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The Calculus of Variations and the Euler-Lagrange Equation: History, Principles, and Applications

av

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The Calculus of Variations and the Euler-Lagrange Equation: History, Principles, and Applications

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Abstract

English: This thesis explores the historical context, foundational principles, and significant applications of the Calculus of Variations, focusing particularly on the Euler-Lagrange equation. By examining these elements, the thesis aims to highlight the relevance and impact of this mathematical discipline in addressing complex optimization problems.

Swedish: Denna avhandling utforskar det historiska sammanhanget, de grundläggande principerna och de betydande tillämpningarna av variationskalkylen, med särskilt fokus på Euler-Lagrange-ekvationen. Genom att granska dessa element syftar avhandlingen till att belysa relevansen och påverkan av denna matematiska disciplin vid hantering av komplexa optimeringsproblem.

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1 Introduction

1.1 Overview

Throughout history, humans have strived for innovation and optimization, leading to many breakthroughs that have greatly improved our lives. This drive for progress is evident in our efforts to improve functionality, encourage creativity, gain recognition in the scientific community, and create economic value. These efforts have transformed our work and lifestyle, creating new opportunities and industries while significantly simplifying our daily lives.

One of the most significant areas of optimization is found in mathematics, specifically in the field of Calculus of Variations. This branch of mathematics focuses on identifying optimal solutions to problems involving functionals, which are essentially functions of functions. A key tool in this field is the Euler-Lagrange equation, which plays a crucial role in solving optimization problems.

1.2 Roadmap

This thesis is structured as follows:

- **Section 1**: Provides a brief history of the Calculus of Variations, tracing its origins and highlighting the contributions of key figures such as Johann Bernoulli, Leonhard Euler, and Joseph-Louis Lagrange. This section sets the historical context for the development of the field.
- **Section 2**: Discusses the fundamental concepts and principles of the Calculus of Variations, including differentiation, extrema, and the distinction between functions and functionals. This section lays the groundwork for understanding more advanced topics.
- **Section 3**: Introduces the Euler-Lagrange equation, detailing its derivation and significance. The section also covers necessary conditions for extremality and the second variation, which are crucial for determining optimality.
- **Section 4**: Explores practical applications of the Euler-Lagrange equation through detailed examples, such as the brachistochrone problem and the catenary. These examples demonstrate how the Calculus of Variations can be applied to solve real-world optimization problems.
- **Section 5**: Summarizes the key findings of the thesis, highlighting the importance of the Calculus of Variations in both theoretical and practical contexts. It also discusses potential future directions and open problems in the field.

By following this structure, the thesis aims to provide a comprehensive understanding of the Calculus of Variations and its applications, demonstrating its value in addressing complex optimization challenges across various disciplines.
The main goal of this study is to explore the history, fundamental principles, and important applications of the Calculus of Variations, with a specific focus on the Euler-Lagrange equation. By looking at its development and various uses, we aim to highlight the importance and impact of this mathematical field in solving complex problems in different areas. These areas include physics, engineering, economics, and biology, where optimization problems often occur.Through the analysis of these applications, we aim to underscore the importance and impact of this mathematical discipline in addressing complex problems across various fields.

1.3 Brief History:

When it comes to the origins of the Calculus of Variations, Johann Bernoulli is often credited with its beginnings. In 1696, he posed the Brachistochrone problem in a letter to European mathematicians, asking which curve between two points would be traversed in the shortest time by a particle under gravity. The motive behind this challenge was said to be Bernoulli's desire to showcase his intelligence and outshine his rival, his brother Jakob, who was also a brilliant mathematician. Johann firmly believed that he was the greatest mathematician of his time, surpassing not only his brother but also other well-known figures such as Leibniz and Isaac Newton.

Figure 1: Johann Bernoulli (1667–1748)

Figure 2: Jakob Bernoulli (1654-1705)

Figure 3: Gottfried Wilhelm Leibniz (1646-1716)

Newton, who was semi-retired from mathematics at the time, was unimpressed at being tested, especially by someone he considered of lesser ability. However, he stayed up all night to solve the problem and submitted the solution anonymously to the *Philosophical Transactions* journal. When Johann saw the anonymous solution, he famously said, "I recognize the lion by his claw," acknowledging Newton's brilliance.

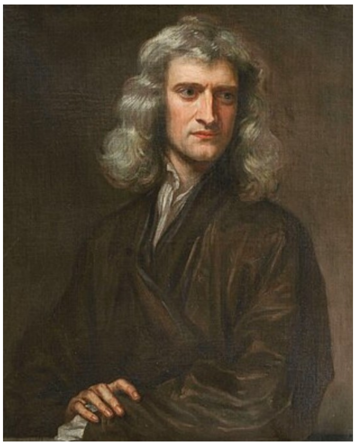

Figure 4: Isaac Newton (1643 - 1727

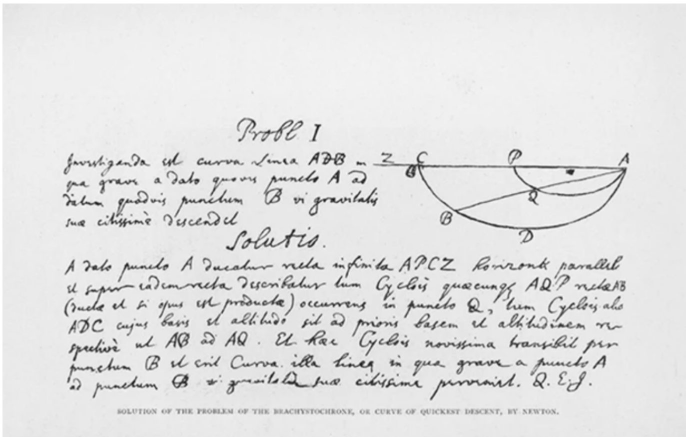

Figure 5: solution for the brachistochrone problem by Newton

Later,In the mid-18th century, Leonhard Euler made significant contributions to the Calculus of Variations, developing fundamental concepts. In 1755, a young Joseph-Louis Lagrange, studied a problem known as the Tautochrone problem, which asks for the curve on which a particle takes the same time to descend regardless of its starting point. Lagrange wrote to Euler about his new methods for solving these problems. Euler was so impressed that he adopted Lagrange's analytical approach, marking a significant advancement in the field.

In 1756, Euler published a paper on the Calculus of Variations, extending and generalizing Lagrange's equations to apply to a much wider range of problems. This collaboration and mutual influence between Euler and Lagrange significantly advanced the field, showing how the Calculus of Variations could be used to solve various theoretical and practical problems.

Figure 6: Leonhard Euler (1707–1783)

Figure 7: Joseph-Louis Lagrange (1736–1813)

These examples illustrate the crucial role the Calculus of Variations plays in both theoretical mathematics and real-world applications. By continuing to investigate and utilize these mathematical techniques, we can drive further advancements in efficiency and innovation across various fields, helping us optimize and enhance many aspects of our daily lives.

2 Fundamental Concepts and Principles

To understand the Calculus of Variations, we must begin with some fundamental concepts from calculus, particularly differentiation and optimization. This section reviews these basics, laying the groundwork for more advanced topics.

2.1 Differentiation and Extrema

In this section, we will revisit and explore the core ideas of differentiation and the concepts of maxima and minima. Providing examples with visual aids will ensure a deeper understanding and pave the way for delving into more advanced topics.

