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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

## Verdier Duality: Generalization of Poincaré Duality via Sheaf Cohomology

av

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# Verdier Duality: Generalization of Poincaré Duality via Sheaf Cohomology

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**Abstract**

Poincaré duality is a relationship of the structure of the homology and cohomology groups of orientable manifolds. This paper discusses a possible generalization of this relationship to a wider set of topological spaces, deriving the so-called Verdier duality. To this end, we will discuss chain complexes over a general abelian category, and derive the homotopy category and the derived category thereof; the derived category arises from a localization of the homotopy category. This allows us to derive relationships that do not necessarily hold at the level of chain complexes. We further show a way to induce functors in the derived category.

We continue by introducing sheaves and study their properties as a category. We discuss several ways in which continuous functions induce functors between categories of sheaves. Further, we derive a way to induce these functors in the derived category of sheaves, yielding relationships at the level of complexes. We finish by presenting Verider duality and show that Poincaré duality is a special case of this relationship.

(Swedish) Poincaré duality är en relation av strukturen mellan de homologiska och kohomologiska grupperna av en orienterbar mångfald. Detta papper diskuterar möjlig generalisering av denna relation till en större grupp av topologiska rum, en härledning av den så kallade Verdier duality. Till detta ändamål kommer vi diskutera kedjekomplex över en generell abelsk kategori, och härleda den homotopiska kategorin och den härledda kategorin därav; den härledda kategorin kommer från en lokalisering av den homotopiska kategorin. Detta möjliggör härledningen av relationer som inte nödvändigtvis gäller för kedjekomplex. Vi fortsätter att visa ett sätt att inducera funktorer i den härledda kategorin.

Vi fortsätter med att introducera kärvar och studerar deras egenskaper som kategori. Vi diskuterar flertalet sätt att inducera kontinuerliga funktioner i topologiska rum till funktorer över kategorier av kärvar. Fortsättningsvis, vi härleder ett sätt att inducera dessa funktorer till den härledda kategorin av kärvar, vilket ger relationer på nivån av komplex. Vi avslutar med att presentera Verdier duality, och visar att Poincaré duality är ett specialfall av denna relation.

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# 1 Introduction

In algebraic topology one associates algebraic structures to topological spaces. It can be seen as a bridge between 'continuous' and 'discrete' mathematics [1]. The algebraic structures are usually invariant under homeomorphisms, and in many cases are invariant under homotopy equivalences. They can help to determine which spaces are homeomorphic to each other and which are genuinely different. Ideally, one wants to find algebraic structures that are sophisticated enough to distinguish between spaces but are simple enough to compute.

One of the first algebraic structures encountered associated with a topological space $X$ is the fundamental group $\pi_1(X)$ [2]. If $X$ is a CW-complex, one can show that $\pi_1(X)$ only depends on the 2-skeleton of $X$ [3]. Because of this, we could say that there is a low-dimensional nature of $\pi_1(X)$; it can not distinguish between spheres of dimension $n \geq 2$. Thus, it is of interest to define higher-dimensional analogs to 1-dimensional loops that can capture higher-dimensional structures. The homotopy groups $\pi_n(X)$, defined via maps from the unit cube $I^n$ into $X$, solve this problem. The drawback, however, is that they are difficult to compute.

To the end of having easily computable algebraic structures that are invariant under homeomorphisms, the homology groups were created [3]. The trick was to abelianize loops when studying the fundamental group and create an appropriate relationship between different dimensional 'structures' in the topology. Given a cellular decomposition of a topological space, the homology groups are relatively easy to compute. However, it turns out that they do not exhibit a lot of 'structure'; they are simple groups. To enrich the theory, cohomological groups enjoy a canonical multiplication making them into rings [3].

A canonical isomorphism exists of the homology groups and cohomology groups given a topological space that is an orientable manifold [3]. This relationship is called Poincaré duality. One consequence of this duality is that if we have an $n$-dimensional manifold, the cohomology groups of higher dimensions than $n$ vanish. Thus, relationships of different algebraic structures are interesting; knowledge about a structure can be inferred from another.

Generalizing homology and cohomology groups can suitably be done with category theory, which is one of the main topics in this document. We will define the necessary properties of a category to be able to define a well-behaved homology functor of a chain complex consisting of objects in that category. Homological relationships can be derived by looking at quotients and localization of the chain complex.

The main category of interest will be sheaves. These are abstractions of associating algebraic structures to open subsets in a given topological space. We will look at different sheaves, and connect the theory thereof to the theory of homological algebra. This connection allows us to derive rich relationships at the level of topology.

# 2 Limits

In this section, we will discuss limits of objects in a category. Informally, take any commutative diagram that consists of objects and morphisms. The limit (or colimit) of the diagram is an object that maps to (or from) the diagram such that any other object mapping similarly will factor uniquely to the limit (or colimit). Interestingly, this generalizes several common operations. For example, disjoint union in **Sets** and direct sum in ${}_R$**Mod** can be described by the same limit operation in their respective category.

The approach to defining limits will be by looking at several special cases. The reason for this is twofold. First, this will make it easier to motivate the definition of limits. Second, some of the special cases will be used explicitly later.

The statements made and the proofs thereof come mostly from Rotman [4]. When studying the direct limit, ideas from Tennison [5] are used.

## 2.1 Categorical constructions

We recall the following.

**Definition 2.1.** Let $\mathcal{C}$ be a category and let $A \in Ob(\mathcal{C})$. $A$ is called an **initial object** if for every object $X \in \mathcal{C}$, there exists a unique morphism $A \to X$.

**Lemma 2.1.** *If $A$ and $A'$ are initial objects in a category $\mathcal{C}$, they are isomorphic.*

*Proof.* Since $A$ is an initial object, there exists a unique morphism $f : A \to A'$, and similarly, there exists a unique morphism $g : A' \to A$. Further, the morphism $A \to A$ is the identity morphism (since there is only one), thus $gf = id_A : A \to A$. Similarly, we find $fg = id_{A'}$, and so $f$ and $g$ are isomorphisms. □

**Definition 2.2.** Let $\mathcal{C}$ be a category and let $\Omega \in Ob(\mathcal{C})$. $\Omega$ is called a **terminal object** if for every object $X \in \mathcal{C}$, there exists a unique morphism $X \to \Omega$.

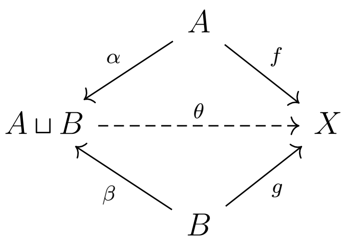
**Lemma 2.2.** *If $\Omega$ and $\Omega'$ are terminal objects in a category $\mathcal{C}$, then they are isomorphic.*

*Proof.* Consider the opposite category $\mathcal{C}^{op}$. Then terminal objects become initial objects, showing that $\Omega$ and $\Omega'$ are isomorphic in $\mathcal{C}^{op}$ by Lemma 2.1, and thus so in $\mathcal{C}$. □

**Definition 2.3.** Let $\mathcal{C}$ be a category and let $A \in Ob(\mathcal{C})$. $A$ is called a **zero object** if it is both an initial object and a terminal object.

### 2.1.1 Coproducts and products

**Definition 2.4.** Let $\mathcal{C}$ be a category and let $A$ and $B$ be objects in $\mathcal{C}$. The **coproduct** of $A$ and $B$ is a triple $(A \sqcup B, \alpha, \beta)$, where $A \sqcup B$ is an object in $\mathcal{C}$, $\alpha : A \to A \sqcup B$, $\beta : B \to A \sqcup B$ are morphisms, that is a solution to the following universal mapping problem: for every object $X \in Ob(C)$ with morphisms $f : A \to X$ and $g : B \to X$, there exists a unique morphism $\theta : A \sqcup B \to X$ making the following diagram commute:




We call $\alpha$ and $\beta$ the structure maps of the coproduct.

It should be noted that the coproduct does not necessarily exist in an arbitrary category. However, we will later show that if it does exist, it is unique up to isomorphism. We now look at examples.

**Example 2.1.** For two sets $A$ and $B$ in **Sets**, their coproduct is their disjoint union $A \sqcup B$, or to be more correct, $(A \sqcup B, p_A, p_B)$, where $\sqcup$ is the disjoint union operator for sets, $p_A : A \sqcup B \to A$ is the projection to the set $A$ and $p_B : A \sqcup B \to B$ is the projection to the set $B$. Here, we define the disjoint union to be given by $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$, where $\times$ is the Cartesian product.

To see this, consider an arbitrary object $X$ in $\mathcal{C}$ and any maps $f : A \to X$ and $g : B \to X$. Let $\theta : A \sqcup B \to X$ be given by $(a, 1) \mapsto f(a)$, $(b, 2) \mapsto g(b)$, which clearly is well-defined. It is also not hard to see that it makes the diagram in the definition of coproduct commute.

Consider another mapping $\psi : A \sqcup B \to X$ such that $\psi\alpha = f$ and $\psi\beta = g$. But then $\psi((a, 1)) = f(a) = \theta((a, 1))$ and $\psi((b, 2)) = g(b) = \theta((b, 2))$, showing $\psi = \theta$. Hence, coproduct in **Sets** exists and is given by the disjoint union of two sets.

Given objects $A$ and $B$ in a category, we will often not mention explicitly the structure maps $p_A$ and $p_B$.

**Example 2.2.** For two left $R$-modules in ${}_R\mathbf{Mod}$, their coproduct exists and is the direct sum $A \oplus B$. The structure maps in the coproduct are given by $\alpha : a \mapsto (a, 0)$ and $\beta : b \mapsto (0, b)$, which are clearly $R$-maps.

Let $X \in Ob(\mathcal{C})$, and let $f : A \to X$ and $g : B \to X$ be $R$-maps. Let $\theta : A \oplus B \to X$ be given by $\theta((a,b)) = f(a) + g(b)$. Now, we see that $\theta(\alpha(a)) = \theta((a,0)) = f(a)$ for $a \in A$, and similarly $\theta(\beta(b)) = \theta((b,0)) = g(b)$. Thus, for a new $R$-map $\psi : A \oplus B \to X$ making the diagram commute, we have $\psi(\alpha(a)) = f(a)$ and $\psi((\beta(b)) = g(b)$ for $a \in A$, $b \in B$. Since $\psi$ is an $R$-map, we have $\psi((a,b)) = \psi((a,0)) + \psi((0,b)) = f(a) + g(b)$, showing $\psi = \theta$. Thus $\theta$ is unique, and the coproduct in ${}_R\mathbf{Mod}$ is given by the direct sum.

We will now prove that coproduct is unique (up to canonical isomorphism).

**Proposition 2.3.** *Let $\mathcal{C}$ be a category and let $A, B$ be objects in $\mathcal{C}$. Any two coproducts of $A$ and $B$, given that they exist, are isomorphic.*

*Proof.* Consider a category $\mathcal{D}$ with objects

$$A \xrightarrow{\gamma} X \xleftarrow{\delta} B,$$

where $\gamma : A \to X$, $\delta : B \to X$ are morphisms, and $X$ is an object in $\mathcal{C}$. The morphisms in $\mathcal{D}$ are given by a triple $(1_A, \theta, 1_B)$, where $\theta$ is a morphism such that the following diagram commutes.

$$\begin{array}{ccccc} A & \xrightarrow{\gamma} & X & \xleftarrow{\delta} & B \\ \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow{\gamma'} & X' & \xleftarrow{\delta'} & B \end{array}$$

Composition of two morphisms $(1_A, \theta, 1_B)$ and $(1_A, \psi, 1_B)$ is given by $(1_A, \theta, 1_B)(1_A, \psi, 1_B) = (1_A, \theta\psi, 1_B)$. Proving that $\mathcal{D}$ is a category is routine.

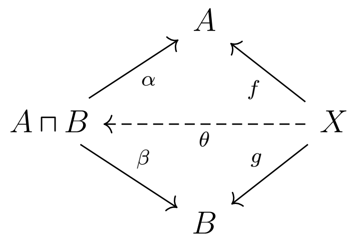
Assume the coproduct of $A$ and $B$ exists, and let $\alpha : A \to A \sqcup B$ and $\beta : B \to A \sqcup B$ be the morphisms of the coproduct. Consider the object

$$A \xrightarrow{\alpha} A \sqcup B \xleftarrow{\beta} B.,$$

Now, for any other object $A \xrightarrow{f} X \xleftarrow{g} B$, by the definition of coproduct, there exists a unique morphism $\theta : A \sqcup B \to X$, hence the morphism $(1_A, \theta, 1_B)$ from $A \xrightarrow{\alpha} A \sqcup B \xleftarrow{\beta} B$ to $A \xrightarrow{f} X \xleftarrow{g} B$ is unique. This shows $A \xrightarrow{\alpha} A \sqcup B \xleftarrow{\beta} B$ is an initial object, and hence by Lemma 2.1, this is unique up to isomorphism. Thus, if there are two coproducts of $A$ and $B$, their corresponding objects in $\mathcal{D}$ are isomorphic, hence also isomorphic in $\mathcal{C}$. $\square$

We now present the dual of coproduct.

**Definition 2.5.** Let $\mathcal{C}$ be a category and let $A$ and $B$ be objects in $\mathcal{C}$. The **product** of $A$ and $B$ is a triple $(A \sqcap B, p, q)$ such that $A \sqcap B$ is an object in $\mathcal{C}$, and $p : A \sqcap B \to A$ and $q : A \sqcap B \to B$ are morphisms called **projections**, that is a solution to the following universal mapping problem: for every object $X \in Ob(\mathcal{C})$ with morphisms $f : X \to A$ and $g : X \to B$, there exists a unique morphism $\theta : X \to A \sqcap B$ making the following diagram commute:




Again, we exemplify with ${}_R\mathbf{Mod}$.

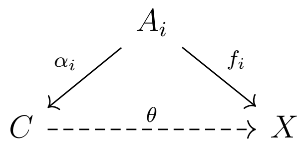
**Example 2.3.** Let $A$ and $B$ be objects in ${}_R\mathbf{Mod}$. Then their product exists and is given by $A \sqcap B = A \oplus B$. Let the morphisms in the product be given by $p : (a, b) \mapsto a$ and $q : (a, b) \mapsto b$. Let $X$ be an object in $\mathcal{C}$ with morphisms $f : X \to A$ and $g : X \to B$. Let $\theta : X \to A \oplus B$ be given by $\theta(x) = (f(x), g(x))$. It is clear that this makes the diagram commute. We need to show that $\theta$ is unique. Let $\psi : X \to A \oplus B$ be another map such that $\alpha\psi = f$ and $\beta\psi = g$. Then, for any $x \in X$, $\alpha(\psi(x)) = f(x)$ and $\beta(\psi(x)) = g(x)$, that is, $\psi(x) = (f(x), g(x)) = \theta(x)$, which shows uniqueness.

**Proposition 2.4.** *Let $\mathcal{C}$ be a category and let $A, B$ be objects in $\mathcal{C}$. Any two products of $A$ and $B$, given that they exist, are isomorphic.*

*Proof.* The proof is analogous to the proof of Proposition 2.3; the product is a terminal object in a suitable category. □

We will now extend the definition of coproduct and product from two objects to arbitrary many objects.

**Definition 2.6.** Let $\mathcal{C}$ be a category and let $(A_i)_{i \in I}$ be a family of objects in $\mathcal{C}$, where $I$ is an index set. A **coproduct** is an ordered pair $(C, (\alpha_i : A_i \to C)_{i \in I})$, with $C$ being an object in $\mathcal{C}$, and $(\alpha_i)_{i \in I}$ is a family of morphisms, called **injections**, that is a solution to the following universal mapping problem: for every object $X$ in $\mathcal{C}$ and morphisms $(f_i : A_i \to X)_{i \in I}$, there exists a unique morphism $\theta : C \to X$ such that the following diagram commutes for each $i \in I$.




The coproduct, should it exist, is typically denoted by $\coprod_{i \in I} A_i$. Furthermore, the coproduct is unique up to isomorphism, which can be shown by defining a suitable category and noting that the coproduct is an initial object.

**Example 2.4.** Let $(A_i)_{i \in I}$ be a family of objects in ${}_R\mathbf{Mod}$. Then the coproduct of $(A_i)_{i \in I}$ exists, and is given by $\bigoplus_{i \in I} A_i$. The morphisms $\alpha_i : A_i \to \bigoplus_{i \in I} A_i$ are given by sending $a_i$ to the element with $i$th coordinate being $a_i$, and the rest zero. It is clear that $\alpha_i$ are $R$-maps.

Now, let $X$ be a left $R$-module, and let $f_i : A_i \to X$ be an $R$-map for each $i \in I$. Choose an element $(a_i) \in \bigoplus_{i \in I} A_i$, and since there are finitely many non-zero elements, we have $(a_i) = \sum_i \alpha_i a_i$. We define $\theta$ to be given by $(a_i) \mapsto \sum_i f_i a_i$. To see that the diagram commutes, we have $a_i \in A_i$, then $\theta \alpha_i a_i = \theta((\alpha_i)) = f_i a_i$.

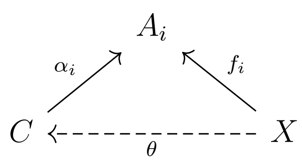
Let $\psi : \bigoplus_{i \in I} A_i \to X$ be another $R$-map making the diagram commute. Then,

$$\begin{aligned} \psi((a_i)) &= \psi(\sum_i \alpha_i a_i) \\ &= \sum_i \psi \alpha_i a_i \\ &= \sum_i f_i(a_i) \\ &= \theta((a_i)), \end{aligned}$$

hence $\psi = \theta$. This shows that $\theta$ is unique, and hence the coproduct of $(A_i)_{i \in I}$ is given by $\bigoplus_{i \in I} A_i$.

We also present an extension of the product.

**Definition 2.7.** Let $\mathcal{C}$ be a category and let $(A_i)_{i \in I}$ be a family of objects in $I$, where $I$ is an index set. A **product** is an ordered pair $(C, (p_i : C \to A_i)_{i \in I})$, where $C$ is an object in $\mathcal{C}$, and $(p_i)_{i \in I}$ is a family of morphisms, called **projections**, that is a solution to the following universal mapping problem: for every object $X$ in $\mathcal{C}$ and morphisms $(f_i : X \to A_i)_{i \in I}$, there exists a unique morphism $\theta : X \to C$ such that the following diagram commutes for each $i$.



The product, should it exist, is typically denoted by $\prod_{i \in I} A_i$. Furthermore, the product is unique up to isomorphism, which can be shown by defining a suitable category and noting that the product is a terminal object. We now provide an example.

**Example 2.5.** Let $(A_i)_{i \in I}$ be a family of objects in ${}_R\mathbf{Mod}$. Then the product of $(A_i)_{i \in I}$ exists and is given by $\prod_{i \in I} A_i$. The projections $p_i : \prod_{i \in I} A_i \to A_i$ are given by $(a_i) \mapsto a_i$ for all $i$. It is clear that the projections are $R$-maps.

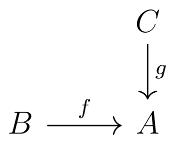
Now, let $X$ be a left $R$-module, and let $f_i : X \to A_i$ be an $R$-map for each $i \in I$. We define $\theta : X \to \prod_{i \in I} A_i$ to be given by $x \mapsto (f_i(x))$. For $x \in X$, we have $p_i \theta(x) = f_i(x)$ showing that the diagram commutes.

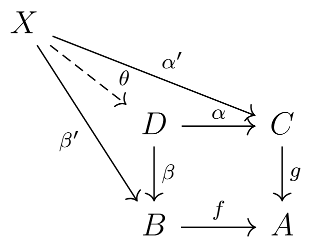
Let $\psi : X \to \prod_{i \in I} A_i$ be another $R$-map making the diagram commute, hence, $p_i \psi(x) = f_i(x)$ for $x \in X$ and all $i \in I$. This means that the $i$th coordinate of $\psi(x)$ is $f_i(x)$, the same as $\theta(x)$, hence $\psi = \theta$, showing uniqueness.

### 2.1.2 Pullback and pushout

We will now give another pair of dual categorical constructions.

**Definition 2.8.** Let $\mathcal{C}$ be a category, $A, B, C$ objects in $\mathcal{C}$, and $f : B \to A$ and $g : C \to A$ morphisms. A **pullback** is a triple $(D, \alpha, \beta)$, where $D$ is an object in $\mathcal{C}$, $\alpha : D \to C$ and $\beta : D \to B$ are morphisms, and $g\alpha = f\beta$, that is a solution to the following universal mapping problem: for every triple $(X, \alpha', \beta')$, with $g\alpha' = f\beta'$, there exists a unique morphism $\theta : X \to D$ making the following diagram commute.




The pullback, should it exist, is unique up to isomorphism. This is shown by defining a suitable category and noting that the pullback is the terminal object in that category. We exemplify with pullbacks in ${}_R\mathbf{Mod}$.

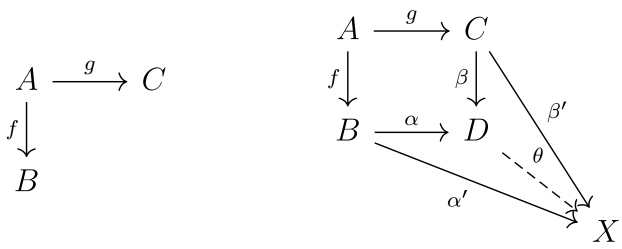
**Example 2.6.** Given three objects $A, B$, and $C$ in ${}_R\mathbf{Mod}$, and morphisms $f : B \to A$, $g : C \to A$, the pullback exists, and is given by the set $D = \{(b, c) \in B \oplus C : f(b) = g(c)\}$. The morphisms of the pullback are the restrictions $\alpha : D \to C$, $(b, c) \mapsto c$, and $\beta : D \to B$, $(b, c) \mapsto b$. It is clear that these morphisms make the diagram commute.

Let the triple $(X, \alpha', \beta')$ be such that $X$ is a left $R$-module, $\alpha' : X \to C$ and $\beta' : X \to B$, and $\alpha' g = \beta' f$. We define $\theta : X \to D$ to be given by $x \mapsto (\beta'(x), \alpha'(x))$. It is clear that $\theta$ makes the diagram commute.

Let $\psi : X \to D$ be another $R$-map making the diagram commute. Then, for $x \in X$, we have $\alpha(\psi(x)) = \alpha'(x)$ and $\beta(\psi(x)) = \beta'(x)$. This means $\psi(x) = (\beta'(x), \alpha'(x))$ and thus $\psi = \theta$. This shows that $\theta$ is unique.

We now define the dual of pullback, namely pushout.

**Definition 2.9.** Let $\mathcal{C}$ be a category, $A, B, C$ objects in $\mathcal{C}$, and $f : A \to B$ and $g : A \to C$ morphisms. A **pushout** is a triple $(D, \alpha, \beta)$, where $D$ is an object in $\mathcal{C}$, $\alpha : B \to D$ and $\beta : C \to D$ are morphisms, and $\beta g = \alpha f$, that is a solution to the following universal mapping problem: for every triple $(X, \alpha', \beta')$, with $\alpha' f = \beta' g$, there exists a unique morphism $\theta : D \to X$ making the following diagram commute.




Pushout, should it exist, is unique up to isomorphism. This can be shown by defining a suitable category and noting that a pushout is an initial object therein.

**Example 2.7.** Given three objects $A$, $B$, and $C$ in ${}_R\mathbf{Mod}$ and morphisms $f : A \to B$, $g : A \to C$, the pushout exists, and is given by $D = (B \oplus C)/S$, with $S = \{(f(a), -g(a)) \in B \oplus C : a \in A\}$ (it is easy to show that $S$ is a submodule of $B \oplus C$, hence the quotient $D$ makes sense). The morphisms of the pushout are given by $\alpha : B \to D$, $b \mapsto (b, 0) + S$ and $\beta : C \to D$, $c \mapsto (0, c) + S$. For $a \in A$, we have $\alpha(f(a)) = (f(a), 0) + S$ and $\beta(g(a)) = (0, g(a)) + S$. Now, $(f(a), 0) + S = (0, g(a)) + S$ since $(f(a), 0) - (0, g(a)) = (f(a), -g(a)) \in S$, showing that the diagram commute.

Consider another triple $(X, \alpha', \beta')$ such that $\alpha' f = \beta' g$. We define $\theta : D \to X$ to be given by $(b, c) + S \mapsto \alpha'(b) + \beta'(c)$. First, $\theta$ is well defined, which can be seen in the following. If $(b', c')$ is another representative of $(b, c) + S$, then $(b - b', c - c') \in C$; that is, there exists an $a \in A$ such that $f(a) = b - b'$ and $-g(a) = c - c'$. By commutativity, we have $\alpha'(f(a)) = \beta'(g(a))$, hence $\alpha'(b - b') - \beta'(-(c - c')) = 0$. Since $\alpha'$ and $\beta'$ are $R$-maps, it follows that

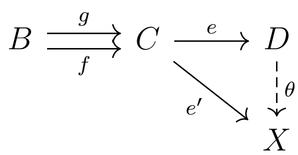
$\alpha'(b) - \alpha'(b') + \beta'(c) - \beta'(c') = 0$, or $\alpha'(b) + \beta'(c) = \alpha'(b') + \beta'(b')$. Second, $\theta$ makes the diagram commute, which can easily be shown.

To prove that $\theta$ is unique, let $\psi : D \to X$ be another $R$-map that makes the diagram commute. By commutativity, let $b \in B$ and $c \in C$, we have $\psi(\alpha(b)) = \psi((b,0)+S) = \alpha'(b)$ and $\psi(\beta(c)) = \psi((0,c)+S) = \beta'(c)$. Since $\psi$ is an $R$-map, we have for $(b,c)+S \in D$, $\psi((b,c)+S) = \psi((b,0)+S)+\psi((0,c)+S) = \alpha'(b) + \beta'(c)$, hence $\psi = \theta$.

### 2.1.3 Coequalizer and equalizer

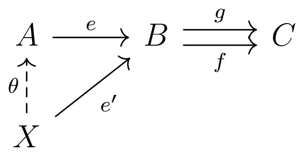
We continue with the last pair of dual categorical constructions, before looking at a generalization.

**Definition 2.10.** Let $\mathcal{C}$ be a category, $B$ and $C$ be objects in $\mathcal{C}$, and $f, g : B \to C$ be two morphisms. A **coequalizer** is an ordered pair $(D, e)$, where $D$ is an object in $\mathcal{C}$ and $e : C \to D$ is a morphism such that $ef = eg$, that is a solution to the following universal mapping problem: for all objects $X$ in $\mathcal{C}$ with a morphism $e' : C \to X$ such that $e'f = e'g$, there exists a unique morphism $\theta : D \to X$ such that $\theta e = e'$.




Coequalizer, given that it exists, is unique up to isomorphism. This can be shown by defining a suitable category and noting that it is an initial object in that category. We continue with defining equalizer, the dual of coequalizer.

**Definition 2.11.** Let $\mathcal{C}$ be a category, $B$ and $C$ be objects in $\mathcal{C}$, and $f, g : B \to C$ be two morphisms. An **equalizer** is an ordered pair $(A, e)$, where $A$ is an object in $\mathcal{C}$ and $e : A \to B$ is a morphism such that $fe = ge$, that is a solution to the following universal mapping problem: for all objects $X$ in $\mathcal{C}$ with a morphism $e' : X \to B$ such that $fe' = ge'$, there exists a unique morphism $\theta : X \to A$ such that $e\theta = e'$.



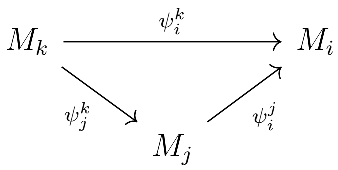

## 2.2 Generalization

We will now discuss two constructions that generalize the abovementioned constructions; inverse limits and direct limits.

### 2.2.1 Inverse limit

The inverse limit is a generalization of products, pullbacks, and equalizers. We start with defining an inverse system.

**Definition 2.12.** Let $\mathcal{C}$ be a category and let $I$ be a partially ordered set. An **inverse system** in $\mathcal{C}$ over $I$ is an ordered pair $((M_i)_{i \in I}, (\psi_i^j)_{j \geq i})$, where $M_i$ are objects in $\mathcal{C}$ for all $i \in I$, and $\psi_i^j : M_j \to M_i$ are morphisms for $i, j \in I$ such that $j \geq i$, and $\psi_i^i = id_{M_i}$ for all $i$. Furthermore, the morphisms make the following diagram commute whenever $k \geq j \geq i$.

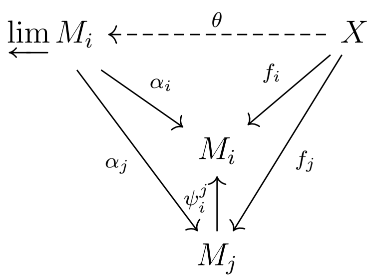



We abbreviate an inverse system with $\{M_i, \psi_i^j\}$. We continue with defining inverse limits.

**Definition 2.13.** Let $\mathcal{C}$ be a category, $I$ a partially ordered set, and $\{M_i, \psi_i^j\}$ an inverse system in $\mathcal{C}$ over $I$. The **inverse limit** is an object $\varprojlim M_i$ and a family of projections $(\alpha_i : \varprojlim M_i \to M_i)_{i \in I}$ such that

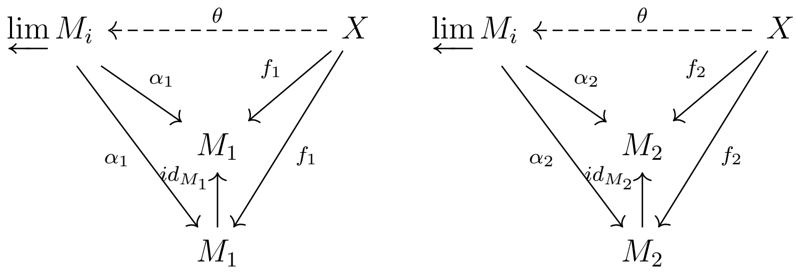
(i) $\psi_i^j \alpha_j = \alpha_i$, whenever $j \geq i$,

(ii) it satisfies the following universal mapping problem: for every object $X$ in $\mathcal{C}$ and all morphisms $f_i : X \to M_i$ satisfying $\psi_i^j f_j = f_i$, $j \geq i$, there exists a unique morphism $\theta : X \to \varprojlim M_i$, making the following diagram commute.

