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Inverse Systems with Applications to Ideals of Projective Points

av

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Abstract

We study Macaulay's concept of an inverse system of a polynomial ideal, largely in the form it was given by Emsalem and Iarrobino in their paper "Inverse system of a symbolic power, I". One of our main goals is to present a version of their theorem giving the inverse system of the intersection of ideals, each of which describes a projective point and is raised to a power. At the same time, we wish to make a deeper exploration of the concepts involved. In particular, we investigate some of the linear algebraic properties of the operators that are used to define inverse systems, and also highlight the simple form these systems take for monomial ideals, as well as for ideals that can be reduced to monomial ones through suitable linear transformations. Finally, we use the results we have gathered to briefly explore two new problems: the inverse system of an ideal of several projective points that is together raised to a power, and the relationship between inverse systems and coordinate rings.

Sammanfattning

Uppsatsen behandlar Macaulays begrepp *inverst system* av ett polynomideal, i stort sett i den form det getts av Emsalem och Iarrobino i artikeln "Inverse system of a symbolic power, I". Ett av våra huvudsakliga mål är att presentera en version av deras teorem för att räkna ut det inversa systemet för snittet av ideal som beskriver projektiva punkter upphöjda till exponenter. Samtidigt gör vi också en djupare undersökning av de begrepp som används. Mer specifikt utforskar vi några av de linjär-algebraiska egenskaper som operatorerna som används för att definiera inversa system har. Vi belyser också den särskilt enkla form dessa system tar för monoma ideal, såväl som för ideal som kan reduceras till monoma genom linjära transformationer. Slutligen använder vi de resultat vi visat för att kort undersöka två nya problem: det inversa systemet för ett ideal av projektiva punkter som tillsammans upphöjts till en exponent, och relationen mellan inversa system och koordinatringar.

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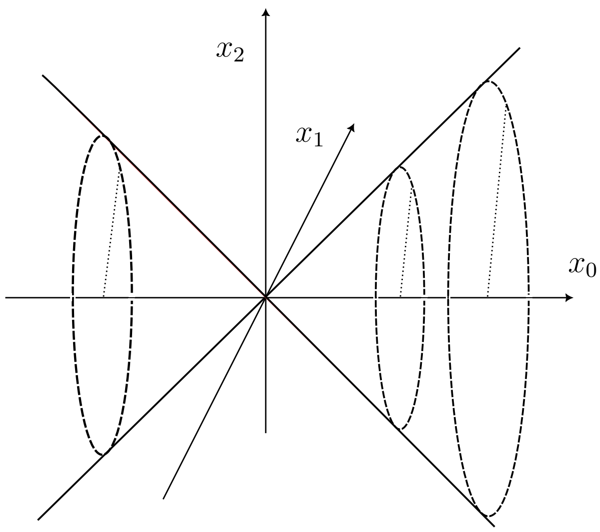
1 Introduction

Polynomial ideals are a central concept in commutative algebra and algebraic geometry. Since the ideal $\langle f_1, \ldots, f_n \rangle$ generated by the polynomials $f_1, \ldots, f_n$ consists of polynomials that are zero on all points where all of $f_1, \ldots, f_n$ are zero, such ideals constitute a powerful tool for studying the roots of sets of polynomials. Seen from the viewpoint of algebraic geometry, these roots make up shapes in various spaces. In the most intuitive case, ideals in the ring $R = \mathbb{R}[x_1, \ldots, x_r]$ describe shapes in the usual n-dimensional affine space, or in $n-1$-dimensional projective space, through their zeros. For example, the zeros common to all polynomials in the ideal $Q = \langle q \rangle = \langle x_1^2 + x_2^2 - x_0^2 \rangle$ in $\mathbb{R}^3$ describe a quadric hypersurface. If we interpret $\mathbb{R}^3$ as affine 3-space, this is a double cone stretching along the x_0 axis. Interpreted in the projective plane it is a curve that could be described as a projective version of a circle, except that it does not have a specific radius:



Thus one way to approach polynomial ideals is through their sets of zeros. Much of the attraction of algebraic geometry comes from how it lets us apply geometric arguments to questions about such ideals. For example, the set of zeros of the intersection $I \cap J$ of two ideals I and J is the union of the sets of zeros of I with those of J. However, there are also many properties of ideals that do not show up in their zeros. For example, the power ideal

$$I^n = \langle f_1 \cdots f_n \mid f_1, \ldots, f_n \in I \rangle$$

always has the same zeros as the ideal I itself, so $Q^2 = \langle x_1^2 + x_2^2 - x_0^2 \rangle^2 = \langle q^2 \rangle$ describes the same shape as $Q = \langle x_1^2 + x_2^2 - x_0^2 \rangle$ as long as as we only consider the roots. However, the difference between them is still geometrically relevant. Calculating the gradients ∇q and ∇q^2 gives us

$$\begin{aligned}\nabla q &= (-2x_0, 2x_1, 2x_2) \\ \nabla q^2 &= \left(-4x_0(x_1^2 + x_2^2 - x_0^2), 4x_1(x_1^2 + x_2^2 - x_0^2), 4x_2(x_1^2 + x_2^2 - x_0^2)\right)\end{aligned}$$

We see that $\nabla q(x) = 0$ iff $x = 0$, so all points of the quadric hypersurface except 0 have non-zero derivatives in most directions. For ∇q^2, however, the way we have written out the gradient makes it clear that it must be 0 on the whole hypersurface: the expression in parenthesis in each of the coordinates is zero there, so all coordinates of the gradient are zero. As any function $h \in Q^2$ is of the form gq^2 for some polynomial g, we furthermore have

$$\begin{aligned}\nabla(gq^2) &= \left(\frac{\partial}{\partial x_0}(gq^2), \frac{\partial}{\partial x_1}(gq^2), \frac{\partial}{\partial x_2}(gq^2)\right) \\ &= \left(\left(\frac{\partial}{\partial x_0}g\right)q^2 + g\frac{\partial}{\partial x_0}q^2, \left(\frac{\partial}{\partial x_1}g\right)q^2 + g\frac{\partial}{\partial x_1}q^2, \left(\frac{\partial}{\partial x_2}g\right)q^2 + g\frac{\partial}{\partial x_2}q^2\right)\end{aligned}$$

Since $q^2(x) = 0$ and $\frac{\partial}{\partial x_i}q(x) = 0$ at all points on the hypersurface, we have that $\nabla gq^2(x) = 0$ whenever $q(x) = 0$ as well, so one difference between Q and Q^2 is that not only are all functions in Q^2 zero on the hypersurface in question, but so are their first-order derivatives. This means that, as x moves away from the hypersurface, the functions in Q^2 move away from 0 at a slower pace than those in Q, at least when arbitrarily close to it. Seen from the point of the functions, the shape described is "thicker" or "fatter", although only in an informal, infinitesimal sense. For individual points this is not too difficult to make rigorous, and if $I(p)$ is the ideal of polynomials that are zero on the point p, we say that the ideals $I(p)^n$ for $n > 1$ are *ideals of fat points.*[1]

There are many ways to analyze ideals, and they differ in character from the

[1]It is important to note that being the ideal of a fat point is a property of the ideal, and not of the point it is zero at in the traditional sense of "point". We will touch on the problem of making more sense of the concept again in Section 2.3.

more algebraic (such as being generated by polynomials with a certain number of terms) to the more geometric (such as having certain dimension, according to some definition of the word). When *inverse systems* were introduced by F.S. Macaulay as a tool for working with ideals in his book [Mac94] in 1916, they were most likely thought of by him in mainly algebraic terms. However, interestingly, after him the idea has shown itself to also have some geometric content, and in fact it is connected to zeros of derivatives, at least in one of its forms.

Although we shall describe both a modern treatment and Macaulay's own approach later, we will try to briefly summarize the main ideas here. A function is *inverse* to another, for Macaulay, if a certain formula of both functions is zero. In his case, at least one of the functions was typically an infinite power series with negative exponents, and the formula said that the degree 0 part of the product of the functions was zero. An inverse *system* of an ideal I is a set of functions such that each member of it is inverse to each member of I (the word "system" was not uncommonly used for "set" in the beginning of the 20th century). Inverse systems are then sets of functions that, when combined with any polynomial in an ideal, give the result zero.

It is worth comparing this with the concept of an *annihilator subspace* in linear algebra. The annihilator $X^\perp$ of a subset X of a vector space V over the field $\mathbb{F}$ is the set of all linear functionals $f : V \to \mathbb{F}$ for which $f(x) = 0$ for all $x \in X$ (see e.g. [MB99, p. 209], [Rom08, p. 102]). The elements of the annihilator thus come from the dual space V^* of V, and the method of combining the functional with the vector is through function application. If V is finite-dimensional, V^* is isomorphic to V, and in vector spaces that support a notion of orthogonality the annihilator of a subset X comes out as isomorphic to the subspace of vectors that are orthogonal to all vectors in X. As we shall see, the similarities between this and the inverse system concept are more than accidental.

In [EI95] Emsalem and Iarrobino pick up Macaulay's idea in the context of projective geometry. They give a different definition of what an inverse system is than Macaulay. In fact, they give two definitions—one which is more similar to Macaulay's, and one which is related to zeros of derivatives, and thus gets its specific geometric meaning from this. In the first half of the paper they show how to derive inverse systems according to both definitions for ideals that correspond to sets of fat points. In the second half they prove a further theorem about the inverse systems of ideals for shapes that are, in a sense, made up of such fat points.

However, Emsalem and Iarrobino's treatment of inverse systems is quite brief, and in proofs they rely on results by Macaulay, which are not always directly transferable to their definitions. In fact, the part of [EI95] up to and including their proof of Theorem 1, which gives the inverse system of a set of fat points, is barely more than 6 pages. Because of this, and because Macaulay's original treatment is not easily readable, often also omits proofs, and as we mentioned relies on different definitions, the main aim of the present text is to describe and prove Theorem 1 at a slower pace, giving more background, proving more of the necessary supplementary theorems, and making a deeper investigation of the tools used. A secondary objective is to use the results obtained to try to say something about inverse systems other than the ones that Emsalem and Iarrobino describe.

The structure of the text is as follows. Section 2 introduces terminology for talking about ideals of polynomials defined in a projective space, and goes on to discuss ideals of projective points, and the concept of an ideal vanishing to a certain order. Section 3 contains results about inverse systems in general, which do not depend on being inverse systems of fat point ideals specifically. It can be seen as a supplement to Macaulay's work that explores consequences of definitions of inverse systems more similar to those that Emsalem and Iarrobino use. Section 4 applies the findings we have obtained so far to ideals of fat points and derives Theorem 1 of [EI95] in two ways. Finally, Section 5 contains some brief investigations of possible other applications of the apparatus we have developed here to the question of inverse systems for other classes of ideals.

2 Projective points and their ideals

2.1 Polynomials in projective space

Let $\mathbb{F}$ be a field of characteristic 0 and let V be a vector space over $\mathbb{F}$. We take the projective space $\mathbf{P}(V)$ of V to be the set of one-dimensional subspaces of V and refer to its elements as *projective points*. The *dimension* of $\mathbf{P}(V)$ is defined to be one less than the dimension of V. When $V = \mathbb{F}^{r+1}$ we will typically refer to its projective space as *the r-dimensional projective space over* $\mathbb{F}$, and write it as $\mathbf{P}^r(\mathbb{F})$.[2]

Projective spaces are not vector spaces, but they nevertheless have rich geometric structure. Although we will not need to rely on most of that structure in this text, the notion of a *linear projective subspace* is sometimes useful for geometric intuition. With the definitions we have adopted here, $\mathbf{P}(V)$ is simply a linear projective subspace of $\mathbf{P}(W)$ iff V is a subspace of W. A 1-dimensional linear projective subspace is called a projective line, and a 2-dimensional linear projective subspace is called a projective plane. Each projective concept of dimension d is determined by (and determines) an affine concept of dimension $d+1$. For example, the vectors in $\mathbb{F}^{r+1}$ that make up the projective points of a projective line describe a plane in $\mathbb{F}^{r+1}$.

Most often, we will specify points in $\mathbf{P}^r(\mathbb{F})$ through non-zero vectors in the space $\mathbb{F}^{r+1}$. Two such vectors $p = (p_0, \dots, p_k)$ and $p' = (p'_0, \dots, p'_k)$ correspond to the same projective point iff $p' = \lambda p$ for some $\lambda \in \mathbb{F}$. If $p \in \mathbb{F}^{r+1} \smallsetminus \{0\}$ we write $\hat{p}$ for the projective point determined by p, i.e.

$$\hat{p} = \{\lambda p \mid \lambda \in \mathbb{F}\}.$$

Since the coordinates of a projective point are only determined up to a constant we will follow the convention of writing coordinates for these as $(x_0 : \dots : x_r)$ rather than $(x_0, \dots, x_r)$ to indicate that it is only the proportion between them that is meaningful. Thus $(x_0 : \dots : x_r) = (\lambda x_0 : \dots : \lambda x_r)$ for any $\lambda \in \mathbb{F} \smallsetminus \{0\}$. We use $\mathsf{e}_0, \dots, \mathsf{e}_r$ to refer to the standard basis of $\mathbb{F}^{r+1}$, i.e. the vectors $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$. These determine $r+1$ projective points $\hat{\mathsf{e}}_0, \dots, \hat{\mathsf{e}}_r$ with coordinates $(1 : 0 : \dots : 0), \dots, (0 : \dots : 0 : 1)$. The interpretation of $\mathbf{P}^r(\mathbb{F})$ in terms of 1-dimensional subspaces of $\mathbb{F}^{r+1}$ means that some vector space concepts carry over: for example, we may call $\hat{p}$ and $\hat{q}$

[2] It is possible, and often preferable, to define projective spaces through their own axioms rather than as constructions made from vector spaces, as we have done here. The present definitions and general approach have been chosen for the sole reason that they are easier to work with for our purposes.

linearly independent or orthogonal if p and q are linearly independent or orthogonal as vectors, since these concepts are invariant under multiplication of p and q with non-zero elements of $\mathbb{F}$. On the other hand, being orthonormal is not a concept that makes sense for projective points since it depends on the specific lengths of vectors.

We want to study ideals of polynomials defined on $\mathbb{F}^{r+1}$ that are interpretable in terms of shapes in the projective space $\mathbb{P}^r$. Let R be a polynomial ring on $\mathbb{F}$ of dimension $r+1$ (i.e. $R = \mathbb{F}[x_0, \dots, x_r]$). For algebraic geometry in projective space, one should consider only homogenous polynomials (i.e. ones such that $f(\lambda x) = \lambda^{\deg f} f(x)$, or equivalently, ones that are generated by polynomials all of whose terms have same degree), since it is only these that have the same zeros under multiplication by elements of $\mathbb{F}$, and which thus determine well-defined 1-dimensional subspaces [CLO15, p. 398]. An ideal generated by homogenous polynomials is called a homogenous ideal. Since we will more or less completely focus on homogenous polynomials and ideals in this text, we make the following standing assumption:

Standing Assumption: With "any polynomial" and "any ideal" will be meant "any *homogenous* polynomial" and "any *homogenous* ideal", unless otherwise stated.

The ring R is graded by the degree of the polynomials in it and we write R_i for the subset consisting of the homogenous polynomials of degree i. Each R_i has the structure of a vector space with multiplication with elements of $\mathbb{F}$ as scalar multiplication. R_0 is isomorphic to $\mathbb{F}$, and R_1 is the dual space ${\mathbb{F}^{r+1}}^*$ of $\mathbb{F}^{r+1}$, consisting of linear functionals $\mathbb{F}^{r+1} \to \mathbb{F}$ [Joh21, p. 61][MB99, p. 207]. Since all these spaces are finite, R_1 is isomorphic to $\mathbb{F}^{r+1}$ itself: each function $p_0 x_0 + \dots + p_r x_r \in R_1$ corresponds to one and only one vector $(p_0, \dots, p_r) \in \mathbb{F}^{r+1}$.

To simplify working with elements of R we use multiindex notation. A multiindex $\alpha = (\alpha_0, \dots, \alpha_r)$ of length $r+1$ is an element of $\mathbb{N}^{r+1}$. Multiindices are added and subtracted elementwise, and the *degree* $|\alpha|$ of a multiindex α is defined to be $\sum_{i=0}^{r} \alpha_i$.

