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## Linear Groups

av


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**Abstract**

We present the theory on linear groups. These are defined as subgroups of the general linear group and consists of invertible matrices. Once the theory has been developed we derive several examples and the main topic of study is the special unitary group of $2 \times 2$ - matrices. We provide a proof that this group is isomorphic to the 3-sphere in $\mathbb{R}^4$ and define the equator of this sphere. By constructing a map $\gamma : SU_2 \longrightarrow \{f : \mathbb{E} \longrightarrow \mathbb{E}\}$ we show that every matrix of $SU_2$ can be represented as an element of the special orthogonal group $SO_3$. This representation is interpreted geometrically as a rotation of the 3-sphere. We conclude by considering a class of differentiable homomorphisms. We prove that the image of these homomorphisms define the one parameter groups.

**Sammanfattning**

Vi presenterar teorin om linjära grupper. Dessa definieras som delgrupper av den allmäna linjära gruppen och består av inverterbara matriser. Varvid teorin har utecklats härleds ett antal exempel där huvudämnet är den speciella linjära gruppen bestående av $2 \times 2$-matriser. Vidare presenterar vi ett bevis på att den här gruppen och 3-sfären i $\mathbb{R}^4$ utgör en gruppisomorfi. Varpå detta har redogjorts definierar vi ekvatorn av $SU_2$. Följaktligen definieras en funktion $SU_2 \longrightarrow \{f : \mathbb{E} \longrightarrow \mathbb{E}\}$ som används för att beskriva hur $SU_2$ kan representeras som ortogonalmatriser i den speciella linjära gruppen $SO_3$. Den här representationen beskriver en rotation av 3-sfären. Avslutningsvis betraktas en klass av deriverbara homomorfismer. Vi bevisar att värdemängden för dessa homomorfismer definierar de så kallade enparametergrupperna.

# Contents

# 1 Introduction

What follows is a collection of definitions and results related to linear groups (or matrix groups). It turns out that by combining matrices with the concept of a group we can define the linear groups. A particularly important example is the group of unitary $2 \times 2$ - matrices with determinant 1, denoted by $SU_2$. The reason why this group is so important to our case becomes apparent when we discover that it can be represented as a unit-sphere in $\mathbb{R}^4$. Since $SU_2$ has the geometric interpretation of a sphere one might wonder whether or not this sphere has any interesting properties. It does! In fact, we will show that the action of conjugation of the matrices in $SU_2$ can be described geometrically as a rotation of the sphere. Once all of this is achieved we end with a brief section on differentiable homomorphisms. The images of these homomorphisms constitute a special class of matrix-valued functions called the one-parameter groups.

# 2 Preliminary

In this section we recall some results that will be referred to throughout the text. The topics included are derived in group theory, topology and linear algebra. Some of the proofs of the results presented here are left out given their elaborate and rather tedious nature.

## 2.1 Group Theory

**Definition 1** (Subgroup)**.** A nonempty set $H$ of a group $G$ is a subgroup if $H$ is a group under the binary operation $(*)$ of G. We write $H \leqslant G$.

**Proposition 1** (The subgroup test)**.** Let $G$ be a group and $H$ a subset of $G$ that is nonempty. Then $H$ is a subgroup of $G$ if $ab^{-1} \in H$ , for every $a, b \in H$.

**Definition 2** (Normal subgroups)**.** A subgroup $N \leqslant G$ is normal if $gNg^{-1} = N$ for all $g \in G$.

Since it will come up when we discuss the latitudes of the group $SU_2$, we include the definition of a transitive group action.

**Definition 3** (Transitive group action)**.** Let $G$ be a group acting on a nonempty set $A$. The action of $G$ on $A$ is called transitive if there is only one orbit, that is, given any pair of elements $a, b \in A$ there exists a $g \in G$ such that $g \cdot b = a$

## 2.2 Linear Algebra

**Theorem 1** (Product rule for determinants)**.** If $A, B \in M_n(\mathbb{F})$, then

$$\det(AB) = (\det A)(\det B)$$

where $M_n(\mathbb{F})$ denotes the set of $n \times n$ - matrices.

*Proof.* See [FIS14, Ch.4, Thm.4.7] □

**Lemma 1.** The matrix $A$ is invertible if and only if $A^T$ is invertible. Where $T$ denotes the transpose operator.

*Proof.* We omit the proof. □

**Lemma 2.** Suppose $A \in GL_n(\mathbb{F})$ (Definition 11), then

$$\det(A) = \det(A^T)$$

*Proof.* Omitted. □

**Corollary 1.** Let $A \in GL_n(\mathbb{F})$ then it holds that

$$(A^T)^{-1} = (A^{-1})^T$$

*Proof.* We know from Lemma 1 that the inverse property of a matrix implies that $\det(A^T) \neq 0$ and hence that $A^T \in GL_n(\mathbb{F})$. We show that $A^T$ behaves as the inverse. Since the inverse of a matrix is unique the result will follow. We simply compute

$$(A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n$$

and

$$A^T (A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$$

□

**Definition 4** (Trace of a matrix)**.** The trace of a matrix $A$, denoted $tr(A)$ is the sum

$$\sum_{j=1}^{k} a_{jj} = a_{11} + a_{22} + ... + a_{nn}$$

where $a_{jj}$ denotes the element in the jth row and jth column of $A$.

**Definition 5.** A matrix $A \in M_n(\mathbb{C})$ is said to be skew-Hermitian if it satisfies the equation

$$-A = A^*$$

where $*$ denotes complex conjugation.

**Lemma 3.** If $A$ and $B$ are square matrices of equal size, then

$$tr(AB) = tr(BA)$$

*Proof.* Omitted but follows from the formula for matrix multiplication. □

If we replace $B$ by $BC$ in the previous Lemma we have that $tr(ABC) = tr(CAB)$ given the associativity property of matrix multiplication.

**Lemma 4.** For any matrix $A \in GL_n(\mathbb{F})$ it holds that

$$\det(A^{-1}) = \det(A)^{-1} \tag{1}$$

*Proof.* For $A \in GL_n(\mathbb{F})$ we have have by Theorem 1 that

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) \tag{2}$$

which reduces to the identity. Hence we multiply (1) on the right by the left side of (2) and the result follows. $\square$

**Lemma 5** (Determinant of orthogonal matrices)**.** If M is an orthogonal matrix then,

$$\det(M) = \pm 1$$

.

*Proof.* Let $M$ be an orthogonal matrix, then by Lemma 2 we have

$$\det(I) = \det(M^T M) = \det(M)^2 = 1$$

The lemma follows by taking square roots. $\square$

### 2.3 Topology

**Definition 6** (Homeomorphism)**.** A continuous bijective map $\phi : X \to Y$ is called a homeomorphism if the inverse map $\psi : Y \to X$ is also continuous.

Roughly speaking, a homeomorphism is an injective map that preserves the topological properties of the function on which it acts.

**Definition 7** (Topological embedding)**.** A map $f : X \to Y$ is called an embedding between topological spaces if the map

$$f^{'} : X \to f(X)$$

is a homeomorphism where we obtain $f(X)$ by restricting the space $X$.

A relevant example is the embedding of a subgroup into a group.

**Definition 8** (Covering)**.** Let X be a topological space. A cover of X is a collection of sets $U_i,\ i = 1, 2, 3, ...$ such that $X \subseteq \cup_{i=1}^{\infty} U_i$.

In other words, a topological cover for a space $X$ is a collection of sets such that $X$ is completely contained in the union of these sets.

**Definition 9** (Path-connected space)**.** A topological space $X$ is called a path-connected space if for any pair of points $(x_0, x_1) \in X$ there exists a continuous map

$$f : [0, 1] \to X$$

with $f(0) = x_0$ and $f(1) = x_1$.

Henceforth a topological space $X$ is said to be path connected if there is a continuous path connecting any two points $x_0, x_1 \in X$ such that the path lies entirely inside $X$.

**Example 1.** The space

$$X = \{z \in \mathbb{C} \mid z \geq 1 \text{ and } z \leq -1\}$$

is not path-connected.
Pick $z_1 \geq 1$ and $z_2 \leq -1$, then there exists no path connecting $z_1$ with $z_2$ without intersecting the infinite strip which does not lie in $X$.

