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The Steinberg Module of the Mapping Class Group

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Abstract

The Steinberg module is a certain homology group of the curve complex of a surface considered as a module over the mapping class group of the surface. In this paper we will define the Steinberg module and examine some of its properties. Most notably we will look at a presentation for the Steinberg module and at a proof that it is cyclic.

Sammanfattning

Steinbergmodulen av en yta är en viss homologigrupp av ytans kurvkomplex sedd som modul över ytans avbildningsklassgrupp. I den här uppsatsen kommer vi definiera Steinbergmodulen och undersöka vissa av dess egenskaper. Framför allt kommer vi att se en presentation av Steinbergmodulen samt att den är cyklisk.

Contents

1 Introduction

The mapping class group is an important object of study in geometric and algebraic topology and related fields. It is an algebraic invariant of a topological space which contains information about the different kinds of automorphisms of the space. It was first studied by Max Dehn and Jakob Nielsen in the first part of the twentieth century.

This paper will focus on connections between the mapping class group of surfaces and a certain simplicial complex associated to a surface called the curve complex. The simplices in the curve complex are collections of curves in the surface which are non-trivial and, in a certain sense, different from each other. John L. Harer established in [7] that the curve complex is homotopy equivalent to a wedge sum of spheres.

The interplay between the mapping class group and the curve complex comes from the fact that there is a natural action of the mapping class group on the complex of curves. This turns the homology groups of the curve complex into a module over the mapping class group. The Steinberg module is defined as a certain homology group of the curve complex.

In this paper we will give a definition of the curve complex and the Steinberg module. We will also look at another simplicial complex associated to a surface with a marked point, which is constructed in a similar way to the curve complex but using loops based at the marked point instead of general curves, this is called the arc complex. A certain subcomplex of the arc complex which is called the arc complex at infinity turns out to be homotopy equivalent to the curve complex, as shown by Harer in [7].

The arc complex is in many ways easier to deal with than the curve complex, one reason for this is that arc systems (the simplices in the arc complex) can be pictorially represented by chord diagrams, which makes calculations easier.

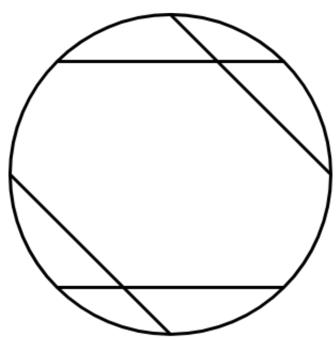

Figure 1: An example of a chord diagram representing a simplex in the arc complex.

We will look at results from Nathan Broaddus in [2] concerning the Steinberg module of closed surfaces and surfaces with one marked point. Most notably we will look at a presentation of the Steinberg module, and we will see that it is cyclic as a module over the mapping class group by giving an explicit generator for it.

We are also going to look at Harer's homotopy equivalence between the curve complex and the arc complex at infinity, and how it can be used to exhibit explicit non-trivial spheres in the curve complex using the generator for the Steinberg module.

Finally we will see how Church, Farb and Putman used Broaddus' work in [4] to deduce a result about the rational cohomology of the mapping class group.

2 Preliminaries

This section will cover notation and some important definitions that will be used throughout the paper. The unit interval $[0, 1]$ will be denoted by I. The notation $A \subset B$ does not exclude the possibility that $A = B$.

Definition 2.1. Let $f, g : X \to Y$ be embeddings. An **isotopy** between f and g is a map $H : X \times I \to Y$ such that $H(-, 0) = f$, $H(-, 1) = g$ and $H(-, t)$ is an embedding for each $t \in I$. If such a map exists we say that f and g are **isotopic**.

Note the similarity between isotopy and homotopy, the difference is that an isotopy requires each member of the family of maps to be an embedding. Obviously isotopic maps are also homotopic, but the converse does not need to be true.

2.1 Surfaces

A surface is a topological space which locally looks like the Euclidean plane or half-plane. Formally we define a surface as follows.

Definition 2.2. Let S be a Hausdorff and second-countable manifold (possibly with boundary) of dimension 2. Then we say that S is a **surface**. If the boundary ∂S is empty we say that S is a **closed surface**.

An important operation on surfaces is the connected sum. If S and S' are surfaces their connected sum $S \# S'$ is constructed by deleting a small disk from both S and S' and then attaching the surfaces together along the boundary of the respective discs. It turns out that the resulting surface is independent of which disks we choose (up to homeomorphism, of course).

There is a well-known theorem called the classification theorem for closed surfaces which states that any closed, connected surface is homeomorphic to either a sphere, a connected sum of some number of tori, or a connected sum of a number of projective planes. In the first two cases the surface is orientable, in the third case it is not.

In this paper we will mostly consider surfaces that are connected sums of tori. We will denote the closed surface which is the connected sum of g tori by Σ_g, and Σ_g^n will denote the same surface but with n marked points.

2.1.1 Genus and Euler characteristic

In this section we will define the genus and Euler characteristic of a surface, and recall some important basic results about these.

Definition 2.3. Let X be a finite cell complex and let X_n be the set of n-cells. Then the **Euler characteristic** of X is defined as the number

$$\chi(X) = |X_0| - |X_1| + |X_2| - |X_3| + ...$$

and if X is a surface (i.e. only has cells of dimension 0, 1 and 2) this just reduces to $\chi(X) = |X_0| - |X_1| + |X_2|$. The Euler characteristic of a simplicial complex is defined similarly (with the number of n-simplices instead of the number of n-cells).

Remark 2.4. The Euler characteristic can be defined for a general topological space as the alternating sum of the ranks of the homology groups of the space, and if the space has a simplicial decomposition or a cell decomposition the general definition will agree with the previous definition. Also note that since it can be defined using the homology groups it is a topological invariant.

The genus of an orientable surface can be defined in terms of the Euler characteristic.

Definition 2.5. The **genus** g of an orientable surface S is defined by the equation

$$\chi = 2 - 2g$$

where χ is the Euler characteristic of S.

Example 2.6. The sphere $\Sigma_0 = S^2$ has a cell decomposition given by starting with a point, attaching an interval to it to get a circle and then attach two disks to the circle. This gives a cell decomposition with one 0-cell, one 1-cell and two 2-cells, so we have $\chi(S^2) = 1 - 1 + 2 = 2$ and by the formula $\chi = 2 - 2g$ this shows that the genus of S^2 is 0.

Example 2.7. Consider the torus $\Sigma_1 = T^2$. If we cut the torus along the two curves going once around the longitude and once around the meridian we obtain a disk. This gives a cell decomposition of the surface with one 0-cell (the intersection of the two curves), two 1-cells (the curves themselves) and a single 2-cell. We can then compute the Euler characteristic to be $\chi(T^2) = 1-2+1 = 0$ and the genus $g = \frac{1}{2}(2 - \chi) = \frac{1}{2}(2 - 0) = 1$.

More generally it is true that Σ_g^n has genus g.

2.1.2 The mapping class group of a surface

Let S be a surface. The mapping class group of a surface is an important algebraic invariant that encodes information about the automorphisms of the surface.

To define the mapping class group, first let $\mathrm{Aut}^+(S)$ be the set of all homeomorphisms from S to itself that preserve the orientation of S and fixes the boundary of S. Then $\mathrm{Aut}^+(S)$ has a natural group structure given by composition.

Furthermore, let $\mathrm{Aut}_0^+(S)$ denote the subgroup of homeomorphisms that are isotopic to the identity.