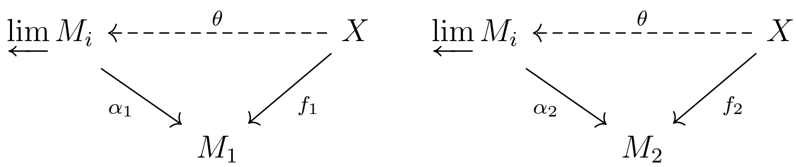



The inverse limit, given that it exists, is a terminal object in a suitable category, hence it is unique up to isomorphism. We continue by exemplifying how the inverse limit is a generalization of the product (for two objects). The reader can think about how the inverse limit also generalizes pullbacks and equalizers.

**Example 2.8.** Let $\mathcal{C}$ be a category and let $I = \{1, 2\}$ be a partially ordered set where each element only relates to itself. Thus, we have two objects $M_1$ and $M_2$, with identity morphisms, that give the following two diagrams.



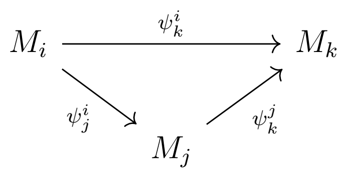

Note that the problem is equivalent to the following two diagrams, as given in the definition of the categorical product.




### 2.2.2 Direct limit

We continue with defining direct limit, which is a generalization of coproducts, pushouts, and coequalizers. We start with defining a direct system.

**Definition 2.14.** Let $\mathcal{C}$ be a category and let $I$ be a partially ordered set. A **direct system** in $\mathcal{C}$ over $I$ is an ordered pair $((M_i)_{i \in I}, (\psi^i_j)_{j \geq i})$, where $M_i$ are objects in $\mathcal{C}$ for all $i \in I$, and $\psi^i_j : M_i \to M_j$ are morphisms for $i, j \in I$ such that $j \geq i$, and $\psi^i_i = id_{M_i}$ for all $i$. Furthermore, the morphisms make the following diagram commute whenever $k \geq j \geq i$.

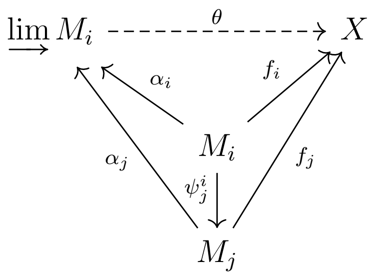



We abbreviate a direct system with $\{M_i, \psi^i_j\}$. Now, the definition of direct limit.

**Definition 2.15.** Let $\mathcal{C}$ be a category, $I$ a partially ordered set, and $\{M_i, \psi^i_j\}$ a direct system in $\mathcal{C}$ over $I$. The **direct limit** is an object $\varinjlim M_i$ and a family of **insertion morphisms** $(\alpha_i : M_i \to \varinjlim M_i)_{i \in I}$ such that

(i) $\alpha_j \psi_j^i = \alpha_i$ whenever $j \geq i$,

(ii) it satisfies the following universal mapping problem: for every object $X$ in $\mathcal{C}$ and all morphisms $f_i : M_i \to X$ satisfying $f_j \psi_j^i = f_i$, $j \geq i$, there exists a unique morphism $\theta : \varinjlim M_i \to X$ making the following diagram commute.



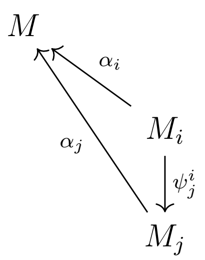

Again, the direct limit, given that it exists, is unique up to isomorphism since it is an initial object in a suitable category.

In our study, we are going to work with directed sets which makes the study of direct limits a little bit easier.

**Definition 2.16.** A directed set $I_D$ is a partially ordered set that satisfies the following: for all $i, j \in I_D$, there exists a $k \in I_D$ such that $k \geq i$ and $k \geq j$.

Before constructing the direct limit using a directed set, we need the following proposition.

**Proposition 2.5.** *Let $\mathcal{C}$ be a category, $I_D$ a directed set, and $\{M_i, \psi_j^i\}$ a direct system in $\mathcal{C}$ over $I_D$. Assume $M$, $(\alpha_i : M_i \to M)_{i \in I_D}$ is a target for the direct system; that is, the following diagram commutes.*



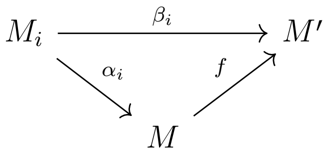

*Further, assume that the following two conditions hold:*

*(i) For all $m \in M$, there exists an $i \in I_D$ such that $m = \alpha_i(m_i)$.*

*(ii) If $i, j \in I_D$, $m_i \in M_i$, $m_j \in M_j$, then $\alpha_i(m_i) = \alpha_j(m_j)$ if and only if there exists a $k \in I_D$ such that $k \geq i$, $k \geq j$, and $\psi_k^i(\alpha_i) = \psi_k^j(\alpha_j)$.*

*Then $M$ is a direct limit of the system.*

*Proof.* Let $M'$, $(\beta_i : M_i \to M')_{i \in I_D}$ be another target for the system. Assume there exists a morphism $f : M \to M'$ such that the following diagram commutes.




Then, for any $m \in M$, by (i) there exists an $i \in I_D$ such that $\alpha_i(m_i) = m$. Hence, $f(m) = \beta_i(m_i)$; that is, $f$ is unique given that it exists.

We now show that $f$ is well-defined. Choose $m \in M$ and let $i, j \in I_D$ be such that $m = \alpha_i(m_i)$ and $m = \alpha_j(m_j)$, which exists by (i). By (ii) there exists a $k \in I_D$ such that $\psi^i_k(m_i) = \psi^j_k(m_j)$. Then, we have

$$\begin{aligned} \beta_i(m_i) &= \beta_k(\psi^i_k(m_i)) \\ &= \beta_k(\psi^j_k(m_j)) \\ &= \beta_j(m_j), \end{aligned}$$

and we conclude that $f$ is well-defined and hence exists. This shows that $M$ is a direct limit of the system. □

We will now give an explicit construction of the direct limit in the category **Group**. Groups will be one of the main studies when we will discuss sheaves. Let $I_D$ be a directed set and let $\{M_i, \psi^i_j\}$ be a direct system in **Group** over $I_D$. Let

$$W = \sqcup_{i \in I_D} M_i$$

be the disjoint union of the groups $M_i$. We define the equivalence relation $\sim$ by the following. Let $m_i \in M_i$ and $m_j \in M_j$. Then $m_i \sim m_j$ if and only if there exists a $k \in I_D$ such that $k \geq i$, $k \geq j$ and $\psi^i_k(m_i) = \psi^j_k(m_j)$.

To see that $\sim$ is an equivalence relation, it is clearly reflexive and symmetric. To show transitivity, let $m_k \in M_k$ and assume $m_i \sim m_j$ and $m_j \sim m_k$. Then there exists an $i'$ and $j'$ such that $\psi^i_{i'}(m_i) = \psi^j_{i'}(m_j)$ and $\psi^j_{j'}(m_j) = \psi^k_{j'}(m_k)$. By the definition of a directed set, there exists a $k' \in I_D$ such that $k' \geq i'$ and $k' \geq j'$. By the definition of a directed system, $\psi^{i'}_{k'}\psi^j_{i'}(m_j) = \psi^{j'}_{k'}\psi^j_{j'}(m_j)$; that is, $\psi^i_{k'}(m_i) = \psi^k_{k'}(m_k)$, which shows $m_i \sim m_k$.

Now, let $M = W/\sim$, and we let $\alpha_i : M_i \to M$ be given by the map $M_i \to W \to W/\sim$. It is not hard to show that $M$ satisfies the conditions of Proposition 2.5 and thus is a direct limit of the system. We get the following proposition.

**Proposition 2.6.** *Let $I_D$ be a directed set and let $\{M_i, \psi^i_j\}$ be a direct system in* **Group** *over $I_D$. Then the direct limit exists and is given by the direct limit in Proposition 2.5.*

# 3 Abelian categories

In this section, we will define a special class of categories, namely abelian categories. These will be the basic building blocks later when studying chain complexes and homology thereof. The reason for this interest is because we want to work with sequences of objects in a category. To do so, we need to define what we mean by 'kernel' and 'image' of a morphism. The property of being abelian is one way to do this.

Most of the ideas in this section can be found in [4]. However, we do not use Rotman's definition of an abelian category, but instead, the one used in both Iversen [6] and Bredon [7], as I find this more intuitive when working with sequences of objects in an abelian category.

We start with defining an additive category, a precursor of abelian categories.

**Definition 3.1.** Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is **additive** if

(i) $\mathrm{Hom}(A, B)$ is equipped with the structure of an abelian group for all $A, B \in Ob(\mathcal{C})$,

(ii) the distributive law over the morphisms holds; that is, given $X, Y \in Ob(\mathcal{C})$ and morphisms according to the diagram

$$X \xrightarrow{a} A \underset{f}{\overset{g}{\rightrightarrows}} B \xrightarrow{b} Y,$$

then

$$b(f+g) = bf + bg \quad \text{and} \quad (f+g)a = fa + ga,$$

(iii) $\mathcal{C}$ has a zero object,

(iv) $\mathcal{C}$ has finite coproducts and finite products.

**Example 3.1.** I claim ${}_R\mathbf{Mod}$, the category of left $R$-modules is an additive category. First, it is clear that for $A, B \in {}_R\mathbf{Mod}$, $\mathrm{Hom}(A, B)$ - the group of $R$-maps - is an abelian group with addition as operator, and that the distributive law holds. ${}_R\mathbf{Mod}$ has a zero object, explicitly $\{0\}$, since all maps from this set has to map to $\{0\}$ in another module (the maps are $R$-maps), and clearly all maps into $\{0\}$ can only map to one element. Lastly, we have seen previously that coproducts and products exist and are finite, which shows that ${}_R\mathbf{Mod}$ is an additive category.

We continue with the definition of an additive functor.

**Definition 3.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories, and let $T : \mathcal{C} \to \mathcal{D}$ be a functor (of either variance), then $T$ is **additive** if for all $A, B \in Ob(\mathcal{C})$ and all morphisms $f, g \in \mathrm{Hom}(A, B)$, we have

$$T(f + g) = Tf + Tg.$$

We need some more data before we can define abelian categories.

**Definition 3.3.** Let $\mathcal{C}$ be a category, $B$ and $C$ objects in $\mathcal{C}$, and $u : B \to C$ a morphism. We say that $u$ is a **monomorphism** if $u$ can be canceled from the left; that is, for all $A \in Ob(\mathcal{C})$ and morphisms $f, g : A \to B$, we have that $uf = ug$ implies $f = g$.

For additive categories, we get a simpler relationship.

**Proposition 3.1.** *Let $\mathcal{C}$ be an additive category, $A, B, C$ objects in $\mathcal{C}$, and $f : A \to B$ and $u : B \to C$ morphisms. Then $u$ is a monomorphism if and only if $uf = 0$ implies $f = 0$.*

*Proof.* First, assume that $u$ is a monomorphism. Then, consider the mapping $0 : A \to B$, where $0$ maps all elements to the zeroth element in $B$. The equation $uf = u0 = 0$ implies $f = 0$ since $u$ is a monomorphism.

For the other implication, assume $uf = 0$ implies $f = 0$. Consider two other morphisms $g_1, g_2 : A \to B$ and study the equation $ug_1 = ug_2$. Since $\mathrm{Hom}(A, C)$ is an abelian group, this implies $ug_1 - ug_2 = 0$. By the distributive law, we get $u(g_1 - g_2) = 0$, which implies by the hypothesis $g_1 - g_2 = 0$, or $g_1 = g_2$; that is, $u$ is a monomorphism and we are done. $\square$

Now the dual of a monomorphism.

**Definition 3.4.** Let $\mathcal{C}$ be a category, $B$ and $C$ objects in $\mathcal{C}$, and $v : B \to C$ a morphism. We say that $v$ is a **epimorphism** if $v$ can be canceled from the right; that is, for all $D \in Ob(\mathcal{C})$ with two morphisms $f, g : A \to B$, we have that $fv = gv$ implies $f = g$.

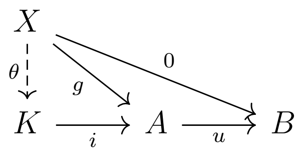
Similar to monomorphisms, a simpler relationship can be obtained when studying epimorphisms in an additive category.

**Proposition 3.2.** *Let $\mathcal{C}$ be an additive category, $B, C, D$ objects in $\mathcal{C}$, and $v : B \to C$ and $f : C \to D$ morphisms. Then $v$ is an epimorphism if and only if $fv = 0$ implies $f = 0$.*

*Proof.* Similar to the proof of Proposition 3.1. $\square$

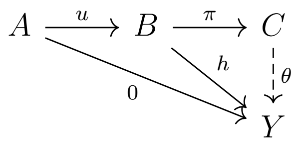
We continue with defining categorical kernels and categorical cokernels.

**Definition 3.5.** Let $\mathcal{C}$ be a (not necessarily additive) category and let $u : A \to B$ be a morphism of two objects $A$ and $B$. The **kernel** of $u$, denoted Ker $u$, is an object $K$ in $Ob(\mathcal{C})$ and a morphism $i : K \to A$, that satisfy the following universal mapping problem: $ui = 0$, and for every object $X$ in $\mathcal{C}$ and all morphisms $g : X \to A$ such that $ug = 0$, there exists a unique morphism $\theta : X \to K$ such that $i\theta = g$.




And here is the dual definition, that of the cokernel.

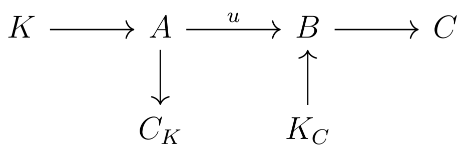
**Definition 3.6.** Let $\mathcal{C}$ be a category and let $u : A \to B$ be a morphism of two objects $A$ and $B$. The **cokernel** of $u$, denoted Coker $u$, is an object $C$ in $Ob(\mathcal{C})$ and a morphism $\pi : B \to C$, that satisfy the following universal mapping problem: $\pi u = 0$, and for every object $Y$ in $\mathcal{C}$ and all morphisms $h : B \to Y$ such that $hu = 0$, there exists a unique morphism $\theta : C \to Y$ such that $\theta\pi = h$.




The kernel and the cokernel might not always exist for a morphism, but if it does, then they are unique up to isomorphism. In an additive category, there is a simple relationship between monomorphisms and kernels, and epimorphisms and cokernels.

We continue with defining the image and the coimage of a morphism.

**Definition 3.7.** Let $\mathcal{C}$ be an additive category and let $u : A \to B$ be a morphism between two objects in $\mathcal{C}$. The **image** of $u$, denoted Im $u$, is the kernel of the cokernel of $u$, given that it exists. Dually, the **coimage** of $u$, denoted Coim $u$, is the cokernel of the kernel of $u$. We get the following diagram.



Here, the $A \to C_K$ is the Coim $u$, and $K_C \to B$ is the Im $u$.

**Proposition 3.3.** *Let $\mathcal{C}$ be an additive category and let $u : A \to B$ be a morphism between two objects in $\mathcal{C}$.*

*(i) $u$ is a monomorphism if and only if* Ker $u = 0$.

*(ii) $u$ is an epimorphism if and only if* Coker $u = 0$.

*Proof.* We will only prove the first statement - the second one is similar.

Let Ker $u$ be given by the morphism $i : K \to A$, where $K$ is an object in $\mathcal{C}$, and assume first that $u$ is a monomorphism. Consider the morphism $0 : K \to A$. Since $ui = 0 = u0$, we have $i = 0$ by Proposition 3.1.

Now, assume Ker $u = i = 0$. Let $g : X \to A$ be a morphism, $X$ an object, such that $ug = 0$. By the definition of the kernel, we have $g = i\theta = 0$, which shows $u$ is a monomorphism again using Proposition 3.1. □

To make it more readable, we will simplify the notation such that if $i : K \to A$ is a kernel of $u : A \to B$, we will denote $K$ by Ker $u$. Similarly with Coker $u$, Coim $u$, and Im $u$.

We are now ready for the definition of an abelian category.

**Definition 3.8.** Let $\mathcal{C}$ be a category. $\mathcal{C}$ is an **abelian category** if it is an additive category such that

(i) for every morphism $u : A \to B$ for $A, B \in Ob(\mathcal{C})$, Ker $u$ and Coker $u$ exists, and

(ii) the canonical morphism Coim $u \to$ Im $u$ is an isomorphism.

In an abelian category, the categorial notion of image and coimage agree with the usual notions.

**Example 3.2.** The category of free abelian groups is additive, but not abelian, since cokernels might not exist.

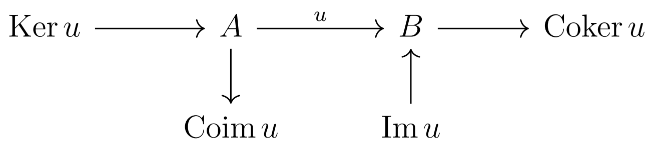
Let us understand where the canonical morphism Coim $u \to$ Im $u$ comes from. First, we show that Ker $u \to A$ is a monomorphism and $B \to$ Coker $u$ is an epimorphism in an additive category.

**Lemma 3.4.** *Let $\mathcal{C}$ be an additive category and let $u : A \to B$ be a morphism of objects $A, B$ in $\mathcal{C}$. Then* Ker $u \to A$ *is a monomorphism, given that it exists. Dually, $B \to$* Coker $u$ *is an epimorphism, given that it exists.*

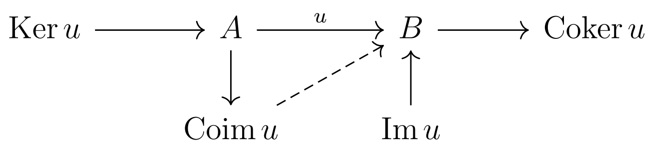
*Proof.* Consider an object $D$ in $Ob(\mathcal{C})$ such that the composed morphism $D \to \operatorname{Ker} u \to A$ is zero. Then the morphism $D \to A \to B$ is zero; there exists a unique morphism from $D \to \operatorname{Ker} u$ by the definition of a kernel. Since the zero morphism $0 : D \to \operatorname{Ker} u$ clearly makes the diagram commute, $D \to \operatorname{Ker} u$ is the zero morphism and $\operatorname{Ker} u$ is a monomorphism by Proposition 3.1.

Dually, it follows that $B \to$ Coker $u$ is an epimorphism. □

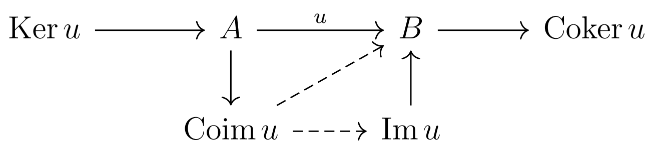
We will now derive the canonical morphism Coim $u \to$ Im $u$. Consider the following diagram.



Since $\operatorname{Ker} u \to A \to B$ is zero, it follows that there exists a unique morphism Coim $u \to B$, which gives the following commutative diagram.



We have that $A \to B \to$ Coker $u$ is zero, hence, $A \to$ Coim $u \to B \to$ Coker $u$ is zero. Since $A \to$ Coim $u (= \operatorname{Coker} \operatorname{Ker} u)$ is an epimorphism, Coim $u \to B \to$ Coker $u$ is zero. But then there is a unique morphism Coim $u \to$ Im $u$. We get the following commutative diagram.



In an abelian category, this unique morphism is an isomorphism, which we will prove later.

From now on, we will denote an abelian category by $\mathcal{A}$. The motivation for defining abelian categories is that we can make a reasonable definition of chain complexes, hence, viewing the objects in a chain complex as objects from $\mathcal{A}$, we retrieve a generalization from the common definition in algebraic topology (where the objects are from **Ab**).

## 3.1 Exact sequences

We are now ready for the definition of exactness.

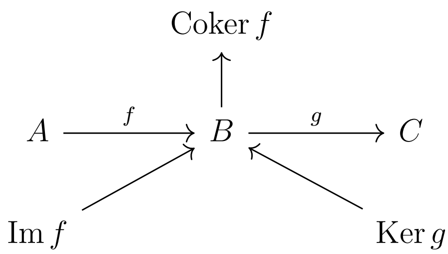
**Definition 3.9.** Let $\mathcal{A}$ be an abelian category, $A, B, C$ objects in $\mathcal{A}$, $f : A \to B$ and $g : B \to C$ morphisms. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

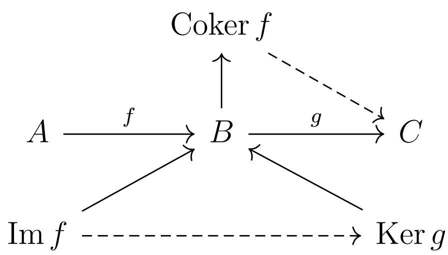
is **exact** if

1. the composition $g \circ f$ is the zero morphism, and
2. the canonical morphism Im $f \to$ Ker $g$ is an isomorphism.

To see the canonical morphism Ker $g \to$ Im $f$, consider the following diagram.




We have that $A \to B \to C$ is zero by assumption, hence there is a unique morphism Coker $f \to C$ making the diagram commute. This shows that Im $f \to B \to C$ is zero since Im $f \to B \to$ Coker $f \to C$ is. Therefore, there exists a unique morphism Im $f \to$ Ker $g$ by the universal mapping problem. We get the following commutative diagram.




Further, whenever $g \circ f = 0$, the canonical morphism Im $f \to \operatorname{Ker} g$ is a monomorphism.

**Example 3.3.** Let $\mathcal{A}$ be an abelian category and let $u : A \to B$ be a morphism of objects in $\mathcal{A}$. The sequence

$$0 \longrightarrow A \xrightarrow{u} B$$

is exact if and only if $u$ is a monomorphism. Dually, the sequence

$$A \xrightarrow{u} B \longrightarrow 0$$

is exact if and only if $u$ is an epimorphism.

We show the first statement; the other can be shown similarly. Assume the sequence is exact, and let $g : D \to A$ be any morphism such that $ug = 0$. By exactness, Ker $u$ is isomorphic to Im $(0 \to A)$, which is zero. Subsequently, there exists a unique morphism from $D \to \operatorname{Ker} u$, which has to be the zero morphism. For clarity, we have the following commutative diagram.

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{u} & B \\ & \nearrow & & \nwarrow & \\ D & & \dashrightarrow & & \operatorname{Ker} u = 0 \end{array}$$

It follows that $D \to A$ is zero.

Now, assume that $u$ is a monomorphism. It is clear that the composition in the sequence is zero. Consider the morphism $\operatorname{Ker} u \to A \to B$. Since $u$ is a monomorphism, $\operatorname{Ker} u \to A$ is zero, and thus isomorphic to Im $(0 \to A)$, the result follows.

**Definition 3.10.** Let $\mathcal{A}$ be an abelian category and let $\{A_i\}_{i \in \mathbb{Z}}$ be objects in $\mathcal{A}$. We say that the sequence of morphisms

$$\ldots \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow A_{i+2} \longrightarrow \ldots$$

is exact, if each sequence $A_i \longrightarrow A_{i+1} \longrightarrow A_{i+2}$ is exact, for all $i \in \mathbb{Z}$.

We now provide an example of an exact sequence.

**Example 3.4.** Let $\mathcal{A}$ be an abelian category and let $u : A \to B$ be a morphism between objects in $\mathcal{A}$. Then

$$0 \longrightarrow \operatorname{Ker} u \longrightarrow A \longrightarrow \operatorname{Im} u \longrightarrow 0,$$

with the obvious morphisms, is exact.

First, it is clear that

$$0 \longrightarrow \operatorname{Ker} u \longrightarrow A$$

is exact, since $\operatorname{Ker} u \to A$ is a monomorphism. To show that

$$A \longrightarrow \operatorname{Im} u \longrightarrow 0$$

is exact, note that the morphism $A \to \operatorname{Im} u$ is given by $A \to \operatorname{Coim} u \to \operatorname{Im} u$. The morphism $A \to \operatorname{Coim} u$ is an epimorphism and the morphism $\operatorname{Coim} u \to \operatorname{Im} u$ is an isomorphism, subsequently $A \to \operatorname{Coim} u \to \operatorname{Im} u$ is an epimorphism, hence

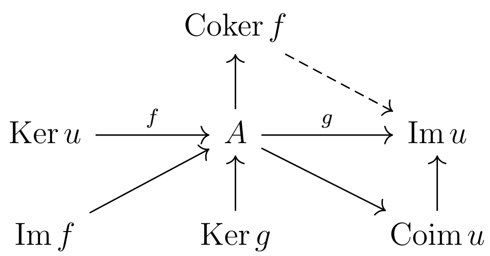
$$A \longrightarrow \operatorname{Im} u \longrightarrow 0$$

is exact.

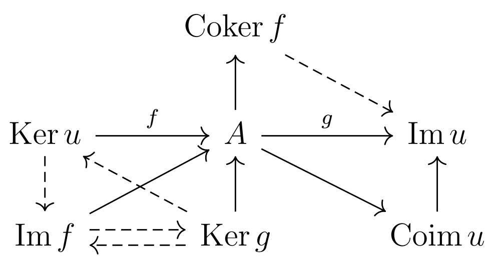
Lastly, we will show that

$$\operatorname{Ker} u \longrightarrow A \longrightarrow \operatorname{Im} u$$

is exact. Call the morphisms $\operatorname{Ker} u \to A$ and $A \to \operatorname{Im} u$, $f$ and $g$ respectively. We get the following commutative diagram.




Since $\operatorname{Im} f \to A \to \operatorname{Coker} f \to \operatorname{Im} u$ is zero, there is a unique morphism $\operatorname{Im} f \to \operatorname{Ker} g$. Further, since $\operatorname{Ker} g \to A \to \operatorname{Im} u$ is zero, there exists a unique morphism $\operatorname{Ker} g \to \operatorname{Ker} u$. Similarly, there exists a unique morphism $\operatorname{Ker} u \to \operatorname{Im} u$; combining the two, we have a unique morphism $\operatorname{Ker} g \to \operatorname{Im} f$. Thus, we get the following commutative diagram.




But then, since the diagram commutes, the unique morphism $\operatorname{Im} f \to \operatorname{Ker} g \to \operatorname{Im} f$ is the identity; so is $\operatorname{Ker} g \to \operatorname{Im} f \to \operatorname{Ker} g$, showing that Im $f$ is isomorphic to Ker $g$.

We finish this section with two examples. First, in an abelian category, if a morphism is a monomorphism and an epimorphism then it is an isomorphism. Second, if we have an additive category that is not abelian, then a morphism that is both a monomorphism and an epimorphism might not be an isomorphism.

**Example 3.5.** Let $\mathcal{A}$ be an abelian category and let $f : X \to Y$ be a morphism of objects in $\mathcal{A}$. If

$$0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$$

is exact, then it is an isomorphism. By the above example, this is equivalent to saying that if $f$ is a monomorphism and an epimorphism, then it is an isomorphism.

To see this, note that $\operatorname{Coim} f = X$ and $\operatorname{Im} f = Y$, which follows from the universal mapping problem. Subsequently, we get the following commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_X} & & \uparrow{\scriptstyle id_Y} & & \\ & & X = \operatorname{Coim} f & \dashrightarrow & \operatorname{Im} f = Y & & \end{array}$$

Since $\mathcal{A}$ is abelian, $X = \operatorname{Coim} f \to \operatorname{Im} f = Y$ is an isomorphism.

**Example 3.6.** Consider the category of divisible abelian groups. Recall that a divisible group $G$ is a group such that for all positive integers $n$ and all $g \in G$, there exists a $g' \in G$ such that $ng' = g$. It is not hard to show that it is an additive category. It is not an abelian category, which can be seen by the following.

Consider the divisible abelian group $\mathbb{Q}$ and its subgroup $\mathbb{Z}$. The quotient map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is clearly an epimorphism. To show it is a monomorphism, let $G$ be a divisible abelian group and assume the morphism $G \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ vanishes. Since $G$ is divisible, $G \to \mathbb{Q}$ vanish. Hence, $G \to \mathbb{Q}$ factors through $G \to \mathbb{Z}$, so the image of the last morphism is a divisible subgroup of $\mathbb{Z}$. But there is only one divisible subgroup of $\mathbb{Z}$, the trivial group, showing that it is a monomorphism.

However, $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is not injective thus not an isomorphism.

# 4 Category of complexes

In this section, we will study the category of complexes. This is most likely very familiar to the reader since the reader should've seen similar definitions and propositions in the category **Ab**. Here, we are generalizing these ideas to abelian categories.

Most of the theory and ideas come from Kachiwara and Shapira [8], [9], but also Rotman [4]. We are going in a similar style as Kashiwara and Schapira, as this makes the transition into defining triangulated and derived categories more natural.

We will assume $\mathcal{C}$ to be an additive category.

**Definition 4.1.** The category of complexes of $\mathcal{C}$, denoted by $C(\mathcal{C})$, is given by the following data.

An object $X$ in $C(\mathcal{C})$ is of the form $\{X^n, d_X^n\}_{n\in\mathbb{Z}}$ such that for all $n$,

$$X^n \in \mathcal{C}, \quad d_X^n \in \mathrm{Hom}_{\mathcal{C}}(X^n, X^{n+1}) \quad \textit{and} \quad d_X^{n+1} d_X^n = 0.$$

A morphism $X \to Y$ in $C(\mathcal{C})$ is a sequence of morphisms $\{f^n\}_{n\in\mathbb{Z}}$ such that $f^n : X^n \to Y^n$ and the following diagram commutes.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & Y^n & \xrightarrow{d_X^n} & Y^{n+1} & \longrightarrow & \cdots \end{array}$$

The composition of morphisms is defined in an obvious way. It is easy to check that $C(\mathcal{C})$ is indeed a category.

There are three interesting full subcategories of $C(\mathcal{C})$.

**Definition 4.2.** The full subcategory...

(i) $C^b(\mathcal{C})$ has as objects complexes $X$ such that $X^n = 0$ for $|n| \gg 0$.

(ii) $C^+(\mathcal{C})$ has as objects complexes $X$ such that $X^n = 0$ for $n \ll 0$.

(iii) $C^-(\mathcal{C})$ has as objects complexes $X$ such that $X^n = 0$ for $n \gg 0$.

**Definition 4.3.** Let $X$ be a complex in $C(\mathcal{C})$. Let $T : C(\mathcal{C}) \to C(\mathcal{C})$ be a functor given by the following. On objects $X = \{X^n, d_X^n\}_{n\in\mathbb{Z}} \in Ob(C(\mathcal{C}))$, we have $T(X) = \{T(X^n), T(d_X^n)\}_{n\in\mathbb{Z}} = \{X^{n+1}, -d_X^{n+1}\}_{n\in\mathbb{Z}}$. On morphisms $f : X \to Y$ in $C(\mathcal{C})$, we have $f^n \mapsto f^{n+1}$.