### 2.4 Stereographic projection

Before proceeding with the main material we intend to define the concept of stereographic projection. In short, this provides a method for projecting an $n+1$ - dimensional sphere onto an $n$ - dimensional surface. The geometric interpretation changes for higher dimensions and becomes harder to vizualize. Nevertheless, the approach is similar. We begin this endeavor by considering the locus

$$\{x_0^2 + x_1^2 + x_2^2 + ... + x_n^2 = 1\} \in \mathbb{R}^{n+1}$$

commonly referred to as the n-dimensional unit sphere $\mathbb{S}^n$. Hence, for example, the unit sphere in $\mathbb{R}^3$ is given by $\mathbb{S}^2$. We shall consider $x_0$ as the vertical vector protruding from the origin. Initially we define stereographic projection for $\mathbb{S}^2$ (sitting in $\mathbb{R}^3$) onto the two-dimensional plane and then extend the definition to $\mathbb{S}^3$.
Denote by $\mathbb{V}$ the plane determined by putting $x_0 = 0$ and let $p = (1, 0, 0)$ be the north pole of the $\mathbb{S}^2$-sphere. If $x$ is any point on the sphere we can define the map

$$\sigma : \mathbb{S}^2 \to \mathbb{V}$$

as the stereographic projection from the $\mathbb{S}^2$-sphere onto $\mathbb{V}$ in the following way. Consider the image $\sigma(x)$ that we obtain by constructing a line $l$ between $p$ and $x$. We define $\sigma(x)$ as the point on $\mathbb{V}$ where it is intersected by $l$. Note that this projection is a bijective correspondence at all points on the sphere except for $p$. We justify this exclusion by the observation that if we let $x$ tend to $p$ on the sphere this will map the image $\sigma(x)$ further and further away on $\mathbb{V}$. Hence we say that $p$ is sent to infinity as $x$ approaches $p$ on the sphere.
We now compute the coordinates for $\sigma(x)$ on $\mathbb{V}$ represented by the point $x$ on the sphere. Since the $x_0$-vector sits on top of the vertical axis in $\mathbb{V} = (x_0, x_1, x_2)$ this space is interpreted as $\mathbb{R}^2$. If a point in $\mathbb{V}$ has coordinates $(v_1, v_2)$ the equation for the line $l$ intersecting $p$ and $x$ is given by

$$u = (0, v_1, v_2) - (1, 0, 0) = (-1, v_1, v_2) \tag{3}$$

From (3) we can write down the parametric representation. This is given by

$$\begin{cases} x_0 = 1 - t \\ x_1 = tv_1 \\ x_2 = tv_2 \end{cases} \tag{4}$$

for $\sigma(x) = (v_1, v_2) \in \mathbb{V}$. From (4) we have that $t = 1 - x_0$. Solving for $v_1$ and $v_2$ in the other two equations and inserting the expression for $t$ yields the coordinates on the $\mathbb{S}^2$-sphere. Namely,

$$\sigma(x) = (v_1, v_2) = \left( \frac{x_1}{1 - x_0}, \frac{x_2}{1 - x_0} \right)$$

*Remark* 1. If we put $x_0 = 1$ in the preceding system the formula is clearly not well defined.

When we later on define the special rotation group $SO_3$ it will be of great value to define the concepts of longitude and latitude which will assist us in understanding the structure of this group. Hence we devote some time to define these concepts here. To get a grasp on the geometry we begin by defining the latitude and longitude for the $\mathbb{S}^2$ unit ball. Deriving these concepts for the $\mathbb{S}^3$ - sphere (which is what we really need them for) will then present no further difficulty.

**Definition 10** (Latitude and longitude)**.** Consider the unit sphere $\mathbb{S}^2$ defined by the point set $\{x_0^2 + x_1^2 + x_2^2 = 1\}$. The horizontal circles $x_0 = c$ with $-1 < c < 1$ are called latitudes. Similarly, the vertical circles that intersect the poles of the $\mathbb{S}^2$-sphere are called longitudes.

For $\mathbb{S}^3$ we define the latitudes to be the surfaces on which the $x_0$-coordinate is fixed. Geometrically these correspond to two-dimensional spheres in $\mathbb{R}^4$ satisfying

$$x_0 = c, \; x_1^2 + x_2^2 + x_3^2 = (1 - c^2), \; -1 < c < 1 \tag{5}$$

If we fix $x_0 = 0$ we obtain the intersection of $\mathbb{S}^3$ with $\mathbb{V}$ which is the plane we obtained in the discussion on stereographic projection. However, this is the $\mathbb{S}^2$-sphere $\{v_1^2 + v_2^2 + v_3^2 = 1\}$. We shall refer to this particular latitude as the *equator* of $\mathbb{S}^3$ and we use the notation $\mathbb{E}$. The equator has important applications and we will devote an entire subsection to it later.
Similarly, we define the longitudes to be the vertical circles that intersect the north pole $p = (1, 0, 0, 0)$. These intersections are unit spheres of two dimensions.
In particular, the latitudes on $\mathbb{S}^3$ are 2-spheres but the longitudes are ordinary circles (1-spheres).
When we later in Section 6.2 establish a bijection between $\mathbb{S}^3$ and the group $SU_2$ we shall return to these concepts. The projection of unit spheres of dimension $n+1$ onto a surface of dimension $n$ will present no additional difficulty to what we have encountered here if one disregards the geometry.

# 3 Linear groups

Linear groups are a special class of matrix groups that have important applications in many fields. The name of these groups suggests a connection between linear algebra and group theory and it will become clear that knowledge from both fields is required. To get a preview, consider the special unitary group of complex valued $2 \times 2$ - matrices, denoted $SU_2$. We will discover that conjugation of the elements of this group describes the rotation of the $\mathbb{S}^3$-sphere. Although these ideas might seem unclear now, we shall fully explain them in section 7. Before that we say a few words on the structure of this thesis after which we formally define linear groups.
The material covered in this paper follows closely the chapter on linear groups presented in [Art10]. In addition some parts of the same chapter in [Art91] have been implemented. Although we deviate somewhat in some sections to include some additional topics that we wished to present. When we construct the homomorphism $\gamma$ in section 7 the older version of Artins book will be used more frequently. Some of the subsequent results have been restated from how they are presented in [Art10] and [Art91]. In addition many of the proofs that Artin presents have been used as inspiration for how they are depicted in this thesis. In regards to all of this credits are due to Artin for his wonderful depiction of linear groups.
Definition 11 and 12 are the foundations for the rest of the subsequent material.

**Definition 11** (General linear group)**.** Let $\mathbb{F}$ be a field. The general linear group is the set

$$GL_n(\mathbb{F}) = \{P \in M_n(\mathbb{F}) \mid \det(\mathrm{P}) \neq 0\}$$

**Definition 12** (Linear groups)**.** Any subgroup $H$ of $GL_n(\mathbb{F})$ is called a linear group.

Extending these definitions permits us to list several examples of linear groups. In the subsequent section we will make frequent use of the fact that linear groups are subgroups of $GL_n(\mathbb{F})$.

## 3.1 Examples

**Definition 13** (The special linear group)**.** The special linear group is defined by the set

$$SL_n(\mathbb{F}) = \{P \in GL_n(\mathbb{F}) \mid \det(P) = 1\}$$

As promised, we shall verify that this is a group.

**Theorem 2.** *$SL_n(\mathbb{F})$ is a subgroup of $GL_n(\mathbb{F})$.*

*Proof.* Let $A, B \in SL_n(\mathbb{F})$, then, by definition

$$\det(A) = \det(B) = 1$$

By properties of the determinant we have that

$$\det(AB) = \det(A)\det(B) \tag{6}$$

Therefore, $\det(A)\det(B) \in SL_n(\mathbb{F})$. Next we need to verify that

$$\det(A)\det(B)^{-1} \in SL_n(\mathbb{F})$$

Now let $A, B \in SL_n(\mathbb{F})$ then $AB \in SL_n(\mathbb{F})$ and by Theorem 1, it immediately follows that $AB^{-1} \in SL_n(\mathbb{F})$. ☐

**Definition 14** (The orthogonal group)**.** The orthogonal group is the set

$$O_n(\mathbb{R}) = \{P \in GL_n(\mathbb{R}) \mid P^t P = I_n\}$$

where $I_n$ denotes the $n \times n$ identity matrix and $t$ is the transpose operator.

Applying the contents of definition 1 we can now prove the following result.

**Theorem 3.** The set $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$

*Proof.* Let $A \in O_n(\mathbb{R})$. Then $A^T A = I$, $\det(A)^2 = 1$ and $\det(A) \neq 0$. It follows that $A \in GL_n(\mathbb{R})$. Furthermore $I^T I = I \cdot I = I$. Henceforth $I \in O_n(\mathbb{R})$. Now if $A \in O_n(\mathbb{R})$ we have that $A^T A = I$, but then, $A^T = A^{-1}$. Hence $I = (A^{-1})^T A^{-1} = ((A^{-1})^T A^{-1})^T$ and $(A^{-1})^T A^{-1} = I$. So we conclude that $A^{-1} \in O_n(\mathbb{R})$.
We now invoke the subgroup test. Let $A, B \in O_n(\mathbb{R})$ we obtain

$$(A^{-1}B)^T(A^{-1}B) = B^T(A^{-1})^T A^{-1}B = B^T I B = B^T B = I$$

Hence $A^{-1}B \in O_n(\mathbb{R})$. It follows that $O_n(\mathbb{R}) \leqslant GL_n(\mathbb{R})$. ☐

By considering the intersection of the orthogonal group with the special linear group we define the special orthogonal group.

**Definition 15** (The special orthogonal group)**.** The special orthogonal group is the set

$$SO_n(\mathbb{R}) = \{P \in GL_n(\mathbb{R}) \mid P^t P = I_n,\ \det(P) = 1\}$$

where $I_n$ denotes the $n \times n$ identity matrix.