Multiindices are compared element by element, so $\alpha \leqslant \beta$ iff $\alpha_i \leqslant \beta_i$ for all i, which makes $\leqslant$ a partial order on the set of multiindices of the same length. To save space under summation signs we usually write $\deg f$, where f is a polynomial, not only for the natural number that gives the polynomial's degree, but also for the set $\{\alpha \mid |\alpha| = \deg f\}$ (i.e. the set of non-negative multiindices with the same degree as f), and rely on context to determine which is meant. We furthermore use the following conventions:

$$\alpha! = \alpha_0! \cdots \alpha_r!$$
$$\binom{\alpha}{\beta} = \frac{\alpha_0! \cdots \alpha_r!}{\beta_0! \cdots \beta_r!}$$
$$x^\alpha = x_0^{\alpha_0} \cdots x_r^{\alpha_r}$$
$$\frac{\partial}{\partial x^\alpha} = \frac{\partial^{\alpha_0}}{\partial x_0^{\alpha_0}} \cdots \frac{\partial^{\alpha_r}}{\partial x_r^{\alpha_r}}.$$

It it quickly verified that if $p = (p_0, \ldots, p_r) \in \mathbb{F}^{r+1}$ then

$$(px)^\alpha = (p_0 x_0, \ldots, p_r x_r)^\alpha = p^\alpha x^\alpha$$

where $p^\alpha = p_0^{\alpha_0} \cdots p_r^{\alpha_r}$, just as for a vector x of variables. Using multiindices any element f of R_i can be written as

$$f(x) = \sum_{\alpha \in \deg f} a_\alpha x^\alpha$$

where $a_\alpha \in \mathbb{F}$ for each $\alpha \in \deg f$. Thus any homogenous polynomial of R is determined by a function from multiindices to $\mathbb{F}$ given by an assignment $a : \mathbb{N}^{r+1} \to \mathbb{F}$ such that $a(\alpha) = 0$ when $|\alpha| \neq i$ (although we will typically write a_α rather than $a(\alpha)$). The assignments of degree i make up a vector space K_i of dimension

$$\dim K_i = \binom{r+i}{r}$$

which contains the possible coefficients of homogenous polynomials of degree i. This space carries a coordinate system where each coordinate axis is determined by the multiindex exponent of the monomial it corresponds to. This means that for K_3 in 2 projective dimensions we have axes such as $(2,0,1), (0,3,0)$ and $(1,1,1)$ which correspond to the monomials $x_0^2 x_2$, x_1^3 and $x_0 x_1 x_2$.

Let $\kappa : R_i \to K_i$ be the function that takes each degree i homogenous polynomial to the vector of its coefficients, and let $\kappa(f)_\alpha$ be the coefficient assigned to the basis vector α. Then we have

$$f(x) = \sum_{\alpha \in \deg f} \kappa(f)_\alpha x^\alpha$$

for any homogenous $f \in R$. It is clear that κ sets up an isomorphism between R_i and K_i: addition of polynomials corresponds to addition of the polynomials' coefficient vectors, and multiplication of polynomials by scalars (elements of $\mathbb{F}$) corresponds to multiplication of the coefficient vectors with the same scalars. Since K_i is spanned by axes α such that $|\alpha| = i$ and these are the images of the monomials of R_i under κ, it follows that R_i has a basis consisting of its i^{th} degree monomials.

Let I be a homogenous ideal of R. We write I_i for $I \cap R_i$, and refer to I_i as degree i of I. Any such degree of an ideal is a subspace of R_i, and the ideal I as a whole is a subspace of R, seen as a vector space. In the latter case, both I (if non-zero) and R are infinite-dimensional. As this sometimes complicates things, we will mostly try to focus on one degree at a time.

As we will work with both vector space and ideal structure in parallel it is important to be careful about notation. Like most texts about commutative algebra we use $I + J$ to denote the sum of I and J as ideals, i.e. the result of closing the set $I \cup J$ under addition and under multiplication with arbitrary elements of R. When X and Y are any subsets of R (including ideals), we write $X +_v Y$ for the vector space generated by $X \cup Y$, i.e. the minimal set containing $X \cup Y$ that is closed under addition and under multiplication with elements of R_0. When both X and Y are vector spaces, this is their vector space sum. Likewise, we mark the difference between the *ideal* generated by a set X of polynomials, which we denote $\langle X \rangle$, and the *vector space* generated by them, which we denote span X. We always have $X +_v Y \subseteq X + Y$ and span $X \subseteq \langle X \rangle$. In particular, if X consists of degree i homogenous polynomials, so does span X, but $\langle X \rangle$ will have polynomials of all degrees $\geqslant i$, not all of which are homogenous.

2.2 Ideals determined by projective points

Any subset A of the projective space $\mathbf{P}^r(\mathbb{F})$ also determines a subset of the affine space $\mathbb{F}^{r+1}$ defined by taking the union of the projective points in A. Conversely, any subset of $\mathbb{F}^{r+1}$ that is closed under multiplication with scalars determines a unique subset in $\mathbf{P}^r(\mathbb{F})$, so such subsets are interchangeable with subsets of $\mathbb{P}^r(\mathbb{F})$. It follows that we can describe shapes in projective space by working with ideals I in $R = \mathbb{F}[x_0, \ldots, x_r]$ such that if $f(x) = 0$ for all $f \in I$, then $f(\lambda x) = 0$ for all $f \in I$ and $\lambda \in \mathbb{F}$. These are exactly the homogenous ideals, which is why we introduced our standing assumption to limit ourselves to these.

Our primary concern in this text will be with ideals of homogenous polynomials that are zero on a finite sets of projective points. For the case of a single projective point $\hat{p}$ we define

$$I(\hat{p}) = \{f \in R \mid f(p') = 0 \text{ for all } p' \in \hat{p}\}.$$

Such an ideal will not depend on which vector $p' \in \hat{p}$ we use to represent $\hat{p}$, i.e. $I(\hat{p}) = I(\hat{q})$ if $\hat{p} = \hat{q}$. The simplest case to calculate is when the projective point is along one of the coordinate axes, as in $\hat{\mathsf{e}}_k = (0 : \ldots : 1 : \ldots : 0)$ with all coordinates except x_k set to 0. Here we can see at once that the ideal $\langle x_0, \ldots, x_{k-1}, x_{k+1}, x_r\rangle$ will contain only polynomials that are zero on the whole of $\hat{\mathsf{e}}_k$, and as will follow from the next theorem, it actually contains all such polynomials. For the general (non-axis-aligned) case, we will write l_p, where $p \in \mathbb{F}^{r+1}$, for the linear polynomial

$$l_p(x) = p_0 x_0 + \cdots + p_r x_r.$$

This is an element of $R_1 \simeq \mathbb{F}^{r+1*}$, and $l : \mathbb{F}^{r+1} \to \mathbb{F}^{r+1*}$ can be seen as giving an isomorphism between that space and $\mathbb{F}^{r+1}$, with the inverse being

$$l_f^{-1} = f(\mathsf{e}_0, \ldots, \mathsf{e}_r)$$

for any $f \in R_1$. We want to show that the ideal $I(\hat{p})$ is, intuitively speaking, generated by linear functions that are orthogonal to l_p. Orthogonality in R_1 as well as in its dual space $\mathbb{F}^{r+1}$ is relative to a bilinear (or possibly sesquilinear) form defined on that space, and consists in a pair of elements of the space having value zero under that form. As we will need to keep the choice of form open for our future applications, we will typically only require that such a form is *nonsingular*: that it makes no vectors except the zero vector come out as orthogonal to all other vectors in the space [Rom08, p. 266].

To proceed we will need a lemma that guarantees that the generators of an ideal change as expected when we apply a linear transformation to the variables. If $L : \mathbb{F}^{r+1} \to \mathbb{F}^{r+1}$ is linear and f is a homogenous polynomial of degree i, then the composition $f \circ L$ is also a homogenous polynomial of degree i which we can think of as f after having performed a linear "change of variables". If I is an ideal, we write $I \circ L$ for the ideal $\{f \circ L \mid f \in I\}$. We have the following:

Lemma 2.2.1. *If $I = \langle f_1, \ldots, f_k\rangle$ and $L : \mathbb{F}^{r+1} \to \mathbb{F}^{r+1}$ is an invertible linear transformation, then $I \circ L = \langle f_1 \circ L, \ldots, f_k \circ L\rangle$.*

Proof. Let $h \in I$. By assumption, $h = g_1 f_1 + \cdots + g_k f_k$ for some $g_i \in R$. Applying L we get $h \circ L = (g_1 \circ L)(f_1 \circ L) + \cdots + (g_k \circ L)(f_k \circ L)$. But this shows at once that $h \circ L$

is in $\langle f_1 \circ F, \ldots, f_k \circ F \rangle$ since the functions $g_i \circ L$ are certainly elements of R. To show, conversely, that $h \circ L \in I \circ L$ implies $h \in I$ so we are guaranteed that applying L does not add any "unwanted" polynomials, simply apply L^{-1} on the right to each of $h, f_1, \ldots, f_k$. □

This lemma can be used to prove what we are after by reducing it to the simple axis-aligned case we considered first.

Theorem 2.2.2. *The ideal $I(\hat{p})$ of polynomials zero on the point $\hat{p}$ in projective r-space is generated by r linear polynomials $l_{q_1}, \ldots, l_{q_r} \in R$ such that $l_{q_i}(p) = 0$. If $\mathbb{F}^{r+1}$ has a nonsingular bilinear form defined on it, the q_i can be chosen to be orthogonal to each other and to p according to that form.*

Proof. That $l_{q_i}(p) = 0$ for all i follows directly from the definition of $I(\hat{p})$. To show that r such polynomials suffice to span $I(\hat{p})_1$, pick $l_1, \ldots, l_r \in R_1$ to be a basis for the annihilator of the subspace $p \in \mathbb{F}^{r+1}$. Since the annihilator is an r-dimensional subspace of the $r+1$-dimensional space R_1, this will always be possible. To associate each basis function l_i with a vector $q_i \in \mathbb{F}^{r+1}$, define each q_i to have coordinates $(l_i(\mathsf{e}_0), \ldots, l_i(\mathsf{e}_r))$, i.e. the result of applying l_i to the standard basis. A straightforward calculation shows that $l_i = l_{q_i}$, where l_{q_i} has the meaning we introduced just before Lemma 2.2.1.

To show that any polynomial $f \in I(\hat{p})$ can be written as $f = g_1 l_{q_1} + \cdots + g_r l_{q_r}$ with $g_i \in R$ so that $I(\hat{p})$ is generated by the l_{q_i} we note that $p, q_1, \ldots, q_r$ can be obtained by applying the linear transform $L : \mathbb{F}^{r+1} \to \mathbb{F}^{r+1}$ given by

$$
\begin{aligned}
x'_0 &= p_1 x_1 + \cdots + p_r x_r \\
x'_1 &= q_{1_1} x_1 + \cdots + q_{1_r} x_r \\
&\vdots \\
x'_r &= q_{r_1} x_1 + \cdots + q_{r_r} x_r
\end{aligned}
$$

to the standard basis $\mathsf{e}_0, \ldots, \mathsf{e}_r$. Since L is invertible (because $p, q_1, \ldots, q_r$ are linearly independent) it follows by the previous lemma that it is sufficient if we prove that $\langle l_{\mathsf{e}_1}, \ldots, l_{\mathsf{e}_r} \rangle$ generate the ideal of polynomials that are zero on $\hat{\mathsf{e}}_0$. It is easy to see that the homogenous polynomials that are zero on $\hat{\mathsf{e}}_0$ are of the form

$$f = \sum_{\alpha \in \deg f} a_\alpha x^\alpha$$

whenever $\alpha_1, \ldots, \alpha_r = 0$. All such polynomials are however generated by $x_1, \ldots, x_r$, which are the functions that $l_{q_1}, \ldots, l_{q_r}$ are taken to by L^{-1}.

Finally, that the q_i can be chosen to be orthogonal to each other and to p follows from the Riesz representation theorem, which entails that a nonsingular bilinear form $\langle \cdot, \cdot \rangle$ induces an isomorphism $\varphi : \mathbb{F}^{r+1} \to R$ defined through $\varphi(p)(q) = \langle p, q \rangle$ [Rom08, pp. 268–269]. This means that any linear functional l corresponds to a unique vector $p \in \mathbb{F}^{r+1}$ such that $l(q) = \langle p, q \rangle$ for all $q \in \mathbb{F}^{r+1}$, so $l(q) = 0$ iff p and q are orthogonal according to the bilinear form $\langle \cdot, \cdot \rangle$. □

As an example we may consider the point $\hat{p}_a = (1 : -1 : 0)$. The theorem lets us find $I(\hat{p}_a)$ easily: all we have to do is to construct two vectors that are orthogonal to p_a. For example, $q_1 = (1, 1, 0)$ and $q_2 = (0, 0, 1)$ will work. Since $I(\hat{p}_a)$ is generated by linear functions with these vectors as directions, we get that $I(\hat{p}_a) = \langle x + y, z \rangle$.

Since $I(\hat{p})$, for any projective point $\hat{p}$, is generated by different degree 1 (and thus irreducible) polynomials, it is always a radical ideal. If $I(\hat{p}_1), \ldots, I(\hat{p}_k)$ are ideals of polynomials that are zero on the projective points $\hat{p}_1, \ldots, \hat{p}_k$, respectively, then $I(\hat{p}_1) \cap \cdots \cap I(\hat{p}_k)$ is the ideal of polynomials that are zero on all of $\hat{p}_1, \ldots, \hat{p}_k$ [CLO15, p. 196]. We will write this ideal, which describes a finite set of points in projective space, as $I(\hat{p}_1, \ldots, \hat{p}_k)$.

2.3 Powers of point ideals and orders of zeros

As we mentioned in the introduction, many ideals vanish on the same points in $\mathbf{P}^r(\mathbb{F})$. Indeed, if $\mathbb{F}$ is algebraically closed, then Hilbert's Nullstellensatz tells us that any two ideals I, J that have the same radical $\sqrt{I} = \sqrt{J}$ have the same set of zeros [Eis04, p. 34][CLO15, p.183]. From this it follows that the zeros of ideals give a fairly coarse-grained classification of them, although one that has the great advantage of having geometric content. For a more fine-grained categorization, but one that stays in the geometric vein, we can also consider the order of the zeros of an ideal's functions, or as it was referred to in the introduction, the vanishing of their derivatives. For this, recall that the *product* IJ of two ideals I, J is the ideal

$$IJ = \langle \{ fg \mid f \in I \text{ and } g \in J \} \rangle$$

and the ideal I *raised to nth power* is

$$I^n = \underbrace{I \cdots I}_{n \text{ copies of } I}.$$

A useful property of products that we will rely on is that the product of two ideals $I = \langle f_1, \dots, f_k \rangle$ and $J = \langle g_1, \dots, g_l \rangle$ is generated by the products of the generators of I and J, i.e. $IJ = \langle f_i g_j \rangle$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$ [CLO15, p. 191].

To see how powers of ideals relate to orders of zeros in an intuitive example, consider the ideal $I(\hat{\mathsf{e}}_0) = \langle x_1 \rangle$ in $\mathbf{P}^1(\mathbb{F})$. It has zeros along the x_0 axis, and so do the ideals $\langle x_1 \rangle^n$ for any $n > 0$. However, the behavior of the functions in these ideals around the x_0-axis is not the same: not only does x_1^n take the value zero at x_0, but so do all of its first $n-1$ x_0-derivatives, i.e.

$$\frac{\partial}{\partial x_0^n} x_1^m(p) = 0 \text{ for all } p \in \hat{\mathsf{e}}_0$$

whenever $m > n$. [Eis04, pp. 103–105] describes this as $\langle x_1 \rangle^2$ also containing an "infinitesimal neighborhood" around the x_0 axis that the ideal $\langle x_1 \rangle$ does not. These kinds of points—where the functions of an ideal have zeros of higher order than 1—are are the ones we have referred to as *fat points*. To make the idea of points that stretch out over an infinitesimal neighborhood precise requires a more subtle handling of ideals than through their vanishing sets, such as the one provided by schemes. As we shall see, the concept of a *differential inverse system* will provide another way to approach the problem in the sense that the differential inverse system encodes some information about zeros of derivatives.

Nothing like the simple correspondence that I^n has zeros of order n at p whenever I is radical and has zeros of order 1 at p can hold in general, though. Instead, a more complex operation called a *symbolic power* is needed, and even with that the correspondence holds only when I is prime and $\mathbb{F}$ is algebraically closed [Eis04, p. 106]. However, for the relatively simple case that we are considering here—ideals generated by linearly independent linear functions—we do have the following:

Theorem 2.3.1. *Let $I = \langle l_1, \dots, l_k \rangle$ where $l_1, \dots, l_k \in R_1$ and are linearly independent. Let $A \subseteq \mathbb{F}^{r+1}$ be the subspace where I vanishes. Then $\frac{\partial}{\partial x^\beta} f(p) = 0$ for all β such that $|\beta| < n$, all $p \in A$, and all homogenous $f \in I^n$.*

Proof. For the case where $k = r + 1$, A is just the origin and the theorem follows trivially by f being assumed to be homogenous. For $k \leqslant r$ we again simplify by reducing to the axis-aligned case using an invertible linear transformation L. This is admissible here because we are looking for points where *all* partial derivatives up to a certain order are zero, and it is obvious that if this holds for $f(p)$ then it will also hold for $f(L(p))$. So assume that $l_1, \ldots, l_k$ are mapped to the axis vectors $\mathsf{e}_0, \ldots, \mathsf{e}_{k-1}$ by L, which entails that A must be mapped to the subspace spanned by the other basis vectors $\mathsf{e}_k, \ldots, \mathsf{e}_r$. Any $f \in I^n$ can be written as a linear combination of terms of the form

$$g \prod_{j=1}^{n} l_{i_j}$$

where each l_{i_j} is one of $l_1, \ldots, l_k$ and $g \in R$. Transformed to our new coordinate system, this means that $f \circ L$ is a linear combination of products of n linear functions along some of the axis vectors $\mathsf{e}_0, \ldots, \mathsf{e}_{k-1}$ and a homogenous polynomial $g \in R_{\deg f - n}$. Write each such term as gx^α, where $|\alpha| = n$ and $\alpha_i = 0$ for $i \geqslant k$. Then we need to show that

$$\frac{\partial}{\partial x^\beta} gx^\alpha(p) = 0$$

whenever the first k coordinates of p are zero. Applying the differentiation using the general Leibniz rule gives us

$$\begin{aligned} \frac{\partial}{\partial x^\beta} gx^\alpha &= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} \frac{\partial}{\partial x^{\beta-\gamma}} g \, \frac{\partial}{\partial x^\gamma} x^\alpha \\ &= \sum_{\gamma \leqslant \min(\alpha,\beta)} \binom{\beta}{\gamma} \left(\frac{\partial}{\partial x^{\beta-\gamma}} g \right) x^{\alpha-\gamma}. \end{aligned}$$

In the terms where $\gamma < \alpha$ this entails that at least one of the coordinates $x_0, \ldots, x_{k-1}$ must be in the product, and since all of these are 0, it follows that these terms are also 0. But $\gamma = \alpha$ would imply that $|\gamma| = |\alpha|$, which contradicts the conditions that $\gamma \leqslant \beta$ and $|\beta| < |\alpha| = n$. So all terms of the sum are 0.