The functor $T$ defined above is called the **shift functor** of degree 1. The shift functor of degree $k \in \mathbb{Z}$ is given by $T^k(X)$; that is, iteratively applying $T$, $k$ times on $X$. We typically write $X[k]$ instead of $T^k(X)$, and $f[k]$ instead of $T^k(f)$.

**Definition 4.4.** Let $X, Y$ be complexes in $C(\mathcal{C})$. A **map of degree** $k$ is a morphism $s$ of chain complexes from $X$ to $Y[k]$.

**Definition 4.5.** Let $f, g : X \to Y$ be two morphisms in $C(\mathcal{C})$. Then $f$ and $g$ are said to be **homotopic**, denoted by $f \simeq g$, if there is a map $s$ of degree $-1$ from $X$ to $Y$ such that

$$f^n - g^n = d_Y^{n+1} s^n + s^{n-1} d_X^n,$$

for all $n \in \mathbb{Z}$.

If $f \simeq 0$, then we say $f$ is **null-homotopic**.

**Proposition 4.1.** *Let $X, Y$ be complexes in $C(\mathcal{C})$. Let $\mathrm{Ht}(X, Y)$ be the subset of $\mathrm{Hom}_{C(\mathcal{C})}(X, Y)$ consisting of null-homotopic morphisms. Then $\mathrm{Ht}(X, Y)$ is a subgroup.*

*Proof.* Clearly, $\mathrm{Ht}(X, Y)$ is not empty, since the null morphism is null-homotopic. We have two show that $\mathrm{Ht}(X, Y)$ is closed under addition, and for each $f \in \mathrm{Ht}(X, Y)$, we have $f^{-1} \in \mathrm{Ht}(X, Y)$.

Let $f, g \in \mathrm{Ht}(X, Y)$, then there exists two maps $s_1, s_2 : X \to Y[-1]$ such that $f = ds_1 + s_1 d$ and $g = ds_2 + s_2 d$ (here we omit the scripts to make it easier to read). Thus, $f + g = d(s_1 + s_2) + (s_1 + s_2)d$, since function composition is distributive over addition. Hence, the map $s_1 + s_2$ shows $f + g$ is null-homotopic.

Note that $f^{-1} = -f$. Using the map $-s_1$ with the above assumptions shows $-f \in \mathrm{Ht}(X, Y)$. This shows $\mathrm{Ht}(X, Y)$ is a subgroup. □

**Definition 4.6.** The **homotopy category of chain complexes**, denoted by $K(\mathcal{C})$, is given by

(i) $Ob(K(\mathcal{C})) = Ob(C(\mathcal{C}))$,

(ii) $\mathrm{Hom}_{K(\mathcal{C})}(X, Y) = \mathrm{Hom}_{C(\mathcal{C})}(X, Y)/\mathrm{Ht}(X, Y)$.

That the homotopy category of chain complexes is a category requires some justification. Note that $\mathrm{Ht}(X, Y) \times \mathrm{Hom}_{C(\mathcal{C})}(X, Y)$ and $\mathrm{Hom}_{C(\mathcal{C})}(X, Y) \times \mathrm{Ht}(X, Y)$ are sent into $\mathrm{Ht}(X, Y)$, thus, $f, g \in \mathrm{Hom}_{C(\mathcal{C})}(X, Y)/\mathrm{Ht}(X, Y)$ are mapped to $f \circ g \in \mathrm{Hom}_{C(\mathcal{C})}(X, Y)/\mathrm{Ht}(X, Y)$, making composition well defined.

## 4.1 Cohomology of a complex

In this section, we assume $\mathcal{A}$ is an abelian category, and use the notation $\mathcal{A}$ instead of $\mathcal{C}$.

**Definition 4.7.** Let $X$ be a complex in $C(\mathcal{A})$, and define $Z^k(X) = \operatorname{Ker} d_X^k$, $B^k(X) = \operatorname{Im} d_X^{k-1}$.

From the discussion of exact sequences, we recall that there exists a canonical monomorphism $B^k(X) \to Z^k(X)$.

We define the $k$**th comhology of the complex** $X$ to be given by

$$H^k(X) = \operatorname{coker}(B^k(X) \to Z^K(X)).$$

*Remark* 4.1. By the definition above, we have

$$H^k(X) = \operatorname{Ker} d_X^k / \operatorname{Im} d_X^{k-1}.$$

This is some abuse of terminology since the kernel and image are objects associated with a morphism. Here we refer to the underlying object, forgetting about the morphism.

**Proposition 4.2.** *Let $f, g : X \to Y$ be two morphisms in $C(\mathcal{A})$, and assume $f \simeq g$. Then the induced morphisms $f^{*k}, g^{*k} : H^k(X) \to H^k(Y)$ are equal, for $k \in \mathbb{Z}$.*

*Proof.* Let $z \in \operatorname{Ker} d_X^k$, then $d_X^k(z) = 0$. Hence, we have

$$f^k(z) - g^k(z) = d_Y^{k+1} s^k(z) + s^{k-1} d_X^k(z) = d_Y^{k+1} s^k(z),$$

that is, $f^k(z) - g^k(z) = d_Y^{k+1} s^k(z) \in B^k(Y)$, and thus $f^{*k} = g^{*k}$. □

We get this result.

**Corollary 4.2.1.** *The functor $H^k : K(\mathcal{A}) \to \mathcal{A}$ is well-defined.*

*Proof.* Let $f : X \in Y$ be a null-homotopic morphism in $C(\mathcal{A})$. Then $H^k(f)$ is the zero morphism by Proposition 4.2. □

**Theorem 4.3.** *Let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

*be an exact sequence in $C(\mathcal{A})$. Then there is a long exact sequence in $\mathcal{A}$ given by*

$$\cdots \longrightarrow H^n(X) \xrightarrow{f^{*n}} H^n(Y) \xrightarrow{g^{*n}} H^n(Z) \xrightarrow{\delta^n} H^{n+1}(X) \longrightarrow \cdots$$

*with $f^*$ and $g^*$ being the induced maps in cohomology of $f$ and $g$, and $\delta^n$ is a morphism $H^n(Z) \to H^{n+1}(X)$ for all $n \in \mathbb{Z}$.*

*Proof.* See [9], page 33. □

We finish this section by defining the truncated complexes.

**Definition 4.8.** Let $X$ be an object in an abelian category $\mathcal{A}$. We define the **truncated complexes** of $X$, denoted $\tau^{\leq n}(X)$ and $\tau^{\geq n}(X)$, to be given by

$$\tau^{\leq n}(X): \cdots \longrightarrow X^{n-2} \longrightarrow X^{n-1} \longrightarrow \operatorname{Ker} d_X^n \longrightarrow 0 \longrightarrow \cdots$$

$$\tau^{\geq n}(X): \cdots \longrightarrow 0 \longrightarrow \operatorname{Coker} d_X^{n-1} \longrightarrow X^{n+1} \longrightarrow X^{n+2} \longrightarrow \cdots$$

**Proposition 4.4.** *Let $X$ be an object in an abelian category $\mathcal{A}$. The following holds.*

*(i) For $k \leq n$, $H^k(\tau^{\leq n}(X))$ is isomorphic to $H^k(X)$. For $k > n$, $H^k(\tau^{\leq n}(X)) = 0$.*

*(ii) For $k \geq n$, $H^k(\tau^{\geq n}(X))$ is isomorphic to $H^k(X)$. For $k < n$, $H^k(\tau^{\geq n}(X)) = 0$.*

*Proof.* The statement is obvious for all integers except $n-1$ and $n$. First, note that since $X$ is a chain complex, the mapping $X^{n-1} \to \operatorname{Ker} d_X^n$ is induced naturally by $d_X^{n-1}$. We have $\ker(X^{n-1} \to \ker d_X^n) = \ker d_X^{n-1}$ showing that $H^{n-1}(\tau^{\leq n}(X))$ is isomorphic to $H^{n-1}(X)$. For $k = n$, we have $H^n(\tau^{\leq n}(X)) = \operatorname{Ker} d_X^n / \operatorname{Im} d_X^{n-1}$ which is clearly isomorphic to $H^n(X)$.

A similar argument shows the second statement. □

**Definition 4.9.** Let $f : X \to Y$ be a morphism of objects in $K(\mathcal{A})$. We say that $f$ is a quasi-isomorphism if $H^n(f)$ is an isomorphism of all $n$.

# 5 Triangulated categories

We will now introduce triangulated categories which will be one of the main objects of study. The usefulness comes in when we localize a category of chain complexes. Localization can be seen by making some objects invertible, similarly in commutative algebra; informally, localizing a ring/module introduces 'denominators'.

The theory and idea of the proofs can be found in [8] and [9].

We will begin by motivating them with an example; triangulated categories arise as an abstraction thereof.

## 5.1 Mapping cones

**Definition 5.1.** Let $\mathcal{C}$ be an additive category and let $f : X \to Y$ be a morphism in $C(\mathcal{C})$. The mapping cone of $f$, denoted by $M(f)$, is an object in $C(\mathcal{C})$ given by the following data:

(i) $M(f) = X[1] \oplus Y$; that is, the object at index $n$ is given by $X^{n+1} \oplus Y$.

(ii) The morphisms $d^n_{M(f)}$ are given by $\begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}$.

Consider the following morphisms. We define $\alpha(f) : Y \to M(f)$ to be given by $\alpha(f)^n = \begin{pmatrix} 0 \\ id_{Y^n} \end{pmatrix}$, and we define $\beta(f) : M(f) \to X[1]$ to be given by $\beta(f)^n = \begin{pmatrix} id_{X^{n+1}} & 0 \end{pmatrix}$. It is routine to show that these are indeed morphisms of complexes.

The next lemma shows some usefulness of the homotopy category of chain complexes; the same lemma would not be true in the category of chain complexes and all chain morphisms.

**Lemma 5.1.** *Let $\mathcal{C}$ be an additive category and let $f : X \to Y$ be a morphism in $C(\mathcal{C})$. Then there exists a morphism $\phi : X[1] \to M(\alpha(f))$ such that the following diagram commutes in $K(\mathcal{C})$.*

$$\begin{array}{ccccccc} Y & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\beta(f)} & X[1] & \xrightarrow{-f[1]} & Y[1] \\ \downarrow{\scriptstyle id_Y} & & \downarrow{\scriptstyle id_{M(f)}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id_{Y[1]}} \\ Y & \xrightarrow[\alpha(f)]{} & M(f) & \xrightarrow[\alpha(\alpha(f))]{} & M(\alpha(f)) & \xrightarrow[\beta(\alpha(f))]{} & Y[1] \end{array}$$

*Moreover, $\phi$ is an isomorphism in $K(\mathcal{C})$.*

*Proof.* First, note that $\alpha(f) : Y \to M(f)$, hence $M(\alpha(f)) = Y[1] \oplus X[1] \oplus Y$. The morphisms of this complex are given by the following:

$$d^n_{M(\alpha(f))} = \begin{pmatrix} d^n_{Y[1]} & 0 \\ \alpha(f)^{n+1} & d^n_{M(f)} \end{pmatrix} = \begin{pmatrix} d^n_{Y[1]} & 0 & 0 \\ 0 & d^n_{X[1]} & 0 \\ id_{Y[1]} & f^{n+1} & d^n_Y \end{pmatrix}.$$

Define $\phi : X[1] \to M(\alpha(f))$ to be given by

$$\phi = \begin{pmatrix} -f[1] \\ id_{X[1]} \\ 0 \end{pmatrix},$$

and let $\psi : M(\alpha(f)) \to X[1]$ be a morphism given by

$$\psi = \begin{pmatrix} 0 & id_{X[1]} & 0 \end{pmatrix}.$$

From the explicit definition of the morphisms in the chain complex $M(\alpha(f))$, it is easy to show that $\phi$ and $\psi$ are morphisms of complexes. Further, note that $\psi \circ \phi = id_{X[1]}$. Also, $\phi \circ \psi = id_{M(\alpha(f))}$ in $K(\mathcal{C})$; that is, $\phi \circ \psi$ is homotopic to $id_{M(\alpha(f))}$. To see this, take the morphism $s^n : M(\alpha(f))^n \to M(\alpha(f))^{n-1}$, given by

$$s^n = \begin{pmatrix} 0 & 0 & id_{Y^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This morphism makes the following identity true:

$$id_{M(\alpha(f)^n} - \phi^n \circ \psi^n = s^{n+1} \circ d_{M(\alpha(f))^n} + d^{n-1}_{M(\alpha(f)} \circ s^n;$$

showing they are chain homotopic.

That the diagram commutes is now easy; simply take the definition of the morphisms and show equality in $C(\mathcal{C})$, which gives equality in $K(\mathcal{C})$ as well. □

**Definition 5.2.** Let $\mathcal{C}$ be an additive category, and let $X, Y, Z \in Ob(K(\mathcal{C}))$. We define a **triangle** in $K(\mathcal{C})$ to be a sequence of morphisms of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1] .$$

A morphism between two triangles $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ and $X' \longrightarrow Y' \longrightarrow Z' \longrightarrow X'[1]$ is given by commutative diagrams in $K(\mathcal{C})$ of the form:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle\phi} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\phi[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

**Definition 5.3.** We call a triangle $(X, Y, Z)$ in $K(\mathcal{C})$ **distinguished**, if it isomorphic to a triangle of the form

$$X' \xrightarrow{f} Y' \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X'[1],$$

with $f : X' \to Y'$ being a morphism in $C(\mathcal{C})$.

**Proposition 5.2.** *Let $\mathcal{C}$ be an additive category. The distinguished triangles in $K(\mathcal{C})$ satisfy the following properties.*

**(TR 0):** *If a triangle is isomorphic to a distinguished triangle, then it is distinguished.*

**(TR 1):** *For any $X$ in $Ob(K(\mathcal{C}))$ the triangle* $X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow X[1]$ *is distinguished.*

**(TR 2):** *For all morphisms $f : X \to Y$, there exists a $Z$ such that* $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$ *is a distinguished triangle.*

**(TR 3):** *The triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ *is distinguished if and only if* $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ *is distinguished.*

**(TR 4):** *Let*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1],$$

$$X' \xrightarrow{f'} Y' \longrightarrow Z' \longrightarrow X'[1]$$

*be two distinguished triangles. Then any commutative diagram of the following form can be embedded in a morphism of triangles.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

**(TR 5):** *Let*

$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow X[1],$$

$$Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow Y[1],$$

$$X \xrightarrow{g\circ f} Z \longrightarrow Y' \longrightarrow X[1],$$

*be distinguished triangles. Then there exists a distinguished triangle*

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow Z'[1]$$

*such that the following diagram commutes.*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & X[1] \\
{\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle id_{X[1]}} \\
X & \xrightarrow{g\circ f} & Z & \longrightarrow & Y' & \longrightarrow & X[1] \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id_Z} & & \downarrow & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & Y[1] \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle id_{X'}} & & \downarrow \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z'[1]
\end{array}$$

*Proof.* (TR 0) is clear, since composition of two isomorphisms is again an isomorphism.

(TR 2) is also clear; take $Z = M(f)$ and we have a distinguished triangle.

(TR 3) follows from Lemma 5.1. To see this, assume

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle; we may assume $Z = M(f)$. Then, we have a distinguished triangle

$$Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\alpha(\alpha(f))} M(\alpha(f)) \xrightarrow{\beta(\alpha(f))} Y[1],$$

which by Lemma 5.1 is isomorphic to

$$Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{-f[1]} Y[1],$$

hence

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is distinguished. The other way is similar.

(TR 1) can be seen by the following. Consider the map $0 \to X$ and construct the triangle $0 \longrightarrow X \xrightarrow{id_X} X \longrightarrow 0[1]$. Since the mapping cone of $f$ is $X$, this triangle is distinguished. Applying (TR 3), we get the desired result.

(TR 4). Since the triangles are distinguished, we can assume that $Z$ and $Z'$ are the mapping cones of $f$ and $f'$ respectively with the obvious morphisms between; that is, we have the distinguished triangles

$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1]$ and

$X' \xrightarrow{f'} Y' \xrightarrow{\alpha(f')} M(f') \xrightarrow{\beta(f')} X'[1]$ . We want to show that the following diagram commutes, for some morphism $w : M(f) \to M(f')$.

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\beta(f)} & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \phi[1]} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[\alpha(f')]{} & M(f') & \xrightarrow[\beta(f')]{} & X'[1] \end{array}$$

Note that the diagram in the assumption is commutative in $K(\mathcal{C})$, which means there exists a morphism $s : X \to Y'$ with degree $-1$ such that $v^n \circ f^n - f'^n \circ u^n = s^{n+1} \circ d_X^n + d_{Y'}^{n-1} \circ s^n$. From this, we define $w$ to be given by

$$w^n = \begin{pmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{pmatrix}.$$

That this is a morphism and makes the diagram commute can be shown by calculations.

(TR 5). Again, we will consider the corresponding distinguished triangles of the sequences. We let $Z' = M(f)$, $X' = M(g)$, and $Y' = M(g \circ f)$. Define the maps $u : M(f) \to M(g \circ f)$ and $v : M(g \circ f) \to M(g)$ to be given by

$$u^n = \begin{pmatrix} id_{X^{n+1}} & 0 \\ 0 & g^n \end{pmatrix},$$
$$v^n = \begin{pmatrix} f^{n+1} & 0 \\ 0 & id_{Z^n} \end{pmatrix}.$$

Further, we define $w : M(g) \to M(f)[1]$ to be given by $\alpha(f)[1] \circ \beta(g)$ (i.e., the composite of $M(g) \xrightarrow{\beta(g)} Y[1] \xrightarrow{\alpha(f)[1]} M(f)[1]$ ). With the relevant morphisms defined, it is now relatively straightforward to show that the diagram of (TR 5) commutes. The last part is showing that the last row is a distinguished triangle; that is, we want to show that

$$M(f) \xrightarrow{u} M(g \circ f) \xrightarrow{v} M(g) \xrightarrow{w} M(f)[1]$$

is a distinguished triangle. Thus, we define the morphism $\phi : M(u) \to M(g)$ and its inverse $\psi : M(g) \to M(u)$ to be given by

$$\phi^n = \begin{pmatrix} 0 & id_{Y^{n+1}} & f^{n+1} & 0 \\ 0 & 0 & 0 & id_{X^n} \end{pmatrix}, \quad \psi^n = \begin{pmatrix} 0 & 0 \\ id_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & id_{X^{n+1}} \end{pmatrix}.$$

Again by calculation, one shows that $\phi$ and $\psi$ are morphisms of complexes and that these make into the commutative diagram

$$\begin{array}{ccccccc} M(f) & \xrightarrow{u} & M(g \circ f) & \xrightarrow{v} & M(g) & \xrightarrow{w} & M(f)[1] \\ \downarrow{\scriptstyle id_{M(f)}} & & \downarrow{\scriptstyle id_{M(g\circ f)}} & & {\scriptstyle \psi}\downarrow\uparrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id_{M(f)[1]}} \\ M(f) & \xrightarrow[u]{} & M(g \circ f) & \xrightarrow[\alpha(u)]{} & M(u) & \xrightarrow[\beta(u)]{} & M(f)[1]. \end{array}$$

Notably, we can identify a morphism of triangles in the diagram. What is left to show is that $\psi$ is an isomorphism, because then we have an isomorphism of triangles.

Consider $s : M(u) \to M(u)[-1]$ given by

$$s^n = \begin{pmatrix} 0 & 0 & id_{X^{n+1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

One can now show $\psi \circ \phi$ is equal to $id_{M(u)}$ in $K(\mathcal{C})$. Further, $\phi \circ \psi = id_{M(u)}$ is true by calculation. This means we have an isomorphism of triangles and that $Z' \longrightarrow Y' \longrightarrow Z' \longrightarrow Z'[1]$ is distinguished. □

## 5.2 Triangulated categories

We can abstract the properties of triangles in $K(\mathcal{C})$.

**Definition 5.4.** Let $\mathcal{C}$ be an additive category. A **triangulated category** of $\mathcal{C}$, consists of an automorphism $T : \mathcal{C} \to \mathcal{C}$, and a family of triangles; that is, a family of sequences of morphisms of the form $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ . Moreover, the triangles satisfy the axioms of (TR 0) to (TR 5), letting $X[1] = T(X)$.

To make it simpler, we will typically denote a triangulated category of an additive category $\mathcal{C}$ by $(\mathcal{C}, T)$, where $T$ is the associated automorphism. Sometimes we might use $\mathcal{C}$ to denote the triangulated category; what is meant will be clear from the context.

**Proposition 5.3.** *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow T(X)$ *be a distinguished triangle. Then* $g \circ f = 0$.

*Proof.* Note that $X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow X[1]$ is distinguished by (TR 1). Thus, by (TR 4), we get the following commutative diagram.

$$\begin{array}{ccccccc} X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & T(X) \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \phi} & & \downarrow \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & T(X) \end{array}$$

We get in particular $g \circ f = \phi \circ 0 = 0$. □

**Definition 5.5.** Let $(\mathcal{C}, T)$ and $(\mathcal{C}', T')$ be a two triangulated categories, and let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor. We say that $F$ is a **functor of triangulated categories**, if $F \circ T$ is isomorphic to $T' \circ F$ and $F$ sends distinguished triangles in $(\mathcal{C}, T)$ to distinguished triangles in $(\mathcal{C}', T')$.

**Definition 5.6.** Let $\mathcal{C}$ be a traingulated category, let $\mathcal{A}$ be an abelian category, and let $F : \mathcal{C} \to \mathcal{A}$ be an additive functor. We say that $F$ is a **cohomological functor** if, for any distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ in $(\mathcal{C}, T)$, the sequence $F(X) \longrightarrow F(Y) \longrightarrow F(Z)$ is exact in $\mathcal{A}$.

**Proposition 5.4.** *Let* $W$ *be an object in* $\mathcal{C}$. *Then* $\mathrm{Hom}_{\mathcal{C}}(W, \cdot)$ *and* $\mathrm{Hom}_{\mathcal{C}}(\cdot, W)$ *are cohomological functors.*

*Proof.* Let $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow T(X)$ be a distinguished triangle. We want to show that the sequence

$$\mathrm{Hom}_{\mathcal{C}}(W, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(W, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(W, Z)$$

is exact (and similarly with $\mathrm{Hom}_{\mathcal{C}}(\cdot, W)$).

We begin by showing that the image of $f_*$ is a subset of the kernel of $g_*$. But this is clear by Proposition 5.3.

Next, we show the other way, that the kernel of $g_*$ is a subset of the image of $f_*$. Let $\phi \in \mathrm{Hom}_{\mathcal{C}}(W, Y)$ be such that $g_*(\phi) = g \circ \phi = 0$. By (TR 1) we have a distinguished triangle of the form $W \xrightarrow{id_W} W \longrightarrow 0 \longrightarrow T(W)$. Combining (TR 3) and (TR 4) shows that the commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle\phi} & & \big\downarrow \\ Y & \longrightarrow & G \end{array}$$

can be completed from the left to a morphism of triangles; that is, there exists a morphism $\psi \in \mathrm{Hom}_{\mathcal{C}}(W, X)$ such that $f_*(\psi) = \phi$. This shows exactness. Similarly we can show that $\mathrm{Hom}_{\mathcal{C}}(\cdot, W)$ is a cohomological functor. □

**Corollary 5.4.1.** *Let* $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ *and* $X' \longrightarrow Y' \longrightarrow Z' \longrightarrow T(X')$ *be distinguished triangles. Assume we have a commutative diagram*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle\psi} & & \big\downarrow{\scriptstyle\theta} & & \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}.$$

*If* $\phi$ *and* $\psi$ *are isomorphisms, then so is* $\theta$*.*

*Proof.* Applying $\mathrm{Hom}_{\mathcal{C}}(W, \cdot)$ for some $W \in Ob(\mathcal{C})$ gives a commutative diagram with exact rows.

$$\begin{array}{cccccc} \mathrm{Hom}_{\mathcal{C}}(W,X) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W,Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W,Z) & \dashrightarrow \\ \big\downarrow{\scriptstyle\phi_*} & & \big\downarrow{\scriptstyle\psi_*} & & \big\downarrow{\scriptstyle\theta} & \\ \mathrm{Hom}_{\mathcal{C}}(W,X') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W,Y') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W,Z') & \dashrightarrow \end{array}$$

$$\begin{array}{cccc} \dashrightarrow & \mathrm{Hom}_{\mathcal{C}}(W,T(X)) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W,T(Y)) \\ & \big\downarrow{\scriptstyle T(\phi)_*} & & \big\downarrow{\scriptstyle T(\psi)_*} \\ \dashrightarrow & \mathrm{Hom}_{\mathcal{C}}(W,T(X')) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W,T(Y')) \end{array}$$

Since the induced morphisms $\phi_*$, $\psi_*$, $T(\phi)_*$, $T(\psi)_*$ are isomorphisms, $\theta_*$ is an isomorphism by the five lemma. But this means $\theta$ is an isomorphism. □

We have another interesting cohomological functor.

**Proposition 5.5.** *Let* $\mathcal{A}$ *be an abelian category. The* 0*th cohomology functor of a complex,* $H^0(\cdot) : K(\mathcal{A}) \to \mathcal{A}$*, is a cohomological functor.*

*Proof.* Let $X \xrightarrow{f} Y \longrightarrow M(f) \longrightarrow X[1]$ be a distinguished triangle. Then in light of (TR 2), it suffices to show that

$$H^0(Y) \longrightarrow H^0(M(f)) \longrightarrow H^0(X[1])$$

is exact. But this follows from the fact that $0 \longrightarrow Y \longrightarrow M(f) \longrightarrow X[1] \longrightarrow 0$ is exact in $C(\mathcal{C})$, and by Theorem 4.3. □

# 6 Derived categories and derived functors

We have seen that by working in $K(\mathcal{C})$ over some additive category $\mathcal{C}$, we can derive interesting relationships that would not be true in the 'bigger' category $C(\mathcal{C})$. However, the category $K(\mathcal{C})$ is still a bit too 'big' for our need; we want morphisms that induce isomorphisms in cohomology to be invertible, which is not necessarily true in $K(\mathcal{C})$.

Subsequently, this opens a larger set of relationships in the chain complex, not only in the homology/cohomology.

The theory and most of the ideas of the proofs come from [9]. I complement with examples and some proofs of statements made without proof.

## 6.1 Categorical localization

We begin by discussing the localization of categories. In this section, let $\mathcal{C}$ be (any) category.

**Definition 6.1.** Let $S$ be a collection of morphisms in $\mathcal{C}$. We say that $S$ is a **multiplicative system** if it satisfies the following axioms.

**(S 1)**: For all objects $X$ in $\mathcal{C}$, $id_X \in S$.

**(S 2)**: For all pairs of morphisms $f, g \in S$, if $g \circ f$ exists, then $g \circ f \in S$.

**(S 3)**: Let $g : Z \to Y \in S$ be a morphism. For all mapping $f : X \to Y$, there exists a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array},$$

with $h \in S$. A similar property hold with the arrows reversed.

**(S 4)**: Let $f, g : X \to Y$ be morphisms. Then the following two conditions are equivalent.

(i) There exists a $t : Y \to Y' \in S$ such that $t \circ f = t \circ g$.

(ii) There exists a $s : X' \to X \in S$ such that $f \circ s = g \circ s$.

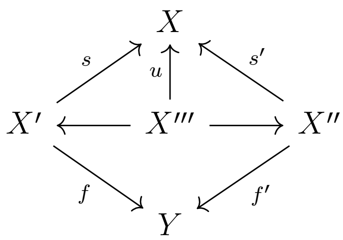
**Definition 6.2.** Let $S$ be a multiplicative system in $\mathcal{C}$. The **localization of $\mathcal{C}$ by** $S$, denoted $\mathcal{C}_S$, is given by the following data:

(i) $Ob(\mathcal{C}_S) = Ob(\mathcal{C})$,

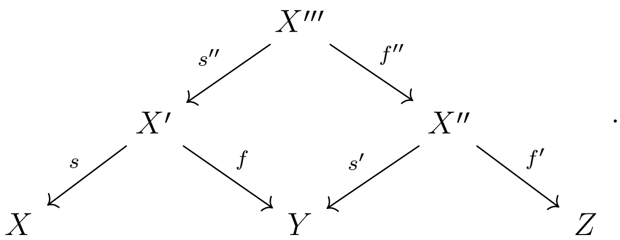
(ii) for all objects $X, Y \in Ob(\mathcal{C})$,

$$\mathrm{Hom}_{\mathcal{C}_S}(X,Y) = \{(Z,s,f) : Z \in Ob(\mathcal{C}),\ s : Z \to X, f : Z \to Y, s \in S\}/\sim,$$

where $(X', s, f) \sim (X'', s', f')$ if there exists an object $X'''$ in $\mathcal{C}$ equipped with morphisms to $X'$ and $X''$, such that the following diagram commutes.



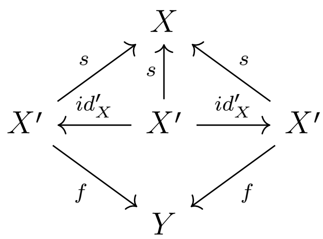

Composition of two morphisms $(X', s, f) \in \mathrm{Hom}_{\mathcal{C}_S}(X,Y)$ and $(X'', s', f') \in \mathrm{Hom}_{\mathcal{C}_S}(Y,Z)$ is given by the following. By (S 3), we get a commutative diagram



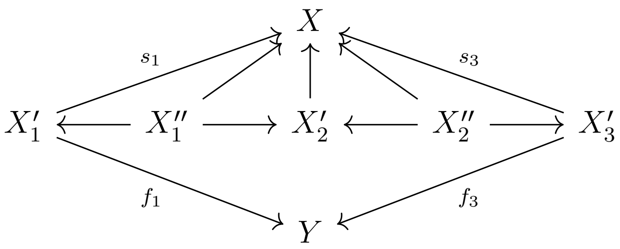

From this, we let $(X'', s', f') \circ (X', s, f) = (X''', s \circ s'', f' \circ f'')$.

We need to show two things for the previous definition to make sense. The first is that $\sim$ is an equivalence relation, and the second is that composition is well-defined.

Let $s : X' \to X$, $f : X' \to Y$ be two morphisms with $s \in S$, and $X', X, Y$ are objects in $\mathcal{C}$. Then the diagram

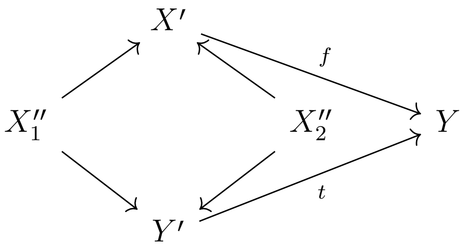



commute, showing that $\sim$ is reflexive. $\sim$ is clearly symmetric. To show transitivity, assume $(X_1', s_1, f_1) \sim (X_2', s_2, f_2)$ and $(X_2', s_2, f_2) \sim (X_3', s_3, f_3)$ with obvious definitions. We get the following commutative diagram.