Since this set is constructed by taking the intersection of two groups that are subgroups of the general linear group it follows from [DF04, Ch.2.4, Prop.8] that $SO_n(\mathbb{R})$ is a group.

*Remark* 2. The geometric interpretation of orthogonal matrices is that of a rotation or a reflection. Since every orthogonal matrix has determinant $\pm 1$ it preserves the length of the linear transformation.

**Definition 16** (The symplectic group)**.** Given the matrix

$$S = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

the symplectic group is the set

$$SP_{2n}(\mathbb{R}) = \{P \in GL_{2n}(\mathbb{R}) \mid P^T SP = S\}$$

where the $I_n$ entries in S denote the identity matrix.

**Theorem 4.** The set $SP_{2n}(\mathbb{R})$ is a linear group.

*Proof.* Let $P \in SP_{2n}(\mathbb{R})$. Then, by definition, $P^T SP = S$. By taking the determinant on both sides of the last equality we have by Lemma 2 and the product rule for determinants that

$$\det(P^T)\det(S)\det(P) = \det(S)$$

Since $\det(S) = 1$ it follows that $\det(P)^2 = 1$ and so $P \in GL_{2n}(\mathbb{R})$. Furthermore

$$I^T SI = ISI = S$$

which proves that $I \in SP_{2n}(\mathbb{R})$. Now suppose $A \in SP_{2n}(\mathbb{R})$ then we have that

$$A^T SA = S$$

which is equivalent to

$$S = (A^{-1})^T SA^{-1}$$

Hence we conclude that $A^{-1} \in SP_{2n}(\mathbb{R})$. Finally, suppose $A, B \in SP_{2n}(\mathbb{R})$. Then

$$(A^{-1}B)^T S(A^{-1}B) = B^T(A^{-1})^T SA^{-1}B = B^T SB$$

Therefore $SP_{2n}(\mathbb{R}) \leqslant GL_n(\mathbb{R})$. □

# 4 The quaternions

We shall describe the quaternions algebraically using matrices. Quaternions consist of four dimensional numbers of the form

$$a + bi + cj + dk \tag{7}$$

where $i, j$ and $k$ are the unit vectors of the coordinate axis in $\mathbb{R}^4$. These are subject to the quaternion relations given by

$$\begin{aligned} \mathrm{i}^2 = j^2 = k^2 = -1, \mathrm{i}j = -j\mathrm{i} = k \\ jk = -kj = \mathrm{i}, k\mathrm{i} = -\mathrm{i}k = j, ijk = -1 \end{aligned} \tag{8}$$

*Remark* 3. The Hamilton relations were discovered by William Rowan Hamilton (1805-1865). Hamilton was studying multiplication of three-dimensional numbers in an attempting to extend the complex number system. He realized that the four-dimensional quaternions was the key to his success. Pleased with his discovery Hamilton carved the relations that he discovered $\mathrm{i}^2 = j^2 = k^2 = \mathrm{i}jk = -1$ into the bridge of Brougham.

## 4.1 Matrix representation of the quaternions

Jumping ahead we shall prove in section 6.2 that there exists a bijection between the elements of $SU_2$ and $\mathbb{S}^3$. By applying the form (16) it can be derived that the north pole of $\mathbb{S}^3$ is the identity matrix $I_2$. Furthermore, since $i = (0,1,0,0)$, $j = (0,0,1,0)$ and $k = (0,0,0,1)$ they are elements in $\mathbb{S}^3$ and, consequently, of $SU_2$. It follows from (16) that

$$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\ j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \text{and } k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Hence we have established the matrix representation of the quaternions. We now prove that these matrices constitute a four-dimensional real valued vector space with entries in $\mathbb{C}$.

**Theorem 5.** The set of matrices $\{i, j, k, I\}$ with coefficients in $\mathbb{C}$ are $\mathbb{R}$-linearly independent.

*Proof.* The $2\times 2$ - matrices with coefficients in $\mathbb{C}$ correspond to an 8-dimensional vector space in $\mathbb{R}$. We select the basis

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix} \right\}$$

If we express the elements $i, j, k$ and $I$ in the basis $\beta$ and put them as columns of a matrix $A$, we obtain

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

It can be verified by reducing the matrix to row echelon form that $A$ has rank 4 and hence attains maximal rank. Therefore the columns are linearly independent in $\mathbb{R}$.

□

*Remark* 4. The basis vectors in $\beta$ are not linearly independent over $\mathbb{C}$. in fact, $\beta$ has dimension 4 in this case.

Since $\{i, j, k, I\}$ are $\mathbb{R}$-linearly independent they span a real vector space $\mathbf{V}$ commonly denoted as the quaternion algebra.
When we later on construct a surjective homomorphism between $SU_2$ and $SO_3$ the basis elements of the quaternion algebra will display their usefulness. We end this section by defining a nonabelian subgroup of $GL_2(\mathbb{C})$ which is obtained by a natural extension of the quaternions.

**Definition 17** (The quaternion group)**.** The quaternion group $\mathbb{H}$ is the set consisting of the 8 matrices

$$Q_8 = \{\pm \mathrm{i}, \pm j, \pm k, \pm I\}$$

where

$$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \; j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We observe that

$$jk = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = kj$$

Hence $Q_8$ is nonabelian.
The proof that $\mathbb{H} \leqslant GL_2(\mathbb{C})$ is similar to that of the symplectic group hence we shall omit the details and instead present an idea of the required steps in conducting this proof.
Clearly $I \in \mathbb{H}$ and it is a matter of tedious, although simple matrix algebra to verify that the set $\{\pm \mathrm{i}, \pm j, \pm k, \pm I\}$ is closed under the group operation (multiplication). Since these matrices all have determinant 1 they are contained in $GL_2(\mathbb{C})$. Finally, proving that $\mathbb{H}$ is closed under the inversion map is just a matter of finding the correct inverse. This is straightforward but tedious and hence we omit the details. In either case, every element of $\mathbb{H}$ has an inverse that is also contained in $\mathbb{H}$ and so $\mathbb{H} \leqslant GL_2(\mathbb{C})$.

# 5 The rotation group $SO_2$

In section 7 we construct a surjective map between $SU_2$ and $SO_3$. Since this construction is important we shall illustrate the approach that is used by establishing a similar connection between the unit circle and the orthogonal matrices of $SO_2$. It turns our that there is a convenient way to represent these matrices as elements on the unit circle ($\mathbb{S}^1$).

## 5.1 The bijection between the unit circle and $SO_2$

Recall that the unit circle is formally defined by

$$\mathbb{S}^1 := x_0^2 + x_1^2 = 1 \tag{9}$$

By a suitable choice of parameters we can express the unit circle as a group. This is obtained by putting $(x_0, x_1) = (\cos\theta, \sin\theta)$ in (9). In fact, there is a natural embedding of the unit circle into $\mathbb{R}^{2\times 2}$ obtained by the map

$$(\cos\theta, \sin\theta) \hookrightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{10}$$

The unit circle is fundamentally connected with the rotation matrices of $SO_2$. This is the contents of the next lemma.

**Lemma 6.** There exists a bijective map

$$\Psi : \mathbb{S}^1 \longrightarrow SO_2(\mathbb{R})$$

*Proof.* Our goal is to construct a bijection between the unit sphere and the special orthogonal group consisting of $2 \times 2$ - matrices. Let

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in SO_2(\mathbb{R})$$

Since $A$ is an element of a special orthogonal group we require

$$AA^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2$$

together with $\det(A) = 1$.
Carrying out this matrix multiplication and comparing the coefficients with the identity matrix we obtain the system

$$\begin{cases} a^2 + c^2 = 1 \\ ab + cd = 0 \\ b^2 + d^2 = 1 \end{cases}$$

By putting $a = \cos\theta$ and $c = \sin\theta$ we obtain the unit circle. Similarly $b = \sin\phi$ and $d = \cos\phi$, also transforms the third equation into the unit circle. Plugging these values into the second equation we obtain

$$0 = ab + cd = \cos\theta \sin\phi + \sin\theta \cos\phi = \sin(\phi + \sin\theta)$$

Hence we see that

$$\sin(\phi + \theta) = 0$$

which reduces to

$$\phi = -\theta + k\pi \tag{11}$$

Substituting (11) into the matrix $A$ yields

$$A = \begin{bmatrix} \cos\theta & \sin(k\pi - \theta) \\ \sin\theta & \cos(k\pi - \theta) \end{bmatrix}$$

Now, if $k$ is even we have that $\sin(k\pi - \theta) = -sin(\theta)$ and $\cos(k\pi - \theta) = \cos(\theta)$. Therefore the matrix becomes

$$A = \begin{bmatrix} \cos\theta & -\sin(\theta) \\ \sin\theta & \cos(\theta) \end{bmatrix}, \ k \in \mathbb{Z}_{2n}$$

Conversely, if $k$ is odd we obtain $\sin(k\pi - \theta) = \sin(\theta)$ and $\cos(k\pi - \theta) = -\cos(\theta)$. Henceforth the matrix $A$ reduces to

$$A = \begin{bmatrix} \cos\theta & \sin(\theta) \\ \sin\theta & -\cos(\theta) \end{bmatrix}, \ k \in \mathbb{Z}_{2n+1}$$

But this matrix is not in $SO_2$ since $\det(A) \neq 1$. Hence we have a bijective map from the unit circle to $SO_2$ when $k \in \mathbb{Z}_{2n}$. □

With that out of the way we now turn our attention to the main topic of this thesis, the special unitary groups.

