mathrm{ev}_X$ and $X = \mathbf{1}$.

By 5.23, there exists a unique isomorphism $\xi : X^{-1} \to X^{\vee}$ so that $\mathrm{ev}_X \circ (\xi \otimes \mathrm{id}_X) = \delta$.

(5.24) gives us the following relations, where the the diagram to the right commutes

$$\begin{tikzcd}
X \arrow[d, dotted, "i_X"'] & X \otimes \underline{\mathrm{Hom}}(X, \mathbf{1}) \arrow[d, dotted, "i_X \otimes \mathrm{id}_{\underline{\mathrm{Hom}}(X,\mathbf{1})}"'] \arrow[dr, "\mathrm{ev}_X \circ \psi_{X,X^{\vee}}"] & \\
\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(X, \mathbf{1}), \mathbf{1}) & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(X, \mathbf{1}), \mathbf{1}) \otimes \underline{\mathrm{Hom}}(X, \mathbf{1}) \arrow[r, "\mathrm{ev}_{X^{\vee}, \mathbf{1}}"] & \mathbf{1}
\end{tikzcd}$$

or with simpler notation, as

$$\begin{tikzcd}
X \arrow[d, dotted, "i_X"'] & & X \otimes X^\vee \arrow[d, dotted, "i_X \otimes \mathrm{id}_{X^\vee}"'] \arrow[rd, "\mathrm{ev}_X \circ \psi_{X,X^\vee}"] & \\
X^{\vee\vee} & & X^{\vee\vee} \otimes X^\vee \arrow[r, "\mathrm{ev}_{X^\vee,\mathbf{1}}"] & \mathbf{1}
\end{tikzcd}$$

We have natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}(-, X^{\vee\vee}) &\simeq \mathrm{Hom}(- \otimes X^\vee, \mathbf{1}) \\
&\simeq \mathrm{Hom}(- \otimes X^{-1}, \mathbf{1}) \\
&\simeq \mathrm{Hom}(-, (X^{-1})^{-1}).
\end{aligned}$$

In the last (natural) isomorphism, we used (5.29) with $X := X^{-1}$. By 5.10, it follows that $(X^{-1})^{-1} \cong X^{\vee\vee}$. We also know that $X \cong (X^{-1})^{-1} \implies X \cong X^{\vee\vee}$.

Hence we have

$$X \otimes X^\vee \xrightarrow{\mathrm{ev}_X \circ \psi_{X,X^\vee}} \mathbf{1}$$

$$X^{\vee\vee} \otimes X^\vee \xrightarrow{\mathrm{ev}_{X^\vee,\mathbf{1}}} \mathbf{1}$$

and natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}(-, X) &\simeq \mathrm{Hom}(-, X^{\vee\vee}) \\
&\simeq \mathrm{Hom}(- \otimes X^\vee, \mathbf{1}) \\
&\simeq \mathrm{Hom}(- \otimes X^{-1}, \mathbf{1}).
\end{aligned}$$

□

For any finite families of objects $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$, there is a morphism

$$\bigotimes_{i\in I} \underline{\mathrm{Hom}}(X_i, Y_i) \to \underline{\mathrm{Hom}}\left(\bigotimes_{i\in I} X_i, \bigotimes_{i\in I} Y_i\right) \tag{5.30}$$

If we take $T = \bigotimes_{i\in I} \underline{\mathrm{Hom}}(X_i, Y_i)$, $X = \bigotimes_{i\in I} X_i$ and $Y = \bigotimes_{i\in I} Y_i$ in diagram 1.6.1 in [10], we see that (5.30) is the unique morphism such that the diagram

$$\begin{tikzcd}
\left(\bigotimes_{i\in I}\underline{\mathrm{Hom}}(X_i,Y_i)\right)\otimes\left(\bigotimes_{i\in I}X_i\right) \arrow[d, "f\otimes\mathrm{id}_{\bigotimes_{i\in I}X_i}"'] \arrow[drr, "g"] & & \\
\underline{\mathrm{Hom}}\left(\bigotimes_{i\in I}X_i,\bigotimes_{i\in I}Y_i\right)\otimes\left(\bigotimes_{i\in I}X_i\right) \arrow[rr, "\otimes\,\mathrm{ev}_{Y_i}"] & & \bigotimes_{i\in I}Y_i
\end{tikzcd}$$

commutes.

Let $X_1 = X, X_2 = \mathbf{1} = Y_1$ and $Y_2 = Y$. By (5.30) we get

$$\underline{\mathrm{Hom}}(X,\mathbf{1})\otimes\underline{\mathrm{Hom}}(\mathbf{1},Y) = X^\vee\otimes\underline{\mathrm{Hom}}(\mathbf{1},Y)\cong X^\vee\otimes Y\to\underline{\mathrm{Hom}}(X\otimes\mathbf{1},\mathbf{1}\otimes Y)\cong\underline{\mathrm{Hom}}(X,Y) \tag{5.31}$$

Remark 5.26. Since

$$\begin{aligned}\mathrm{Hom}(-,Y) &\simeq \mathrm{Hom}(-\otimes\mathbf{1},Y)\\ &\simeq \mathrm{Hom}(-,\underline{\mathrm{Hom}}(\mathbf{1},Y))\end{aligned}$$

and the isomorphisms are all natural, we see that $\mathrm{Hom}(-,Y)\simeq\mathrm{Hom}(-,\underline{\mathrm{Hom}}(\mathbf{1},Y))$ in $\mathbf{PSh}(\mathcal{C})$. Recall that $X\longmapsto\mathrm{Hom}(-,X)$ is fully faithful by 5.10.

Using 5.25, we find that

$$\mathrm{Hom}(-,Y)\simeq\mathrm{Hom}(-,\underline{\mathrm{Hom}}(\mathbf{1},Y)) \implies Y\cong\underline{\mathrm{Hom}}(\mathbf{1},Y). \tag{5.32}$$

This explains the first isomorphism in (5.31).

If we take $Y_i\equiv\mathbf{1}$, for all $i\in I$, and use (5.30), we see that

$$\bigotimes_{i\in I}X_i^\vee = \bigotimes_{i\in I}\underline{\mathrm{Hom}}(X_i,\mathbf{1})\to\underline{\mathrm{Hom}}\left(\bigotimes_{i\in I}X_i,\mathbf{1}\right) = \left(\bigotimes_{i\in I}X_i\right)^\vee.$$

6 Rigid tensor categories

Definition 6.1. Let $(\mathcal{C},\otimes)$ be a tensor category. Then we call $(\mathcal{C},\otimes)$ a **rigid tensor category** if

1. $\underline{\mathrm{Hom}}(A,B)$ exists for all $A,B\in\mathcal{C}$.
2. Morphisms $\underline{\mathrm{Hom}}(A_1,B_1)\otimes\underline{\mathrm{Hom}}(A_2,B_2)\to\underline{\mathrm{Hom}}(A_1\otimes A_2,B_1\otimes B_2)$ are *isomorphisms* for all $A_1,A_2,B_1,B_2\in\mathcal{C}$.
3. All objects in $\mathcal{C}$ are *reflexive*.

If $(C, \otimes)$ is a *rigid tensor category*, so that the conditions in 6.1 are fulfilled, then the morphisms in (5.30) are all isomorphisms. This is clear from the fact that I is finite in 5.30. So, for example, let $I = \{1, 2, 3\}$. Then

$$\begin{aligned}(\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2)) \otimes \underline{\mathrm{Hom}}(X_3, Y_3) &\cong \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2) \otimes \underline{\mathrm{Hom}}(X_3, Y_3) \\ &\cong \underline{\mathrm{Hom}}(X_1 \otimes X_2 \otimes X_3, Y_1 \otimes Y_2 \otimes Y_3).\end{aligned}$$

The same idea applies more generally for any finite set I: Put an arbitrary ordering on I, so that $I = \{i_1, \ldots, i_n\}$. Let f_{i_j, i_ℓ} denote the isomorphism

$$\underline{\mathrm{Hom}}(X_{i_j}, X_{i_j}) \otimes \underline{\mathrm{Hom}}(X_{i_\ell}, Y_{i_\ell}) \cong \underline{\mathrm{Hom}}(X_{i_j} \otimes X_{i_\ell}, Y_{i_j} \otimes Y_{i_\ell})$$

and let $f_{i_1 \otimes \cdots \otimes i_\ell, i_{\ell+1}} \otimes \mathrm{id}$ denote the isomorphism

$$\underline{\mathrm{Hom}}(X_{i_1} \otimes \cdots \otimes X_{i_\ell}, Y_{i_1} \otimes \cdots \otimes Y_{i_\ell}) \otimes \bigotimes_{i \in I \setminus \{i_\ell, \ldots, i_n\}} \underline{\mathrm{Hom}}(X_i, Y_i)$$

$$\cong \underline{\mathrm{Hom}}(X_{i_1} \otimes \cdots \otimes X_{i_{\ell+1}}, Y_{i_1} \otimes \cdots \otimes Y_{i_{\ell+1}}) \otimes \bigotimes_{i \in I \setminus \{i_{\ell+1}, \ldots, i_n\}} \underline{\mathrm{Hom}}(X_i, Y_i).$$

If $\ell = n - 1$ then we define $f_{i_1 \otimes \cdots \otimes i_{n-1}, i_n}$ as the isomorphism

$$\left(\underline{\mathrm{Hom}}\left(\bigotimes_{i \in I \setminus \{i_n\}} X_i, \bigotimes_{i \in I \setminus \{i_n\}} Y_i\right) \otimes \underline{\mathrm{Hom}}(X_{i_n}, Y_{i_n})\right) \cong \underline{\mathrm{Hom}}\left(\bigotimes_{i \in I} X_i, \bigotimes_{i \in I} Y_i\right).$$

$$\begin{aligned}\rightsquigarrow \bigotimes \underline{\mathrm{Hom}}(X_i, Y_i) &\overset{f_{i_1, i_2} \otimes \mathrm{id}}{\cong} \underline{\mathrm{Hom}}(X_{i_1} \otimes X_{i_2}, Y_{i_1} \otimes Y_{i_2}) \otimes \bigotimes_{i \in I \setminus \{i_1\}} \underline{\mathrm{Hom}}(X_i, Y_i) \\ &\overset{f_{i_1 \otimes i_2, i_3} \otimes \mathrm{id}}{\cong} \underline{\mathrm{Hom}}(X_{i_1} \otimes X_{i_2} \otimes X_{i_3}, Y_{i_1} \otimes Y_{i_2} \otimes Y_{i_3}) \otimes \bigotimes_{i \in I \setminus \{i_1, i_2\}} \underline{\mathrm{Hom}}(X_i, Y_j) \\ &\vdots \\ &\overset{f_{i_1 \otimes \cdots \otimes i_{n-1}, i_n}}{\cong} \underline{\mathrm{Hom}}\left(\bigotimes_{i \in I} X_i, \bigotimes_{i \in I} Y_i\right)\end{aligned}$$

Example 6.2. Let G be a group and let $\Bbbk$ be a field. Then the category $\mathbf{Rep}_{\Bbbk}^{\mathrm{fd}}(G)$ of finite dimensional representations of G is *rigid*. One has that

$$V^\vee := \mathrm{Hom}_{\mathbf{Rep}_{\Bbbk}^{\mathrm{fd}}(G)}(V, \Bbbk)$$

for each object (V, ρ_V) in $\mathbf{Rep}_{\Bbbk}^{\mathrm{fd}}(G)$, and where we define $\rho_{V^\vee}(g) := (\rho_V(g)^{-1})^\vee$.

We have maps $\mathrm{ev}_V V \otimes V^\vee \to \Bbbk$ defined by $k \otimes \varphi \longmapsto \varphi(k)$, and $\mathrm{coev}_V : \Bbbk \to V^\vee \otimes V$ defined by $k \longmapsto k \cdot \sum_{i=1}^{n} v_i^* \otimes e_i$ where $\{e_1, \ldots, e_n\}$ is a basis for V and $\{v_1^*, \ldots, v_n^*\}$ is a basis for $V^\vee$, and $v_i^*(v_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.

For more details, see [12].

Lemma 6.3. *Any functor $F : \mathcal{C} \to \mathcal{D}$ that is full, faithful and essentially surjective, defines an equivalence of categories.*

Remark 6.4. Compare 6.3 with 4.7.

Remark 6.5. Note that in the proof of 6.3, we need to assume *Axiom of Choice.*

Proof. Assume that $F : \mathcal{C} \to \mathcal{D}$ is full, faithful and essentially surjective. Since F is essentially surjective, we know that for each $D \in \mathcal{D}$, the set A_D consisting of elements C' in $\mathcal{C}$ such that $F(D) \cong C'$ is *non*-empty. Then we can form the family of sets $(A_D)_{D\in\mathcal{D}}$. Assuming *Axiom of Choice* ([5, p. 909]), there exists a choice function $G : \mathrm{Ob}(\mathcal{D}) \to \bigcup\limits_{D\in\mathcal{D}} A_D$ such that $G(D) \in A_D \implies G(D)$ is such that $F(G(D)) \cong D$. By essential surjectivity, the choice-function also defines an isomorphism $\epsilon_D : F(G(D)) \cong D$. Let $\ell \in \mathrm{Hom}_{\mathcal{D}}(D, D')$ be arbitrary. Then by similar reasoning as in the proof of 4.8 there is a unique morphism f such that

$$\begin{tikzcd}
F(G(D)) \arrow[r, "\epsilon_D", "\cong"'] \arrow[d, dotted, "\exists! f"'] & D \arrow[d, "\ell"] \\
F(G(D')) \arrow[r, "\epsilon_{D'}", "\cong"'] & D'
\end{tikzcd}$$

commutes. Explicitly, we want $\ell \circ \epsilon_D = \epsilon_{D'} \circ f \implies f = \epsilon_{D'}^{-1} \circ \ell \circ \epsilon_D$. Since F is fully faithful, the map $\mathrm{Hom}_{\mathcal{C}}(G(D), G(D')) \to \mathrm{Hom}_{\mathcal{D}}(F(G(D)), F(G(D')))$ is bijective, so there is a *unique* morphism $G\ell \in \mathrm{Hom}_{\mathcal{C}}(G(D), G(D'))$ such that $F(G(\ell)) = f$.

From this, we see that the morphisms ϵ_D assemble into the components of a *natural transformation* $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.

We need to prove that G is *functorial* (remember, G was originally a *choice function*). We note that by the functoriality of F, one has $F\left(\mathrm{id}_{G(D)}\right) = \mathrm{id}_{FG(D)}$.

Then we see that for $\ell = \mathrm{id}_D$ one has

$$\begin{aligned}
\mathrm{id}_D \circ \epsilon_D &= \epsilon_D \circ F(G(\mathrm{id}_D)) \\
\implies \epsilon_D &= \epsilon_D \circ F(G(\mathrm{id}_D)) \\
\implies \mathrm{id}_D &= F(G(\mathrm{id}_D)).
\end{aligned}$$

But we also have

$$\begin{aligned}
\epsilon_D \circ F\left(\mathrm{id}_{G(D)}\right) &= \epsilon_D \circ \mathrm{id}_{F(G(D))} \\
&= \epsilon_D \\
&= 1_D \circ \epsilon_D
\end{aligned}$$

Diagrammatically, this is

$$\begin{tikzcd}
F(G(D)) \arrow[r, "\epsilon_D"] \arrow[d, dotted, "F(G(\mathrm{id}_D)) \text{ or } F\left(\mathrm{id}_{G(D)}\right)"'] & D \arrow[d, "\mathrm{id}_D"] \\
F(G(D)) \arrow[r, "\epsilon_D"] & D
\end{tikzcd}$$

commutes, for both $F(G(\mathrm{id}_D))$ and $F\left(\mathrm{id}_{G(D)}\right)$ By uniqueness of $G\ell$, we need $G(\mathrm{id}_D) = \mathrm{id}_{G(D)}$.