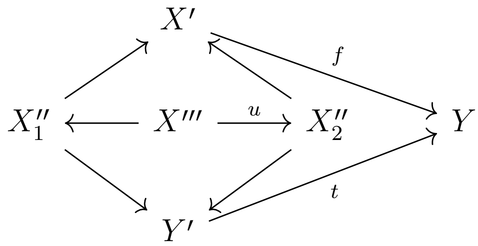



Composing $X''_1$ with $X''_2$ yields the desired equivalence, showing $(X'_1, s_1, f_1) \sim (X'_3, s_3, f_3)$.

We continue by showing that composition is well-defined. To this end, let $(X''_1, s''_1, f''_1)$ and $(X''_2, s''_2, f''_2)$ denote two possible extensions of $(X', s, f) \circ (Y', t, g)$. This gives the following commutative diagram.




This can be extended by (S 3) to the following commutative diagram.




with $u \in S$. We get a morphism $X''' \to X'$ by the composition of $X''' \to X''_2 \to X'$. That the diagram commutes below follows from (S 4).

**Proposition 6.1.** *$\mathcal{C}_S$ is a category.*

*Proof.* That it contains the necessary data to be a category is clear. We need to show that for each object $X$, there exists an identity morphism in $\mathrm{Hom}_{\mathcal{C}_S}(X, X)$, and that composition of morphisms is associative.

The identity morphism is given by $(X, id_X, f)$, which is easy to see. Further, that associativity holds is clear by the symmetry of the equivalence relation. □

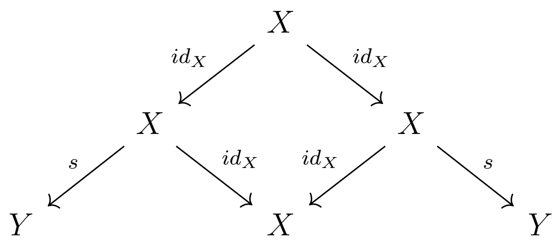
**Definition 6.3.** Let $S$ be a multiplicative system in $\mathcal{C}$. The **localization functor** $Q : \mathcal{C} \to \mathcal{C}_S$ is given by $Q(X) = X$ for $X \in Ob(\mathcal{C})$, and $Q(f) = (X, id_X, f)$ for a morphism $f : X \to Y$.

**Proposition 6.2.** *Let $S$ be a multiplicative system in $\mathcal{C}$.*

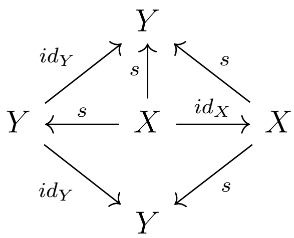
*(i) For all $s \in S$, $Q(s)$ is an isomorphism in $\mathcal{C}_S$.*

*(ii) Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor from $\mathcal{C}$ to another category. If $F(s)$ is an isomorphism for all $s \in S$, then $F$ factors uniquely through $Q$.*

*Proof.* (i) Let $s : X \to Y \in S$, we have $Q(s) = (X, id_X, s)$. Then, the morphism $(X, s, id_X)$ is an inverse. To see this, it is clear that $(X, id_X, s) \circ (X, s, id_X) = (X, id_X, id_X)$. To show $(X, s, id_X) \circ (X, id_X, s) = (Y, id_Y, id_Y)$, first note that we have a commutative diagram

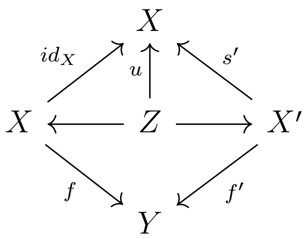



which means $(X, s, id_X) \circ (X, id_X, s) = (X, s, s)$. This is equivalent to $(Y, id_Y, id_Y)$ which is seen by the following commutative diagram.




Hence, $(X, s, id_X)$ is an inverse to $Q(s)$.

(ii) Consider a morphism $f : X \to Y$ in $\mathcal{C}$. Then $Q(f) = (X, id_X, f)$. Let $(X', s', f')$ be another morphism equivalent to $Q(f)$. Then there exists another object $Z$ and a morphism $u \in S$ such that the following diagram commutes.




Applying $F$ to the commutative diagram, using that $F(s)$ is an isomorphism for each $s \in S$, we get that $F(Z) \cong F(X) \cong F(X')$. It follows from commutativity that $F(f) = F(f')$, hence $F$ factors uniquely through $Q$.

□

**Proposition 6.3.** *Let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$ and let $S$ be a multiplicative system in $\mathcal{C}$. Let $S' \subseteq S$ be a family of morphisms belonging to $\mathcal{C}'$. Assume $S'$ is a multiplicative system in $\mathcal{C}'$ and assume that at least one of the following conditions is true.*

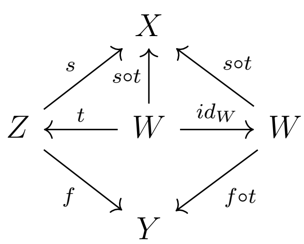
*(i) If $f : X \to Y \in S$, with $Y \in Ob(\mathcal{C}')$, then there exists a morphism $g : W \to X$, $W \in Ob(\mathcal{C}')$ such that $f \circ g \in S$.*

*(ii) If $f : X \to Y \in S$, with $Y \in Ob(\mathcal{C}')$, then there exists a morphism $g : Y \to W$, $W \in Ob(\mathcal{C}')$ such that $g \circ f \in S$.*

*Then $\mathcal{C}'_{S'}$ is a full subcategory of $\mathcal{C}_S$.*

*Proof.* Since $S'$ is assumed to be a multiplicative system, it follows that $\mathcal{C}'_{S'}$ is a category. We need to show it is a full subcategory of $\mathcal{C}_S$.

Assume (i) in the Proposition is true. Let $X$ and $Y$ be objects in $\mathcal{C}'_{S'}$. Take an element $(Z, s, f) \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ such that $s : Z \to X$, $f : Z \to Y$. Then by (i), there exists an object $W$ in $\mathcal{C}'_{S'}$ and a morphism $t : W \to Z$ such that $s \circ t \in S$. By commutative diagram




it follows that $(Z, s, f) \sim (W, s \circ t, f \circ t)$. But since $s \circ t \in S$, we have $s \circ t \in S'$ by construction, and thus $(Z, s, f) \sim (W, s \circ t, f \circ t) \in \mathrm{Hom}_{\mathcal{C}'_{S'}}(X, Y)$. □

We will now see how we can construct a multiplicative system from a family of objects in a triangulated category $\mathcal{C}$; that is, we associate a multiplicative system to such a family.

**Definition 6.4.** Let $\mathcal{C}$ be a triangulated category and let $\mathcal{N}$ be a subfamily of the objects of $\mathcal{C}$. We say that $\mathcal{N}$ is a **null system** if it satisfies the following axioms.

**(N 1)**: $0 \in \mathcal{N}$.

**(N 2)**: For an object $X$ in $\mathcal{C}$, $X \in \mathcal{N}$ if and only if $T(X) \in \mathcal{N}$.

**(N 3)**: If $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ is a distinguished triangle with $X, Y \in \mathcal{N}$, then $Z \in \mathcal{N}$.

**Proposition 6.4.** *Let $\mathcal{C}$ be a triangulated category and let $\mathcal{N}$ be a null system. The collection of morphisms*

$$S(\mathcal{N}) = \{f : X \to Y \mid \text{there exists a distinguished triangle} \\ X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X) \text{ such that } Z \in \mathcal{N}\}$$

*is a multiplicative system.*

*Proof.* Since $0 \in \mathcal{N}$ by (N 1), and by (TR 1) there exists a distinguished triangle of the form

$$X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow T(X) \ ,$$

we conclude $id_X \in S(\mathcal{N})$; (S 1) is satisfied.

For (S 2), let $X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow T(X)$ and $Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow T(Y)$ be two distinguished triangles with $Z', X' \in \mathcal{N}$. Such exists by the previous argument. By (TR 2), there exists a distinguished triangle $X \xrightarrow{g \circ f} Z \longrightarrow Y' \longrightarrow T(X)$ , hence by (TR 5) we have another distinguished triangle $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow T(Z')$ . Applying (TR 3) twice, we get the distinguished triangle $X' \longrightarrow T(Z') \longrightarrow T(Y') \longrightarrow T(X')$ . Since $X', Z' \in \mathcal{N}$ by assumption, it follows that $Y' \in \mathcal{N}$ after applying (N 2) and (N 3). Thus, $g \circ f \in \mathcal{N}$.

Now we address (S 3). Let $g \in S(\mathcal{N})$; that is, there exists a distinguished triangle $Z \xrightarrow{g} Y \xrightarrow{k} X' \longrightarrow T(Z)$ with $X' \in \mathcal{N}$. Let $f : X \to Y$. We can then construct a distinguished triangle $W \longrightarrow X \xrightarrow{k \circ f} X' \longrightarrow T(W)$ by combining (TR 2) and (TR 3). Applying (TR 3) and (TR 4), we get a morphism of distinguished triangles:

$$\begin{array}{ccccccc}
W & \xrightarrow{h} & X & \longrightarrow & X' & \longrightarrow & T(W) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle id_{X'}} & & \downarrow \\
Z & \longrightarrow & Y & \longrightarrow & X' & \longrightarrow & T(Z)
\end{array}$$

By assumptions $X' \in \mathcal{N}$, hence by definition $h \in S(\mathcal{N})$. A similar argument proves that the statement holds with reversed arrows, showing (S 3).

We now show that (S 4) is satisfied. Let $f : X \to Y$ be a morphism, and assume there exists a morphism $t : Y \to Y'$, $t \in S(\mathcal{N})$, such that $t \circ f = 0$. By assumption, there exists a distinguished triangle

$$Y \xrightarrow{t} Y' \longrightarrow Z \longrightarrow T(Y),$$

with $Z \in \mathcal{N}$. By applying (TR 3), we get a distinguished triangle

$$Z \longrightarrow Y \xrightarrow{t} Y' \longrightarrow T(Z),$$

with $Z \in \mathcal{N}$. We denote the morphism $Z \to Y$ by $g$. By (TR 1) and (TR 4), we get the commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{id_X} & X \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{g} & Y.
\end{array}$$

This shows $f = g \circ h$. By (TR 2) and (TR 3), we can embed $h$ into a distinguished triangle,

$$X' \xrightarrow{s} X \xrightarrow{h} Z \longrightarrow T(X').$$

But then, composing $f$ with $s$, we have $f \circ s = (g \circ h) \circ s = g \circ (h \circ s) = 0$, since composition of consecutive morphisms in a distinguished triangle is zero. Hence, $s$ satisfies our needs. Analogously, we can show the other direction, and (S 4) follows. □

Further on, we will denote the localization of a triangulated category $\mathcal{C}$ by the multiplicative system $S(\mathcal{N})$ by $\mathcal{C}/\mathcal{N}$, instead of the usual $\mathcal{C}_{S(\mathcal{N})}$.

**Proposition 6.5.** *Let $\mathcal{C}$ be a triangulated category and let $\mathcal{N}$ be a null system.*

*(i) Say that a triangle in $\mathcal{C}/\mathcal{N}$ is distinguished if it is isomorphic to the image of a distinguished triangle in $\mathcal{C}$. Then $\mathcal{C}/\mathcal{N}$ is a trianguled category.*

*(ii) If $X \in \mathcal{N}$, then $Q(X) \cong 0$, where $Q$ is the localization functor.*

*(iii) Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor of triangulated categories. If for all $X \in \mathcal{N}$ we have $F(X) \cong 0$, then $F$ factors uniquely through $Q$.*

*Proof.* To show (i), one first shows $T$ is well-defined in the quotient category, which is an easy exercise. The rest is straight forward.

(ii) and (iii) are clear by Proposition 6.2. □

The study later will be on truncated complexes. Subsequently, the following proposition shows that null systems and localization "are well behaved" under the truncation operation. We start with a definition.

**Definition 6.5.** Let $\mathcal{C}$ be a triangulated category and let $\mathcal{N}$ be a null system in $\mathcal{C}$. We call a subcategory $\mathcal{C}'$ of $\mathcal{C}$ a **full triangulated subcategory** if for any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$$

in $\mathcal{C}$ such that $X, Y \in Ob(\mathcal{C}')$, then it is a distinguished triangle in $\mathcal{C}'$.

**Proposition 6.6.** *Let $\mathcal{C}$ be a triangulated category, $\mathcal{N}$ a null system in $\mathcal{C}$, and $\mathcal{C}'$ a full triangulated subcategory of $\mathcal{C}$. Denote $\mathcal{N} \cap Ob(\mathcal{C}')$ by $\mathcal{N}'$. Then the following holds.*

*(i) $\mathcal{N}'$ is a null system in $\mathcal{C}'$.*

*(ii) If any morphism $Y \to Z$ in $\mathcal{C}$, with $Y \in \mathcal{C}'$ and $Z \in \mathcal{N}$, factors through an object in $\mathcal{N}'$, then $\mathcal{C}'/\mathcal{N}'$ is a full subcategory of $\mathcal{C}/\mathcal{N}$.*

*Proof.* (i) (N 1) and (N 2) follows immediately from $\mathcal{C}'$ being a triangulated category. (N 3) follows from the definition of a full triangulated subcategory, showing that $\mathcal{N}'$ is a null system in $\mathcal{C}'$.

(ii) We will show this with the help of Proposition 6.3. Let $f \in S(\mathcal{N})$ and consider the associated distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$$

with $Z \in \mathcal{N}$ and $Y \in Ob(\mathcal{C}')$. By the assumption, we have that the morphism $Y \to Z$ factors through $Y \to Z' \to Z$, where $Z' \in \mathcal{N}'$. Subsequently, we get a distinguished triangle

$$Y \longrightarrow Z' \longrightarrow W \longrightarrow T(Y)$$

by applying (TR 5) to the morphisms $Y \to Z'$ and $Z' \to Z$. It follows from (TR 5) that $T^{-1}(W) \to Y$ factors to $T^{-1}(W) \to X \xrightarrow{f} Y$, and we are done. □

## 6.2 Derived category

In this section, we will define the derived category and look at some properties thereof. The motivation behind the derived category is to make quasi-isomorphisms invertible, which allows for deeper relationships. In this section, let $\mathcal{A}$ denote an abelian category, and let $K(\mathcal{A})$ denote the homotopy category of $\mathcal{A}$.

Consider the set

$$\mathcal{N} = \{X \in K(\mathcal{A}); H^n(X) \cong 0 \text{ for all } n \in \mathbb{N}\}. \tag{6.2.1}$$

Then $\mathcal{N}$ is clearly a null system, and thus $S(\mathcal{N})$ is a multiplicative system consisting of quasi-isomorphisms. To see the last statement, note that $H^0$ is a cohomological functor, and by applying (TR 3), so is $H^n$. We have that $S(\mathcal{N})$ consist of morphisms $f : X \to Y$ such that

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$$

is a distinguished triangle with $Z \in \mathcal{N}$. But then

$$H^n(X) \xrightarrow{H^n(f)} H^n(Y) \longrightarrow 0$$

is exact, and so is

$$0 \longrightarrow H^n(X) \xrightarrow{H^n(f)} H^n(Y)$$

after applying (TR 3); that is,

$$0 \longrightarrow H^n(X) \xrightarrow{H^n(f)} H^n(Y) \longrightarrow 0$$

is exact. This means $H^n(f)$ is an isomorphism, thus $f$ is a quasi-isomorphism.

We are ready for the definition of the derived category.

**Definition 6.6.** The derived category of $\mathcal{A}$ is given by $D(\mathcal{A}) = K(\mathcal{A})/\mathcal{N}$.

Similar to the discussion of complexes, there are three interesting subcategories:

(i) $D^b(\mathcal{A}) = K^b(\mathcal{A})/(\mathcal{N} \cap K^b(\mathcal{A}))$,

(ii) $D^+(\mathcal{A}) = K^+(\mathcal{A})/(\mathcal{N} \cap K^+(\mathcal{A}))$, and

(iii) $D^-(\mathcal{A}) = K^-(\mathcal{A})/(\mathcal{N} \cap K^-(\mathcal{A}))$.

These definition makes sense since $(\mathcal{N} \cap K^*(\mathcal{A}))$ is a null system by Proposition 6.6 for $* \in \{b, +, -\}$.

We ask if these are full subcategories, as was the case for the category of complexes and the homotopy category. The answer is yes, and is given by the following proposition. First, note that since $S(\mathcal{N})$ consists of quasi-isomorphisms, $H^n(f)$ is an isomorphism for all $f \in S(\mathcal{N})$. Applying Proposition 6.2, the functor $H^n : K(\mathcal{A}) \to \mathcal{A}$ factors uniquely through $D(\mathcal{A})$; we call this functor once again $H^n$.

**Proposition 6.7.** *The categories $D^b(\mathcal{A})$, $D^+(\mathcal{A})$, and $D^-(\mathcal{A})$ are full subcategories of $D(\mathcal{A})$. Moreover, they consists of objects $X$ in the usual way; that is, $D^b(\mathcal{A})$ has $H^n(X) = 0$ for $|n| \gg 0$, and similarly with $D^+(\mathcal{A})$ and $D^-(\mathcal{A})$.*

*Proof.* It is clear that the prerequisites in Proposition 6.6 (ii) are satisfied, hence the categories are full subcategories. Since $H^n : D(\mathcal{A}) \to \mathcal{A}$ comes from the unique factorization of $H^n : K(\mathcal{A}) \to \mathcal{A}$, we have the last of the statement as well. □

It can be difficult working with the derived category directly, and even so when studying how to induce functors to this. The following proposition shows an important equivalence, which will help in subsequent analysis.

**Proposition 6.8.** *Let $\mathcal{I}$ be a full additive subcategory of $\mathcal{A}$ such that for any object $X$ in $\mathcal{A}$, there exists an object $X'$ in $\mathcal{I}$ and a monomorphism $X \to X'$. Then for each object $X \in Ob(K^+(\mathcal{A}))$, there exists a quasi-isomorphism $f : X \to X'$, where $X' \in Ob(K^+(\mathcal{I}))$.*

*Furthermore, let $\mathcal{N}$ be given as in 6.2.1 and let $\mathcal{N}' = \mathcal{N} \cap K^+(\mathcal{I})$. Then the canonical functor*

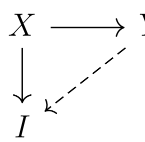
$$K^+(\mathcal{I})/\mathcal{N}' \to D^+(\mathcal{A})$$

*is an equivalence of categories.*

*Proof.* See Kashiwara and Schapira [8], page 325-326. □

We remember the definition of an injective object.

**Definition 6.7.** Let $\mathcal{C}$ be a category, and let $I$ be an object in $\mathcal{C}$. $I$ is said to be injective, if for every monomorphism $X \to Y$ and every morphism $X \to I$, there exists a morphism $Y \to I$ such that the following diagram commutes.



**Definition 6.8.** Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ has enough injectives if for any $X \in Ob(\mathcal{C})$, there exists an injective object $X'$ in $\mathcal{C}$ and a monomorphism $X \to X'$.

**Proposition 6.9.** *Let $\mathcal{A}$ be an abelian category with enough injectives, and let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ consisting of injective objects. Then, the natural functor from $K^+(\mathcal{I}) \to D^+(\mathcal{A})$ is an equivalence of categories.*

*Proof.* Let $\mathcal{N}$ be as in 6.2.1. In light of Proposition 6.8, we only need to show that

$$\mathcal{N} \cap K^+(\mathcal{I}) = 0;$$

that is, for any $X \in C^+(\mathcal{I})$ such that $H^n(X)$ for all $n$ is homotopic to zero.

Consider an object

$$0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

in $C^+(\mathcal{I})$. Denote the morphisms by $d_X^n$. Then we have an short exact sequences

$$0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \mathrm{Ker} d_X^2$$

and

$$0 \longrightarrow \mathrm{Ker} d_X^n \xrightarrow{i^n} X^n \xrightarrow{j^n} \mathrm{Ker} d_X^{n+1}.$$

It follows by induction that $\mathrm{Ker} d_X^n$ is injective for all $n$. This means the sequences

$$0 \longrightarrow \mathrm{Ker} d_X^n \xrightarrow{i^n} X^n \xrightarrow{j^n} \mathrm{Ker} d_X^{n+1}.$$

split and we have morphisms $f^n : X^n \to \mathrm{Ker} d_X^n$ and $g^n : \mathrm{Ker} d_X^{n+1} \to X^n$ such that $f^n \circ i^n = id_{\mathrm{Ker} d_X^n}$, $j^n \circ g^n = id_{\mathrm{Ker} d_X^{n+1}}$, $f^n \circ g^n = 0$, and $i^n \circ f^n + g^n \circ j^n = id_{X^n}$. But then $s^n = g^{n-1} \circ f^n : X^n \to X^{n-1}$ gives the desired homotopy; we have $id_{X^n} = d_X^{n-1} \circ s^n + s^{n+1} \circ d_X^n$.

□

## 6.3 Derived functor

We will now define derived functors, which is a way to induce a functor defined over an abelian category into the corresponding derived category. The definition of the derived functor is technical and I will follow the one given in Kashiwara and Schapira [9]. Note that there are alternatives, such as those given in Rotman [4] and Iversen [6] respectively. After the technical definition, we will show how the definition in Rotman and Iversen naturally falls out; this gives an easier tool for calculating the derived functor.

Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories, and let $F : C(\mathcal{A}) \to C(\mathcal{A}')$ be an additive functor. Consider the induced functor $K(F) : K(\mathcal{A}) \to K(\mathcal{A}')$ between homotopy categories, given in an obvious way. To see that it is a functor, we need to check that it is well-defined on morphisms and that it follows the axioms of a functor. Given that it is well-defined, it is easy to see the latter, so we only show it is well-defined.

Consider two homotopic morphisms $f, g \in Hom_{C(\mathcal{A})}(X, Y)$. We want to show that $F(f)$ and $F(g)$ are homotopic in $Hom_{C(\mathcal{A}')}(F(X), F(Y))$. Let $d_X$ and $d_Y$ be the boundary morphisms in the chain complexes $X$ and $Y$ respectively, and let $s$ be a chain map such that

$$f^n - g^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n,$$

which exists because $f \sim g$. Applying $F$ and using the additive property, we get

$$F(f^n) - F(g^n) = F(s^{n+1}) \circ F(d_X^n) + F(d_Y^{n-1}) \circ F(s^n);$$

that is, $F(f) \sim F(g)$.

Similarly, we are tempted to let $F$ induce a functor in the natural way in the derived category, but we immediately run into problems; this functor does not necessarily preserve quasi-isomorphisms, and subsequently is not well-defined.

**Example 6.1.** Consider the category of abelian groups, **Ab**, and the two chain complexes in $C(\mathbf{Ab})$

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots ,$$

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/n \longrightarrow 0 \longrightarrow \cdots .$$

We have the following quasi-isomorphism

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle mod\, n} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/n & \longrightarrow & 0 & \longrightarrow & \cdots \end{array},$$

However, applying the additive functor $Hom(\mathbb{Z}/n, \cdot)$ on the chain complexes, noting that $Hom(\mathbb{Z}/n, \mathbb{Z}) = 0$, and $Hom(\mathbb{Z}/n, \mathbb{Z}/n) = \mathbb{Z}/n$, we get the following commutative diagram.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/n & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Clearly, we do not have a quasi-isomorphism anymore, so additive functors do not necessarily preserve this property. Subsequently, the natural way to induce a functor in the derived category may not be proper; it is not necessarily well-defined.

To overcome this, we consider the 'best' functor in the derived categories to represent $F : \mathcal{A} \to \mathcal{A}'$, and is given by the following definition.

**Definition 6.9.** Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor of abelian categories, and let $T : D^+(\mathcal{A}) \to D^+(\mathcal{A}')$ be a functor of triangulated categories. Let

$$s : Q \circ K^+(F) \to T \circ Q$$

be a morphism of functors, and assume that for any functor of triangulated categories $U : D^+(\mathcal{A}) \to D^+(\mathcal{A}')$, the morphism

$$\mathrm{Hom}(U,T) \to \mathrm{Hom}(Q \circ K^+(F), U \circ Q)$$

is an isomorphism. Then $(T, s)$ is called the **right derived functor** of $F$ and is denoted by $RF$.

The condition of

$$\mathrm{Hom}(U,T) \to \mathrm{Hom}(Q \circ K^+(F), U \circ Q)$$

being an isomorphism makes $(T, s)$ unique.

Note that $T$ is the *left Kan extension* of $Q(\mathcal{A}') \circ K^+(F)$ along $Q(\mathcal{A})$.

We will now give some easier-to-check criterium for when the right derived functor exists. At the same time, we provide a tool for calculating the right derived functor.

**Definition 6.10.** Let $F : \mathcal{A} \to \mathcal{A}'$ be a left exact additive functor of abelian categories and let $\mathcal{I}$ be a full additive subcategory of $\mathcal{A}$ (not necessarily consisting of injective objects). Then we say that $\mathcal{I}$ is **injective** with respect to $F$, or $F$**-injective** if the following conditions hold.

(i) For all $X \in Ob(\mathcal{A})$, there exists an $X' \in \mathcal{I}$ and a monomorphism $X \to X'$.

(ii) If $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is an exact sequence in $\mathcal{A}$, and if $X', X \in Ob(\mathcal{I})$, then so is $X''$.

(iii) If $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is an exact sequence in $\mathcal{A}$, and if $X', X, X'' \in Ob(\mathcal{I})$, then the sequence $0 \longrightarrow F(X') \longrightarrow F(X) \longrightarrow F(X'') \longrightarrow 0$ is exact.

The definition of an $F$-injective subcategory is useful because it gives us a way to compute the right derived functor of $F$, which will be clear by the two following propositions.

**Proposition 6.10.** *Let $F : \mathcal{A} \to \mathcal{A}'$ be a left exact additive functor of abelian categories and let $\mathcal{I}$ be a full additive subcategory of $\mathcal{A}$ that is $F$-injective. Then the induced functor $K^+(F) : K^+(\mathcal{I}) \to K^+(\mathcal{A}')$ transforms objects quasi-isomorphic to zero to objects into objects quasi-isomorphic to zero.*

*Proof.* See [8], page 356. □

Consider the setting of the last proposition. Then, we have a composition of morphisms

$$K^+(\mathcal{I}) \xrightarrow{K^+(F)} K^+(\mathcal{A}') \xrightarrow{Q} D^+(\mathcal{A}')$$

such that by Proposition 6.5, this factors through $K^+(\mathcal{I})/(\mathcal{N} \cap Ob(K^+(\mathcal{I})))$. Moreover, by Proposition 6.8, $K^+(\mathcal{I})/(\mathcal{N} \cap Ob(K^+(\mathcal{I})))$ is equivalent to $D^+(\mathcal{A})$ and hence we have the following.

**Proposition 6.11.** *Let $F : \mathcal{A} \to \mathcal{A}'$ be a left exact additive functor of abelian categories and let $\mathcal{I}$ be a full additive subcategory of $\mathcal{A}$ that is $F$-injective. Then unique functor $K^+(\mathcal{I})/(\mathcal{N} \cap Ob(K^+(\mathcal{I}))) \to D^+(\mathcal{A}')$ given above is the right derived functor of $F$. In particular, the right derived functor exists.*

**Proposition 6.12.** *Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ be three abelian categories, and let $F : \mathcal{A} \to \mathcal{A}'$, $F' : \mathcal{A}' \to \mathcal{A}''$. Assume we have two full additive subcategories $\mathcal{I}$, $\mathcal{I}'$ of $\mathcal{A}$, $\mathcal{A}'$ respectively, where $\mathcal{I}$ is $F$-injective and $\mathcal{I}'$ is $F'$-injective. Further, assume $F$ maps objects in $\mathcal{I}$ into objects in $\mathcal{I}'$.*

*Then $\mathcal{I}$ is $(F' \circ F)$-injective and we have the following equality:*

$$R(F' \circ F) = RF' \circ RF.$$

*Proof.* This is routine by Proposition 6.11. □

# 7 Sheaves

We will now look at sheaves, which informally can be seen as defining local algebraic structures in a topological space. This can be seen as a way to generalize singular homology of topological spaces. We will have two definitions of a sheaf: *etale-sheaf of abelian groups* and *sheaf of abelian groups*. We will show that these are equivalent, but sheaves will be the more emphasized object going forward; etale-sheaves will serve as a way of "sheafifying close-to-be sheaves".

The ideas presented here come from several sources that complement each other. The beginning of looking at the definition of sheaves and etale-sheaves of abelian groups is based mostly on Rotman [4]. The discussion on stalks and sheafification is more detailed in Tennison [5], and I've chosen to take this path as well.

## 7.1 Protosheaves and Etale-sheaves

**Definition 7.1.** Let $E$ and $X$ be topological spaces. A continuous map $p : E \to X$ is called a **local homeomorphism** if, for each $e \in E$, there is an open neighborhood $S$ of $e$ such that $p(S)$ is open and $p|_S : S \to p(S)$ is a homeomorphism. We call $S$ a **sheet**.

If the local homeomorphism $p$ is surjective, we call the triple $(E, p, X)$ a **protosheaf**.

We typically call $E$ a **sheaf space**, $p$ a **projection**, and $X$ the **base space**. Further, the fiber $p^{-1}(x)$ for $x \in X$ is called the **stalk** over $x$ and is denoted by $E_x$. Here are some basic properties of protosheaves.

**Proposition 7.1.** *Let $(E, p, X)$ be a protosheaf.*

*(i) The sheets form a base for $E$.*

*(ii) $p$ is an open map.*

*(iii) Each stalk is discrete.*

*Proof.* (i) Consider the set of all sheets $\{S\}_S$ of the elements of $E$. This, by definition, forms an open cover of $E$. Let $U \subseteq E$ be an open set, and note that $U = \bigcup_S (U \cap S)$. Since every open subset of a sheet is also a sheet, $U$ is a union of sheets.

(ii) By the previous, we have $f(U) = f(\bigcup_S (U \cap S)) = \bigcup_S f(U \cap S)$; that is, we take a union of open sets which is open.

(iii) Consider $e \in E_x$ for some $x$ and let $S$ be a sheet over $e$. For another $e' \in E_x$ such that $e' \neq e$, then $e' \notin S$. This can be seen because $p$ is injective on $S$, and if both are in $S$ then $p(e) = x = p(e')$, a contradiction. Thus, we have $S \cap E_x = \{e\}$; $E$ is discrete.

□

We continue with two properties of continuous maps.