Definition 2.8. We define the **mapping class group** of S to be the quotient group

$$\mathrm{Mod}(S) = \mathrm{Aut}^+(S)/\mathrm{Aut}_0^+(S).$$

Note that $\mathrm{Mod}(S)$ is in fact a group since if g is isotopic to the identity by the isotopy H and $f \in \mathrm{Aut}^+(S)$ then $f \circ H \circ (f^{-1}, \mathrm{id}_I)$ gives an isotopy between fgf^{-1} and the identity, so $\mathrm{Aut}_0^+(S)$ is a normal subgroup of $\mathrm{Aut}^+(S)$.

There is an important exact sequence called the Birman exact sequence which relates $\mathrm{Mod}(\Sigma_g^{n+1})$ with $\mathrm{Mod}(\Sigma_g^n)$ if $g \geq 2$. A proof of this can be found in for example [5].

Theorem 2.9. *Let x denote the additional marked point in Σ_g^{n+1} and let $g \geq 2$. Then there is an exact sequence of groups*

$$1 \to \pi_1(\Sigma_g^n, x) \to Mod(\Sigma_g^{n+1}) \to Mod(\Sigma_g^n) \to 1.$$

The map $\mathrm{Mod}(\Sigma_g^{n+1}) \to \mathrm{Mod}(\Sigma_g^n)$ in the Birman exact sequence is given by simply forgetting the additional marked point.

The map from $\pi_1(\Sigma_g^n, x)$ to $\mathrm{Mod}(\Sigma_g^{n+1})$ is a bit more complicated, but essentially it takes a loop $\alpha \in \pi_1(\Sigma_g^n, x)$ and considers it as an isotopy from the point x to itself (or more explicitly an isotopy from f to itself where $f : \{*\} \to \Sigma_g^n$ is defined by $f(*) = x$), and then we extend this isotopy to the whole surface and define the image of α to be the homeomorphism of Σ_g^1 at the end of this isotopy.

Definition 2.10. The subgroup of $\mathrm{Mod}(\Sigma_g^{n+1})$ which is the image of the inclusion $\pi_1(\Sigma_g^n, x) \to \mathrm{Mod}(\Sigma_g^{n+1})$ from the Birman exact sequence (or the trivial group in the case of $g = 1$) is called the **point-pushing subgroup** of $\mathrm{Mod}(\Sigma_g^{n+1})$.

2.2 Group (co)homology and duality groups

2.2.1 Group modules

Here we will define modules over a group and recall some of their properties.

Definition 2.11. Let G be a group. A (left) ***G*-module** is an abelian group M together with a group action $\cdot : G \times M \to M$ such that

$$g \cdot (x + y) = g \cdot x + g \cdot y$$

for all $g \in G$ and $x, y \in M$. Or equivalently an abelian group M together with a group homomorphism $G \to \mathrm{End}(M)$.

Example 2.12. Let M be any abelian group and G any group. Then M is a G-module with the G-module structure defined by $g \cdot x = x$ for all $g \in G$ and $x \in M$, this is called the trivial G-module structure.

There is a certain ring denoted $\mathbb{Z}[G]$ such that modules over that ring correspond exactly to G-modules. When we write $A \otimes_G B$ for G-modules A and B it will mean $A \otimes_{\mathbb{Z}[G]} B$. The ring $\mathbb{Z}G$ is defined as follows.

Definition 2.13. Let G be a group. Define $\mathbb{Z}[G]$ as the set of formal linear combinations of elements in G over $\mathbb{Z}$, addition is defined in the obvious way and the multiplication is defined so that the distributive laws hold.

Example 2.14. Let G be the infinite cyclic group generated by an element x, so $G \cong \mathbb{Z}$. Then the group ring $\mathbb{Z}[G]$ is isomorphic to $\mathbb{Z}[x, x^{-1}]$.

Morphisms between G-modules are called G-equivariant maps, and are defined as follows.

Definition 2.15. Let M and N be G-modules. A function $f : M \to N$ is called a ***G*-equivariant map** if

$$f(g \cdot m) = g \cdot f(m)$$

for all $g \in G$ and $m \in M$.

2.2.2 Group homology and cohomology

There are many different ways to define homology and cohomology of a group, for our purposes it will be easiest to define it in terms of the Tor and Ext functors.

Definition 2.16. Let G be a group and M a G-module. We define the homology groups of G with coefficients in M by

$$H_n(G; M) = \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M)$$

and similarly we define the cohomology groups by

$$H^n(G; M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$$

where in both cases we view $\mathbb{Z}$ as a G-module with the trivial G-module structure.

To be a bit more explicit, to compute $H_n(G; M)$ and $H^n(G, M)$ we would start with a projective resolution of $\mathbb{Z}$ with the trivial G-module structure, in other words an exact sequence of the form

$$\cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

where the P_i are projective G-modules, and then we remove $\mathbb{Z}$ to get the chain complex

$$\cdots \to P_1 \to P_0 \to 0.$$

Now to compute $H_n(G; M)$ we would tensor the chain complex above with M to get

$$\cdots \to P_1 \otimes_G M \to P_0 \otimes_G M \to 0$$

and then $H_n(G; M)$ would be the n:th homology of this chain complex. If we apply $\mathrm{Hom}(-, M)$ instead of $- \otimes_G M$ to the chain complex we get

$$0 \to \mathrm{Hom}_G(P_0, M) \to \mathrm{Hom}_G(P_1, M) \to \cdots$$

and the n:th cohomology of this chain complex would be $H^n(G; M)$.

2.2.3 Duality groups

There is a generalisation of Poincaré duality defined by Bieri and Eckmann in [1]. The usual Poincaré duality for groups is defined so that G is a Poincaré duality group if there is an integer n and a G-module structure on $\mathbb{Z}$ such that

$$H^k(G; A) \cong H_{n-k}(G; \mathbb{Z} \otimes_{\mathbb{Z}} A)$$

for all G-modules A, where the G-module structure on $\mathbb{Z} \otimes_{\mathbb{Z}} A$ is defined by $g(z \otimes a) = gz \otimes ga$. This can be generalised by replacing $\mathbb{Z}$ with an arbitrary module.

Definition 2.17. Let G be a group. We say that G is a **Bieri-Eckmann duality group** if there is a G-module D and an integer n such that

$$H^k(G; A) \cong H_{n-k}(G; D \otimes_{\mathbb{Z}} A)$$

for all G-modules A, where again the G-module structure on $D \otimes_{\mathbb{Z}} A$ is defined by $g(d \otimes a) = gd \otimes ga$. The module D is called the dualizing module for G.

2.3 The long exact sequence of a triple

Consider a triple of spaces (X, Y, Z), where $Z \subset Y \subset X$. Then we have the following long exact sequence of their relative homology groups.

Lemma 2.18. *For a triple of spaces (X, Y, Z) the sequence*

$$\cdots \to H_n(Y, Z; \mathbb{Z}) \to H_n(X, Z; \mathbb{Z}) \to H_n(X, Y; \mathbb{Z}) \to H_{n-1}(Y, Z; \mathbb{Z}) \to \cdots$$

is exact.

Proof. There is an exact sequence of relative chain complexes

$$0 \to C_\bullet(Y, Z) \to C_\bullet(X, Z) \to C_\bullet(X, Y) \to 0$$

where the map $C_\bullet(Y, Z) \to C_\bullet(X, Z)$ is induced by the inclusion $Y \hookrightarrow X$ and the map $C_\bullet(X, Z) \to C_\bullet(X, Y)$ is the quotient map (which makes sense since we can view $C_\bullet(X, Y)$ as $C_\bullet(X, Z)/C_\bullet(Y, Z)$). A short exact sequence of chain complexes induces a long exact sequence in homology, which in this case gives the long exact sequence above. □

We will later make use of the following special case.