Definition 2.1 (Derivative)**.** *Differentiation is the process of finding the derivative of a function. The derivative of a function $f(x)$ at a point x is defined as the limit:*

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The derivative, denoted as $f'(x)$ or $\frac{df}{dx}$, represents the rate at which the function $f(x)$ is changing with respect to x. (Figure 8)

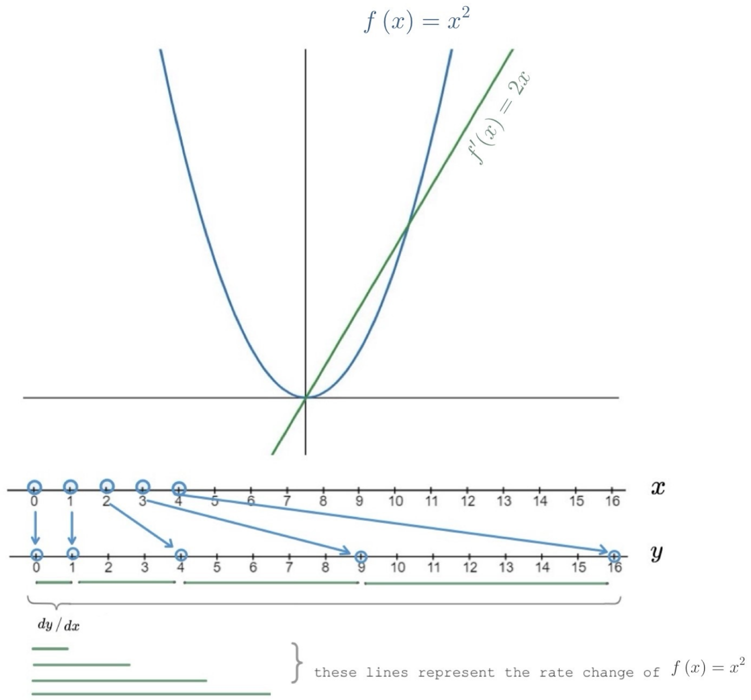

Figure 8: The rate of change of $f(x) = x^2$

Differentiation helps us understand how a function behaves at any given point by revealing the slope of the tangent line to the graph of $f(x)$ at that point (Figure 9). By examining the rate of change, we can also identify points of local maxima and minima. These are points where the function reaches local extremes, and they occur where the derivative $f'(x)$ is zero.

Definition 2.2 (Local Extrema)**.** *Local extrema are points in a function where the function reaches a local maximum or minimum. A point c is a local minimum of $f(x)$ if there exists an $\epsilon > 0$ such that for all x with $\|x - c\| < \epsilon$, $f(c) \leq f(x)$. Similarly, c is a local maximum if $f(c) \geq f(x)$ for all x in the neighborhood.*

Theorem 2.3 (Fermat's Theorem)**.** *Let $f : (a, b) \to \mathbb{R}$. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

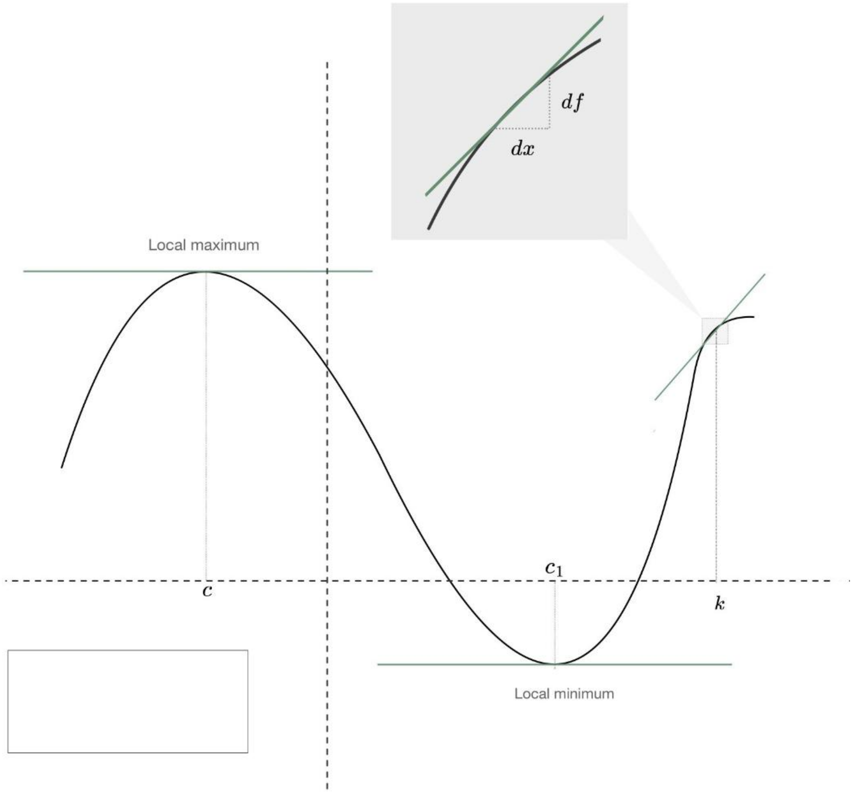

Figure 9

Identifying local minima and maxima involves finding points where the function values are either lower or higher than those at nearby points. These points, called stationary or critical points, occur where the slope (derivative) is zero (Figure 9). This is critical for understanding the function's behavior and for solving optimization problems in calculus and mathematical analysis.

Motivations:

Optimization: In real-world problems, finding local extremes helps in optimizing processes, costs, and performances.

Understanding Functions: Analyzing local extremes aids in comprehending the function's shape and behavior over its domain.

Conditions and Rules:

1. **Differentiability**: Generally, for a function to have a local minimum or maximum at a point, it should be differentiable in a neighborhood around that point.
2. **Critical Points Identification**: A critical point of a function $f(x)$ occurs where its derivative $f'(x)$ is zero or undefined. These points are potential candidates for local minima or maxima.

2.1.1 The First Derivative Test

The first derivative test involves analyzing the sign of the derivative before and after the critical point. If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a local maximum. If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a local minimum.

Closed Interval Method: When finding local minima or maxima on a closed interval $[a, b]$, besides critical points within the interval, check the function's values at the endpoints, a and b, as potential minima or maxima.

2.1.2 The Second Derivative Test

This method is used to determine whether a critical point of a function is a local maximum, a local minimum, or inconclusive. Given a function $f(x)$, this test examines the second derivative at the critical point.

- If $\frac{d^2f}{dx^2} > 0$ at a critical point, the function is concave up at that point, indicating a local minimum.
- If $\frac{d^2f}{dx^2} < 0$ at a critical point, the function is concave down at that point, indicating a local maximum.
- If $\frac{d^2f}{dx^2} = 0$ at a critical point, the test is inconclusive. Higher-order derivatives or other methods may be needed to determine the nature of the critical point.

This test is a local property, meaning it applies to individual points rather than the entire function. It helps us understand the behavior of functions near those critical points, which is essential for optimization problems discussed later in this thesis. The concepts of maxima and minima, and the methods used to find them, are foundational in many applications of the Calculus of Variations.
To make this more concrete, consider the example of a moving car traveling from point A to point B.

Example 2.4. *We can analyze the motion of a car traveling from point A to point B by looking at two different figures:*

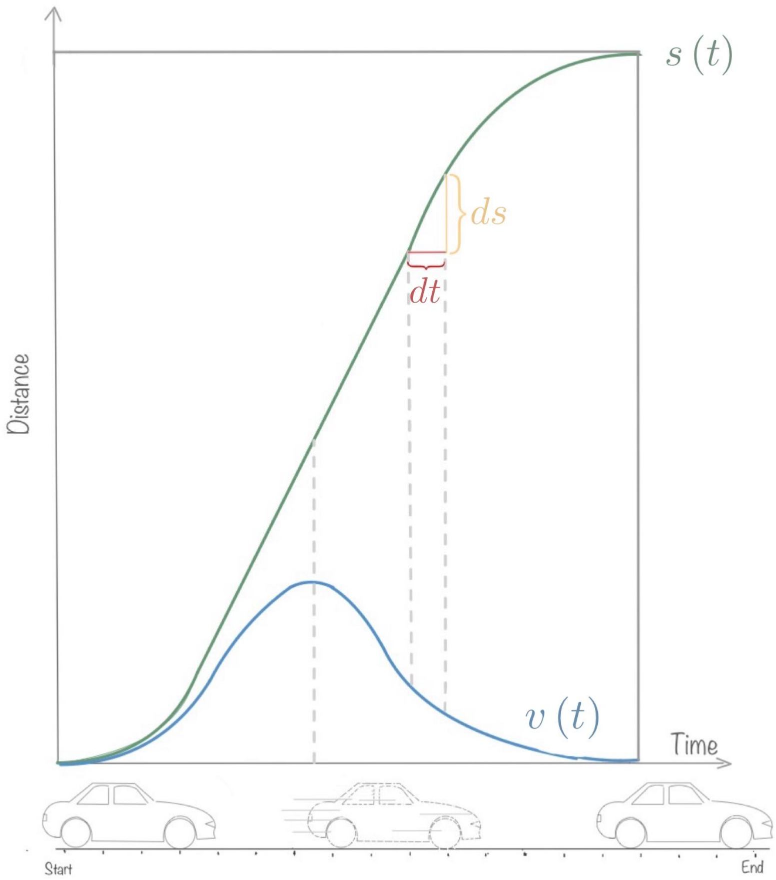

Figure 10: Graphs representing the motion of a car from point A to point B.