**Proposition 7.2.** *Let $X$ be a space.*

*(i) Let $\{U_i\}_{i \in I}$ be an open cover of some open set $U \subseteq X$. If $f, g : U \to Y$ are two maps to some space $Y$, and if $f$ and $g$ agrees on $U_i$ for $i \in I$, then $f = g$.*

*(ii) Let $\{U_i\}_{i \in I}$ be an open cover of some open set $U \subseteq X$. let $f_i : U_i \to Y$ be a continuous map for $i \in I$. If $f_i \mid_{U_i \cap U_j} = f_j \mid_{U_i \cap U_j}$, then there exists a unique continuous map $f : U \to Y$ such that $f \mid_{U_i} = f_i$ for all $i \in I$.*

*Proof.* (i) If $x \in U$, then $x \in U_i$ and we have $f(x) = f|_{U_i}(x) = g|_{U_i}(x) = g(x)$; that is, $f = g$.

(ii) If $x \in U$, then $x \in U_i$ for some $i \in I$. We define $f : U \to Y$ to be given by $f(x) = f_i(x)$. The assumption that $f_i$ and $f_j$ agree on $U_i \cap U_j$ shows that $f$ is well-defined, and it is clear that is the unique function satisfying $f \mid_{U_i} = f_i$. We need to show that it is continuous.

Let $V \subseteq Y$ be an open subset We have

$$\begin{aligned} f^{-1}(V) &= U \cap f^{-1}(V) \\ &= (\bigcup_i U_i) \cap f^{-1}(V) \\ &= \bigcup_i (U_i \cap f^{-1}(V)) \\ &= \bigcap_i f_i^{-1}(V), \end{aligned}$$

which is a union of open sets hence open. This shows that $f$ is continuous.

□

We will now define etale-sheaf.

**Definition 7.2.** Let $\mathcal{S} = (E, p, X)$ be a protosheaf. We say that $\mathcal{S}$ is an **etale-sheaf of abelian groups** if

(i) the stalk $E_x$ is an abelian group for each $x \in X$, and

(ii) inversion and addition are continuous.

The condition (ii) of an etale-sheaf of abelian groups needs some clarification. What is meant with inversion $e \mapsto -e$ for $e \in E$ is clear. For addition, define $E + E = \bigcup_{x \in X} E_x \times E_x = \{(e, e') : p(e) = p(e')\}$. Addition $\alpha : E + E \to E$ is defined by $(e, e') \mapsto e + e'$. It is continuous if for every open neighborhood $V$ of $e + e'$, there exists an open neighborhood $U$ of $(e, e')$ such that $\alpha(U) \subseteq V$.

We are now ready to define the category of etale-sheaves.

**Definition 7.3.** Let $\mathcal{S} = (E, p, X)$ and $\mathcal{S}' = (E', p', X')$ be two etale-sheaves over a space $X$. An **etale-map** $\psi : \mathcal{S} \to \mathcal{S}'$ is a continuous map $\psi : E \to E'$ such that $p' \circ \psi = p$, and $\psi|_{E_x}$ is a homomorphism for each $x \in X$.

We define the category of etale-sheaves of abelian groups over a topological space $X$, denoted by $\mathbf{Sh}_{\mathrm{et}}(X, \mathbf{Ab})$, to be given by the following:

(i) The objects of $\mathbf{Sh}_{\mathrm{et}}(X, \mathbf{Ab})$ are etale-shaves of abelian groups.

(ii) The morphisms are given by the etale-maps.

We denote the set of etale-maps from two etale-sheaves $\mathcal{S}$ and $\mathcal{S}'$ by $\mathrm{Hom}_{\mathrm{et}}(\mathcal{S}, \mathcal{S}')$.

It is routine to show that $\mathbf{Sh}_{\mathrm{et}}(X, \mathbf{Ab})$ is a category. We continue with some properties of $\mathrm{Hom}_{\mathrm{et}}(\mathcal{S}, \mathcal{S}')$.

**Proposition 7.3.** *Let $\mathcal{S} = (E, p, X)$ and $\mathcal{S}' = (E', p', X')$ be two etale-sheaves over a space $X$.*

*(i)* $\mathrm{Hom}_{\mathrm{et}}(\mathcal{S}, \mathcal{S}')$ *is an additive abelian group, where addition is given by* $\psi + \phi : E \to E'$, $e \mapsto \psi(e) + \phi(e)$ *for* $\phi, \psi \in \mathrm{Hom}_{\mathrm{et}}(\mathcal{S}, \mathcal{S}')$.

*(ii) The distributive law holds; that is, given two etale-sheaves $\mathcal{X}$ and $\mathcal{Y}$, the etale-maps*

$$\mathcal{X} \xrightarrow{\alpha} \mathcal{S} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \mathcal{S}' \xrightarrow{\beta} \mathcal{Y},$$

*makes the following equalities hold:*

$$\beta(\psi + \phi) = \beta\psi + \beta\phi \quad \text{and} \quad (\psi + \phi)\alpha = \psi\alpha + \phi\alpha.$$

*(iii) Every etale-map $\psi : \mathcal{S} \to \mathcal{S}'$ is an open map $\psi : E \to E'$.*

*Proof.* The first two statements are routine to prove. For the last, we note that the sheets form a basis of the topology of $E$, and $p'\psi = p$. □

Let's look at one example of an etale-sheaf, and a non-example.

**Example 7.1.** Let $X$ be a topological space and let $Y$ be an abelian group with the discrete topology. Let $E = X \times Y$ be a space with the product topology, and study the triple $\mathcal{S} = (E, p, X)$, with $p : (x, y) \mapsto x$ being the projection. This is an etale-sheaf which can be seen by the following.

$\mathcal{S}$ is a protosheaf: $p$ is a local homeomorphism since for any open set $U \subseteq X$, $U \times \{y\}$ is open in $X \times Y$ for $y \in y$, and $p|_{U \times \{y\}}$ is clearly a homeomorphism $U \times \{y\} \to p(U \times \{y\})$. Further, $p$ is surjective since it is the projection.

For a given $x \in X$, we have $E_x = \{x\} \times Y$ which is isomorphic to $Y$, hence can be seen as an abelian group. Continuity of inversion and addition follows immediately since $Y$ is discrete and $p$ is the projection. We conclude that $\mathcal{S}$ is an etale-sheaf.

**Example 7.2.** A non-example is the protosheaf $(\mathbb{R}, p, S^1)$, with $S^1$ being the circle and $p : \mathbb{R} \to S^1$ is the map $x \mapsto e^{2\pi i x}$, which is a local homeomorphism. It is not an abelian group, and hence not an etale-sheaf. To see this, note that the stalk at $S^1 \ni s \neq 1$ is not an abelian group.

**Definition 7.4.** Let $\mathcal{S} = (E, p, X)$ and $\mathcal{S}' = (E', p', X')$ be two etale-sheaves. We call $\mathcal{S}'$ an **subetale-sheaf** of $\mathcal{S}$ if $E' \subseteq E$ and the inclusion $E' \to E$ is an etale-map.

## 7.2 Presheaves and Sheaves

We will now continue with presheaves and sheaves, which will be the main objects we will work with when studying the cohomology of sheaves. We begin with a "categorification" of a topology.

**Example 7.3.** Consider a topological space $X$ with topology $\mathcal{U}$. View $\mathcal{U}$ as a category by the following. Let $Ob(\mathcal{U}) = \mathcal{U}$, and let $\mathrm{Hom}(U, V)$ be either the empty set if $U \nsubseteq V$, or the set consisting of the inclusion $U \hookrightarrow V$ if $U \subseteq V$. The composition of morphisms is defined as expected. It is clear that $\mathcal{U}$ is a category.

**Definition 7.5.** Let $\mathcal{C}$ be a category and let $\mathcal{U}$ be the category of a topology of a topological space $X$. A **presheaf** is defined to be a contravariant functor $\mathcal{F} : \mathcal{U} \to \mathcal{C}$. Further, this gives a set of morphisms called **restriction morphisms**. Whenever $U \subseteq V$ for two open subsets $U, V$, there is a restriction morphism $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$.

We denote a presheaf with $\{\mathcal{F}, \rho_U^V\}$, or simply $\mathcal{F}$.

Sometimes we are only interested in a subset of $X$.

**Definition 7.6.** Let $U \subseteq X$ be an open subset and let $\mathcal{F}$ be a presheaf. We define the **restriction of** $\mathcal{F}$ to $U$, denoted $\mathcal{F}|_U$, to be given by the presheaf

$$U \supseteq V \mapsto \mathcal{F}(U),$$

where $V$ is open in $U$ (and so in $X$).

We call a presheaf a **presheaf of abelian groups** if $\mathcal{C} = \mathbf{Ab}$. Let's look at an example of a presheaf.

**Example 7.4.** Let $X$ be a topological space, $x \in X$ an element, $\mathcal{U}$ the topology category, and $A$ an abelian group. We define the **skyscraper presheaf** $x_*A : \mathcal{U} \to \mathbf{Ab}$ to be given by

$$x_*A(U) = \begin{cases} A & \text{if } x \in U, \\ \{0\} & \text{otherwise.} \end{cases}$$

For $U \subseteq V$, with $U$ and $V \in \mathcal{U}$, then the inclusion map is mapped by $x_*A$ to either $1_A$ or 0, depending on if $x \in X$. It is not hard to show that $x_*A$ is a contravariant functor, and thus, a presheaf.

We will later show, after providing with the definition, that $x_*A$ is a sheaf.

Before defining a sheaf, we will define the so-called equalizer condition, from which a sheaf is defined. For a given presheaf $\{\mathcal{F}, \rho_U^V\}$, and element $\sigma \in \mathcal{F}(V)$, we will sometimes abbreviate $\rho_U^V(\sigma)$ by $\sigma|_U$.

**Definition 7.7.** Let $\{\mathcal{F}, \rho_U^V\}$ be a presheaf of abelian groups on a topological space $X$ with topology $\mathcal{U}$. The presheaf $\{\mathcal{F}, \rho_U^V\}$ satisfy the **equalizer condition** if

(i) (**Uniqueness**) for every open set $U \in \mathcal{U}$ and open cover $\{U_i\}_{i \in I}$, if $\sigma, \tau \in \mathcal{F}(U)$ satisfy $\sigma|_{U_i} = \tau|_{U_i}$ for all $i \in I$, then $\sigma = \tau$.

(ii) (**Gluing**) for every open set $U \in \mathcal{U}$ and open cover $\{U_i\}_{i \in I}$, if $\sigma_i \in \mathcal{F}(U_i)$ satisfy $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique $\sigma \in \mathcal{F}(U)$ such that $\sigma|_{U_i} = \sigma_i$ for all $i \in I$.

At first glance, it might seem like the equalizer condition is satisfied by all presheaf, but this is not the case, which is illustrated by the next example.

**Example 7.5.** Let $X$ be the $2D$ euclidean plane with regular topology, and define the presheaf $\mathcal{F}$ by

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} | \, f \text{ is constant}\} \quad \text{for } U \in \mathcal{U}.$$

Now, consider two disjoint open sets $U_1, U_2 \in \mathcal{U}$, and let $U = U_1 \cup U_2$, hence $\{U_1, U_2\}$ is an open cover of $U$. Consider the mappings $\sigma_1 : \mathcal{F}(U_1) \to \mathbb{R}$ to be given by $\sigma_1(u_1) = 0$ for $u_1 \in U_1$. Similarly, we define $\sigma_2(u_2) = 1$ for $u_2 \in U_2$. Since $U_1 \cap U_2 = \varnothing$, we have $\sigma_1|_{U_1 \cap U_2} = \sigma_2|_{U_1 \cap U_2}$. However, it is easy to see that there is no constant map $\sigma \in \mathcal{F}(U)$ such that $\sigma|_{U_1} = 0$ and $\sigma|_{U_2} = 1$, hence the gluing condition is not satisfied.

We are now ready for the definition of a sheaf.

**Definition 7.8.** Let $\{\mathcal{F}, \rho_U^V\}$ be a presheaf of abelian groups. Then $\{\mathcal{F}, \rho_U^V\}$ is called a **sheaf of abelian groups** if it satisfies the equalizer condition.

Sheaves are interesting as they describe some local properties of a space. And by using sheaf cohomology, as we will do in a coming section, we will "make this data global". This is useful for generalizing Poincaré duality.

We bring up the example of the skyscraper presheaf again and show that it is a sheaf.

**Example 7.6.** Let the skyscraper presheaf be as defined in Example 7.4. To show it is a sheaf, we need to show it satisfies the equalizer condition.

(**Uniqueness**) Let $U$ be an open set and let $\{U_i\}$ be an open cover of $U$. Consider two elements $\sigma, \tau \in x_*\mathcal{A}(U)$ such that $\sigma|_{U_i} = \tau|_{U_i}$. Then, if $x \notin U$, both elements are trivial and hence equal. If $x \in U$, then there exists some $U_i \ni x$ where $\sigma|_{U_i} = \tau|_{U_i}$. But since the restriction map is the identity, $\sigma = \tau$.

(**Gluing**) Let $\sigma_i \in x_*\mathcal{A}(U_i)$, $\sigma_j \in x_*\mathcal{A}(U_j)$ and assume $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$ for all $i, j$. If $x \notin U$, then there clearly exists a unique element, 0, the trivial element, which satisfy $0|_{U_i} = \sigma_i$. If $x \in U$, then we have two cases: either no intersection of two open subsets in $\{U_i\}$ contain $x$, or such exists. In the former case, all restrictions to an intersection of open subsets is trivial, and we arrive at the same situation as for the uniqueness condition. For the latter, if some intersection $U_i \cap U_j$ contains $x$, the sections are mapped using an identity thereto. This means that there exists a unique section in $x_*\mathcal{A}(U_i \cup U_j)$ which is mapped to $\sigma_i$ and $\sigma_j$ respectively. Continuing inductively gives the result.

Now, let us construct the category of presheaves. To do so, we need morphisms.

**Definition 7.9.** Let $\{\mathcal{F}, \rho_U^V\}$ and $\{\mathcal{G}, \rho'^V_U\}$ be presheaves of abelian groups over a space $X$. A **sheaf map**, $\phi : \mathcal{F} \to \mathcal{G}$ is a natural transformation; that is, we have a commutative diagram whenever $U \subseteq V$ for $U, V \in \mathcal{U}$:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \\ {\scriptstyle \rho_U^V}\big\downarrow & & \big\downarrow{\scriptstyle \rho'^V_U} \\ \mathcal{F}(U) & \xrightarrow[\phi_U]{} & \mathcal{G}(U). \end{array}$$

From this, we can define the category of presheaves. To be more correct, we should have it as a proposition, but since the proof is routine, we keep it as a definition.

**Definition 7.10.** We define the category of presheaves of abelian groups, denoted by $\mathbf{pSh}(X, \mathbf{Ab})$, to consists of presheaves $\mathcal{F} : \mathcal{U} \to \mathbf{Ab}$, with $\mathcal{U}$ being the category of the topology of $X$, as objects, and $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Nat}(\mathcal{F}, \mathcal{G})$ as the morphisms.

We continue with defining the category of sheaves.

**Definition 7.11.** Define $\mathbf{Sh}(X, \mathbf{Ab})$ to be the full category of $\mathbf{pSh}(X, \mathbf{Ab})$ generated by the sheaves over a space $X$.

Lastly, we define the support of a section.

**Definition 7.12.** Let $\mathcal{F} \in Ob(\mathbf{pSh}(X, \mathbf{Ab}))$, $U$ be open set in $X$, and $\sigma \in \mathcal{F}(U)$. The **support** of the section $\sigma$ in $U$ is defined to be the complementary in $U$ of the open sets $V$ such that $\sigma|_V = 0$. The support of $\sigma$ is denoted by $supp(\sigma)$.

## 7.3 Stalks

Before looking at the connection between etale-sheaves and sheaves, we will define stalks. This will help us with the understanding of this connection. Informally, we have seen how presheaves are only defined on open sets. Stalks is a way to increase this domain, to assign an abelian group to each point in the topological space. The reason for making this is that some propositions and theorems of presheaves and sheaves can be transformed into questions about stalks, which can potentially be easier to work with.

**Definition 7.13.** Let $\mathcal{F} \in \mathbf{pSh}(X, \mathbf{Ab})$. The **stalk** $\mathcal{F}_x$ of $\mathcal{F}$ at $x$ is given by

$$\mathcal{F}_x = \varinjlim_{U \ni x} F(U).$$

An element in $\mathcal{F}_x$ is called a **germ**.

Let $\sigma \in \mathcal{F}(U)$ with $x \in U$. We sometimes abbreviate the image of $\sigma$ in $\mathcal{F}_x$ by $\sigma_x$.

By Proposition 2.6, $\mathcal{F}_x$ exists for all $x$. We have the following useful proposition.

**Proposition 7.4.** *Let* $\mathcal{F} \in \mathbf{pSh}(X, \mathbf{Ab})$.

*(i) For each germ $\tau \in \mathcal{F}_x$, there exists an open neighborhood $U$ of $x$ and a section $\sigma \in \mathcal{F}(U)$ such that $\tau = \sigma_x$.*

*(ii) Let $\sigma_x, \tau_x \in \mathcal{F}_x$ be two germs such that $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ for open neighborhoods $x \in U$, $x \in V$. Then $\sigma_x = \tau_x$ if and only if there exists an open neighborhood $W \subseteq U \cap V$ with $\rho^U_W(\sigma) = \rho^V_W(\tau)$.*

*Proof.* This follows from Proposition 2.5 and 2.6. □

**Example 7.7.** We will now give two examples of stalks.

(i) Let $\mathcal{F} \in \mathbf{pSh}(X, \mathbf{Ab})$ such that $\mathcal{F}(U) = A$ for all open sets $U$, where $A$ is an abelian group. This is called the **constant presheaf**. It is clear that for any $x$, $A$ is a target of the system and satisfies the conditions in Proposition 2.5. Hence, $\mathcal{F}_x = A$ for all $x$.

(ii) Let $X$ be a topological space with more than one point having the discrete topology and consider the presheaf $\mathcal{P}$ given by $\mathcal{P}(X) = \mathbb{Z}$, and $\mathcal{P}(U) = 0$ for $U \neq X$, where 0 is the trivial abelian group. Then we have $\mathcal{P}_x = 0$ for all $x$, but $\mathcal{P} \neq 0$. Thus we see that the local behavior on elements is not enough to derive the global behavior for presheaves.

Let $x \in X$ and consider two presheaves $\mathcal{F}, \mathcal{G} \in \mathbf{pSh}(X, \mathbf{Ab})$ and a morphism $\phi : \mathcal{F} \to \mathcal{G}$. This map induces a map of stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ such that whenever we have another presheaf $\mathcal{H} \in \mathbf{pSh}(X, \mathbf{Ab})$ and a morphism $\psi : \mathcal{G} \to \mathcal{H}$, we have

$$(\psi \circ \phi)_x = \psi_x \circ \phi_x.$$

We define $\phi_x$ as follows. Consider an element $\sigma_x \in \mathcal{F}_x$ with an open set $x \in U$, and $\sigma \in \mathcal{F}(U)$. We define

$$\phi_x(\sigma_x) = (\phi(U)(\sigma))_x$$

. We need to show that this is well-defined. Assume we have another open set $x \in V$, an element $\tau \in \mathcal{F}(V)$ such that $\sigma_x = \tau_x$. Then there exists an open set $x \in W \subseteq U \cap V$ and $\rho^U_W(s) = \rho^V_W(t)$. But this means we have

$$\begin{aligned} \rho^U_W(\phi(U)(\sigma)) &= \phi(W)\rho^U_W(\sigma) \\ &= \phi(W)\rho^V_W(\tau) \\ &= \rho^V_W(\phi(V)(\tau)); \end{aligned}$$

that is, $(\phi(U)(\sigma))_x = (\phi(V)(\tau))_x$, hence $\phi_x$ is well-defined. It is routine to check the functorial property $(\psi \circ \phi)_x = \psi_x \circ \phi_x$.

We continue with a useful implication of morphisms of stalks.

**Proposition 7.5.** *Let $\mathcal{F}, \mathcal{G} \in$ **Sh**$(X, $**Ab**$)$ be two sheaves of abelian groups and let $\phi, \psi : \mathcal{F} \to \mathcal{G}$ be two morphisms. If $\phi_x = \psi_x$ for all $x \in X$, then $\phi = \psi$.*

*Proof.* Let $U$ be an open neighborhood of $X$, and let $\sigma \in \mathcal{F}(U)$. By assumption, we have that for all $x$, $\phi_x(\sigma_x) = \psi_x(\sigma_x)$, which means $(\phi(U)(\sigma))_x = (\psi(U)(\sigma))_x$. Then there exists an open neighborhood of $x \in U_x \subseteq U$ such that $\rho^U_{U_x}(\phi(U)(\sigma)) = \rho^U_{U_x}(\psi(U)(\sigma))$. Applying the equalizer condition (uniqueness), we get $\phi = \psi$. □

*Remark* 7.1. Note that in the proof of Proposition 7.5, we only used that $\mathcal{G}$ satisfies the uniqueness condition.

**Proposition 7.6.** *Let $\mathcal{F} \in$ **Sh**$(X, $**Ab**$)$ and let $\sigma, \sigma' \in \mathcal{F}(U)$ for an open set $U$. Then $\sigma = \sigma'$ if and only if $\sigma_x = \sigma'_x$ for all $x \in U$.*

*Proof.* The only if is clear. Assume $\sigma_x = \sigma'_x$ for all $x \in X$. Then there exists an open neighborhood $x \in U_x$ such that $\rho^U_{U_x}(\sigma) = \rho^U_{U_x}(\sigma')$ for all $x$. Applying the uniqueness condition on the cover $\{U_x\}_{x \in U}$ we get $\sigma = \sigma'$. □

## 7.4 Connecting etale-sheaves and sheaves

We are now ready to look at the connection between etale-sheaves and sheaves.

**Definition 7.14.** Let $\mathcal{S} = (E, p, X)$ be an etale sheaf of abelian groups and $U \subseteq X$ be an open set. A **section over** $U$ is a continuous map $\sigma : U \to E$ such that $p \circ \sigma = id_U$. $\sigma$ is called a **global section** if $U = X$.

We denote the set of sections over $U$ with $\Gamma(U, \mathcal{S})$, and we set $\Gamma(\varnothing, \mathcal{S}) = \{0\}$.

**Proposition 7.7.** *Let $\mathcal{S} = (E, p, X)$ be an etale-sheaf of abelian groups, and let $\Gamma(\cdot, \mathcal{S})$ be the functor defined above.*

*(i) $\Gamma(U, \mathcal{S})$ is an abelian group for all open sets $U \subseteq X$.*

*(ii) $\Gamma(\cdot, \mathcal{S})$ is a presheaf of abelian groups.*

*(iii) The function $z : X \to E$, $x \mapsto 0_x \in E_x$ is a global section.*

We call the presheaf $\Gamma(\cdot, \mathcal{S})$ the **sheaf of global sections**. The function $z$ is called the **zero section**.

*Proof.* (i) First, we need to show that $\Gamma(U,\mathcal{S}) \neq \varnothing$ for any open set $U \subseteq X$. For the empty set, $U = \varnothing$, we have by definition $\Gamma(U,\mathcal{S}) = \{0\}$. If $U$ is not empty, let $x \in U$ and take a sheet $S$ of $e \in E_x$. By Proposition 7.1, $p$ is an open map, and so $p(S) \cap U$ is an open neighborhood of $x$. Since $p$ is a homeomorphism, we have a section $(p|_S)^{-1} : p(S) \to S$; define $\sigma_S$ to be the restriction of this section to $p(S) \cap U$. But this construction can be done for all elements in $U$, hence, we have an open cover of set $p(S) \cap U$ of $U$. All the sections agree on overlap, thus by Proposition 7.1 again, there is a section $U \to E$ coming from 'gluing' the constructed sections.

Now we show that it is an abelian group. Fix an open set $U \subseteq X$, and let $\sigma, \tau \in \Gamma(U,\mathcal{S})$. The map $(\sigma,\tau) : x \mapsto (\sigma(x), \tau(x))$ is a continuous map $U \to E + E$. Composing with addition, which is continuous, we get $\sigma + \tau : x \mapsto \sigma(x) + \tau(x) \in \Gamma(U,\mathcal{S})$. Lastly, the inverse is continuous, hence for $\sigma \in \Gamma(U,\mathcal{S})$ we have $-\sigma \in \Gamma(U,\mathcal{S})$.

(ii) By (i), it is clear that it maps to abelian groups, hence on objects seems to be a contravariant functor. On morphisms, if $U \subseteq V$, we get the restriction $\sigma \to \sigma|_U$ as the group homomorphism $\Gamma(V,\mathcal{S}) \to \Gamma(U,\mathcal{S})$.

(iii) Study $\Gamma(X,\mathcal{S})$, which is an abelian group by (i), hence has an identity. But this is the zero section, and we are done.

□

By Proposition 7.1, we see that the presheaf $\Gamma(\cdot,\mathcal{S})$ for an etale-sheaf $\mathcal{S}$ satisfies the equalizer condition, thus it is a sheaf. This fact is important and will be the bridge between etale-shaves and sheaves. We state it explicitly.

**Proposition 7.8.** *Let $\mathcal{S}$ be an etale-sheaf of abelian groups and $\Gamma(\cdot,\mathcal{S})$ be the sheaf of sections. Then $\Gamma(\cdot,\mathcal{S})$ is a sheaf.*

*Proof.* Follows from Proposition 7.1. □

**Proposition 7.9.** *The sheaf of sections defines a functor $\Gamma : \mathbf{Sh}_{et}(X,\mathbf{Ab}) \to \mathbf{Sh}(X,\mathbf{Ab})$.*

*Proof.* Let $\mathcal{S}$ be an etale-sheaf. Then by Proposition 7.8, $\Gamma\mathcal{S} : U \mapsto \Gamma(U,\mathcal{S})$ is a sheaf.

Let $\mathcal{S}'$ be another etale-sheaf, and let $\psi : \mathcal{S} \to \mathcal{S}'$ be an etale-map. The morphism $\Gamma\psi$ is given by $\sigma \mapsto \psi \circ \sigma$, for $\sigma \in \Gamma(U,\mathcal{S})$. It is routine to check that $\Gamma$ satisfies the axioms of being a functor. □

This means we have a functorial way of associating a sheaf to an etale-sheaf. We can also construct an etale-sheaf from a presheaf. Before doing so, we need the following.

**Lemma 7.10.** *Let* $\mathcal{S} = (E, p, X) \in Ob(\mathbf{Sh}_{et}(X, \mathbf{Ab}))$*. If* $U$ *is an open subset in* $X$*, and* $\sigma \in \Gamma\mathcal{S}(U) = \Gamma(U, \mathcal{S})$*, then* $\sigma(U) \subseteq E$ *is open.*

*Proof.* Let $e \in \sigma(U)$. Since $p$ is a local homeomorphism, there exists an open neighborhood $W \subseteq E$ such that $p|_W$ is a homeomorphism. Moreover, the image $V$ of $p|_W$ is open by Proposition 7.1. This means $p|_W$ maps $W \cap \sigma(U)$ bijectively to the open set $U \cap V$. Thus, $W \cap \sigma(U)$ is an open neighborhood of $e$, but since $\sigma(U) \subseteq W \subseteq \sigma(U)$, we see that we can decompose $U$ into subsets which are sent to open sets; since a union of open sets is open, $\sigma(U)$ is open. ☐

**Proposition 7.11.** *Let* $\mathcal{S} = (E, p, X) \in Ob(\mathbf{Sh}_{et}(X, \mathbf{Ab}))$*. The stalk of* $\Gamma\mathcal{S}$ *at* $x \in X$ *is the fibre* $p^{-1}(x)$*, with the discrete topology.*

*Proof.* Let $U$ be an open neighborhood of $x$ and consider the map $\Gamma\mathcal{S}(U) \to p^{-1}(x)$, given by $\sigma \mapsto \sigma(x)$. It is clear that these maps make a commutative diagram with the restriction morphisms. We will show that this is the direct limit using Proposition 2.5.

First, let $e \in p^{-1}(x)$. Since $p$ is a local homomorphism, there exists a neighborhood $W$ of $e$ such that $p|_W$ is a homeomorphism; the inverse $(p|_W)^{-1}$ exists and is an element of $\Gamma\mathcal{S}(U)$ such that $(p|_W)^{-1}(x) = e$. This shows (i) in Proposition 2.5.

Second, assume we have $\sigma \in \Gamma\mathcal{S}(U)$ and $\tau \in \Gamma\mathcal{S}(U)$ such that $\sigma(x) = \tau(x)$. By Lemma 7.10, we have that $W = \sigma(U) \cap \tau(V)$ is open in $E$. Further, $p(W)$ is open by Proposition 7.1, and $\sigma$ and $\tau$ agree on $p(W)$ since they are inverses of $p|_W$. Thus, $\rho^U_{p(W)}(\sigma) = \rho^V_{p(W)}(\tau) \in \Gamma\mathcal{S}(p(W))$, which shows (ii) in Proposition 2.5. Thus, we get

$$p^{-1}(x) \cong \varinjlim_{U \ni x} \Gamma\mathcal{S}(U).$$

Lastly, $p^{-1}(x)$ is a discrete subspace of $E$. This can be seen by taking the $W$ defined above for $e \in p^{-1}(x)$, noting that $W$ is open by Lemma 7.10, and we have $W \cap p^{-1}(x) = \{e\}$. ☐

We will now look at the construction of an etale-sheaf given a presheaf. Let $\mathcal{F} \in Ob(\mathbf{pSh}(X, \mathbf{Ab}))$, and we define the following functor $\mathrm{L} : \mathbf{pSh}(X, \mathbf{Ab}) \to \mathbf{Sh}_{\mathrm{et}}(X, \mathbf{Ab})$.

We let $E = \sqcup_{x \in X} \mathcal{F}_x$, and define $p : E \to X$ to be the natural projection. We give $E$ the following topology. Let $U \subseteq X$ be an open set and let $\sigma \in \mathcal{F}(U)$.

This gives a map $\hat{\sigma} : U \to E$ such that $x \mapsto \sigma_x$, and we let $\hat{\sigma}(U) = \{\sigma_x \in E; x \in U\}$ be an open set in $E$.

Now, consider the topology generated by $\{\hat{\sigma}(U); \sigma \in \mathcal{F}(U)\}$. Then, the intersection of two such sets $\hat{\sigma}(U) \cap \hat{\sigma}'(V)$, for $\sigma' \in \mathcal{F}(V)$, is either empty or an element of the same form. Choose an element $e \in \hat{\sigma}(U) \cap \hat{\sigma}'(V)$. By construction, $\sigma_x = \sigma'_x$ at $p(e) = x$, which means there exists a neighborhood $W \subseteq U \cap V$ of $x$ such that $\rho^U_W(\sigma) = \rho^V_W(\sigma')$. Thus, we get a neighborhood $\widehat{\rho^U_W(\sigma)}(W) = \widehat{\rho^V_W(\sigma')}(W)$ of $e$ which is contained in $\hat{\sigma}(U) \cap \hat{\sigma}'(V)$. This means $\{\hat{\sigma}(U); s \in \mathcal{F}(U)\}$ forms a basis of the topology it generates.