## 6 The special unitary group $SU_2$

**Definition 18** (The unitary group)**.** The unitary group is defined to be

$$UL_n(\mathbb{C}) = \{P \in GL_n(\mathbb{C}) \mid P^*P = I_n\}$$

where $*$ denotes the conjugate transpose of the matrix $P$.

**Theorem 6.** The set $UL_n(\mathbb{C})$ is a group.

*Proof.* Let $A, B \in UL_n(\mathbb{C})$. We make the observation that

$$(AB)(AB)^* = I_n$$

and hence $AB$ is unitary. Since $A^{-1} = A^*$ the result follows by copying the last step of the proof of Theorem 3. □

By taking the intersection of the unitary group with the special linear group we obtain the special unitary group.

**Definition 19** (The special unitary group)**.** The set

$$SU_n(\mathbb{C}) = \{P \in GL_n(\mathbb{C}) \mid P^*P = I_n,\ \det(\mathrm{P}) = 1\}$$

is called the special unitary group.

Since $SU_n(\mathbb{C})$ was constructed by taking the intersection of two subgroups of the general linear group it follows from [DF04, Ch.2.4, Prop.8] that $SU_2(\mathbb{C})$ is a subgroup of the general linear group.

It is worth investing some time to establish some properties of $SU_2(\mathbb{C})$. Since it is clear from the context that we are working with the complex field we shall simply write $SU_2$ in place of $SU_2(\mathbb{C})$ going forward. In fact, this group can be represented in several different ways.

### 6.1 Matrix representation

We propose the following.

**Proposition 2.** The elements of $SU_2$ are matrices P of the form

$$P = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \text{ with } a\bar{a} + b\bar{b} = 1 \tag{12}$$

*Proof.* We shall present a proof that follows closely the argument used in [Art10]. Let

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since $P \in SU_2$ the equations $P^* = P^{-1}$ and $\det(P) = 1$ must hold.
Furthermore, we have that

$$P^* = \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix} \tag{13}$$

Recall from linear algebra that the inverse of $P$ is given by the formula

$$P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{14}$$

Now, $P$ is an element of $SU_2$ and so it's determinant must be equal to 1. But $\det P = ad - bc$ and hence (14) yields

$$P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{15}$$

But the matrices (13) and (15) have to be equal and therefore $c = -\overline{b}$ and $d = \overline{a}$.
This proves the proposition. □

## 6.2 Unit sphere representation

When we later on describe the properties of the special orthogonal group $SO_3$ we will make use of the unit sphere representation of $SU_2$. Since $SU_2$ has complex entries we can put $a = x_0 + ix_1$ and $b = x_2 + ix_3$ in (12). We have that

$$P = \begin{bmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{bmatrix} \tag{16}$$

but then

$$\det P = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

which we recognize as the $\mathbb{S}^3$-sphere embedded into $\mathbb{R}^4$. This alternative representation of the special unitary group as points on the $\mathbb{S}^3$-sphere is clearly a bijective map. Hence the group $SU_2$ is isomorphic to the unit sphere $\mathbb{S}^3$ and the north pole $p = (1, 0, 0, 0)$ is mapped to $I_2 \in SU_2$
We require a few results from linear algebra before we develop the theory any further. The first result establishes a strong classification which holds for all unitary matrices.

**Theorem 7.** Suppose $\lambda$ is an eigenvalue of a unitary matrix $U$, then

$$|\lambda| = 1$$

which denotes the modulus of the eigenvalue.

*Proof.* Suppose $\lambda$ is an eigenvalue of the unitary matrix $U$ with associated eigenvector $v \neq 0$. Consistent with traditional notation we shall denote by $\langle , . , \rangle$ the inner product. Then by properties of inner product spaces we have that

$$\begin{aligned} \langle v, v \rangle &= \langle v, UU^* v \rangle = \langle Uv, Uv \rangle \\ \langle Uv, Uv \rangle &= \langle \lambda v, \lambda v \rangle \\ \langle \lambda v, \lambda v \rangle &= \lambda \langle v, \lambda v \rangle = \lambda \overline{\lambda} \langle v, v \rangle \end{aligned} \tag{17}$$

By assumption $v \neq 0$ and so

$$(1 - \lambda\overline{\lambda}) = 0$$

which reduces to

$$\lambda\overline{\lambda} = |\lambda|^2 = 1$$

Since $|\lambda|^2 \geq 0$ we can proceed by taking square roots. This completes the proof. $\square$

This is a very powerful result since it puts restrictions on every unitary matrix and its associated eigenvalues. We now apply Theorem 7 to prove the following lemma.

**Lemma 7.** ([Art10, Lem.9.3.4]).
The eigenvalues of P in (16) are complex conjugate numbers, except for the matrices $\pm I$. The eigenvalues of $P \in SU_2$ have modulus 1.

We derive a proof which is inspired by the techniques used in [Art10].

*Proof.* The characteristic polynomial of $P$ is given by

$$\det \begin{bmatrix} (x_0 + ix_1) - \lambda & x_2 + ix_3 \\ -x_2 + ix_3 & (x_0 - ix_1) - \lambda \end{bmatrix}$$

which reduces to

$$\lambda^2 - 2x_0\lambda + (x_0^2 + x_1^2 + x_2^2 + x_3^2) \tag{18}$$

Recall that $SU_2$ is isomorphic to $\mathbb{S}^3$-sphere and hence $(x_0^2 + x_1^2 + x_2^2 + x_3^2) = 1$. Therefore we have from (18) that the characteristic polynomial reduces to

$$\lambda^2 - 2x_0\lambda + 1 = 0$$

or equivalently

$$\lambda = x_0 \pm \sqrt{x_0^2 - 1} \tag{19}$$

Since $(x_0, x_1, x_2, x_3)$ is on the $\mathbb{S}^3$-sphere it must hold that $-1 \leq x_0 \leq 1$ and the result follows.
Since $SU_2 \leqslant U_2$ it follows from 7 that every matrix of $SU_2$ has eigenvalues of modulus 1. $\square$

Hence we are permitted to alternate between the matrix and vector representations of $SU_2$ which turns out to be quite useful. Our next objective is to admit a relationship between diagonal matrices and the complex matrices of $SU_2$. To achieve this we need the following theorem.

**Theorem 8** (Diagonalizability of $SU_2$)**.** Every matrix of $SU_2$ is diagonalizable. That is, for every $P \in SU_2$ there exists a $Q \in SU_2$ and a diagonal matrix $A$ such that $P = QAQ^*$.

*Proof.* From Lemma 7 we know that the eigenvalues of every element $P \in SU_2$ are complex conjugate numbers except for the special matrices $\pm I$. Furthermore Lemma 7 states that every matrix $P \in SU_2$ has complex conjugate eigenvalues except for $\pm I$. The eigenvalues admit distinct linearly independent eigenvectors. Then it follows from [FIS14, Ch.5, Thm.5.9] that $P$ is indeed diagonalizable. Finally, the matrices $\pm I$ are diagonal and hence diagonalizable for every $Q \in SU_2$.

□

The next theorem along with Theorem 10 is central to the theory that shall be developed in the subsequent sections.

**Theorem 9.** For any matrix $P \in SU_2$, $tr(P)$ is equal to the sum of the eigenvalues. Furthermore, if the degree one term of the characteristic polynomial of the matrix P is zero, then $tr(P) = 0$.

*Proof.* For every $P \in SU_2$ we know from Theorem 8 that there exists a matrix $Q \in SU_2$ and a diagonal matrix $A$ such that

$$P = Q^* \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} Q, \text{ where } A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

According to Lemma 7, $\lambda_1 = \overline{\lambda}_2$. Then by Lemma 3 we have that

$$tr(P) = tr(Q^* A Q) = tr(A) = \lambda_1 + \lambda_2$$

Furthermore, note that the characteristic polynomial of $P$ is a degree-two polynomial in $\mathbb{C}$ with roots $\lambda_1$ and $\lambda_2$. Hence we obtain

$$\begin{aligned} \det(P - \lambda I) &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \\ &= \lambda^2 - (tr(P))\lambda + \det(A) \end{aligned} \tag{20}$$

If the degree 1 term is zero then $tr(P) = 0$. In particular, the trace of $P$ is the sum of the eigenvalues. □

Note that the diagonal matrix $A$ can be obtained by conjugation of any matrix $P \in SU_2$. This is obtained by

$$QPQ^* = A$$

where $Q$ and $Q^*$ in Theorem 9 have been interchanged.
We make some general observations regarding the matrix $A$ before we proceed. Now recall from Theorem 9 that

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \ \lambda_1\lambda_2 = 1$$

Then by Lemma 7, $\lambda_1 = \overline{\lambda}_2$ and the notation in (16) we have that $A$ is of the form

$$T = \left\{ \begin{bmatrix} x_0 + ix_1 & 0 \\ 0 & x_0 - ix_1 \end{bmatrix} \middle| \; x_0^2 + x_1^2 = 1, \; x_2 = x_3 = 0 \right\} \tag{21}$$

Recalling the definition we recognize the subset $T$ as the particular longitude described geometrically by unit circles intersecting the north and south pole of $\mathbb{S}^3$.