Corollary 2.19. *For a pair of spaces (X, Y) the sequence*

$$\cdots \to \tilde{H}_n(Y; \mathbb{Z}) \to \tilde{H}_n(X; \mathbb{Z}) \to H_n(X, Y; \mathbb{Z}) \to \tilde{H}_{n-1}(Y; \mathbb{Z}) \to \cdots$$

is exact.

Proof. Let $x \in Y \subset X$. If we consider the triple $(X, Y, \{x\})$ we get from the above lemma the long exact sequence

$$\cdots \to H_n(Y, \{x\}; \mathbb{Z}) \to H_n(X, \{x\}; \mathbb{Z}) \to H_n(X, Y; \mathbb{Z}) \to H_{n-1}(Y, \{x\}; \mathbb{Z}) \to \cdots$$

which since $\tilde{H}_n(X; \mathbb{Z}) \cong H_n(X, \{x\}; \mathbb{Z})$ and $\tilde{H}_n(Y; \mathbb{Z}) \cong H_n(Y, \{x\}; \mathbb{Z})$ gives that

$$\cdots \to \tilde{H}_n(Y; \mathbb{Z}) \to \tilde{H}_n(X; \mathbb{Z}) \to H_n(X, Y; \mathbb{Z}) \to \tilde{H}_{n-1}(Y; \mathbb{Z}) \to \cdots$$

is exact. □

Remark 2.20. In the case where (X, Y) is a good pair we have the isomorphism $H_n(X, Y; \mathbb{Z}) \cong \tilde{H}_n(X/Y; \mathbb{Z})$, so the long exact sequence above becomes the following long exact sequence in reduced homology

$$\cdots \to \tilde{H}_n(Y; \mathbb{Z}) \to \tilde{H}_n(X; \mathbb{Z}) \to \tilde{H}_n(X/Y; \mathbb{Z}) \to \tilde{H}_{n-1}(Y; \mathbb{Z}) \to \cdots .$$

3 The Steinberg module

The Steinberg module of Σ_g^1 and Σ_g will be defined as a certain homology group of a simplicial complex associated to the surface. In this section we will define this simplicial complex and then the Steinberg module, and we will also discuss some properties of the Steinberg module.

3.1 The curve complex and the arc complex

Here we will define some important simplicial complexes associated with Σ_g^1 and Σ_g, which will then be used to define and to study the Steinberg modules of the two surfaces. The simplices in the complexes will be certain collection of curves and arcs as defined below.

Definition 3.1. Let $*$ be the marked point in Σ_g^1. A **curve** in Σ_g^1 will be defined as an isotopy class of embedded loops in $\Sigma_g^1 - \{*\}$, and similarly in Σ_g it will be defined as an isotopy class of embedded loops in Σ_g.

An **arc** in Σ_g^1 will be defined as an isotopy class of embedded loops in Σ_g^1 based at $*$. We will exclude curves and arcs that bound a disk or a once punctured disk from the definition.

We can now define the curve complex and the arc complex as in [2].

Definition 3.2. A **curve system** is a set of curves such that we can choose isotopy representatives of each of the curves that are disjoint.

The **curve complex** of Σ_g^1, which will be denoted by $\mathcal{C}(\Sigma_g^1)$, is defined as the simplicial complex with n-simplices consisting of curve systems with $n+1$ curves and inclusion as face relation. The curve complex of Σ_g, denoted $\mathcal{C}(\Sigma_g)$, is defined in the same way.

Definition 3.3. An **arc system** is a set of arcs such that we can choose isotopy representatives of each of the arcs that only intersect at the marked point.

The **arc complex** of Σ_g^1, denoted $\mathcal{A}(\Sigma_g^1)$, is defined as the simplicial complex with n-simplices given by arc systems with $n+1$ arcs and inclusion as face relation.

There is also an important subcomplex of $\mathcal{A}(\Sigma_g^1)$. We first need to define the notion of a filling arc system.

Definition 3.4. Let $a \in \mathcal{A}(\Sigma_g^1)$. We say that a **fills** Σ_g^1 if all the connected components of $\Sigma_g^1 - \cup a$ are disks.

We will call an arc system a a ***k*-filling system** if a is a filling system with $2g+k$ arcs.

The reason for the terminology "k-filling system" is that a k-filling system cuts the surface into exactly $k+1$ disks, and in particular a 0-filling system cuts the surface into a single disk.

This can be seen by a computation using Euler characteristic, if we have a k-filling system it gives a cell decomposition of the surface where the number of

0-cells is one, the number of 1-cells is $2g+k$ (the number of arcs) and where we let d be the number of 2-cells (the number of disks the arcs cut the surface into). Then this cell decomposition gives that the Euler characteristic of the surface is $d-2g-k+1$ but on the other hand we know that the Euler characteristic of Σ_g^1 is $2-2g$, so $d=k+1$ and thus if we have a k-filling system it has to cut the surface into $k+1$ disks.

We now define the subcomplex of the arc complex called the arc complex at infinity.

Definition 3.5. Let $\mathcal{A}_\infty(\Sigma_g^1)$ be the subcomplex of $\mathcal{A}(\Sigma_g^1)$ consisting of all arc systems that do not fill Σ_g^1.

Remark 3.6. To simplify notation we will often denote $\mathcal{A}(\Sigma_g^1)$ and $\mathcal{A}_\infty(\Sigma_g^1)$ by simply $\mathcal{A}$ and $\mathcal{A}_\infty$, respectively.

3.2 The Steinberg module

We can now give the definition of the Steinberg module.

Definition 3.7. Let S be either Σ_g or Σ_g^1. We define the Steinberg module $\mathrm{St}(S)$ by

$$\mathrm{St}(S) = \tilde{H}_{2g-2}(\mathcal{C}(S);\mathbb{Z}).$$

We have that $\mathrm{St}(S)$ is a $\mathrm{Mod}(S)$-module since $f \in \mathrm{Mod}(S)$ acts on $\mathcal{C}(S)$ (a homeomorphism will map a curve system to a curve system), and thus we get an action of $\mathrm{Mod}(S)$ on $\mathrm{St}(S)$ as well. Furthermore, $\mathrm{St}(\Sigma_g)$ is also a $\mathrm{Mod}(\Sigma_g^1)$-module since any homeomorphism of Σ_g^1 is also a homeomorphism of Σ_g. Harer shows in [7] that there is a homotopy equivalence $\mathcal{C}(\Sigma_g^1) \simeq \mathcal{C}(\Sigma_g)$ which is $\mathrm{Mod}(\Sigma_g^1)$-equivariant, and thus we have the following lemma.

Lemma 3.8. *As Mod(Σ_g^1)-modules, St(Σ_g^1) $\cong$ St(Σ_g).*

Note that a consequence of this is that the action of $\mathrm{Mod}(\Sigma_g^1)$ on $\mathrm{St}(\Sigma_g^1)$ factors through $\mathrm{Mod}(\Sigma_g)$ since $\mathrm{Mod}(\Sigma_g)$ is isomorphic to a quotient of $\mathrm{Mod}(\Sigma_g^1)$ by the Birman exact sequence, so we also have

Lemma 3.9. *As Mod(Σ_g)-modules, St(Σ_g^1) $\cong$ St(Σ_g).*

Another characterisation of $\mathrm{St}(\Sigma_g^1)$ which will be useful later is the following, which follows from the homotopy equivalence between $\mathcal{C}(\Sigma_g^1)$ and $\mathcal{A}_\infty(\Sigma_g^1)$ given by Harer in [7]. We will later give an overview of what this homotopy equivalence looks like.