We define $\mathrm{L} : \mathbf{pSh}(X, \mathbf{Ab}) \to \mathbf{Sh}_{\mathrm{et}}(X, \mathbf{Ab})$ to be given by $\mathcal{F} \mapsto (E, p, X)$ with the notations used above.

**Proposition 7.12.** *Let $\mathcal{F} \in \mathbf{pSh}(X, \mathbf{Ab})$ and let $\mathrm{L}$ be as defined above. Then $\mathrm{L}\mathcal{F}$ is an etale-sheaf.*

*Proof.* Denote the created object by $(E, p, X) = \mathrm{L}\mathcal{F}$. It is clear that $(E, p, X)$ has the data of being a protosheaf. Further, $p$ is surjective, so we need to show that it is local homomorphism.

Take an element $e \in E$, then $e = \sigma_x$ for some $\sigma \in \mathcal{U}$ and $x \in X$, where $x \in U \subseteq X$ is open. But then $\hat{\sigma}(U) = \{\sigma_x \in E; x \in U\}$ is open in $E$ by definition and thus is an open neighborhood of $e$. It is easy to see that $p|_{\hat{\sigma}(U)}$ is a bijection, and it is continuous by definition. Further, the inverse is also continuous, since the topology on $E$ is generated by the sets $\hat{\sigma}(U) = \{\sigma_x \in E; x \in U\}$; taking the inverse of these sets yields a union of open sets in $X$. We conclude that $p$ is a local homeomorphism.

Each stalk is an abelian group, since $p^{-1}(x) \cong \mathcal{F}_x$. The proof that addition and inversion are continuous can be found in [4], page 281-283. □

In order for L to be a functor, we need to define what happens on morphisms of presheaves. We start with a useful lemma.

**Lemma 7.13.** *Let $\mathcal{S} = (E, p, X)$ and $\mathcal{S}' = (E', p', X)$ be two objects in $\mathbf{Sh}_{et}(X, \mathbf{Ab})$ and let $\psi : \mathcal{S} \to \mathcal{S}'$ be an etale-map. If the diagram*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\psi} & \mathcal{S}' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array}$$

*commutes, then the following are equivalent.*

*(i) $\psi$ is continuous.*

*(ii) $\psi$ is open.*

*(iii) $\psi$ is a local homeomorphism.*

*Proof.* By definition, if $\psi$ is a local homeomorphism, then it is continuous. Further, since $p$ is open by Proposition 7.1, it follows that $\psi$ is open by commutativity ($p'$ is continuous). This shows (iii) $\implies$ (i), (ii).

Next, we show (i) $\implies$ (iii). Let $e \in E$, $\psi(e) \in E'$. Let $S'$ be a sheet of $\psi(e)$, hence $p'|_{S'}$ is a homeomorphism. Since $\psi$ is continuous, $e \in \psi^{-1}(S')$ is open in $E$. Since $p$ is a local homeomorphism, we can find a sheet $S$ of $e$ such that $S \subseteq \psi^{-1}(S')$. Thus, both $p|_S$ and $p'|_{\psi(S)\subseteq S'}$ are homeomorphisms, thus so is $\psi|_S$, and the result follows.

Lastly, for (ii) $\implies$ (iii), we can do a similar analysis as above. We first choose a sheet $S$ of $e$, and we note that $\psi(S)$ is open in $E'$. The rest is routine. □

We now define L on morphisms. Let $\psi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves of abelian groups, and let $\mathrm{L}\mathcal{F} = (E, p, X)$, $\mathrm{L}\mathcal{G} = (E', p', X)$. We have a stalk map $\psi_x : \mathcal{F}_x \to \mathcal{G}_x$ for each $x \in X$, which gives a map $\mathrm{L}\psi : \mathrm{L}\mathcal{F} \to \mathrm{L}\mathcal{G}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{L}\mathcal{F} & \xrightarrow{\mathrm{L}\psi} & \mathrm{L}\mathcal{G} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array}$$

Note that on stalks, we have $\mathrm{L}\psi|_{E_x} = \mathrm{L}\psi|_{\mathcal{F}_x} = \psi_x$, which is a homomorphism. The last thing we need to show is that $\mathrm{L}\psi$ is continuous. But this follows from Lemma 7.13 since $\mathrm{L}\psi$ is an open map; we have $\mathrm{L}\psi(\hat{\sigma}(U)) = \psi(\hat{U})(s)(U)$, where $\psi(\hat{U})(s)(U)$ is open in $E'$ by construction.

**Theorem 7.14.** *The map* $\mathrm{L} : \mathbf{pSh}(X, \mathbf{Ab}) \to \mathbf{Sh}_{et}(X, \mathbf{Ab})$ *is a functor.*

*Proof.* Clearly, L maps objects in $\mathbf{pSh}(X, \mathbf{Ab})$ to objects in $\mathbf{Sh}_{\mathrm{et}}(X, \mathbf{Ab})$. We need to show it satisfies the properties of a functor.

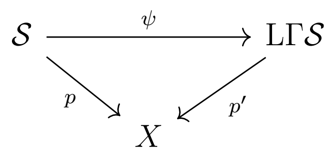
$\mathrm{L}(\psi \circ \phi) = \mathrm{L}(\psi) \circ \mathrm{L}(\phi)$: Let $\phi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be two morphisms in $\mathbf{pSh}(X, \mathbf{Ab})$. Choose an element $e \in \mathrm{L}\mathcal{F}$. Then there exists an $x \in X$ such that $e \in \mathcal{F}_x$, thus there exists an open neighborhood $U \subseteq X$ of $x$ such that $\sigma \in \mathcal{F}(U)$ and $\sigma_x = e$. We have $\mathrm{L}(\psi \circ \phi)(e) = (\psi \circ \phi)_x(\sigma_x) = \psi_x \circ \phi_x(\sigma_x)$. Further, we have $\mathrm{L}(\psi) \circ \mathrm{L}(\phi)(e) = \mathrm{L}(\psi) \circ \phi_x(\sigma_x)$. Since $\phi_x$ maps element into $G_x$, we get $\mathrm{L}(\psi) \circ \phi_x(\sigma_x) = \psi_x \circ \phi_x(\sigma_x)$ and we conclude $\mathrm{L}(\psi \circ \phi) = \mathrm{L}(\psi) \circ \mathrm{L}(\phi)$.

$\mathrm{L}(id) = id$: With the notations above, we have $\mathrm{L}(id)(e) = id_x(\sigma_x) = (id(U)(\sigma))_x = \sigma_x = e$, hence $\mathrm{L}(id) = id$, and we are done. □

We have now constructed two functors, $\Gamma : \mathbf{Sh}_{\mathrm{et}}(X, \mathbf{Ab}) \to \mathbf{Sh}(X, \mathbf{Ab})$ and $\mathrm{L} : \mathbf{pSh}(X, \mathbf{Ab}) \to \mathbf{Sh}_{\mathrm{et}}(X, \mathbf{Ab})$, giving us a way to work with either etale-sheaves or (pre)sheaves. But then it is interesting what happens if we apply these functors after each other.

**Theorem 7.15.** *Let $\mathcal{S} = (E, p, X) \in \mathbf{Sh}_{et}(X, \mathbf{Ab})$. Then $\mathcal{S}$ is isomorphic to $\mathrm{L}\Gamma\mathcal{S}$ in $\mathbf{Sh}_{et}(X, \mathbf{Ab})$.*

*Proof.* We will construct an isomorphism $\psi : \mathcal{S} \to \mathrm{L}\Gamma\mathcal{S}$ in the following way. Let $\mathrm{L}\Gamma\mathcal{S} = (E', p', X)$. By Proposition 7.11, for $x \in X$, there is a bijection between the fibre $p^{-1}(x)$ and the stalk of $\Gamma\mathcal{S}$ at $x$. By the same Proposition, there is a bijection of $p'^{-1}(x)$ and the stalk of $\Gamma\mathcal{S}$ at $x$. Subsequently, this gives a bijection $\psi : \mathcal{S} \to \mathrm{L}\Gamma\mathcal{S}$ such that the following diagram commutes.




Now, $\psi$ is open. This can be seen by taking an open set $U$ in $X$, and a section $\sigma \in \Gamma(U, E)$, we have

$$\psi(\sigma(U)) = \{\sigma_x \in E'; x \in U\} = \hat{\sigma}(U).$$

But then, by Lemma 7.13, $\psi$ is also continuous, thus a homeomorphism. □

We will finish this section by showing that if $\mathcal{F} \in \mathbf{Sh}(X, \mathbf{Ab})$, then there exists an isomorphism $\mathcal{F} \to \Gamma\mathrm{L}\mathcal{F}$. Before proving this, we have the following useful proposition.

**Proposition 7.16.** *Let $\psi : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathbf{pSh}(X, \mathbf{Ab})$. Then $\psi$ is an isomorphism of preshaves if and only if for all open subsets $U$ of $X$, $\psi(U)$ is bijective.*

*Proof.* Only if: Assume $\psi$ is an isomorphism, then there exists a morphism $g : \mathcal{G} \to \mathcal{F}$ such that $\psi \circ g = id_{\mathcal{G}}$ and $g \circ \psi = id_{\mathcal{F}}$. By definition, this means that for all open sets $U$ of $X$, $\psi(U) \circ g(U) = id_{\mathcal{G}(U)}$ and $g(U) \circ \psi(U) = id_{\mathcal{F}(U)}$; that is, $\psi(U)$ is an isomorphism and hence bijective.

If: Assume that for all open sets $U$ of $X$, $\psi(U)$ is a bijection. Thus we can find an inverse $\psi^{-1}(U)$, where we need to check this is compatible with the restrictions. Let $V \subseteq U$ be an open subset, and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\psi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\psi(V)} & \mathcal{G}(V) \end{array} .$$

Since $\psi(U)$ and $\psi(V)$ are bijective, we get another commutative diagram, showing that the inverses are compatible with the restrictions.

$$\begin{array}{ccc} \mathcal{F}(U) & \xleftarrow{\psi^{-1}(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xleftarrow{\psi^{-1}(V)} & \mathcal{G}(V) \end{array}$$

Thus, we get that $\psi^{-1} : \mathcal{G} \to \mathcal{F}$ is a morphism in $\mathbf{pSh}(X, \mathbf{Ab})$, and $\psi \circ \psi^{-1} = id_{\mathcal{G}}$ and $\psi^{-1} \circ \psi = id_{\mathcal{F}}$; that is, $\psi$ is an isomorphism. □

**Theorem 7.17.** *Let $\mathcal{F} \in \mathbf{Sh}(X, \mathbf{Ab})$. Then $\mathcal{F}$ is isomorphic to $\Gamma \mathrm{L}\mathcal{F}$ in $\mathbf{Sh}(X, \mathbf{Ab})$.*

*Proof.* By Proposition 7.16, it suffices to find a bijective map $\mathcal{F}(U) \to \Gamma\mathrm{L}\mathcal{F}(U)$ on all open subsets $U$ of $X$. Consider the map $\sigma \mapsto \hat{\sigma}$. We need to show it is injective and surjective.

For injectivity, let $\sigma, \sigma' \in \mathcal{F}$ and assume $\hat{\sigma} = \hat{\sigma}'$. Then, for all $x \in U$, $\sigma_x = \sigma'_x$. By Proposition 7.6, $\sigma = \sigma'$.

To show it is surjective, let $\tau \in \Gamma\mathrm{L}\mathcal{F}(U) = \Gamma(U, \mathrm{L}\mathcal{F})$. Seen as a section, $\tau(U)$ is open in $\mathrm{L}\mathcal{F}$ by Lemma 7.10. Thus, for each $x \in X$, there exists an open neighborhood $\hat{\sigma}_x(U_x) \subseteq \tau(U)$ of $\tau(x)$ for some open set $U_x \subseteq U$ and element $\sigma_x \in \mathcal{F}(U_x)$, since the collection of sets $\{\hat{\sigma}\}_{\sigma \in \mathcal{F}(U)}$ is a basis of the topology of the sheaf space of $\mathrm{L}\mathcal{F}$.

Consider another point $y \in U$. Similarly, there exists an open neighborhood of of $\tau(y)$ on the form $\hat{\sigma}_y(U_y)$, with $U_y$ open and $\sigma_y \in \mathcal{F}(U_y)$. Further, on the set $V = U_x \cap U_y$, the elements $\psi_V^{U_x}(\sigma_x)$ and $\psi_V^{U_y}(\sigma_y)$ are both mapped to the germ $\tau(z)$ for $z \in V$, and thus are equal on $V$ by Proposition 7.6. Applying the equalizer condition, there exists an element $\sigma \in \mathcal{F}(U)$ such that $\hat{\sigma} = \{\sigma_x; x \in U\} = \{\tau(x); x \in U\} = \tau$, and we are done. □

We finish this section by showing that the sheafification functor $\Gamma\mathrm{L}$ satisfy a universal property, which will come to use when looking at the 'abelianess' of $\mathbf{pSh}(X, \mathbf{Ab})$ and $\mathbf{Sh}(X, \mathbf{Ab})$.

Let $\mathcal{F} \in Ob(\mathbf{pSh}(X, \mathbf{Ab}))$, we define the following morphism $n_{\mathcal{F}} : \mathcal{F} \to \Gamma\mathrm{L}\mathcal{F}$: Let $U \subseteq X$ be an open set and let $\sigma \in \mathcal{F}(U)$. Then $\sigma$ induces a morphism

$$\hat{\sigma} : U \to \mathrm{L}\mathcal{F}$$

given by $x \mapsto \sigma_x$. By the above construction of the functors $\Gamma$ and L, we saw $\hat{\sigma} \in \Gamma(U, \mathrm{L}\mathcal{F})$, which gives the morphism $n_{\mathcal{F}}(U) : \sigma \mapsto \hat{\sigma}$. We have the following lemma.

**Lemma 7.18.** *Let $\mathcal{F} \in Ob(\mathbf{pSh}(X, \mathbf{Ab}))$. Then for all $x \in X$, the induced morphism*

$$n_{\mathcal{F},x} : \mathcal{F}_x \to (\Gamma\mathrm{L}\mathcal{F})_x$$

*is an isomorphism.*

*Proof.* The stalk of $\Gamma\mathrm{L}\mathcal{F}$ over $x$ is by construction the fiber of $\mathrm{L}\mathcal{F}$ over $x$, which is $\mathcal{F}_x$. The induced morphism $n_{\mathcal{F},x}$ map $s_x$ to $s_x$, hence is an isomorphism. □

We now look at the universal property.

**Theorem 7.19.** *Let $\mathcal{F} \in Ob(\mathbf{pSh}(X, \mathbf{Ab}))$, $\mathcal{G} \in Ob(\mathbf{Sh}(X, \mathbf{Ab}))$, and $\psi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Then $\psi$ factors uniquely through $n_{\mathcal{F}}$.*

*Proof.* We will first show that there exists a morphism $\Gamma\mathrm{L}\mathcal{F}$ making the following triangle commute:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\psi} & \mathcal{G} \\ & \searrow^{n_{\mathcal{F}}} & \uparrow \\ & & \Gamma\mathrm{L}\mathcal{F} \end{array}$$

First, we apply $\Gamma\mathrm{L}$ to the morphism $\psi : \mathcal{F} \to \mathcal{G}$ to get $\Gamma\mathrm{L}\psi : \Gamma\mathrm{L}\mathcal{F} \to \Gamma\mathrm{L}\mathcal{G}$. By Theorem 7.17, there is an isomorphism $\Gamma\mathrm{L}\mathcal{G} \to \mathcal{G}$, and we get a morphism $\Gamma\mathrm{L}\mathcal{F} \to \Gamma\mathrm{L}\mathcal{G} \to \mathcal{G}$. From the construction of the isomorphism in Theorem 7.17, it is not hard to see that the diagram above commutes.

We now show that the morphism constructed is the only one making the diagram commute. Let $g : \Gamma\mathrm{L}\mathcal{F} \to \mathcal{G}$ be any morphism making the triangle above commute. By Lemma 7.18, $n_{\mathcal{F},x} : \mathcal{F}_x \to (\Gamma\mathrm{L}\mathcal{F})_x$ is an isomorphism, hence the inverse exists. Thus, the stalk map $g_x$ is uniquely determined by

$$(\Gamma\mathrm{L}\mathcal{F})_x \to \mathcal{F}_x \to \mathcal{G}_x.$$

Since $g$ is a map of sheaves, Proposition 7.5 implies $g$ is unique. □

# 8 Functors of sheaves

We will now look at functors of sheaves that are induced by continuous maps of topological spaces. The motivation is probably clear from algebraic topology; inducing continuous maps as morphisms of algebraic structures can give information about the underlying topological spaces. E.g., a homeomorphism between topological spaces induces isomorphism in singular homology, given that it exists.

There are several interesting sources here, see for example [7] and [6]. However, I have gone for a similar discussion as given in [9], and most proofs stem from this source. An alternative was [5] as was used in the previous chapter, and I highly recommend this source, but the arguments are more concrete and longer. Also, since subsequent analysis will be more abstract, introducing a more abstract thinking of sheaves can be beneficial, instead of working with 'concrete' sections. Therefore, [9] was a suitable source.

## 8.1 Sheaf of solutions and tensor product

We will now expand our view on sheaves. Some sheaves we typically use have a richer structure than just being an abelian group.

**Definition 8.1.** A presheaf over a topological space $X$ with values in **Ring** is called a **presheaf of rings**, and the category thereof is denoted by $\mathbf{pSh}(X, \mathbf{Ring})$. If it is a sheaf, we call it a sheaf of rings and denote it by $\mathbf{Sh}(X, \mathbf{Ring})$.

**Definition 8.2.** Let $\mathcal{R}$ be an object in $\mathbf{Sh}(X, \mathbf{Ring})$. An **$\mathcal{R}$-module** $M$ is a sheaf $M$ such that for each open set $U \subseteq X$, $M(U)$ is a left $\mathcal{R}(U)$-module. Furthermore, the restriction morphisms are compatible with the structure of the module; that is, if $V \subseteq U$, then for any $r \in \mathcal{R}(U)$ and $\sigma \in M(U)$, we have $\rho_V^U(r\sigma) = \rho_V^U(r)\rho_V^U(\sigma)$. We will denote the sheaf of left $\mathcal{R}$-modules by $\mathbf{Mod}(\mathcal{R})$.

Let $\mathbb{Z}_x$ denote the constant sheaf $U \mapsto \mathbb{Z}$. Then $\mathbb{Z}_x$ is a sheaf of rings, and we have

$$\mathbf{Sh}(X, \mathbf{Ab}) = \mathbf{Mod}(\mathbb{Z}_x).$$

Therefore, statements in $\mathbf{Mod}(\mathcal{R})$ also holds in $\mathbf{Sh}(X, \mathbf{Ab})$.

**Definition 8.3.** Let $U \subseteq X$ be an open subset, $\mathcal{F}, \mathcal{G} \in Ob(\mathbf{Sh}(X, \mathbf{Ab}))$ and $\mathcal{R} \in Ob(\mathbf{Sh}(X, \mathbf{Ring}))$. We define the **sheaf of solutions** of $\mathcal{F}$ in $\mathcal{G}$ over $\mathcal{R}$ to be given by

$$U \mapsto \mathrm{Hom}_{\mathcal{R}|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

We denote this presheaf by $\mathscr{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$.

By construction, it is clear that $\Gamma(X, \mathscr{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$.

We now define tensor product.

**Definition 8.4.** Let $\mathcal{R}$ be a sheaf of rings on $X$, $\mathcal{F}$ a right $\mathcal{R}$-module, and $\mathcal{G}$ a left $\mathcal{R}$-module. We define the **tensor product** of $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{R}$, denoted $\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G}$, to be the sheafification of the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{R}(U)} \mathcal{G}(U).$$

We denote the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{R}(U)} \mathcal{G}(U)$ by $\mathcal{F} \otimes^{\vee}_{\mathcal{R}} \mathcal{G}$.

**Proposition 8.1.** *Let $\mathcal{R}$ be a sheaf of rings on $X$, $\mathcal{F}, \mathcal{G} \in Ob(\mathbf{Sh}(X, \mathbf{Ab}))$, and $x \in X$. Then*

$$(\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{R}_x} \mathcal{G}_x.$$

*Proof.* We start by constructing a map $\phi$ from $\mathcal{F}_x \otimes_{\mathcal{R}_x} \mathcal{G}_x$ to $(\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G})_x$. Let $\sigma_x \otimes \tau_x \in \mathcal{F}_x \otimes_{\mathcal{R}_x} \mathcal{G}_x$. Then there exists open neighborhoods $U, V \subseteq X$ of $x$ such that $\sigma \in \mathcal{F}(U)$ and $\tau \in \mathcal{G}(V)$. Since $x \in W = U \cap V$ is open, we get $\rho^U_W(\sigma) \otimes \rho^V_W(\tau) \in \mathcal{F}(W) \otimes_{\mathcal{R}(W)} \mathcal{G}(W)$; we can view $\sigma, \tau$ as elements of $\mathcal{F}(W), \mathcal{G}(W)$ respectively. This is then mapped to $(\sigma \otimes \tau)_x$.

We start by showing that $\phi$ is well-defined. Let $\sigma_x \otimes \tau_x = \sigma'_x \otimes \tau'_x$ such that $\sigma \otimes \tau \in \mathcal{F}(W) \otimes_{\mathcal{R}(W)} \mathcal{G}(W)$ and $\sigma' \otimes \tau' \in \mathcal{F}(W') \otimes_{\mathcal{R}(W')} \mathcal{G}(W')$. Then there exists an open neighborhood $U_x$ such that $\rho^W_{U_x}(\sigma) \otimes \rho^W_{U_x}(\tau) = \rho^{W'}_{U_x}(\sigma') \otimes \rho^{W'}_{U_x}(\tau')$. Then, we have

$$\begin{aligned} \phi(\sigma_x \otimes \tau_x) &= (\sigma \otimes \tau)_x \\ &= (\rho^W_{U_x}(\sigma) \otimes \rho^W_{U_x}(\tau))_x \\ &= (\rho^W_{U_x}(\sigma') \otimes \rho^W_{U_x}(\tau'))_x \\ &= (\sigma' \otimes \tau')_x. \end{aligned}$$

Surjectivity is clear, since for any element $(\sigma \otimes \tau)_x \in (\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G})_x$, there exists an open neighborhood $W$ of $x$ such that $\sigma \otimes \tau \in \mathcal{F}(W) \otimes_{\mathcal{R}(W)} \mathcal{G}(W)$, and so $\phi(\sigma_x \otimes \tau_x) = (\sigma \otimes \tau)_x$.

For injectivity, let $\sigma_x \otimes \tau_x$ and $\sigma'_x \otimes \tau'_x$ be such that $(\sigma \otimes \tau)_x = (\sigma' \otimes \tau')_x$. Then there exists an open neighborhood $W$ of $x$ such that $\sigma \otimes \tau = \sigma' \otimes \tau' \in \mathcal{F}(W) \otimes_{\mathcal{R}(W)} \mathcal{G}(W)$, and so $\sigma_x \otimes \tau_x = \sigma'_x \otimes \tau'_x$. □

**Corollary 8.1.1.** *Let $\mathcal{R}$ be a sheaf of rings on $X$. The functor $\cdot \otimes_{\mathcal{R}} \cdot$ is right exact in each of its arguments.*

*Proof.* Let $\mathcal{G}$ be a left $\mathcal{R}$-module. Let $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ be right $\mathcal{R}$-modules such that the following sequence is exact.

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

Then

$$0 \longrightarrow \mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x \longrightarrow 0$$

is exact, and thus

$$\mathcal{G}_x \otimes_{\mathcal{R}_x} \mathcal{F}'_x \longrightarrow \mathcal{G}_x \otimes_{\mathcal{R}_x} \mathcal{F}_x \longrightarrow \mathcal{G}_x \otimes_{\mathcal{R}_x} \mathcal{F}''_x \longrightarrow 0$$

is exact since $\mathcal{G}_x \otimes_{\mathcal{R}_x} \cdot$ is right exact in the category of $\mathcal{R}_x$-modules. □

**Proposition 8.2.** *Let $\mathcal{R}$ be a sheaf of rings, $\mathcal{I}$ a sheaf of commutative rings, and $\mathcal{I} \to \mathcal{R}$ a morphism of sheaves such that the image is contained in the center of $\mathcal{R}$. Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{R}$-modules and $\mathcal{H}$ an $\mathcal{I}$-module. Then there are canonical isomorphisms:*

$$\begin{aligned} \mathscr{H}om_{\mathcal{R}}(\mathcal{H} \otimes_{\mathcal{I}} \mathcal{F}, \mathcal{G}) &\cong \mathscr{H}om_{\mathcal{R}}(\mathcal{F}, \mathscr{H}om_{\mathcal{I}}(\mathcal{H}, \mathcal{G}) \\ &\cong \mathscr{H}om_{\mathcal{I}}(\mathcal{H}, \mathscr{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{G}). \end{aligned}$$

*Proof.* By the tensor-hom adjunction, we have for all open sets $U \subseteq X$

$$\begin{aligned} \mathrm{Hom}_{\mathcal{R}(U)}(\mathcal{H}(U) \otimes_{\mathcal{I}(U)} \mathcal{F}(U), \mathcal{G}(U)) &\cong \mathrm{Hom}_{\mathcal{R}(U)}(\mathcal{F}(U), \mathrm{Hom}_{\mathcal{I}(U)}(\mathcal{H}(U), \mathcal{G}(U)) \\ &\cong \mathrm{Hom}_{\mathcal{I}(U)}(\mathcal{H}(U), \mathrm{Hom}_{\mathcal{R}(U)}(\mathcal{F}(U), \mathcal{G}(U)). \end{aligned}$$

Subsequently, we get the following isomorphisms of *presheaves.*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{R}}(\mathcal{H} \otimes^{\vee}_{\mathcal{I}} \mathcal{F}, \mathcal{G}) &\cong \mathrm{Hom}_{\mathcal{R}}(\mathcal{F}, \mathrm{Hom}_{\mathcal{I}}(\mathcal{H}, \mathcal{G})) \\ &\cong \mathrm{Hom}_{\mathcal{I}}(\mathcal{H}, \mathscr{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{G})). \end{aligned}$$

By applying sheafification, we get $\mathcal{H} \otimes^{\vee}_{\mathcal{I}} \mathcal{F} \cong \mathcal{H} \otimes_{\mathcal{I}} \mathcal{F}$, and the result follows. □

## 8.2 Direct and inverse image

Further on, we let $X$ and $Y$ denote topological spaces and $f : X \to Y$ a continuous map between them.

**Definition 8.5.** Let $\mathcal{F}$ be a presheaf in $\mathbf{pSh}(X, \mathbf{Ab})$. We define the **direct image** of $\mathcal{F}$ by $f : X \to Y$, denoted $f_*\mathcal{F}$, to be given by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)),$$

for an open set $U \subseteq Y$. The restriction morphisms are given in the obvious way.

Note that the direct image of a presheaf changes the base space of the sheaf; since $\mathcal{F} \in Ob(\mathbf{pSh}(X, \mathbf{Ab}))$, we get $f_*\mathcal{F} \in Ob(\mathbf{pSh}(Y, \mathbf{Ab}))$.

**Proposition 8.3.** *If $\mathcal{F} \in Ob(\mathbf{Sh}(X, \mathbf{Ab}))$, then $f_*\mathcal{F} \in Ob(\mathbf{Sh}(Y, \mathbf{Ab}))$.*

*Proof.* Let $U \subseteq Y$ be an open subset and let $\{U_i\}_{i\in I}$ be an open cover of $U$ with some indexing set $I$. Let $\sigma, \tau \in f_*\mathcal{F}(U)$ be such that $\sigma|_{U-i} = \tau|_{U_i}$ for some $i \in I$.

Now, let $V = f^{-1}(U)$, and note that $f^{-1}(U_i)_{i\in I}$ is an open cover of $V$. Thus, we have $\sigma, \tau \in \mathcal{V}$ such that $\sigma|_{f^{-1}(U_i)} = \tau|_{f^{-1}(U_i)}$ in $\mathcal{V}$. Since $\mathcal{F}$ is a sheaf, $\sigma = \tau$, which shows the uniqueness condition. Similarly, we can show the gluing condition, hence $f_*\mathcal{F}$ is a sheaf. □

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. We define $f_*\phi$ in an obvious way; that is, $f_*\phi(U) = \phi(f^{-1}(U))$. Denote the direct image of the morphism $\phi$ by $\phi_*$. This makes $f_* : \mathbf{Sh}(X, \mathbf{Ab}) \to \mathbf{Sh}(Y, \mathbf{Ab})$ into a functor, since for two morphisms $\phi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$, we have $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ and $(id_{\mathcal{F}})_* = id_{f_*\mathcal{F}}$.

We can go the other way, that is, if we have a presheaf $\mathcal{G} \in Ob(\mathbf{pSh}(Y, \mathbf{Ab}))$, there exists a functor $f^* : \mathbf{pSh}(Y, \mathbf{Ab}) \to \mathbf{pSh}(X, \mathbf{Ab})$.

**Definition 8.6.** Let $\mathcal{G}$ be a presheaf in $\mathbf{pSh}(Y, \mathbf{Ab})$. We define the **inverse image** of $\mathcal{G}$ by $f$, denoted $f^*\mathcal{G}$, to be given by

$$V \mapsto \varinjlim_{U \supseteq f(V)} \mathcal{G}(U),$$

where $V$ is an open set in $X$, and $U$ are open in $Y$.

Again, if $\mathcal{G}$ is a sheaf, then so is $f^*\mathcal{G}$. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves in $\mathbf{pSh}(Y, \mathbf{Ab})$, and let $V \subseteq X$ be an open subset. Consider an element $\sigma_{f(V)} \in f^*\mathcal{F}(V)$, and so there exists an open set $U \in Y$ such that $\sigma \in \mathcal{F}(U)$. From this, we map $\sigma$ to $\phi(U)(\sigma)$, and further into $f^*\mathcal{G}(U)$. Because $\phi$ is compatible with restrictions, this map is well-defined (to see this, compare with the proof that the stalk map is well-defined). Subsequently, $f^* : \mathbf{Sh}(Y, \mathbf{Ab}) \to \mathbf{Sh}(X, \mathbf{Ab})$ is a functor.

It is not hard to see that both $f_*$ and $f^*$ induce functors in $\mathbf{Mod}(\mathcal{R})$. We conclude this in a proposition.

