SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Kernels of maps between subsets of quotient rings

av

Ludvig Nyman

2024 - No K2

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Kernels of maps between subsets of quotient rings

Ludvig Nyman

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Samuel Lundqvist

2024

Abstract

This paper explores polynomial quotient rings with monomial ideals. Special interest is taken in the kernels for maps between sets of degree d and $d+1$ where d maximises the value of the Hilbert function. The two classes of ideals that are covered in detail are those on the form $\mathbf{I} = \langle x_1^d, x_2^d, x_3^d, x_1^{d/2} x_2^{d/2} \rangle$ for $d = 2 + 6n$ as well as $\mathbf{I} = \langle x_1^2, x_2^2, \ldots, x_n^2 \rangle$ with a new result regarding the vector basis of the kernel for the latter. Minor results regarding a formula for the Hilbert function and it's maximum value for the first class of ideals are also discussed and proven.

Abstrakt

I denna uppsats kommer vi att utforska polynomiska kvotringar med monomiska ideal. Av särskilt intresse är kärnorna för avbildingarna mellan delmängderna av grad d och grad $d+1$ där d maximerar värdet för Hilbert funktionen. De två klasser av ideal som täcks i synnerhet är de på formen $\mathbf{I} = \langle x_1^d, x_2^d, x_3^d, x_1^{d/2}x_2^{d/2} \rangle$ för $d = 2+6n$ samt $\mathbf{I} = \langle x_1^2, x_2^2, \ldots, x_n^2 \rangle$ med nya resultat angående vektorbasen till kärnan för den sistnämnda klassen. Andra resultat angående en formel för Hilbert funktionen och dess maxvärde för första klassens ideal diskuteras och bevisas också.

Acknowledgement

I would like to thank my supervisor Samuel Lundqvist for his constant support and for getting me interested in this subject in the first place. It has been a pleasure working alongside him as this is his current research topic. I would also like to thank my parents for their unconditional and overwhelming support throughout the years. Without them I could never have been able to finish my degree at all. And finally my closest friends for keeping my spirits up when I needed it most. Thank you all.

Contents

1 Introduction

In this paper we will be primarily searching for kernels of maps over a polynomial quotient ring with monomial ideals from one degree to the next. We will only be using the map induced by multiplication with $\ell = x_1 + x_2 + ... + x_n$, and we will go through definitions and give illustrative examples of kernels arising from this map as we dive deeper into this topic. The Hilbert function and the sequenc eof numbers it generates of monomial ideals are of central importance, and basic linear algebra will be used in the cases of multi-dimensional kernels. We will be taking a practical approach to the subject with multiple examples and not so much on any theoretical background.

1.1 Definitions and terminology

Definition 1.1. A **ring** is a set $\boldsymbol{R}$ equipped with 2 binary operations, addition and multiplication, satisfying the following 3 axioms

- $\boldsymbol{R}$ is an abelian group under addition.
- $\boldsymbol{R}$ is a monoid under multiplication.
- Multiplication is distributive with respect to addition.

Definition 1.2. An **ideal I** is a subgroup of a ring $\boldsymbol{R}$ such that for every $r \in \boldsymbol{R}$ and every $x \in \mathbf{I}$, the product rx is in $\mathbf{I}$. For commutative rings, ideals are two-sided. Every element in the ideal equals "zero" in the corresponding quotient $\boldsymbol{R}/\mathbf{I}$, so it can be easily compared with the concept of modulo and congruence classes. For quotients of polynomial rings, the ideals generally have an infinite number of elements. It is sufficient to express the elements that *generate* the ideal, for example $\mathbf{I} = \langle x_1^{d_1}, x_2^{d_2}, ..., x_n^{d_n} \rangle$.

Definition 1.3. A polynomial ring $\boldsymbol{R} = \mathbb{C}[x_1, x_2, \ldots, x_n]$, where $\mathbb{C}$ is the field of complex numbers, together with an ideal $\mathbf{I} = \langle x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n} \rangle$ form a **quotient ring $\boldsymbol{R}/\mathbf{I}$**. With the addition of ideals that are generated by terms where one or more is a monomial and not a single variable, these are the only quotient rings we will be discussing in this paper. For simplicity we can view them as the set of all polynomials with terms that are not multiples of any element that generates the ideal, as any such polynomial would be congruent to 0.

For a given quotient ring, and for any degree $d \geq 0$, there is a finite set of monomials, up to scalar multiplication, that are not part of the ideal. A monomial is a polynomial with a single term. For monomial ideals, the **Hilbert function HF($\boldsymbol{R}$/I, d)** is a function from $\mathbb{N}$ to $\mathbb{N}$ that counts the number of monomials of degree **d**. This function gives rise to a sequence that will be of great interest, that we will be referring to as the **Hilbert sequence**.

An important theorem to remember throughout most of this paper is the **Rank-nullity theorem**. This is a theorem from linear algebra that argues for the existence of kernels and the size of their dimensions. It is formally defined as follows:

Theorem 1.4. *Rank-nullity theorem The number of columns of a matrix M is the sum of the* ***rank*** *of M and the* ***nullity*** *of M, and the dimension of the domain of a linear transformation ℓ is the sum of the rank of ℓ and the nullity of ℓ.*

The *rank* here refers to dimension of the vector space that is *spanned* by the columns of the matrix. That is, the maximal number of linearly independent columns. The same is true for the rows of the matrix, there are equally many. *Nullity* refers to the dimension of the kernel, or the columns in the matrix that are a linear combination of 2 or more other columns if they exist. For the purpose of this paper, it states that the dimension of a linear maps image and the dimension of its kernel add up to the rank of the domain. The kernels dimension is what's important here, as the following chapter shows.

1.2 An introductory example

Consider the ring $\boldsymbol{R} = \mathbb{C}[x_1, x_2, x_3]$ and the ideal $\mathbf{I} = \langle x_1^3, x_2^3, x_3^3 \rangle$, and let's create the Hilbert sequence for the quotient ring $\boldsymbol{R}/\mathbf{I}$. Since any monomial containing a variable of degree 3 or higher is in the ideal, the maximum degree in the Hilbert sequence is $2 \cdot 3 = 6$. Since 1 is technically a monomial, we end up with 7 numbers in the sequence. The process of calculating the value of the Hilbert function for any given degree d can be a tricky combinatorical one. In this simple case, there is a single element of degree 0, namely 1. For $d = 1$ we have 3 options, x_1, x_2 and x_3. For $d = 2$ we can combine any 2 variables, so we get $\binom{3}{2} = 3$ as well as x_1^2, x_2^2 and x_3^2, totaling 6 elements. Continuing this process we arrive at the sequence $[1, 3, 6, 7, 6, 3, 1]$. Notice that this sequence is symmetrical, a property that will be discussed in the next chapter. Another thing to note is that this sequence has a so called “sharp” peak. The sequence corresponding to the ideal $\mathbf{I} = \langle x_1^2, x_2^2, x_3^2 \rangle$ is $[1, 3, 3, 1]$ and is called a “flat” peak, since the sequence’s highest value appears twice in a row.