Lemma 3.10. *We have an isomorphism St(Σ_g^1) $\cong \tilde{H}_{2g-2}(\mathcal{A}_\infty(\Sigma_g^1);\mathbb{Z})$.*

3.3 The Steinberg module is a dualizing module

Let Σ be either Σ_g or Σ_g^1. Harer showed in [7] that any finite index subgroup of $\mathrm{Mod}(\Sigma)$ is a Bieri-Eckmann duality group and that $\mathrm{St}(\Sigma)$ is the dualizing module, or in other words, that for a finite index subgroup $\Gamma \subset \mathrm{Mod}(\Sigma)$ we have

$$H^k(\Gamma; A) \cong H_{d-k}(\Gamma; \mathrm{St}(\Sigma) \otimes_{\mathbb{Z}} A)$$

for all Γ-modules A and natural numbers k, where d is $4g-3$ for Σ_g^1 and $4g-5$ for Σ_g.

3.4 A presentation of the Steinberg module

We will now exhibit presentations for $\mathrm{St}(\Sigma_g^1)$ and $\mathrm{St}(\Sigma_g)$. To do this we will first give a resolution of $\mathrm{St}(\Sigma_g^1)$ as a $\mathrm{Mod}(\Sigma_g^1)$-module as in [2]. We will use the following two results which were established by Harer in [7] and [6].

Theorem 3.11. *The arc complex $\mathcal{A}(\Sigma_g^1)$ is contractible.*

Theorem 3.12. *The arc complex at infinity $\mathcal{A}_\infty(\Sigma_g^1)$ is homotopy equivalent to a wedge sum of spheres of dimension $2g-2$.*

We will first use this to prove the following lemma which gives yet another characterisation of the Steinberg module.

Lemma 3.13. *We have an isomorphism $St(\Sigma_g^1) \cong H_{2g-1}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z})$.*

Proof. Consider the long exact sequence in reduced homology for the pair of spaces $(\mathcal{A}, \mathcal{A}_\infty)$ from corollary 2.19, in particular we have that the sequence

$$\tilde{H}_{k+1}(\mathcal{A}; \mathbb{Z}) \to H_{k+1}(\mathcal{A}, \mathcal{A}_\infty; \mathbb{Z}) \to \tilde{H}_k(\mathcal{A}_\infty; \mathbb{Z}) \to \tilde{H}_k(\mathcal{A}; \mathbb{Z})$$

is exact for all $k \geq 0$. But by 3.11 and since $H_{k+1}(\mathcal{A}, \mathcal{A}_\infty; \mathbb{Z}) \cong \tilde{H}_{k+1}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z})$ this just becomes

$$0 \to \tilde{H}_{k+1}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z}) \to \tilde{H}_k(\mathcal{A}_\infty; \mathbb{Z}) \to 0$$

and thus we get the isomorphism

$$\tilde{H}_k(\mathcal{A}_\infty; \mathbb{Z}) \cong \tilde{H}_{k+1}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z})$$

for all $k \geq 0$, and therefore

$$\mathrm{St}(\Sigma_g^1) \cong \tilde{H}_{2g-2}(\mathcal{A}_\infty; \mathbb{Z}) \cong \tilde{H}_{2g-1}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z}) = H_{2g-1}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z}).$$

□

Now we can give the resolution. Let

$$\mathcal{C}_k = C_{2g-1+k}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z})$$

and consider the chain complex $\mathcal{C}_\bullet$ with the same boundary maps as the chain complex for $\mathcal{A}/\mathcal{A}_\infty$.

Lemma 3.14. *Let $\mathcal{C}_\bullet$ be the chain complex as defined above, then there is a resolution*

$$0 \to \mathcal{C}_{4g-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}_1 \xrightarrow{\partial} \mathcal{C}_0 \xrightarrow{q} St(\Sigma_g^1) \to 0.$$

Proof. We follow the general idea of the proof from [2]. By definition $\mathcal{C}_\bullet$ is the cellular chain complex for $\mathcal{A}/\mathcal{A}_\infty$ with shifted indices, so we can calculate it's homology by

$$H_k(\mathcal{C}_\bullet) = H_{2g-1+k}(\mathcal{A}/\mathcal{A}_\infty;\mathbb{Z}) \cong \tilde{H}_{2g-2+k}(\mathcal{A}_\infty;\mathbb{Z})$$

so $H_0(\mathcal{C}_\bullet) \cong \mathrm{St}(\Sigma_g^1)$ by 3.13 and by 3.12 all other homology is trivial. Now since $H_0(\mathcal{C}_\bullet) = \mathcal{C}_0/\mathrm{im}(\partial)$ we can let q be the projection map, then the sequence

$$\cdots \to \mathcal{C}_{4g-2} \to \mathcal{C}_{4g-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}_1 \xrightarrow{\partial} \mathcal{C}_0 \xrightarrow{q} \mathrm{St}(\Sigma_g^1) \to 0$$

is exact since by definition of q we have $\ker(q) = \mathrm{im}(\partial)$ and q is surjective, and we already know that all higher homology of the sequence is trivial.

Finally, an arc system with as many arcs as possible would be a one-vertex triangulation of the surface since otherwise we could add another arc to the arc system, and if a is the number of arcs in a one-vertex triangulation then the number of triangles would be $\frac{2}{3}a$ since each edge meets two of the triangles, thus by the Euler characteristic we get $a = 6g-3$ which means that an arc system can have at most $6g-3$ arcs, and thus $\mathcal{A}$ has dimension $6g-4$ which means that $\mathcal{A}/\mathcal{A}_\infty$ also has dimension $6g-4$ and thus $\mathcal{C}_k = 0$ when $2g-1+k > 6g-4$ which is equivalent to $k > 4g-3$, this completes the proof since setting $\mathcal{C}_k = 0$ for $k > 4g-3$ in the sequence above gives the resolution. □

Since $\mathcal{C}_k = C_k(\mathcal{A}/\mathcal{A}_\infty;\mathbb{Z})$ the elements of $\mathcal{C}_k$ will be $\mathbb{Z}$-linear combinations of cells of dimension $2g-1$ in $\mathcal{A}/\mathcal{A}_\infty$, but such a cell is by definition a k-filling arc system. Furthermore, $\mathcal{C}_k$ is actually finitely generated if we consider it as a $\mathrm{Mod}(\Sigma_g^1)$-module, as the following lemma shows.

Lemma 3.15. *The $\mathrm{Mod}(\Sigma_g^1)$-module $\mathcal{C}_k$ is generated by finitely many of k-filling systems.*

Proof. A k-filling system in Σ_g^1 gives a one-vertex cell decomposition of Σ_g^1, but there's only finitely many topologically distinct (and by this we mean that two such decompositions would not be distinct if a mapping class maps one to the other), and for all such decompositions which are topologically the same type there has to exist an element of $\mathrm{Mod}(\Sigma_g^1)$ taking one to the other, so we only need a single representative from each of these classes of arc systems. This implies that $\mathcal{C}_k$ has to be finitely generated as a $\mathrm{Mod}(\Sigma_g^1)$-module. □

Before giving the presentations we need one more lemma concerning the stabilizers of 0-filling arc systems.