Let $\ell' : D' \to D''$. Then we know that $G(\ell' \circ \ell)$ is such that

$$\begin{tikzcd}
FG(D) \arrow[r, "\epsilon_D", "\cong"'] \arrow[d, "F(G(\ell'\circ \ell))"'] & D \arrow[d, "\ell'\circ \ell"] \\
F(G(D'')) \arrow[r, "\epsilon_{D''}", "\cong"'] & D''
\end{tikzcd}$$

commutes. We contemplate the following diagram

$$\begin{tikzcd}
F(G(D)) \arrow[r, "\epsilon_D", "\cong"'] \arrow[d, "F(G(\ell))"'] & D \arrow[d, "\ell"] \\
F(G(D')) \arrow[r, "\epsilon_{D'}", "\cong"'] \arrow[d, "F(G(\ell'))"'] & D' \arrow[d, "\ell'"] \\
F(G(D'')) \arrow[r, "\epsilon_{D''}", "\cong"'] & D''
\end{tikzcd}$$

We know that the upper and lower constituent squares of the diagram commutes. We find that

$$\begin{aligned}
\ell' \circ \ell \circ \epsilon_D &= \ell' \circ (\ell \circ \epsilon_D) \\
&= \ell' \circ (\epsilon_{D'} \circ F(G(\ell))) \\
&= (\ell' \circ \epsilon_{D'}) \circ F(G(\ell)) \\
&= (\epsilon_{D''} \circ F(G(\ell'))) \circ F(G(\ell)) \\
&= \epsilon_{D''} \circ (F(G(\ell')) \circ F(G(\ell))) \\
&= \epsilon_{D''} \circ F(G(\ell') \circ G(\ell)).
\end{aligned}$$

By uniqueness, we see that $G(\ell' \circ \ell) = G(\ell') \circ G(\ell)$. We conclude that G is indeed a functor.

Lemma 6.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor. Then F both **reflects** and **creates** isomorphisms.*

Definition 6.7. A functor $F : \mathcal{C} \to \mathcal{D}$ **reflects** isomorphisms if $F(f)$ is *conservative* (5.12).

Definition 6.8. A functor F **creates** isomorphisms if $F(X) \cong F(Y)$ in $\mathcal{D} \implies X \cong Y$ in $\mathcal{C}$.

Proof. For reflection, see the proof of 5.14. For creation, see proof of 5.25. ☐

By fullness and faithfulness of F, we see that F both *reflects* and *creates* isomorphisms. Then we can define $\eta_C : C \to GF(C)$ by finding isomorphisms $F\eta_C : F(C) \to FGF(C)$. Let $F\eta_C := \epsilon^{-1}_{F(C)}$. Then η_C will be defined as the *unique* morphism $\eta_C \in \mathrm{Hom}_{\mathcal{C}}(C, GF(C))$ such that its image under F is $\epsilon^{-1}_{F(C)}$. We see that it follows from 6.6 that η_C is an isomorphism.

Let $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$, and consider the following diagram

$$\begin{tikzcd}
F(C) \arrow[r, "F\eta_C"] \arrow[d, "F(f)"'] & FGF(C) \arrow[r, "\epsilon_{F(C)}"] \arrow[d, "FGF(f)"] & F(C) \arrow[d, "F(f)"] \\
F(C') \arrow[r, "F\eta_{C'}"] & FGF(C') \arrow[r, "\epsilon_{F(C')}"] & F(C')
\end{tikzcd}$$

We have

$$\begin{aligned}
F(f) \circ \epsilon_{F(C)} \circ F\eta_C &= F(f) \circ \left(\epsilon_{F(C)} \circ F\eta_C\right) \\
&= F(f) \circ \left(\epsilon_{F(C)} \circ \epsilon^{-1}_{F(C)}\right) \\
&= F(f) \circ \mathrm{id}_{F(C)} \\
&= F(f)
\end{aligned}$$

and

$$\begin{aligned}
\epsilon_{F(C')} \circ F\eta_{C'} \circ F(f) &= \left(\epsilon_{F(C')} \circ F\eta_{C'}\right) \circ F(f) \\
&= \left(\epsilon_{F(C')} \circ \epsilon^{-1}_{F(C')}\right) \circ F(f) \\
&= \mathrm{id}_{F(C')} \circ F(f) \\
&= F(f)
\end{aligned}$$

so that the outer rectangle commutes.

The right square commutes since ϵ is a natural transformation (applied to the morphism $F(f)$ in $\mathcal{D}$).

Then we see that

$$\begin{aligned}
F(f) \circ \epsilon_{F(C)} \circ F\eta_C &= \epsilon_{F(C')} \circ F\eta_{C'} \circ F(f) \\
&= \epsilon_{F(C')} \circ (FGF(f) \circ F\eta_C).
\end{aligned}$$

Lemma 6.9. *Let $\mathcal{C}$ be a category. If $f \in \mathrm{Mor}(\mathcal{C})$ is an isomorphism, then f is monic.*

Proof. Let $f : C \to C'$ with inverse $g : C' \to C$, and assume that $h, k : A \rightrightarrows C$, such that

$$f \circ h = f \circ k. \tag{6.1}$$

Applying g on the left side of both sides of (6.1), we see that

$$\begin{aligned}
& g \circ (f \circ h) = g \circ (f \circ k) \\
\iff & (g \circ f) \circ h = (g \circ f) \circ k \\
\iff & \mathrm{id}_C \circ h = \mathrm{id}_C \circ k \\
\iff & h = k.
\end{aligned}$$

□

Since $\epsilon_{F(C')}$ is an isomorphism, by 6.9, $\epsilon_{F(C')}$ is monic.

$$\begin{aligned}\epsilon_{F(C')} \circ (F\eta_{C'} \circ F(f)) = \epsilon_{F(C')} \circ (FGF(f) \circ F\eta_C) \\ \implies F\eta_{C'} \circ F(f) = FGF(f) \circ F\eta_C.\end{aligned}$$

Hence the left-hand square also commutes, and we have a commutative rectangle where all constituent squares commutes. Then faithfulness and functoriality of F gives us that

$$\begin{aligned}F\eta_{C'} \circ F(f) &= F(\eta_{C'} \circ f) \\ &= F(GF(f) \circ \eta_C) \\ &= FGF(f) \circ F\eta_C \\ \implies \eta_{C'} \circ f &= GF(f) \circ \eta_C.\end{aligned}$$

That is,

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[d, "f"'] & GF(C) \arrow[d, "GF(f)"] \\
C' \arrow[r, "\eta_{C'}"] & GF(C')
\end{tikzcd}$$

commutes, for all $C \in \mathcal{C}$. We conclude that $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ is a natural transformation.

As we saw that ϵ_D and η_C were isomorphisms for all $D \in \mathcal{D}$ and all $C \in \mathcal{C}$, we have that $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ are indeed natural isomorphisms. ☐

Proposition 6.10. *Let* $(\mathcal{C}, \otimes)$ *be a* rigid *tensor category. Then the (contravariant) functor* $F : \mathcal{C} \to \mathcal{C}$ *defined explicitly by* $X \mapsto X^\vee$ *for* $X \in \mathcal{C}$ *and* $f \mapsto {}^t f$ *for* $f \in \mathrm{Mor}(\mathcal{C})$ *is an equivalence of categories.*

Proof. Going back to the (contravariant) functor $(-)^\vee : \mathcal{C} \to \mathcal{C}$ covered in (5.19), (5.20), (**??**), defined by taking objects $X \in \mathcal{C}$ to their dual $X^\vee$, and morphisms $f : X \to Y$ to ${}^t f : Y^\vee \to X^\vee$, we see that in a rigid tensor category we have:

$$\begin{tikzcd}
X^{\vee\vee} \arrow[r, mapsto, "i_X^{-1}", "\cong"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
{}^t({}^t f) : X^{\vee\vee} \to Y^{\vee\vee} \arrow[r, mapsto] & f : X \to Y
\end{tikzcd}$$

We aim to show that $(-)^\vee$ yields an equivalence of categories. By 6.3 it is enough to show that $(-)^\vee$ is full, faithful and essentially surjective. Essentially surjective is clear, since for each object $X \in \mathcal{C}$, we know that $X^{\vee\vee} \cong X$. So for any $X \in \mathcal{C}$, we can choose $X^\vee$ such that $X^\vee \mapsto X^{\vee\vee} \cong X$. We aim to show that the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(Y^\vee, X^\vee)$ is a bijection.

By (5.32) and 6.1 we see that

$$\begin{aligned}X^\vee \otimes Y &\cong \underline{\mathrm{Hom}}(X, \mathbf{1}) \otimes \underline{\mathrm{Hom}}(1, Y) \\ &\cong \underline{\mathrm{Hom}}(X \otimes \mathbf{1}, \mathbf{1} \otimes Y) \\ &\cong \underline{\mathrm{Hom}}(X, Y).\end{aligned}$$

So

$$\begin{aligned}\underline{\mathrm{Hom}}(X,Y) &\cong X^\vee \otimes Y \\ &\cong X^\vee \otimes Y^{\vee\vee} \\ &\cong Y^{\vee\vee} \otimes X \\ &\cong \underline{\mathrm{Hom}}(Y^\vee, X^\vee)\end{aligned}$$

where we in the last isomorphism used (5.31).

By (5.12) we find

$$\begin{tikzcd}
\mathrm{Hom}(\mathbf{1}, \underline{\mathrm{Hom}}(X,Y)) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathbf{1}, \underline{\mathrm{Hom}}(Y^\vee, X^\vee)) \arrow[d, "\cong"] \\
\mathrm{Hom}(X,Y) \arrow[r, "\cong"] & \mathrm{Hom}(Y^\vee, X^\vee)
\end{tikzcd}$$

□

Definition 6.11. For any $X \in \mathcal{C}$, we have $f : \underline{\mathrm{Hom}}(X,X) \underbrace{\cong}_{(5.31)} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbf{1}$. Applying the functor $\mathrm{Hom}(\mathbf{1}, -)$ to f we get

$$\mathrm{Hom}(\mathbf{1}, f) := \mathrm{tr}_X : \mathrm{Hom}(\mathbf{1}, \underline{\mathrm{Hom}}(X,X)) \cong \mathrm{Hom}(X,X) \longrightarrow \mathrm{Hom}(\mathbf{1},\mathbf{1})$$

or in other notation $\mathrm{tr}_X : \mathrm{End}(X) \to \mathrm{End}(\mathbf{1})$. We call tr_X the **trace morphism**.

Definition 6.12. For any $X \in \mathcal{C}$, the **rank** of X, denoted $\mathrm{rank}(X)$, is defined as

$$\mathrm{rank}(X) := \mathrm{tr}_X(\mathrm{id}_X).$$

Remark 6.13. [10] warns us that 6.12 only makes sense in characteristic 0.

We have the following set of equations:

$$\begin{cases}\mathrm{tr}_{X\otimes X'}(f \otimes f') = \mathrm{tr}(f) \cdot \mathrm{tr}(f') \\ \mathrm{tr}_{\mathbf{1}}(f) = f\end{cases} \tag{6.2}$$

Applying (6.2) we have

$$\begin{aligned}\mathrm{rank}(X \otimes X') &= \mathrm{tr}_{X\otimes X'}(\mathrm{id}_X \otimes \mathrm{id}_{X'}) \\ &= \mathrm{tr}(\mathrm{id}_X) \cdot \mathrm{tr}(\mathrm{id}_{X'}) \\ &= \mathrm{rank}(X) \cdot \mathrm{rank}(X').\end{aligned}$$

and

$$\begin{aligned}\mathrm{rank}(\mathbf{1}) &= \mathrm{tr}_{\mathbf{1}}(\mathrm{id}_{\mathbf{1}}) \\ &= \mathrm{id}_{\mathbf{1}}\,.\end{aligned}$$

7 Tensor functors

We let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be tensor categories. Let's introduce a functor related to tensor categories:

Definition 7.1. A **tensor functor** $(\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ is a pair (F, c) where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, and $c_{-,-} : - \otimes - \to F(- \otimes -)$ is a *natural isomorphism.* (F, c) has the following properties:

- For all $X, Y, Z \in \mathcal{C}$, the following diagram commutes

$$\begin{tikzcd}
F(X) \otimes' (F(Y) \otimes F(Z)) \arrow[r, "\mathrm{id}_{F(X)} \otimes c_{F(X),F(Y)}"] \arrow[d, "\phi'_{F(X),F(Y),F(Z)}"] & F(X) \otimes' F(X \otimes Y) \arrow[r, "c_{F(X),F(X \otimes Y)}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\phi_{X,Y,Z})"] \\
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "c_{F(X),F(Y)} \otimes \mathrm{id}_{F(Y)}"] & F(X \otimes Y) \otimes' F(Z) \arrow[r, "c_{F(X \otimes Y),F(Z)}"] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

- For all $X, Y \in \mathcal{C}$

$$\begin{tikzcd}
F(X) \otimes' F(Y) \arrow[r, "c_{F(X),F(Y)}"] \arrow[d, "\psi'_{F(X),F(Y)}"] & F(X \otimes Y) \arrow[d, "F(\psi_{X,Y})"] \\
F(Y) \otimes' F(X) \arrow[r, "c_{F(Y),F(X)}"] & F(Y \otimes X)
\end{tikzcd}$$

  commutes.

- If (U, u) is an identity object of $(\mathcal{C}, \otimes)$, then $(F(U), F(u))$ is an identity object of $(\mathcal{C}', \otimes')$.

The above conditions are those given in [10]. We will add the following two conditions, coming from [7] (1.29, 1.30 on p. 47): The squares below must commute

$$\begin{tikzcd}
F(X) \otimes' \mathbf{1}' \arrow[r, "(r'_{F(X)})^{-1}"] \arrow[d, "\mathrm{id}_{F(X)} \otimes a'"] & F(X) \arrow[d, "F(r_X)"] & \mathbf{1}' \otimes' F(X) \arrow[r, "(l'_{F(X)})^{-1}"] \arrow[d, "a' \otimes \mathrm{id}_{F(X)}"] & F(X) \arrow[d, "F(l_X)"] \\
F(X) \otimes' F(\mathbf{1}) \arrow[r, "c"] & F(X \otimes \mathbf{1}) & F(\mathbf{1}) \otimes' F(X) \arrow[r, "c"] & F(\mathbf{1} \otimes X)
\end{tikzcd} \tag{7.1}$$

where a' is the canonical isomorphism coming from 2.23 applied to $F(\mathbf{1})$ and $\mathbf{1}'$ in $\mathcal{C}'$ (using the third condition in 7.1, i.e. that $F(\mathbf{1})$ is an identity object).

Together, these conditions gives us, for any every finite family of objects $(X_i)_{i \in I}$, an isomorphism

$$c : \bigotimes_{i \in I}{}' F(X_i) \overset{\cong}{\to} F\left(\bigotimes_{i \in I} X_i\right)$$

Furthermore, for any map $\alpha : I \to J$, where I, J is finite, the following diagram commutes

$$\begin{tikzcd}
\bigotimes'_{i\in I} F(X_i) \arrow[rr, "c"] \arrow[d, "\chi'(\alpha)"] & & F\left(\bigotimes_{i\in I} X_i\right) \arrow[d, "F(\chi(\alpha))"] \\
\bigotimes'_{j\in J}\left(\bigotimes'_{i\mapsto j} F(X_i)\right) \arrow[r, "c"] & \bigotimes'_{j\in J}\left(F\left(\bigotimes_{i\mapsto j} X_i\right)\right) \arrow[r, "c"] & F\left(\bigotimes_{j\in J}\left(\bigotimes_{i\mapsto j} X_i\right)\right)
\end{tikzcd}$$

(F, c) takes inverse objects to inverse objects. Let $X, Y \in \mathcal{C}$, if $\underline{\mathrm{Hom}}(X, Y)$ exists, then

$$F(\mathrm{ev}_{X,Y}) : F(\underline{\mathrm{Hom}}(X, Y)) \otimes' F(X) \to F(Y)$$

affords us with morphisms

$$F_{X,Y} : F(\underline{\mathrm{Hom}}(X, Y)) \to \underline{\mathrm{Hom}}(F(X), F(Y)).$$

To expand on this point: Recall that $\mathrm{Hom}(T \otimes X, Y) \simeq \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y))$. $\mathcal{C}'$ is a tensor category, so assuming $\underline{\mathrm{Hom}}(X, Y)$ and $\underline{\mathrm{Hom}}(FX, FY)$ exists, we have

$$\mathrm{Hom}_{\mathcal{C}'}(F(\underline{\mathrm{Hom}}(X, Y)) \otimes' FX, FY) \simeq \mathrm{Hom}_{\mathcal{C}'}(F(\underline{\mathrm{Hom}}(X, Y)), \underline{\mathrm{Hom}}(FX, FY)).$$

Then we see, that there is a *unique* morphism $F_{X,Y}$ corresponding to $F(\mathrm{ev}_{X,Y}) \circ c$ such that the rightmost diagram below commutes (cf. (5.18)).

$$\begin{tikzcd}
F(\underline{\mathrm{Hom}}(X, Y)) \arrow[dd, dotted, "F_{X,Y}"'] & & F(\underline{\mathrm{Hom}}(X, Y)) \otimes' FX \arrow[dd, dotted, "F_{X,Y}\otimes'\mathrm{id}_{FX}"'] \arrow[ddr, "F(\mathrm{ev}_{X,Y})\circ c"] & \\
& \leftrightsquigarrow & & \\
\underline{\mathrm{Hom}}(FX, FY) & & \underline{\mathrm{Hom}}(FX, FY) \otimes' FX \arrow[r, "\mathrm{ev}_{FX,FY}"] & FY
\end{tikzcd} \tag{7.2}$$

Letting $Y = \mathbf{1}$, we get

$$\begin{tikzcd}
F(\underline{\mathrm{Hom}}(X, \mathbf{1})) = F(X^\vee) \arrow[dd, dotted, "F_X"'] & & F(\underline{\mathrm{Hom}}(X, \mathbf{1})) \otimes' FX \arrow[dd, dotted, "F_X\otimes'\mathrm{id}_{FX}"'] \arrow[ddr, "F(\mathrm{ev}_{X,Y})\circ c"] & \\
& \leftrightsquigarrow & & \\
\underline{\mathrm{Hom}}(FX, F\mathbf{1}) = F(X)^\vee & & \underline{\mathrm{Hom}}(FX, F\mathbf{1}) \otimes' FX \arrow[r, "\mathrm{ev}_{FX}"] & F\mathbf{1}
\end{tikzcd} \tag{7.3}$$

where we have used the last condition in 7.1, i.e. that $F\mathbf{1}$ is an identity object of $\mathcal{C}'$.