In Figure 10, we see the distance-time graph $s(t)$. This graph shows how the car's distance from the starting point changes over time. At any time t, The height of the graph at any given time t indicates the total distance the car has traveled. A steeper slope in the graph corresponds to an increase in the distance traveled in a specific time interval as the car accelerates. Conversely, the curve becomes less steep when the car decelerates. Plotting the rate of change of the function $s(t)$ gives us the velocity curve $v(t)$, which is the change in distance divided by the change in time $\frac{ds}{dx}$. In the middle of the journey, the car reaches a maximum velocity, corresponding to the larger change on the distance curve $s(t)$, and a maximum point on the curve $v(t)$.

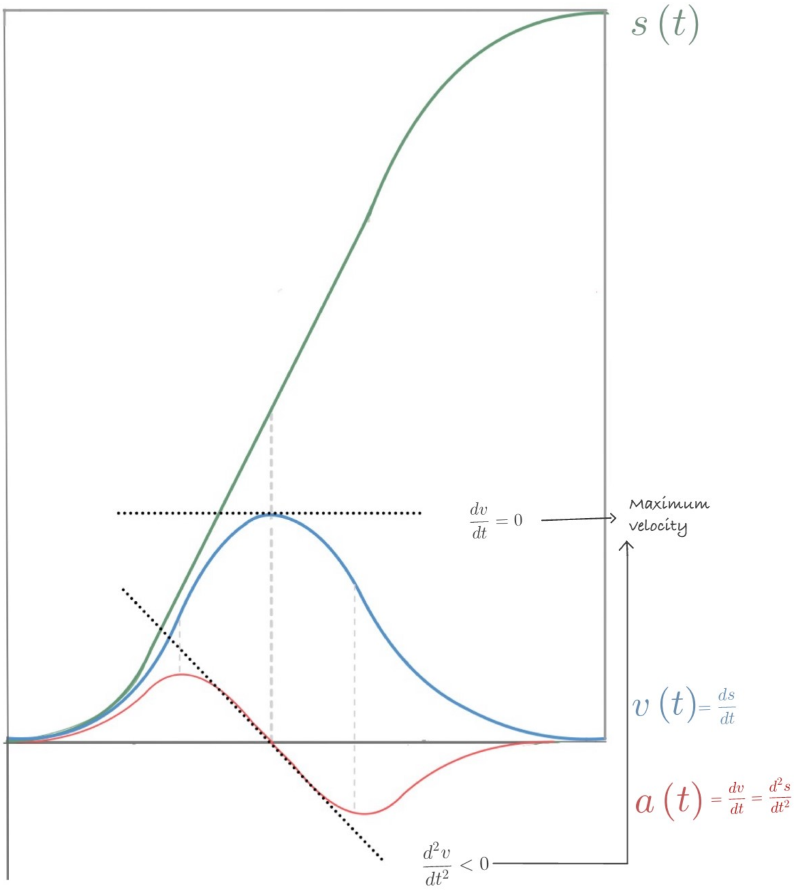

Figure 11: Distance, velocity, and acceleration vs. time.

In Figure 11, we show three graphs: $s(t)$, $v(t)$, and $a(t)$. Here, we have added the graph showing the car's acceleration, $a(t)$, which is the change in velocity divided by the change in time $\frac{dv}{dt}$. When the acceleration $a(t)$ is zero, it indicates that the velocity is not changing at that instant, which marks a stationary point on the curve $v(t)$. We can see in the figure that this point is a maximum, and observing the graph of $a(t)$, we see whether the car's speed is increasing (positive acceleration) or decreasing (negative acceleration). This aligns with the first derivative test (Section 2.1.1). By taking the second derivative of $v(t)$ at this point, which is the derivative of $a(t)$, we see that the slope $\frac{d^2v}{dt^2} < 0$, confirming it as a local maximum according to the second derivative test (Section 2.1.2).

The principles of differentiation, specifically the first and second derivatives, offer deeper insights into maxima and minima. Understanding these basic concepts is essential as we move forward to more complex topics in the Calculus of Variations.

2.2 Function and Functional

In mathematics, understanding the distinction between functions and functionals is crucial, especially in the context of the Calculus of Variations.
A function is a relation between a set of inputs and a set of permissible outputs, where each input is related to exactly one output. Mathematically, a function f from a set X to a set Y is defined as:

$$f : X \to Y$$

This means that for every element $x \in X$, there exists a unique element $y \in Y$ such that $y = f(x)$.
Example: Consider the function $f(x) = x^2$, which maps real numbers to real numbers. For each input x, the output is the square of x. Here, $X = Y = \mathbb{R}$, the set of real numbers.
In contrast, a functional maps functions to real numbers. In other words, a functional is a function of

functions. Formally, a functional J can be expressed as:

$$J[y] = \int_a^b F(x, y(x), y'(x))\, dx$$

where $y(x)$ is a function, $y'(x)$ is its derivative, and F is a given function of x, $y(x)$, and $y'(x)$.
Example: Consider the functional

$$J[y] = \int_0^1 (y'(x))^2\, dx$$

This functional takes a function $y(x)$ and maps it to a real number, which is the integral of the square of its derivative over the interval $[0, 1]$.
The distinction between a function and a functional is analogous to the difference between a vector and a vector field. Just as a vector field assigns a vector to each point in space, a functional assigns a real number to each function within its domain.
Example: For the function $y(x) = x^2$ defined on the interval $[0, 1]$, the functional

$$J[y] = \int_0^1 (y'(x))^2\, dx$$

can be evaluated by first finding the derivative $y'(x) = 2x$ and then computing the integral:

$$J[y] = \int_0^1 (2x)^2\, dx = \int_0^1 4x^2\, dx = \left[\frac{4x^3}{3}\right]_0^1 = \frac{4}{3}$$

Thus, the functional J maps the function $y(x) = x^2$ to the real number $\frac{4}{3}$.

2.3 Leibniz's Rule for Differentiation Under the Integral

Leibniz's rule is a fundamental theorem that allows us to differentiate an integral whose limits and integrand depend on a variable. This rule is particularly useful in the Calculus of Variations when deriving the Euler-Lagrange equation and handling integrals that involve parameters.

Theorem 2.5 (Leibniz's Rule)**.** *Let $\alpha(x)$ and $\beta(x)$ be differentiable functions, and let $f(x, t)$ be a function that is continuous and differentiable with respect to x and t. Then,*

$$\frac{d}{dx}\left(\int_{\alpha(x)}^{\beta(x)} f(x,t)\, dt\right) = f(x, \beta(x)) \cdot \beta'(x) - f(x, \alpha(x)) \cdot \alpha'(x) + \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} f(x,t)\, dt. \tag{2.1}$$

Proof. Consider the integral:

$$I(x) = \int_{\alpha(x)}^{\beta(x)} f(x,t)\, dt.$$

To find the derivative of $I(x)$ with respect to x, we express it as:

$$\frac{dI}{dx} = \lim_{h\to 0} \frac{1}{h}\left(\int_{\alpha(x+h)}^{\beta(x+h)} f(x+h,t)\, dt - \int_{\alpha(x)}^{\beta(x)} f(x,t)\, dt\right).$$

This difference can be decomposed into three integrals:

$$\frac{dI}{dx} = \lim_{h\to 0} \frac{1}{h}\left(\int_{\beta(x)}^{\beta(x+h)} f(x+h,t)\, dt - \int_{\alpha(x)}^{\alpha(x+h)} f(x+h,t)\, dt\right) + \lim_{h\to 0} \frac{1}{h}\int_{\alpha(x)}^{\beta(x)} (f(x+h,t) - f(x,t))\, dt.$$