Lemma 3.16. *For a 0-filling arc system α in Σ_g^1 its stabilizer is finite and cyclic.*

Proof. A 0-filling system cuts the surface into a single $4g$-gon, so the stabilizer of the 0-filling system has to be a subgroup of the group of rotational symmetries of a regular $4g$-gon. □

There is another definition we need to look at before we can give the presentations.

Definition 3.17. Let M be a G-module and let $H \subset G$ be a subgroup. We define the ***H*-co-invariants** of M, denoted M_H, as the quotient

$$M/\{hm - m : h \in H, m \in M\}.$$

Note that if $H \subset G$ is a normal subgroup and M is a G-module then M_H will also be a G/H-module, since if $g, g' \in G$ are in the same coset of H, i.e. $g = g'h$ for $h \in G$, then

$$gm = g'hm = g'(hm - (hm - m)) = g'm$$

for all $m \in M$, so the action of G on M_H respects the equivalence relation which we quotient G by to get G/H.

We can now give presentations for $\mathrm{St}(\Sigma_g^1)$ and $\mathrm{St}(\Sigma_g)$. The proof will follow [2] closely.

Theorem 3.18. *Let $g \geq 1$. By 3.15 we can choose oriented representatives $\phi_0, ..., \phi_n$ for each of the orbits of 0-filling systems. Let $h_i \in Mod(\Sigma_g^1)$ be a generator for the stabilizer of the arc system ϕ_i and let e_i be the sign of the permutation that h_i induces on the set of arcs in ϕ_i. Similarly choose oriented representatives $\rho_0, ...\rho_m$ for each of the orbits of 1-filling systems. Then $St(\Sigma_g^1)$ has a presentation*

$$\langle \phi_0, ..., \phi_n | \partial\rho_0, ..., \partial\rho_m, (1 - e_0h_0)\phi_0, ..., (1 - e_nh_n)\phi_n \rangle$$

and if we send the coefficients in this presentation to their images in $\mathbb{Z}Mod(\Sigma_g)$ under the homomorphism $\mathbb{Z}[Mod(\Sigma_g^1)] \to \mathbb{Z}[Mod(\Sigma_g)]$ defined by forgetting the marked point we get a presentation for $St(\Sigma_g)$.

Proof. By lemma 3.14 we have the isomorphism

$$\mathrm{St}(\Sigma_g^1) \cong \mathcal{C}_0/\partial\mathcal{C}_1$$

and we know that every (oriented) 0-filling system in Σ_g^1 has to be of the form $\pm h\phi_i$ for some $h \in \mathrm{Mod}(\Sigma_g^1)$, so $\{\phi_0, ..., \phi_n\}$ spans $\mathcal{C}_0$ as a $\mathrm{Mod}(\Sigma_g^1)$-module. The only linear dependencies in $\mathcal{C}_0$ are those that arise from stabilizers (which we know from lemma 3.16 are finite cyclic), so for instance we should have that $e_0h_0\phi_0 = \phi_0$. Consequently, we have that

$$\mathcal{C}_0 = \langle \phi_0, ..., \phi_n | (1 - e_0h_0)\phi_0, ..., (1 - e_nh_n)\phi_n \rangle$$

and similarly that $\mathcal{C}_1$ is generated by $\rho_0, ..., \rho_m$, therefore it follows that

$$\langle \phi_0, ..., \phi_n | \partial\rho_0, ..., \partial\rho_m, (1 - e_0h_0)\phi_0, ..., (1 - e_nh_n)\phi_n \rangle.$$

For $\mathrm{St}(\Sigma_g)$ it's not quite as simple since we don't have a resolution of $\mathrm{St}(\Sigma_g)$ as a $\mathrm{Mod}(\Sigma_g)$-module, but to show that the presentation holds in this case as well we start by considering the Birman exact sequence which was described in theorem 2.9. In the case where $g > 1$ we get that the sequence

$$1 \to \pi_1(\Sigma_g, *) \to \mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_g) \to 1$$

is exact, where $*$ is the marked point in Σ_g^1. Now let P be the point-pushing subgroup of $\mathrm{Mod}(\Sigma_g^1)$, i.e. the kernel of the map $\mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_g)$ in the exact sequence for $g > 1$ or the trivial group in the case where $g = 1$.

Now recall the exact sequence

$$\mathcal{C}_1 \to \mathcal{C}_0 \to \mathrm{St}(\Sigma_g^1) \to 0$$

and take P-co-invariants to get the sequence

$$(\mathcal{C}_1)_P \to (\mathcal{C}_0)_P \to \mathrm{St}(\Sigma_g^1)_P \to 0$$

which will still be exact since taking co-invariants is right exact (this is shown for example in II.2 of [3]).

Now we can use lemma 3.9 to conclude that

$$\mathrm{St}(\Sigma_g^1)_P \cong \mathrm{St}(\Sigma_g^1) \cong \mathrm{St}(\Sigma_g)$$

as $\mathrm{Mod}(\Sigma_g)$-modules, so we have from the exact sequence that

$$\mathrm{St}(\Sigma_g) \cong (\mathcal{C}_0)_P/(\partial(\mathcal{C}_1))_P.$$

The presentation for $\mathrm{St}(\Sigma_g)$ then follows since taking P-co-invariants of $\mathcal{C}_0$ and $\mathcal{C}_1$ will send the coefficients in the presentation for $\mathrm{St}(\Sigma_g^1)$ to their equivalence classes in the quotient $\mathrm{Mod}(\Sigma_g) \cong \mathrm{Mod}(\Sigma_g^1)/P$. □

3.5 Chord diagrams

A filling arc system can be represented by a chord diagram, and this makes it easier to work with them and describe them. We will now define chord diagrams and explain their connection to arc systems, and later we will be able to use this to show that $\mathrm{St}(\Sigma_g^1)$ is a cyclic $\mathrm{Mod}(\Sigma_g^1)$-module as in [2].

Definition 3.19. Let $*$ be the marked point in Σ_g^1. A chord diagram is a regular $2n$-gon with the vertices paired up in a way such that no two adjacent vertices are paired.

We will draw the $2n$-gon as a circle, and two paired vertices will be connected by a chord. Each chord will be labelled with an element of $\pi_1(\Sigma_g, *)$ representing an arc in a filling arc system.

Given a filling arc system in Σ_g^1 with n arcs we construct its chord diagram by choosing a neighborhood of $*$ in the shape of a $2n$-gon so that each arc leaves and reenters the neighborhood at a vertex of the $2n$-gon. For each arc we then

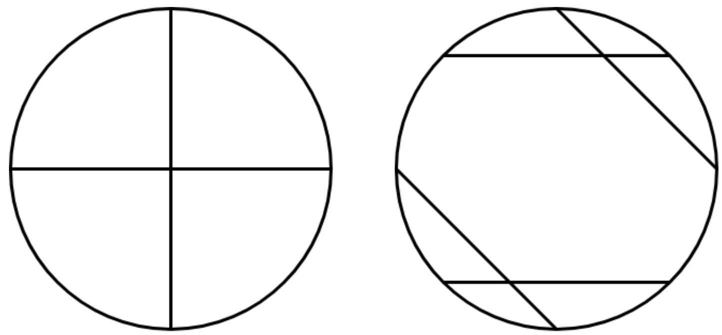

Figure 2: Two chord diagrams representing arc systems.

draw a chord between the vertices where it leaves and reenters the neighborhood and label the chord by the corresponding element of $\pi_1(\Sigma_g, *)$.