**Theorem 10.** The subset $T$ is a subgroup of $SU_2$.

*Proof.* If we put $x_0 = 1$ and $x_1 = 0$ in (21) we obtain the identity matrix. Hence $I \in T$. Futhermore, let $A, B \in T$, then

$$A = \begin{bmatrix} x_0 + ix_1 & 0 \\ 0 & x_0 - ix_1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} x_2 + ix_3 & 0 \\ 0 & x_2 - ix_3 \end{bmatrix}$$

Now the condition invoked on $T$ yields $x_0^2 + x_1^2 = 1$ and $x_2^2 + x_3^2 = 1$. We obtain

$$\begin{aligned} AB &= \begin{bmatrix} x_0 + ix_1 & 0 \\ 0 & x_0 - ix_1 \end{bmatrix} \begin{bmatrix} x_2 + ix_3 & 0 \\ 0 & x_2 - ix_3 \end{bmatrix} \\ &= \begin{bmatrix} (x_0 + ix_1)(x_2 + ix_3) & 0 \\ 0 & (x_0 - ix_1)(x_2 - ix_3) \end{bmatrix} \end{aligned} \tag{22}$$

Taking the determinant of (22) yields

$$(x_0^2 + x_1^2)(x_2^2 + x_3^2) = 1$$

So $AB \in T$ and since the product $AB$ is symmetric, it is commutative (in particular, diagonal matrices commute). Therefore $BA \in T$ and $T$ is closed under multiplication.
Let $A$ be the matrix we defined above. It follows by applying the inverse formula (14) that

$$A^{-1} = \begin{bmatrix} x_0 - ix_1 & 0 \\ 0 & x_0 + ix_1 \end{bmatrix}$$

Clearly $A^{-1} \in T$ and if we replace $B$ by $A^{-1}$ in (22) the theorem follows. $\square$

### 6.3 The conjugacy classes of $SU_2$

It will become apparent in the next section that the conjugacy classes of $SU_2$ can be used to identify the matrices of the special orthogonal group $SO_3$. Henceforth some time will be devoted to classify these latitudes. The next theorem contains some of the contents of [Art10, Prop.9.3.5]. However we shall derive the center of $SU_2$ and the conjugacy classes of the matrices $\{\pm I\}$ later.

**Proposition 3.** [Art10, Prop.9.3.5] The latitudes in $SU_2$ are conjugacy classes. In particular, for a given $-1 < c < 1$ the latitude defined by $x_0 = c$ are the matrices $P$ of $SU_2$ with $tr(P) = 2c$.

*Proof.* By Lemma 7 we have that the characteristic polynomial of a matrix $P \in SU_2$ is given by

$$\lambda^2 - 2x_0\lambda + 1 = 0 \tag{23}$$

Furthermore, Theorem 9 tells us that the trace of $P$ is the sum of the eigenvalues. In particular, for a given $x_0 = c$, $tr(P) = 2x_0 = 2c$. We know that the eigenvalues are complex conjugate numbers with modulus 1 and that the trace is completely determined by the choice of latitude. By altering the value of $-1 < c < 1$ we obtain a different pair of complex conjugate eigenvalues and, consequently, a matrix that lies on a different latitude.
Hence any given latitude $x_0 = c$ contains the matrices $P \in SU_2$ such that $tr(P) = \lambda + \overline{\lambda} = 2c$ where $\lambda$ and $\overline{\lambda}$ are the eigenvalues of $P$.
Furthermore, every matrix $P \in SU_2$ is contained in exactly one such latitude. To conclude the proof we need to verify that every latitude $x_0 = c$ contains every $P \in SU_2$ with $tr(P) = 2c$. But this follows directly from Theorem 9 and the following observation.
Recall that any matrix $P \in SU_2$ with eigenvalues $\lambda$ and $\overline{\lambda}$ can be written in the form $P = QAQ^*$ where

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}$$

Since $P$ is obtained via conjugation in this way, the matrices $P$ and $A$ lie in the same conjugacy class of $SU_2$, determined by $x_0 = c$. In particular, the matrix $P$ was arbitrary, and hence every $P \in SU_2$ with eigenvalues $\lambda$ and $\overline{\lambda}$ can be obtained by conjugating $A$ with some $Q \in SU_2$. This proves the theorem. □

*Remark* 5. For the matrices $\{\pm I\}$ the element $A$ in Proposition 3 corresponds to $I$ and $-I$, respectively. Hence the north and south pole of $\mathbb{S}^3$, determined by $c = 1$ and $c = -1$, contain only the matrices $\{I\}$ and $\{-I\}$.

*Remark* 6. Given the form (5) we observe that the latitudes of $SU_2$ correspond to 2-dimensional spheres in $\mathbb{R}^4$.

Conjugation of matrices correspond to orbits which we denote by

$$SU_2 \cdot P = \{QPQ^{-1} \mid Q \in SU_2\}$$

The orbits of $SU_2$ are determined by its conjugacy classes. In particular, the orbits of $SU_2$ define a partition of the group where each subset of the partition contains the matrices with the same trace. Since the orbits and conjugacy classes are the same, conjugation defines a transitive group action operating on the latitudes of $SU_2$.

## 6.4 The equator

We recall that the equator of (16) is given by the set

$$\mathbb{E} = \{x_0, x_1, x_2, x_3 \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 = 1,\ x_0 = 0\} \tag{24}$$

We have established that every matrix of $SU_2$ sits in a unique latitude on $\mathbb{S}^3$. In particular, any point on the equator satisfies $x_0 = 0$ and by inserting this into (16) we obtain the matrix form of an element on the equator. Namely,

$$Q = \begin{bmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{bmatrix} \tag{25}$$

If we take the trace of $Q$ we obtain

$$tr(Q) = ix_1 + (-ix_1) = 0 \tag{26}$$

Hence a necessary condition for any matrix $Q \in SU_2$ on the equator is that its trace must be equal to zero. This result will turn out to be useful later when we derive a representation of the special rotation group $SO_3$. Before tending to this we need the following proposition which contains some of the theory presented in [Art10, Prop.9.3.8]. However we do not make use of the last part of this proposition and given economy of space we will not include it.

**Proposition 4.** For a matrix $P \in SU_2$ the following are equivalent.
$(i)$ $P$ is on the equator,
$(ii)$ $tr(P) = 0$,
$(iii)$ The eigenvalues of $P$ are $i$ and $-i$.

*Proof.* We proved that $(i)$ implies $(ii)$ in the preceding section. For the reverse implication note that if $tr(P) = 0$ then the sum of the eigenvalues is zero by Theorem 9. Let $\lambda = a + ib$ then by Lemma 7 it follows that

$$0 = a + ib + a - ib = 2Re(\lambda)$$

Hence the eigenvalues are purely imaginary and by (19) we have that $x_0 = 0$.
Suppose that $(iii)$ holds then the eigenvalues of P are $\pm i$. Recall from Theorem 9 that the trace of $P$ is the sum of the eigenvalues and hence $tr(P) = i+(-i) = 0$ so $(iii)$ implies $(ii)$.
For $(iii)$ let $P \in SU_2$ and suppose that $(ii)$ holds. Then again by Theorem 9 the characteristic polynomial is given by

$$\lambda^2 + 1 \tag{27}$$

which clearly has eigenvalues $i$ and $-i$. So $(ii)$ implies $(iii)$.
Finally, suppose that $(iii)$ holds, then the eigenvalues of $P$ are $\pm i$. Since $P \in SU_2$ it is diagonalizable by Theorem 8. Hence we have that the characteristic polynomial of $P$ is a degree two polynomial. It follows that it is of the form

$$(\lambda + i)(\lambda - i)$$

which clearly reduces to (27). $\square$

We end this section with the observation that the matrices contained in the equator are completely determined by the conjugacy class $x_0 = 0$.

# 7 The rotation group $SO_3$

Our goal in this section is to establish a connection between the elements of $SU_2$ and the orthogonal matrices of $SO_3$. Our strategy for achieving this goal will be devoted to constructing a surjective homomorphism from $SU_2$ to a morphism (structure preserving map) on $\mathbb{E}$ to itself. As it turns out this map describes the representation of elements in $SU_2$ as real-valued orthogonal matrices with determinant 1.
Now consider the conjugation action of a matrix on the equator 24 by a matrix $P \in SU_2$. As the theory of this section unfolds it will become apparent that conjugation actually operates on the latitudes of $\mathbb{S}^3$ by rotating them.