1. Contribution from the variable limits:
- For the upper limit $\beta(x)$:

$$\int_{\beta(x)}^{\beta(x+h)} f(x+h,t)\, dt \approx f(x, \beta(x)) \cdot (\beta(x+h) - \beta(x)) = f(x, \beta(x)) \cdot \beta'(x) \cdot h.$$

- For the lower limit $\alpha(x)$:

$$\int_{\alpha(x)}^{\alpha(x+h)} f(x+h,t)\,dt \approx f(x,\alpha(x))\cdot(\alpha(x+h)-\alpha(x)) = f(x,\alpha(x))\cdot\alpha'(x)\cdot h.$$

2. Contribution from the integrand:
- The integrand term, expanded using Taylor's theorem:

$$\int_{\alpha(x)}^{\beta(x)} (f(x+h,t)-f(x,t))\,dt \approx \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}(x,t)\cdot h\,dt.$$

Combining these results, we have:

$$\frac{dI}{dx} = f(x,\beta(x))\cdot\beta'(x) - f(x,\alpha(x))\cdot\alpha'(x) + \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}(x,t)\,dt.$$

Thus, Leibniz's rule is established. □

To understand Leibniz's rule better, let's break it down step by step.
1. Variable Limits of Integration: The first part of Leibniz's rule deals with the differentiation of the integral when the limits of integration are functions of x. This involves evaluating the integrand at the upper and lower limits and multiplying by the derivatives of those limits.
If $\alpha(x)$ and $\beta(x)$ are the lower and upper limits of the integral respectively, and both are functions of x, the contributions from these moving limits are given by

$$f(x,\beta(x))\cdot\beta'(x) - f(x,\alpha(x))\cdot\alpha'(x). \tag{2.2}$$

2. Differentiation of the Integrand: The second part of Leibniz's rule accounts for the differentiation of the integrand itself with respect to x within the limits $\alpha(x)$ and $\beta(x)$. This is given by

$$\int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} f(x,t)\,dt. \tag{2.3}$$

Let's consider a practical example to see how this rule is applied.

Example 2.6 (Differentiating an Integral with Variable Limits)**.** *Suppose we have the integral*

$$I(x) = \int_0^x e^{xt}\,dt. \tag{2.4}$$

We want to find the derivative of $I(x)$ with respect to x.
Using Leibniz's rule, we identify:

- *The lower limit $\alpha(x)=0$ with $\alpha'(x)=0$.*
- *The upper limit $\beta(x)=x$ with $\beta'(x)=1$.*
- *The integrand $f(x,t)=e^{xt}$.*

Applying Leibniz's rule

$$\frac{d}{dx}\left(\int_0^x e^{xt}\,dt\right) = e^{x\cdot x}\cdot 1 - e^{x\cdot 0}\cdot 0 + \int_0^x \frac{\partial}{\partial x} e^{xt}\,dt. \tag{2.5}$$

Simplifying the first two terms

$$\frac{d}{dx}\left(\int_0^x e^{xt}\,dt\right) = e^{x^2} + \int_0^x t e^{xt}\,dt. \tag{2.6}$$

Here, the term $\frac{\partial}{\partial x}e^{xt} = te^{xt}$ arises from differentiating the integrand with respect to x.
This example illustrates how Leibniz's rule allows us to differentiate an integral with respect to a parameter, accounting for both variable limits and the change in the integrand.

2.4 Function Spaces and Norms

To understand the calculus of variations, we first need to grasp the fundamental concepts of function spaces and norms. These concepts are essential for analyzing the behavior of functionals, which are mappings from a space of functions to real numbers.

Definition 2.7 (Norms in Function Spaces). *A norm on a function space assigns a non-negative length or size to each function within the space. A norm $\|\cdot\|$ satisfies the following properties for any functions f and g, and any scalar α:*

1. ***Non-negativity:*** $\|f\| \geq 0$ *and* $\|f\| = 0$ *if and only if* $f = 0$.
2. ***Scalar Multiplication:*** $\|\alpha f\| = |\alpha|\|f\|$.
3. ***Triangle Inequality:*** $\|f + g\| \leq \|f\| + \|g\|$.

In the space of continuous functions defined on a closed interval $[a, b]$, known as $C[a, b]$, the norm, often called the *supremum norm* or *infinity norm*, is defined as:

$$\|f\|_C = \sup_{x\in[a,b]} |f(x)| = \max_{x\in[a,b]} |f(x)|. \tag{2.7}$$

For the space of continuously differentiable functions on $[a, b]$, denoted by $C^1[a, b]$, the norm is given by:

$$\|f\|_{C^1} = \sup_{x\in[a,b]} (|f(x)| + |f'(x)|) = \max_{x\in[a,b]} (|f(x)| + |f'(x)|). \tag{2.8}$$

The *distance* between two functions f and g in these spaces is determined using these norms, defined as:

$$d(f, g) = \|f - g\|. \tag{2.9}$$

These norms allow us to quantify how "large" a function is and how "close" two functions are to each other, providing a rigorous framework for analyzing variations in functionals.

2.5 The fundamental lemma of the calculus of variations

The Fundamental Lemma of Calculus of Variations indicates that if the integral of the product of a continuous function $f(x)$ with any arbitrary smooth function $\eta(x)$ that vanishes at the interval boundaries is zero, then the function $f(x)$ itself must be identically zero within that interval.
Formally, the lemma, which is essential for validating the Euler-Lagrange equation in the calculus of variations, can stated as follows:

Lemma 2.8 (The Fundamental Lemma of Calculus of Variations). *If $f(x)$ is a continuous function on a closed interval $[x_1, x_2]$, and if for every smooth function $\eta(x)$ satisfying $\eta(x_1) = \eta(x_2) = 0$, the integral condition*

$$\int_{x_1}^{x_2} f(x)\eta(x)\, dx = 0$$

is met, then $f(x)$ must identically be zero for all $x \in [x_1, x_2]$.

Proof. To prove this lemma by contradiction, we suppose contrary to our claim, that there exists a point $a \in [x_1, x_2]$ where $f(a) > 0$. Since $f(x)$ is continuous, there exists a subinterval $[a_1, a_2] \subset [x_1, x_2]$ around a where $f(x) > 0$. Define a function $\eta(x)$ such that

$$\eta(x) = \begin{cases} 0, & x < a_1 \text{ or } x > a_2, \\ (x - a_1)^2(x - a_2)^2, & a_1 \leq x \leq a_2. \end{cases}$$

The integral of interest then becomes

$$I = \int_{x_1}^{x_2} f(x)\eta(x)\, dx = \int_{a_1}^{a_2} f(x)(x - a_1)^2(x - a_2)^2\, dx > 0.$$

Since $I > 0$, this contradicts the assumption that the integral of $f(x)\eta(x)$ over $[x_1, x_2]$ is zero for all such $\eta(x)$. Therefore, by contradiction, $f(x)$ must be zero everywhere on $[x_1, x_2]$.

Thus, the lemma is proven by contradiction.

□

3 The Euler-Lagrange Equation

In this section, we delve into the calculus of variations, focusing on the Euler-Lagrange equation. We introduce essential concepts such as stationary paths, and optimal paths, followed by a rigorous derivation of the Euler-Lagrange equation. Additionally, we discuss the necessary conditions for extremality and provide a detailed examination of the second variation.

3.1 The First Variation

With an understanding of function spaces and norms, we can now explore how these concepts apply to the calculus of variations. Our focus will be on deriving the Euler-Lagrange equation, a fundamental result that provides the necessary conditions for a functional to have a stationary point. This equation is crucial for identifying optimal paths and solutions within a given function space.