For example, the left chord diagram in figure 2 corresponds to the 0-filling arc system in Σ_1^1 consisting of one arc going once around the longitude of the torus and one arc going around a meridian, and the right chord diagram in figure 2 corresponds to a 0-filling arc system in Σ_2^1.

We will now define what a cycle in a chord diagram is. Choose a point just inside the outer edge of the diagram, and start walking in a clockwise direction so that the edge is on the left side until you encounter a chord, turn right and then walk along the chord until you get back to the outer edge (again always keeping the line on the left side), continue until you get back to the point where you started, this is a cycle.

More formally, we will define a cycle as follows.

Definition 3.20. Consider a chord diagram. A **cycle** in the chord diagram is an alternating sequence of chords and outer edges obtained by the process described above.

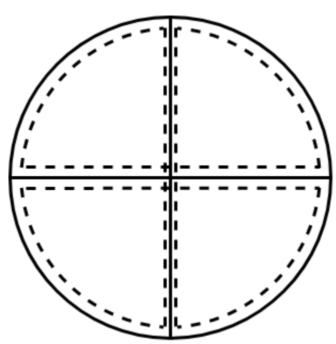

Figure 3: A cycle in a chord diagram.

Definition 3.21. If two chords in a chord diagram bound a rectangular cycle together with two outer edges, we say that the chords are **parallel**.

We can also recover a surface from a given chord diagram. If we take a chord diagram with n chords and glue a disk along each of its cycles, we obtain a surface which has the arc system corresponding to the chord diagram as a filling arc system. This also gives a cell decomposition of the surface with one 0-cell, one 1-cell for each chord, and if c is the number of cycles in the chord diagrams it will also be the number of 2-cells in the cell decomposition. Thus the Euler characteristic of the surface is $1 - n + b$ and the genus

$$g = \frac{2 - (1 - n + b)}{2} = \frac{n - 1 + b}{2}.$$

From this we can see that a chord diagram with $2g$ chords corresponds to a 0-filling system in a surface of genus g if and only if the chord diagram has a single cycle.

More generally, we can identify k-filling systems in Σ_g^1 with chord diagrams which have $2g + k$ chords, $k + 1$ cycles and no parallel chords (since arc systems can't have parallel arcs).

3.6 The Steinberg module is cyclic

We know from lemma 3.14 that

$$\mathrm{St}(\Sigma_g^1) \cong \mathcal{C}_0 / \partial \mathcal{C}_1.$$

Surprisingly, it turns out that $\mathrm{St}(\Sigma_g^1)$ is in fact cyclic, meaning that it will be generated by the class of a certain 0-filling arc system whose chord diagram (with $2g$ chords) is pictured to the left in figure 4, and to the right we can see how it would look for $g = 2$.

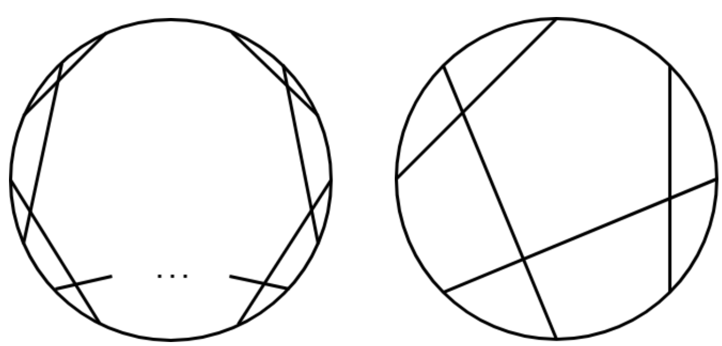

Figure 4: A generator for $\mathrm{St}(\Sigma_g^1)$, and the special case where g=2.

To prove this we will first need to prove a few lemmas about chord diagrams. The proof of the theorem and the lemmas are based on the proofs in [2]. We will define a certain type of chord diagram where each of its connected components has the same form as a part of the chord diagram in figure 4, then we will show that the set of all such chord diagrams generate $\mathrm{St}(\Sigma_g^1)$ and that the one in figure 4 is the only such chord diagram which represents a non-trivial equivalence class in the quotient $\mathcal{C}_0/\partial\mathcal{C}_1$.

Definition 3.22. A chord diagram such that each of its connected components look like a part of the chord diagram in figure 4 will be called a **salient** chord diagram. If only a part of the chord diagram looks like that, and this part consists of n chords, we will say that the chord diagram has a **salient tail** of length n.

Now we will prove that disconnected chord diagrams represent the trivial class in $\mathrm{St}(\Sigma_g^1)$, from which is follows that $\mathrm{St}(\Sigma_g^1)$ is generated by all the connected chord diagrams.

Lemma 3.23. *Let α be a chord diagram representing a 0-filling system in Σ_g^1 and assume that α is disconnected. Then $[\alpha] = 0$ in $St(\Sigma_g^1)$.*

Proof. We need to show that α is in $\partial\mathcal{C}_1$. Let α_c be the chord system obtained by adding a chord c to α which does not cross any other chords (which exists since α is disconnected). Note that α_c will be a 1-filling system as long as c is not parallel to any chord in α, but if there was a chord c' in α parallel to c then c' would not cross any of the other chords in α, so α would have at least two different cycles, contradicting that α is a 0-filling system. Therefore α_c has to be a 1-filling system.

Furthermore, the only way to obtain a 0-filling system by removing a chord from α_c is to remove c, since by the same argument removing any other chord would give a chord diagram with two different cycles so it can't be a 0-filling system. We thus have that $\partial\alpha_c = \pm\alpha$ and consequently $\alpha = \partial(\pm\alpha_c) \in \partial\mathcal{C}_1$ showing that $[\alpha] = 0$ in $\mathrm{St}(\Sigma_g^1)$. □

Lemma 3.24. *The Steinberg module $St(\Sigma_g^1)$ is generated by the equivalence classes of all the salient 0-filling systems.*

Proof. Let α be a chord diagram representing a 0-filling system. There's some $n \geq 0$ such that α has a salient tail of length n, and if we add a chord c we can extend the salient tail of length n to one of length $n+1$, let α_c be the resulting chord diagram.

Then we have that $\partial\alpha_c$ is the sum of $\pm\alpha$ and other terms which are obtained by removing a chord from α_c, but removing a chord other than c from the salient tail in α_c would give a disconnected chord diagram, so these other terms will either be trivial (by the previous lemma) or have a salient tail of length $n+1$. It thus follows that

$$[\partial\alpha_c] = 0 = \pm[\alpha] \pm [\beta_1] \pm ... \pm [\beta_m]$$

so that

$$[\alpha] = \pm[\beta_1] \pm ... \pm [\beta_m]$$

where each β_i has a salient tail of length $n+1$. A 0-filling system has $2g$ chords, so if we iterate this process at the most $2g$ times we will have written the class of α as linear combinations of chord diagrams with salient tails of length $2g$, i.e. salient chord diagrams. Thus all 0-filling systems is generated by salient ones. □

Now that we have established these two lemmas the fact that $\mathrm{St}(\Sigma_g^1)$ follows almost directly.