Lemma 7.2. *Let* $(F, c) : (C, \otimes) \to (C', \otimes')$ *be a tensor functor of* rigid *tensor categories. Then* F preserves duals.

Proof. As in [10], we want to show that F *preserves duality*

$$\rightsquigarrow$$

$$\begin{tikzcd}
X \arrow[d, "l_X"] \arrow[ddddd, bend right, "\mathrm{id}_X"'] & & F(X) \arrow[d, "F(l_X)"] \arrow[ddddd, bend right, "F(\mathrm{id}_X)=\mathrm{id}_{F(X)}"'] \\
\mathbf{1} \otimes X \arrow[d, "\epsilon \otimes \mathrm{id}_X"] & & F(\mathbf{1} \otimes X) \arrow[d, "F(\epsilon \otimes \mathrm{id}_X)"] \\
(X \otimes Y) \otimes X \arrow[d, "\phi^{-1}_{X,Y,X}"] & & F((X \otimes Y) \otimes X \arrow[d, "F(\phi^{-1}_{X,Y,X})"] \\
X \otimes (Y \otimes X) \arrow[d, "\mathrm{id}_X \otimes \mathrm{ev}"] & & F(X \otimes (Y \otimes X)) \arrow[d, "F(\mathrm{id}_X \otimes \mathrm{ev})"] \\
X \otimes \mathbf{1} \arrow[d, "r_X^{-1}"] & & F(X \otimes \mathbf{1}) \arrow[d, "F(r_X^{-1})"] \\
X & & F(X)
\end{tikzcd}$$

(7.4)

We rewrite the right diagram above the following way.

⇝

$$\begin{tikzcd}
F(X) \arrow[rr, "l'_{F(X)}"] \arrow[dd, "F(l_X)"] \arrow[dddddddddd, bend right=40, "F(\mathrm{id}_X)=\mathrm{id}_{F(X)}"'] & & \mathbf{1}'\otimes' F(X) \arrow[dd, "a'\otimes'\mathrm{id}_{F(X)}"] & & \\
& (1) & & & \\
F(\mathbf{1}\otimes X) \arrow[dd, "F(\epsilon\otimes\mathrm{id}_X)"] & & F(\mathbf{1})\otimes' F(X) \arrow[ll, "c"'] \arrow[dd, "F(\epsilon)\otimes'\mathrm{id}_{F(X)}"] & & \\
& (2) & & & \\
F((X\otimes Y)\otimes X) \arrow[dd, "F(\phi^{-1}_{X,Y,X})"] & & F(X\otimes Y)\otimes' F(X) \arrow[ll, "c"] \arrow[r, "c^{-1}\otimes'\mathrm{id}_{F(X)}"] & (F(X)\otimes' F(Y))\otimes' F(X) \arrow[dr, "\phi'_{F(X),F(Y),F(X)}"] & \\
& & (3) & & F(X)\otimes'(F(Y)\otimes' F(X)) \arrow[dl, "\mathrm{id}_{F(X)}\otimes' c"] \\
F(X\otimes(Y\otimes X)) \arrow[dd, "F(\mathrm{id}_X\otimes\mathrm{ev})"] & & & F(X)\otimes' F(Y\otimes X) \arrow[lll, "c"] \arrow[dd, "\mathrm{id}_{F(X)}\otimes' F(\mathrm{ev})"] & \\
& & (4) & & \\
F(X\otimes\mathbf{1}) \arrow[dd, "F(r_X^{-1})"] & & & F(X)\otimes' F(\mathbf{1}) \arrow[lll, "c"] \arrow[dd, "\mathrm{id}_{F(X)}\otimes a'^{-1}"] & \\
& & (5) & & \\
F(X) & & & F(X)\otimes'\mathbf{1}' \arrow[lll, "(r'_{F(X)})^{-1}"] &
\end{tikzcd}
\tag{7.5}$$

We see that subdiagram (1) in (7.5) above, commutes by (7.1), (2) commutes by naturality of c, (3) commutes by the first condition in definition 7.1, (4) commutes by naturality of c, and (5) commutes by (7.1).

We let

$$\begin{aligned}
\epsilon'_{FX} &:= \mathbf{1}' \xrightarrow{a'} F(\mathbf{1}) \xrightarrow{F(\epsilon_X)} F(X\otimes Y) \xrightarrow{c^{-1}} F(X)\otimes' F(Y) \\
\mathrm{ev}'_{FX} &:= F(Y)\otimes' F(X) \xrightarrow{c} F(Y\otimes X) \xrightarrow{F(\mathrm{ev}_X)} F(\mathbf{1}) \xrightarrow{a'^{-1}} \mathbf{1}'.
\end{aligned}\tag{7.6}$$

Then we see that ϵ'_{FX} and ev'_{FX} are the corresponding duality data for $F(X)$, by following the rightmost outer circuit in (7.5), and using that

$$c^{-1}\otimes' \mathrm{id}_{F(X)} \circ F(\epsilon)\otimes' \mathrm{id}_{F(X)} \circ a' \otimes' \mathrm{id}_{F(X)} = \epsilon'_{FX}\otimes' \mathrm{id}_{F(X)}$$

and similarly with ev';

$$\mathrm{id}_{F(X)}\otimes' a'^{-1} \circ \mathrm{id}_{F(X)}\otimes' F(\mathrm{ev}) \circ \mathrm{id}_{F(X)}\otimes' c = \mathrm{id}_{F(X)}\otimes' \mathrm{ev}'_{FX}\,.$$

By similar reasoning, we see that the following diagram commutes

$$\begin{tikzcd}
F(Y) \arrow[r, "r'_{F(Y)}"] \arrow[d, "F(r_Y)"] \arrow[dr, phantom, "(1)"] \arrow[dddddd, bend right, "F(\mathrm{id}_Y)=\mathrm{id}_{F(Y)}"'] & F(Y)\otimes' \mathbf{1}' \arrow[d, "\mathrm{id}_{F(Y)}\otimes' a'"] & & \\
F(Y\otimes\mathbf{1}) \arrow[d, "F(\mathrm{id}_Y\otimes\epsilon)"] \arrow[dr, phantom, "(2)"] & F(Y)\otimes' F(\mathbf{1}) \arrow[l, "c"'] \arrow[d, "\mathrm{id}_{F(Y)}\otimes' F(\epsilon)"] & & \\
F(Y\otimes(X\otimes Y)) \arrow[dd, "F(\phi^{-1}_{Y,X,Y})"] & F(Y)\otimes' F(X\otimes Y) \arrow[l, "c"'] \arrow[r, "\mathrm{id}_{F(Y)}\otimes' c^{-1}"] \arrow[dd, phantom, "(3)"] & F(Y)\otimes'(F(X)\otimes' F(Y)) \arrow[dr, "\phi'_{F(Y),F(X),F(Y)}"] & \\
 & & & (F(Y)\otimes' F(X))\otimes' F(Y) \arrow[dll, "c\otimes\mathrm{id}_{F(Y)}"] \\
F((Y\otimes X)\otimes Y) \arrow[d, "F(\mathrm{ev}\otimes\mathrm{id}_Y)"] \arrow[dr, phantom, "(4)"] & F(Y\otimes X)\otimes' F(Y) \arrow[l, "c"'] \arrow[d, "F(\mathrm{ev})\otimes'\mathrm{id}_{F(Y)}"] & & \\
F(\mathbf{1}\otimes Y) \arrow[d, "F(l_Y^{-1})"] \arrow[dr, phantom, "(5)"] & F(\mathbf{1})\otimes' F(Y) \arrow[l, "c"'] \arrow[d, "(a')^{-1}\otimes'\mathrm{id}_{F(Y)}"] & & \\
F(Y) & \mathbf{1}'\otimes' F(Y) \arrow[l, "(l'_{F(Y)})^{-1}"'] & &
\end{tikzcd} \tag{7.7}$$

where (1) commutes by (7.1), (2) commutes by naturality, (3) commutes by the first condition in 7.1, (4) commutes by naturality, and (5) commutes by (7.1). Hence $(F(Y), \epsilon'_X, \mathrm{ev}'_X)$ is the duality data for the image of X under the tensor functor F (cf. 5.13, 5.14). □

Definition 7.3. Assuming that $\mathcal{C}, \mathcal{D}$ are *locally small* categories. Then an **adjunction** consists of a *pair* of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ such that

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \tag{7.8}$$

for each pair of objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. We call F **left adjoint** to G and G **right adjoint** to F, and we usually denote this by $F \dashv G$, or equivalently $G \vdash F$.

By the isomorphism (7.8), we get pairs of morphisms

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \ni f^{\sharp} \leftrightsquigarrow f^{\flat} \in \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \tag{7.9}$$

that we call **adjunct** or **transposes** of each other.

Diagrammatically, this is the assertion

$$\begin{tikzcd}
\mathcal{C}^{\text{opp}} \times \mathcal{D} \arrow[rr, bend left, "\mathrm{Hom}_{\mathcal{D}}(F(-),-)"] \arrow[rr, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,G(-))"'] & \Downarrow \cong & \mathbf{Set}
\end{tikzcd}$$

Following [13], we can say a bit more. *Naturality* in $\mathcal{D}$ amounts to the assertion that for *any* morphism $k : D \to D'$ in $\mathcal{D}$, the following diagram commutes in **Set**

$$\begin{tikzcd}
D \arrow[d, "k"] & {} \arrow[d, phantom, "\rightsquigarrow"] & \mathrm{Hom}_{\mathcal{D}}(F(X), D) \arrow[r, "\cong"'] \arrow[d, "k_*"'] & \mathrm{Hom}_{\mathcal{C}}(X, G(D)) \arrow[d, "G(k)_*"'] \\
D' & {} & \mathrm{Hom}_{\mathcal{D}}(F(X), D') \arrow[r, "\cong"'] & \mathrm{Hom}_{\mathcal{C}}(X, G(D'))
\end{tikzcd}$$

So for any $f^\sharp \in \mathrm{Hom}_{\mathcal{D}}(F(X), D)$ and $k : D \to D'$, we have that

$$\begin{aligned}
(k_* \circ f^\sharp)^\flat &= (k \circ f^\sharp)^\flat \\
&= G(k)_* \circ f^\flat \\
&= G(k) \circ f^\flat .
\end{aligned}$$

Naturality in $\mathcal{C}$, on the other hand, means that for each $h : X \to X'$, the diagram below commutes in **Set**

$$\begin{tikzcd}
X \arrow[d, "h"] & {} \arrow[d, phantom, "\rightsquigarrow"] & \mathrm{Hom}_{\mathcal{D}}(F(X'), D) \arrow[r, "\cong"'] \arrow[d, "F(h)^*"'] & \mathrm{Hom}_{\mathcal{C}}(X', G(D)) \arrow[d, "h^*"'] \\
X' & {} & \mathrm{Hom}_{\mathcal{D}}(F(X), D) \arrow[r, "\cong"'] & \mathrm{Hom}_{\mathcal{C}}(X, G(D'))
\end{tikzcd}$$

What this means, is that for any $h : X \to X'$ and $f^\sharp \in \mathrm{Hom}_{\mathcal{D}}(F(X'), D)$, we have

$$\begin{aligned}
(F(h)^* \circ f^\sharp)^\flat &= (f^\sharp \circ F(h))^\flat \\
&= h^* \circ f^\flat \\
&= f^\flat \circ h .
\end{aligned}$$

Example 7.4. In particular, $- \otimes X \dashv \underline{\mathrm{Hom}}(X, -)$ is an adjunction, in a rigid tensor category $(\mathcal{C}, \otimes)$.

Similar to as in [13], we can pictorially represent an adjunction $F \dashv G$ as

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=2, "G"]
\end{tikzcd} \qquad \mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd) \qquad .$$

If we fix $c \in \mathcal{C}$, we get that Fc *represents* the (covariant) functor $\mathrm{Hom}_{\mathcal{C}}(c, G(-)) : \mathcal{D} \to \mathbf{Set}$ (since the latter is naturally isomorphic to $\mathrm{Hom}_{\mathcal{D}}(Fc, -)$). Then, by Yoneda lemma (5.6), we see that

$$\mathbf{Nat}(\mathrm{Hom}_{\mathcal{D}}(Fc, -), \mathrm{Hom}_{\mathcal{C}}(c, G(-))) \cong \mathrm{Hom}_{\mathcal{C}}(c, G(Fc))$$

gives that the natural isomorphism $\alpha : \mathrm{Hom}_{\mathcal{D}}(Fc, -) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(c, G(-))$ corresponds to $\alpha_{Fc}(\mathrm{id}_{Fc}) := \eta_c$. This motivates the following.

Lemma 7.5. *Let $F \dashv G$ be an adjunction; then there is a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$, called the* ***unit*** *of the adjunction, with components $\eta_X : X \to GF(X)$, defined to be the transpose of the identity morphism $\mathrm{id}_{F(X)}$.*

Proof. Naturality of η:

Consider the diagrams

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & GF(X) \arrow[d, "GF(f)"] & F(X) \arrow[r, "\mathrm{id}_{F(X)}"] \arrow[d, "F(f)"'] & F(X) \arrow[d, "F(f)"] \\
Y \arrow[r, "\eta_Y"] & GF(Y) & F(Y) \arrow[r, "\mathrm{id}_{F(Y)}"] & F(Y)
\end{tikzcd} \tag{7.10}$$

That the right hand diagram commutes follows trivially; For the left hand diagram, we use the following lemma.

Lemma 7.6. *Consider a pair of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ with isomorphisms*

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \qquad (\forall X \in \mathcal{C}, \forall Y \in \mathcal{D}). \tag{7.11}$$

Then we have that: If naturality of the families of isomorphisms in (7.11) holds $\Longleftrightarrow$ for any morphisms $h : c \to c' \in \mathrm{Mor}(\mathcal{C})$ and $k : d \to d' \in \mathrm{Mor}(\mathcal{D})$, the right-hand diagram below commutes if and only if the left-hand diagram below commutes

$$\begin{tikzcd}
Fc \arrow[r, "f^\sharp"] \arrow[d, "Fh"'] & d \arrow[d, "k"] & & c \arrow[r, "f^\flat"] \arrow[d, "h"'] & Gd \arrow[d, "Gk"] \\
Fc' \arrow[r, "g^\sharp"'] & d' \arrow[r, phantom, "\leftrightsquigarrow"] & & c' \arrow[r, "g^\flat"'] & Gd'
\end{tikzcd}$$

Proof. We will only prove $\Longrightarrow$, since this is what we need.