□

Since the ideals $I(\hat{p})$ that we have been studying here are generated by linear functions we can infer that their powers $I(\hat{p})^n$ can be used to describe single fat projective points in this sense. For ideals such as $I(\hat{p}_1, \ldots \hat{p}_k)$ which involve several points the situation is far more complicated, and we will return to it briefly in Section 5.1.

3 Inverse systems

3.1 Contractive and differential operators

We will follow the authors of [EI95] in studying inverse systems based on two operators: one similar to the one Macaulay presents in [Mac94], and one that ties in to differentiation. Unlike them, we will simplify our discussion somewhat by defining both to be binary operators on the polynomial ring $R = \mathbb{F}[x_0, \ldots, x_r]$ rather than to take elements from different rings.

For the first operator, define the *contraction* $x^\beta \triangleright_{\mathrm{c}} x^\alpha$ of a monomial x^α by another monomial x^β as

$$x^\beta \triangleright_{\mathrm{c}} x^\alpha = \begin{cases} x^{\alpha-\beta} \text{ if } \alpha \geqslant \beta \\ 0 \text{ otherwise} \end{cases}$$

and extend it to all homogenous polynomials of R linearly, i.e. if

$$f = \sum_{\alpha \in \deg f} a_\alpha \, x^\alpha \qquad\qquad h = \sum_{\beta \in \deg h} b_\beta \, x^\beta$$

then

$$\begin{aligned} h \triangleright_{\mathrm{c}} f &= \sum_{\alpha \in \deg f} \sum_{\beta \in \deg h} a_\alpha b_\beta \, (x^\beta \triangleright_{\mathrm{c}} x^\alpha) \\ &= \sum_{\alpha \in \deg f} \sum_{\substack{\beta \in \deg h \\ \beta \leqslant \alpha}} a_\alpha b_\beta \, x^{\alpha-\beta} \end{aligned}$$

We define the operator $\triangleright_{\mathrm{d}}$ by setting

$$x^\beta \triangleright_{\mathrm{d}} x^\alpha = \begin{cases} \binom{\alpha}{\alpha-\beta} \, x^{\alpha-\beta} \text{ if } \alpha \geqslant \beta \\ 0 \text{ otherwise} \end{cases}$$

and extending it linearly to

$$
\begin{aligned}
h \rhd_{\mathrm{d}} f &= \sum_{\alpha \in \deg f} \sum_{\beta \in \deg h} a_\alpha b_\beta \left(x^\beta \rhd_{\mathrm{d}} x^\alpha\right) \\
&= \sum_{\alpha \in \deg f} \sum_{\substack{\beta \in \deg h \\ \beta \leqslant \alpha}} \binom{\alpha}{\alpha - \beta} a_\alpha b_\beta \, x^{\alpha - \beta}
\end{aligned}
$$

for all homogenous $h, f \in R$. This is equivalent to

$$
h \rhd_{\mathrm{d}} f = \sum_{\beta \in \deg h} \kappa(h)_\beta \frac{\partial}{\partial x^\beta} f
$$

so applying h to f with $\rhd_{\mathrm{d}}$ is the same as interpreting h as a differential operator with constant coefficients and applying this operator to f, and we may therefore refer to $h \rhd_{\mathrm{d}} f$ as the *differentiation* of f by h. This interpretation is what gives $\rhd_{\mathrm{d}}$ its specific geometric content.

The following are some example applications of these two operators:

$$
\begin{aligned}
x_0 \rhd_{\mathrm{c}} x_0^2 &= x_0 & \qquad x_0 \rhd_{\mathrm{d}} x_0^2 &= 2x_0 \\
x_0^2 \rhd_{\mathrm{c}} x_0 x_1^2 &= 0 & \qquad x_0^2 \rhd_{\mathrm{d}} x_0 x_1^2 &= 0 \\
2x_0^2 - 3x_0x_1 \rhd_{\mathrm{c}} x_0^2 x_1 &= 2x_1 - 3x_0 & \qquad 2x_0^2 - 3x_0x_1 \rhd_{\mathrm{d}} x_0^2 x_1 &= 4x_1 - 6x_0
\end{aligned}
$$

Since differential operators with constant coefficients commute we always have

$$
h_2 \rhd_{\mathrm{d}} (h_1 \rhd_{\mathrm{d}} f) = (h_1 h_2) \rhd_{\mathrm{d}} f = h_1 \rhd_{\mathrm{d}} (h_2 \rhd_{\mathrm{d}} f)
$$

and the same property for $\rhd_{\mathrm{c}}$ follows for much the same algebraic reason, so the result of applying several such operators in sequence is the same as applying their product as polynomials.

When we discuss properties that hold for both $\rhd_{\mathrm{c}}$ and $\rhd_{\mathrm{d}}$ we will sometimes use a "generic" version of the operator written as $\rhd$ to stand for either. We will mainly be interested in zeros of these operators for given left-hand arguments, i.e. the polynomials f such that $h \rhd f = 0$. The following lemma gives a useful characterization of these through the polynomials' coefficients:

Lemma 3.1.1. *Let $f, h \in R$ be homogenous polynomials and let $a = \kappa(f)$ and $b =$*

$\kappa(h)$. *Then*

$$h \rhd_{\mathrm{c}} f = 0 \textit{ iff } \sum_{\beta \in \deg h} a_{\gamma+\beta} b_\beta = 0 \textit{ for all } \gamma \textit{ such that } |\gamma| = \deg f - \deg h$$
$$h \rhd_{\mathrm{d}} f = 0 \textit{ iff } \sum_{\beta \in \deg h} \binom{\gamma+\beta}{\gamma} a_{\gamma+\beta} b_\beta = 0 \textit{ for all } \gamma \textit{ such that } |\gamma| = \deg f - \deg h.$$

Proof. We sketch the proof for $\rhd_{\mathrm{d}}$; it is essentially the same for $\rhd_{\mathrm{c}}$. The value of $h \rhd_{\mathrm{d}} f$ is zero iff $h \rhd_{\mathrm{d}} f$ is a sum of monomials $c_\gamma x^\gamma$ such that $c_\gamma = 0$ for all γ. From the definition, it follows at once that the only values of γ such that $c_\gamma \neq 0$ is possible must be ones where $|\gamma| = \deg f - \deg h$. By writing $\alpha = \gamma + \beta$ we arrive at

$$c_\gamma = \sum_{\beta \in \deg h} \binom{\gamma+\beta}{\gamma} a_{\gamma+\beta} b_\beta.$$

□

3.2 Contraction, differentiation, and vector space structure

Lemma 3.1.1 can be seen as bringing out the fundamentally linear algebraic nature of the contractive and differential operators. This is clearest when $\deg f = \deg h$, in which case $\rhd$ works like a pairing operation on a vector space:

Theorem 3.2.1. *If* $\deg f = \deg h = i$ *then*

$$h \rhd_{\mathrm{c}} f = \sum_{|\alpha|=i} \kappa(f)_\alpha \kappa(h)_\alpha$$
$$h \rhd_{\mathrm{d}} f = \sum_{|\alpha|=i} \alpha! \kappa(f)_\alpha \kappa(h)_\alpha.$$

Furthermore, $\rhd$ *is a perfect pairing on* $R_i \times R_i$*, i.e. there is an isomophism of vector spaces* $\varphi : R_i \to (R_i \to \mathbb{F})$ *such that* $\varphi(h)(f) = h \rhd f$*.*

Proof. The equalities are proved the same way as Lemma 3.1.1, by taking $|\gamma| = 0$. The function φ is an isomorphism because $\rhd$ is bilinear and, furthermore, the only $f \in R_i$ such that $h \rhd f = 0$ for all $h \in R_i$ is $f = 0$ (see [MB99, p. 210], where the term "dual pairing" is used for this). □

The pairing is not perfect for $R_i \times R_j$ when $i \neq j$. For $i > j$, $h \rhd f = 0$ for all h, f, and for $i < j$, φ would have to be an isomorphism between R_i and the space of linear functions $R_i \to R_{i-j}$, which is impossible since the spaces have different dimensionalities.

That $\triangleright$ sets up a perfect pairing lets us talk about orthogonality for pairs of polynomials of the same degree in the sense that h is orthogonal to f when $h \triangleright f = 0$. However, this notion of orthogonality does not always work like one would intuitively expect it to. In particular when $\mathbb{F} = \mathbb{C}$, we have polynomials that are orthogonal to themselves, such as $x_0 + ix_1$. Vectors that are self-orthogonal are called *isotropic*, and the one-dimensional subspace each of them spans is *totally singular* in the sense that all vectors in the subspace are orthogonal to all others [Rom08, pp. 265–266]. The problem is that although $\triangleright$, when restricted to a given degree, is a symmetric bilinear form, it is not an inner product since it is not positive definite. Since it will sometimes make things easier for us to have a more well-behaved product, we will say that $\mathbb{F}$ has an *inverse system-compatible automorphism* if it comes with an automorphism $\bar{\cdot}$ that, when extended to R in the sense that

$$\bar{f} = \sum_{\alpha \in \deg f} \overline{\kappa(f)_\alpha} x^\alpha$$

satisfies the property

$$\overline{f} \triangleright f = 0 \Rightarrow f = 0.$$

This is not as strong as being positive definite since we haven't required $\triangleright$ to take values in an ordered field, and so cannot require it to be non-negative. It will however be enough for our purposes here. For fields like $\mathbb{Q}$ and $\mathbb{R}$ we will assume $\bar{\cdot}$ to be the identity automorphism (as that is the only one that exists), and for $\mathbb{C}$ we will assume it to be the regular complex conjugate. In both these spaces, we could strengthen the geometric flavor of the operators we are studying by making slight modifications to the definitions $\triangleright_c$ and $\triangleright_d$, obtaining variants $\overline{\triangleright_c}$ and $\overline{\triangleright_d}$ as follows:

$$b_\beta x^\beta \,\overline{\triangleright_c}\, a_\alpha x^\alpha = \overline{b_\beta} x^\beta \triangleright_c a_\alpha x^\alpha$$
$$b_\beta x^\beta \,\overline{\triangleright_d}\, a_\alpha x^\alpha = \overline{b_\beta} x^\beta \triangleright_d a_\alpha x^\alpha$$

Doing this makes both $\overline{\triangleright_c}$ and $\overline{\triangleright_d}$ into sesquilinear forms on $\mathbb{C}$ rather than linear ones, and causes them to be inner products on $R_i \times R_i$. Using these rather than $\triangleright_c$ and

$\triangleright_{\mathrm{d}}$ would let us approach the concept of orthogonality in a more intuitive and easy-to-work with way, which would be a benefit since, as we will see, orthogonality plays a central role in the theory. However, to not stray too far from the existing literature (and in particular [EI95]) we will not adopt this definition generally. Instead, we will point out where we have to apply the automorphism $\bar{\cdot}$ as we go. However, we will in the future tacitly assume that the automorphism in question exists for any of the fields $\mathbb{F}$ we are considering:

Standing Assumption: *Unless otherwise stated, $\mathbb{F}$ is assumed to have an inverse system-compatible automorphism $\bar{\cdot}$.*

3.3 Definitions and basic properties of inverse systems

By the *inverse systems* $\operatorname{CAnn} I$ and $\operatorname{DAnn} I$ of a homogenous ideal I we mean the sets of annihilators of the left-hand arguments of the operators $\triangleright_{\mathrm{c}}$ and $\triangleright_{\mathrm{d}}$, i.e.

$$\operatorname{CAnn} I = \{f \in R \mid h \triangleright_{\mathrm{c}} f = 0 \text{ for all } h \in I\}$$
$$\operatorname{DAnn} I = \{f \in R \mid h \triangleright_{\mathrm{d}} f = 0 \text{ for all } h \in I\}$$

or, spelled out, the set of polynomials that are taken to 0 by the application of $h \triangleright_{\mathrm{c}}$ or $h \triangleright_{\mathrm{d}}$ for all elements $h \in I$. We refer to these as the *contractive inverse system* and the *differential inverse system* of I. When we discuss results that are valid for either of these we use the symbol Ann, and we will assume that CAnn matches with $\triangleright_{\mathrm{c}}$ and DAnn with $\triangleright_{\mathrm{d}}$ when we use both Ann and $\triangleright$. Since we will often be interested in specific degrees of inverse systems we will also write $\operatorname{Ann}_i I$ for $(\operatorname{Ann} I) \cap R_i$.

From the fact we have defined Ann on ideals it follows that the sets $\operatorname{Ann}_i I$ cannot vary arbitrarily for different i. In particular, we have that whether a homogenous polynomial of degree i is in $\operatorname{Ann} I$ or not is determined fully by which elements are in I_i. The following theorem and its corollaries will be some of the most commonly used ones in this section:

Theorem 3.3.1. *If $f \in \operatorname{Ann}_i I_i$ then $f \in \operatorname{Ann}_i I_j$ for all j, or equivalently,*

$$\operatorname{Ann}_i I = \operatorname{Ann}_i I_i.$$

Proof. For $j > i$, $h \triangleright f = 0$ for every $h \in R_j$, so $f \in \operatorname{Ann}_i I_j$ trivially. For $j < i$ we will show that if $f \notin \operatorname{Ann}_i I_j$ then $f \notin \operatorname{Ann}_i I_i$ for any f. Assume that there is a $h \in I_j$

such that $h \triangleright f \neq 0$ (if $I_j = \varnothing$, then all f are in $\operatorname{Ann} I_j$ vacuously). We want to show that there is a $h' \in \operatorname{Ann} I_i$ such that $h' \triangleright f \neq 0$.

Define $h' = \overline{(h \triangleright f)}h$. Then $h' \in I_i$ since $\deg \overline{(h \triangleright f)} = i - j$. Furthermore

$$h' \triangleright f = (\overline{(h \triangleright f)}h) \triangleright f = \overline{(h \triangleright f)} \triangleright (h \triangleright f) \neq 0$$

where the last equality holds because of the assumption we made on the automorphism $\bar{\cdot}$.

For the equivalent version, assume that $f \in R_i$ and that $h_j \triangleright f = 0$ for all $h_j \in I_j$ and all j. Then it follows that $h \triangleright f = 0$ for all $h \in \bigcup_j I_j$, and since $I = \operatorname{span}\{\bigcup_j I_j\}$ and $\triangleright$ is bilinear, we get that $h \triangleright f = 0$ for all $h \in I$. ☐

Corollary 3.3.2. *If I is an ideal of R then for all $i \geqslant 1$*

$$\dim(\operatorname{Ann}_i I) = \dim R_i - \dim I_i.$$

Proof. From $\triangleright$ being a perfect pairing (Th. 3.2.1) it follows that $\operatorname{Ann}_i I_i$ is the regular annihilator subspace (in the linear algebraic sense) of the subspace I_i of R_i. Thus $\dim(\operatorname{Ann}_i I_i) = \dim(R_i) - \dim(I_i)$ (see e.g. [MB99, p. 211]). But by the previous theorem $\operatorname{Ann}_i I_i = \operatorname{Ann}_i I$, so $\dim(\operatorname{Ann}_i I) = \dim(R_i) - \dim(I_i)$. ☐

Corollary 3.3.3. *For any $f \in R_i$ and any ideal $I \subseteq R$,*

$$f \in \operatorname{CAnn} I \ \textit{iff} \sum_{\alpha:|\alpha|=i} \kappa(f)_\alpha \kappa(h)_\alpha = 0 \textit{ for all } h \in I_i$$
$$f \in \operatorname{DAnn} I \ \textit{iff} \sum_{\alpha:|\alpha|=i} \alpha!\, \kappa(f)_\alpha \kappa(h)_\alpha = 0 \textit{ for all } h \in I_i.$$

Proof. Follows from Theorem 3.3.1 together with Theorem 3.2.1. ☐

Theorem 3.3.1 lets us approach inverse systems degree by degree. In fact, a further simple corollary of it is that

$$\operatorname{Ann} I = \bigcup_i \operatorname{Ann}_i I_i.$$

It is worth noting that although the difference between $\triangleright_{\mathrm{c}}$ and $\triangleright_{\mathrm{d}}$ seems to be just about which constants are being multiplied with, this results in concrete differences in their inverse systems. For example, $x^2 + 2xy$ is in the differential inverse system of the principal ideal $\langle x^2 - xy \rangle$, but in not the contractive inverse system of the same ideal (which contains $x^2 + xy$ instead). Corollary 3.3.3 gives a particularly clear way to view the difference: f is in $\operatorname{Ann}_i I$ iff f is orthogonal to h for all $h \in I_i$ with respect to the bilinear form $\triangleright$. This says that $\operatorname{Ann}_i I$ is precisely the orthogonal complement of I_i in the space R_i. We get the two variants $\triangleright_{\mathrm{c}}$ and $\triangleright_{\mathrm{d}}$ by imposing different weighting of the coordinate axes: equal weighting for $\triangleright_{\mathrm{c}}$ and giving the axis x^α the weight $\alpha!$ for $\triangleright_{\mathrm{d}}$.