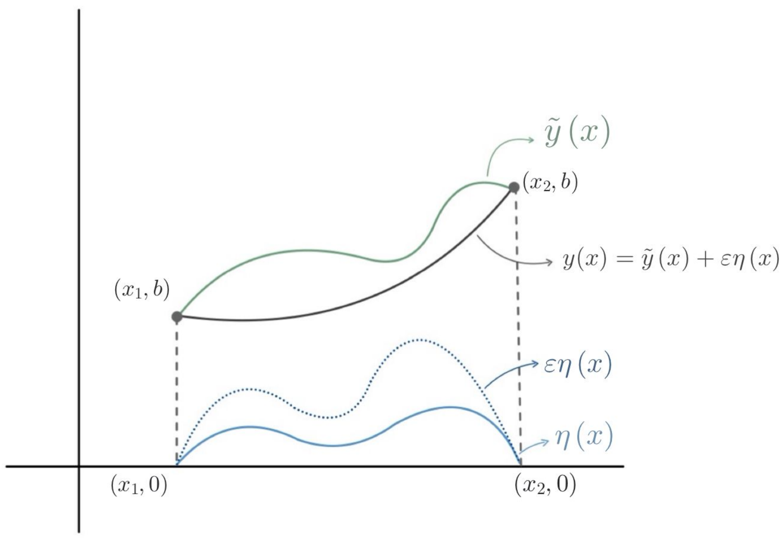

Figure 12: Illustration of the Euler-Lagrange Equation

Theorem 3.1 (Euler–Lagrange Equation). *Let $J[y] = \int_{x_1}^{x_2} F(x, y, y')\,dx$ be a functional where $F(x, y, y')$ is continuously differentiable with respect to y and y'. If $y(x)$ is a function that makes $J[y]$ stationary, and $y(x)$ satisfies the boundary conditions $y(x_1) = A$ and $y(x_2) = B$, then $y(x)$ satisfies the Euler-Lagrange equation:*

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0.$$

This provides the necessary condition for $y(x)$ to be an extremal of J.

Proof. Let's start with two given points, represented by x_1 and x_2. Our goal is to find a function of x that makes the following functional—a function of functions—stationary. The Euler-Lagrange equation is fundamental in the calculus of variations, aiming to determine the path, $y(x)$, that either minimizes or maximizes a specified functional, $J[y(x)]$. This functional is defined as:

$$J[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx \tag{3.1}$$

where F is the Lagrangian, dependent on x, the function $y(x)$, and its derivative $y'(x)$.

Definition 3.2 (Stationary and Optimal Paths). *In the calculus of variations, we consider paths that make a functional $J[y]$ stationary or optimal. Below are the key definitions and concepts involved:*

- ***y(x)*** *represents the optimal path. It is the particular function that makes the functional stationary, meaning the first variation of the functional vanishes, indicating an extremum. Specifically, a path $y(x)$ is stationary if:*

$$\left.\frac{dJ[y+\varepsilon\eta]}{d\varepsilon}\right|_{\varepsilon=0} = 0,$$

  for all variations $\eta(x)$ satisfying $\eta(x_1) = \eta(x_2) = 0$. An optimal path further satisfies the second variation condition, ensuring it is a minimum or maximum.

- ***η(x)*** *represents an arbitrary variation function with boundary conditions $\eta(x_1) = \eta(x_2) = 0$, ensuring the variations vanish at the endpoints. This condition allows us to explore how small changes in the path affect the functional.*

- ***ỹ(x)*** *indicates a variation of the optimal path $y(x)$, expressed as:*

$$\tilde{y}(x) = y(x) + \varepsilon\eta(x) \tag{3.2}$$

  This represents a family of possible curves around the stationary path, where ε is a small parameter.

It is assumed that y, $\tilde{y}$, and η all have continuous second derivatives in the interval $[x_1, x_2]$. This smoothness ensures that the necessary calculus operations are well-defined and applicable for deriving conditions like the Euler-Lagrange equation.

From (3.2), we see that $\tilde{y}(x)$ represents a family of functions deviating from the original path $y(x)$ by a scaled variation $\varepsilon\eta(x)$. Here, $\eta(x)$ is an arbitrary, yet smooth, function that vanishes at the boundary points, and ε is a small parameter scaling the deviation.
As ε modifies $\tilde{y}(x)$, it inherently adjusts $J[\tilde{y}]$, making $J[\tilde{y}]$ a function of ε. To find the conditions under which $J[\tilde{y}]$ is stationary, its derivative with respect to ε must equal zero at $\varepsilon = 0$. This evaluation point is chosen because $\varepsilon = 0$ corresponds to the optimal function $y(x)$ without any variation:

$$\left.\frac{dJ[\tilde{y}]}{d\varepsilon}\right|_{\varepsilon=0} = 0. \tag{3.3}$$

This implies that for the functional J to be stationary at $y(x)$, the first-order change in J with respect to ε must vanish when ε is zero. This ensures that $y(x)$ is a function for which $J[y(x)]$ has an extremal value (either a minimum or maximum), leading to the Euler-Lagrange equation. Thus, the evaluation at $\varepsilon = 0$ is crucial for determining the stationary condition of the functional.
Applying this to our functional yields:

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_{x_1}^{x_2} F(x, \tilde{y}, \tilde{y}')\, dx = 0, \tag{3.4}$$

By applying *Leibniz's rule* (2.5) for differentiation under the integral, we arrive at:

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left.\frac{d}{d\varepsilon}F(x, \tilde{y}, \tilde{y}')\right|_{\varepsilon=0} dx = 0. \tag{3.5}$$

Expanding the derivative inside the integral via the chain rule:

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left.\left[\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial \varepsilon} + \frac{\partial F}{\partial \tilde{y}}\cdot\frac{\partial \tilde{y}}{\partial \varepsilon} + \frac{\partial F}{\partial \tilde{y}'}\cdot\frac{\partial \tilde{y}'}{\partial \varepsilon}\right]\right|_{\varepsilon=0} dx = 0. \tag{3.6}$$

From (3.2), we see that:

$$\frac{\partial \tilde{y}}{\partial \varepsilon} = \eta, \quad \frac{\partial \tilde{y}'}{\partial \varepsilon} = \eta',$$

And given $\frac{\partial x}{\partial \varepsilon} = 0$ since x is independent of ε, substituting and simplifying yields:

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left.\left[\frac{\partial F}{\partial \tilde{y}}\cdot\eta + \frac{\partial F}{\partial \tilde{y}'}\cdot\eta'\right]\right|_{\varepsilon=0} dx = 0. \tag{3.7}$$

Acknowledging that $\varepsilon = 0$ leads to $\tilde{y} = y$:

$$\frac{dJ}{d\varepsilon} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \cdot \eta + \frac{\partial F}{\partial y'} \cdot \eta' \right] dx \bigg|_{\varepsilon=0} = 0. \tag{3.8}$$

Integrating by parts $\frac{\partial F}{\partial y'} \cdot \eta'$, considering the boundary conditions for η, yields:

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y'} \cdot \eta' \right) dx = \frac{\partial F}{\partial y'} \eta \bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \cdot \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) dx = \int_{x_1}^{x_2} \eta \cdot \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) dx. \tag{3.9}$$

Substituting (3.9) into (3.8):

$$\frac{dJ}{d\varepsilon} = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right) \eta \right] dx = 0. \tag{3.10}$$

By applying the *fundamental lemma of the calculus of variations* (2.8), we arrive at the Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0, \tag{3.11}$$

establishing the necessary condition for $J[y(x)]$ to be stationary, providing the necessary criterion for $y(x)$ to be an extremal of J. □

3.2 Spacial Case (The Beltrami Identity)

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The Beltrami Identity is a special case of the Euler-Lagrange equation, applicable when $F(x, y, y')$ does not explicitly depend on x.

Theorem 3.3 (Beltrami Identity)**.** *If the Lagrangian $F(x, y, y')$ does not explicitly depend on x, then the Euler-Lagrange equation can be simplified to:*

$$F - y' \frac{\partial F}{\partial y'} = C \tag{3.12}$$

where C is a constant.