Theorem 3.25. *Let ϕ_0 be the generator in figure 4. Then $St(\Sigma_g^1)$ is generated by $[\phi_0]$.*

Proof. We know by lemma 3.24 that $\mathrm{St}(\Sigma_g^1)$ is generated by salient 0-filling systems, but by lemma 3.23 all of these are trivial in $\mathrm{St}(\Sigma_g^1)$ except $[\phi_0]$ since ϕ_0 is the unique connected salient 0-filling system, so $\mathrm{St}(\Sigma_g^1)$ is generated by just $[\phi_0]$. □

Corollary 3.26. *The class of ϕ_0 is nontrivial in $St(\Sigma_g^1)$ for all $g \geq 1$.*

Proof. If $[\phi_0] = 0$ we would have $\mathrm{St}(\Sigma_g^1) = 0$ by theorem 3.25, but this is not possible since $\mathrm{St}(\Sigma_g^1)$ is the dualizing module for any finite-index subgroup of $\mathrm{Mod}(\Sigma_g^1)$.

To be explicit, if we let $G \subset \mathrm{Mod}(\Sigma_g^1)$ be a finite-index subgroup and we assume that $\mathrm{St}(\Sigma_g^1) = 0$ we would have

$$H^k(G; M) = H_{4g-3-k}(G; 0 \otimes_G M) = H_{4g-3-k}(G; 0) = \mathrm{Tor}_{4g-3-k}^{\mathbb{Z}G}(\mathbb{Z}; 0) = 0$$

for any G-module M. But on the other hand, letting $M = \mathbb{Z}$ with the trivial G-module structure we get

$$H^0(G; \mathbb{Z}) = \mathrm{Ext}^0_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Hom}_G(\mathbb{Z}, \mathbb{Z}) = 0$$

which clearly can't be true, for example $f : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x$ for all $x \in \mathbb{Z}$ is a nonzero map which is G-equivariant since $f(g \cdot x) = f(x) = g \cdot f(x)$ due to both copies of $\mathbb{Z}$ having the trivial G-module structure. □

Remark 3.27. Note that from the proof of theorem 3.4 we see that $\mathrm{St}(\Sigma_g)$ is also cyclic, and is generated by the class of ϕ_0, since we had that $\mathrm{St}(\Sigma_g) \cong \mathrm{St}(\Sigma_g^1)_P$.

4 Some consequences

4.1 Spheres in the curve complex

Recall from 3.12 that the arc complex at infinity $\mathcal{A}_\infty(\Sigma_g^1)$ is homotopy equivalent to a wedge sum of spheres of dimension $2g-2$. In this section we will see that the curve complex $\mathcal{C}(\Sigma_g^1)$ is homotopy equivalent to $\mathcal{A}_\infty(\Sigma_g^1)$, and thus in turn also homotopy equivalent to a wedge sum of spheres of dimension $2g-2$.

The homotopy equivalence between them can then be used to find explicit examples of spheres in $\mathcal{C}(\Sigma_g^1)$.

4.1.1 The homotopy equivalence between $\mathcal{C}(\Sigma_g^1)$ and $\mathcal{A}_\infty(\Sigma_g^1)$

In this section we will give an overview of Harer's homotopy equivalence between the arc complex at infinity and the curve complex. Instead of defining the homotopy equivalences directly between them we will define it as a map

$$\Psi : \mathcal{A}_\infty^{\circ\circ}(\Sigma_g^1) \to \mathcal{C}^\circ(\Sigma_g^1)$$

where S° denoted the barycentric subdivision of S (which is always homotopy equivalent to S).

To construct Ψ we need to consider the details of how the barycentric subdivision is defined formally, so we will start by recalling a precise definition of the barycentric subdivision.

Definition 4.1. Let S be a simplicial complex. The **barycentric subdivision** of S, denoted S° is a simplicial complex where the n-simplices are chains of simplices is S ordered by inclusion and of length $n+1$, and the face maps are given by removing one of the elements of a chain.

To illustrate why this definition actually gives a barycentric subdivision, we will look at an example.

Example 4.2. Let S be the simplicial complex consisting of three 0-simplices $\{x, y, z\}$, three 1-simplices $\{a, b, c\}$ and one 2-simplex $\{A\}$, arranged as a triangle. Then the set of 0-simplices in S° is chains of just a single simplex in S, so the vertex set of S° is $\{x, y, z, a, b, c, A\}$. If we think of a, b and c as adding vertices to S in the middle of the respective 1-simplices, and A as a point in the barycenter of the triangle, we can see that S° looks like in figure 5. For example each 2-simplex will be a chain with a 0-simplex, a 1-simplex and a 2-simplex, there are six chains of this form, each corresponding to one of the small triangles in the figure.

Now consider $\mathcal{A}_\infty^{\circ\circ}(\Sigma_g^1)$. A vertex in $\mathcal{A}_\infty^{\circ\circ}(\Sigma_g^1)$ is a simplex in $\mathcal{A}_\infty^{\circ}(\Sigma_g^1)$, which is a chain of non-filling arc systems, and a simplex in $\mathcal{A}_\infty^{\circ\circ}(\Sigma_g^1)$ is a sequence of such chains so that going one step down the sequence corresponds to removing one of the arc systems in the chain, for example if we have non-filling arc systems $\alpha_0 \subset \alpha_1 \subset \alpha_2$ then

$$(\alpha_0) \subset (\alpha_0 \subset \alpha_2) \subset (\alpha_0 \subset \alpha_1 \subset \alpha_2)$$

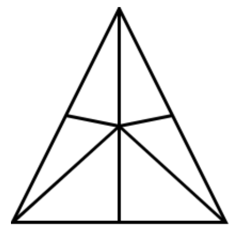

Figure 5: The barycentric subdivision of S from example 4.2.

is an example of a 2-simplex if $\mathcal{A}_\infty^{\circ\circ}(\Sigma_g^1)$.

For a vertex v in $\mathcal{A}_\infty^{\circ\circ}(\Sigma_g^1)$, say

$$\alpha_0 \subset \alpha_1 \subset ... \subset \alpha_n$$

where the α_i are non-filling arc systems. If we remove a small regular neighborhood of the arcs in α_i we get a surface $\Sigma(i)$ with boundary, and the boundary components give a curve system in Σ_g^1 if we omit duplicates and trivial curves, let c_i be the curve system obtained in this way from α_i.

Harer defines $\Psi(v)$ as $\bigcup_{i=0}^n c_i$ where we again omit any redundancies. Note that the union is in fact still a curve system, because if $\alpha_i \subset \alpha_j$ we have $\Sigma(j) \subset \Sigma(i)$ which means that we can choose the representatives such that the curves in c_i are disjoint from the curves in c_j. Thus $\bigcup_{i=0}^n c_i$ is a curve system, and the curve systems are the vertices in $\mathcal{C}^\circ(\Sigma_g^1)$. We can then extend Ψ to all of $\mathcal{A}_\infty^{\circ\circ}(\Sigma_g^1)$ simplicially, meaning that a simplex with vertices $v_1, ..., v_n$ gets mapped to a simplex with vertices $\Psi(v_1), ..., \Psi(v_n)$, respecting the simplicial structure.

The 0-filling system ϕ_0 from before is nontrivial as we saw before, and recall from lemma 3.13 that

$$\mathrm{St}(\Sigma_g^1) \cong H_{2g-1}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z}).$$

Also recall from the proof of lemma 3.13 that we had an isomorphism of homology groups

$$H_{2g-1}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z}) \cong \tilde{H}_{2g-2}(\mathcal{A}_\infty; \mathbb{Z}).$$

What this means is that ϕ_0 represents a non-trivial simplex of dimension $2g-1$ in $\mathcal{A}$ and that $\partial[\phi_0]$ is an element of $H_{2g-2}(\mathcal{A}_\infty; \mathbb{Z}) \cong \mathrm{St}(\Sigma_g^1)$ which is represented by the boundary of a simplex of dimension $2g-1$, which means that it is a sphere of dimension $2g-2$ in $\mathcal{A}_\infty$.