**Proposition 8.4.** *Let $\mathcal{R} \in Ob(\mathbf{Sh}(Y, \mathbf{Ring}))$ and $\mathcal{I} \in Ob(\mathbf{Sh}(X, \mathbf{Ring}))$. Then $f^*\mathcal{R} \in Ob(\mathbf{Sh}(X, \mathbf{Ring}))$, $f_*\mathcal{I} \in Ob(\mathbf{Sh}(Y, \mathbf{Ring}))$, and $f^*$ and $f_*$ induces functors in $\mathbf{Mod}(\mathcal{R})$:*

$$f^* : \mathbf{Mod}(\mathcal{R}) \to \mathbf{Mod}(f^*\mathcal{R})$$
$$f_* : \mathbf{Mod}(\mathcal{I}) \to \mathbf{Mod}(f_*\mathcal{I}).$$

**Proposition 8.5.** *Let $\mathcal{G}$ be a sheaf over $Y$ (in either $\mathbf{Sh}(Y, \mathbf{Ab})$ or $\mathbf{Mod}(\mathcal{R})$) and let $x \in X$. Then $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

*Proof.* See proof on page 12 in [7]. □

There are natural morphisms

$$f_* f^* \to id$$

and

$$id \to f^* f_*.$$

We begin by showing the first one. Consider a sheaf $\mathcal{F}$ over $X$. Let $U \subseteq X$ be an open set, and take a section $\sigma \in f_* f^* \mathcal{F}(U)$. Expanding $\sigma \in f_* f^* \mathcal{F}(U)$, we get

$$\begin{aligned} f_* f^* \mathcal{F}(U) &= \varinjlim_{V \supseteq f(U)} f^* \mathcal{F}(V) \\ &= \varinjlim_{V \supseteq f(U)} \mathcal{F}(f^{-1}(V)). \end{aligned} \tag{8.2.2}$$

Since $V \supseteq f(U)$ if and only if $f^{-1}(V) \supseteq U$, we get

$$f_* f^* \mathcal{F}(U) = \varinjlim_{f^{-1}(V) \supseteq U} \mathcal{F}(f^{-1}(V)).$$

Subsequently, there exists a $W \supseteq U$ in $X$ and a $\sigma' \in \mathcal{F}(W)$ such that $\sigma'$ gets mapped to $\sigma$ in $f_* f^* \mathcal{F}(U)$. Thus, we get the map $\sigma \mapsto \sigma'|_U$. It is not hard to show that this is well-defined. A similar argument holds for $id \to f^* f_*$.

For example, a result that follows is that the functor $f_* : \mathbf{Mod}(f^*\mathcal{R}) \to \mathbf{Mod}(f_* f^* \mathcal{R})$ induces a functor $f_* : \mathbf{Mod}(f^*\mathcal{R}) \to \mathbf{Mod}(\mathcal{R})$.

Using these results, we have the following proposition.

**Proposition 8.6.** *Consider the functors* $f^* : \mathbf{Mod}(\mathcal{R}) \to \mathbf{Mod}(f^*\mathcal{R})$ *and* $f_* : \mathbf{Mod}(f^*\mathcal{R}) \to \mathbf{Mod}(\mathcal{R})$*. Then* $f^*$ *is a left adjoint to* $f_*$*. That is, if we let* $\mathcal{F}$ *be a sheaf in* $\mathbf{Mod}(\mathcal{R})$ *and* $\mathcal{G}$ *be a sheaf in* $\mathbf{Mod}(f^*\mathcal{R})$*, then*

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{F}, f_*\mathcal{G}) \cong \mathrm{Hom}_{f^*\mathcal{R}}(f^*\mathcal{F}, \mathcal{G}).$$

*Proof.* First, we have a morphism

$$\alpha : \mathrm{Hom}_{\mathcal{R}}(\mathcal{F}, f_*\mathcal{G}) \to \mathrm{Hom}_{f^*\mathcal{R}}(f^*\mathcal{F}, f^* f_* \mathcal{G})$$

defined in the obvious way. Second, note that for an open set $V \subset Y$, $f \circ f^{-1}(V) = V$, which means $f^* \circ f_* \mathcal{G}(V) = \mathcal{G}(V)$. It follows we have a morphism

$$\beta : \mathrm{Hom}_{f^*\mathcal{R}}(f^*\mathcal{F}, f^* f_* \mathcal{G}) \to \mathrm{Hom}_{f^*\mathcal{R}}(f^*\mathcal{F}, \mathcal{G}).$$

On the other hand, we have an obvious morphism

$$\gamma : \mathrm{Hom}_{f^*\mathcal{R}}(f^*\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{f_*f^*\mathcal{R}}(f_*f^*\mathcal{F}, f_*\mathcal{G}).$$

Further, note that for an open set $U \subseteq X$, we have $U \subseteq f^{-1} \circ f(U)$. By the definition of direct limit, this implies we have a unique morphism $f_*f^*\mathcal{F}(U) \to F(U)$, which induces the morphism

$$\delta : \mathrm{Hom}_{f_*f^*\mathcal{R}}(f_*f^*\mathcal{F}, f_*\mathcal{G}) \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{F}, f_*\mathcal{G}).$$

It is straightforward to show they are inverses of each other. □

**Corollary 8.6.1.** *Let $\mathcal{F} \in Ob(\mathbf{Mod}(\mathcal{R}))$ and $\mathcal{G} \in Ob(\mathbf{Mod}(f^*\mathcal{R}))$. Then*

$$\mathscr{H}om_{\mathcal{R}}(\mathcal{F}, f_*\mathcal{G}) \cong f_* \mathscr{H}om_{f^*\mathcal{R}}(f^*\mathcal{F}, \mathcal{G}).$$

*Proof.* Let $U \subseteq X$ be an open subset. First, note that from the definition of direct limit, $f_* \mathscr{H}om_{f^*\mathcal{R}}(f^*\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{f^*\mathcal{R}|_{f^{-1}(U)}}(f^*\mathcal{F}|_{f^{-1}(U)}, \mathcal{G}|_{f^{-1}(U)})$. This gives

$$\begin{aligned}\Gamma(U, f_* \mathscr{H}om_{f^*\mathcal{R}}(f^*\mathcal{F}, \mathcal{G})) &= \mathrm{Hom}_{f^*\mathcal{R}|_{f^{-1}(U)}}(f^*\mathcal{F}|_{f^{-1}(U)}, \mathcal{G}|_{f^{-1}(U)}) \\ &= \mathrm{Hom}_{\mathcal{R}|_{f(U)}}(\mathcal{F}|_{f(U)}, f_*\mathcal{G}|_{f(U)}) \\ &= \Gamma(U, \mathscr{H}om_{\mathcal{R}}(\mathcal{F}, f_*\mathcal{G})).\end{aligned}$$

□

We finish this subsection by showing that the inverse image commutes with the tensor product.

**Proposition 8.7.** *Let $\mathcal{F}$ be a right $\mathcal{R}$-module, and let $\mathcal{G}$ be a left $\mathcal{R}$-module. Then there is a canonical isomorphism:*

$$f^*\mathcal{F} \otimes_{f^*\mathcal{R}} f^*\mathcal{G} \cong f^*(\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G}).$$

*Proof.* Let $U \subseteq X$ be an open set. The canonical morphism is induced by

$$\mathcal{F}(U) \otimes_{\mathcal{R}(U)} \mathcal{G}(U) \to (\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G})(U).$$

Note that this morphism is given by the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{R}(U)} \mathcal{G}(U)$.

Now, since $f^*\mathcal{F} \otimes_{f^*\mathcal{R}} f^*\mathcal{G}$ and $f^*(\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G})$ are both sheaves, it suffices to show that they are isomorphic on stalks. Let $x \in X$, and set $y = f(x)$. We have

$$\begin{aligned}(f^*\mathcal{F} \otimes_{f^*\mathcal{R}} f^*\mathcal{G})_x &\cong (f^*\mathcal{F})_x \otimes_{(f^*\mathcal{R})_x} (f^*\mathcal{G})_x \\ &\cong \mathcal{F}_y \otimes_{\mathcal{R}_y} \mathcal{G}_y \\ &\cong (\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G})_y.\end{aligned}$$

□

## 8.3 Sheaves on subsets

We will now look at how subsets $Z$ of $X$ give rise to new sheaves.

**Definition 8.7.** Let $\mathcal{F}$ be sheaf in $\mathbf{Sh}(X, \mathbf{Ab})$, and let $Z \subseteq X$ be a subset. Let $j : Z \to X$ be the inclusion. We define the sheaf $\mathcal{F}|_Z$, called the **restriction of** $\mathcal{F}$ to $Z$, to be given by

$$\mathcal{F}|_Z = j^* \mathcal{F}.$$

Note that this is in agreement with the defined sheaf restricted to an open subset $Z$ of $X$. Further, we extend the domain of $\Gamma$, such that

$$\Gamma(Z, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z).$$

We have a natural morphism $\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F})$ and we denote the image of $\sigma \in \Gamma(X, \mathcal{F})$ by $\sigma|_Z$.

When $Z$ is *locally closed*, new sheaves can be defined. We remember the definition of locally closed.

**Definition 8.8.** A subset $Z$ of $X$ is said to be locally closed if it can be written as an intersection of an open and closed set of $X$.

Further on in this section, assume $\mathcal{F}$ is a sheaf in $\mathbf{Sh}(X, \mathbf{Ab})$ and $Z \subseteq X$ is a locally closed set.

We will define a new sheaf on $Z$ through a series of steps. Let $Z = U \cap A$ for some open set $U \subseteq X$ and closed set $A \subseteq X$. Denote the inclusion for all subsets by $j$. For the closed set $A$, we define

$$\mathcal{F}_A = j_* j^* \mathcal{F},$$

and so $\mathcal{F}_A$ is a sheaf. For open $U$, we define

$$\mathcal{F}_U = \mathrm{Ker}(\mathcal{F} \to \mathcal{F}_{X \setminus U}),$$

which also is a sheaf. Finally, we set

$$\mathcal{F}_Z = (\mathcal{F}_U)_A.$$

We will later show this is well-defined.

**Proposition 8.8.** *Take an open set $U \subseteq X$ and a closed set $A \subseteq X$ such that $Z = U \cap A$. Then for any open set $V \subseteq Z$, $\mathcal{F}_Z|_Z(V) = \mathcal{F}_Z(V)$, and $\mathcal{F}_Z|_{X \setminus Z} = 0$.*

*Proof.* Showing that $\mathcal{F}_U$ and $\mathcal{F}_A$ satisfy the equalities implies $(\mathcal{F}_U)_A$ does as well. It is straightforward to prove the results for the two cases. □

**Proposition 8.9.** *Let $\mathcal{F}_Z$ be a sheaf satisfying $\mathcal{F}_Z|_Z(V) = \mathcal{F}_Z(V)$ for an open set $V \subseteq Z$, and $\mathcal{F}_Z|_{X\setminus Z} = 0$. Then $(\mathcal{F}_Z)_x = \mathcal{F}_x$ if $x \in Z$, and $(\mathcal{F}_Z)_x = \mathcal{F}_x = 0$ otherwise.*

*Proof.* This is clear by Proposition 8.8. □

By the next proposition, we see that $\mathcal{F}_Z$ is well-defined; it does not depend on the choice of $U$ and $A$.

**Corollary 8.9.1.** *Let $\mathcal{F}_Z$ be a sheaf satisfying $\mathcal{F}_Z|_Z(V) = \mathcal{F}_Z(V)$ for an open set $V \subseteq Z$, and $\mathcal{F}_Z|_{X\setminus Z} = 0$. Then $F_Z$ is unique up to isomorphism.*

*Proof.* By Proposition 8.9, any two sheaves satisfying the equalities have the same stalks, and so are isomorphic. □

**Proposition 8.10.** *The mapping $(\cdot)_Z : \mathbf{Sh}(X, \mathbf{Ab}) \to \mathbf{Sh}(Z, \mathbf{Ab})$ given by $\mathcal{F} \mapsto \mathcal{F}_Z$ on objects, and in the obvious way to morphisms, is a functor. Further, $(\cdot)_Z$ is exact.*

*Proof.* Showing it is a functor is routine. To show it is exact, consider a set of sheaves $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ in $\mathbf{Sh}(X, \mathbf{Ab})$ such that we have a short exact sequence.

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

Then, for $x \in Z$, we have an exact sequence

$$0 \longrightarrow \mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x \longrightarrow 0$$

and hence

$$0 \longrightarrow (\mathcal{F}'_Z)_x \longrightarrow (\mathcal{F}_Z)_x \longrightarrow (\mathcal{F}''_Z)_x \longrightarrow 0$$

is exact. For $x \in X\setminus Z$, we clearly have

$$0 \longrightarrow (\mathcal{F}'_Z)_x \longrightarrow (\mathcal{F}_Z)_x \longrightarrow (\mathcal{F}''_Z)_x \longrightarrow 0,$$

therefore

$$0 \longrightarrow \mathcal{F}'_Z \longrightarrow \mathcal{F}_Z \longrightarrow \mathcal{F}''_Z \longrightarrow 0$$

is exact. □

If $Z'$ is another locally closed subset of $X$, it is easy to show that $(\mathcal{F}_Z)_{Z'} = \mathcal{F}_{Z\cap Z'}$.

**Proposition 8.11.** *Let $Z' \subseteq Z$ be a locally closed subset. Then we have a short exact sequence*

$$0 \longrightarrow \mathcal{F}_{Z\backslash Z'} \longrightarrow \mathcal{F}_Z \longrightarrow \mathcal{F}_{Z'} \longrightarrow 0.$$

*Proof.* Since $Z$ is locally closed in $X$, and $Z'$ is locally closed in $Z$, it follows $Z\backslash Z'$ is locally closed in $X$. Considering the stalks of elements $x \in Z'$, $x \in Z\backslash Z'$, and $x \in X\backslash Z$ respectively, we get short exact sequences, and the result follows. □

There is another sheaf on $Z$ that is of interest.

**Definition 8.9.** Let $U$ be an open subset of $X$ such that $Z$ is a closed subset of $U$. We define the sheaf $\Gamma_Z(U,\mathcal{F})$ by

$$\Gamma_Z(U,\mathcal{F}) = \mathrm{Ker}(\mathcal{F}(U) \to \mathcal{F}(U\backslash Z)).$$

Note that $\Gamma_Z(U,\mathcal{F})$ is a subgroup of $\Gamma(U,\mathcal{F})$, and it consists of sections with support in $Z$.

**Proposition 8.12.** *Let $U \subseteq V$ be open subsets containing $Z$. The canonical morphism*

$$\Gamma_Z(U,\mathcal{F}) \to \Gamma_Z(V,\mathcal{F})$$

*is an isomorphism.*

*Proof.* The morphism is given by the restriction. To see that it is well-defined, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U\backslash Z) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V\backslash Z), \end{array}$$

showing that a section $\sigma \in \Gamma_Z(U,\mathcal{F})$ is taken to an element $\rho^U_V(\sigma)$ such that $\rho^U_{V\backslash Z}(\sigma) = 0$.

Injectivity follows from the uniqueness condition of sheaves. To see this, let $\sigma,\tau \in \mathcal{F}$, and assume $\rho^U_V(\sigma) = \rho^U_V(\tau)$. Further, both are zero in $U\backslash Z$ and since $V \cup (U\backslash Z) = U$, $\sigma = \tau$.

Surjectivity follows from the gluing condition. Let $\sigma|_V \in \Gamma_Z(V,\mathcal{F})$. Then for $0 \in \mathcal{F}(U\backslash Z)$, we have $\rho^V_{V\cap(U\backslash Z)}(\sigma|_V) = \rho^V_{V\backslash Z)}(\sigma|_V) = 0$, and $\rho^V_{V\cap(U\backslash Z)}(0) = 0$, hence $\sigma|_V$ and $0$ agree on intersection. Therefore there exists a (unique) section $\sigma$ in $\mathcal{F}(U)$ such that $\rho^U_V(\sigma) = \sigma|_V$ and $\rho^U_{U\backslash Z}(\sigma) = 0$; that is, $\sigma \in \Gamma_Z(U,\mathcal{F})$ and we are done. □

Note that the presheaf $U \mapsto \Gamma_{Z\cap U}(U,\mathcal{F})$ is a sheaf. We define the following.

**Definition 8.10.** We define the **sheaf of sections of $\mathcal{F}$ supported by** $Z$, denoted $\Gamma_Z(\mathcal{F})$, to be given by the sheaf

$$U \mapsto \Gamma_{Z\cap U}(U, \mathcal{F}).$$

**Proposition 8.13.** *The functors* $\Gamma_Z(X, \cdot) : \mathbf{Sh}(X, \mathbf{Ab}) \to \mathbf{Ab}$ *and* $\Gamma_Z(\cdot) : \mathbf{Sh}(X, \mathbf{Ab}) \to \mathbf{Sh}(X, \mathbf{Ab})$ *are left exact. Moreover, we have*

$$\Gamma_Z(X, \cdot) = \Gamma(X, \cdot) \circ \Gamma_Z(\cdot).$$

*Proof.* That they are left exact follows $\Gamma(X, \cdot)$ being left exact. It is easy to see that the equality holds. □

Let $Z'$ be another locally closed subset of $X$. Then it is easy to see that $\Gamma_{Z'}(\cdot) \circ \Gamma_Z(\cdot) = \Gamma_{Z'\cap Z}(\cdot)$. If $Z$ is open, we get a nice view of $\Gamma_Z(\cdot)$.

**Proposition 8.14.** *Let $Z'$ be another locally closed subset of $X$. Then the following sequence is exact.*

$$0 \longrightarrow \Gamma_{Z'}(\mathcal{F}) \longrightarrow \Gamma_Z(\mathcal{F}) \longrightarrow \Gamma_{Z\setminus Z'}(\mathcal{F})$$

*Proof.* Straightforward by studying the stalks. □

## 8.4 Direct image with proper support

We will now define a subsheaf of $f_*\mathcal{F}$, where $f : X \to Y$ is a continuous map of topological spaces, and $\mathcal{F}$ is a sheaf on $X$. This will be one of the main functors of interest since it can be used to derive several interesting relationships in the category of sheaves. First, some topological prerequisites.

**Definition 8.11.** Let $X$ be a topological space. $X$ is said to be **locally compact** if every point $x \in X$ admits a compact neighborhood.

In this subsection, we assume that $X$ and $Y$ are locally compact and Hausdorff.

**Definition 8.12.** Let $X$ be a topological space. $X$ is said to be **paracompact**, if for every open cover of $X$, there exists a locally finite open refinement; that is, if $\{U_i\}_{i\in I}$ is an open cover of $X$, then there exists another open cover $\{V_j\}_{j\in J}$ of $X$ such that:

(i) For every $V_j$, there exists an $U_i$ such that $V_j \subseteq U_i$, and

(ii) For every point $x \in X$, there exists a neighborhood $W$ such that the number of non-trivial intersections with elements in $\{V_j\}_{j\in J}$ is finite.

**Definition 8.13.** Let $f : X \to Y$ be a continuous map of topological spaces. We say that $f$ is **proper**, if it is closed and the fibers are compact.

We also have that $f$ is proper if the preimage of a compact set is compact. We are now ready for the definition of the direct image with proper support.

**Definition 8.14.** Let $f : X \to Y$ be a continuous map and let $\mathcal{G}$ be a sheaf on $X$. The **direct image with proper supports** of $\mathcal{G}$, denoted $f_!\mathcal{G}$, is given by

$$\Gamma(U, f_!\mathcal{G}) = \{\sigma \in f_*\mathcal{G}(U); f : \mathrm{supp}(\sigma) \to U \text{ is proper}\}.$$

Since being proper is a local property on $X$, it is not hard to see that $f_!\mathcal{G}$ is a subsheaf of $f_*\mathcal{G}$. Subsequently, we get a functor $f_! : \mathbf{Sh}(X, \mathbf{Ab}) \to \mathbf{Sh}(Y, \mathbf{Ab})$ that maps $\mathcal{G}$ to $f_!\mathcal{G}$. For $\mathcal{R}$-modules, where $\mathcal{R}$ is an object in $\mathbf{Sh}(Y, \mathbf{Ring})$, we have a functor $f_! : \mathbf{Mod}(f^*\mathcal{R}) \to \mathbf{Mod}(\mathcal{R})$.

**Definition 8.15.** Let $X$ be a topological space and let $\mathcal{F}$ be a sheaf on $X$. We define the **sections of $\mathcal{F}$ with compact support**, denoted $\Gamma_c(X, \mathcal{F})$, to be given by

$$\Gamma_c(X, \mathcal{F}) = \{\sigma \in \Gamma(X, \mathcal{F}); \mathrm{supp}(\sigma) \text{ is compact and Hausdorff}\}.$$

**Example 8.1.** Let $a : X \to \{\mathrm{pt}\}$ be a map mapping to a point, and let $\mathcal{F}$ be a sheaf on $X$. Then, for $U = \{\mathrm{pt}\}$ we have

$$\begin{aligned} a_!\mathcal{F}(U) &= \{\sigma \in a_*\mathcal{F}(U); a : \mathrm{supp}(\sigma) \to U \text{ is proper}\} \\ &= \{\sigma \in \mathcal{F}(X); a : \mathrm{supp}(\sigma) \to \{\mathrm{pt}\} \text{ is proper}\}. \end{aligned}$$

Now, $a$ is clearly closed, thus $a : \mathrm{supp}(\sigma) \to \{\mathrm{pt}\}$ is proper if and only if the preimage of $\{\mathrm{pt}\}$ is compact. Since $X$ is assumed to be Hausdorff, and so the support of every section is as well, we get $a_!\mathcal{F}(\{\mathrm{pt}\}) \cong \Gamma_c(X, \mathcal{F})$.

In the general case, we have $\Gamma_c(X, f_!\mathcal{G}) \cong \Gamma_c(Y, \mathcal{G})$. We have the following proposition which is useful in the study of the direct image of proper support.

**Proposition 8.15.** *Given the situation in the above paragraph. Consider the morphism*

$$\psi : \varinjlim_{U \supseteq Z} \Gamma(U, \mathcal{F}) \to \Gamma(Z, \mathcal{F}).$$

*We have:*

*(i) $\psi$ is injective.*

*(ii) If $Z$ is open, then $\psi$ is an isomorphism.*

*(iii) If $X$ is Hausdorff and $Z$ is compact, then $\psi$ is an isomorphism.*

*(iv) If $X$ is paracompact and $Z$ is closed, then $\psi$ is an isomorphism.*

*Proof.* Consider an element $\sigma \in \Gamma(U, \mathcal{F})$ such that the image of $\sigma$ in $\Gamma(Z, \mathcal{F})$ is zero. Then $\sigma_x = 0$ for all $x \in X$, so there exists a open neighborhood $V$ of $Z$ such that $\sigma|_V = 0$. This shows (i).

(ii) is obvious.

For (iii) and (iv), we need to show surjectivity. To this end, let $\sigma \in \Gamma(Z, \mathcal{F})$. Then there exists an open cover $\{U_i\}_{i \in I}$ of $Z$ and $\sigma_i \in U_i$ such that $\sigma|_{U_i \cap Z} = \sigma_i|_{U_i \cap Z}$. If $Z$ is compact (in (iii)), we can assume $I$ is finite, and if $X$ is paracompact (in (iv), we can assume $\{U_i\}_{i \in I}$ is a locally finite covering of $X$. Thus, we can find another open cover $\{V_i\}_{i \in I}$ of $Z$, which is locally finite, such that $Z \cap \overline{V}_i \subseteq U_j$.

Now, for each $x \in X$, we define $I(x) = \{i \in I; x \in \overline{V}_i\}$ and $W = \{x \in \cup V_i; \sigma_{i,x} = \sigma_{j,x}, i, j \in I(x)\}$. It is clear that $Z \subseteq W$. Since $\{V_i\}_{i \in I}$ is locally finite, $I(x)$ is finite. Moreover, each $x$ has an open neighborhood $W_x$ such that for all $y \in W_x$, we have $I(y) \subseteq I(x)$, which implies $W$ is open. By construction, we have $\sigma_i|_{W \cap V_i \cap V_j} = \sigma_j|_{W \cap V_i \cap V_j}$ which means there exists a section $\sigma' \in \Gamma(W, \mathcal{F})$ such that $\sigma'|_{W \cap V_i} = \sigma_i|_{W \cap V_i}$. But then $\psi(\sigma') = \sigma$, and we are done. □

**Proposition 8.16.** *Let $\mathcal{F}$ be a sheaf on $X$. Then, for all $y \in Y$, we have a canonical isomorphism*

$$(f_! \mathcal{F})_y \cong \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}).$$

*Proof.* We begin by showing that it is injective. Let $\tau_y \in (f_! \mathcal{F})_y$ and assume the image of $\tau_y$ is zero in $\Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$ under the canonical morphism. Then there exists an open neighborhood $U \subseteq Y$ of $y$ such that $\tau \in f_! \mathcal{F}(U)$ and $\tau$ is mapped to $\tau_y$ in $(f_! \mathcal{F})_y$. This means $\tau$ is defined by a section $\sigma \in \mathcal{F}(f^{-1}(U))$ with $f : \operatorname{supp}(\sigma) \to U$ being proper. Further, since $\sigma$ is mapped to zero in $\Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$, there exists an open neighborhood of $f^{-1}(y)$ wherein $\sigma$ is zero; it implies that $\operatorname{supp}(\sigma)$ and $f^{-1}(y)$ are disjoint. But then $y \notin f(\operatorname{supp}(\sigma))$, and since $f(\operatorname{supp}(\sigma))$ is closed ($f$ is proper on $\operatorname{supp}(\sigma)$), there exists a neighborhood of $y$ such that $\tau$ is zero therein. It follows that $\tau_y = 0$.

We now show it is surjective. Let $\sigma \in \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$. By definition, $K = \operatorname{supp}(\sigma)$ is compact, and since $X$ is Hausdorff, we have that $\varinjlim_{U \supseteq K} \Gamma(U, \mathcal{F}) \to \Gamma(K, \mathcal{F})$ is an isomorphism by Proposition 8.15. This means there exists an open set $U$ and a section $\tau \in \Gamma(U, \mathcal{F})$ such that $\tau|_K = \sigma|_K$. To continue, since $Y$ is locally compact, there exists a relatively compact open neighborhood $V$ of $K$ such that $\overline{V} \subseteq U$. Hence, since

$y$ is not contained in $f(\overline{V} \cap \text{supp}(\tau)\backslash V)$, there exists an open neighborhood $W$ of $y$ such that $f^{-1}(W) \cap \overline{V} \cap \text{supp}(\tau) \subseteq V$. Therefore, we can define the following section $\tilde{\sigma} \in \Gamma(f^{-1}(W), \mathcal{F})$: on $f^{-1}(W)\backslash(\overline{V} \cap \text{supp}(\tau))$ $\tilde{\sigma}$ is zero, and on $f^{-1}(W) \cap V$ it is $\tau|_{f^{-1}(W)\cap V}$. Since $\text{supp}(\tilde{\sigma})$ is contained in $f^{-1}(W) \cap \overline{V} \cap \text{supp}(\tau)$, we have that $f$ is proper on this domain, and thus, $\tilde{\sigma}|_{f^{-1}(y)} = \sigma$, and we are done. □

**Proposition 8.17.** *Let $Z$ be a locally closed subset of $Y$ and let $j : Z \to Y$ denote the inclusion. The functor $j_!$ is exact.*

*Proof.* Let $\mathcal{G}$ be a sheaf on $Y$. Then we have $(j_!\mathcal{G})_y \cong \mathcal{G}_y$ if $y \in Z$, else zero. Hence, $j_!$ is exact. □

## 8.5 Relationships of functors

We will now state some important relationships between the defined functors. I will provide proof of three of them, for the others, see for example [7].

**Proposition 8.18.** *Let $\mathcal{F}$ be an object in* $\mathbf{Mod}(\mathcal{R})$, *and let $Z$ be a locally closed subset of $X$. Then we have a natural isomorphism*

$$\mathcal{R}_Z \otimes_{\mathcal{R}} \mathcal{F} \cong \mathcal{F}_Z$$

.

*Proof.* Restricting to $Z$, we have

$$\begin{aligned}(\mathcal{R}_Z \otimes_{\mathcal{R}} \mathcal{F})|_Z &\cong (\mathcal{R}_Z)|_Z \otimes_{\mathcal{R}|_Z} \mathcal{F}|_Z \\ &\cong \mathcal{R}|_Z \otimes_{\mathcal{R}|_Z} \mathcal{F}|_Z \\ &\cong \mathcal{F}|_Z.\end{aligned}$$

Restricting the sheaf on $X\backslash Z$, we note that $(\mathcal{R}_Z)|_{X\backslash Z} = 0$, hence $(\mathcal{R}_Z \otimes_{\mathcal{R}} \mathcal{F})|_{X\backslash Z} = 0$. By the uniqueness of $\mathcal{F}_Z$, the isomorphism follows. □

**Proposition 8.19.** *Let $\mathcal{F}$ be an object in* $\mathbf{Mod}(\mathcal{R})$, *and let $Z$ be a locally closed subset of $Y$. Then*

$$f^*\mathcal{F}_Z \cong (f^*\mathcal{F})_{f^{-1}(Z)}.$$

*Proof.* If $Z$ is locally closed, then so is $f^{-1}(Z)$, hence the right-hand side is defined.