Proof. Given that $F(x, y, y')$ does not explicitly depend on x, we start with the Euler-Lagrange equation:

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0. \tag{3.13}$$

Since F is independent of x, the total derivative of F with respect to x is given by:

$$\frac{dF}{dx} = \frac{\partial F}{\partial y} \frac{dy}{dx} + \frac{\partial F}{\partial y'} \frac{dy'}{dx}. \tag{3.14}$$

Substitute $\frac{dy}{dx} = y'$ and $\frac{dy'}{dx} = y''$ into the above equation:

$$\frac{dF}{dx} = \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y''. \tag{3.15}$$

Using the Euler-Lagrange equation $\frac{\partial F}{\partial y} = \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right)$, we get:

$$\frac{dF}{dx} = \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) y' + \frac{\partial F}{\partial y'} y''. \tag{3.16}$$

Simplifying, we have:

$$\frac{dF}{dx} = \frac{d}{dx} \left(\frac{\partial F}{\partial y'} y' \right). \tag{3.17}$$

This implies that:

$$\frac{d}{dx} \left(F - y' \frac{\partial F}{\partial y'} \right) = 0. \tag{3.18}$$

Integrating both sides with respect to x, we obtain:

$$F - y' \frac{\partial F}{\partial y'} = C, \tag{3.19}$$

where C is a constant of integration. □

The Beltrami Identity is useful for solving practical problems in the field of Calculus of Variations, particularly when used to generate solutions for the brachistochrone problem and the catenary in the *Applications of the Euler-Lagrange Equation* section. By leveraging this identity, we can simplify the process of finding optimal paths in various scenarios

3.3 The Second Variation and Optimality Conditions

In calculus, we determine the nature of extrema using the first and second derivatives of functions. Similarly, in the calculus of variations, we use the first variation to identify stationary points and the second variation to assess whether these points are minima, maxima, or neither.
The Euler-Lagrange equation provides the necessary condition for a functional $J[y(x)]$ to be stationary, meaning it provides the potential extrema (or extremals). For a functional of the form:

$$J[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx,$$

the Euler-Lagrange equation identifies candidate extremals, but further analysis is required to determine whether these extremals are minima, maxima, or saddle points.

3.3.1 Derivation of the Second Variation

To establish the nature of a solution to the Euler-Lagrange equation, we examine the second variation of the functional, denoted as $\delta^2 J$. The second variation is derived by considering the second-order term in the expansion of $J[y + \varepsilon\eta]$ using Taylor's theorem:

$$J[y + \varepsilon\eta] = J[y] + \varepsilon\delta J[y] + \frac{\varepsilon^2}{2}\delta^2 J[y] + O(\varepsilon^3), \tag{3.20}$$

where $\eta(x)$ is an arbitrary smooth function that vanishes at the boundaries, i.e., $\eta(x_1) = \eta(x_2) = 0$. The second variation $\delta^2 J$ is given by:

$$\delta^2 J = \int_{x_1}^{x_2} \left[\frac{\partial^2 F}{\partial y^2}\eta^2 + 2\frac{\partial^2 F}{\partial y \partial y'}\eta\eta' + \frac{\partial^2 F}{\partial (y')^2}(\eta')^2 \right] dx. \tag{3.21}$$

3.3.2 Necessary and Sufficient Conditions for Extrema

For the functional $J[y]$ to have a local minimum or maximum at $y(x)$, the second variation $\delta^2 J$ provides insight:

Definition 3.4 (Local Minima and Maxima)**.** *A function $y(x)$ is a local minimum of $J[y]$ if $\delta^2 J \geq 0$ for all admissible variations $\eta(x)$. It is a local maximum if $\delta^2 J \leq 0$.*

Theorem 3.5 (Conditions for Local Extrema)**.** *Let $J[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx$ be a functional where $y(x)$ is a solution of the Euler-Lagrange equation. The nature of $y(x)$ as a local extremum is determined by the second variation $\delta^2 J$.*

- *Necessary Condition for Minima: $\delta^2 J \geq 0$ for all admissible $\eta(x)$.*
- *Sufficient Condition for Minima: $\delta^2 J > 0$ for all non-zero admissible $\eta(x)$.*
- *Necessary Condition for Maxima: $\delta^2 J \leq 0$ for all admissible $\eta(x)$.*
- *Sufficient Condition for Maxima: $\delta^2 J < 0$ for all non-zero admissible $\eta(x)$.*

Proof. To prove these conditions, consider the Taylor expansion of $J[y + \varepsilon\eta]$:

$$J[y + \varepsilon\eta] = J[y] + \varepsilon\delta J[y] + \frac{\varepsilon^2}{2}\delta^2 J[y] + O(\varepsilon^3).$$

For $y(x)$ to be a local minimum or maximum, the first variation $\delta J[y]$ must vanish. The second variation $\delta^2 J[y]$ determines the nature of the extremum:
- If $\delta^2 J > 0$, then the quadratic term dominates, ensuring $J[y + \varepsilon\eta] > J[y]$ for small ε, indicating a local minimum. - If $\delta^2 J < 0$, then $J[y + \varepsilon\eta] < J[y]$, indicating a local maximum.
The conditions on $\delta^2 J$ are necessary because, for a local extremum, the functional must exhibit the appropriate second-order behavior. They are sufficient because the positivity or negativity of $\delta^2 J$ across all variations confirms the extremal nature of the solution.

To derive $\delta^2 J$, we perform the second differentiation:

$$\frac{d^2}{d\varepsilon^2} J[y + \varepsilon\eta]\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right] dx\right)\bigg|_{\varepsilon=0} \tag{3.22}$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial^2 F}{\partial y^2}\eta^2 + 2\frac{\partial^2 F}{\partial y \partial y'}\eta\eta' + \frac{\partial^2 F}{\partial (y')^2}(\eta')^2\right] dx. \tag{3.23}$$

This derivation aligns with the need for $\eta(x_1) = \eta(x_2) = 0$ to ensure boundary terms vanish. The signs of these integrals determine whether the stationary solution is a minimum or maximum. □

Thus, the integrand in $\delta^2 J$ helps determine whether the stationary function is a local minimum or maximum. These conditions provide a comprehensive framework for assessing optimality in the calculus of variations.

4 Applications of the Euler-Lagrange Equation

In this section, we will explore several practical applications of the Euler-Lagrange equation, demonstrating its utility in solving real-world problems. These examples highlight the versatility and power of the Calculus of Variations.

4.1 shortest distance

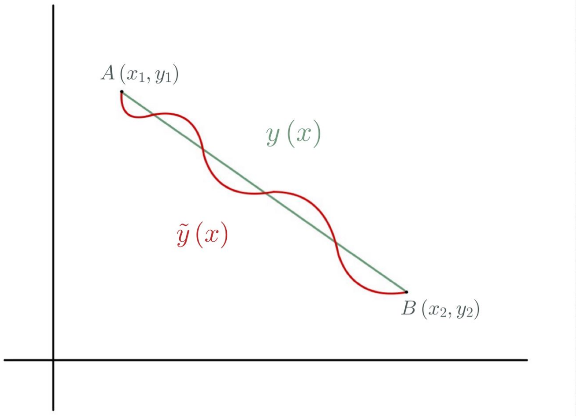

Figure 13

Given two points, $A(x_1, y_1)$ and $B(x_2, y_2)$, we seek to find a function $y(x)$ such that the distance between A and B is minimized. This optimization problem can be framed as a problem in the calculus of variations.

$$L = \int_{x_1}^{x_2} F(x, y, y')\, dx \tag{4.1}$$

The length of a smooth curve between two points can be expressed as

$$L = \int_{A}^{B} ds, \tag{4.2}$$

where ds represents the differential arc length.
According to the Pythagorean theorem, the differential arc length squared, ds^2, is given by

$$ds^2 = dx^2 + dy^2 = dx^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right].$$

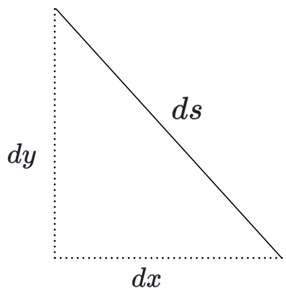

Figure 14

Hence, the differential arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{4.3}$$

Substituting this into the integral for L, we have

$$L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx, \tag{4.4}$$

For the length of the path L to be minimal, its variation must be equal to zero, i.e., $\delta L = 0$, satisfying the Euler-Lagrange equation (3.11)

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \tag{4.5}$$

where $F(x, y, y') = \sqrt{1 + (y')^2}$.

Since F does not explicitly depend on y, we have

$$\frac{\partial F}{\partial y} = 0. \tag{4.6}$$

Thus, the Euler-Lagrange equation simplifies to

$$\frac{\partial F}{\partial y'} = C, \tag{4.7}$$

where C is a constant. Differentiating F with respect to y' yields

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}} = C. \tag{4.8}$$

The only way for this expression to be a constant is if y' is constant. Therefore, we have

$$y' = c, \tag{4.9}$$

which integrates to

$$y = cx + c_1, \tag{4.10}$$

where c_1 is the integration constant. This equation represents a straight line, confirming the geometric intuition that the shortest path between two points is a line.