## 7.1 The orthogonal representation

Let $\gamma$ be the map

$$\gamma : SU_2 \longrightarrow \{f : \mathbb{E} \longrightarrow \mathbb{E}\}$$

where

$$p \mapsto \gamma_P$$

defined by

$$\gamma_P(U) = PUP^*$$

with $U \in \mathbb{E}$ and $f$ a morphism from the equator to itself.
We recall from section 6.4 that the equator of $SU_2$ is $\mathbb{S}^2$, embedded into $\mathbb{R}^4$.
Note that if $P \in \mathbb{E}$ Proposition 4 tells us that $tr(P) = 0$. In particular, every matrix on the equator is skew-Hermitian and the map $\gamma$ preserves this property. This is the contents of the next lemma.

**Lemma 8.** Any matrix $Q \in \mathbb{E}$ is Skew-Hermitian. In particular, the matrix $QPQ^*$ with $P \in SU_2$ is skew-Hermitian.

*Proof.* if we take the conjugate transpose of (25) we obtain

$$Q^* = \begin{bmatrix} -ix_1 & -x_2 - ix_3 \\ x_2 - ix_3 & ix_1 \end{bmatrix}$$

Henceforth $Q^* = -Q$ and we conclude that $Q$ is skew-Hermitian.
Next we verify that $(P^*QP)^*$ is skew-Hermitian. Since $Q$ is skew-Hermitian we have that

$$(P^*QP)^* = P^*Q^*P = (P^*(-Q)P) = -(P^*QP)$$

and hence $(P^*QP)^*$ is also skew-hermitian. $\square$

**Lemma 9.** For any matrix $P \in \mathbb{E}$ it holds that

$$tr(QPQ^*) = 0$$

*Proof.* Pick an element $Q \in \mathbb{E}$. Then, by definition, $tr(P) = 0$. Furthermore, let $P$ be any element in $SU_2$ not necessarily on $\mathbb{E}$. Then by Lemma 3 we obtain

$$tr(QPQ^*) = tr(Q^*QP) = tr(P)$$

This completes the proof. $\square$

**Lemma 10.** The conjugation $PUP^*$ preserves the determinant for every matrix $P \in SU_2$ and every $U \in \mathbb{E}$.

*Proof.* Since we have that $P \in SU_2$, $\det(U) = 1$. By associativity and Theorem 1 we obtain

$$\det(PUP^*) = \det(P)\det(U)\det(P^*) = 1$$

$\square$

Before we prove one of the major results of this thesis we require the contents of another theorem which states that the skew-Hermitian matrices of trace zero form a real vector space. In fact, this vector space contains the three elements that define the quaternion relations (8).

**Theorem 11.** The $2 \times 2$ skew-Hermitian matrices that have trace zero define a vector space with dimension three over $\mathbb{R}$.

*Proof.* Let $P$ be the matrix with complex entries given by

$$P = \begin{bmatrix} x_0 + ix_1 & x_4 + ix_5 \\ x_2 + ix_3 & x_6 + ix_7 \end{bmatrix}$$

Now $tr(P) = 0$ implies that $x_0 = -x_6$ and $ix_1 = -ix_7$. Inserting this into the above expression for $P$ yields

$$P = \begin{bmatrix} x_0 + ix_1 & x_4 + ix_5 \\ x_2 + ix_3 & -x_0 - ix_1 \end{bmatrix}$$

By conjugation we obtain

$$P^* = \begin{bmatrix} x_0 - ix_1 & x_2 - ix_3 \\ x_4 - ix_5 & -x_0 + ix_1 \end{bmatrix}$$

On the other hand, $P$ is skew-Hermitian and hence

$$P^* = \begin{bmatrix} x_0 - ix_1 & x_2 - ix_3 \\ x_4 - ix_5 & -x_0 + ix_1 \end{bmatrix} = \begin{bmatrix} -x_0 - ix_1 & -x_4 - ix_5 \\ -x_2 - ix_3 & x_0 + ix_1 \end{bmatrix} = -P$$

For this equation to hold we must have that $x_0 = 0$, $x_2 = -x_4$ and $x_3 = x_5$. Putting all of this together we conclude that

$$P = \begin{bmatrix} ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & -ix_1 \end{bmatrix}$$

from which it is clear that $P \in \mathbb{E}$.
If we now let $x_1 = 1$, $x_2 = 1$ and $x_3 = 1$ in $P$ and recall the quaternion relations (8) we obtain

$$S = \begin{bmatrix} i & -1+i \\ 1+i & -i \end{bmatrix} = i - j + k$$

Note that we have expressed the matrix $S$ as a linear combination of the vectors $i, j$ and $k$. We conclude that $S$ (and consequently, $P$) are contained in the vector space defined by the basis

$$B = \left\{ \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}$$

We established above that these matrices are $\mathbb{R}$-linearly independent. Hence the set of skew-Hermitian, trace-zero matrices span a vector space of dimension 3 over $\mathbb{R}$. □

*Remark* 7. These matrices are actually linearly independent over $\mathbb{C}$ . However, we are attempting to construct a map whose image is $\mathbb{R}$ and hence this fact will be of little use.

We denote by $\mathbb{V}'$ the three-dimensional vector space of skew-Hermitian, trace-zero matrices. It will become clear that the equator $\mathbb{E}$ is the unit sphere contained in this space. Clearly, $\gamma$ operates on the whole of the vector space $\mathbb{V}'$ by the results of Lemma 8 and Lemma 9 and it maps matrices in $\mathbb{V}'$ to other matrices in $\mathbb{V}'$.
Denote by $\gamma_P(U)$ the image of the matrix $U$ obtained by applying the map $\gamma$ by means of conjugation. We proved that $\mathbb{V}'$ has dimension 3 above and therefore it is isomorphic to $\mathbb{R}^3$ (the basis elements in $B$ are the unit vectors of $\mathbb{R}^3$). Moreover, the map $\gamma$ is a linear operator (we shall prove this shortly) from $\mathbb{V}'$ to itself. From all of these deliberations we conclude that the image $\gamma_P(U)$ must be a matrix of dimension 3 with entries in $\mathbb{C}$.

*Remark* 8. It is possible to write down the matrix of $\gamma$ explicitly which is a more efficient way of verifying that it represents a rotation in $\mathbb{R}^3$. However, this calculation is neither particularly enlightening or necessary for our cause and therefore we will not give it.

The classifications that we have established concerning skew-Hermitian, trace-zero matrices will now be applied when we state one of the major results of this thesis. In Theorem 12 we apply the results above to prove the existence of a surjective homomorphism between $SU_2$ and the set of isomorphisms from the vector space $\mathbb{V}'$ to itself, denoted $\mathrm{Isom}(\mathbb{V}', \mathbb{V}')$.

*Remark* 9. The vector space $\mathbb{V}'$ is not a group under multiplication. Hence we have to be careful about how we construct the homomorphism $\gamma$. The necessary adjustment is made in the following theorem.

**Theorem 12.** The map

$$\gamma : SU_2 \longrightarrow GL_3(\mathbb{V}')$$

is a surjective homomorphism with kernel $\{\pm I\}$.

*Proof.* Let $P \in SU_2$ and let $U, V \in \mathbb{V}'$. Our goal is to prove that $\gamma : SU_2 \longrightarrow \mathrm{Isom}(\mathbb{V}', \mathbb{V}')$ is a group homomorphism. We begin by verifying that $\gamma_P$ is linear. We obtain

$$\begin{aligned}\gamma_P(U+V) &= P(U+V)P^* \\ &= PUP^* + PVP^* \\ &= \gamma_P(U) + \gamma_P(V)\end{aligned}$$

and

$$\gamma_P(c(U)) = P(c(U))P^* = cPUP^* = c\gamma_P(U)$$

for some constant $c$.
Next we need to check that $\gamma_P$ is injective. Suppose that

$$PUP^* = PU'P^*$$

for some $U, U^* \in \mathbb{V}'$. Then we find that

$$P^*PUP^* = P^*PU'P^*$$

which yields

$$UP^*P = U'P^*P$$

so $U = U'$. Moreover, we know that $\gamma_P$ acts transitively on the elements of $\mathbb{V}'$. Hence for every pair of matrices $U, V \in \mathbb{V}'$ there exists a matrix $P \in SU_2$ such that $PVP^* = U$ therefore $\gamma_P$ is surjective and hence it is an isomorphism.
Now let $P, P' \in SU_2$ then we have that

$$\begin{aligned}\gamma_P\gamma_{P'}(U) &= PP'U(P')^*(P)* \\ &= (PP')U(PP')^* = \gamma_{PP'}U\end{aligned} \tag{28}$$

This proves that $\gamma$ is a group homomorphism.
For the second part of the theorem we apply a technique used in the proof of [Art91, Ch.8, Lem 3.15] to verify that the kernel contains nothing but the identity matrix and its negative counterpart.
Recall the kernel of a homomorphism, denoted ker($\gamma$). For $\gamma$ this is given by the set

$$ker(\gamma) = \{P \in SU_2 \mid PUP^* = U, \text{ for all } U \in \mathbb{V}'\}$$

This condition is equivalent to $PU = UP$. Hence the kernel consists of all elements in $SU_2$ that commute with every skew-Hermitian, trace zero matrix. We established in Theorem 11 that these matrices are given by the basis

$$B = \left\{ \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}$$

Recalling the general form (12) of an element in $SU_2$ we conjugate the basis for $\mathbb{V}'$ to determine the kernel of the homomorphism $\gamma$. Carrying out these calculations establishes that the only possible entries for $P$ are $a = \overline{a}$ and $b = 0$.