1.3 Example of a simple kernel

Of central interest in this paper are **kernels** of maps between two sets of monomials with degrees differing by one. A map can be viewed as a function applied to every element of a set. The map in question, the one that will be used throughout this paper, is one induced by multiplication with the sum of all variables present in the ring. We will mostly be investigating the kernels for the maps where the domain correspond to the peak of the Hilbert sequence.

Using the previous example with the ring $\boldsymbol{R} = \mathbb{C}[x_1, x_2, x_3]$ and the ideal $\mathbf{I} = \langle x_1^3, x_2^3, x_3^3 \rangle$. The peak of the Hilbert sequence for $\boldsymbol{R}/\mathbf{I}$ is 7, and is the number of monomials of degree 3. These monomials are $x_1^2x_2, x_1x_2^2, x_1x_2x_3, x_1^2x_3, x_2^2x_3, x_1x_3^2$ and $x_2x_3^2$. Multiplying each of these elements with $\ell = x_1 + x_2 + x_3$ and removing any resulting term that is in I yields

$$\ell \cdot x_1^2x_2 = x_1^3x_2 + x_1^2x_2^2 + x_1^2x_2x_3 = x_1^2x_2^2 + x_1^2x_2x_3$$

$$\ell \cdot x_1x_2^2 = x_1^2x_2^2 + x_1^2x_2^3 + x_1x_2^2x_3 = x_1^2x_2^2 + x_1x_2^2x_3$$

$$\ell \cdot x_1x_2x_3 = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2$$

$$\ell \cdot x_1^2x_3 = x_1^3x_3 + x_1^2x_2x_3 + x_1^2x_3^2 = x_1^2x_2x_3 + x_1^2x_3^2$$

$$\ell \cdot x_2^2x_3 = x_1x_2^2x_3 + x_2^3x_3 + x_2^2x_3^2 = x_1x_2^2x_3 + x_2^2x_3^2$$

$$\ell \cdot x_1x_3^2 = x_1^2x_3^2 + x_1x_2x_3^2 + x_1x_3^3 = x_1^2x_3^2 + x_1x_2x_3^2$$

$$\ell \cdot x_2x_3^2 = x_1x_2x_3^2 + x_2^2x_3^2 + x_2x_3^3 = x_1x_2x_3^2 + x_2^2x_3^2$$

As is shown, none of the monomials by themselves are in the kernel for this map, but since we have 7 expressions with only 6 distinct terms, basic linear algebra tells us that that at least one expression is a linear combination of the others. This can be done through setting up a system of linear equations, or simply by trying to find common terms to cancel out. In fact, we can determine that adding result 1, 5 and

6 together and subtracting result 2, 4 and 7 gives us a polynomial congruent with 0. This tells us that the polynomial $x_1^2x_2 + x_2^2x_3 + x_1x_3^2 - x_1x_2^2 - x_1^2x_3 - x_2x_3^2$ is in the kernel of the map induced by multiplication with ℓ. There is a clear pattern in this polynomial which at a glance explains why this polynomial is in the kernel. We multiply the polynomial by each variable once and add the results together, and we can see that every positive term, when multiplying with a variable that would not immediately make it part of the kernel, is accompanied by a negative term that when multiplied by another variable will cancel out the first. These patterns can become quite intriguing as we will show later, as they often seem to follow some logic even if we cannot explicitly explain or prove it.

1.4 Multi-dimensional kernels

It is important to ask if this is the only element in the kernel, or in a more general case, if a kernel must exist at all? Let $\mathbf{I}$ be an ideal generated by monomials. An algebra $\mathbf{R} = \mathbb{C}[x_1, x_2 ... x_n]/\mathbf{I}$ is said to have the **Weak Lefschetz property** (WLP) if the map induced by multiplication with $\ell = x_1 + x_2 + \ldots + x_n$ from the set of elements with degree d to $d+1$ for all possible d, is either injective or surjective. The study of this property for graded algebras has been a central subject in commutative algebra in recent years. There are many results for when this property holds with a key one being that $\mathbf{R}$ has the WLP if $\mathbf{I} = \langle x_1^{d_1}, x_1^{d_2}, \ldots, x_n^{d_n} \rangle$. This result is due to Richard P. Stanley [1]. By this property, the map from the previous subchapter must be surjective between degrees 3 and 4 since the first set has more elements, thus no other polynomial can be in the kernel. Let's continue by taking a look at a case where the kernel is multi-dimensional.

An example of a quotient ring where the kernel is a sub-space is that when $\boldsymbol{R} = \mathbb{C}[x_1, x_2, x_3, x_4]$ and $\mathbf{I} = \langle x_1^2, x_2^2, x_3^2, x_4^2 \rangle$. This class of quotient rings is discussed in greater detail in chapter 4, and the existence of linearly independent polynomials that span the entire null-space that make up the kernel is the central result of this paper. $\boldsymbol{R}/\mathbf{I}$ has a Hilbert sequence with a peak equal to $\binom{4}{2} = 6$, and the set that ℓ maps it to has $\binom{4}{3} = 4$ elements. The sequence in it's entirety is $[1, 4, 6, 4, 1]$. The set containing the 6 elements have degree 2, and are all pairs of 2 distinct variables. Multiplication of these elements with ℓ creates the map with image that is the set of elements of degree 3, and we can look for kernels the same way as before. A quick demonstration of how to do this by solving a system of linear equations follows:

$$\ell \cdot (\mathrm{A}x_1x_2 + \mathrm{B}x_1x_3 + \mathrm{C}x_1x_4 + \mathrm{D}x_2x_3 + \mathrm{E}x_2x_4 + \mathrm{F}x_3x_4) \tag{1}$$