4.2 The Brachistochrone Problem

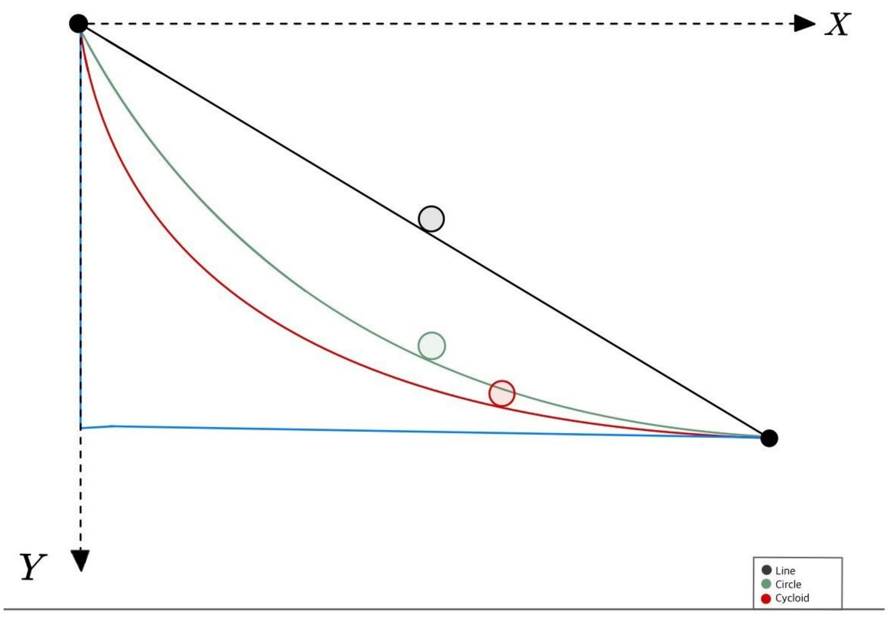

Figure 15

The term "brachistochrone" comes from the Greek language, meaning "shortest time." In variational calculus, it refers to a problem posed by Johann Bernoulli: Given two points, A and B, and an object moving under the influence of gravity along a slope connecting these points, what is the optimal path that allows the object to travel from A to B in the shortest possible time?

We seek to minimize the travel time of the object. This involves finding a balance between two factors: minimizing the distance, akin to a straight line, and maximizing the object's speed under gravity, which

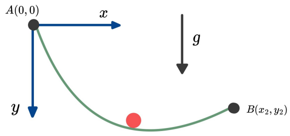

Figure 16

necessitates starting with a steep descent, thereby increasing the path length. The challenge is to find the curve that achieves this balance, minimizing the time taken for the journey.

To solve this minimization problem, we need to find the stationary function of the total time, representing the time taken:

$$T = \int_0^t dt\,. \tag{4.11}$$

Because time (t) is a function of displacement (ds) and velocity (v):

$$t = \frac{ds}{v}\,. \tag{4.12}$$

We rewrite the integral T as

$$T = \int_{x_1}^{x_2} \frac{ds}{v}\,. \tag{4.13}$$

We set up a coordinate system with the origin at point A and the x-axis in its usual horizontal direction, while the y-axis points downwards. (figure 16).

Physics Principles

Before delving into the solution, let us refresh the key physics formulas concerning the conservation of energy that are essential for analyzing the problem:

- Kinetic energy (K):

$$K = \frac{1}{2}mv^2\,. \tag{4.14}$$

- Gravitational potential energy (U):

$$U = mgy\,. \tag{4.15}$$

To determine the velocity v, we use the conservation of energy, which states that the kinetic energy of a moving object equals the loss of potential energy:

$$\begin{aligned} \frac{1}{2}mv^2 &= mgy\,, \\ \Rightarrow \quad v &= \sqrt{2gy}\,. \end{aligned} \tag{4.16}$$

Now, substituting the velocity v and the path length $ds = \sqrt{1+y'^2}\,dx$ into the integral T gives

$$T = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}}\,. \tag{4.17}$$

Simplifying further, we obtain

$$T = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx\,. \tag{4.18}$$

Thus, we define the functional F as

$$F = \sqrt{\frac{1+y'^2}{y}}\,. \tag{4.19}$$

Noticing that this functional does not explicitly depend on x, we use the first integral of the Euler-Lagrange equation, the Beltrami identity, and we get

$$F - y'\frac{\partial F}{\partial y'} = C\,. \tag{4.20}$$

Differentiating F with respect to y' gives

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{y}\sqrt{1+y'^2}} . \tag{4.21}$$

By substituting F and $\frac{\partial F}{\partial y'}$ into (4.20), we get

$$\sqrt{\frac{1+y'^2}{y}} - \frac{y'^2}{\sqrt{y(1+y'^2)}} = C . \tag{4.22}$$

Multiplying by $\sqrt{y(1+y'^2)}$ and simplifying, we find

$$1 = \sqrt{y(1+y'^2)} \cdot C . \tag{4.23}$$

Dividing by C and then squaring both sides gives

$$k = y + yy'^2 . \tag{4.24}$$

where $k = \frac{1}{C^2}$ is a constant. Solving for the derivative y' yields

$$\frac{dy}{dx} = \sqrt{\frac{k-y}{y}} . \tag{4.25}$$

We separate the variables to solve this first-order differential equation

$$dx = \sqrt{\frac{y}{k-y}}\, dy . \tag{4.26}$$

Integrating both sides gives

$$x + c_2 = \int \sqrt{\frac{y}{k-y}}\, dy . \tag{4.27}$$

For some constant c_2. To evaluate this integral, we use the trigonometric substitution, so we substitute the variables as follows

$$y = k \sin^2 \theta . \tag{4.28}$$

$$dy = 2k \sin\theta \cos\theta \, d\theta . \tag{4.29}$$

Thus

$$x = \int_0^\theta \sqrt{\frac{k \sin^2\theta}{k - k\sin^2\theta}} \cdot 2k \sin\theta \cos\theta \, d\theta - c_2 . \tag{4.30}$$

Simplifying yields

$$x = \int_0^\theta \sqrt{\frac{k \sin^2\theta}{k \cos^2\theta}} \cdot 2k \sin\theta \cos\theta \, d\theta - c_2 . \tag{4.31}$$

We proceed

$$x = 2k \int_0^\theta \sin^2\theta \, d\theta - c_2 . \tag{4.32}$$

Rewriting that using trigonometric identities and then integrating the right side gives

$$\begin{aligned} x &= k \int_0^\theta (1 - \cos 2\theta)\, d\theta - c_2 , \\ &= k\left(\theta - \frac{1}{2}\sin 2\theta\right) - c_2 . \end{aligned} \tag{4.33}$$

We rewrite y from (4.28) using trigonometric identities, and we obtain

$$y = \frac{k}{2}(1 - \cos 2\theta) . \tag{4.34}$$

Where k and c_2 can be found from the boundary conditions.
Now we see that these two parametric equations for x and y represent a cycloid. The cycloid is the path traced by a point on the perimeter of a wheel as it rolls along a straight line. In conclusion, the function that makes the total time functional stationary is segment on a cycloid.