This interpretation points to a method for translating between the two systems that will be useful later on, since it allows us to prove theorems for the kind of system that it is easiest for, and then translate the result to the other kind. Let $\phi_{c\to d} : R \to R$ be defined as

$$\phi_{c\to d}(f) = \sum_{\alpha \in \deg f} \alpha!\, \kappa(f)_\alpha x^\alpha.$$

This means that $\phi_{c\to d}(f)$ is like f, except that each term $a_\alpha x^\alpha$ is multiplied by $\alpha!$. This function, which is linear on each degree, clearly has an inverse that consists in dividing each term with $\alpha!$ rather than multiplying with it. We will refer to this inverse with the symbol $\phi_{d\to c}$. The following is another corollary of Theorem 3.3.1:

Corollary 3.3.4. *For any ideal I*

$$\operatorname{CAnn} I = \phi_{d\to c}[\operatorname{DAnn} I]$$
$$\operatorname{DAnn} I = \phi_{c\to d}[\operatorname{CAnn} I].$$

Proof. Follows from Corollary 3.3.3 of Theorem 3.3.1. □

3.4 Inverse system structure

Even though it is a subset of R, an inverse system is typically not an ideal. More specifically, it is not guaranteed to be closed under multiplication with general elements of R. It is, however, closed under linear combinations (so it is a module of the base field, and even a vector space in the cases we are considering here) and under applications of its defining operator to the right-hand side.

Theorem 3.4.1. *The sets $\operatorname{DAnn} I$ and $\operatorname{CAnn} I$ are closed under the following, for any ideal I:*

- *If $f, g \in \operatorname{Ann} I$ then $\lambda_1 f + \lambda_2 g \in \operatorname{Ann} I$ for $\lambda_1, \lambda_2 \in \mathbb{F}$.*

- *If $f \in \operatorname{Ann} I$ then $g \triangleright f \in \operatorname{Ann} I$ for all $g \in R$.*

Proof. Linearity is a direct consequence of $\triangleright_{\mathrm{c}}$ and $\triangleright_{\mathrm{d}}$ being bilinear. Closure under $g \triangleright_{\mathrm{c}} f$ or $g \triangleright_{\mathrm{d}} f$ follows from I being closed under multiplication with elements of R together with the rule $h_2 \triangleright (h_1 \triangleright f) = (h_1 h_2) \triangleright f$. □

Corollary 3.4.2. *The only ideals that are also inverse systems of any ideal are the zero ideal $\{0\}$ and the unit ideal R.*

Proof. It is trivial that the zero ideal is also an inverse system. Thus let X be an inverse system and ideal which is non-zero. Then $f \neq 0$ for some $f \in X$, so $\bar{f} \triangleright f \neq 0$ by the assumption on $\bar{\cdot}$, and $\bar{f} \triangleright f \in X$ by Theorem 3.4.1. Since $\deg \bar{f} = \deg f$, $\bar{f} \triangleright f$ must be a non-zero constant. As X is closed under multiplication with scalars, it follows that $1 \in X$, so $X = R$ since X is an ideal. □

For CAnn, closure under application of the defining operator to the right-hand side is equivalent to CAnn being closed under the lowering of the degree of any variable. For DAnn, the situation is slightly different. For example, $3x^2 + 2xy$ is in every differential inverse system that contains $x^3 + xy^2$, but it is not guaranteed to be in a contractive inverse system with the same element.

For a principal ideal the inverse system is given directly by that of the generating polynomial:

Theorem 3.4.3. *For any $h \in R$ we have* $\operatorname{Ann}\langle h \rangle = \{f \in R \mid h \triangleright f = 0\}$.

Proof. That $\{f \mid h \triangleright f = 0\} \supseteq \operatorname{Ann}\langle h \rangle$ follows trivially from the definition of Ann. For the converse we need to show that if $h \triangleright f = 0$ and $h' \in \langle h \rangle$ then $h' \triangleright f = 0$. But h' being in $\langle h \rangle$ means that $h' = gh$ for some $g \in R$, and because $(gh) \triangleright f = g \triangleright (h \triangleright f)$ and $h \triangleright_{\mathrm{d}} 0 = h \triangleright_{\mathrm{c}} 0 = 0$, it follows that $(gh) \triangleright f = 0$. □

To understand inverse systems of non-principal ideals it is useful to inspect the definitions of the Ann operators closer. We can then see that they are one half of a *polarity* (a type of antitone Galois connection, see [Bir73, pp. 122–124]) between the sets of ideals $\mathcal{I} \subseteq \wp(R)$ and inverse systems $\mathcal{S} \subseteq \wp(R)$ of R, each ordered by set inclusion. The other half of the polarity would be given by

$$\operatorname{Ann}^* X = \{h \in R \mid h \triangleright f \text{ for all } f \in X\}$$

i.e for $\triangleright_{\mathrm{d}}$, the function that assigns to the inverse system X the set of differential operators under which all functions in X are zero. From Ann and Ann* making up a polarity a number of important properties follow. In particular,

$$I \subseteq J \Leftrightarrow \operatorname{Ann} J \subseteq \operatorname{Ann} I.$$

Furthermore, bounds on $\mathcal{I}$ and $\mathcal{S}$ map to each other. Because we have the least upper and greatest lower bounds of the lattice of ideals—least upper bound is ideal sum and greatest lower bound is intersection—the least upper and greatest lower bounds of inverse systems are also determined:

Theorem 3.4.4. *If I, J are ideals then*

$$\begin{aligned}\operatorname{Ann}(I + J) &= \operatorname{Ann}(I) \cap \operatorname{Ann}(J)\\ \operatorname{Ann}(I \cap J) &= \operatorname{Ann}(I) +_v \operatorname{Ann}(J).\end{aligned}$$

Proof. Working from the definition of the inverse system as a polarity we have

$$\begin{aligned}f \in \operatorname{Ann}(I) \cap \operatorname{Ann}(J) &\Leftrightarrow h \triangleright f = 0 \text{ for all } h \in I \text{ and } h \triangleright f = 0 \text{ for all } h \in J\\ &\Leftrightarrow h \triangleright f = 0 \text{ for all } h \in (I \cup J)\\ &\Leftrightarrow f \in \operatorname{Ann}(I \cup J)\end{aligned}$$

from which it follows that $\operatorname{Ann}(I + J) \subseteq \operatorname{Ann}(I) \cap \operatorname{Ann}(J)$ since $I \cup J \subseteq I + J$ and Ann is order reversing.[3]

To show the reverse inclusion we need to prove that $\operatorname{Ann}(I \cup J) \subseteq \operatorname{Ann}(I + J)$. Let $h_1, h_2 \in I \cup J$. Assume that $f \in \operatorname{Ann}(I \cup J)$ and pick arbitrary $h_1 \in I$ and $h_2 \in J$. This means that $h_1 \triangleright f = h_2 \triangleright f = 0$, so $h_1 + h_2 \triangleright f = 0$ since $\triangleright$ is bilinear. This gives at once that $f \in \operatorname{Ann}(I + J)$.

To show that $\operatorname{Ann}(I) +_v \operatorname{Ann}(J) \subseteq \operatorname{Ann}(I \cap J)$, let $f = f_1 + f_2$ with $f_1 \in \operatorname{Ann} I$ and $f_2 \in \operatorname{Ann} J$ (we do not need to care about the scale factors since each inverse system is closed under multiplication with a constant). Then $h_1 \triangleright f_1 = 0$ for all $h_1 \in I$ and $h_2 \triangleright f_2 = 0$ for all $h_2 \in J$, so $(h \triangleright f_1) + (h \triangleright f_2) = h \triangleright (f_1 + f_2) = 0$ for all h in both I

[3]Note that we have allowed ourselves to apply Ann to a non-ideal in the last equality despite our having only strictly defined it for ideals. However, due to its definition as a polarity, it makes sense for any set.

and J. But this is the same as $f = (f_1 + f_2) \in \mathrm{Ann}(I \cap J)$.

For the reverse direction, note that we for purely linear algebraic reasons always have

$$\dim(I_i \cap J_i) = \dim(I_i) + \dim(J_i) - \dim(I_i +_v J_i)$$

for any degree i simply because I_i and J_i are subspaces of R_i. Applying Corollary 3.3.2 gives us

$$\begin{aligned}\dim(R_i) - \dim(\mathrm{Ann}_i(I \cap J)) = &\dim R_i - \dim(\mathrm{Ann}_i I)\\ &+ \dim R_i - \dim(\mathrm{Ann}_i J)\\ &- (\dim R_i - \dim(\mathrm{Ann}_i(I + J))\end{aligned}$$

from which we may infer

$$\begin{aligned}\dim(\mathrm{Ann}_i(I \cap J)) &= \dim(\mathrm{Ann}_i I) + \dim(\mathrm{Ann}_i J) - \dim(\mathrm{Ann}_i(I + J))\\ &= \dim(\mathrm{Ann}_i I) + \dim(\mathrm{Ann}_i J) - \dim((\mathrm{Ann}_i I) \cap (\mathrm{Ann}_i J))\\ &= \dim(\mathrm{Ann}_i I +_v \mathrm{Ann}_i J).\end{aligned}$$

But if the subspaces $\mathrm{Ann}_i(I \cap J)$ and $\mathrm{Ann}_i I +_v \mathrm{Ann}_i J$ have the same dimension and one contains the other they must be equal, so $\mathrm{Ann}_i(I \cap J) = \mathrm{Ann}_i I +_v \mathrm{Ann}_i J$. Since i was arbitrary, the result follows.

□

From the fact that $\mathrm{Ann}(I + J) = \mathrm{Ann}(I) \cap \mathrm{Ann}(J)$ we can derive some useful further properties. For example, the inverse system of any ideal is given by the intersection of the inverse systems of its generators:

Corollary 3.4.5. *For any $h_1, \dots h_k \in R$,*

$$\mathrm{Ann}\langle h_1, \dots, h_k\rangle = \bigcap_{i=1}^{k} \{f \in R \mid h_i \triangleright f = 0\}.$$

Proof. This follows from $\langle h_1, \dots, h_k\rangle = \langle h_1\rangle + \cdots + \langle h_k\rangle$ together with Theorem 3.4.3.

□

We also have that Ann is invertible, so the lattice $\mathcal{S}$ of inverse systems is isomorphic to the lattice $\mathcal{I}$ of ideals of R with the subsethood relation reversed. The inverse of Ann is the other half of the polarity, which goes from $\mathcal{S}$ back to $\mathcal{I}$.

Theorem 3.4.6. *The functions* Ann *are invertible, and* $\operatorname{Ann}^{-1} = \operatorname{Ann}^*$.

Proof. We need to show that $\operatorname{Ann}^*(\operatorname{Ann} I) = I$ and that $\operatorname{Ann}(\operatorname{Ann}^* X) = X$. The second of these is straightforward: since we have *defined* $\mathcal{S}$ to be the image of $\mathcal{I}$ under Ann, Ann is clearly surjective, so each $X \in \mathcal{S}$ can be written as $X = \operatorname{Ann} I$ for some I. We furthermore always have that $\operatorname{Ann}(\operatorname{Ann}^*(\operatorname{Ann} I)) = \operatorname{Ann} I$ from Ann and Ann^* making up a Galois connection [Bir73, p. 123], so replacing $\operatorname{Ann} I$ with X gives us $\operatorname{Ann}(\operatorname{Ann}^* X) = X$.

To prove injectivity of Ann, note that if $I \neq J$ then there must be some degree i such that $I_i \neq J_i$. Since different subspaces have different orthogonal complements for any nonsingular bilinear form (see e.g. [Rom08, p. 270] where it is proved that orthogonal complement is an involution) and both $\triangleright_{\mathrm{c}}$ and $\triangleright_{\mathrm{d}}$ are nonsingular by Theorem 3.2.1 it follows from Theorem 3.3.1 that $\operatorname{Ann}_i I \neq \operatorname{Ann}_i J$, so Ann^{-1} exists. From Ann and Ann^* being a Galois connection we then have that $\operatorname{Ann}(\operatorname{Ann}^*(\operatorname{Ann} I)) = \operatorname{Ann} I$. Applying Ann^{-1} to both sides then gives us $\operatorname{Ann}^*(\operatorname{Ann} I) = I$, so Ann^* is the two-sided inverse of Ann. ☐

3.5 Calculating the inverse system of a quadric

What we have proved so far is sometimes sufficient for calculating the inverse system of an ideal degree by degree. To make this section a bit more concrete, we will here give an example of how one can determine the inverse system of the ideal $Q = \langle q \rangle = \langle x_1^2 + x_2^2 - x_0^2 \rangle$ that we discussed in the introduction. We start by noting that $\operatorname{Ann}_0 Q$ and $\operatorname{Ann}_1 Q$ must be the whole of R_0 and R_1 since the least degree of the functions in Q is 2, and $h \triangleright f = 0$ whenever $\deg h > \deg f$.

For Ann_2 Q, we begin by looking at the vector space structure. R_2 is spanned by the monomials $x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2$ and x_1x_2, and using these as our coordinate system makes $\kappa(q)$ come out as the vector $(-1, 1, 1, 0, 0, 0)$. Since degree i of the inverse system of Q is the set of degree i polynomials orthogonal to all functions in Q_i and $Q_2 = \{\lambda q \mid \lambda \in \mathbb{F}\}$, $\operatorname{Ann}_2 Q$ must be spanned by the degree 2 polynomials that are orthogonal to $x_1^2 + x_2^2 - x_0^2$ according to the bilinear form $\triangleright$ that we are using. Expressed in terms of coefficients, these are the polynomials whose coefficient vectors are orthogonal to $(-1, 1, 1, 0, 0, 0)$. One example of a set of coefficient vectors satisfying this is $(1,1,0,0,0,0)$, $(1,0,1,0,0,0)$, $(0,0,0,1,0,0)$, $(0,0,0,0,1,0)$ and $(0,0,0,0,0,1)$, so

$$\operatorname{Ann}_2 Q = \operatorname{span}\{x_0^2 + x_1^2, x_0^2 + x_2^2, x_0x_1, x_0x_2, x_1x_2\}.$$

This holds both for CAnn and DAnn. Although they have different weights for the basis vectors (equal for CAnn and twice the weight of the first three vectors compared to the last three vectors for DAnn) the fact that q contains no mixed terms of the form x_kx_l for $k \neq l$ entails that this makes no difference for degree 2. Unfortunately this is not true for higher degrees since these are spanned by the functions $x^\beta q$ where β is an arbitrary exponent.