$\underline{\Longrightarrow}$: Assume that the isomorphisms in (7.11) are natural, and furthermore, assume that the left-hand diagram above commutes.

Then we have

$$g^\sharp \circ Fh = k \circ f^\flat. \tag{7.12}$$

Then, by naturality, (7.12) and reasoning as in 7.8, we see that

$$\begin{aligned}
Gk \circ f^\flat &= (k \circ f^\sharp)^\flat \\
&= (g^\sharp \circ Fh)^\flat \\
&= g^\flat \circ h,
\end{aligned}$$

so that the right-hand square commutes.

On the other hand, if the right-hand square commutes, we have

$$Gk \circ f^{\flat} = g^{\flat} \circ h. \tag{7.13}$$

Then, we see that

$$\begin{aligned}(k \circ f^{\sharp})^{\flat} &= Gk \circ f^{\flat} \\ &= g^{\flat} \circ h \\ &= (g^{\sharp} \circ Fh)^{\flat}.\end{aligned}$$

But $(-)^{\flat}$ is bijective, hence $k \circ f^{\sharp} = g^{\sharp} \circ Fh$. ☐

It follows from 7.6, by the way η:s components was choosen, and the obvious commutativity of the right diagram above, that the left-diagram commutes, so that $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow FG$ is *natural.* ☐

Dually, given an adjunction $F \dashv G$, one can, by fixing $d \in \mathcal{D}$, and using the contravariant Yoneda lemma (5.8) find that the object $Gd \in \mathcal{C}$ *represents* the functor $\mathrm{Hom}_{\mathcal{D}}(F(-), d) : \mathcal{C}^{\mathrm{opp}} \to \mathbf{Set}$, and that the natural isomorphism $\beta : \mathrm{Hom}_{\mathcal{C}}(F(-), d) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(-, Gd)$ corresponds to an element $\beta_{Gd}(\mathrm{id}_{Gd}) := \varepsilon_d$. By dualizing 7.5, we get that the ε_d gives us a family of morphisms who assemble into the components of a natural transformation $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.

We state the dual lemma.

Lemma 7.7. *Let $F \dashv G$ be an adjunction. Then there is a natural transformation $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$, called the* ***counit*** *of the adjunction, such that $\varepsilon_X : FG(X) \to X$ is defined to be the transpose of the identity $\mathrm{id}_{G(X)}$.*

Lemma 7.8. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=2, "G"] \end{tikzcd}$$

be an adjunction, with unit η and counit ε. Then the following diagrams commute, in the functor category $[\mathcal{C}, \mathcal{D}]$ (leftmost diagram) and $[\mathcal{D}, \mathcal{C}]$ (rightmost diagram), respectively.

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon F"] & & G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, Rightarrow, "\mathrm{id}_G"'] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\ & F & & & G \end{tikzcd} \tag{7.14}$$

Remark 7.9. By εF we mean, that for each $X \in \mathcal{C}$, we have $\varepsilon FX = \epsilon_{FX}$, and similarly for ηG.

Definition 7.10. We call the identities in (7.14) the **triangle identities for adjunctions**.

Proposition 7.11. *If we have*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=2, "G"] & \text{and} & \mathcal{C} \arrow[r, shift left=2, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left=2, "G"] \end{tikzcd}$$

so that F, F' are left adjoint to G, then $F \cong F'$, and there exists a unique natural isomorphism $\theta : F \Rightarrow F'$ such that both diagrams below commutes in their respective functor categories , $[\mathcal{C}, \mathcal{D}]$ and $[\mathcal{D}, \mathcal{C}]$, for the left and right diagram below, respectively

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{C}} \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta'"'] & GF \arrow[d, Rightarrow, "G\theta"] & & FG \arrow[r, Rightarrow, "\varepsilon"] \arrow[d, Rightarrow, "\theta G"'] & \mathrm{id}_{\mathcal{D}} \\
& GF' & & F'G \arrow[ur, Rightarrow, "\varepsilon'"'] & .
\end{tikzcd}$$

Proof. [13, proposition 4.4.1, p. 132]. □

Remark 7.12. By dualizing 7.11, we get a dual lemma, where we have two right adjoints F, F', and one left adjoint G, such that the following pair of diagrams commutes

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{C}} & GF \arrow[l, Rightarrow, "\eta"'] & & FG & \mathrm{id}_{\mathcal{D}} \arrow[l, Rightarrow, "\epsilon"'] \arrow[dl, Rightarrow, "\epsilon'"] \\
& GF' \arrow[ul, Rightarrow, "\eta'"] \arrow[u, Rightarrow, "G\theta"'] & & F'G \arrow[u, Rightarrow, "\theta G"] & .
\end{tikzcd}$$

We see that η, η' corresponds to the two counits.

We introduce an equivalent definition of an *internal hom* object.

Definition 7.13. Let $(\mathcal{C}, \otimes)$ be a tensor category. An **internal hom** in $\mathcal{C}$ is a functor

$$\underline{\mathrm{Hom}}(-,-) : \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \to \mathcal{C}$$

such that $\forall X \in \mathcal{C}$, we have a pair of *adjoint functors*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left=2, "-\otimes X"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left=2, "\underline{\mathrm{Hom}}(X,-)"] .
\end{tikzcd}$$

Proposition 7.14. *Let $(F, c) : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ be a tensor functor (7.1) of* rigid tensor categories *(6.1). Then*

$$F_{X,Y} : F(\underline{\mathrm{Hom}}(X, Y)) \to \underline{\mathrm{Hom}}(F(X), F(Y)) \tag{7.15}$$

is an isomorphism for all $X, Y \in \mathcal{C}$.

Remark 7.15. Note that there is a typo in (7.15) above, in [10, proposition 1.9, page 11].

Proof. We proceed similarly to [2].

Lemma 7.16. *If $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ such that $F \dashv G$, where $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ are the counit and unit respectively, then the leftmost diagram below commutes $\iff$ the rightmost diagram below commute.*

$$\begin{tikzcd}
A \arrow[r, "f^\flat"] & GD & & FA \arrow[r, "f^\sharp"] & D \\
B \arrow[u, "b"] \arrow[r, "c"'] & C \arrow[u, "g^\flat"'] & \leftrightsquigarrow & FB \arrow[u, "Fb"] \arrow[r, "Fc"'] & FC \arrow[u, "g^\sharp"']
\end{tikzcd}$$

Proof. Assume that the leftmost diagram commutes. Since functors preserve commutative diagrams, we get the leftmost diagram below

$$\begin{tikzcd}
FA \arrow[r, "Ff^\flat"] & FGD & & FA \arrow[r, "Ff^\flat"] \arrow[rr, bend left, "f^\sharp"] & FGD \arrow[r, "\varepsilon_D"] & D \\
FB \arrow[u, "Fb"] \arrow[r, "Fc"'] & Fc \arrow[u, "Fg^\flat"'] & \rightsquigarrow & FB \arrow[u, "Fb"] \arrow[r, "Fc"'] & Fc \arrow[u, "Fg^\flat"'] \arrow[ur, "g^\sharp"'] &
\end{tikzcd}$$

leading to the rightmost diagram above, which also commutes, since by definition, we have $f^\sharp := \varepsilon_D \circ Fg^\flat$ (and similarly for $g^\sharp$), and since we know that

$$\begin{aligned}
& Ff^\flat \circ Fb = Fg^\flat \circ Fc \\
\iff \; & \varepsilon_D \circ Ff^\flat \circ Fb = \varepsilon_D \circ Fg^\flat \circ Fc \\
\iff \; & f^\sharp \circ Fb = g^\sharp \circ Fc,
\end{aligned}$$

which is what we wanted to show.

On the other hand, if the rightmost diagram commutes, by applying the functor G to the rightmost diagram in 7.16, we get the leftmost diagram below, leading to the rightmost diagram below

$$\begin{tikzcd}
GFA \arrow[r, "Gf^\sharp"] & GD & & A \arrow[r, "\eta_A"] \arrow[rr, bend left, "f^\flat"] \arrow[dr, phantom, "(1)"] & GFA \arrow[r, "Gf^\sharp"] & GD \\
GFB \arrow[u, "GFb"] \arrow[r, "GFc"'] & GFC \arrow[u, "Gg^\sharp"'] & \rightsquigarrow & B \arrow[u, "b"] \arrow[r, "\eta_B"'] \arrow[d, "c"'] \arrow[dr, phantom, "(2)"] & GFB \arrow[u, "GFb"'] \arrow[r, "GFc"'] \arrow[d, "GFc"] & GFC \arrow[u, "Gg^\sharp"'] \\
& & & C \arrow[r, "\eta_C"'] & GFC \arrow[ur, "\mathrm{id}_{GFC}"'] &
\end{tikzcd}$$

where the rightmost diagram commutes since G preserves commutative diagrams, basic properties of categories, and naturality of η (in (1) and (2)). We have also used that $f^\flat = Gf^\sharp \circ \eta_A$ (and similarly for $g^\flat$).

By following the down-right path from B, we see that $g^\flat \circ c = f^\flat \circ b$, which is what we wanted to show. $\square$

Applying 7.16 to the adjunction $- \otimes FX \dashv \underline{\mathrm{Hom}}(FX, -)$ and the diagram to the left below, we get the diagram to the right below

$$\begin{tikzcd}
F(\underline{\mathrm{Hom}}(X,Y)) \arrow[r, "F_{X,Y}"] & \underline{\mathrm{Hom}}(FX,FY) \\
F(X^\vee \otimes Y) \arrow[u, "\alpha"] \arrow[r, "\gamma"'] & F(X)^\vee \otimes' FY \arrow[u, "\beta"']
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
F(\underline{\mathrm{Hom}}(X,Y)) \otimes' FX \arrow[r, "F(\mathrm{ev}_{X,Y})\circ c"] & FY \\
F(X^\vee \otimes Y) \otimes FX \arrow[u, "\alpha\otimes\mathrm{id}_{FX}"] \arrow[r, "\gamma\otimes\mathrm{id}_{FX}"'] & F(X)^\vee \otimes' F(Y) \otimes' FX \arrow[u, "\mathrm{ev}^{2'}_{FX,FY}"']
\end{tikzcd}
\tag{7.16}$$

We will utilize the following set of natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}'}(- \otimes FX, FY) &\simeq \mathrm{Hom}_{\mathcal{C}'}(-, \underline{\mathrm{Hom}}(FX, FY)) \\
&\simeq \mathrm{Hom}_{\mathcal{C}'}(-, F(X)^\vee \otimes' FY) \\
&\simeq \mathrm{Hom}_{\mathcal{C}'}(-, F(X^\vee) \otimes' FY) \\
&\simeq \mathrm{Hom}_{\mathcal{C}'}(-, F(X^\vee \otimes Y)).
\end{aligned}$$

By 7.12 we find that there is a unique such β so that the diagram below commutes, and that β is an isomorphism.

$$\begin{tikzcd}
F(X)^\vee \otimes' FY \otimes' FX \arrow[d, dashed, "\beta\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{ev}^{2'}_{FX,FY}"] & \\
\underline{\mathrm{Hom}}(FX,FY) \otimes' FX \arrow[r, "\mathrm{ev}^{1'}_{FX,FY}"'] & FY
\end{tikzcd}
\tag{7.17}$$

We choose γ as the *unique* morphism making the following diagram commute:

$$\begin{tikzcd}
F(X^\vee \otimes Y) \arrow[d, dashed, "\gamma"'] \\
F(X)^\vee \otimes' FY
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X^\vee \otimes Y) \otimes' FX \arrow[d, dashed, "\gamma\otimes\mathrm{id}"'] \arrow[dr, "F(\mathrm{ev}^1_{X,Y})\circ c"] & \\
F(X)^\vee \otimes' FY \otimes' FX \arrow[r, "\mathrm{ev}^{2'}_{FX,FY}"'] & FY
\end{tikzcd}
\tag{7.18}$$

coming from 7.12. It follows that γ is an isomorphism.

α in 7.16 was choosen to be the image under F of the unique isomorphism b (again, by 7.12, but now in $\mathcal{C}$) such that the diagram below commutes

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(X,Y)\otimes X \arrow[r, "\mathrm{ev}_{X,Y}"] & Y \\
X^\vee\otimes Y\otimes X \arrow[u, "b\otimes\mathrm{id}"] \arrow[ur, "\mathrm{ev}^1_{X,Y}"'] &
\end{tikzcd}$$

Since F preserves isomorphisms, $F(b)=\alpha$ is an isomorphism. Applying F to the commutative diagram above, we again get a commutative diagram

$$\begin{tikzcd}
F(\underline{\mathrm{Hom}}(X,Y)\otimes X) \arrow[r, "F(\mathrm{ev}_{X,Y})"'] & FY \\
F(X^\vee\otimes Y\otimes X) \arrow[u, "F(b\otimes\mathrm{id})"'] \arrow[ur, "F(\mathrm{ev}^1_{X,Y})"'] &
\end{tikzcd}$$

By naturality of c, we also get the following diagram

$$\begin{tikzcd}
F(X^\vee\otimes Y)\otimes' FX \arrow[r, "c"] \arrow[d, "\alpha\otimes\mathrm{id}"'] & F(X^\vee\otimes' Y\otimes X) \arrow[d, "F(b\otimes\mathrm{id})"] \\
F(\underline{\mathrm{Hom}}(X,Y))\otimes' FX \arrow[r, "c"'] & F(\underline{\mathrm{Hom}}(X,Y)\otimes X)
\end{tikzcd}$$

Then we have

$$\begin{aligned}
& F(\mathrm{ev}_{X,Y})\circ F(b\otimes\mathrm{id}) = F(\mathrm{ev}^1_{X,Y}) \\
\iff\ & F(\mathrm{ev}_{X,Y})\circ c\circ\alpha\otimes\mathrm{id}\circ c^{-1} = F(\mathrm{ev}^1_{X,Y}) \\
\iff\ & F(\mathrm{ev}_{X,Y})\circ c\circ\alpha\otimes\mathrm{id} = F(\mathrm{ev}^1_{X,Y})\circ c
\end{aligned}$$

By (7.18) we have $F(\mathrm{ev}^1_{X,Y})\circ c = \mathrm{ev}^{2'}_{FX,FY}\circ\gamma\otimes\mathrm{id}$. Hence the rightmost diagram in diagram (7.16) commutes, so by lemma 7.16 the leftmost diagram commutes. Since β,γ,α are isomorphisms, it follows that $F_{X,Y}$ is an isomorphism.

□

Definition 7.17. A *tensor functor* (7.1) $(F,c):(\mathcal{C},\otimes)\to(\mathcal{C}',\otimes')$ is a **tensor equivalence** (or an **equivalence of tensor categories**) if $F:\mathcal{C}\to\mathcal{C}'$ yields an equivalence of categories (2.3).

Proposition 7.18. *Let $(F,c):(\mathcal{C},\otimes)\to(\mathcal{C}',\otimes')$ be a tensor equivalence (7.17). Then there exists a tensor functor $(F',c'):\mathcal{C}'\to\mathcal{C}$ and isomorphisms of functors $F'\circ F\Rightarrow\mathrm{id}_{\mathcal{C}}$ and $F\circ F'\Rightarrow\mathrm{id}_{\mathcal{C}'}$ that commutes with tensor products.* □

8 Morphism of tensor functors

Definition 8.1. Let $(F, c), (G, d) : \mathcal{C} \rightrightarrows \mathcal{C}'$ be *tensor functors* (7.1). Then we say that a **morphism of tensor functors** $(F, c) \to (G, c')$ is a natural transformation $\lambda : F \Rightarrow G$ such that, for all *finite* index sets I, and families $(X_i)_{i\in I}$ of objects $X_i \in \mathcal{C}$, the following diagram commutes

$$\begin{tikzcd}
\bigotimes_{i\in I} F(X_i) \arrow[r, "c"] \arrow[d, "\bigotimes_{i\in I} \lambda_{X_i}"] & F\left(\bigotimes_{i\in I} X_i\right) \arrow[d, "\lambda_{\bigotimes_{i\in I} X_i}"] \\
\bigotimes_{i\in I} G(X_i) \arrow[r, "d"] & G\left(\bigotimes_{i\in I} X_i\right)
\end{tikzcd} \tag{8.1}$$

[10] points out, that it is enough to require that (8.1) is commutative when $I = \{1, 2\}$ or when I is the empty set. If $I = \varnothing$, then 8.1 becomes

$$\begin{tikzcd}
\mathbf{1}' \arrow[r, "\simeq"] \arrow[d, equal] & F(\mathbf{1}) \arrow[d, "\lambda_{\mathbf{1}}"] \\
\mathbf{1}' \arrow[r, "\simeq"] & G(\mathbf{1})
\end{tikzcd} \tag{8.2}$$

We see that when (8.2) commutes, and $\alpha : \mathbf{1}' \xrightarrow{\simeq} F(\mathbf{1})$ and $\beta : \mathbf{1}' \xrightarrow{\simeq} G(\mathbf{1})$, with $\mathrm{id}_{\mathbf{1}'} : \mathbf{1}' \to \mathbf{1}'$ we have that $\lambda_{\mathbf{1}} = (\beta \circ \mathrm{id}_{\mathbf{1}}) \circ \alpha^{-1}$ so that $\lambda_{\mathbf{1}}$ is an isomorphism.