$$= (\mathrm{A}+\mathrm{B}+\mathrm{D})x_1x_2x_3 + (\mathrm{A}+\mathrm{C}+\mathrm{E})x_1x_2x_4 + (\mathrm{B}+\mathrm{C}+\mathrm{F})x_1x_3x_4 + (\mathrm{D}+\mathrm{E}+\mathrm{F})x_2x_3x_4$$

and the problem has been reduced to finding values of the coefficients A through F so that each of the above sums equal 0. Doing this, one notices that some coefficients can be written as the sum or difference of 2 others, or that some of them are equal. In fact we get that $\mathrm{A} = \mathrm{F}, \mathrm{B} = \mathrm{E}$, and $\mathrm{C} = \mathrm{D} = -\mathrm{A} - \mathrm{E}$. Using this, (1) can be simplified and factorised, giving us the kernel $\varphi = \mathrm{A}(x_1 - x_3)(x_2 - x_4) + \mathrm{B}(x_1 - x_2)(x_3 - x_4)$. It is easy to verify that both terms are elements of the kernel and that are linearly independent, the latter coming from the fact that the first polynomial has a term with x_1x_2 while the second one does not. Since the difference between the number of monomials of degree 2 and 3 is 2, finding 2 polynomials that span the kernel is expected. However, this way of expressing the kernel is not unique. It could also be expressed as $\varphi = \mathrm{A}(x_1 - x_3)(x_2 - x_4) + \mathrm{B}(x_1 - x_4)(x_2 - x_3)$, but adding another factorisation to either expression would make one term a linear combination of the other 2. You could of course add a lot of things to kernels while keeping them congruent to 0, but as they serve no purpose the simplest way is the best. This also makes it easier to verify that we have the right number of linearly independent polynomials and if we can see any patterns. This neat way of writing the complete kernel to a quotient ring of this class, that class being those where the ideal is generated by squared variables, is not due to coincidence and will be central to proving that this can be done for any number of variables in chapter 4.

2 Symmetry in the Hilbert sequence

In the introduction, we noticed that the sequences of numbers generated by the Hilbert function were symmetrical. As stated then, this is no coincidence. However it is important to note, that this is only true for ideals as defined below. For ideals that are generated by a product of 2 distinct variables, the symmetry does not necessarily need to hold. The indexing of the variables in the following proof is unorthodox but ultimately is easier to read.

Theorem 2.1. *Let $\boldsymbol{R} = \mathbb{C}[x_0, x_1, ..., x_n]$ be a polynomial ring with ideal $\boldsymbol{I} = \langle x_0^{d_0}, x_1^{d_1}, ..., x_n^{d_n} \rangle$, where $d_i > 0$. Then the Hilbert sequence associated with the quotient ring $\boldsymbol{R/I}$ will be symmetrical.*

Proof of Theorem 2.1. The single monomial with the highest degree not in $\mathbf{I}$ is $x_0^{d_0-1} x_2^{d_1-1} \cdots x_n^{d_n-1}$ and will have degree $\boldsymbol{d} = \sum_{i=0}^{n}(d_i - 1)$. Likewise, the single monomial with the lowest degree will be 1, which has degree 0. With the indexing of the numbers in the Hilbert sequence starting from 0, the $\boldsymbol{k}$'th number will equal the number of integer solutions $(k_0, k_1, \ldots, k_n)$ to the equation $k_0 + k_1 + \ldots + k_n = \boldsymbol{k}$ such that $0 \leq k_i \leq d_i - 1$ for all $0 \leq i \leq n$. A monomial of of this degree would be on the form $x_0^{k_0} x_1^{k_1} \cdots x_n^{k_n}$. From here we can create a bijection between this set of monomials and the set of monomials on the form $x_0^{d_0-k_0-1} x_1^{d_1-k_1-1} \ldots x_n^{d_n-k_n-1}$, since the number of integer solutions $(k_0, k_1, \ldots, k_n)$ will be the same for both cases. Thus we can see a link between the number of monomials of degree $(k_0 + k_1 + \ldots + k_n)$ and those of degree $(\boldsymbol{d} - (k_0 + k_1 + \ldots + k_n))$. The sets of monomials of both degrees are clearly each others reflection across the middle of the sequence, and thus the sequence must be symmetrical.

$\square$

3 The class $\mathbf{I} = \langle x_1^d, x_2^d, x_3^d, x_1^{d/2} x_2^{d/2} \rangle$

We will be taking a detailed look of two different classes of ideals. The first being ideals on the form $\mathbf{I} = \langle x_1^d, x_2^d, x_3^d, x_1^{d/2} x_2^{d/2} \rangle$ for $d = 2 + 6n$, $n = 1, 2, 3....$ The goal is to find the singular kernel for the map induced by multiplication with $\ell = x_1 + x_2 + x_3$ from the peak of the Hilbert sequence to the next degree, as their difference happens to be 1 for such values of d.

The second class of ideals are those generated by the square of the variables $\mathbf{I} = \langle x_1^2, x_2^2 ... x_n^2 \rangle$, and finding the kernel for every value of n. More specifically, finding a set of polynomials which span the null-space for any value of n. We begin with the first class.

This class has multiple interesting properties that will be discussed, one of them being that the WLP holds due to a result by David Cook II and Uwe Nagel [2], and thus the map is surjective from the peak of the Hilbert sequence to the next degree, which happen to differ by exactly 1. As such a one-dimensional kernel is expected.

3.1 Deriving a formula for the Hilbert function of $\mathbb{C}[x_1, x_2, x_3]/\mathbf{I}$

Working over the ideal $\mathbf{I} = \langle x_1^d, x_2^d, x_3^d, x_1^{d/2} x_2^{d/2} \rangle$ we want to determine the number of monomials of degree r that are not congruent to zero. We determine a formula for this using the inclusion-exclusion principle.

First, we find that all possible monomials of degree r equals $\binom{r+2}{r}$, since we have 3 different variables to which we distribute the r degrees among. Any monomial where the degree of any of the 3 variables is greater than or equals d is in the ideal, and should not be counted. After allocating d degrees to either of the 3 variables, the remaining $r - d$ degrees can be distributed in $\binom{r-d+2}{r-d}$ different ways. Since we have 3 different variables that can have a degree of d or higher we remove 3 times this value from our total. Now, any monomial where d degrees have been allocated equally between x_1 and x_2 is also in the ideal, and should also be removed from the total. Once again, there are $\binom{r-d+2}{r-d}$ such monomials, for the same reason.