Writing Q_3 as span $\{x_0q, x_1q, x_2q\} = x_0q +_v x_1q +_v x_2q$ and applying Theorem 3.4.4 lets us calculate $\operatorname{Ann}_3 Q$ as

$$\operatorname{Ann}_3 Q = \operatorname{Ann}_3\langle x_0q\rangle \cap \operatorname{Ann}_3\langle x_1q\rangle \cap \operatorname{Ann}_3\langle x_2q\rangle.$$

As a vector space R_3 has 10 dimensions which are spanned by the functions x_0^3, x_1^3, x_2^3, $x_0^2x_1$, $x_0^2x_2$, $x_0x_1^2$, $x_0x_2^2$, $x_1^2x_2$, $x_1x_2^2$ and $x_0x_1x_2$. Writing out the coefficients of the generators $h_1 = x_0q$, $h_2 = x_1q$ and $h_3 = x_2q$ of Q_3 in terms of these gives us the following table:

	x_0^3	x_1^3	x_2^3	$x_0^2x_1$	$x_0^2x_2$	$x_0x_1^2$	$x_0x_2^2$	$x_1^2x_2$	$x_1x_2^2$	$x_0x_1x_2$
h_1	−1	0	0	0	0	1	1	0	0	0
h_2	0	1	0	-1	0	0	0	0	1	0
h_3	0	0	1	0	-1	0	0	1	0	0

The subspace $\operatorname{Ann}_3 Q$ is spanned by any 7 linearly independent 3[rd] degree homogenous polynomials orthogonal to these. One is directly visible in the table: $x_0x_1x_2$. For the others, we can start by renaming the dimensions of R_3 to $y_1, \ldots, y_{10}$ (so e.g. $x_0^3 = y_1$ and $x_0x_1x_2 = y_{10}$). We can then find the coefficients of the orthogonal complement of Q_3 by solving the linear system of equations

$$-y_1 + y_6 + y_7 = 0$$
$$y_2 - y_4 + y_9 = 0$$
$$y_3 - y_5 + y_8 = 0$$

The vectors that span the solution will be precisely the ones that span $\operatorname{CAnn}_3 Q$. In the present case, however, we can take a shortcut by trying some of the elements of $\operatorname{Ann}_2 Q$ and raising their degree by multiplying them with x_0, x_1 and x_2. Doing this for the polynomials $x_0^2 + x_1^2$ and $x_0^2 + x_2^2$ and adding the function $x_0x_1x_2$ that we found by direct inspection gives us the following table of coefficients for $\operatorname{CAnn}_3 Q$:[4]

	x_0^3	x_1^3	x_2^3	$x_0^2x_1$	$x_0^2x_2$	$x_0x_1^2$	$x_0x_2^2$	$x_1^2x_2$	$x_1x_2^2$	$x_0x_1x_2$
f_1	1	0	0	0	0	1	0	0	0	0
f_2	1	0	0	0	0	0	1	0	0	0
f_3	0	1	0	1	0	0	0	0	0	0
f_4	0	0	0	1	0	0	0	0	1	0
f_5	0	0	1	0	1	0	0	0	0	0
f_6	0	0	0	0	1	0	0	1	0	0
f_7	0	0	0	0	0	0	0	0	0	1

These can be checked to be linearly independent by row elimination, which shows the rank of this matrix to be 7. To get $\operatorname{DAnn}_3 Q$ rather than $\operatorname{CAnn}_3 Q$ we have to divide the coefficients of x_0^3, x_1^3 and x_2^3 by $3! = 6$ and the coefficients of $x_0^2x_1$, $x_0^2x_2$, $x_0x_1^2$, $x_0x_2^2$, $x_1^2x_2$ and $x_1x_2^2$ by $2! = 2$ (i.e apply the function $\phi_{c\to d}$). Doing this results in the following expression:

$$\begin{aligned}\operatorname{DAnn}_3 Q = \operatorname{span}\{&x_0^3/6 + x_0x_1^2/2, x_0^3/6 + x_0x_2^2/2,\\ &x_1^3/6, +x_0^2x_1/2, x_0^2x_1/2 + x_1x_2^2/2,\\ &x_2^3/6 + x_0^2x_2/2, x_0^2x_2/2 + x_1^2x_2/2,\\ &x_0x_1x_2\}.\end{aligned}$$

[4]Note that this is not always guaranteed to work: although $x^\beta h \triangleright_c x^\beta f = h \triangleright_c f$ for all β, h and f, as is easily shown, $x_kq \in \operatorname{Ann} Q$ requires x_kq to be annihilated by *all* of x_0h, x_1h and x_2h. Finding useful criteria for when $f \in \operatorname{Ann}_i I$ implies $x_kf \in \operatorname{Ann}_{i+1} I$ for arbitrary I is an open problem.

Continuing to degree 4, we have a vector space R_4 with 15 dimensions, and Q_4 has 6 dimensions which are spanned by $x_0^2q, x_1^2q, x_2^2q, x_0x_1q, x_0x_2q$ and x_1x_2q, so $\operatorname{Ann}_4 Q$ has 9 dimensions. Finding an expression for $\operatorname{Ann}_4 Q$ can, as in the 3rd degree case, be done by solving a linear system of equations.

For an arbitrary degree $i \geqslant 2$, Q_i is spanned by the polynomials $x^\beta q$ where $|\beta| = i - 2$, which means that Q_i is $\binom{3+i-2}{3}$-dimensional. As the whole of R_i is $\binom{3+i}{3}$-dimensional, $\operatorname{Ann}_i Q$ is spanned by $\binom{3+i}{3} - \binom{1+i}{3}$ polynomials, which can again be calculated from the basis of Q_i by solving the corresponding system of linear equations.

3.6 Inverse systems for monomial and related ideals

As we saw in the last section, the linear algebraic nature of $\triangleright$ helps a lot when calculating a given degree of an inverse system. However, there is still no clear common way to express the different degrees systematically, or to describe the inverse system as a whole. There of course no guarantee that this is possible in general: some ideals might just have too complicated inverse systems for us to be able to give a simple formula for them. This section will focus on a class of ideals where we can give clear answers, however, and it will also introduce a theorem that can be used to reduce certain other ideals to this case. In particular, the theorem in question will be our main tool for handling ideals of fat points in projective space.

The simplest ideals to calculate inverse systems for are undoubtedly those generated by monomials; this corresponds to the axis-aligned cases we have taken as examples and used in proofs in the preceding section. In fact, the monomials that are in the inverse system of a monomial ideal are exactly the ones that are not in that ideal, and the inverse system in question is precisely the linear span of these monomials, both for $\triangleright_{\mathrm{c}}$ and $\triangleright_{\mathrm{d}}$:

Theorem 3.6.1. *If I is a monomial ideal then* $\operatorname{Ann} I = \operatorname{span} \{x^\alpha \mid x^\alpha \notin I\}$.

Proof. Assume that $I = \langle x^{\beta_1}, \ldots, x^{\beta_k} \rangle$ for some monomials $x^{\beta_1}, \ldots, x^{\beta_k}$. We first show that $x^\alpha \notin \operatorname{Ann} I$ iff $x^\alpha \in I$. From the definitions of $\triangleright_{\mathrm{c}}$ and $\triangleright_{\mathrm{d}}$ it follows that $x^{\beta_i} \triangleright_{\mathrm{c}} x^\alpha \neq 0$ iff $\alpha \geqslant \beta_i$. But this is exactly the same condition under which $x^\alpha = x^{\beta_i} x^\gamma$ for some x^γ in R, i.e. the condition under which $x^\alpha \in \langle x^{\beta_i} \rangle$. Since the inverse system of a sum of principal ideals is the intersection of the individual inverse systems, we also have that $x^\alpha \notin \operatorname{Ann} I$ iff there is no i such that $x^\alpha \in \langle x^{\beta_i} \rangle$, or, contrapositively, that $x^\alpha \in \operatorname{Ann} I$ iff $x^\alpha \notin I$.

Since inverse systems are always closed under linear spans, span $\{x^\alpha \mid x^\alpha \notin I\} \subseteq \operatorname{Ann} I$. Assume that this is not an identity. Then there must be a further monomial $x^\gamma \in R$ such that $x^\gamma \in \operatorname{Ann} I$. But we have just shown that each monomial is either in I or in $\operatorname{Ann} I$, so this gives a contradiction. It follows that $\operatorname{Ann} I = \operatorname{span} \{x^\alpha \mid x^\alpha \notin I\}$. □

This typically makes it easy to find the inverse system of a monomial ideal, or at least not more difficult than determining the members of the ideal itself. To extend the applicability of this theorem we would like to also be able to reduce other cases to the monomial one, e.g. by using a linear change of variables of the kind we've been employing in the proofs of the last section to reduce to the axis aligned case. This is possible for $\triangleright_{\mathrm{d}}$, but not for $\triangleright_{\mathrm{c}}$, as we will see later. Proving it for $\triangleright_{\mathrm{d}}$ will require three lemmas, the first of which also holds for $\triangleright_{\mathrm{c}}$. Some further notation will also be useful. Let L be an $(r+1) \times (r+1)$ matrix

$$L = \begin{bmatrix} L_{00} & \cdots & L_{0r} \\ \vdots & \ddots & \vdots \\ L_{r0} & \cdots & L_{rr} \end{bmatrix}$$

where $L_{ij} \in \mathbb{F}$ and write L_k for row k of L. We now extend the usage of multiindex exponents from variables to linear functions of variables as follows:

$$\begin{aligned} (L(x))^\alpha &= (L_0 x)^{\alpha_0} \cdots (L_r x)^{\alpha_r} \\ &= (L_{00}x_0 + \cdots + L_{0r}x_r)^{\alpha_0} \cdots (L_{r0}x_0 + \cdots + L_{rr}x_r)^{\alpha_r} \end{aligned}$$

Thus, each row L_k of L gives the coefficients for the linear function that is to be raised to the power α_k. Let L^C be the contragredient matrix of L, i.e. $L^C = (L^{-1})^T$. Then we have:

Lemma 3.6.2. *If L is invertible then $L^C_j x \triangleright L_i x = \delta_{ij}$ where δ_{ij} is the Kronecker delta, i.e. $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.*

Proof. For both $\triangleright_{\mathrm{d}}$ and $\triangleright_{\mathrm{c}}$ we have that

$$\begin{aligned} L^C_j x \triangleright L_i x &= (L^C_{j\,0} x_0 + \dots + L^C_{j\,r} x_r) \triangleright (L_{i0} x_0 + \dots + L_{ir} x_r) \\ &= L^C_{j\,0} L_{i0} + \dots + L^C_{j\,r} L_{ir} \\ &= L^C_j \cdot L_i. \end{aligned}$$

But the last expression must be equal to the Kronecker delta due to fact that the columns of a matrix inverse are orthonormal to the rows of the matrix it is an inverse of, so the rows of L^C are orthonormal to the rows of L. □

We will also need a lemma to explain what happens when we apply a power of a linear function to a power of a linear function using $\triangleright_\mathrm{d}$.

Lemma 3.6.3.

$$(L^C_j x)^m \triangleright_\mathrm{d} (L_i x)^n = \begin{cases} \frac{n!}{(n-m)!}(L_i x)^{n-m} & \textit{if } n \geqslant m \textit{ and } i = j \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Begin by writing out both $(L^C_j x)^m$ and $(L_i x)^n$ as products:

$$(L^C_j x)^{m-1}(L^C_j x) \triangleright_\mathrm{d} (L_i x)(L_i x)^{n-1}$$

We begin with the case $i = j$ and $m > n$. Because $h_1 h_2 \triangleright_\mathrm{d} f = h_2 \triangleright_\mathrm{d} (h_1 \triangleright_\mathrm{d} f)$ we can apply the m left-hand factors one by one. To find the result of such an application, apply the standard product differentiation rule repeatedly:

$$\begin{aligned} & L^C_j x \triangleright_\mathrm{d} (L_i x)(L_i x)^{n-1} \\ &\quad = (L^C_j x \triangleright_\mathrm{d} L_i x)(L_i x)^{n-1} + (L_i x)\left(L^C_j x \triangleright_\mathrm{d} (L_i x)^{n-1}\right) \\ &\quad = (1)(L_i x)^{n-1} + (L_i x)\left(L^C_j x \triangleright_\mathrm{d} \left((L_i x)(L_i x)^{n-2}\right)\right) \\ &\quad = (L_i x)^{n-1} + (L_i x)\left(\left((L^C_j x \triangleright_\mathrm{d} L_i x)(L_i x)^{n-2}\right) + (L_i x)\left(L^C_j x \triangleright_\mathrm{d} (L_i x)^{n-2}\right)\right) \\ &\quad = (L_i x)^{n-1} + (L_i x)\left((1)(L_i x)^{n-2} + (L_i x)\left(L^C_j x \triangleright_\mathrm{d} (L_i x)^{n-2}\right)\right) \\ &\quad = (L_i x)^{n-1} + (L_i x)^{n-1} + (L_i x)^2\left(L^C_j x \triangleright_\mathrm{d} (L_i x)^{n-2}\right) \\ &\quad = 2(L_i x)^{n-1} + (L_i x)^2\left(L^C_j x \triangleright_\mathrm{d} \left((L_i x)(L_i x)^{n-3}\right)\right) \\ &\quad = \dots \\ &\quad = n(L_i x)^{n-1} \end{aligned}$$

where the third line follows because of the $i = j$ case of Lemma 3.6.2. Through applying this procedure for each factor of $(L_j^C x)^m$ it follows that if $m \leqslant n$, then $(L_j^C x)^m \triangleright_{\mathrm{d}} (L_i x)^n = \frac{n!}{(n-m)!}(L_i x)^{n-m}$.

If $i \neq j$, the result of applying $L_j^C x \triangleright_{\mathrm{d}} L_i x$ in the third line in the above derivation will be 0, also by Lemma 3.6.2, and the whole sum will eventually become 0. If $n > m$, some factor of $(L_j^C x)^m$ will still be left when the right-hand side reduces to a constant, which means that applying that factor will give 0. □

The next step is to extend this argument to multiindex powers.

Lemma 3.6.4. *If L is invertible, then*

$$\left(L^C(x)\right)^\beta \triangleright_{\mathrm{d}} (L(x))^\alpha = \begin{cases} \binom{\alpha}{\alpha-\beta} (L(x))^{\alpha-\beta} & \textit{if } \alpha \geqslant \beta \\ 0 & \textit{otherwise} \end{cases}$$

Proof. As in the proof of the previous lemma, write out both sides of the consequent as products:

$$\left(L^C(x)\right)^\beta \triangleright_{\mathrm{d}} (L(x))^\alpha = (L_0^C x)^{\beta_0} \cdots (L_r^C x)^{\beta_r} \triangleright_{\mathrm{d}} (L_0 x)^{\alpha_0} \cdots (L_r x)^{\alpha_r}.$$

Again, we can apply the left-hand factors one by one, which gives us

$$\begin{aligned}
&(L_k^C x)^{\beta_k} \triangleright_{\mathrm{d}} (L_0 x)^{\alpha_0} \cdots (L_r x)^{\alpha_r} \\
&\quad= \left((L_k^C x)^{\beta_k} \triangleright_{\mathrm{d}} (L_0 x)^{\alpha_0}\right) (L_1 x)^{\alpha_1} \cdots (L_r x)^{\alpha_r} \\
&\qquad+ (L_0 x)^{\alpha_0} \left((L_k^C x)^{\beta_k} \triangleright_{\mathrm{d}} (L_1 x)^{\alpha_1}\right) (L_2 x)^{\alpha_2} \cdots (L_r x)^{\alpha_r} \\
&\qquad+ \cdots \\
&\qquad+ (L_0 x)^{\alpha_0} \cdots (L_{r-2} x)^{\alpha_{r-2}} \left((L_k^C x)^{\beta_k} \triangleright_{\mathrm{d}} (L_{r-1} x)^{\alpha_{r-1}}\right) (L_r x)^{\alpha_r} \\
&\qquad+ (L_0 x)^{\alpha_0} \cdots (L_{r-1} x)^{\alpha_{r-1}} \left((L_k^C x)^{\beta_k} \triangleright_{\mathrm{d}} (L_r x)^{\alpha_r}\right) \\
&\quad= (L_0 x)^{\alpha_0} \cdots (L_{k-1} x)^{\alpha_{k-1}} \left((L_k^C x)^{\beta_k} \triangleright_{\mathrm{d}} (L_k x)^{\alpha_k}\right) (L_{k+1} x)^{\alpha_{k+1}} \cdots (L_r x)^{\alpha_r} \\
&\quad= \frac{\alpha_k!}{(\alpha_k - \beta_k)!} (L_0 x)^{\alpha_0} \cdots (L_{k-1} x)^{\alpha_{k-1}} (L_k x)^{\alpha_k - \beta_k} (L_{k+1} x)^{\alpha_{k+1}} \cdots (L_r x)^{\alpha_r}
\end{aligned}$$

if $\beta_k \leqslant \alpha_k$ and zero otherwise, where the next-to-last equality follows because of Lemma 3.6.2 and the last one because of Lemma 3.6.3.

Doing this for each of $(L_0^C x)^{\beta_0}, \ldots, (L_r^C x)^{\beta_r}$ gives us that

$$\left(L^C(x)\right)^\beta \triangleright_{\mathrm{d}} (L(x))^\alpha = \frac{\alpha_0!}{(\alpha_0-\beta_0)!}(L_0x)^{\alpha_0-\beta_0}\cdots\frac{\alpha_r!}{(\alpha_r-\beta_r)!}(L_rx)^{\alpha_r-\beta_r}$$
$$= \binom{\alpha}{\alpha-\beta}(L(x))^{\alpha-\beta}$$

as long as $\beta \leqslant \alpha$. □

We are now ready to prove our theorem: to find the inverse system of an ideal after a linear change of variables, we can find the inverse system of the untransformed ideal and then apply the contragredient change to the inverse system obtained. $X \circ L$, for any $X \subseteq R$, as before, is defined to be the result of composing each element of X on the right with L, or to connect with the notation of the last lemma, $X \circ L = \{f(L(x)) \mid f(x) \in X\}$.