Hence the matrices $P \in SU_2$ that commute with every element of $\mathbb{V}'$ are the diagonal matrices

$$P = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix},\ a \in \mathbb{R} \right\}$$

Since we must have $a = \overline{a}$, the only options for the kernel are the matrices $\{\pm I\}$ and given that the kernel is a subgroup of $SU_2$, we must have that $\pm I \in ker(\gamma)$. In particular, $ker(\gamma) = \{\pm I\}$ for these matrices were the only possibilities. Hence the theorem has been proved. □

*Remark* 10. The matrices of the kernel are precisely those matrices of $SU_2$ that commute with every other matrix of $SU_2$. This set is the center of $SU_2$ defined by

$$Z(SU_2) = \{P \in SU_2 \mid PQ = QP, |\text{for all } Q \in SU_2\}$$

Hence the kernel of the homomorphism $\gamma$ is the center of $SU_2$.

Since we have obtained the kernel of the homomorphism we may write down its cosets, explicity. Recall that the action of taking cosets partitions the elements of a group into disjoint subsets. In particular, since $ker(\gamma) = \{\pm I\}$ the associated cosets are given by $\{\pm P\}$. Hence the map $\gamma$ associates every matrix in $\mathbb{V}'$ with a pair of matrices $\pm P$ in $SU_2$.
These pairs of matrices $\{\pm P\}$ are called antipodal points of the equator. For any given conjugacy class ($x_0 = c$) they can be identified by placing a line segment at the center of the latitude (a sphere) and identifying the points of intersection with its boundary. In topology, this construction is called a double covering.
Before we close out this section with two final theorems let us summarize what we have learned so far.
We have a map $\gamma : SU_2 \longrightarrow \text{Isom}(\mathbb{V}', \mathbb{V}')$ which operates on the matrices of $\mathbb{V}'$ by conjugation. We proved that this map is a surjective homomorphism and that its kernel is given by $Z(SU_2)$.
Now recall the contents of Theorem 11 which tells us that $\mathbb{V}'$ is a real-valued vector space of dimension 3. Since the equator is the unit sphere in this space $\gamma$ preserves vectors of length 1 by Lemma 10. Furthermore, Theorem 12 established that the map

$$\begin{aligned} \gamma_P &: \mathbb{V}' \longrightarrow \mathbb{V}' \\ U &\longrightarrow PUP^* \end{aligned} \tag{29}$$

is linear. In particular, the vector space $\mathbb{V}'$ is isomorphic to $\mathbb{R}^3$ and hence $SU_2 \longrightarrow GL_3(\mathbb{R})$ is a group homomorphism by Theorem 12. We are now ready to prove that the image of this map is contained in the special orthogonal group.

**Theorem 13.** [Art91, Ch.8, Lem.3.13] For $P \in SU_2$ and $U \in \mathbb{V}'$ the image $\gamma_P(U) \in SO_3$. Therefore $P \mapsto \gamma_P(U)$ defines a homomorphism $SU_2 \longrightarrow SO_3$.

*Proof.* We will adopt the techniques used [Art91]. In addition, some steps of the proof for [Art10, Lem.9.4.4] have also been included.
Our strategy will be to invoke the properties of the ordinary dot product on

the vector space $\mathbb{R}^3$ (since $\mathbb{V}'$ is isomorphic to $\mathbb{R}^3$, this is perfectly viable). Note that any element $V \in \mathbb{V}'$ can be written as a linear combination of the basis vectors $i, j$ and $k$. That is if $V = (v_1, v_2, v_3) \in \mathbb{V}'$ then it has the form $V = iv_1 + jv_2 + kv_3$. Now let $U = u_1 i + u_2 j + u_3 k$ be some other element in $\mathbb{V}'$. We define $\langle U, V\rangle$ as the bilinear form (linear in each component)

$$\langle U, V\rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{30}$$

We compute $UV$ in $\mathbb{R}^3$ and together with the quaternion relations (8) we obtain

$$\begin{aligned} &UV = (iu_1 + ju_2 + ku_3)(iv_1 + jv_2 + kv_3) \\ &= -(u_1 v_1 + u_2 v_2 + u_3 v_3)I + U \times V \end{aligned} \tag{31}$$

The notation $U \times V$ defines the cross product for vectors in $\mathbb{R}^3$ which is given by the formula

$$U \times V = (u_2 v_3 - u_3 v_2)i + (u_3 v_1 - u_1 v_3)j + (u_1 v_2 - u_2 v_1)k$$

But then from Theorem 11 we have that $i, j$ and $k$ all have trace zero (they are elements of $\mathbb{V}'$) and so

$$tr(U \times V) = 0$$

Now since $tr(I) = 2$ we have from (31) that

$$\langle U, V\rangle = -\frac{1}{2} tr(UV) = -2(u_1 v_1 + u_2 v_2 + u_3 v_3) \tag{32}$$

Hence given any matrices $Q, U \in \mathbb{V}'$ we find from (32) that

$$\begin{aligned} &\langle PQP^*, PUP^*\rangle = -\frac{1}{2} tr(PQP^* PUP^*) \\ &= -\frac{1}{2} tr(PQUP^*) = -\frac{1}{2} tr(QU) = \langle Q, U\rangle \end{aligned}$$

Therefore the dot product in $\mathbb{R}^3$ is invariant under the linear transformation $\gamma_P$ and hence $\gamma_p(U) \in O_3$.
To complete the proof we need to verify that the image $\gamma_P(U)$ has determinant 1. We know that every orthogonal matrix has determinant $-1$ or 1 by Lemma 4. So all we have to do is rule out the value $-1$.
Given that $SU_2$ is a sphere a theorem from topology asserts that it is path-connected (any curve lying entirely on $\mathbb{S}^3$ can be continuously contracted into a point contained in $\mathbb{S}^3$). Moreover, the determinant is a continuous function and hence can only attain one of the values $\pm 1$. Clearly $\gamma_p(I_2) = I_3 \in O_3$ which has determinant 1 and we conclude that every matrix $P \in O_3$ has determinant 1. It follows that $\gamma_p(U) \in SO_3$. □

Our final theorem of this section verifies that the image of the map $\gamma$ contains nothing more than the matrices of $SO_3$.

**Theorem 14.** [Art91, Ch.8, Lem.3.16]
The image $\gamma_P(U)$ of the homomorphism $\gamma$ is equal to $SO_3$.

*Proof.* We shall follow the proof of [Art91].
Recall the subgroup $T$ of diagonal matrices that we defined in Theorem 10. Let $P \in T$ with diagonal entries $\lambda$ and $\overline{\lambda}$ and let $U \in \mathbb{V}'$ ($z = x_3 + ix_4$). Then $PUP^*$ is given by

$$\begin{aligned} &\begin{bmatrix} a & 0 \\ 0 & \overline{a} \end{bmatrix} \begin{bmatrix} ix_2 & z \\ -\overline{z} & -ix_2 \end{bmatrix} \begin{bmatrix} \overline{a} & 0 \\ 0 & a \end{bmatrix} \\ &= \begin{bmatrix} ix_2 & a^2 z \\ -\overline{a}^2 \overline{z} & -ix_2 \end{bmatrix} \end{aligned} \tag{33}$$

We note that conjugation of elements of $T$ fixes the first coordinate in the first vector. Moreover, $P$ was assumed to be an element of $T$ and hence $|a| = 1$. But then we can make the substitution $a = e^{i\theta}$ which yields $a^2 = e^{2i\theta}$. The matrix $\gamma_P(U)$ then becomes

$$\begin{bmatrix} ix_2 & (e^{2i\theta})z \\ -(e^{2i\theta})\overline{z} & -ix_2 \end{bmatrix}$$

which we identify as a rotation of the z-plane by an angle of $2\theta$. The set consisting of these rotations about the origin is a subgroup of $SU_2$. This is obvious by the form of the matrix $\gamma_p(U)$ in (33).
In fact, these rotations are about the point $(1, 0, 0)$ which is the unit sphere in $\mathbb{V}'$. This is realized by expressing $\gamma_P(U)$ as an element of $SO_3$. We apply the basis $B$ of $\mathbb{V}'$ to the matrix $\gamma_P(U)$ and obtain

$$\gamma_P(U) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{bmatrix}$$

From this depiction it is clear that $\gamma_P(U) \in SO_3$ contains the subgroup of all these rotations. We shall denote this subgroup of rotations about $(1, 0, 0)$ by $K$. Observe that the point $(1, 0, 0)$ in the basis for $\mathbb{V}'$ is given by matrix

$$i = \mathbb{E} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Since this matrix represents the equator in $\mathbb{V}'$ by Proposition 4 it corresponds to a conjugacy class. We know that $SU_2$ acts on it transitively. In particular, there exists a matrix $Q \in SU_2$ such that $Q\mathbb{E}Q^* = U$ since $U \in \mathbb{V}'$.
Note that the subgroup $\gamma_p(U)K\gamma_p(U)^*$ is in the image of the map $\gamma$ by Lemma 8. Furthermore, every element of $SO_3$ is a rotation and hence the conjugation action contains every element in the image $\gamma_P(U)$. So there is nothing else in the image of $\gamma$ and hence $Im(\gamma) = SO_3$. □

We have successfully constructed a map that describes the complex matrices of $SU_2$ as rotations about the point $(1, 0, 0)$ in $\mathbb{R}^3$. In defining the vector space

$\mathbb{V}'$ we have inadvertently shown (Theorem 11) that the group $SU_2$ is contained inside the quaternions (Section 4).
By explicitly writing down the coordinate representation of $i, j$ and $k$ in $\mathbb{R}^3$ we see that the basis vectors of $\mathbb{V}'$ are the set of unit vectors contained in three-dimensional space.