We now have arrived at the formula $\binom{r+2}{r} - 4\binom{r-d+2}{r-d}$. For certain values of r however, this will not be correct. We must employ the inclusion part of the inclusion-exclusion principle because we have "double counted" certain monomials. For instance, any monomial where $\deg(x_1), \deg(x_2) \geq d$ has been counted 3 different times. First we add back any monomial where $\deg(x_1), \deg(x_2) \geq \frac{d}{2}$ and $\deg(x_1) \geq d$ or $\deg(x_2) \geq d$. We must allocate at least $\frac{3}{2}d$ degrees to get such a monomial, and the remaining $r-\frac{3}{2}d$ degrees can be distributed in $\binom{r-\frac{3}{2}d+2}{r-\frac{3}{2}d}$ different ways. Since we have a choice of x_1 and x_2 to exceed d, we multiply this number by 2 and add it to our total. Lastly, we add back any monomial where 2 out of 3 variables have a degree exceeding d as they were also double-counted. We count a total of $3\binom{r-2d+2}{r-2d}$ such monomials. In total, we then have $\binom{r+2}{r} - 4\binom{r-d+2}{r-d} + 2\binom{r-\frac{3}{2}d+2}{r-\frac{3}{2}d} + 3\binom{r-2d+2}{r-2d}$ monomials of degree r not in $\mathbf{I}$. Since every other possible monomial have been taken into account during this process, there are no more to include or exclude. Note that the last 2 terms may be equal to 0 for certain values of r, giving us a shorter formula that will prove much more manageable. We have arrived at the result that the number of monomials of degree r not in $\mathbf{I}$ equals

Lemma 3.1. *The number of monomials of degree r can be calculated using the formula* $\binom{r+2}{r} - 4\binom{r-d+2}{r-d} + 2\binom{r-\frac{3}{2}d+2}{r-\frac{3}{2}d} + 3\binom{r-2d+2}{r-2d}$.

A note on these particular values of d

You may ask why we limit the values of d to be on the form $2 + 6n$. As briefly mentioned, these are the only values of d where the Hilbert sequence has a peak that is one greater than the number following it, thus giving ℓ a one-dimensional kernel. For other values of d the peaks have different shapes that are on a rotation when increasing d by 2. With p denoting the maximum value of the sequence, the peak rotates between the 3 different forms $[p-2, p, p-1]$, $[p-1, p-p-2]$, and $[p-3, p, p, p-3]$ in that order. Verifying these claims using the aforementioned shortened formula from 3.1 is quite simple and this part is only included as an interesting tidbit about this class. Certainly these other values of d could have properties of interest but we will stick with $d = 2 + 6n$ in this paper.

3.2 The peak of the Hilbert sequence

Now that we can calculate the degree of the peak of the Hilbert sequence, we can begin looking for kernels to the map ℓ. In order to do this, we must first determine which degree has the highest number of monomials not in $\mathbf{I}$, as this would be the peak. This is done by finding the value of r where the formula from lemma **3.1** has it's maximum value. Calculating this can be tricky, but we can actually use the shorter version of the formula where we ignore the last 2 terms, as they will equal 0 for any degree $r < \frac{3}{2}d$. We know this will be the case for the maximum, because for ideals on the form $\langle x_1^d, x_2^d, x_3^d \rangle$ its maximum can be found for degree $r = \frac{3d-3}{2}$. Since the ideal we are working over has the extra element $x_1^{d/2} x_2^{d/2}$, some elements that are not in $\langle x_1^d, x_2^d, x_3^d \rangle$ would be in $\mathbf{I}$. Then the Hilbert sequence peak must have a smaller index.

After expanding the binomial coefficients we can reduce the formula to something simpler to work with, namely

$$\frac{1}{2}(r+1)(r+2) - 2(-d+r+1)(-d+r+2).$$

From here we can take the partial derivative with respect to r to find a maximum value. We get

$$\frac{\partial}{\partial r}(\frac{1}{2}(r+1)(r+2) - 2(-d+r+1)(-d+r+2)) = 4d - 3r - \frac{9}{2} = 0,$$

and if we isolate r we get $r = \frac{1}{6}(8d-9)$. For the values of d we are interested in, this is not an integer, but it is very close. We know that $d = 6n+2$, so we can rewrite the expression as $r = \frac{48n+16-9}{6} = \frac{48n+7}{6}$. Subtracting $\frac{1}{6}$ will make sure that r is an integer, simplified to $r = \frac{4d-5}{3}$. Plugging this value into the reduced form of the binomial expression we get the value $\frac{1}{3}(2d^2+1)$, the number of monomials of degree r that are not congruent to 0. To ensure that this is indeed the maximum, we must consider the fact that we arbitrarily chose to subtract a number to make r an integer. If instead of subtracting $\frac{1}{6}$, we add $\frac{5}{6}$ which gives us the closest integer value that is higher than the actual maximum, we get the value $\frac{1}{3}(2d^2-1)$ when plugging it into the formula, which is clearly less. Since the function takes a form similar to a parabola, the maximum value must be $\frac{1}{3}(2d^2+1)$ and is found when

$r = \frac{4d-5}{3}$. This is an improvement of lemma **5.3** in [3] which states that the value of the Hilbert function at degree $\lfloor \frac{3d-3}{2} - 1 \rfloor$ is strictly greater than the value at degree $\lfloor \frac{3d-3}{2} \rfloor$.

Lemma 3.2. *For* $\boldsymbol{I} = \langle x_1^d, x_2^d, x_3^d, x_1^{d/2} x_2^{d/2} \rangle$ *the Hilbert function has a maximum value of* $\frac{1}{3}(2d^2+1)$ *when* $r = \frac{4d-5}{3}$.