Theorem 3.6.5. *Let I be a homogenous ideal of $R = \mathbb{F}[x_0, \ldots, x_r]$ and let and let $L : \mathbb{F}^{r+1} \to \mathbb{F}^{r+1}$ be a linear transformation. If L is invertible then*

$$\operatorname{DAnn}(I \circ L^C) = (\operatorname{DAnn} I) \circ L.$$

Proof. We first show that if $h \circ L^C \triangleright_{\mathrm{d}} f \circ L = 0$ for all $h \in I$ then $h \triangleright_{\mathrm{d}} f = 0$ for all $h \in I$, and vice versa. By Theorem 3.3.1, I being an ideal means that it suffices if we consider h such that $\deg h = \deg f = i$. Let $a = \kappa(f)$ and $b = \kappa(g)$. Writing out the application of $\triangleright_{\mathrm{d}}$ using its linearity and applying Lemma 3.6.4, we have

$$\begin{aligned} h \circ L^C \triangleright_{\mathrm{d}} f \circ L &= \sum_{|\alpha|=i}\sum_{|\beta|=i} a_\alpha b_\beta \ \left((L^Cx)^\beta \triangleright_{\mathrm{d}} (Lx)^\alpha\right) \\ &= \sum_{|\alpha|=i}\sum_{\substack{|\beta|=i \\ \beta\leqslant\alpha}} \binom{\alpha}{\alpha-\beta} a_\alpha b_\beta \ (Lx)^{\alpha-\beta} \\ &= \sum_{|\gamma|=i} = \gamma! a_\gamma b_\gamma \\ &= h \triangleright_{\mathrm{d}} f \end{aligned}$$

where the third line follows because $|\alpha| = |\beta|$. The theorem then follows by a series of equivalences:

$$
\begin{aligned}
f \in \operatorname{DAnn} I &\Leftrightarrow h \rhd_{\mathrm{d}} f = 0 \text{ for all } h \in I \\
&\Leftrightarrow (h \circ L^C) \rhd_{\mathrm{d}} (f \circ L) = 0 \text{ for all } h \in I \\
&\Leftrightarrow h' \rhd_{\mathrm{d}} (f \circ L) = 0 \text{ for all } h' \in (I \circ L^C) \\
&\Leftrightarrow f \circ L \in \operatorname{DAnn}(I \circ L^C).
\end{aligned}
$$

□

Theorem 3.6.5 expands the number of ideals we can calculate the differential inverse system of as if they were monomial. Any set of linearly independent linear functions $l_0 \dots, l_r$ span R_1, and the i-ary products of these functions span R_i. We can write such a product as l^β, where $l = (l_0, \dots, l_r)$ and $|\beta| = i$. Since $l_0, \dots, l_r$ are linearly independent, we can set up an invertible linear transformation $L : \mathbb{F}^{r+1} \to \mathbb{F}^{r+1}$ such that $l_k \circ L = x_k$, and more generally $l^\beta \circ L = x^\beta$. Theorem 3.6.5 then lets us treat ideals generated by such products as if they were "monomials in linear functions".

So far, we have only shown the theorem for differential inverse systems, and if we try to prove it for $\rhd_{\mathrm{c}}$ we quickly run into difficulties. The main problem appears in proving lemmas 3.6.3 and 3.6.4 since we used the product rule there, and the product rule does nor apply to $\rhd_{\mathrm{c}}$—if it did, we would get different results from $x_0 \rhd_{\mathrm{c}} x_0^2$ and $x_0 \rhd_{\mathrm{c}} x_0 x_0$. In fact, the kind of simple relationship that holds between I and $\operatorname{DAnn} I$ when we apply a linear change of variables L is not possible for $\rhd_{\mathrm{c}}$, even when L is not only invertible but even assumed to be orthonormal:

Theorem 3.6.6. *Let $R = \mathbb{F}[x_0, \dots, x_r]$ with $r > 0$ and let $\mathcal{L}_{\mathbb{F}^{r+1}}$ be the set of linear functions $\mathbb{F}^{r+1} \to \mathbb{F}^{r+1}$. Then there is is no function $\Phi : \mathcal{L}_{\mathbb{F}^{r+1}} \to \mathcal{L}_{\mathbb{F}^{r+1}}$ such that* $\operatorname{CAnn}(I \circ \Phi(L)) = (\operatorname{CAnn} I) \circ L$ *for all orthonormal $L \in \mathcal{L}_{\mathbb{F}^{r+1}}$ and all homogenous ideals $I \subseteq R$.*

Proof. We first show that Φ has to satisfy $\Phi(L) = DL^C$, where D is a diagonal matrix. Let $L \in \mathcal{L}_{\mathbb{F}^{r+1}}$ and let $L' = \Phi(L)$. Consider the first-degree case: it is clear that we here must have that $x_j \rhd_{\mathrm{c}} x_i = 0$ iff $x_j \circ L' \rhd_{\mathrm{c}} x_i \circ L = 0$, which means that $x_j \circ L' \rhd_{\mathrm{c}} x_i \circ L$ must be zero iff $i \neq j$. But $x_j \circ L' = L'_j \circ x$, where L'_j is the jth row of L', so this says that the rows of L' must be orthogonal to the rows of L, except when $i = j$. This entails that each row j of L' is a constant λ_j times row j of L^C. Writing this relationship as a matrix product gives us that $L' = DL^C$, where D is diagonal with $\lambda_0, \dots, \lambda_r$ as entries.

Now let $r = 2$, let $I = \langle x_1^2 \rangle$, and let L and $L' = \Phi(L)$ be the matrices

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad L' = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix} \cdot L^C = \sqrt{2} \begin{bmatrix} \lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{bmatrix}$$

where neither λ_0 nor λ_1 are zero. Assume for contradiction that $\mathrm{CAnn}(I \circ \Phi(L)) = (\mathrm{CAnn}\, I) \circ L$, which is equivalent to the following condition:

$$h \triangleright_c f = 0 \text{ for all } h \in I \text{ iff } (h \circ \Phi(L)) \triangleright_c (f \circ L) = 0 \text{ for all } h \in I.$$

Considering $x_0^2 \in \mathrm{CAnn}\langle x_1^2 \rangle$, we then have

$$\begin{aligned} x_1^2 \triangleright_c x_0^2 = 0 \Leftrightarrow \\ \sqrt{2}\lambda_1 (x_0 - x1)^2 \triangleright_c \frac{1}{\sqrt{2}}(x_0 + x1)^2 = 0 \Leftrightarrow \\ \lambda_1(x_0^2 - x_0 x1 + x_1^2) \triangleright_c x_0^2 + x_0 x1 + x_1^2 = 0 \Leftrightarrow \\ \lambda_1(1 + 4 - 1) = 0 \Leftrightarrow \\ 4\lambda_1 = 0. \end{aligned}$$

But the last line cannot be true since we assumed λ_1 to not be zero. Thus this gives a counterexample to the existence of a function Φ such as described in the theorem. □

Intuitively, this can perhaps be seen as one way in which CAnn is less "geometric" than DAnn: it does not behave well even under something as simple as a rotation, so it is more tightly tied to a specific coordinate system. Of course, we can always use Corollary 3.3.4 to translate an inverse system from differential to contractive, and in that way, we can also find what happens to a contractive inverse system when we subject an ideal to a linear transformation. The function $\phi_{d \to c}$ used in that theorem is not as simple as a linear change of variables, however, and the last theorem shows that it cannot be.

3.7 Historical note: Macaulay's inverse systems

The inverse system is one of the things that Macaulay claims is a new creation of his in [Mac94]. Among the things he uses it for is to prove theorems related to what are now called Gorenstein rings [Mac94, p. xxvi] through his concept of "principal systems", by which he means ideals whose inverse systems are generated by a single element.

For Macaulay, the elements of an inverse system are negative power series $s = \sum_\alpha (c_\alpha x^{-\alpha})$, and s is inverse to h iff degree 0 of sh is zero [Mac94, p. 64]. While using negative power series introduces some complexities, it also has some nice side effects:

- As mentioned, the "combination" of a polynomial with its inverse simply consists in multiplying them and then throwing away the nonconstant terms.
- An inverse system, as a set of inverse functions, comes out as closed under linear combinations and products with elements of R, so it has the same closure conditions as an ideal [Mac94, p. 69].
- Allowing infinite series lets Macaulay prove that every inverse system has a "finite" basis, by which he means that it is generated by finitely many series through addition and multiplication with elements of R [Mac94, p. 91].

At first, it may seem like these differences make his concept very different from any of ours—even the contractive inverse system, which Emsalem and Iarrobino say "is Macaulay's inverse system" [EI95, p. 1083]. In particular, the fact that Macaulay only considers the constant terms of the product when deciding whether s is inverse to h appears to make his definition weaker than ours, since we require that *all* terms are zero, not only the constant ones. However, because of Theorem 3.3.1, there is a sense in which Macaulay's concept can be used to describe our contractive inverse systems too:

Theorem 3.7.1. *Let R^{-1} be the power series ring $\mathbb{F}[[x_0^{-1}, \dots, x_r^{-1}]]$, and let $\mu : R \to R^{-1}$ take each element $f \in R$ to*

$$\mu(f) = \sum_\alpha \kappa(f)_\alpha x^{-\alpha}.$$

where α ranges over all multiindices of length r. Then $f \in \operatorname{CAnn} I$ iff the degree 0 term of $h\mu(f)$ is 0 for all $h \in I$.

Proof. Let $[h\mu(f)]_i$, where $i \in \mathbb{Z}$, be the sum of the degree i terms of $h\mu(f)$. It is easily seen that $[h\mu(f)]_0 = h \rhd_c f$ when $\deg f = \deg h$, since the right–hand side as a whole is degree 0 then. Now assume that $h \rhd_c f = 0$. Then $[h\mu(f)]_i = 0$ for all $i \geqslant 0$, so $[h\mu(f)]_0 = 0$. Conversely, assume that $[h\mu(f)]_0 = 0$ for all $h \in I$. Then this holds for all $h \in I_{\deg f}$ as well, so $f \in \operatorname{CAnn} I$ by Corollary 3.3.3 and our observation that $[h\mu(f)]_0 = h \rhd_c f$ when $\deg f = \deg h$. □

While the function μ of the last theorem is clearly injective, it is far from being surjective, which means that there are many sets of elements of R^{-1} that are Macaulay-inverse to an ideal I which do not not correspond to any of ours. However, Macaulay's inverse systems are not defined as polarities, but in terms of a generator-like construction that builds on his concept of a "dialytic array". Although it would take us too far to go into how this works, it means that his inverse systems are not easily comparable with ours in a more general sense, even if we can express our conditions in his terms.

Macaulay mentions several of the results that we have proved here, and in some cases also discusses proofs of them. For example, he gives a proof that f is in the inverse system of $I_1 + \cdots + I_k$ iff it is in that of each I_i, but does not mention the dual (that the inverse system of $I_1 \cap \cdots \cap I_k$ is the vector sum of the inverse systems of $I_1, \ldots, I_k$) [Mac94, p. 70].

He also gives a formula for how an inverse system transforms under a linear change of variables for the ideal, but his general solution does not guarantee that the transform needed is itself a linear change of variables (and indeed we have proved that it cannot be). And while he does say that if the change of variables does not need to move the origin it can be undone with a contragredient transformation, he does so for a different kind of inverse systems than the one he otherwise employs, which are more similar to the differential inverse systems we have been using here [Mac94, pp. 71–73]. In fact, Macaulay's solution for a linear change of variables without change of origin is equivalent to using our Theorem 3.6.5 and then applying the function $\phi_{d\to c}$ to translate the result to a contractive inverse system.

4 The inverse system of a finite set of fat points

4.1 Simple points

We now return to focusing on ideals of the form $I(\hat{p})$, where $\hat{p}$ is a projective point, which we studied in section 2. Our aim is to find out what their inverse systems are like. As usual, we start with the simplest case: the point $\hat{\mathsf{e}}_0 = \{\lambda \mathsf{e}_0 \,|\, \lambda \in \mathbb{F}\}$ aligned with the axis e_0. Then

$$I(\hat{\mathsf{e}}_0) = \langle l_{\mathsf{e}_1}, \ldots, l_{\mathsf{e}_r} \rangle = \langle x_1, \ldots, x_r \rangle$$

where we have used the notation $l_p = p_0 x_0 + \cdots + p_r x_r$ that we introduced in section 2.2. Beginning with $\triangleright_{\mathrm{d}}$ and its corresponding inverse system DAnn, the generators of $I(\hat{\mathsf{e}}_0)$ become the regular first-order partial differentiation operators $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}$. It follows that the inverse system $\operatorname{DAnn} I(\hat{\mathsf{e}}_0)$ contains the functions f such that $\frac{\partial}{\partial x_k} f = 0$ for $1 \leqslant k \leqslant r$. In R_1, these are just the multiples of x_0. Likewise, in R_2, every monomial that has one of $x_1, \ldots, x_r$ as a factor is excluded from $\operatorname{DAnn}_2 I(\hat{\mathsf{e}}_0)$. This applies in every degree, so

$$\operatorname{DAnn} I(\hat{\mathsf{e}}_0) = \{\lambda x_0^n \,|\, \lambda \in \mathbb{F} \text{ and } n \in \mathbb{N}\}$$

as the powers of x_0 are the only monomials that do not contain any factors of $x_1, \ldots, x_r$. Since $\triangleright_{\mathrm{c}}$ is identical to $\triangleright_{\mathrm{d}}$ on R_1, this is also the inverse system $\operatorname{CAnn} I(\hat{\mathsf{e}}_0)$.

For projective points in general, we have by Theorem 2.2.2 that

$$I(\hat{p}) = \langle l_{q_1}, \ldots, l_{q_r} \rangle$$

where $q_1, \ldots, q_r$ are orthogonal to each other and to p. Rather than partial differentiation along the coordinate axes, the generators here work like unnormalized directional derivatives when used with $\triangleright_{\mathrm{d}}$:

$$l_{q_k} \triangleright_{\mathrm{d}} f = \nabla_{q_k} f.$$

The differential inverse system then consists precisely of the functions whose derivatives in the directions of $q_1, \ldots, q_r$ are all zero. The following lemma will help us when trying to reduce the problem of determining the inverse system of ideals like this to the axis aligned case:

Lemma 4.1.1. *If $L : \mathbb{F}^{r+1} \to \mathbb{F}^{r+1}$ is an invertible linear transformation then*

$$\mathrm{DAnn}\, I(\widehat{Lp}_1, \ldots, \widehat{Lp}_k)^n = \mathrm{DAnn}\, I(\hat{p}_1, \ldots, \hat{p}_k)^n \circ L^T$$

where $\widehat{Lp} = \{\lambda Lp \mid \lambda \in \mathbb{F}\}$.

Proof. Let $I(A)$, where A is an algebraic subset of $\mathbf{P}^r(\mathbb{F})$, be the ideal of polynomials that are zero on all of A, and let $L(A)$ be defined as

$$L(A) = \{\lambda Lp \mid \lambda \in \mathbb{F} \text{ and } p \in \hat{p} \text{ for some } \hat{p} \in A\}$$

i.e. the set of projective points resulting from applying L to each vector p in any projective point of A. By a standard result we have that $I(L(A)) = I(A) \circ L^{-1}$ (see e.g. [CLO15, p. 437]). Since the set of points $\{\hat{p}_1, \ldots, \hat{p}_k\}$ is algebraic, this can be applied to our case to give that $I(\widehat{Lp}_1, \ldots, \widehat{Lp}_k) = I(\hat{p}_1, \ldots, \hat{p}_k) \circ L^{-1}$. Since we furthermore have that $(I \circ L)^n = I^n \circ L$ for any ideal I, we get $I(\widehat{Lp}_1, \ldots, \widehat{Lp}_k)^n = I(\hat{p}_1, \ldots, \hat{p}_k)^n \circ L^{-1}$. Applying Theorem 3.6.5 allows us to infer

$$\begin{aligned} \mathrm{DAnn}\, I(\widehat{Lp}_1, \ldots, \widehat{Lp}_k)^n &= I(\widehat{p}_1, \ldots, \widehat{p}_k)^n \circ (L^{-1})^C \\ &= I(\hat{p}_1, \ldots, \hat{p}_k)^n \circ L^T \end{aligned}$$

because $(L^{-1})^C = ((L^{-1})^{-1})^T = L^T$. □

This lemma simplifies the application of Theorem 3.6.5 for sets of simple and fat projective points. To begin with, it allows us to easily find the inverse system of the ideal of a single arbitrary projective point:

Theorem 4.1.2. *For any projective point $\hat{p}$,*

$$\mathrm{DAnn}\, I(\hat{p}) = \{\lambda (l_p)^n \mid \lambda \in \mathbb{F} \text{ and } n \in \mathbb{N}\}.$$

Proof. Let L be the $(r+1) \times (r+1)$ matrix

$$L = \begin{bmatrix} p_0 & p_1 & \cdots & p_r \\ q_{10} & q_{11} & \cdots & q_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ q_{r0} & q_{r1} & \cdots & q_{rr} \end{bmatrix}$$

where $q_1, \dots, q_r$ again are any vectors that are orthogonal to p and to each other. Because of the orthogonality, $L^{-1} = L^T$, and $L^C = (L^{-1})^T = L$. We have that $p = L^T \mathsf{e}_0$ and $q_j = L^T \mathsf{e}_j$ for $1 \leqslant j \leqslant r$, and also that $l_p = x_0 \circ L$ and $\langle l_{q_1}, \dots, l_{q_r} \rangle = \langle x_1, \dots x_r \rangle \circ L$. By Lemma 4.1.1 the differential inverse system comes out as

$$\begin{aligned} \operatorname{DAnn} I(\hat{p}) &= \operatorname{DAnn}(I(\widehat{L^T \mathsf{e}_0})) \\ &= (\operatorname{DAnn} I(\hat{\mathsf{e}}_0)) \circ L \\ &= \{\lambda x_0^n \mid \lambda \in \mathbb{F} \text{ and } n \in \mathbb{N}\} \circ L \\ &= \{\lambda (l_p)^n \mid \lambda \in \mathbb{F} \text{ and } n \in \mathbb{N}\} \end{aligned}$$

where the second line is the inverse system we got for $I(\hat{\mathsf{e}}_0)$ before, composed with L.