Definition 8.2. $\mathrm{Hom}^{\otimes}(F, G) := \{\lambda \mid \lambda \text{ is a morphism of tensor functors}\}$.

Proposition 8.3. *Let $(F, c), (G, d) : \mathcal{C} \rightrightarrows \mathcal{C}'$ be tensor functors (7.1). If $\mathcal{C}$ and $\mathcal{C}'$ are rigid tensor categories (6.1), then every morphism of tensor functors (8.1) λ is an isomorphism.*

Proof. We give a proof sketch for how to proceed. The *natural transformation* $\mu : G \Rightarrow F$ making the following diagram commute

$$\begin{tikzcd}
F(X^\vee) \arrow[r, "\lambda_{X^\vee}"] \arrow[d, "\simeq"'] & G(X^\vee) \arrow[d, "\simeq"] \\
F(X)^\vee \arrow[r, "{}^t(\mu_X)"'] & G(X)^\vee
\end{tikzcd} \tag{8.3}$$

for all $X \in \mathcal{C}$ is an inverse of λ. Recall that in a *rigid* tensor category, each object is of the form $X^\vee$, so we can assume that all objects are of the form $X^\vee$, so that $\lambda_{X^\vee} = \lambda_X$ since $(X^\vee)^\vee = X$ (up to canonical isomorphism).

We have seen, in the proof of (7.2), that a *tensor functor* (7.1) of *rigid* (6.1) tensor categories $\mathcal{C}, \mathcal{C}'$, *preserve duality*; i.e. if X is a dualizable object, then $F(X)$ is a dualizable. So, suppose we are given $\lambda : F \Rightarrow G$, such that λ is a *morphism of tensor functors*, that is, fulfills the condition given in 8.1.

Let X be a *dualizable* object. We claim that $\lambda_X : F(X) \to G(X)$ is *invertible*, with inverse the *transpose* ${}^t\lambda_{X^\vee} : G(X^\vee) \to F(X^\vee)$.

Let $\mathrm{ev} : X^\vee \otimes X \to \mathbf{1}$ and $\epsilon : \mathbf{1} \to X \otimes X^\vee$ be the duality data for X. Furthermore, let $c_{F,X,Y} : F(X) \otimes' F(Y) \overset{\cong}{\to} F(X \otimes Y)$ and $c_{G,X,Y} : G(X) \otimes' G(Y) \to G(X \otimes Y)$ be the natural isomorphisms coming from 7.1 for F, G respectively. Let $a_F : \mathbf{1'} \overset{\cong}{\to} F(\mathbf{1})$ and $a_G : \mathbf{1'} \overset{\cong}{\to} G(\mathbf{1})$ be the unique isomorphism between identity objects in $\mathcal{C}'$, coming from 2.23.

As in [17], we get the following diagram

$$\begin{tikzcd}
 & \mathbf{1'} \otimes' GX \arrow[dl, bend right, "a_F \otimes \mathrm{id}_{GX}"'] \arrow[d, "a_G \otimes \mathrm{id}_{GX}"] \arrow[dr, phantom, "(1)"] & GX \arrow[l, "l'_{GX}"'] \arrow[d, "G(l'_{GX})"] \arrow[dr, bend left, "\mathrm{id}_{GX}"] & & & \\
F(\mathbf{1}) \otimes' GX \arrow[ur, phantom, "(1)'"] \arrow[r, "\lambda_{\mathbf{1}} \otimes \mathrm{id}_{GX}"] \arrow[d, "F(\epsilon) \otimes' \mathrm{id}_{GX}"'] \arrow[dr, phantom, "(2)"] & G(\mathbf{1}) \otimes' GX \arrow[r, "c_G"] \arrow[d, "G(\epsilon) \otimes' \mathrm{id}_{GX}"] \arrow[dr, phantom, "(3)"] & G(\mathbf{1} \otimes X) \arrow[d, "G(\epsilon \otimes \mathrm{id}_X)"] \arrow[dr, phantom, "(4)"] & GX & GX \arrow[l, "\mathrm{id}_{GX}"] & \\
F(X \otimes X^\vee) \otimes' GX \arrow[r, "\lambda_{X \otimes X^\vee} \otimes' \mathrm{id}_{GX}"] \arrow[d, "c_F^{-1} \otimes' \mathrm{id}_{GX}"'] \arrow[dr, phantom, "(7)"] & G(X \otimes X^\vee) \otimes' GX \arrow[r, "c_G"] \arrow[d, "c_G^{-1} \otimes' \mathrm{id}_{GX}"] \arrow[dr, phantom, "(8)"] & G(X \otimes X^\vee \otimes X) \arrow[r, "G(\mathbf{1} \otimes \mathrm{ev}_X)"] \arrow[dr, phantom, "(9)"] & G(X \otimes \mathbf{1}) \arrow[u, "G(r_X^{-1})"] \arrow[r, phantom, "(5)"] & GX \otimes' \mathbf{1'} \arrow[u, "r'^{-1}_{GX}"'] \arrow[r, phantom, "(6)"] & FX \arrow[ul, "\lambda_X"'] \\
F(X) \otimes' F(X^\vee) \otimes' GX \arrow[r, "\lambda_X \otimes' \lambda_{X^\vee} \otimes' \mathrm{id}_{GX}"] \arrow[dr, bend right, "\mathrm{id}_{FX} \otimes \lambda_{X^\vee} \otimes \mathrm{id}_{GX}"'] \arrow[dr, phantom, "(2)'"] & GX \otimes' G(X^\vee) \otimes' GX \arrow[r, "\mathrm{id}_{GX} \otimes' c_G"] \arrow[dr, phantom, "(11)"] & GX \otimes' G(X^\vee \otimes X) \arrow[u, "c_G"'] \arrow[r, "\mathrm{id}_{GX} \otimes' G(\mathrm{ev})"] \arrow[dr, phantom, "(12)"] & GX \otimes' G(\mathbf{1}) \arrow[u, "c_G"'] \arrow[ur, "\mathrm{id}_{GX} \otimes' a_G^{-1}"] \arrow[r, phantom, "(13)"] & FX \otimes' \mathbf{1'} \arrow[u, "\lambda_X \otimes' \mathrm{id}_{\mathbf{1'}}"'] \arrow[ur, "r'^{-1}_{FX}"'] & \\
 & F(X) \otimes' G(X^\vee) \otimes' G(X) \arrow[u, "\lambda_X \otimes \mathrm{id} \otimes \mathrm{id}"] \arrow[r, "\mathrm{id}_{FX} \otimes' c_G"] & FX \otimes' G(X^\vee \otimes X) \arrow[u, "\lambda_X \otimes' \mathrm{id}"'] \arrow[r, "\mathrm{id}_{FX} \otimes' G(\mathrm{ev})"] & FX \otimes' G(\mathbf{1}) \arrow[u, "\lambda_X \otimes' \mathrm{id}_{GX}"] \arrow[ur, "\mathrm{id}_{FX} \otimes' a_G^{-1}"'] & &
\end{tikzcd} \tag{8.4}$$

We then see that

- (1) commutes by 7.1.
- (1)′ commutes by 8.2.
- (2) commutes by naturality of λ.
- (2)′ commutes by bifunctoriality of $\otimes$.
- (3) commutes by naturality of c_G.
- (4) commutes by the *snake equation* (5.13), if we unpack the suppressed ϕ, and functoriality of G.
- (5) commutes by 7.1.
- (6) commutes by naturality of r'.
- (7) commutes by the constraint 8.1.
- (8) commutes by the structural constraint for tensor functors in 7.1 (one has to “unpack” the suppressed associators ϕ and $F(\phi)$ to see this).

- (9) commutes by naturality of c_G.
- $(11), (12)$ commutes by naturality of λ.
- (13) commutes by bifunctoriality of $\otimes$.

It follows that the diagram above commutes. Then we see that, if we follow the *outer perimeter* of the diagram, by (5.21) and (7.6), we have $\lambda_X \circ (\mathrm{id} \otimes \mathrm{ev}'_{G(X)}) \circ (\mathrm{id} \otimes \lambda_{X^\vee} \otimes \mathrm{id}) \circ (\epsilon'_{FX} \otimes \mathrm{id}) = \mathrm{id}$, but $(\mathrm{id} \otimes \mathrm{ev}'_{G(X)}) \circ (\mathrm{id} \otimes \lambda_{X^\vee} \otimes \mathrm{id}) \circ (\epsilon'_{FX} \otimes \mathrm{id}) = {}^t(\lambda_{X^\vee})$ so that $\lambda_X \circ {}^t(\lambda_{X^\vee}) = \mathrm{id}$.

According to [17], a similar diagram shows that ${}^t(\lambda_{X^\vee}) \circ \lambda_X = \mathrm{id}$.

□

9 Tensor subcategories

Recall the definition of 1.16.

Definition 9.1. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called **replete** if for any $D \in \mathcal{D}$, we have that if $f : D \cong Y$ for $f \in \mathrm{Hom}_{\mathcal{C}}(D, Y)$, then $Y \in \mathcal{D}$ and $f \in \mathrm{Hom}_{\mathcal{D}}(D, Y)$.

Definition 9.2. Let $\mathcal{C}$ be a category. We say that a subcategory $\mathcal{D}$ of $\mathcal{C}$ is a **strictly full subcategory** if $\mathcal{D}$ is *full*, and *replete*.

Definition 9.3. Let $\mathcal{D}$ be a *strictly full* subcategory of a tensor category $\mathcal{C}$. We call $\mathcal{D}$ a **tensor subcategory** of $\mathcal{C}$ if it is closed under finite tensor-products. That is, if $A, B \in \mathcal{D} \implies A \otimes B$. One could also define $\mathcal{D}$ as a tensor subcategory if it contains an identity object for $\mathcal{C}$, and if $A \otimes B \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$.

Definition 9.4. Let $(\mathcal{C}, \otimes)$ be a *rigid* tensor category. Then we call a tensor subcategory $\mathcal{D}$ a **rigid tensor subcategory** if for all objects D in $\mathcal{D}$, one has that $D^\vee$ is in $\mathcal{D}$.

Subcategories that fit the descriptions of either 9.2 or 9.3 become tensor categories in their own right, with the tensor product as bifunctor.

10 Abelian tensor categories

10.1 Buildup; introducing definitions

In this section, we build up the constructions we need, in order to finally be able to define what an **abelian category** is. This is a special *type* of category, modelled on the prototypical example **Ab**, where objects are *abelian groups* and morphisms are *group homomorphisms*.

Definition 10.1. We call a category $\mathcal{C}$ **preadditive**, if for all objects $A, B \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has the structure of an abelian group, and if there is a composition-operation $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ that is bilinear. What *bilinarity* amounts to is the following:

- Let $f \in \mathrm{Hom}(B, C)$ and let $g, h \in \mathrm{Hom}(A, B)$. Then $f \circ (g + h) = f \circ g + f \circ h$, where $+$ is the group-operation in $\mathrm{Hom}(A, B)$ on the left hand side, and the group-operation in $\mathrm{Hom}(A, C)$ on the right hand side.
- Let $f, g \in \mathrm{Hom}(B, C)$ and let $h \in \mathrm{Hom}(A, B)$. Then $(f + g) \circ h = f \circ h + g \circ h$ (again, with $+$ the

group-operation in the obvious respective groups on the left hand side and the right hand side).

Remark 10.2. Another way to phrase definition 10.1 is to say that $\mathcal{C}$ is *preadditive* $\iff$ $\mathcal{C}$ is **enriched** over **Ab**, in the sense that all hom-sets have the structure of an *abelian group*, and where composition is bilinear, in the sense given above.

Definition 10.3. Let $\mathcal{C}$ be a category and let A be an object in $\mathcal{C}$. Then the functor $F : \mathcal{J} \to \mathcal{C}$ defined by $F(j) \equiv A$ for all objects j in $\mathcal{J}$, and $F(f) \equiv \mathrm{id}_A$ for all morphisms f in $\mathcal{J}$, is the **constant functor at** A. We will denote the specified constant functor as ΔA.

Definition 10.4. Let $\mathcal{C}$ be a category and $F : \mathcal{J} \to \mathcal{C}$ a diagram, and let A be an object in $\mathcal{C}$, with constant constant functor ΔA. Let $\lambda : \Delta A \to \mathcal{C}$ be a natural transformation. We then call the natural transformation λ the **cone over** the diagram F with **summit** or **apex** A and the components

$$(\lambda_j : \Delta A(j) = A \to F(j))_{j \in \mathcal{J}}$$

the **legs** of λ.

We get the following commutative diagrams, for each $f \in \mathrm{Hom}_{\mathcal{J}}(i, j)$ with $i, j \in \mathcal{J}$

$$\begin{tikzcd}
i \arrow[dd, "f"] & \Delta A(i) = A \arrow[r, "\lambda_i"] \arrow[dd, equal, "\Delta A(f)=\mathrm{id}_A"] & F(i) \arrow[dd, "F(f)"] & & & A \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
 & & & \rightsquigarrow & F(i) \arrow[rr, "F(f)"] & & F(j) \\
j & \Delta A(j) = A \arrow[r, "\lambda_j"] & F(j) & & & &
\end{tikzcd}$$

where $\rightsquigarrow$ follows from the fact that the leftmost diagram commutes, hence we have

$$\begin{aligned} &\lambda_j \circ \mathrm{id}_A = F(f) \circ \lambda_i \\ \iff &\lambda_j = F(f) \circ \lambda_i. \end{aligned}$$

We also want to define a dual concept.

Definition 10.5. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram, let A be an object in $\mathcal{C}$ and let $\lambda : F \to \Delta A$ be a natural transformation. We call λ a **cone under** F with **nadir** A and **legs** $(\lambda_j : F(j) \to \Delta A(j) = A)_{j \in \mathcal{J}}$.