3.3 Finding kernels

Now that we can easily find the peak of the Hilbert sequence that arises from the ideal $\mathbf{I} = \langle x_1^d, x_2^d, x_3^d, x_1^{d/2} x_2^{d/2} \rangle$, we can begin to look for kernels. As previously mentioned, this ideal does have the WLP. However, there is no obvious way for us to try to find a kernel, but for $d = 8$ my thesis supervisor Samuel Lundqvist gave the answer for me to investigate. From there I was able to extrapolate how it was constructed and what a kernel for general values of $d = 2 + 6n$ might look like. Plugging in the value $d = 8$ in our formula for the degree that corresponds with the peak of the Hilbert sequence, we get that the peak has degree 9. The kernel to the familiar map $\ell = x_1 + x_2 + x_3$ is then a polynomial where every term has degree 9. The kernel is expressed in the following lemma:

Lemma 3.3. *The kernel for* $d = 8$ *can be expressed as* $\varphi = (f - 42g)$ *where*

$$f = x_3(7x_1^2 - 16x_1x_2 + 7x_2^2)\frac{(x_1+x_2)^7 + x_3^7}{x_1+x_2+x_3}$$
$$g = x_1^2x_2^2(x_1 - x_2)(x_1^4 - x_2^4).$$

Since the rational expression in f can be written as $(x_1+x_2)^6 - (x_1+x_2)^5x_3 + (x_1+x_2)^4x_3^2 + \ldots + x_3^6$, f is indeed a polynomial. These expressions look quite pleasant, and we can immediately see that it follows some sort of pattern. If we try to multiply $(f - 42g)$ with ℓ, we will see a more general method that can be used to find kernels when d is larger. We begin by applying the map to the f term, which just so happens to have the polynomial that induces the map as a denominator, giving us

$$\ell \cdot x_3(7x_1^2 - 16x_1x_2 + 7x_2^2)\frac{(x_1+x_2)^7 + x_3^7}{x_1+x_2+x_3}$$
$$= x_3(7x_1^2 - 16x_1x_2 + 7x_2^2)((x_1+x_2)^7 + x_3^7)$$
$$= x_3^8(7x_1^2 - 16x_1x_2 + 7x_2^2) + (7x_1^2 - 16x_1x_2 + 7x_2^2)(x_1+x_2)^7.$$

Since the first term of the last expression is a multiple of x_3^8, which is in the ideal, the entire term is congruent to 0 and can be ignored. In the second term, we start by expanding the exponential term. A polynomial written on this form will, when expanded, result in 7 binary choices of variables and therefore the resulting polynomial will be

$$(x_1 + x_2)^7 = \binom{7}{0} x_1^7 + \binom{7}{1} x_1^6 x_2 + \ldots + \binom{7}{6} x_1 x_2^6 + \binom{7}{7} x_2^7.$$

Multiplying this with $(7x_1^2 - 16x_1x_2 + 7x_2^2)$ can be seen as multiplying the expression with a single term 3 separate times and adding them together. Viewing it from this perspective is beneficial since we can easily see which resulting terms belong to the ideal and can be ignored. When multiplying with $7x_1^2$, every term where the exponent of x_1 equals 6 or more can be ignored as the product would have an exponent 8 or greater. Likewise, multiplying with $-16x_1x_2$ lets us remove any term with an exponent equal to 7 for either variable, and for $7x_2^2$ the same principle as for $7x_1^2$ applies. The below illustration showcases the coefficients that arise for each monomial as they can be products of multiple factors. Each row represents each of the three terms in the factor $(7x_1^2 - 16x_1x_2 + 7x^2)$ and the colums are summed up to get the number of such monomials (when multiplied with the appropriate coefficients)

$$\begin{array}{cccccccccc}
x_1^9 & x_1^8x_2 & x_1^7x_2^2 & x_1^6x_2^3 & x_1^5x_2^4 & x_1^4x_2^5 & x_1^3x_2^6 & x_1^2x_2^7 & x_1x_2^8 & x_2^9 \\
\binom{7}{0} & \binom{7}{1} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & \binom{7}{7} & & \\
 & \binom{7}{0} & \binom{7}{1} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & \binom{7}{7} & \\
 & & \binom{7}{0} & \binom{7}{1} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & \binom{7}{7}
\end{array}$$

This way of constructing the kernel practically removes the variable x_3 during any calculations, and simplifies the process as x_3 is different than the other variables in the ideal. We now get the coefficients for each term by adding the columns in the illustration above where the monomials are not part of the ideal, so for $x_1^7x_2^2$ the coefficient equals

$$7\binom{7}{0} - 16\binom{7}{1} + 7\binom{7}{2} = 42.$$

Likewise for $x_1^6x_2^3$ we have the coefficient -42, and the symmetry with the terms $x_1^3x_2^6$

and $x_1^2x_2^7$ means they equal -42 and 42 respectively. -42 is the coefficient before the expression g, as well as the greatest common multiple of 7 and 16. Looking at the g term, the 2 parenthesis expand to the expression $x_1^5 - x_1x_2^4 - x_1^4x_2 + x_2^5$, which when multiplied by $x_1^2x_2^2$ yields the same result as f after multiplying with 42, and thus the polynomial $(f - 42g) = 0$ and is a kernel of this map.

You can use the same type of method to find a kernel for $d = 14$ and beyond, but with additional and different coefficients. The g term does not always factor as nicely either, but through testing it appears that for every other ideal it does while the ones in between it don't. What does not change however, is the symmetry of the first polynomial in the f term. Since the other factor is symmetrical and x_1 and x_2 are interchangeable, there has to be a symmetrical solution. We can then switch our focus to the question of the existence of a kernel that is on a similar form to this one. Since this is a pretty straightforward process we would prove that we can find a kernel for any $d = 2 + 6n$. In this example we had the coefficients 7 and -16, which is a solution to the following system

$$\begin{bmatrix} \binom{7}{0} + \binom{7}{2} & \binom{7}{1} \\ \binom{7}{1} + \binom{7}{3} & \binom{7}{2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For greater values of d, this matrix of binomial coefficients grows larger and in order to ensure that such a method of constructing a kernel works, the system must be solvable. That is to say the matrix must be invertible, or in other words have a determinant that is non-zero. This proves to be a very difficult problem and remains unsolved, meaning that this method of finding kernels is not guaranteed to work for every value of $d = 2 + 6n$. As such the problem of finding a method to generate kernels for any of these values of d remains unsolved. An example of a larger matrix is that of $d = 20$:

$$\begin{bmatrix} \binom{19}{0} + \binom{19}{6} & \binom{19}{1} + \binom{19}{5} & \binom{19}{2} + \binom{19}{4} & \binom{19}{3} \\ \binom{19}{1} + \binom{19}{7} & \binom{19}{2} + \binom{19}{6} & \binom{19}{3} + \binom{19}{5} & \binom{19}{4} \\ \binom{19}{2} + \binom{19}{8} & \binom{19}{3} + \binom{19}{7} & \binom{19}{4} + \binom{19}{6} & \binom{19}{5} \\ \binom{19}{3} + \binom{19}{9} & \binom{19}{4} + \binom{19}{8} & \binom{19}{5} + \binom{19}{7} & \binom{19}{6} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

It's easy to see the difficulty in showing that the generalised matrices have a non-zero determinant. If you reconstruct the binomial coefficients to polynomials you can find some interesting patterns for the exponents and coefficients when calculating the determinant, something we unfortunately won't get to the bottom of in this paper.