□

Since the ideal that vanishes on a finite set of points is the intersection of the ideals of the individual points, and Ann maps ideal intersections to inverse system vector sums, we have for a set $\hat{p}_1, \dots, \hat{p}_k$ of points that

$$\operatorname{DAnn} I(\hat{p}_1, \dots, \hat{p}_k) = \{\lambda_1 (l_{p_1})^n + \dots + \lambda_k (l_{p_k})^n \mid \lambda_1, \dots, \lambda_k \in \mathbb{F} \text{ and } n \in \mathbb{N}\}$$

Unfortunately there is so far no standardized and efficient way to present inverse systems like there is for ideals, which can be presented through their generators (and even more systematically through generators that make up Gröbner bases). Still, that each degree of an inverse system is a finite dimensional vector space means that a degree-by-degree presentation can often be useful. In the case of $\operatorname{Ann} I(\hat{p}_1, \dots, \hat{p}_k)$, this kind of presentation comes out as

$$\operatorname{DAnn}_i(I(\hat{p}_1, \dots, \hat{p}_k)) = l_{p_1}^i +_v \dots +_v l_{p_k}^i.$$

Translating this argument to CAnn rather than DAnn is made difficult by Theorem 3.6.5 not holding for CAnn. However, the relative simplicity of the inverse system for $\triangleright_{\mathrm{d}}$—that it is generated by powers of linear functions—means that we can still represent the inverse system for $\triangleright_{\mathrm{c}}$ concisely. For this, we have to consider a different kind of power for contractive inverse systems, introduced in [EI95, p.

1082]:[5]

$$l_p^{[n]} = \sum_{|\alpha|=n} p^\alpha x^\alpha.$$

This lets us translate inverse systems containing powers of linear functions from $\triangleright_\mathrm{d}$ to $\triangleright_\mathrm{c}$:

Lemma 4.1.3. *For any $h \in R$, $p \in \mathbb{F}^{r+1}$ and $n \in \mathbb{N}$,*

$$h \triangleright_\mathrm{d} l_p^n = 0 \text{ iff } h \triangleright_\mathrm{c} l_p^{[n]} = 0.$$

Proof. Writing out l_p^n using the multinomial theorem gives us

$$l_p^n = \sum_{|\alpha|=n} \frac{n!}{\alpha!} p^\alpha x^\alpha.$$

From this we can see that $l_p^{[n]} = \phi_{c\to d}((l_p)^n)/n!$, where $\phi_{c\to d}$ is the function translating contractive to differential inverse systems that we introduced at the end of subsection 3.3. The lemma follows from applying Corollary 3.3.4 to Lemma 3.6.3, since the constant factor $n!$ does not affect if the right-hand side is zero or not. □

Applying this lemma to the result we got for $\operatorname{DAnn} I(\hat{p}_1, \dots, \hat{p}_k)$ gives us

$$\operatorname{CAnn}_i(I(\hat{p}_1, \dots, \hat{p}_k)) = l_{p_1}^{[i]} +_v \dots +_v l_{p_k}^{[i]}.$$

4.2 Fat points

With the results we have gathered so far, determining the inverse systems for ideals of fat points—point ideals raised to a power—causes no special difficulties. Beginning again with the axis-aligned case $\hat{\mathsf{e}}_0 = \{\lambda \mathsf{e}_0 \,|\, \lambda \in \mathbb{F}\}$, we have

$$I(\hat{\mathsf{e}}_0)^n = \langle x_1, \dots, x_r \rangle^n$$

which means that $I(\hat{\mathsf{e}}_0)^n$ is generated by monomials of degree n, and in particular those monomials that do not contain x_0. Since the monomials generated are those of the form $x^\beta x^\gamma$ where $x^\beta \in I(\hat{\mathsf{e}}_0)^n_n$ and γ is arbitrary, this comes out as

[5] Although it uses the same symbolism, this notation should not be confused with that of a "bracket power" as implemented in e.g. Macaulay2. The power presented here applies to linear functions, while bracket powers are powers of ideals that specifically involve their generators. See e.g. [MRSW18, pp. 64–69].

$$I(\hat{\mathsf{e}}_0)^n = \text{span}\,\{x^\beta \mid |\beta| \geqslant \beta_0 + n\}$$

so

$$\text{DAnn}(I(\hat{\mathsf{e}}_0)^n) = \text{span}\,\{x^\alpha \mid |\alpha| < \alpha_0 + n\}.$$

Written out degree by degree, this is equivalent to

$$\text{DAnn}_i(I(\hat{\mathsf{e}}_0)^n) = \begin{cases} R_i \text{ for } i < n \\ R_{n-1}x_0^{i-n+1} \text{ for } i \geqslant n. \end{cases}$$

Handling non-axis-aligned points again boils down to applying Theorem 3.6.5:

Theorem 4.2.1. *For any projective point $\hat{p}$, $i \in \mathbb{N}$ and $n \geqslant 1$, we have*

$$\text{DAnn}_i(I(\hat{p})^n) = \begin{cases} R_i \textit{ for } i < n \\ R_{n-1}{l_p}^{i-n+1} \textit{ for } i \geqslant n \end{cases}$$

Proof. Going through the same steps as in the proof of Theorem 4.1.2 (and using the same linear transformation L) gives us

$$\text{DAnn}(I(\hat{p})^n) = \text{span}\,\{l_p^{\alpha_0} l_{q_1}^{\alpha_1} \cdots l_{q_r}^{\alpha_r} \mid |\alpha| < \alpha_0 + n\}$$

For degrees $i < n$, this entails that all α such that $|\alpha| = i$ can appear on the right-hand side, so in total we get a span of $\binom{r+i}{r}$ vectors—the same as the number of dimensions of R_i. Since these vectors are all linearly independent, they must span R_i, from which it follows that $\text{DAnn}_i\, I(\hat{p})^n = R_i$ for $i < n$.[6]

For $i \geqslant n$ the condition $|\alpha| = i < \alpha_0 + n$ entails that $\alpha_0 > i - n$, which is equivalent to $\sum_{k=1}^r \alpha_k < n$. This means that after degree $n-1$, $\alpha_1, \ldots, \alpha_r$ do not increase anymore, but only α_0. Another way of saying this is that $\text{DAnn}_i(I(\hat{p})^n) = R_{n-1}{l_p}^{i-n+1}$ when $i > n-1$. □

For several ideals of points raised to different powers—like $I(\hat{p}_1)^{n_1} \cap \cdots \cap I(\hat{p}_k)^{n_k}$—it is useful to adopt the convention that $R_{n-1}l_p^{i-n+1} = R_i$ when $i - n + 1 < 0$. Using this

[6] Of course, this can also be realized just by noting that $I(\hat{p})^n$ contains nothing below degree n, so its inverse system must be spanned by all monomials in those degrees.

and applying Theorem 3.4.4 we then get Theorem 1 of [EI95] as a simple corollary of Theorem 4.2.1 (with the aid of Lemma 4.1.3 of the previous section for CAnn) :

Corollary 4.2.2. *For any projective points $\hat{p}_1, \ldots, \hat{p}_k$, any strictly positive $n_1, \ldots, n_k$, and any $i \in \mathbb{N}$,*

$$\mathrm{DAnn}_i(I(\hat{p}_1)^{n_1} \cap \cdots \cap I(\hat{p}_k)^{n_k}) = R_{n_1-1} l_{p_1}^{i-n_1+1} +_v \cdots +_v R_{n_k-1} l_{p_k}^{i-n_k+1}$$
$$\mathrm{CAnn}_i(I(\hat{p}_1)^{n_1} \cap \cdots \cap I(\hat{p}_k)^{n_k}) = R_{n_1-1} l_{p_1}^{[i-n_1+1]} +_v \cdots +_v R_{n_k-1} l_{p_k}^{[i-n_k+1]}$$

4.3 A second proof

The way we have proved Corollary 4.2.2 largely follows the ideas presented in the first proof of Theorem 1 of [EI95] insofar as it rests on the procedure of translating the problem to the axis-aligned case and proving it there. Emsalem and Iarrobino give in total 3 different proofs (or proof outlines) for the theorem, however. This section describes a proof based on second one, since this goes via a different route than the first (as does the third, which we will not relate here since it does not introduce any major new concepts).

We start by giving a lemma that can be seen as a strengthening of Lemma 3.6.3 in that it tells us how to apply $h \triangleright$ to powers of linear functions, but does not assume that h itself is a power of a linear function but only that it is homogenous. It appears as the lemma before Theorem 1 in [EI95] together with its corollary, which we also present here.

Lemma 4.3.1. *Let $h \in R_i$, and $n \geqslant i$. Then*

$$h \triangleright_{\mathrm{d}} l_p^n = \frac{n!}{(n-i)!} l_p^{n-i} h(p)$$
$$h \triangleright_{\mathrm{c}} l_p^{[n]} = l_p^{[n-i]} h(p).$$

Proof. We begin with $\triangleright_{\mathrm{d}}$. Let $b = \kappa(h)$ and apply $h \triangleright_{\mathrm{d}}$ termwise as a differential operator:

$$h \triangleright_{\mathrm{d}} l_p^n = \sum_{|\beta|=i} b_\beta \frac{\partial}{\partial x^\beta} l_p^n.$$

Now expand l_p^n in each term of the sum on the right hand side using the multinomial theorem:

$$b_\beta \frac{\partial}{\partial x^\beta}(l_p)^n = b_\beta \frac{\partial}{\partial x^\beta} \sum_{|\alpha|=n} \frac{n!}{\alpha!} p^\alpha x^\alpha = b_\beta \sum_{|\alpha|=n} \frac{n!}{\alpha!} \frac{\partial}{\partial x^\beta} p^\alpha x^\alpha.$$

Applying the differentiation and rearranging gives

$$\begin{aligned}
b_\beta \sum_{|\alpha|=n} \frac{n!}{\alpha!} \frac{\partial}{\partial x^\beta} p^\alpha x^\alpha &= b_\beta \sum_{\substack{|\alpha|=n \\ \alpha\geq\beta}} \frac{n!}{\alpha!} \frac{\alpha!}{(\alpha-\beta)!} p^\alpha x^{\alpha-\beta} \\
&= b_\beta p^\beta \sum_{\substack{|\alpha|=n \\ \alpha\geq\beta}} \frac{n!}{(\alpha-\beta)!} p^{\alpha-\beta} x^{\alpha-\beta} \\
&= \frac{n!}{(n-i)!} b_\beta p^\beta \sum_{\substack{|\alpha|=n \\ \alpha\geq\beta}} \frac{(n-i)!}{n!} \frac{n!}{(\alpha-\beta)!} p^{\alpha-\beta} x^{\alpha-\beta} \\
&= \frac{n!}{(n-i)!} b_\beta p^\beta \sum_{\substack{|\alpha|=n \\ \alpha\geq\beta}} \frac{(n-i)!}{(\alpha-\beta)!} p^{\alpha-\beta} x^{\alpha-\beta}.
\end{aligned}$$

The sum in the last expression is however just a reindexing of

$$\sum_{|\gamma|=n-i} \frac{(n-i)!}{\gamma!} p^\gamma x^\gamma$$

where each γ appears in one sum for each $\alpha - \beta$ in the other, and vice versa. This expression, in turn, is equal to l_p^{n-i} by the multinomial theorem, so the last right hand side simplifies to

$$\frac{n!}{(n-i)!} b_\beta p^\beta (l_p)^{n-i}$$

which we can plug into the first equation to get

$$\begin{aligned}
h \triangleright_{\mathrm{d}} l_p^n &= \sum_{|\beta|=i} \frac{n!}{(n-i)!} b_\beta p^\beta l_p^{n-i} \\
&= \frac{n!}{(n-i)!} l_p^{n-i} h(p).
\end{aligned}$$

The version for $\triangleright_{\mathrm{c}}$ can be shown by using the definition of $l_p^{[n]}$ rather than the multinomial theorem. The derivation is similar to the one for $\triangleright_{\mathrm{d}}$, but slightly simpler.

□

Corollary 4.3.2. *If* $\deg h \leqslant n$ *then* $h \triangleright_{\mathrm{d}} l_p^n = 0$ *iff* $h(p) = 0$.

This corollary lets us construct a quick alternate proof of Theorem 4.1.2. $I(\hat{p})$, by definition, consists precisely of those h for which $h(p) = 0$. The corollary thus entails that $h \in I(\hat{p})_i$ iff $h \triangleright_{\mathrm{d}} l_p^i = 0$, which is equivalent to what Theorem 4.1.2 says when presented degree-by-degree.

To apply this to fat points one can proceed by expanding the ring we are working in. Let $R[p] = R[p_0, \dots, p_r]$. This means that we treat the coefficients of l_p as variables as well, which makes l_p a bilinear function of two $r+1$-dimensional vectors rather than a linear functional of one such vector. It also allows us to apply partial differentiation operators $\partial/\partial p_i$ for $0 \leqslant i \leqslant r$, and more generally higher partial differentiation

$$\frac{\partial}{\partial p^\beta} = \frac{\partial}{\partial p_0^{\beta_0} \cdots \partial p_r^{\beta_r}}$$

just as for the variables $x_0, \dots, x_r$. Each such operator corresponds to an element $p^\beta \in R[p]$ through the correspondence

$$p^\beta \triangleright_{\mathrm{d}} f = \frac{\partial}{\partial p^\beta} f.$$

Lemma 4.3.3. *In* $R[p]$,

$$p^\beta \triangleright_{\mathrm{d}} l_p^n = \frac{n!}{(n - |\beta|)!} x^\beta l_p^{n-|\beta|}$$

for $|\beta| \leqslant n$.

Proof. This can be shown by writing out $\partial/\partial p^\alpha$ as a product of first-order differential operators and then applying each of the resulting $\partial/\partial p_k$ operators one by one, using either the chain rule or the product rule. □

Theorem 4.3.4. *In* $R[p]$, $(\mathrm{DAnn}_i\, I(\hat{p})^n) \cap R = \mathrm{span}\,(R_{n-1} l_p^{i-n+1})$.

Proof. Assume that $\deg h = i$. Apply $p^\beta \triangleright_{\mathrm{d}}$, where $|\beta| < n$, to both sides of the equality in Lemma 4.3.1 to get

$$p^\beta \rhd_{\mathrm{d}} (h \rhd_{\mathrm{d}} l_p^n) = p^\beta \rhd_{\mathrm{d}} \frac{n!}{(n-i)!}(l_p^{n-i}\, h(p)) \Leftrightarrow$$
$$h \rhd_{\mathrm{d}} (p^\beta \rhd_{\mathrm{d}} l_p^n) = \frac{n!}{(n-i)!}\sum_{\gamma \leqslant \beta}\binom{\beta}{\gamma}(\frac{\partial}{\partial p^\gamma} l_p^{n-i})(\frac{\partial}{\partial p^{\beta-\gamma}} h(p)) \Leftrightarrow$$
$$h \rhd_{\mathrm{d}} (x^\beta l_p^{n-|\beta|}) = \frac{(n-|\beta|)!}{(n-i)!}\sum_{\gamma \leqslant \beta}\binom{\beta}{\gamma}(\frac{\partial}{\partial p^\gamma} l_p^{n-i})(\frac{\partial}{\partial p^{\beta-\gamma}} h(p))$$

where the second line follows from the general Leibniz rule, and the third from the lemma we just proved. Now, all terms of the sum on the right hand side must be zero, because the last factor $\frac{\partial}{\partial p^{\beta-\gamma}} h(p)$ is equal to $\frac{\partial}{\partial x^{\beta-\gamma}} h(x)$ when $x = p$, and all partial derivatives of h of degree below n are zero at p, according to Theorem 2.3.1. Since β was arbitrary we have that $h \rhd_{\mathrm{d}} (x^\beta l^{n-|\beta|}) = 0$ for all β such that $|\beta| < n$. But the functions x^β span the degree $R_{|\beta|}$ of R, so the functions $x^\beta l_p^{n-|\beta|}$ span $R_{|\beta|} l_p^{n-|\beta|}$. Let $|\beta| = i - 1$. It then follows that $h \rhd_{\mathrm{d}} f = 0$ for $f \in R_{n-1} l_p^{i-n+1}$, so

$$(\mathrm{DAnn}_i\, I(p)^n) \cap R = R_{n-1} l_p^{i-n+1}.$$

□

Since $R = R[p] \cap R$, the result also holds for R, and we get Theorem 4.2.1.

5 Some additional investigations

5.1 Changing the order of intersection and exponentiation

So far, we have studied ideals of sets of fat points – ones such as $I(\hat{p}_1)^n \cap \dots \cap I(\hat{p}_k)^n$. What if we instead apply the intersection first, and raise the resulting ideal to a power, as in the ideal $I(\hat{p}_1, \dots, \hat{p}_k)^n$? Such a procedure would amount to "fattening" not the individual points, but the set of points as a whole. There are some general things we can say about the possible result, such as that if we do it that way, we will get a bigger inverse system than if we apply exponentiation first and then intersection. This follows from a general lemma about ideals:

Lemma 5.1.1. *For any ideals $I_1, \dots, I_k$, $(I_1 \cap \dots \cap I_k)^n \subseteq I_1^n \cap \dots \cap I_k^n$.*

Proof. We naturally have that $I_1 \cap \dots \cap I_k \subseteq I_i$ for $1 \leqslant i \leqslant k$. Since raising to a power is a monotonic operation on the set of ideals ordered by inclusion, it follows that $(I_1 \cap \dots \cap I_k)^n \subseteq I_i^n$. But since this holds for all i, we must also have that $(I_1 \cap \dots \cap I_k)^n \subseteq I_1^n \cap \dots \cap I_k^n$. □

Corollary 5.1.2. *For any projective points $\hat{p}_1, \dots, \hat{p}_k$, any $n \geqslant 1$ and any $i \in \mathbb{N}$,*

$$\mathrm{DAnn}_i\, I(\hat{p}_1, \dots, \hat{p}_k)^n \supseteq R_{n-1}(l_{p_1}^{i-n+1} +_v \dots +_v l_{p_k}^{i-n+1}).$$

For certain specific classes of points there is more we can say. For example, requiring $p_1, \dots p_k$ to be linearly independent means that we can use Theorem 3.6.5 to answer the question, much as we could for systems of fat points.