In this way the generator ϕ_0 for the Steinberg module together with Harer's homotopy equivalence Ψ can be used to find explicit spheres of dimension $2g-2$ in $\mathcal{C}^\circ(\Sigma_g^1)$ and also in $\mathcal{C}^\circ(\Sigma_g)$. If we take the image of the sphere $\partial\phi_0$ under Ψ we obtain an explicit sphere of dimension $2g-2$ in $\mathcal{C}^\circ(\Sigma_g^1)$, and from there we can get a sphere in $\mathcal{C}^\circ(\Sigma_g)$ by forgetting the marked point. We can then use this to find spheres in the unbarycentrically subdivided curve complexes $\mathcal{C}(\Sigma_g^1)$ and $\mathcal{C}(\Sigma_g)$.

In [2] Broaddus gives an explicit picture of a sphere in $\mathcal{C}(\Sigma_2^1)$ and also in $\mathcal{C}(\Sigma_2)$ which he constructs by the method outlined above.

4.2 Rational cohomology of the mapping class group

Finally we will look at a proof of the following theorem which is from Church, Farb and Putman in [4].

Theorem 4.3. *For all $g \geq 2$*

$$H^{4g-5}(Mod(\Sigma_g); \mathbb{Q}) = 0.$$

They prove this using chord diagrams and results from Broaddus. First we will recall that

$$(\mathrm{St}(\Sigma_g))_{\mathrm{Mod}(\Sigma_g)} \cong (\mathcal{C}_0/\partial C_1)_{\mathrm{Mod}(\Sigma_g)}.$$

Now, the co-invariants of the chain complex $\mathcal{C}_\bullet$ form a chain complex $\mathcal{U}_\bullet$ which can be described as follows: $\mathcal{U}_k$ is the free abelian group generated by $\mathrm{Mod}(\Sigma_g)$-orbits of k-filling systems thought of as a $2g+k$-tuple of arcs. We also quotient by the relation $s \cdot \alpha = \mathrm{sgn}(s)\alpha$ where $s \in S_{2g+k}$ acts on a k-filling system α by permuting the arcs, this sign is added to make the boundary map work. For a k-filling system $\alpha = (\alpha_1, ..., \alpha_{2g+k})$ we let $\partial_i \alpha$ be $(\alpha_1, ..., \hat{\alpha_i}, ..., \alpha_{2g+k})$ if this is a $k-1$-filling system and 0 otherwise, and then

$$\partial \alpha = \sum_{i=0}^{2g+k} (-1)^{i+1} \partial_i \alpha.$$

With this description of the chain complex of the co-invariants, we can prove the theorem.

Proof of theorem 4.3. Recall the generator $\phi_0 = (x_1, ..., x_{2g})$ for the Steinberg module. Since $(\mathrm{St}(\Sigma_g))_{\mathrm{Mod}(\Sigma_g)}$ is a quotient of $\mathrm{St}(\Sigma_g)$ the class of ϕ_0 in the quotient must also generate $(\mathrm{St}(\Sigma_g))_{\mathrm{Mod}(\Sigma_g)}$, so showing that $[\phi_0] = 0$ in $\mathcal{U}_0/\partial\mathcal{U}_1$ would imply that

$$(\mathrm{St}(\Sigma_g))_{\mathrm{Mod}(\Sigma_g)} \cong H_0(\mathrm{Mod}(\Sigma_g); \mathrm{St}(\Sigma_g)) = 0.$$

Let y be the arc corresponding to the shorter dashed line in figure 6 and z be the arc corresponding to the other dashed line, so y intersects only x_{2g} and z intersects both x_1 and x_{2g}. Let $\phi_0^y = (x_1, ..., x_{2g}, y)$ and $\phi_0^z = (x_1, ..., x_{2g}, z)$.

If we compute $\partial_i \phi_0^y$ we see that for $1 < i < 2g+1$ the arc system we get by removing x_i is disconnected, and if we remove either x_1 or y we get an arc system which is in the same orbit as ϕ_0 itself, so we get $\partial \phi_0^y = 2\phi_0$ and thus $2[\phi_0] = 0$ in $\mathcal{U}_0/\partial\mathcal{U}_1$.

Next we compute $\partial_1 \phi_0^z$ we again get an arc system in the same orbit as ϕ_0, so $\partial_1 \phi_0^z = \phi_0$, and then we get $\partial_2 \phi_0^z = -\phi_0$, $\partial_3 \phi_0^z = \phi_0$, and so on. Taken together we get that $\partial \phi_0^z = (2g+1)\phi_0$, so $(2g+1)[\phi_0] = 0$ in $\mathcal{U}_0/\partial\mathcal{U}_1$.

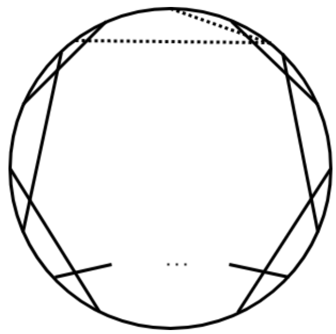

Figure 6: The new arcs y and z.

Lastly we can compute that $[\phi_0] = g(2[\phi_0]) + [\phi_0] = (2g+1)[\phi_0] = 0$, showing that

$$H_0(\mathrm{Mod}(\Sigma_g); \mathrm{St}(\Sigma_g)) = 0$$

and therefore

$$H_0(\mathrm{Mod}(\Sigma_g); \mathrm{St}(\Sigma_g) \otimes \mathbb{Q}) \cong H^{4g-5}(\mathrm{Mod}(\Sigma_g); \mathbb{Q}) = 0$$

where the isomorphism above follows from $\mathrm{St}(\Sigma_g)$ being the dualizing module for $\mathrm{Mod}(\Sigma_g)$ and the connection between $\mathrm{Mod}(\Sigma_g)$ and $\mathcal{M}_g$, the moduli space of genus g surfaces, see [4] for a more thorough explanation of this isomorphism. $\square$

Remark 4.4. The same argument would also work for $\mathrm{Mod}(\Sigma_g^1)$ since it also has the Steinberg module as its dualizing module and is generated by ϕ_0 as well, the only difference is that we would have $4g-3$ instead of $4g-5$.

References

[1] Bieri and Eckmann. Groups with homological duality generalizing poincaré duality. *Inventiones mathematicae*, 20:103–124, 1973.

[2] Nathan Broaddus. Homology of the curve complex and the Steinberg module of the mapping class group. *Duke Math. J.*, 161(10):1943–1969, 2012.

[3] Kenneth S. Brown. *Cohomology of Groups*. Springer New York, 1982.

[4] Thomas Church, Benson Farb, and Andrew Putman. The Rational Cohomology of the Mapping Class Group Vanishes in its Virtual Cohomological Dimension. *International Mathematics Research Notices*, 2012(21):5025–5030, 12 2011.

[5] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton University Press, 2012.

[6] John L. Harer. Stability of the homology of the mapping class groups of orientable surfaces. *Annals of Mathematics*, 121(2):215–249, 1985.

[7] John L. Harer. The virtual cohomological dimension of the mapping class group of an orientable surface. *Inventiones mathematicae*, 84:157–176, 1986.