4 Ideals with square elements

We now switch our attention to quotient rings with ideals on the form $\mathbf{I} = \langle x_1^2, x_2^2, \ldots, x_n^2 \rangle$. We want to determine the kernel of the same map we have used before, that is multiplying with $\ell = x_1 + x_2 + ... + x_n$ the set of elements corresponding to the peak of the Hilbert sequence to the next degree. Since this quotient ring has the WLP and the codomain has fewer elements then the domain, the map is surjective and a kernel does exist. By the rank-nullity theorem, we also know that the kernel is multi-dimensional (in almost all cases) and exactly how many dimensions it has. Calculating every number in the sequence is simple. Since we have a binary choice between including or not including each variable in any monomial the r'th number in the sequence will be $\binom{n}{r}$. The index for the maximum value in the sequence is then $\lceil \frac{n}{2} \rceil$. We take the ceiling function because for odd n we have 2 subsequent numbers that are equal, and starting from the second one ensures that the following number in the sequence is lower. For the sake of simplicity, we will assume that n is even and that $n = 2m$. Odd values of n will be discussed later in the chapter.

In 1.4 we looked at the kernel for $n = 4$ which can be written as $c_1(x_1 - x_2)(x_3 - x_4) + c_2(x_1 - x_3)(x_2 - x_4)$ where c_1 and c_2 are arbitrary constants. As a reminder, these 2 linearly independent polynomials span the entire kernel. The fact that there are 2 of them is expected since $\binom{4}{2} - \binom{4}{3} = 2$. It turns out that we can write any element in the kernel as a factorised polynomial on the form $(x_{a_1} - x_{a_2}) \cdots (x_{a_{n-1}} - x_{a_n})$ with no repeat of variables, as that would mean that it already is in the ideal.

Theorem 4.1. *Let $\boldsymbol{R} = \mathbb{C}[x_1, x_2, ..., x_n]$ and $\boldsymbol{I} = \langle x_1^2, x_2^2, ..., x_n^2 \rangle$. The kernel of the map induced by multiplication with $\ell = x_1 + ... + x_n$ from the set of monomials not in $\boldsymbol{I}$ of degree $\binom{n}{m}$ to degree $\binom{n}{m+1}$ will contain any polynomial that can be factorised as $(x_{a_1} - x_{a_2}) \cdots (x_{a_{n-1}} - x_{a_n})$.*

Proof of Theorem 4.1. Consider the following expression:

$$(x_1 + ... + x_a + ... + x_b + ...x_n)(x_a - x_b)P(\boldsymbol{x})$$

where $P(\boldsymbol{x})$ is the rest of some arbitrary polynomial factorisation on the form mentioned, and x_a, x_b are any 2 variables that are terms in the same factor. This can be rewritten as

$$(x_1 + \ldots + x_{a-1} + x_{a+1} + \ldots + x_{b-1} + x_{b+1} +_n)P(\boldsymbol{x}) + (x_a + x_b)(x_a - x_b)$$
$$= M(\boldsymbol{x})P(\boldsymbol{x}) + (x_a^2 - x_b^2),$$

and since the term factor on the right hand side is congruent to 0 it can be removed. Because x_a, x_b are any 2 variables that are terms of the same factor, along with the fact that every variable appears once, you can repeat the same process for any 2 variables of the same factor and the result would be congruent to 0. $\square$

In order to find every possible polynomial in the kernel, we need to find enough linearly independent factorisations that span the entire null-space of the kernel. The number of factorisations we need is $\binom{n}{m} - \binom{n}{m+1}$, since that is the difference between the number of monomials in the domain and in the codomain of ℓ. These numbers happen to be known as the **Catalan numbers**, usually denoted C_m. The Catalan numbers on their own have very many interesting combinatorial properties, with one of them being of central importance to this problem.

4.1 Linear independence

So how can we determine if a large number of polynomials are linearly independent? We will be using some properties of matrices to do this. The following 2 definitions are needed to understand how we can use linear algebra to show properties of polynomials.

Definition 4.2. A **unit triangular matrix** is a special kind of square matrix. It's diagonal entries are all equal to 1 and for an **upper unit triangular matrix** every entry below the diagonal is 0. Similarly, a **lower unit triangular matrix** has every entry above the diagonal equal 0. Every unit triangular matrix is invertible and therefore it's columns are linearly independent.

Definition 4.3. A **sub-matrix** is a matrix obtained by removing a number of columns and/or rows from a larger matrix.

The key takeaway from these two definitions is that if a matrix M has a unit triangular sub-matrix with rank r, then the rank of M is at least r.

In order to proceed, we will also need a way to order monomials and polynomial factorisations, and the way we will do this is defined below.

Definition 4.4. The **lexicographical order** is a generalization of the alphabetical ordering that can be applied to **totally ordered sets**. Such a set is made up of elements that are comparable in some way, such that if $\mathbf{S}$ is totally ordered and $a, b \in \mathbf{S}$, then $a \leq b$ or $a \geq b$.

For the purposes of the content of this chapter, every set of monomials will be ordered lexicographically, and may be referred to as just being **ordered**. What this means is that each monomial will always be internally ordered, so that their indices are monotonically increasing from left to right, and we compare two monomials $x_1^{a_1}, \ldots, x_n^{a_n} > x_1^{b_1}, \ldots, x_n^{b_n}$ if the first non-zero element in $(a_1 - b_1, \ldots, a_n - b_n)$ is positive. The first monomial comes before the second in the ordering and is referred to as being **larger**.

In order to find C_m factorisations that are linearly independent, we will convert every possible unique factorisation to a vector of size $\binom{n}{m}$, where every element equals 1, 0 or -1, depending on the coefficient that arise when expanding the factorisation. Combining all of these vectors as columns of a matrix, is it sufficient to prove that through regular matrix operations, we can find a unit triangular submatrix of size C_m. The columns of this matrix are the factorisations we need. This is the key takeaway from definitions **4.2** and **4.3** mentioned earlier. We now defininе a map between monomials and polynomial factorisations that will be the foundation of the method we use to find these linearly independent factorisations.