Theorem 5.1.3. *For any projective points $\hat{p}_1, \dots, \hat{p}_k$, any $n \geqslant 1$ and any $i < n$, $\mathrm{DAnn}_i\, I(\hat{p}_1, \dots, \hat{p}_k)^n = R_i$. If $p_1, \dots, p_k$ are linearly independent, then for $i \geqslant n$ we have*

$$\mathrm{DAnn}_i\, I(\hat{p}_1, \dots, \hat{p}_k)^n = \mathrm{span}\left\{ R_{n-1} l_{p_1}^{m_1} \cdots l_{p_k}^{m_k} \mid \sum_{k'=1}^{k} m_{k'} = i - n + 1 \right\}.$$

Proof. As usual we begin with an axis-aligned case: $I(\hat{\mathsf{e}}_0, \dots, \hat{\mathsf{e}}_{k-1})$. An ideal $I(\hat{\mathsf{e}}_j)$ is generated by the set of all x_i where $i \neq j$. This means that $I(\hat{\mathsf{e}}_0, \dots, \hat{\mathsf{e}}_{k-1})^n$ is generated by n-ary products of $x_k, \dots, x_r$, which in turn entails that

$$I(\hat{\mathsf{e}}_0, \dots, \hat{\mathsf{e}}_{k-1})^n = \mathrm{span}\,\{x^\beta \mid \beta_k + \dots + \beta_r \geqslant n\}.$$

Since the inverse system of a monomial ideal is the linear span of the set-theoretic complement of the monomials in that ideal, we then have

$$\begin{aligned}\operatorname{Ann} I(\hat{\mathsf{e}}_0, \ldots, \hat{\mathsf{e}}_{k-1})^n &= \operatorname{span}\{x^\alpha \mid \alpha_k + \cdots + \alpha_r < n\} \\ &= \operatorname{span}\{x^\alpha \mid \alpha_0 + \cdots + \alpha_{k-1} > |\alpha| - n\}\end{aligned}$$

which presented degree-by-degree comes out as

$$\operatorname{Ann}_i I(\hat{\mathsf{e}}_0, \ldots, \hat{\mathsf{e}}_{k-1})^n = \begin{cases} R_i \text{ if } i < n \\ \operatorname{span}\left\{R_{n-1} x^{\alpha_0} \cdots x^{\alpha_{k-1}} \mid \sum_{k'=0}^{k-1} \alpha_{k'} = i - n + 1\right\} \text{ if } i \geqslant n. \end{cases}$$

To translate this to $I(\hat{p}_1, \ldots, \hat{p}_k)^n$, let $q_{k+1}, \ldots, q_{r+1}$ be $r+1-k$ points of $\mathbb{F}^{r+1}$ that are linearly independent with $p_1, \ldots, p_k$ and with each other, so $p_1, \ldots, p_k, q_{k+1}, \ldots q_{r+1}$ together span $\mathbb{F}^{r+1}$. Let L be the $(r+1) \times (r+1)$ matrix with $p_1, \ldots, p_k$, $q_{k+1}, \ldots q_{r+1}$ as rows. We then have that $x_j \circ L = l_{p_j}$ and $L^T \mathsf{e}_j = p_j$ for $j \leqslant k$, and $x_j \circ L = l_{q_j}$ and $L^T \mathsf{e}_j = q_j$ for $j > k$. Applying Lemma 4.1.1 to the expression for $\operatorname{Ann} I(\hat{\mathsf{e}}_0, \ldots, \hat{\mathsf{e}}_{k-1})^n$ that we just derived gives us

$$\begin{aligned}\operatorname{DAnn} I(\hat{p}_1, \ldots, \hat{p}_k)^n &= \operatorname{DAnn} I(\widehat{L^T \mathsf{e}}_0, \ldots, \widehat{L^T \mathsf{e}}_{k-1})^n \\ &= (\operatorname{DAnn} I(\hat{\mathsf{e}}_0, \ldots, \hat{\mathsf{e}}_{k-1})^n) \circ L \\ &= \operatorname{span}\{x^\alpha \mid \alpha_0 + \cdots + \alpha_{k-1} > |\alpha| - n\} \circ L \\ &= \operatorname{span}\{l_{p_1}^{\alpha_0} \cdots l_{p_k}^{\alpha_{k-1}} l_{q_{k+1}}^{\alpha_k} \cdots l_{q_{r+1}}^{\alpha_r} \mid \alpha_0 + \cdots + \alpha_{k-1} > |\alpha| - n\}.\end{aligned}$$

When $i = |\alpha| < n$, the linear functions on the right hand side span the whole of R_i. When $i \geqslant n$ we get

$$\operatorname{DAnn}_i I(\hat{p}_1, \ldots, \hat{p}_k)^n = \operatorname{span}\{R_{n-1} l_{p_1}^{\alpha_0} \cdots l_{p_k}^{\alpha_{k-1}} \mid \alpha_0 + \cdots + \alpha_{k-1} = i - n + 1\}.$$

□

5.2 More on monomial inverse systems

Theorem 3.6.1 established that the inverse system of a monomial ideal is the linear span of the monomials not in that ideal. Here we will take a further look at some consequences of this. In particular we will use the ease of calculation that monomial ideals provide to investigate questions about products in their inverse systems.

We have noted that $\operatorname{Ann} I$ is not generally closed under products: $x_0 \in \operatorname{Ann}\langle x_0^2 \rangle$, but $x_0 x_0 = x_0^2 \notin \operatorname{Ann}\langle x_0^2 \rangle$. For monomial ideals, the question of which inverse systems are closed under products can be given a definite answer:

Theorem 5.2.1. *If I is a non-empty monomial ideal, then $\operatorname{Ann} I$ is closed under products iff I is prime.*

Proof. First note that the non-empty prime monomial ideals are precisely the ones of the form $\langle x_{i_1}, \ldots, x_{i_k} \rangle$ where $0 \leqslant i_j \leqslant r$, i.e. ones generated by sets of single variables, such as $\langle x_0 \rangle$ and $\langle x_0, x_2, x_3 \rangle$. Assume that I is non-prime; then there is some monomial x^α such that $|\alpha| \neq 1$ in its minimal sets of generators. This set is unique since I is monomial [Eis04, p. 324][MRSW18, p. 19]. Let x^γ be such a generating monomial. Then $x^\gamma \triangleright x^\alpha = 0$ and $x^\gamma \triangleright x^\beta = 0$ for each non-zero α, β such that $\alpha + \beta = \gamma$, but $x^\gamma \triangleright x^\alpha x^\beta \neq 0$, so I is not closed under products. Conversely, if I is generated by single-variable first-degree monomials, then $\operatorname{Ann} I$ contains precisely the monomials that contain none of the variables in I. It is easy to see that this set is closed under products. □

It is also possible to modify the definition of product to guarantee closure under it, and as we shall see, this will lead to another connection with an important concept related to ideals. Let $I = \langle x^{\gamma_1}, \ldots, x^{\gamma_k} \rangle$, and define

$$x^\alpha \star_I x^\beta = \begin{cases} 0 \text{ if } \alpha + \beta \geqslant \gamma \text{ for some } x^\gamma \in I \\ x^{\alpha+\beta} \text{ otherwise} \end{cases}$$

i.e $x^\alpha \star_I x^\beta$ equals $x^\alpha x^\beta$ except when $x^\alpha x^\beta \in I$, in which case it is zero. Extend to the whole of $\operatorname{Ann} I$ linearly by defining

$$f \star_I g = \sum_{\alpha \in \deg f} \sum_{\beta \in \deg g} \kappa(f)_\alpha \kappa(g)_\beta (x^\alpha \star_I x^\beta).$$

Lemma 5.2.2. *If I is monomial and $f, g \in \operatorname{Ann} I$, then $f \star_I g \in \operatorname{Ann} I$.*

Proof. Assume that $x^\gamma \triangleright f = 0$ and $x^\gamma \triangleright g = 0$; we want to show that $x^\gamma \triangleright f \star_I g = 0$. Assuming that $f = a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n}$ and $g = b_1 x^{\beta_1} + \cdots + b_m x^{\beta_m}$ we have that $x^\gamma \triangleright f = 0$ iff $\alpha_i \not\geqslant \gamma$ for $1 \leqslant i \leqslant n$, and likewise for g. This means that it is sufficient if we prove is that $x^{\alpha_i} \star_I x^{\beta_j} \in \operatorname{Ann} I$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. But if $\alpha_i + \beta_j \geqslant \gamma$ for some $x^\gamma \in I$, then $x^{\alpha_i} \star_I x^{\beta_j} = 0$ by definition, and is trivially in $\operatorname{Ann} I$. And if $\alpha_i + \beta_j \not\geqslant \gamma$ for all $x^\gamma \in I$, then $x^\gamma \triangleright (x^{\alpha_i} \star_I x^{\beta_j}) = x^\gamma \triangleright x^{\alpha_i + \beta_j} = 0$, so $x^{\alpha_i} \star_x{}^{\beta_j} \in \operatorname{Ann} I$. □

We call an inverse system $\operatorname{Ann} I$ with this product defined on it a *multiplicative inverse system.*

Theorem 5.2.3. *If I is a non-unit monomial ideal of R, then* $\operatorname{Ann} I$*, seen as a multiplicative inverse system, is a ring with + as addition, $\star_I$ as product,* 0 *as zero and* 1 *as multiplicative identity, and is isomorphic to the quotient ring R/I.*

Proof. Checking that $\operatorname{Ann} I$ is a ring with respect to the specified operators consists in verifying the axioms. 0 is in all inverse systems, and 1, together with all other constants, are in inverse systems for non-unit ideals. Since + is the same addition as for polynomials, its axioms are proved the same way. $\star_I$'s associativity follows from the associativity of + as used when adding multiindices in exponents. Distributivity is a consequence of the linear way we have defined $\star_I$ on non-monomials.

For the isomorphism claim, recall that R/I consists of equivalence classes of R under the equivalence relation $f \sim g$ iff $f - g \in I$. Let $[f]_I$ be the equivalence class that f belongs to. We first show that $[\cdot]_I$, restricted to $\operatorname{Ann} I$, is a ring homomorphism from $\operatorname{Ann} I$ to R/I. Since + is the same on $\operatorname{Ann} I$ as on R, this is trivial for addition. For $\star_I$, we want to show that $[f \star_I g]_I = [f]_I[g]_I = [fg]_I$. Since $[f+g]_I = [f]_I + [g]_I$, it is enough if we prove that $[ax^\alpha \star_I bx^\beta]_I = [ax^\alpha]_I[bx^\beta]_I$. If $\alpha + \beta \not\geqslant \gamma$ for all $x^\gamma \in I$, $\star_I$ is the same as the regular polynomial product, and the claim follows because $[fg]_I = [f]_I[g]_I$. If $\alpha + \beta \geqslant \gamma$ for some $x^\gamma \in I$, then $abx^\alpha x^\beta \in I$, and so is in $[0]$, just like $ax^\alpha \star_I bx^\beta$.

Proving that $[\cdot]_I$ is an isomorphism requires showing that it is also bijective. But every element of R/I is a linear combination of elements $[x^\alpha]_I$, where $x^\alpha \notin I$, which is the same as for $\operatorname{Ann} I$ by Theorem 3.6.1. So $x^\alpha \in \operatorname{Ann} I$ iff $[x^\alpha]_I \in R/I$, and furthermore $x^\alpha \neq x^\beta$ iff $[x^\alpha]_I \neq [x^\beta]_I$ whenever x^α or x^β are not in I. □

Corollary 5.2.4. *If the ideal $I(V)$ of functions vanishing on a non-empty projective variety V is monomial, then* $\operatorname{Ann} I(V)$*, as a multiplicative inverse system, is isomorphic to the coordinate ring of V.*

Since monomial ideals are equivalent to ones generated by linearly independent linear functions by Theorem 3.6.5, these results apply to such ideals as well. To what degree they can be also generalized to other kinds of non-monomial ideals is an open question.

6 Conclusion and further work

We have described inverse systems, focusing on a form of them quite close to that given in [EI95], but not exactly the same. In particular, that we are defining $\triangleright_{\mathrm{c}}$ and $\triangleright_{\mathrm{d}}$ as binary operators on the same ring rather than on two different rings lets us consider some properties of them, such as how repeated application works, in a natural way. We have also very much focused on the linear algebraic aspects of these operators: between homogenous polynomials of the same degree they are very similar to dot products (and even identical to them in some cases, such as for $\triangleright_{\mathrm{c}}$ on polynomial rings over $\mathbb{R}$).

One of the most striking consequences of this is that $h \triangleright f = f \triangleright h$ when $\deg f = \deg h$. This is not something that could hold for different degrees: in that case, at least one of $h \triangleright f$ or $f \triangleright h$ must then be zero, so unless both are zero they are never equal. Instead, between different degrees $\triangleright$ also sometimes works more like a kind of division. One example is given by monomials, where we have that

$$x^{\beta} \triangleright x^{\alpha} = c\frac{x^{\alpha}}{x^{\beta}}$$

for a constant c (equal to 1 for $\triangleright_{\mathrm{c}}$ and to $\binom{\alpha}{\alpha-\beta}$ for $\triangleright_{\mathrm{d}}$) if $\alpha \geqslant \beta$. One additional feature which has a critical effect on the inverse system, when seen from this perspective, is that $x^{\beta} \triangleright x^{\alpha}$ is defined to be zero if $\alpha \ngeqslant \beta$; if we instead had allowed negative exponents (so e.g $x_0^2 \triangleright_{\mathrm{c}} x_0 = x_0^{-1}$) we would have had much smaller inverse systems where the linear algebraic aspect would be the only determining factor.

As they have been defined here, inverse systems are however determined by both these aspects, and it is how they interact that causes the complexities sometimes involved in calculating them. When it comes to ideals generated by projective points, which were our primary area where we wanted to apply inverse systems in this text, we were lucky that many of these can be linearly transformed to monomial ideals, where one of the aspects (the linear algebraic one) is suppressed. It seems worthwhile to investigate if similar but more general transformations could be developed that allow one to treat even more ideals this way.

In distinction to this, calculating inverse systems degree by degree, as we did in the example of Section 3.5, relies wholly on the linear algebraic aspect since we then only need to consider polynomials of the same degree. The main difficulty here is to find meaningful generalizations across degrees: although we, in theory, can calculate the inverse system of I for any degree i for which we have a linear spanning set of polynomials, this generally tells us nothing about the overall shape of the inverse system as a whole, or even of other degrees. This is another area that would be interesting to explore further: what can we say about $\operatorname{Ann}_{i+1} I$ when we already know $\operatorname{Ann}_i I$? And what would be a good coordinate system to calculate and express $\operatorname{Ann}_{i+1} I$ in, given that we have a good one for $\operatorname{Ann}_i I$? We saw in Section 3.6 that any set of $r+1$ linearly independent linear functions gives us coordinate systems for all degrees of R that, if well chosen, can be used to simplify and structure the calculation of inverse systems. Are there ways to generalize this, maybe starting with coordinate systems for a higher degree than 1, and inductively making new coordinate systems for higher degrees?

Related to the question of coordinate systems is one we touched on briefly in Section 4.1: the presentation of inverse systems. While we have shown inverse systems to be in one-to-one correspondence with ideals, which means that we can always present them as the inverse systems of those ideals, this does not help us in the cases where the inverse systems in question are difficult to compute. And while Macaulay's approach to inverse systems lets him define them finitely using his method of dialytic arrays, these are hardly easy to work with, and would also require us to allow polynomial series in the inverse systems like Macaulay does. Instead, what would be especially useful to have is something similar to the concept of an ideal basis, or even a Gröbner basis, for inverse systems: something finitely presented that easily lets us determine if two inverse systems are equal, and also to algorithmically work out arbitrary degrees of them without having to reference the ideals they are inverse systems of. Since the polynomial ideals that determine the inverse systems are finitely generated and thus finitely presentable, this should hopefully not be impossible to achieve.

References

[Bir73] G. Birkhoff. *Lattice Theory.* American Mathematical Society Colloquium Publications, third edition, 1973.

[CLO15] D.A. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms.* Springer, fourth edition, 2015.

[EI95] J. Emsalem and A. Iarrobino. Inverse system of a symbolic power, I. *Journal of Algebra*, 174:1080–1090, 1995.

[Eis04] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry.* Springer, 2004.

[Joh21] N. Johnston. *Advanced Linear and Matrix Algebra.* Springer, 2021.

[Mac94] F. S. Macaulay. *The Algebraic Theory of Modular Systems.* Cambridge Mathematical Library, 1994.

[MB99] S. MacLane and G. Birkhoff. *Algebra.* AMS Chelsea Publishing, third edition, 1999.

[MRSW18] W. F. Moore, M. Rogers, and S. Sather-Wagstaff. *Monomial Ideals and Their Decompositions.* Springer, 2018.

[Rom08] S. Roman. *Advanced Linear Algebra.* Springer, third edition, 2008.