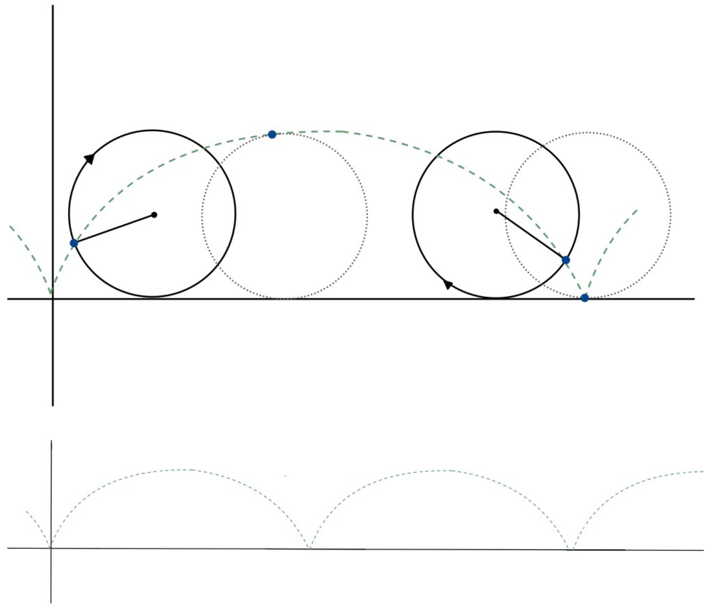

Figure 17

4.3 The Catenary

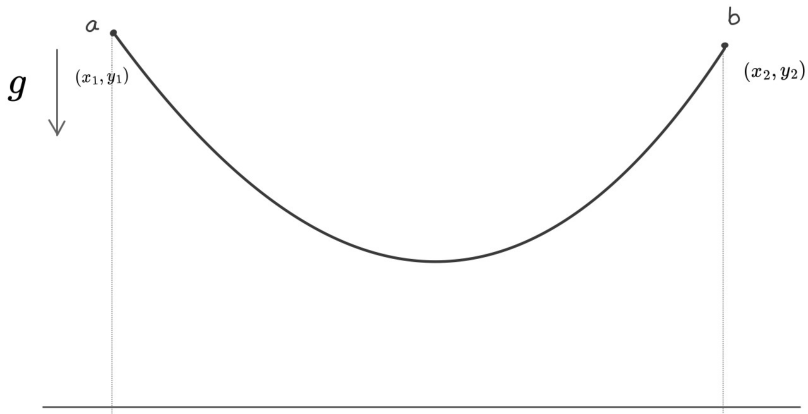

Figure 18

The term "catenary" comes from the Latin word "catena," meaning chain. In the Calculus of Variations, the catenary problem is a classic example that demonstrates how to find the shape of a hanging flexible chain under the influence of gravity. This problem was first posed and solved by mathematicians such as Leibniz, Huygens, and Johann Bernoulli in the late 17th century.
Consider a chain suspended by its endpoints at (x_1, y_1) and (x_2, y_2). The goal is to find the function $y(x)$ that describes the shape of the chain under the influence of gravity, which minimizes the potential energy of the system.

To derive the equation of the catenary, we start by considering the potential energy U of the chain. The potential energy is given by:

$$U = \int_{x_1}^{x_2} \rho g y ds \tag{4.35}$$

where ρ is the linear density of the chain, g is the acceleration due to gravity, and $\int_{x_1}^{x_2} ds \ = l$ is the length of the chain.
Substituting we obtain

$$U = \int_{x_1}^{x_2} \rho g y \sqrt{1 + (y')^2}\, dx, \tag{4.36}$$

where ρ is the linear density of the chain, g is the acceleration due to gravity, and y' is the derivative of y with respect to x.
The function we seek to minimize is the integrand $F(x, y, y') = \rho g y \sqrt{1 + (y')^2}$. To find the extremal function $y(x)$, we use the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{4.37}$$

First, we compute the partial derivatives

$$\frac{\partial F}{\partial y} = \rho g \sqrt{1 + (y')^2}, \tag{4.38}$$

and

$$\frac{\partial F}{\partial y'} = \rho g y \frac{y'}{\sqrt{1 + (y')^2}}. \tag{4.39}$$

Next, we compute the total derivative of $\frac{\partial F}{\partial y'}$ with respect to x

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left(\rho g y \frac{y'}{\sqrt{1 + (y')^2}}\right). \tag{4.40}$$

Applying the chain rule, we get

$$\frac{d}{dx}\left(\rho g y \frac{y'}{\sqrt{1 + (y')^2}}\right) = \rho g \left(\frac{y'}{\sqrt{1 + (y')^2}} + y \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right)\right). \tag{4.41}$$

To simplify, notice that

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = \frac{(1 + (y')^2)y'' - y'(y'y'')}{(1 + (y')^2)^{3/2}} = \frac{y''}{(1 + (y')^2)^{3/2}}. \tag{4.42}$$

Therefore,

$$\frac{d}{dx}\left(\rho g y \frac{y'}{\sqrt{1 + (y')^2}}\right) = \rho g \left(\frac{y'}{\sqrt{1 + (y')^2}} + y \frac{y''}{(1 + (y')^2)^{3/2}}\right). \tag{4.43}$$

Substituting these into the Euler-Lagrange equation, we get

$$\rho g \sqrt{1 + (y')^2} - \rho g \left(\frac{y'}{\sqrt{1 + (y')^2}} + y \frac{y''}{(1 + (y')^2)^{3/2}}\right) = 0. \tag{4.44}$$

Simplifying, we find

$$\sqrt{1 + (y')^2} - \left(\frac{y'}{\sqrt{1 + (y')^2}} + y \frac{y''}{(1 + (y')^2)^{3/2}}\right) = 0. \tag{4.45}$$

Multiplying through by $(1 + (y')^2)^{3/2}$, we obtain

$$(1 + (y')^2)\sqrt{1 + (y')^2} - (y'(1 + (y')^2) + yy'') = 0. \tag{4.46}$$

This simplifies to

$$(1 + (y')^2) - yy'' = 0. \tag{4.47}$$

Dividing by $(1+(y')^2)$, we get

$$1 - y\frac{y''}{1+(y')^2} = 0, \tag{4.48}$$

or equivalently,

$$\frac{y''}{1+(y')^2} = \frac{1}{y}. \tag{4.49}$$

This is a second-order nonlinear differential equation. To solve it, we use the substitution $v = y'$, so that $v' = y''$

$$\frac{v'}{1+v^2} = \frac{1}{y}. \tag{4.50}$$

Separating variables, we have

$$v' = \frac{1+v^2}{y}. \tag{4.51}$$

Integrating both sides with respect to x

$$\int \frac{dv}{1+v^2} = \int \frac{dx}{y}. \tag{4.52}$$

The integral on the left side is

$$\arctan(v) = \int \frac{dx}{y}. \tag{4.53}$$

Rewriting $v = y'$

$$\arctan(y') = \int \frac{dx}{y}. \tag{4.54}$$

Taking the tangent of both sides, we get

$$y' = \tan\left(\int \frac{dx}{y}\right). \tag{4.55}$$

Let $u = \int \frac{dx}{y}$. Then

$$y' = \tan(u). \tag{4.56}$$

Integrating with respect to x

$$y = a\cosh\left(\frac{x-b}{a}\right), \tag{4.57}$$

where a and b are constants determined by the boundary conditions. This equation describes the shape of a catenary, which is the curve that minimizes the potential energy of the hanging chain.
The catenary curve is the solution that minimizes the potential energy of the chain, resulting in the characteristic hyperbolic cosine shape. This solution illustrates how the Calculus of Variations can be used to find optimal solutions in physics and engineering problems.

5 Summary

In this thesis, we have explored the historical development, core concepts, and key applications of the Calculus of Variations. Starting with a brief history, we highlighted the contributions of pioneering mathematicians such as Johann Bernoulli, Leonhard Euler, and Joseph-Louis Lagrange. Their work laid the groundwork for this mathematical discipline, which focuses on finding optimal solutions to problems involving functionals. We then reviewed fundamental concepts such as differentiation, maxima and minima, and the distinction between functions and functionals. These basics are crucial for understanding the more advanced topics in the Calculus of Variations. Through the derivation and application of the Euler-Lagrange equation, we demonstrated how to find extremal functions that make a given functional stationary. Detailed examples, including the brachistochrone problem and the catenary, illustrated the practical applica- tions of these principles. The brachistochrone problem showed how to find the path of quickest descent under gravity, while the catenary problem revealed the optimal shape of a hanging chain. Both examples underscore the power and versatility of the Calculus of Variations in solving real-world optimization problems. Leibniz's rule for differentiation under the integral was also discussed, providing essential tools for handling integrals with variable limits and integrands. This rule is particularly important in the context of deriving the Euler-Lagrange equation and solving variational problems. Overall, this thesis highlights the importance of the Calculus of Variations in both theoretical and practical contexts. By continuing to explore and apply these mathematical techniques, we can drive further advance- ments in efficiency and innovation across various fields, optimizing and enhancing many aspects of our daily lives

6 Sources

This thesis relies on several key resources to explore the Calculus of Variations and its applications. The following references have been instrumental in providing the foundational knowledge and historical context necessary for this work.

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