We find the following commutative diagrams, for each $f \in \mathrm{Mor}(\mathcal{J})$

f

$$\begin{tikzcd}
i \arrow[dd] & F(i) \arrow[r, "\lambda_i"] \arrow[dd, "F(f)"] & \Delta A(i) = A \arrow[dd, equal, "\Delta A(f)=\mathrm{id}_A"] & & F(i) \arrow[rr, "F(f)"] \arrow[dr, "\lambda_i"'] & & F(j) \arrow[dl, "\lambda_j"] \\
 & & & \rightsquigarrow & & A & \\
j & F(j) \arrow[r, "\lambda_j"] & \Delta A(j) = A & & & &
\end{tikzcd}$$

where $\mathrm{id}_A \circ \lambda_i = \lambda_i$ so that

$$\begin{aligned} & \mathrm{id}_A \circ \lambda_i = \lambda_j \circ F(f) \\ \iff & \lambda_i = \lambda_j \circ F(f). \end{aligned}$$

Example 10.6. As in [13], to illustrate 10.4, let F be a functor indexed by the poset-category $(\mathbb{Z}, \leq)$. Then a *cone over* F with *summit* c consists of morphisms $(\lambda_n : c \to F(n))_{n\in\mathbb{Z}}$, so that for each pair λ_m, λ_n with $n \leq m$, and morphism $F(n) \to F(m)$, their respective triangles in the diagram below, *commutes*

$$\begin{tikzcd}
 & & & c \arrow[dlll, "\cdots"'] \arrow[dll, "\lambda_{-2}"'] \arrow[dl, "\lambda_{-1}"'] \arrow[d, "\lambda_0"] \arrow[dr, "\lambda_1"] \arrow[drr, "\lambda_2"] \arrow[drrr, "\cdots"] & & & \\
\ldots \arrow[r] & F(-2) \arrow[r] & F(-1) \arrow[r] & F(0) \arrow[r] & F(1) \arrow[r] & F(2) \arrow[r] & \ldots
\end{tikzcd}$$

Example 10.7. To illustrate 10.5, we again take the poset $(\mathbb{Z}, \leq)$ with morphisms $(\lambda_n : F(n) \to c)_{n\in\mathbb{Z}}$, where, for each pair of objects $n, m \in \mathbb{Z}$ such that $n \leq m$, morphisms λ_n, λ_m, and $F(n) \to F(m)$, the triangle they constitute in the diagram below, *commutes*

$$\begin{tikzcd}
\ldots \arrow[r] \arrow[drrr, "\cdots"'] & F(-2) \arrow[r] \arrow[drr, "\lambda_{-2}"'] & F(-1) \arrow[r] \arrow[dr, "\lambda_{-1}"'] & F(0) \arrow[r] \arrow[d, "\lambda_0"] & F(1) \arrow[r] \arrow[dl, "\lambda_1"] & F(2) \arrow[r] \arrow[dll, "\lambda_2"] & \ldots \arrow[dlll, "\cdots"] \\
 & & & c & & &
\end{tikzcd}$$

We are now ready to define *limits* and *colimits*.

Definition 10.8. Assume that $\mathcal{J}$ is small and $\mathcal{C}$ is locally small. Let $F : \mathcal{J} \to \mathcal{C}$ be a *diagram* (1.20). Then let

$$\text{Cone}(-, F) : \mathcal{C}^{\text{opp}} \to \textbf{Set} \tag{10.1}$$

be the functor that sends $X \in \mathcal{C}$ to the *set of cones* over F with *summit* c (recall 10.4).

A **limit** of F is a *representation* for $\text{Cone}(-, F)$. By 5.6, a limit consists of an object $\lim F \in \mathcal{C}$ together with a *universal cone* $\lambda : \lim F \Rightarrow F$, called the **limit cone**, which defines the natural isomorphism

$$\text{Hom}_{\mathcal{C}}(-, \lim F) \simeq \text{Cone}(-, F). \tag{10.2}$$

Definition 10.9. Again, assume that $\mathcal{J}$ is small and $\mathcal{C}$ is locally small. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram. Let

$$\text{Cone}(F, -) : \mathcal{C} \to \textbf{Set} \tag{10.3}$$

be the functor that sends $X \in \mathcal{C}$ to the *set of cones under* F (see 10.5). Then a **colimit** of F is a *representation* for $\text{Cone}(F, -)$. As in 10.8, by 5.6, a colimit consists of an object $\text{colim } F \in \mathcal{C}$, together with a *universal cone* $\lambda : F \Rightarrow \text{colim } F$, called the **colimit cone**, giving us a natural isomorphism

$$\text{Hom}_{\mathcal{C}}(\text{colim } F, -) \simeq \text{Cone}(F, -). \tag{10.4}$$

One might ask what we mean by *universal cone* in 10.8 and 10.9. The following definition aims to make this clear.

Definition 10.10. We say that a cone $\lambda : X \Rightarrow F$ *from the constant functor at* X *to the functor* $F : \mathcal{J} \to \mathcal{C}$ is **universal** if for any other cone $\eta : Y \Rightarrow F$ (Y again a constant functor), there is a *unique* morphism $u : X \to Y$ such that for all objects A, η_A factors through u and λ_A. The diagram below illustrates what we mean

$$\begin{tikzcd}
& X \arrow[ddl, bend right, "\eta_A"'] \arrow[ddr, bend left, "\eta_B"] \arrow[d, dashed, "\exists! u"] & \\
& Y \arrow[dl, "\lambda_A"'] \arrow[dr, "\lambda_B"] & \\
F(A) \arrow[rr, "F(f)"] & & F(B)
\end{tikzcd}$$

In the diagram above, we have that

$$\begin{aligned} \lambda_A \circ u &= \eta_A \\ \lambda_B \circ u &= \eta_B \\ F(f) \circ \lambda_A &= \lambda_B \end{aligned}$$

holds for arbitrary objects $A, B \in \mathcal{J}$.

Dualizing 10.10, we get the following definition.

Definition 10.11. A cone $\lambda : F \Rightarrow X$ *from the functor* $F : \mathcal{J} \to \mathcal{C}$ *to the constant functor at* X is **universal** if for any other cone $\eta : F \Rightarrow Y$ (Y again the constant functor at Y), there is a *unique* morphism $u : Y \Rightarrow X$ such that the following diagram commutes

$$\begin{tikzcd}
F(A) \arrow[rr, "F(f)"] \arrow[dr, "\lambda_A"'] \arrow[ddr, bend right, "\eta_A"'] & & F(B) \arrow[dl, "\lambda_B"] \arrow[ddl, bend left, "\eta_B"] \\
& Y \arrow[d, dashed, "\exists ! u"] & \\
& X &
\end{tikzcd}$$

so that

$$u \circ \lambda_A = \eta_A$$
$$u \circ \lambda_B = \eta_B$$
$$\lambda_B \circ F(f) = \lambda_A.$$

Definition 10.12. Let X, Y be objects in a category $\mathcal{C}$. Then a (binary) **product** of X, Y (if it exists), which one can denote as $X \times Y$, is an object in $\mathcal{C}$, that comes equipped with a pair of morphisms $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$ that have the following *universal property* (5.20):

- For every other object $Z \in \mathcal{C}$ and every pair of morphisms $f_X : Z \to X$ and $f_Y : Z \to X$, there is a *unique* morphism $f : Z \to X \times Y$ such that the following diagram *commutes*

$$\begin{tikzcd}
& Z \arrow[dl, "f_X"'] \arrow[d, dashed, "\exists ! f"] \arrow[dr, "f_Y"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}$$

Dualizing 10.12, we get the following definition.

Definition 10.13. Let $\mathcal{C}$ be a category and let $X, Y \in \mathcal{C}$. Then a (binary) **coproduct** of X, Y (if it exists), which we denote as $X \coprod Y$, is an object in $\mathcal{C}$ together with morphisms $i_X : X \to X \coprod Y$ and $i_Y : Y \to X \coprod Y$ that satisfies the following *universal property*:

- For any other object Z, and for every pair of morphisms $f_X : X \to Z$ and $f_Y : Y \to Z$, there is a *unique* map f such that the following diagram *commutes*

$$\begin{tikzcd}
& Z & \\
X \arrow[ur, "f_X"] \arrow[r, "i_X"'] & X \coprod Y \arrow[u, dotted, "\exists ! f"] & Y \arrow[ul, "f_Y"'] \arrow[l, "i_Y"]
\end{tikzcd}$$

There is also a definition when 10.12 and 10.13 coincide. To describe this object, we need a few more definitions.

Definition 10.14. Let $\mathcal{C}$ be a category, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. If for any object $Z \in \mathcal{C}$ and any pair of morphisms $g, h : Z \rightrightarrows X$ it holds that $f \circ g = f \circ h$, then we call f a **constant morphism**.

Definition 10.15. Let $\mathcal{C}$ be a category, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. If for any object $Z \in \mathcal{C}$ and any pair of morphisms $g, h : Y \rightrightarrows Z$ it holds that $g \circ f = h \circ f$, then we call f a **coconstant morphism**.

Definition 10.16. If $\mathcal{C}$ is a category, and $f : X \to Y$ is a morphism in $\mathcal{C}$ that is both a *constant morphism* (10.14) and a *coconstant morphism* (10.15), then we call f a **zero morphism**.

We can put the above three definitions into the context of a certain property an arbitrary category $\mathcal{C}$ can possess.

Definition 10.17. Let $\mathcal{C}$ be a category. Then we say that $\mathcal{C}$ is a **category with zero morphisms** if

- for every pair of objects $X, Y \in \mathcal{C}$, there is a *zero morphism* (10.16) $0_{XY} : X \to Y$, giving us what we can call a *system* $0_{-,-}$ such that it gives us a zero morphism, for every pair (X, Y) of objects in C.
- For all objects X, Y, Z and morphisms $f : X \to Y$ and $g : Y \to Z$, there is a zero morphism 0_{XZ} such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "0_{XY}"'] \arrow[dr, "0_{XZ}"] & Y \arrow[d, "0_{YZ}"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Definition 10.18. Let X, Y be objects in $\mathcal{C}$, for a category $\mathcal{C}$, and let $\mathcal{C}$ be a *category with zero morphisms* (10.17). Then we say that $X \oplus Y$ (if it exists) is a **binary biproduct** if it holds that

- There are *projection morphisms* $\pi_X : X \oplus Y \to X, \pi_Y : X \oplus Y \to Y$, together with *embedding morphisms* $i_X : X \to X \oplus Y, i_Y : Y \to X \oplus Y$ satisfying
    (a) $\pi_X \circ i_X = \mathrm{id}_X$.
    (b) $\pi_X \circ i_Y = 0_{XY}$.
- $(X \oplus Y, \pi_X, \pi_Y)$ is a *product* (10.12).
- $(X \oplus Y, i_X, i_Y)$ is a *coproduct* (10.13).

Definition 10.19. A **equalizer of a parallel pair of morphisms** $f, g : X \rightrightarrows Y$ is a system $(E, e : E \to X)$ such that the following holds:

- $f \circ e = g \circ e$.
- For any other such system $(E', e' : E' \to X)$, there is a *unique* map u such that $e' = e \circ u$. Or,

in diagrammatic form, as

$$\begin{tikzcd}
E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
E' \arrow[u, dotted, "\exists ! u"] \arrow[ur, "e'"'] & &
\end{tikzcd}$$

Definition 10.20. A **coequalizer of a parallel pair of morphisms** $f, g : X \rightrightarrows Y$ is a system $(Q, q : Y \to Q)$ such that the following holds:

- $q \circ f = q \circ g$.
- Given any other pair $(Q', q' : Y \to Q')$ with the same property, there is a *unique* morphism $u : Q \to Q'$ such that $q' = u \circ q$. This is usally written in diagrammatic form as

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"] & Q \arrow[d, dashed, "\exists ! u"] \\
& & Q
\end{tikzcd}$$

Definition 10.21. Let $\mathcal{C}$ be a *category with zero morphisms* (10.17). Let $f : X \to Y$ be an arbitrary morphism in $\mathcal{C}$. Then we say that a **kernel** of f is an object K and a morphism $k : K \to X$, defining a pair (K, k), such that the following holds:

- The following diagram *commutes*

$$\begin{tikzcd}
X \arrow[dr, "f"] & \\
K \arrow[u, "k"] \arrow[r, "0_{KY}"] & Y
\end{tikzcd}$$

  so that $f \circ k = 0_{KY}$.

- Let $k' : K' \to X$ be any other morphism such that $f \circ k' = 0_{K'Y}$. Then there is a *unique* morphism u such that the following diagram *commutes*

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
& K \arrow[u, "k"] \arrow[r, "0_{KY}"] & Y \\
K' \arrow[uur, bend left, "k'"] \arrow[ur, dashed, "\exists ! u"] \arrow[urr, bend right, "0_{K'Y}"'] & &
\end{tikzcd}$$

k then has the property of being a *monomorphism*. It is easy to show that any two kernels (K, k) and (K', k') of f then gives rise to a canonical isomorphism $K \cong K'$, via the induced unique maps u, v that comes from the diagram above (v induced from permuting the two systems of kernels of f in the diagram above).

In for example an abelian category (see 10.24), we can also more succintly define $\ker(f)$ as $\text{eq}(f, 0_{X,Y})$, the *equalizer* (10.19) of f and $0_{X,Y}$.

Dualizing 10.21, we get the following definition.

Definition 10.22. Let $\mathcal{C}$ be a *category with zero morphisms* (10.17), and let $f : X \to Y$ be an arbitrary morphism. Then the **cokernel** of f is an object Q together with a morphism $q : Y \to Q$ such that the following holds:

- The following diagram *commutes*

$$\begin{tikzcd}
 & Y \arrow[d, "q"] \\
X \arrow[ur, "f"] \arrow[r, "0_{XQ}"] & Q
\end{tikzcd}$$

  such that $q \circ f = 0_{XQ}$.

- For any other object Q' and morphism $q' : Y \to Q'$ in $\mathcal{C}$ such that $q' \circ f = 0_{XQ'}$, there exists a *unique* morphism $u : Q \to Q'$ such that the following diagram commutes

$$\begin{tikzcd}
 & Y \arrow[d, "q"] \arrow[ddr, bend left, "q'"] & \\
X \arrow[ur, "f"] \arrow[r, "0_{XQ}"] \arrow[drr, bend right, "0_{XQ'}"'] & Q \arrow[dr, dashed, "\exists! u"] & \\
 & & Q'
\end{tikzcd}$$

The map q then also has the property of being an epimorphism. In a similar fashion as we mentioned for the kernel, it is then easy to see that there is an induced canonical isomorphism $Q \cong Q'$ for any two systems of cokernels (Q, q) and (Q', q') of f, coming from the existence of unique maps u, v in the diagram above (by just permuting the two systems we get a unique $v : Q' \to Q$).

In for example an abelian category (see 10.24), we can also more succintly define $\text{coker}(f)$ as $\text{coeq}(f, 0_{X,Y})$, the *coequalizer* (10.20) of f and $0_{X,Y}$.

Remark 10.23. If each morphism $f : X \to Y$ in $\mathcal{C}$ has a kernel (10.21) and cokernel (10.22), then we say that $\mathcal{C}$ *has all kernels and cokernels.*

10.2 Completion; the definition of an abelian category

Definition 10.24. We call a category $\mathcal{C}$ an **abelian category** if

1) It has a *zero object* (1.24).

2) It has all *binary biproducts* (10.18).

3) It has all *kernels* (10.21) and *cokernels* (10.22).

4) Every *monomorphism* (1.18) is the *kernel* of some morphism, and every *epimorphism* (1.19) is the *cokernel* of some morphism.

10.3 Additive categories; Abelian tensor categories; End(1)

After some interlude exploring abelian categories, we will specifically look at *abelian tensor categories.* We will assume that such categories are *additive.* To say what we mean by *additive*, we first introduce a related notion.

Definition 10.25. We say that a category $\mathcal{C}$ **admits all finitary products** if for any finite set of objects $X_1, \ldots, X_n \in \mathcal{C}$ ($n \in \mathbb{Z}_{>0}$), there is an object $X_1 \times \ldots \times X_n \in \mathcal{C}$.

Definition 10.26. Let $\mathcal{C}$ be a *preadditive* (10.1) category. Then we say that $\mathcal{C}$ is **additive** if $\mathcal{C}$ *admits all finitary products* (10.25).

Definition 10.27. If $\mathcal{C}$ is an abelian category (10.24), then we call a sequence

$$\cdots \xrightarrow[f_{n+2}]{} X_{n+1} \xrightarrow[f_{n+1}]{} X_n \xrightarrow[f_n]{} X_{n-1} \xrightarrow[f_{n-1}]{} X_{n-2} \xrightarrow[f_{n-2}]{} \cdots$$

exact if $\ker(f_n) = \mathrm{im}(f_{n+1})$.

Lemma 10.28. *Right adjoints (7.8) preserve limits (10.8).*

Proof. [13, Theorem 4.5.2]. □

Dualizing 10.28, we get the following corollary.

Corollary 10.29. *Left adjoints preserves colimits.*

Proposition 10.30. *Let* $(\mathcal{C}, \otimes)$ *be a* rigid tensor category *(6.1). If* $\mathcal{C}$ *is* abelian *(10.24), then* $\otimes$ *is* biadditive *and commutes with direct and inverse limits in each variable; in particular, it is exact in each variable.*

Proof. As we have seen, $\mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y, Z))$ for all objects $X, Y, Z \in \mathcal{C}$ (since by rigidity, $\underline{\mathrm{Hom}}(X, Y)$ exists for all pairs of objects $X, Y \in \mathcal{C}$), so $- \otimes Y$ has a right adjoint $\underline{\mathrm{Hom}}(Y, -)$. By 10.29, it follows that $- \otimes Y$ preserves colimits.