Definition 4.5. Let $\mathbf{R}$ have $2m$ variables and define $\boldsymbol{\psi}$ as a map from monomials of degree m to polynomial factorisations with m factors. Let the variables of the monomial be lexicographically ordered, then the factorisation will contain the factors on the form $(x_a - x_b)$ that will be ordered in the same manner where x_a is a variable in the monomial, and x_b a variable that isn't. The second terms of each factor will also be ordered lexicographically, so the first factor $(x_a - x_b)$ will have the property that a is the lowest index of any variable in the monomial, and b is the lowest of any variable not in the monomial. We can call $\boldsymbol{\psi}$ the **factorisation map**.

As an example, if $\mathbf{R}$ has 6 variables, $\boldsymbol{\psi}$ maps the monomial $x_1x_2x_3$ to the factorisation $(x_1 - x_4)(x_2 - x_5)(x_3 - x_6)$. Now we need to introduce a connection between monomial orderings and the number of factorisations needed to form the basis of the null-space.

4.2 Dyck paths

One property of the Catalan numbers is that C_m is the number of **Dyck paths** along the edges of a $m \times m$ grid. A Dyck path is a lattice path that does not cross the diagnonal of the square, and goes from the bottom left corner $(0, 0)$ to the top right corner (m, m). These paths are said to have length $2m$. We will give a short overview why this relationship between Dyck paths and Catalan numbers hold and why it is important for finding the basis of the kernel. Let's first take a look at an example, all Dyck paths for a 4×4 grid.

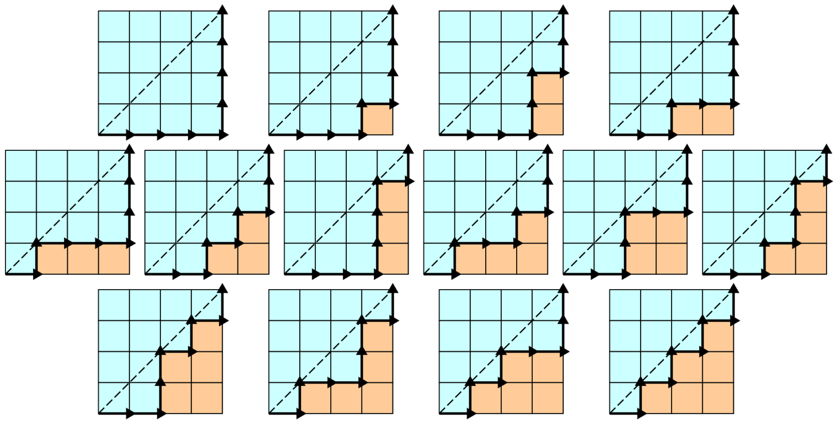

Figure 1: Graphical illustration of Dyck paths across a 4×4 grid, not ordered. Image taken from [4]

We can clearly see that there are $C_4 = 14$ different Dyck paths. One definition of the Catalan numbers are that they follow the recurrence relation

$$C_0 = 1, C_{m+1} = \sum_{k=0}^{m} C_k C_{m-k+1} \text{ for } m > 0.$$

If we assume that C_m does in face count the number of Dyck paths over a $m \times m$ grid, consider a Dyck path of length $2(m+1)$. Now let (k, k) be the first point after $(0, 0)$ where the path touches the diagonal. Between the points $(0, 0)$ and (k, k) is a shorter Dyck path of length $2k$, and there are C_k such Dyck paths from our assumption. Then from the point (k, k) to $(m + 1, m + 1)$ is another Dyck path of length $2(m-k+1)$ and likewise there are C_{m-k+1} such Dyck paths. Letting k range from 0 to m gives us the recurrence relation above.

From here we will transform these Dyck paths to sequences of numbers, with the number in the k'th position being equal to the height of the k'th column as seen in the picture, or the number of "up-steps" that the path has taken before that point. In this example of a 4×4 grid, the first path in the image correlates with the sequence $[0,0,0,0]$ and the last with $[0,1,2,3]$. More generally, a path can be described as a sequence $[a_0, a_1, a_2, ..., a_n]$, with the conditions that $a_k \leq k$ and $a_{k-1} \leq a_k$ for all $1 \leq k \leq n$. From this set of sequences we make a simple bijection by letting $a_k = a_k + k + 1$. In the given example the sequences would then range from $[1,2,3,4]$ to $[1,3,5,7]$, where the number with index i does not exceed $2(i+1)-1$ and every number is unique and always appear in a monotonically increasing order.

We can now transform these sequences into monomials, with the numbers in the sequence referencing the indices of the variables present in the internally ordered monomial, and let the indices follow the sequence from left to right. Thus we have C_m monomials of m variables with indices ranging from 1 to $2m-1$ and that all follow the conditions above. We say that such a monomial fulfills the **Dyck path criteria**.

4.3 An illustrative example

In the interest of brevity, before formally proving a method for finding linearly independent factorisations we will take a look at the specific case of $n = 8$. That is to say there are 8 variables and we need to find $C_4 = 14$ linearly independent polynomial factorisations, as they would span the entire kernel. We start with the 14 monomials that follow the Dyck path criteria and take the factorisations that they are mapped to by $\boldsymbol{\psi}$.

Let the monomials be ordered, with the first monomial equaling $x_1x_2x_3x_4$, and it's corresponding factorisation being $(x_1 - x_5)(x_2 - x_6)(x_3 - x_7)(x_4 - x_8)$. Now we will introduce a different notation to make the process of finding these factorisation easier to read. Since every monomial is defined only by their indices, and the factorisations only have negative signs inside each factor, we will remove the variable letter x and negative signs completely. Instead we express the fact that the monomial and factorisation above correspond like this:
(1 2 3 4) - (1 5)(2 6)(3 7)(4 8)
Remember that the monomial on the left is made up of the first variables of each factor on the right. Let's continue with the monomials in the order previously men-