# 8 The one parameter groups

## 8.1 Differentiable homomorphisms

We now turn to a special class of linear groups called the one-paramter groups. These consist of differentiable homomorphisms defined to be matrix valued functions. For $A \in M_n(\mathbb{C})$ consider the series

$$e^{tA} = I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + ... \tag{34}$$

which converges to a matrix in $M_n(\mathbb{C})$.

*Remark* 11. One should verify that the series (34) converges. This is proved in introductory courses on differential equations. We will not pursue the theory of differential equations any further. The only thing we need to recall is that the derivative of a power series can be obtained by means of differentiating term by term. The details are presented in [AB19, Ch.2.1, Lem.3].

As it turns out the series (34) can be used to define a special type of of linear groups called the one-parameter groups. Before we derive some general facts regarding these groups we do have to define them properly.

**Definition 20.** A one-parameter group is a differentiable homomorphism defined for one of the following two maps

$$\begin{aligned} \mathbb{R} &\longrightarrow GL_n(\mathbb{R}) \\ \mathbb{R} &\longrightarrow GL_n(\mathbb{C}) \end{aligned} \tag{35}$$

Since (34) is a convergent power series we obtain its derivative by differentiating term-wise hence obtaining the series

$$0 + A + A^2t + \frac{t^2A^3}{2!} + ... = Ae^{tA} \tag{36}$$

**Theorem 15.** ([Art10, Thm.9.5.2]).
(a) Let A be any real or complex matrix and consider $GL_n(\mathbb{F})$ where F is $\mathbb{R}$ or $\mathbb{C}$. Then the map $\psi : \mathbb{R}^+ \to GL_n(\mathbb{F})$ defined by $\psi(t) = e^{tA}$ is a homomorphism.

(b) Let $\psi : \mathbb{R}^+ \to GL_n(\mathbb{F})$ be a differentiable map that is a homomorphism and denote by A its derivatives $\psi'(0)$ centered at the origin. Then $\psi(t) = e^{tA}$ for all t.

*Proof.* We present a proof which is similar to that of [Art10].
When $s, r \in \mathbb{R}$ it is certainly true that

$$e^{(r+s)A} = e^{rA}e^{sA}$$

which proves that $\psi(t)$ is a homomorphism.
Let $\psi : \mathbb{R}^{+} \longrightarrow GL_n(\mathbb{F})$ be a differentiable homomorphism. Then we have that

$$\psi(\Delta t + t) = \psi(\Delta t)\psi(t)$$

and if we let $\Delta t \to 0$ we obtain $\psi(t) = \psi(0)\psi(t)$. From this we have that

$$\frac{\psi(\Delta t + t) - \psi(t)}{\Delta t} = \frac{\psi(\Delta t)\psi(t) - \psi(0)\psi(t)}{\Delta t} = \frac{\psi(\Delta t) - \psi(0)}{\Delta t}\psi(t)$$

Given the definition this last limit is equal to $A$ and hence the above equation reduces to

$$\psi'(t) = \psi'(0)\psi(t) = A\psi(t)$$

if we let $\Delta t \to 0$. Therefore $\psi(t)$ solves the equation $\psi' = A\psi$. In particular, the function $e^{tA}$ solves the differential equation and both of these solutions equal the identity matrix $I$ for $t = 0$. A result obtained by putting $t = 0$ in (34). To derive that $\psi(t) = e^{tA}$ for all values of $t$ we have to invoke a result from the theory of differential equations. We will not pursue this any further since we wish to maintain an algebraic standpoint. Nevertheless, the details can be found in [AB19, Ch.1, Thm.1].
By this theorem we know that a first-order, linear differential equation with an initial condition has a unique solution. But then $e^{tA}$ is the only solution to the equation $\psi' = A\psi$. □

**Example 2.** Recall the bijective map $\Psi : \mathbb{S}^1 \longrightarrow SO_2$ that was constructed in Lemma 6. Using the matrix (10) we shall verify that $SO_2$ is a one-parameter subgroup of $GL_2(\mathbb{R})$. We define the map $\psi$ in the following way;

$$\psi(\theta) = e^{\theta A},\ \theta \in \mathbb{R}$$

Our goal is to construct a differentiable homomorphism $\psi : \mathbb{R} \longrightarrow SO_2$. We assign

$$e^{\theta A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in GL_2(\mathbb{R})$$

and recall the derivative of a matrix $M$ which is obtained by differentiating the elements of $M$ term by term. Applying this to $e^{tA}$ yields

$$\frac{d}{d\theta}[e^{\theta A}] = Ae^{\theta A} = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

By the notation of Theorem 15, we have that $A = \psi'(0)$ and so

$$A = \frac{d}{d\theta}\left[e^{\theta A}(0)\right] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Furthermore, note that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

Hence the matrix $e^{\theta A}$ is as solution of the differential equation $\psi'(\theta) = A\psi(\theta)$. By Theorem 15 it is the only solution.
It is immediately clear that the two matrices $\theta A, \rho A \in SO_2$ commute and we verify directly that our assigned map defines a homomorphism. Being familiar with the rules of matrix multiplication we find that

$$\begin{aligned} \psi(\theta)\psi(\rho) &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\rho - \sin\theta\sin\rho & -(\cos\theta\sin\rho + \sin\theta\cos\rho) \\ \sin\theta\cos\rho + \cos\theta\sin\rho & \cos\theta\cos\rho - \sin\theta\sin\rho \end{bmatrix} \end{aligned}$$

But this matrix can be rewritten using the addition and subtraction formulas for the sine and cosine functions. This reduces the matrix to the form

$$\psi(\theta)\psi(\rho) = \begin{bmatrix} \cos(\theta+\rho) & -\sin(\theta+\rho) \\ \sin(\theta+\rho) & \cos(\theta+\rho) \end{bmatrix}$$

Hence $\psi(\theta)\psi(\rho) = \psi(\theta+\rho)$ and so the image of the map

$$\begin{aligned} &\psi : \mathbb{R} \longrightarrow SO_2 \\ &\mathbb{R} \ni \theta \mapsto e^{\theta A} \in SO_2 \end{aligned}$$

is a one-parameter subgroup of $GL_2(\mathbb{R})$

We proved in Theorem 15 that the one-parameter groups contained in $GL_n(\mathbb{F})$ are matrix valued functions of the form $\psi(t) = e^{tA}$.
In fact, every one-parameter group contained in the general linear group is a matrix valued functions of the form $e^{tA}$. To put it differently, it is the unique solution to the differential equation $\psi(t) = e^{tA}$.
We end this thesis by classifying the one-parameter subgroups of $O_n$ and $U_n$. The proof of these two results are identical and hence we only derive the one for the orthogonal case. Artin presents the classification of the $U_n$ and $O_n$ one-parameter groups in a single proposition [Art10, Prop.9.5.8]. Although the material we present is similar we shall state it as two separate propositions.

**Proposition 5.** Let $\psi(s) = e^{sA}$ denote the one-parameter groups of $GL_n(\mathbb{R})$. Then $\psi(s) \in O_n$ if and only if $A$ is skew-symmetric. .

*Proof.* The proof is inspired by [Art10, Prop.9.5.8] in the case that $A$ is skew-symmetric.
Suppose that $\psi(s) \in O_n$. Then we have that $(e^{sA})^t = (e^{sA})^{-1}$ from which it follows that $(e^{sA^t}) = (e^{-sA})$ The implication follows by differentiating and putting $s = 0$.

Conversely, if $A$ is skew-symmetric then $A^t = -A$ and $sA^t = -sA$ which implies that

$$(e^{sA})^t = (e^{sA})^{-1}$$

so $e^{sA} \in O_n$. Hence the proof is complete. $\square$

**Proposition 6.** For the one-parameter groups of $GL_n(\mathbb{C})$, $\psi(t) \in U_n$ if $A$ is skew-Hermitian.

*Proof.* By replacing $A^t$ by $A^*$ in the proof above the proposition follows. $\square$

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