We also want to show that $- \otimes Y$ has a left-adjoint. We consider

$$\begin{aligned} \mathrm{Hom}(X \otimes Y^\vee, W) &\simeq \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y^\vee, W)) \\ &\simeq \mathrm{Hom}(X, Y \otimes W) \\ &\simeq \mathrm{Hom}(X, W \otimes Y), \end{aligned}$$

where the last canonical isomorphism above is using the functorial isomorphism

$$\psi_{A,B} : A \otimes B \to B \otimes A \qquad (\forall A, B \in \mathcal{C}).$$

We have also used that in a rigid abelian tensor category, all objects $X \in \mathcal{C}$ are of the form $Z^\vee$ for some $Z \in \mathcal{C}$ and that $\underline{\mathrm{Hom}}(X, Y) \simeq X^\vee \otimes Y$. Hence $- \otimes Y$ has a left-adjoint $- \otimes Y^\vee$, so preserves limits.

Furthermore, we note that

$$\begin{aligned} \mathrm{Hom}(\underline{\mathrm{Hom}}(X^\vee, Y^\vee), Z) &\simeq \mathrm{Hom}(X \otimes Y^\vee, Z) \\ &\simeq \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y^\vee, Z)) \\ &\simeq \mathrm{Hom}(X, Y \otimes Z), \end{aligned}$$

so that $Y \otimes -$ has a left-adjoint $\underline{\mathrm{Hom}}(-, Y^\vee)$. Hence $Y \otimes -$ preserves limits.

We also have

$$\begin{aligned} \mathrm{Hom}(Y \otimes X, Z) &\simeq \mathrm{Hom}(X \otimes Y, Z) \\ &\simeq \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y, Z)). \end{aligned}$$

so that $Y \otimes -$ has a right adjoint $\underline{\mathrm{Hom}}(Y, -)$, and so preserves colimits.

It follows that $\otimes$ is exact in both variables. And this in turn implies that $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is additive (see e.g. [16]).

$\square$

We will introduce two definitions of the *image* (in a categorical setting) of a morphism f. We start with the most general definition.

Definition 10.31. Let $\mathcal{C}$ be a category, and let $f : X \to Y$ be an arbitrary morphism. Then the **image** of f, denoted $\mathrm{im}(f)$, if it exists, is defined as a *monomorphism* $m : I \to X$ from some object I, such that:

- There exists some morphism $e : X \to I$ such that $f = m \circ e$.
- m satisfies the universal property that if there is some other morphism $e' : I' \to X$ such that $f = m \circ e'$, then there is a *unique* map v such that $m = m' \circ v$.

Pictorially, we represent this as

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "e"'] \arrow[ddr, bend right] & & Y \\
& I \arrow[ur, hook, "m"'] \arrow[d, dotted, "\exists! v"] & \\
& I' \arrow[uur, bend right] &
\end{tikzcd}$$

In an **abelian category**, we can define the image as given below, although when suitable, we use the general definition given above.

Definition 10.32. In an abelian category $\mathcal{C}$, we define the **image**, $\operatorname{im}(f)$, of a morphism f as the kernel of its cokernel, $\operatorname{im}(f) = \ker(\operatorname{coker}(f))$.

Definition 10.33. An **additive functor** $F : \mathcal{C} \to \mathcal{D}$ between *additive tensor categories* $\mathcal{C}, \mathcal{D}$ is a functor fulfilling any of the two equivalent definitions below (cf. [16]):

1. $F(X \otimes Y) \cong F(X) \otimes F(Y) \qquad (\forall X, Y \in \mathcal{C})$.
2. $F(X) \otimes F(Y) \cong F(X \otimes Y) \qquad (\forall X, Y \in \mathcal{C})$.

Definition 10.34. Let $\mathcal{C}$ be an *abelian* category. Then a **short exaqt sequence** is a sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

such that $\operatorname{im}(f) = \ker(g)$ and the sequence is *exact* at X, Y, Z.

Definition 10.35. A **subobject** X of an object $Y \in \mathcal{C}$ is a *monomorphism* $X \hookrightarrow Y$.

Proposition 10.36. *Let* $(\mathcal{C}, \otimes)$ *be a rigid abelian tensor category. If* U *is a subobject of* $\mathbf{1}$*, then* $\mathbf{1} = U \oplus U^{\perp}$*. Therefore,* $\mathbf{1}$ *is a simple object if* $\operatorname{End}(\mathbf{1})$ *is a field.*

Remark 10.37. In 10.36, we have $U^{\perp} := \ker(\mathbf{1} \to U^{\vee})$.

Proof. By assumption, we have a monomorphism $\iota : U \hookrightarrow \mathbf{1}$. We let $V = \operatorname{coker}(\iota)$.

We consider

$$0 \to U \overset{\iota}{\hookrightarrow} \mathbf{1} \overset{q}{\twoheadrightarrow} V \to 0. \tag{10.5}$$

Here, p is the map associated with the cokernel V. It follows that $p \circ \iota = 0_{UV}$. In an abelian category, we have $\operatorname{im}(\iota) = \ker(\operatorname{coker}(\iota))$, but $\operatorname{coker}(\iota) = p$ so that $\operatorname{im}(\iota) = \ker(p)$.

By 10.30 we know that tensoring with $- \otimes U$ is *exact*, hence we get a short exact sequence

$$0 \to U \otimes U \to U \otimes \mathbf{1} \to U \otimes V \to 0.$$

Since right adjoints preserve monomorphisms, we see that tensoring $U \hookrightarrow \mathbf{1}$ with $T \otimes -$, for any object T, gives us a monomorphism $T \otimes U \to T \otimes \mathbf{1} \simeq T$, and since every isomorphism is mono and epi, it follows that the composite of the canonical isomorphism with the induced (from tensoring) mono is mono. So, in particular, tensoring $U \hookrightarrow \mathbf{1}$ with $U \otimes -$ gives us a mono $U \otimes U \hookrightarrow U$, and tensoring $U \hookrightarrow \mathbf{1}$ with $V \otimes -$ gives us a mono $v : V \otimes U \hookrightarrow V \otimes \mathbf{1} \simeq V$.

Consider the following diagram

$$\begin{tikzcd}
U \arrow[r, hook, "\iota"] & \mathbf{1} \arrow[r, two heads, "p"] & V \\
U \otimes U \arrow[u, hook, "\iota'"] \arrow[r, hook, "\iota'"] & U \arrow[u, hook, "\iota"] \arrow[r, two heads, "p'"] \arrow[ur, dotted, "0_{UV}"] & V \otimes U \arrow[u, hook, "v"']
\end{tikzcd}$$

(10.6)

where i', p' are the induced maps from tensoring with $- \otimes U$ (up to natural isomorphism $U \otimes \mathbf{1} \simeq U$). Both the lower and upper sequence is exact, and it is clear that the leftmost diagram commutes. That the righthand square commutes follows by the diagram below, where the rightmost upper diagram and lower leftmost diagrams commutes by naturality, and the lower rightmost diagram commutes by bifunctoriality.

$$\begin{tikzcd}
 & \mathbf{1} \arrow[r, "p"] & V \\
\mathbf{1} \arrow[r, "l_1"] & \mathbf{1} \otimes \mathbf{1} \arrow[u, "\simeq"'] \arrow[r, "p\otimes \mathrm{id}"] & V \otimes \mathbf{1} \arrow[u, "\simeq"] \\
U \arrow[u, "\iota"] \arrow[r, "l_U"] & \mathbf{1} \otimes U \arrow[u, "\mathrm{id} \otimes \iota"] \arrow[r, "p\otimes \mathrm{id}"] & V \otimes U \arrow[u, "\mathrm{id} \otimes \iota"]
\end{tikzcd}$$

Note that v is a monomorphism, so that since $v \circ p' = 0$, it follows that $p' = 0$. Therefore, (using that p' is epi) any distinct two maps $g, h : V \otimes U \to D$, for arbitrary D, are such that

$$\begin{aligned} g \circ p &= h \circ p \\ &= 0. \end{aligned}$$

This in turn implies that there is a unique map $V \otimes U \to V \otimes U$, and this must be the zero map! Therefore, $V \otimes U = 0$. We also note that $U \otimes U$ is a *subobject* of $\mathbf{1} \simeq \mathbf{1} \otimes \mathbf{1}$ (a composition of monomorphisms is a monomorphism!). Since we then have an exact sequence $0 \to U \otimes U \to U \to 0$, this implies that $\mathrm{im}(\iota') = \ker(0)$, but $\ker(0) = \mathrm{id}$, and $\mathrm{im}(\iota') = \iota'$ if ι' is a monomorphism, hence ι' is an isomorphism, such that $\iota \circ \iota' = \phi$, where $\phi : U \otimes U \hookrightarrow \mathbf{1}$. Hence $U \otimes U = U$ as subobjects of $\mathbf{1}$.

We claim that $T \otimes U = 0 \iff T \otimes U \to T$ is zero. $\Longleftarrow$ follows from the fact that $T \otimes U \to T$ is mono, and any mono which is the zero-morphism must have 0 as domain, hence $T \otimes U = 0$. $\Longrightarrow$ follows by definition.

Furthermore, we have the following sequence of natural isomorphisms:

$$\begin{aligned} \mathrm{Hom}(T \otimes U, T) &\simeq \mathrm{Hom}(T \otimes U \otimes T^{\vee}, \mathbf{1}) \\ &\simeq \mathrm{Hom}(T, U^{\vee} \otimes T). \end{aligned}$$

where the first (natural) isomorphism comes from $\mathrm{Hom}(T, X^{\vee}) \simeq \mathrm{Hom}(T \otimes X, \mathbf{1})$, and the second one comes from noting that $U^{\vee} \otimes T \simeq \underline{\mathrm{Hom}}(U, T)$ (by rigidity), and so that $\mathrm{Hom}(T \otimes U, T) \simeq \mathrm{Hom}(T, \underline{\mathrm{Hom}}(U, T))$.

So we in fact have the following equivalences

$$T \otimes U = 0 \iff T \otimes U \to T \text{ is zero} \iff T \to U^{\vee} \otimes T \text{ is zero} \tag{10.7}$$

The above set of equivalences comes from that the associated natural isomorphism respects the **Ab**-enrichment, hence are abelian group homomorphism. It follows that a zero-map must be sent to zero, just as was written in (10.7).

Taking arbitrary object $X \in \mathcal{C}$, and subobject T of X, then if we postcompose $\mathrm{Hom}(T, U^\vee \otimes T)$ with $\mathrm{id}_{U^\vee} \otimes \iota$, where $\iota : T \hookrightarrow X$ identifies T as a subobject of X, it follows that if $T \to U^\vee \otimes T$ is zero $\iff$ $T \to U^\vee \otimes X$ is zero (since post-composition is an abelian group homomorphism). In fact, the kernel of $f : X \to U^\vee \otimes X$ must be the *largest subobject* T (with mono $\iota : T \hookrightarrow X$) of X such that $f \circ \iota = 0$. It follows that

$$T = \ker(X \to U^\vee \otimes X).$$

Then we have (since $\mathrm{im}(\iota) = \iota$ for a monomorphism in an abelian category) an SES

$$0 \to U^\perp \to \mathbf{1} \to U^\vee \to 0.$$

Upon tensoring with $- \otimes X$, which is exact by 10.30, we get an SES

$$0 \to U^\perp \otimes X \to \mathbf{1} \otimes X \simeq X \to U^\vee \otimes X \to 0.$$

Then we see that $U^\vee \otimes X \simeq T$ as subobjects of X. If we let $X = V$, then we see that $U^\vee \otimes V \simeq V$, since $V \otimes U = 0$, so that $T = V$.

If we instead let $X = U$, and noting that $U \otimes U = U$, we see that $T \simeq U^\perp \otimes U$ so

$$\begin{aligned} & T \otimes U = 0 \\ \iff & (U^\perp \otimes U) \otimes U = 0 \\ \iff & U^\perp \otimes (U \otimes U) = 0 \\ \iff & U^\perp \otimes U = 0. \end{aligned}$$

Upon tensoring (10.5) with $U^\perp \otimes -$ we get an exact sequence

$$0 \to U^\perp \otimes U \to U^\perp \to U^\perp \otimes V \to 0.$$

Since $U^\perp \otimes U = 0$, it follows that $\ker(U^\perp \to U^\vee \otimes V)$ is a monomorphism (since its kernel is zero by exactness at $U^\perp$). Since $U^\perp \to U^\perp \otimes V$ is also an epimorphism, it follows that

$$\begin{aligned} U^\perp &\cong U^\perp \otimes V \\ &= V, \end{aligned}$$

where the last equality comes from our earlier result. One should then show that $U \otimes U^\perp \cong \mathbf{1}$. ☐

Lemma 10.38. *An exact functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories $\mathcal{C}, \mathcal{D}$ preserves images.*

Proof. Note that an exact functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories *preserves* kernels and cokernels, and that $\mathrm{im}(f) = \ker(\mathrm{coker}(f))$, hence

$$\begin{aligned} F(\mathrm{im}(f)) &= F(\ker(\mathrm{coker}(f)) \\ &= \ker(\mathrm{coker}(F(f)) \\ &= \mathrm{im}(F(f)). \end{aligned}$$

☐

One property of additive functors $F : \mathcal{C} \to \mathcal{D}$ between abelian categories, is that they act like group homomorphism, in the sense that if $f, g \in \mathrm{Mor}(\mathcal{C})$, then

$$\begin{aligned} F(f) &= F(g) \\ \iff F(f) - F(g) &= 0 \\ \iff F(f-g) &= 0. \end{aligned}$$

Proposition 10.39. *Let $F : \mathcal{C} \to \mathcal{D}$ be an* exact *functor between abelian categories. Then*

$$F(X) = 0 \implies X = 0 \qquad (\forall X \in \mathcal{C})$$

implies that F is faithful.

Proof. Note that

$$\begin{aligned} F(f) &= F(g) \\ F(f-g) &= 0. \end{aligned}$$

We want to show that if $FX = 0 \implies X = 0$, then

$$\begin{aligned} F(f-g) &= 0 \\ \implies f - g &= 0 \\ \iff f &= g. \end{aligned}$$

Lemma 10.40. *If $\mathcal{C}$ is an abelian category, then for arbitrary morphism $f : X \to Y$ in $\mathcal{C}$, it holds that*

$$f = 0 \iff \mathrm{im}(f) = 0.$$

Proof. If $f = 0$, then one finds that $\mathrm{coker}(f) = \mathrm{id}$, and in turn that $\ker(\mathrm{id}) = 0$. Since $\mathrm{im}(f) = \ker(\mathrm{coker}(f))$, it follows that $\mathrm{im}(f) = 0$.

On the other hand, if $\mathrm{im}(f) = 0$, then from 10.31, we see that f factors as

$$\begin{aligned} f &= 0_{0,Y} \circ 0_{X,Y} \\ &= 0_{X,Y}. \end{aligned}$$

Hence $f = 0$. □

Therefore,

$$\begin{aligned} F(f) &= 0 \\ \iff \mathrm{im}(F(f)) &= 0 && \text{(by 10.40)} \\ \iff F(\mathrm{im}(f)) &= 0 && \text{(by 10.38)} \qquad (10.8) \\ \mathrm{im}(f) &= 0 \\ f &= 0. \end{aligned}$$

Here, we have used that $\mathrm{im}(F(f)) = F(\mathrm{im}(f))$, and that $\mathrm{im}(F(f))$ is a monomorphism. If a monomorphism is 0, the *domain* of the monomorphism is 0. I.e. then we see that the system $(\mathrm{im}(F(f)), m)$ with m mono, is such that $\mathrm{im}(F(f)) = 0$. From our assumption that $FX = 0 \implies X = 0$, then we see that $\mathrm{im}(f) = 0$, and so $f = 0$ by 10.40. Hence (10.8). □

Theorem 10.41. *Let $(\mathcal{C}, \otimes), (\mathcal{C}', \otimes')$ be two rigid abelian tensor categories, and let $\mathbf{1}, \mathbf{1}'$ be identity objects of $\mathcal{C}$ and $\mathcal{C}'$, respectively. If* $\operatorname{End}(\mathbf{1})$ *is a field, and* $\mathbf{1}' \neq 0$, *then every* exact *tensor functor* $F : \mathcal{C} \to \mathcal{C}'$ *is faithful.*

Proof. Assuming that if $\operatorname{End}(\mathbf{1})$ is a field then $\mathbf{1}$ is a simple object, we want to show that $X \neq 0 \iff X \otimes X^\vee \to \mathbf{1}$ is an epimorphism, and that this is respected by F, and that this implies that F is faithful, where $X \otimes X^\vee \to \mathbf{1}$ is the map $\operatorname{ev} \circ \psi_{X^\vee, X}$. We will just write this as ev going forward.