tioned
(1 2 3 5) - (1 4)(2 6)(3 7)(5 8)
(1 2 3 6) - (1 4)(2 5)(3 7)(6 8)
(1 2 3 7) - (1 4)(2 5)(3 6)(7 8)
(1 2 4 5) - (1 3)(2 6)(4 7)(5 8)
The first of these factorisations has indices 1 and 4 in the same factor, which means that expanding the factorisation will not result in a term equal to the previous monomial. On the second row 1 and 4 are still in the same factor and so is 2 and 5, so that this factorisation cannot expand into either of the 2 previous monomials. In the last row, the 3 factors that were required to ensure that the factorisations will not generate any previous monomial can be swapped with (1 3), since every previous monomial has both indices 1 and 3, but this monomial does not. Continuing this way we get the rest of the monomials
(1 2 4 6) - (1 3)(2 5)(4 7)(6 8)
(1 2 4 7) - (1 3)(2 5)(4 6)(7 8)
(1 2 5 6) - (1 3)(2 4)(5 7)(6 8)
(1 2 5 7) - (1 3)(2 4)(5 6)(7 8)
(1 3 4 5) - (1 2)(3 6)(4 7)(5 8)
(1 3 4 6) - (1 2)(3 5)(4 7)(6 8)
(1 3 4 7) - (1 2)(3 5)(4 6)(7 8)
(1 3 5 6) - (1 2)(3 4)(5 7)(6 8)
(1 3 5 7) - (1 2)(3 4)(5 6)(7 8)
And those are all 14 monomials and respective factorisations that span the kernel. Once again, on row number 5 the factor (1 2) makes an appearance as every previous monomial has variables with these 2 indices, but the monomial on that row does not. In order to loop back to the concept of linear independence, let's create the

appropriate sub-matrix for these polynomials.

$$
\begin{array}{r}
x_1x_2x_3x_4 \\
x_1x_2x_3x_5 \\
x_1x_2x_3x_6 \\
x_1x_2x_3x_7 \\
x_1x_2x_4x_5 \\
x_1x_2x_4x_6 \\
x_1x_2x_4x_7 \\
x_1x_2x_5x_6 \\
x_1x_2x_5x_7 \\
x_1x_3x_4x_5 \\
x_1x_3x_4x_6 \\
x_1x_3x_4x_7 \\
x_1x_3x_5x_6 \\
x_1x_3x_5x_7
\end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

These are the first 14 rows of the matrix that arise when transforming every factorisation we generated into a vector of coefficients to each possible monomial and order them correctly, meaning the first 14 rows correspond with the monomials that fulfill the Dyck path criteria and are lexicographically ordered. Each vector makes up a column of the matrix and each row correspond with the monomial shown on the left side of the matrix. The columns are ordered from left to right in the same fashion as their respective monomials. This sub-matrix is lower triangular and by definition **4.2** and **4.3**, the columns are linearly independent.

4.4 A method to find the basis of a kernel

Now we are ready to prove that, for $n = 2m$ we can find C_m polynomial factorisations that are linearly independent and as such will span the entire kernel. The method we will use is described in the theorem and following proof.

Theorem 4.6. *Take the set of all monomials of degree m that fulfill the Dyck path criteria, ψ will map that set to a set of linearly independent polynomial factorisations. As there are C_m such monomials, these factorisations will span the entire kernel of ℓ.*

Proof of Theorem 4.6. Let P_d be the set of polynomial factorisations that arise from the map $\boldsymbol{\psi}$ when applied to the monomials fulfilling the Dyck path critera. When expanded, these polynomials can be represented as vectors with elements corresponding to the coefficients to every possible monomial of degree m. Let these vectors be columns of a matrix ordered from left to right the same way the monomials they were mapped from would be ordered. In addition, let the first C_m rows be ordered in the same manner. For column number k, that we call p_k, it's k'th row would be the first row with a non-zero element, as as this would correspond to the highest ordered monomial that is generated by the polynomial factorisation. Due to the construction of the factorisation, this element is 1. No monomial that is a term in the expansion of the factorisastion can have a higher ordering than the monomial $\boldsymbol{\psi}^{-1}(p_k)$, since it is made up of the variables with the lowest indices of each of the factors. From here we can then create a lower trianglular sub-matrix of size $C_m \times C_m$ and by definition **4.2** and **4.3** the columns must be linearly independent. With the columns being a one-to-one representation of the factorisations P_d, they are also linearly independent. Finally, by theorem **4.1** these factorisations are themselves in the kernel and together they span the entire sub-space that is the kernel. ☐

At the start of this chapter, we assumed that n was an even number as it was more convenient. If n is an odd number, say $n = 2m - 1$ nothing about the process of finding linearly independant polynomial factorisations that span the kernel changes, as the variable x_{2m} is never used in any monomial. If we instead let the last variable of each monomial be a lone factor in the polynomial factorisation the exact same method can be applied for odd values of n. Since if $n = 2m$ we have the equality $\binom{n}{m} - \binom{n}{m+1} = \binom{n-1}{m} - \binom{n-1}{m+1}$, the kernels also have the same dimensions.

To summarize, we began by showing that certain factorisations were elements of the kernel to the map ℓ. We preceeded to use a property of the Catalan numbers to create monomials that would map onto specific factorisations such that they were linearly independent. A final but important remark is that the map $\boldsymbol{\psi}$ is not the only map that would have this property. In fact, the only restriction on the factorisations that are associated with the set of ordered monomials is in each factor, the second term have the higher index. This ensures the original monomial is the maximum in the lexicographical ordering.

Final thoughts

When I began to work on this thesis I had very little knowledge of the subject as my supervisor Samuel Lundqvist introduced it to me. I found it very intriguing due to the fact that it was not very heavy with theory but had a lot of profound ideas. There are still problems related to this work I want to keep working on, namely the class of ideals in chapter 3. I am very proud to have been able to stumble upon the class of ideals with square elements and the connections to prove a method for finding linearly independent polynomials that for a basis for the kernel. As a likely previously unknown result, the details and connections between different areas of mathematics were very rewarding to uncover and I hope I can keep the same spark of interest in my further studies. Doing something original for my bachelor's thesis has been very meaningful to me. It has inspired me to keep learning more than any other project, and made me want to keep studying for as long as I can, because discovering these niche things feel like validation that I'm doing something different.

References

[1] R. P. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property. SIAM J. Algebraic Discrete Methods, 1 no. 2 (1980), 168–184.

[2] D. Cook II, U. Nagel, Syzygy bundles and the weak Lefschetz property of monomial almost complete intersections, J. Commut. Algebra 13 (2021), no. 2, 157–178.

[3] N. Altafi, S. Lundqvist, Forcing the weak Lefschetz property for equigenerated monomial ideals, to appear in Transactions of the AMS, preprint avaiable at https://arxiv.org/abs/2306.03188

[4] Public Domain, https://commons.wikimedia.org/w/index.php?curid=913301