We claim that the following shows that if $X = 0$, then $X \otimes X^\vee \to \mathbf{1}$ is not an epimorphism. We have $0 \otimes 0^\vee \to \mathbf{1}$. Without thinking about what $0^\vee$ is; note that

$$\operatorname{Hom}(0 \otimes 0^\vee, \mathbf{1}) \simeq \operatorname{Hom}(0, \underline{\operatorname{Hom}}(0^\vee, \mathbf{1})), \tag{10.9}$$

and that the latter is the one-element set consisting of the zero morphism. But (10.9) is an isomorphism! So $\operatorname{Hom}(0 \otimes 0^\vee, \mathbf{1})$ must consist of the zero morphism $0_{(0 \otimes 0^\vee), \mathbf{1}} : 0 \otimes 0^\vee \to \mathbf{1}$. Consider that $0_{0 \otimes 0^\vee, \mathbf{1}}$ is *not* an epimorphism, since it is coconstant (10.15) so that we have, for $Y \neq Z$, $0_{Y, 0 \otimes 0^\vee} \circ 0_{0 \otimes 0^\vee, \mathbf{1}} = 0_{Z, 0 \otimes 0^\vee} \circ 0_{0 \otimes 0^\vee, \mathbf{1}}$ but $0_{Y, 0 \otimes 0^\vee} \neq 0_{Z, 0 \otimes 0^\vee}$.

For the other direction, we proceed as in [15]: Assume that ev is *not* an epimorphism. Then there are morphisms $\alpha, \beta : \mathbf{1} \to Z$ such that $\alpha \circ \operatorname{ev} = \beta \circ \operatorname{ev}$ but $\alpha \neq \beta$. Since we are in an abelian category, $\operatorname{Hom}(\mathbf{1}, Z)$ has the structure of an abelian group, so $\alpha - \beta$ is defined. Then we can form $\ker(\alpha - \beta)$. Consider the following diagram.

$$\begin{tikzcd}
 & \mathbf{1} \arrow[dr, "\alpha-\beta"] & \\
 & \ker(\alpha - \beta) \arrow[u, "k"] \arrow[r, "0"'] & Z \\
X \otimes X^\vee \arrow[uur, "\operatorname{ev}"] \arrow[ur, dashed, "\exists! u"] \arrow[urr, "0"'] & &
\end{tikzcd}$$

The diagram describes a well-defined situation, since by *preadditivity* of $\mathcal{C}$, $\alpha \circ \operatorname{ev} = \beta \circ \operatorname{ev} \iff (\alpha - \beta) \circ \operatorname{ev} = 0$. Then we see that there is a *unique* morphism $u : X \otimes X^\vee \to \ker(\alpha - \beta)$ such that $\operatorname{ev} = k \circ u$, where $k : \ker(\alpha - \beta) \to \mathbf{1}$ is a monomorphism (see 10.21). This means that $\ker(\alpha - \beta)$ is a *subobject* of $\mathbf{1}$. By assumption, $\operatorname{End}(\mathbf{1})$ is a field, so by 10.30, $\mathbf{1}$ is a *simple object*. Therefore, we know that $\ker(\alpha - \beta) = 0$ or $\ker(\alpha - \beta) = \mathbf{1}$.

If $\ker(\alpha - \beta) = \mathbf{1}$, then k must be an isomorphism, since $\operatorname{End}(\mathbf{1})$ is a field. It follows that $k \circ (\alpha - \beta) = 0 \iff (\alpha - \beta) = k^{-1} \circ 0$. Since 0 work as the additive identity in an abelian group, we have

$$\begin{aligned}
& k^{-1} \circ 0 = k^{-1} \circ (0 + 0) \\
\iff & k^{-1} \circ 0 = 0 \\
\implies & \alpha - \beta = 0 \\
\implies & \alpha = \beta \qquad \text{(contradiction!)}.
\end{aligned}$$

So $\ker(\alpha - \beta) = 0$. Therefore, $k : 0 \to \mathbf{1}$ is the zero morphism.

Then we get the following diagram

$$\begin{tikzcd}
& \mathbf{1} \arrow[rd, "\alpha-\beta"] & \\
& 0 \arrow[u, "k=0"'] \arrow[r, "0"'] & Z \\
X \otimes X^\vee \arrow[ruu, "\mathrm{ev}"] \arrow[ru, dashed, "\exists!u"] \arrow[rru, "0"'] & &
\end{tikzcd}$$

so that $\mathrm{ev} = 0 \circ u$. Hence ev must be the zero morphism, since any zero morphism factors as $A \to 0 \to B$ ([8, chapter 2, VIII]), so in particular $\mathrm{ev} : X \otimes X^\vee \overset{u}{\longrightarrow} 0 \overset{k}{\longrightarrow} \mathbf{1} = 0_{X \otimes X^\vee, \mathbf{1}}$.

The chain of natural isomorphisms $\mathrm{Hom}(X \otimes X^\vee, \mathbf{1}) \simeq \mathrm{Hom}(X^\vee \otimes X, \mathbf{1}) \simeq \mathrm{Hom}(X^\vee, X^\vee)$ is in fact an *isomorphism of abelian groups*[1]. Note that map between hom-sets defined by pre-or-post-composition by some map f is an abelian group homomorphism. This explains why the first natural isomorphism above is a group homomorphism. The second natural isomorphism is an *adjunction*. We see that $\mathrm{ev} \mapsto 0$ under this isomorphism. This follows from the fact that $\mathrm{Hom}(X \otimes X^\vee, \mathbf{1}) \to \mathrm{Hom}(X^\vee \otimes X, \mathbf{1})$ is explicitly defined by $f \mapsto \tilde{f} := f \circ \psi$. So $\mathrm{Hom}(X \otimes X^\vee, \mathbf{1}) \to \mathrm{Hom}(X^\vee \otimes X, \mathbf{1}) \to \mathrm{Hom}(X^\vee, X^\vee)$ maps $\mathrm{ev} \circ \psi$ as follows:

$$\mathrm{ev} \circ \psi \mapsto \mathrm{ev} \circ \psi^2 = \mathrm{ev} \mapsto \mathrm{id}\,.$$

But recall that $\mathrm{ev} = 0$, hence the identity morphism for $X^\vee$ must in fact be the zero-morphism (since a group homomorphism takes zero to zero). It follows that $X^\vee$ is the zero-object. Since $\mathcal{C}$ is rigid, each object is reflexive, so $X^{\vee\vee} \cong X \implies 0^\vee \cong X$. One checks that $0^\vee = 0$ (using that the associated evaluation and coevaluation maps are zero-morphisms, together with the snake-identity), hence $X = 0$.

To prove the theorem, note that $X \otimes X^\vee, F(X \otimes X^\vee)$ are internal hom-objects, and that F preserves the associated counit from $X \otimes X^\vee$. Therefore, if $X \neq 0$, then $\mathrm{ev} \circ \psi : X \otimes X^\vee \to \mathbf{1}$ is an epimorphism, and F preserves epimorphisms, so $F(\mathrm{ev} \circ \psi) : F(X \otimes X^\vee) \to F(1) \cong \mathbf{1'}$ is an epimorphism, and hence $FX \neq 0$. This follows from the fact that if $FX = 0$, then $F(f)$ would be the zero-morphism $0 \to \mathbf{1'}$, which is *not* an epimorphism. The criterion $\mathbf{1'} \neq 0$ ensures that this holds, since otherwise, we would have that $FX = 0$ would imply that the counit was the *unique* map $0 \to 0$, which must be an isomorphism, hence epi (this shows why this fails unless $\mathbf{1'} \neq 0$).

But the argument in the preceding paragraph is the contrapositive of $FX = 0 \implies X = 0$. Hence F is faithful by 10.39. ☐

Definition 10.42. When $(\mathcal{C}, \otimes)$ is an abelian tensor category, then we say that a family of objects $\{X_i\}_{i \in I}$ is a **tensor generating family** for $\mathcal{C}$, if all objects of $\mathcal{C}$ are isomorphic to a subquotient of $P(X_i)$ for some $P(t_i) \in \mathbb{N}[t_i]_{i \in I}$.

Remark 10.43. Note that $P(t_i)$ is some *polynomial* in the variables t_i for $i \in I$, interpreted so that if e.g. $I = \{1, 2, 3\}$ with $P(t_i) = t_1^2 + t_2 + t_1 t_3 \in \mathbb{N}[t_1, t_2, t_3]$, then

$$P(X_i) = (X_1 \otimes X_1) \oplus (X_2) \oplus (X_1 \otimes X_3).$$

Furthermore, we have an action of $R := \mathrm{End}(\mathbf{1})$ on objects X in $\mathcal{C}$ as in the diagram below.

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[1] See e.g. [19].

$$\begin{tikzcd}
\mathbf{1} \arrow[d, "\varphi"'] & \rightsquigarrow & X \arrow[r, "l_X"] \arrow[d, "\psi"] & \mathbf{1} \otimes X \arrow[d, "\varphi\otimes\mathrm{id}"] \\
\mathbf{1} & & X & \mathbf{1} \otimes X \arrow[l, "l_X^{-1}"]
\end{tikzcd}$$

where $\psi := l_X^{-1} \circ \varphi \otimes \mathrm{id} \circ l_X \in \mathrm{End}(X)$. So we have an action $\mathcal{A} : R \times X \to X$ defined so that

$$\mathcal{A}(\varphi, -) : X \to X.$$

For arbitrary $f \in \mathrm{End}(X)$ and ψ induced from $\mathcal{A}$, we have

$$\begin{aligned}
f \circ \psi &= f \circ (l_X^{-1} \circ \varphi \otimes \mathrm{id} \circ l_X) \\
&= l_X^{-1} \circ \mathrm{id} \otimes f \circ \varphi \otimes \mathrm{id} \circ l_X \\
&= l_X^{-1} \circ \varphi \otimes \mathrm{id} \circ \mathrm{id} \otimes f \circ l_X \\
&= (l^{-1} \circ \varphi \otimes \mathrm{id} \circ l_X) \circ f \\
&= \psi \circ f,
\end{aligned}$$

where we have used naturality of l, l^{-1} and bifunctoriality of $\otimes$. By applying this with $X = \mathbf{1}$, and using the commutativity of the diagram below

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "l_{\mathbf{1}}=e"] \arrow[d, "\varphi"'] & \mathbf{1} \otimes \mathbf{1} \arrow[d, "\varphi\otimes\mathrm{id}"] \\
\mathbf{1} & \mathbf{1} \arrow[l, "l_{\mathbf{1}}^{-1}=e^{-1}"]
\end{tikzcd}$$

then we see that $R = \mathrm{End}(\mathbf{1})$ is in fact a commutative ring.

Remark 10.44. To see that R is a ring, note that R has an abelian group structure coming from the **Ab**-enrichment, and $\circ$ acts as multiplication, with $\mathrm{id}_{\mathbf{1}}$ the multiplicative identity.

If we let $f : A \to B$ be an arbitrary morphism in an abelian tensor category $(\mathcal{C}, \otimes)$, and $r \in R$, then we can define

$$r \cdot f := l_Y^{-1} \circ (r \otimes f) \circ l_X : X \to Y.$$

By biadditivity of $\otimes$ and the left and right distributivity of $\circ$, this defines an action of R on the hom-sets of $(\mathcal{C}, \otimes)$. For example, we note that, again for arbitrary morphism $f : X \to Y$, we have

$$\mathrm{id}_{\mathbf{1}} \cdot f = l_Y^{-1} \circ (\mathrm{id} \otimes f) \circ l_X = f$$

by naturality of l. This endows $\mathrm{Hom}_{\mathcal{C}}(A, B)$ with an R-module structure.

One checks that the action defined is such that $\circ$ is R-bilinear with respect to it. For example, if $r \in R$ and $f : X \to Y$ and $g : Y \to Z$, then

$$\begin{aligned}(r \cdot g) \circ f &= l_Z^{-1} \circ (r \otimes g) \circ l_Y \circ f \\ &= l_Z^{-1} \circ (r \otimes g) \circ (\mathrm{id}_{\mathbf{1}} \otimes f) \circ l_X \\ &= l_Z^{-1} \circ (r \otimes gf) \circ l_X \\ &= r \cdot (g \circ f),\end{aligned}$$

where we have used naturality of l and bifunctoriality of $\otimes$.

11 Criterion for rigid abelian tensor categories

Definition 11.1. Let $\mathbf{Vect}_{\Bbbk}$ denote the *category* consisting of objects as finite-dimensional vector spaces over a field $\Bbbk$, with morphisms as *linear maps*.

As in [10], we present the following proposition, which characterizes rigid abelian tensor categories.

Proposition 11.2. *Let $\mathcal{C}$ be a $\Bbbk$-linear abelian tensor category, such that $\Bbbk$ is a field, and let $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a $\Bbbk$-bilinear functor. Suppose that we have an exact, $\Bbbk$-linear, faithful functor $F : \mathcal{C} \to \mathbf{Vect}_{\Bbbk}$, a natural isomorphism $\phi_{X,Y,Z}$ (associator), and a natural isomorphism $\psi_{X,Y}$ (commutator) such that the following holds:*

(a) $F \circ \otimes = \otimes \circ (F \times F)$.

(b) $F(\phi_{X,Y,Z})$ *is the associativity isomorphism in* $\mathbf{Vect}_{\Bbbk}$.

(c) $F(\psi_{X,Y})$ *is the commutativity isomorphism in* $\mathbf{Vect}_{\Bbbk}$.

(d) There exists an identity object U in $\mathcal{C}$, such that $\Bbbk \to \mathrm{End}(U)$ is an isomorphism, and $F(U)$ has dimension one.

(e) If $F(L)$ has dimension one, then there is an object $L^{-1} \in \mathcal{C}$ such that $L \otimes L^{-1} = U$.

Then it follows that $(\mathcal{C}, \otimes, \phi, \psi, U)$ is a rigid abelian tensor category. □

12 Main theorem

We arrive at the main theorem of this article, as presented in [10]. We will not prove this theorem, only state it. We will introduce two more definitions, before stating the theorem.

Definition 12.1. A rigid abelian tensor category $\mathcal{C}$ with $\Bbbk = \mathrm{End}(\mathbf{1})$ a field, is called a **neutral tannakian category** over $\Bbbk$, if it admits an *exact*, *faithful*, $\Bbbk$-linear tensor functor (7.1) $\omega : \mathcal{C} \to \mathbf{Vect}_{\Bbbk}$.

Definition 12.2. Any such functor ω as in 12.1 is called a **fibre functor**.

Theorem 12.3. *Let $(\mathcal{C}, \otimes)$ be a* rigid *(6.1),* abelian *(10.24) tensor category, such that $\mathrm{End}(\mathbf{1}) = \Bbbk$, for a field $\Bbbk$, and let $\omega : \mathcal{C} \to \mathbf{Vect}_{\Bbbk}$ be an exact, faithful, $\Bbbk$-linear tensor functor. Then the functor $\mathcal{C} \to \mathrm{Rep}_{\Bbbk}(G)$ defined by ω, is an* equivalence of categories.

Proof. See [10, theorem 2.11.(b), p. 21]. □

Remark 12.4. Although we did state in our introduction that G was some group, G in 12.3 is really some *affine algebraic group scheme.*

What the statement above then claims, is that any neutral tannakian category $\mathcal{C}$ that admits a fibre functor from $\mathcal{C}$ to $\mathbf{Vect}_{\Bbbk}$ induces an equivalence of categories between $\mathcal{C}$ and the category of linear representations of some affine algebraic group scheme G, on finite dimensional $\Bbbk$-vector spaces.

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