Restricting to $f^{-1}(Z)$, we have $(f^*\mathcal{F}_Z)|_{f^{-1}(Z)} \cong f^*\mathcal{F}_Z|_Z \cong f^*\mathcal{F}|_Z \cong (f^*\mathcal{F})|_{f^{-1}(Z)}$. Similarly, restricting to $f^{-1}(X\backslash Z)$, we get $(f^*\mathcal{F}_Z)|_{f^{-1}(Z)} = 0$, hence by uniqueness, we have $f^*\mathcal{F}_Z \cong (f^*\mathcal{F})_{f^{-1}(Z)}$. □

**Proposition 8.20.** *Let $\mathcal{G}$ be an object in* $\mathbf{Sh}(Y, \mathbf{Ab})$*, and let $Z$ be a locally closed subset of $Y$, $j : Z \to Y$ being the inclusion. Then*

$$\mathcal{G}_Z \cong j_! \circ j^*(\mathcal{G}).$$

*Proof.* This follows immediately from uniqueness: we have $(j_! \circ j^*\mathcal{G})|_Z \cong j^*\mathcal{G}(\cong \mathcal{G}|_Z)$ and $(j_! \circ j^*\mathcal{G})|_{Y\setminus Z} \cong 0$, and we are done. □

I will now state more useful isomorphisms and morphisms. For the following, let $\mathcal{R}$ be sheaf of rings, $f : X \to Y$ a continuous map, $Z$ a locally closed subset of $Y$, $j : Z \to Y$ the inclusion, $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ be sheaves of $\mathcal{R}$-modules, and $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ be sheaves of $f^*\mathcal{R}$-modules. Then we have isomorphisms or morphisms (including the ones proved above):

$$\mathcal{R}_Z \otimes_{\mathcal{R}} \mathcal{F} \cong \mathcal{F}_Z \tag{8.5.3}$$

$$\mathscr{H}om_{\mathcal{R}}(\mathcal{R}_Z, \mathcal{F}) \cong \Gamma_Z(\mathcal{F}) \tag{8.5.4}$$

$$(\mathcal{F}_1 \otimes_{\mathcal{R}} \mathcal{F}_2)_Z \cong \mathcal{F}_1 \otimes_{\mathcal{R}} (\mathcal{F}_2)_Z \cong (\mathcal{F}_1)_Z \otimes_{\mathcal{R}} \mathcal{F}_2 \tag{8.5.5}$$

$$\mathscr{H}om_{\mathcal{R}}((\mathcal{F}_1)_Z, \mathcal{F}_2) \cong \mathscr{H}om_{\mathcal{R}}(\mathcal{F}_1, \Gamma_Z(\mathcal{F}_2)) \cong \Gamma_Z(\mathscr{H}om_{\mathcal{R}}(\mathcal{F}_1, \mathcal{F}_2)) \tag{8.5.6}$$

$$f^*\mathcal{F}_Z \cong (f^*\mathcal{F})_{f^{-1}(Z)} \tag{8.5.7}$$

$$\Gamma_Z(f_*\mathcal{G}) \cong f_*\Gamma_{f^{-1}(Z)}(\mathcal{G}) \tag{8.5.8}$$

$$f_*\mathcal{G} \otimes_{\mathcal{R}} \mathcal{F} \to f_*(\mathcal{G} \otimes_{f^*\mathcal{R}} f^*\mathcal{F}) \tag{8.5.9}$$

$$f_*\mathcal{G}_1 \otimes_{\mathcal{R}} f_*\mathcal{G}_2 \to f_*(\mathcal{G}_1 \otimes_{f^*\mathcal{R}} \mathcal{G}_2) \tag{8.5.10}$$

$$f_* \mathscr{H}om_{f^*\mathcal{R}}(\mathcal{G}_1, \mathcal{G}_2) \to \mathscr{H}om_{\mathcal{R}}(f_*\mathcal{G}_1, f_*\mathcal{G}_2) \tag{8.5.11}$$

$$f^* \mathscr{H}om_{\mathcal{R}}(\mathcal{F}_1, \mathcal{F}_2) \to \mathscr{H}om_{f^*\mathcal{R}}(f^*\mathcal{F}_1, f^*\mathcal{F}_2) \tag{8.5.12}$$

$$\mathcal{G}_Z \cong j_! \circ j^*\mathcal{G} \tag{8.5.13}$$

# 9 Properties of sheaves

We recall that if we have a left-exact additive functor $F$ and if there exists an injective full subcategory with respect to $F$ (Definition 6.10), then the derived functor exist and can be calculated using an injective resolution (Proposition 6.11). Thus, the property of being $F$-injective - there exists an injective full subcategory with respect to $F$ - implies existence of the right derived functor of $F$. Further, Proposition 6.12 allows us to induce relationships in the derived category (with the derived functors) whenever we have a relationship in the category of sheaves.

To the end of finding injective subcategories we will define different properties, namely *injective*, *flabby*, *flat*, and *c-soft*. These will construct injective full subcategories to different functors, in turn giving us the results stated in the previous paragraph.

In this section, we assume $X, Y$ are topological spaces, $f : X \to Y$ a continuous map between them, and $\mathcal{R}$ is a sheaf of rings on $X$.

## 9.1 Injective sheaves

**Definition 9.1.** Let $\mathcal{F}$ be an $\mathcal{R}$-module on $Y$. We say that $\mathcal{F}$ **is $\mathcal{R}$-injective** if $\mathcal{F}$ is injective in the category $\mathbf{Mod}(\mathcal{R})$.

**Proposition 9.1.** *Let $U \subseteq X$ be an open subset and assume $\mathcal{F}$ is $\mathcal{R}$-injective. Then $\mathcal{F}|_U$ is $\mathcal{R}|_U$-injective.*

*Proof.* Let $j : U \to X$ denote the inclusion, and let $\mathcal{G}$ be an $\mathcal{R}|_U$-module. By Eq. 8.5.6, we have $\mathrm{Hom}_{\mathcal{R}|_U}(\mathcal{G}, \mathcal{F}|_U) \cong \mathrm{Hom}_{\mathcal{R}|_U}((j_*\mathcal{G})|_U, \mathcal{F}|_U) \cong \mathrm{Hom}_{\mathcal{R}}((j_*\mathcal{G})|_U, \mathcal{F}|_U)$. Since $\mathcal{G} \mapsto (j_*\mathcal{G})$ is exact, so is $\mathcal{G} \mapsto \mathrm{Hom}_{\mathcal{R}|_U}(\mathcal{G}, \mathcal{F}|_U)$, hence $\mathcal{F}|_U$ is $\mathcal{R}|_U$-injective. □

**Proposition 9.2.** *Let $\mathcal{R}$ be a sheaf of rings on $Y$, and let $\mathcal{G}$ be an $f^*\mathcal{R}$-injective sheaf on $X$. Then $f_*\mathcal{G}$ is $\mathcal{R}$-injective.*

*Proof.* Let $\mathcal{F}$ be an $f_*\mathcal{R}$-module on $X$. By Proposition 8.6, we have

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{F}, f_*\mathcal{G}) \cong \mathrm{Hom}_{f^*\mathcal{R}}(f^*\mathcal{F}, \mathcal{G}).$$

Since $f^*$ is exact, the result follows. □

**Corollary 9.2.1.** *Let $\mathcal{F}$ be an $\mathcal{R}$-injective sheaf. Then $\mathscr{H}om_{\mathcal{R}}(\cdot, \mathcal{F})$ is exact.*

We have the following important proposition.

**Proposition 9.3.** *The category $\mathbf{Mod}(\mathcal{R})$ has enough injectives.*

*Proof.* Let $\hat{X}$ be the set $X$ with the discrete topology, and let $f : \hat{X} \to X$ be the natural map. Let $\mathcal{F}$ be an object in $Mod(f^*R)$. Since the category of modules have enough injectives, there exists for each $x \in X$ an injective module $I_x$ such that the sequence

$$0 \longrightarrow \mathcal{F}_x \longrightarrow I_x$$

is exact. Thus, $\prod_{x \in X} I_x$ is an injective sheaf of $\hat{X}$, and we have that

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_{x \in X} I_x$$

is exact.

Now, let $\mathcal{G}$ be an object in $Mod(R)$. Then $f^*\mathcal{G}$ is an object in $Mod(f^*R)$ and by above there exists an injective object $I$ and a monomorphism $f^*\mathcal{G} \to I$. Applying $f_*$ and using the morphism $id \to f_* \circ f^*$, we get an exact sequence

$$0 \longrightarrow G \longrightarrow f_*I,$$

since $f_*$ is left exact. Since $f_*I$ is injective by Proposition 9.2, the result follows. □

## 9.2 Flabby sheaves

We will now define the property of a flabby sheaf.

**Definition 9.2.** Let $\mathcal{F}$ be an $\mathcal{R}$-module on $Y$. We say that $\mathcal{F}$ is **flabby**, for any subset $U \subseteq Y$, the restriction morphism $\Gamma(Y, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is surjective.

**Proposition 9.4.** *Let $\mathcal{F}$ be a flabby sheaf on $Y$.*

*(i) Let $U \subseteq Y$ be an open subset. Then the sheaf $\mathcal{F}|_U$ is flabby on $U$.*

*(ii) The sheaf $f_*\mathcal{F}$ is flabby.*

*(iii) Let $Z$ be a locally closed subset of $Y$. Then $\Gamma_Z(\mathcal{F})$ is flabby.*

*(iv) Let $Z$ be a locally closed subset of $Y$ and $Z'$ a closed subset of $Z$. Then the following sequence is exact.*

$$0 \longrightarrow \Gamma_{Z'}(\mathcal{F}) \longrightarrow \Gamma_Z(\mathcal{F}) \longrightarrow \Gamma_{Z \setminus Z'}(\mathcal{F}) \longrightarrow 0.$$

*Proof.* (i) and (ii) are obvious.

For (iii), we may assume $Z$ is closed in $Y$ by replacing $Y$ with an open set $U$ where $Z$ is closed in $U$. We want to show that $\Gamma(Y, \Gamma_Z(\mathcal{F})) \to \Gamma(U, \Gamma_Z(\mathcal{F}))$ is surjective for an open subset $U$ of $Y$; that is, $\Gamma_Z(Y, \mathcal{F}) \to \Gamma_{Z \cap U}(U, \mathcal{F})$ is surjective. Let $\sigma \in \Gamma_{Z\cap U}(U,\mathcal{F})$, then $\sigma \in \mathcal{F}(U)$ and $\sigma|_{U\backslash Z} = 0$. Consider the element $0 \in \mathcal{F}(Y\backslash Z)$. Then this element and $\sigma$ agree on overlap, hence there exist an element $\sigma' \in \mathcal{F}(Y)$ such that $\sigma'|_{Y\backslash Z} = 0$ and $\sigma'|_U = \sigma$; that is, $\sigma' \in \Gamma(Y, \Gamma_Z(\mathcal{F}))$ and so $\Gamma_Z(\mathcal{F})$ is flabby.

Lastly, we show (iv). By Proposition 8.14, the sequence

$$0 \longrightarrow \Gamma_{Z'}(\mathcal{F}) \longrightarrow \Gamma_Z(\mathcal{F}) \longrightarrow \Gamma_{Z\backslash Z'}(\mathcal{F})$$

is exact. Let $U$ be an open set, then $\Gamma(U, \Gamma_{Z\backslash Z'}(\mathcal{F}) \cong \Gamma(U\backslash Z', \Gamma_Z(\mathcal{F}))$, and so $\Gamma(U,\Gamma_Z(\mathcal{F})) \to \Gamma(U, \Gamma_{Z\backslash Z'}(\mathcal{F})$ is surjective by (iii), and the result follows. □

**Proposition 9.5.** *Let $\mathcal{R}$ be an object in $\mathbf{Sh}(Y, \mathbf{Ring})$, $\mathcal{G}$ an $\mathcal{R}$-module, and $\mathcal{H}$ an $\mathcal{R}$-injective module. Then $\mathscr{H}om_{\mathcal{R}}(\mathcal{G},\mathcal{H})$ is flabby.*

*Proof.* Let $U \subseteq Y$ be an open set and consider the exact sequence

$$0 \longrightarrow \mathcal{G}_U \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}_{Y\backslash U} \longrightarrow 0.$$

By Corollary 9.2.1, $\mathscr{H}om_{\mathcal{R}}(\cdot,\mathcal{H})$ is exact, so we have an exact sequence

$$0 \longrightarrow \mathscr{H}om_{\mathcal{R}}(\mathcal{G}_U,\mathcal{H}) \longrightarrow \mathscr{H}om_{\mathcal{R}}(\mathcal{G},\mathcal{H}) \longrightarrow \mathscr{H}om_{\mathcal{R}}(\mathcal{G}_{Y\backslash U},\mathcal{H}) \longrightarrow 0.$$

Since $\mathrm{Hom}_{\mathcal{R}}(\cdot,\mathcal{H})$ is also exact, the result follows. □

**Proposition 9.6.** *Let $\mathcal{F}', \mathcal{F}$, and $\mathcal{F}''$ be objects in $\mathbf{Sh}(X, \mathbf{Ab})$, and let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

*be an exact sequence. If $\mathcal{F}'$ is flabby, then*

$$0 \longrightarrow \Gamma(X,\mathcal{F}') \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X,\mathcal{F}'') \longrightarrow 0$$

*is exact.*

*Proof.* Since $\Gamma(X,\cdot)$ is left exact, it suffices to show that $\Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}'')$ is surjective. To this end, let $\sigma'' \in \Gamma(X,\mathcal{F}'')$, and consider the set $S$ consisting of pairs $(U,\sigma)$ such that $\sigma \in \Gamma(U,\mathcal{F})$ and $\sigma$ is mapped to $\sigma''|_U$. We give $S$ a partial order by letting $(U,\sigma) \leq (V,\tau)$, whenever $u \subseteq V$. This order is inductive hence for every chain there exists a maximal element; let $(U,\sigma)$ be one such and assume $U \neq X$.

Now, let $x \in X\backslash U$. Then there exists an open neighborhood $V$ of $x$ and a section $\tau \in \Gamma(V,\mathcal{F})$ such that $\tau$ is sent to $\sigma''|_V$. We have $\sigma - \tau \in \Gamma(U\cap V, \mathcal{F}')$, and since $\mathcal{F}'$ is flabby, there exists a section $\sigma' \in \Gamma(X,\mathcal{F}')$ such that $\sigma'|_{U\cap V} = (\sigma-\tau)|_V$. If we replace $\tau$ with $\tau - \sigma'|_V$, we get $(\sigma - \tau - \sigma'|_V)|_{U\cap V} = 0$, so we may assume $\sigma = \tau$ on $U\cap V$. But then $\sigma$ can be extended on $U\cup V$, which contradicts the maximality of $(U,\sigma)$, thus $U = X$, and the result follows. □

**Proposition 9.7.** *Let $\mathcal{F}', \mathcal{F}$, and $\mathcal{F}''$ be objects in* $\mathbf{Sh}(X, \mathbf{Ab})$*, and let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

*be an exact sequence. Let $Z$ be a locally closed subset of $X$. If $\mathcal{F}'$ is flabby, then the following sequences are exact:*

$$0 \longrightarrow \Gamma_Z(X, \mathcal{F}') \longrightarrow \Gamma_Z(X, \mathcal{F}) \longrightarrow \Gamma_Z(X, \mathcal{F}'') \longrightarrow 0$$

$$0 \longrightarrow \Gamma_Z(\mathcal{F}') \longrightarrow \Gamma_Z(\mathcal{F}) \longrightarrow \Gamma_Z(\mathcal{F}'') \longrightarrow 0$$

*Proof.* Let $U$ be an open subset of $X$ such that $U \cap Z$ is closed in $U$. By Proposition 9.6, the two short-exact sequences

$$0 \longrightarrow \Gamma(U, \mathcal{F}') \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}'') \longrightarrow 0,$$

and

$$0 \longrightarrow \Gamma(U \backslash Z, \mathcal{F}') \longrightarrow \Gamma(U \backslash Z, \mathcal{F}) \longrightarrow \Gamma(U \backslash Z, \mathcal{F}'') \longrightarrow 0$$

are exact. Further, we have an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\mathcal{H}(U) \to \mathcal{H}(U \backslash Z)) \cong \Gamma_{Z \cap U}(U, \mathcal{H}) \longrightarrow \Gamma(U, \mathcal{H}) \longrightarrow \Gamma(U \backslash Z, \mathcal{H}) \longrightarrow 0$$

for $\mathcal{H} = \mathcal{F}', \mathcal{F}, \mathcal{F}''$. This fits into a larger commutative diagram, with the second and third rows being exact, and the columns being exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{Z \cap U}(U, \mathcal{F}') & \longrightarrow & \Gamma_{Z \cap U}(U, \mathcal{F}) & \longrightarrow & \Gamma_{Z \cap U}(U, \mathcal{F}'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(U, \mathcal{F}') & \longrightarrow & \Gamma(U, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F}'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(U \backslash Z, \mathcal{F}') & \longrightarrow & \Gamma(U \backslash Z, \mathcal{F}) & \longrightarrow & \Gamma(U \backslash Z, \mathcal{F}'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

It is easy to show that the top row is also exact given the others, and we are done. □

**Corollary 9.7.1.** *Let $\mathcal{F}', \mathcal{F}$, and $\mathcal{F}''$ be objects in* $\mathbf{Sh}(X, \mathbf{Ab})$*, and let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

*be an exact sequence. If $\mathcal{F}'$ and $\mathcal{F}$ are flabby, then so is $\mathcal{F}''$.*

*Proof.* By Proposition 9.7, the morphism $\Gamma(U,\mathcal{F}) \to \Gamma(U,\mathcal{F}'')$ is surjective for any open set $U$ of $X$. Since $\mathcal{F}$ is flabby, the morphism $\Gamma(X,\mathcal{F}) \to \Gamma(U,\mathcal{F})$ is also surjective. Since the diagram

$$\begin{array}{ccc} \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{F}'') \\ \downarrow & & \downarrow \\ \Gamma(U,\mathcal{F}) & \longrightarrow & \Gamma(U,\mathcal{F}'') \end{array}$$

commutes, it follows that $\Gamma(X,\mathcal{F}'') \to \Gamma(U,\mathcal{F}'')$ is surjective. □

It turns out that being flabby and injective are local properties. That is, if we have an open cover $\{U_i\}_{i\in I}$ of $X$ such that $\mathcal{F}|_{U_i}$ is flabby or $\mathcal{R}|_{U_i}$-injective for all $i \in I$, then $\mathcal{F}$ is flabby or $\mathcal{R}$-injective. See proof in [9].

## 9.3 Flat sheaves

We will now define flat sheaves, which can be used to design an exact functor by tensor product.

**Definition 9.3.** Let $\mathcal{R}$ be an object in $\mathbf{Sh}(X,\mathbf{Ring})$ and let $\mathcal{F}$ be an $\mathcal{R}$-module. $\mathcal{F}$ is called $\mathcal{R}$-flat if the functor $\cdot \otimes_{\mathcal{R}} \mathcal{F} : \mathbf{Mod}(\mathcal{R}) \to \mathbf{Sh}(X,\mathbf{Ab})$ is exact.

By Proposition 8.1, we see that $\mathcal{F}$ is $\mathcal{R}$-flat if and only if $\mathcal{F}_x$ is $\mathcal{R}_x$-flat. This shows the similarity between the definition in sheaves and the definition in $R$-modules, where $R$ is a ring.

**Proposition 9.8.** *Let $\mathcal{R}$ be an object in $\mathbf{Sh}(X,\mathbf{Ring})$ and let $\mathcal{F}$ be an $\mathcal{R}$-module. Then there exists an $\mathcal{R}$-flat module $\mathcal{P}$ such that $\mathcal{P} \to \mathcal{F}$ is an epimorphism.*

*Proof.* Consider the following $\mathcal{R}$-module. Let $S = \{(U,\sigma); U \text{ is open in } X, \sigma \in \Gamma(U,\mathcal{F})\}$, and define

$$\mathcal{P} = \bigoplus_{(U,\sigma)\in S} \mathcal{R}_U.$$

Now, consider the morphism $\mathcal{R}_U \to \mathcal{F}_U$, $1 \mapsto \sigma$ for some pair $(U,\sigma) \in S$. Composing with the morphism $\mathcal{F}_U \to \mathcal{F}$, we get an epimorphism $\mathcal{P} \to \mathcal{F}$.

Lastly, to see that $\mathcal{P}$ is flat, we note that $\mathcal{P}_x$ is a free $\mathcal{R}_x$-module, and so is a flat $\mathcal{R}_x$-module. □

**Proposition 9.9.** *Let $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ be $\mathcal{R}$-modules for some object $\mathcal{R} \in Ob(\mathbf{Sh}(X, \mathbf{Ring}))$, and assume*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

*is exact. If $\mathcal{F}$ and $\mathcal{F}''$ are $\mathcal{R}$-flat, then so is $\mathcal{F}'$.*

*Proof.* Let $x \in X$, we have an exact sequence:

$$0 \longrightarrow \mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x \longrightarrow 0$$

Since $\mathcal{F}_x$ and $\mathcal{F}''_x$ are flat $\mathcal{R}_x$-modules, so is $\mathcal{F}'_x$, and the result follows. □

## 9.4 c-soft sheaves

We will now look at c-soft sheaves.

Before going into the definition, we have the following proposition. Let $Z$ be a subspace of a topological space $X$ and let $j : Z \to X$ denote the inclusion. Consider a map $a_X : X \to \{pt\}$, and let $\mathcal{F}$ be a sheaf on $X$. Then, the morphism $\mathcal{F} \to j_*j^*\mathcal{F}$ gives a morphism $a_{X*}\mathcal{F} \to a_{X*}j^*\mathcal{F}$, mapping sections from $\Gamma(X, \mathcal{F})$ to $\Gamma(Z, \mathcal{F})$. Replacing $X$ with some open subset, we get a nautral morphism:

$$\varinjlim_{U \supseteq Z} \Gamma(U, \mathcal{F}) \to \Gamma(Z, \mathcal{F}).$$

This morphism is an isomorphism under some constrictions on the topological space $X$.

**Definition 9.4.** Let $\mathcal{F}$ be an object in $\mathbf{Sh}(X, \mathbf{Ab})$. We say that $\mathcal{F}$ is c-soft if for all compact subsets $K$ of $X$, the morphism $\Gamma(X, \mathcal{F}) \to \Gamma(K, \mathcal{F})$ is surjective.

**Proposition 9.10.** *Let $\mathcal{F}$ be an object in $\mathbf{Sh}(X, \mathbf{Ab})$. Then $\mathcal{F}$ is c-soft if and only if for all closed subsets $K$ of $X$, the morphism $\Gamma_c(X, \mathcal{F}) \to \Gamma_c(K, \mathcal{F}|_K)$ is surjective.*

*Proof.* We start with sufficiency. Assume $\Gamma_c(X, \mathcal{F}) \to \Gamma_c(K, \mathcal{F}|_K)$ is surjective for all closed subsets $K$. If we let $K$ be compact, then $\Gamma_c(K, \mathcal{F}|_K) = \Gamma(K, \mathcal{F})$ since the support is closed in $K$, and hence compact. It follows that $\Gamma(X, \mathcal{F}) \to \Gamma(K, \mathcal{F})$ is surjective since $\Gamma_c(C, \mathcal{F}) \subseteq \Gamma(X, \mathcal{F})$.

Now for necessity. Assume $\mathcal{F}$ is c-soft and take a section $\sigma \in \Gamma(K, \mathcal{F}|_K)$ with compact support $K_\sigma$. Take a relatively compact open neighborhood $U$ of $K$; that is, the closure of $U$ is compact. Note that $\partial U \cup (K \cap \overline{U})$ is closed in $X$, and take an element $\sigma'$ such that $\sigma'|_{K \cap \overline{U}} = \sigma$, and $\sigma'|_{\partial U} = 0$ ($\partial U$ is the boundary of $U$). Thus, there exists a section $\tau \in \Gamma(X, \mathcal{F})$ such that

$\tau|_{\partial U \cup (K \cap \overline{U})} = \sigma'$. Since $\tau$ is zero on an open neighborhood of $\partial U$, we may assume $\text{supp}(\tau) = \overline{U}$, and so $\tau \in \Gamma_c(X, \mathcal{F})$. □

We have the following important proposition, allowing us to derive relationships in the derived category.

**Proposition 9.11.** *Let $Z \subseteq X$ be locally closed and let $\mathcal{F}$ be a c-soft sheaf on $X$.*

*(i) $f_!\mathcal{F}$ is c-soft.*

*(ii) $\mathcal{F}|_Z$ is c-soft.*

*(iii) $\mathcal{F}_Z$ is c-soft.*

*Proof.* (i) Let $K$ be a compact subset of $Y$. Since $\mathcal{F}$ is c-soft, it follows by Proposition 9.10 that $\Gamma_c(X, \mathcal{F}) \to \Gamma_c(f^{-1}(K), \mathcal{F})$ is surjective. Noting that $\Gamma_c(X, \mathcal{F}) = \Gamma_c(Y, f_!\mathcal{F})$ and $\Gamma_c(f^{-1}(K), \mathcal{F}) = \Gamma(K, f_!\mathcal{F})$, the result follows.

(ii) If $Z$ is open, the result is trivial. If it is closed, it follows from Proposition 9.10.

(iii) This is immediate from the two above because if we let $j : Z \to X$ denote the inclusion, we have $\mathcal{F}_Z = j_!(\mathcal{F}|_Z)$.

□

**Proposition 9.12.** *Let $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ be sheaves on $X$ where $\mathcal{F}'$ c-soft, and assume the sequence*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

*is exact. Then*

$$0 \longrightarrow f_!\mathcal{F}' \longrightarrow f_!\mathcal{F} \longrightarrow f_!\mathcal{F}'' \longrightarrow 0$$

*and*

$$0 \longrightarrow \Gamma_c(X, \mathcal{F}') \longrightarrow \Gamma_c(X, \mathcal{F}) \longrightarrow \Gamma_c(X, \mathcal{F}'') \longrightarrow 0$$

*are exact.*

*Proof.* For all $y \in Y$, we have that $\mathcal{F}'|_{f^{-1}(y)}$ is c-soft on $f^{-1}(y)$. Hence, by Proposition 8.16, it is enough to show the result in the particular case $f : X \to \{pt\}$

We begin by showing that $\Gamma_c(X, \mathcal{F}) \to \Gamma_c(X, \mathcal{F}'')$ is surjective. Let $\sigma'' \in \Gamma_c(X, \mathcal{F}'')$ and thus $\mathrm{supp}(\sigma'')$ is compact; let $U$ be an open neighborhood thereof that is relatively compact. If we replace $\mathcal{F}'$, $\mathcal{F}$, and $\mathcal{F}''$ by $\mathcal{F}'_U$, $\mathcal{F}_U$, and $\mathcal{F}''_U$ respectively, and $X$ by $\overline{U}$, we can assume $X$ is compact. Thus, let $\{K_i\}_{i=1}^n$ be a compact cover of $X$ such that we have $\sigma_i \in \mathcal{F}(K_i)$ with $\sigma_i = \sigma''|_{K_i}$. Now, the result follows by induction. To see this, let $n \geq 2$. We have that $\sigma_1 - \sigma_2 \in \Gamma(K_1 \cap K_2, \mathcal{F}')$ can be extended to $\sigma' \in \Gamma(X, \mathcal{F}')$. This means we can assume $\sigma_1|_{K_1 \cap K_2} = \sigma_2|_{K_1 \cap K_2}$ by replacing $\sigma_2$ by $\sigma_2 + \sigma'$. This means there exists a $\tau \in \Gamma(K_1 \cup K_2, \mathcal{F})$ such that $\tau|_{K_i} = \sigma_i$ for $i = 1, 2$, and the result follows by induction. □

**Corollary 9.12.1.** *Let $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ be sheaves on $X$ and assume the sequence*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

*is exact. If $\mathcal{F}'$ and $\mathcal{F}$ are c-soft, then so is $\mathcal{F}''$.*

*Proof.* Similar proof as done in Corollary 9.7.1. □

# 10 Verdier Duality

In this section, we will derive a right adjoint $f^!$ to the functor $Rf_!$, where $f : X \to Y$ is a continuous map of topological spaces with suitable restrictions to both the spaces and the map. Following the derivation in Kachiwara and Schapira, we will study this over sheaves of $A_X$-modules where $A$ is a commutative ring with finite global dimension. Remember that if $A$ has a finite global dimension, this means it has an injective/projective/flat resolution of finite length.

We also assume $X$ and $Y$ are locally compact, and $f_! : \mathbf{Mod}(\mathbb{Z}_X) \to \mathbf{Mod}(\mathbb{Z}_Y)$ has finite cohomological dimension; that is, there exists an integer $r \geq 0$ such that $R^j f_! = 0$ for $j > r$. We have the following definition.

**Definition 10.1.** Let $\mathcal{F}$ be a sheaf on $X$. $\mathcal{F}$ is called $f$-soft if for any $y \in Y$, $\mathcal{F}|_{f^{-1}(y)}$ is c-soft.

Subsequently, we state without proof the following. If $f_!$ has a cohomological dimension of $r$, then for any $\mathcal{F} \in \mathbf{Sh}(X, \mathbf{Ab})$, there exists an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \cdots \longrightarrow \mathcal{F}^r \longrightarrow 0,$$

where the $\mathcal{F}^j$'s are $f$-soft. Furthermore, we also have: For any exact sequence

$$\mathcal{F}^0 \longrightarrow \cdots \longrightarrow \mathcal{F}^r \longrightarrow 0$$

in $\mathbf{Sh}(X, \mathbf{Ab})$ such that the $\mathcal{F}^j$'s are $f$-soft for $j < r$, then $\mathcal{F}^r$ is $f$-soft.

Let $\mathbb{Z}_X$ be the constant ring on $X$; that is, $\mathbb{Z}_X(U) = \mathbb{Z}$ for all open sets $U \subseteq X$. Let $K$ be a $\mathbb{Z}_X$-module and let $\mathcal{G}$ be a $\mathbb{A}_Y$-module. For an open set $U \subseteq X$, we define the presheaf $f^!_K \mathcal{G}$ to be given by

$$(f^!_K \mathcal{G})(U) = \mathrm{Hom}_{\mathbf{Mod}(A_Y)}(f_!(A_X \otimes_{\mathbb{Z}_X} K_U), \mathcal{G}).$$

For an open subset $V \subseteq U$, we have a natural morphism $f_!(A_X \otimes_{\mathbb{Z}_X} K_V) \to f_!(A_X \otimes_{\mathbb{Z}_X} K_U)$ which gives the restriction morphism $(f^!_K \mathcal{G})(U) \to (f^!_K \mathcal{G})(V)$.

The reason for defining this is because this is the desired right-derived functor of $Rf_!$ that we will soon show. To do so, we need to show that it is exact under suitable restrictions on $K$ (and thus can be seen as a functor in the derived category) and that it is a sheaf. We have the following.

**Lemma 10.1.** *Let $K$ be a flat and $f$-soft $\mathbb{Z}_X$-module. Then the functor $f_!(\cdot \otimes_{\mathbb{Z}_X} K) : \mathbf{Mod}(\mathbb{Z}_X) \to \mathbf{Mod}(\mathbb{Z}_Y)$ is an exact functor.*

*Proof.* We will prove this by showing that $\cdot \otimes_{\mathbb{Z}_X} K$ is $f$-soft, and thus, if $\mathcal{F}$ is a sheaf in $\mathbf{Mod}(\mathbb{Z}_X)$, we have $f_!(\mathcal{F} \otimes_{\mathbb{Z}_X} K)|_{f^{-1}(y)}$ is c-soft for all $y$ hence $f_!(\cdot \otimes_{\mathbb{Z}_X} K)$ is exact.

Consider an arbitrary sheaf $\mathcal{F}$ in $\mathbf{Mod}(\mathbb{Z}_X)$. By the proof of Proposition 9.8, there exists a resolution of $\mathcal{F}$ consisting of the direct sum of sheaves $\mathbb{Z}_U$, where $U$ is an open subset of $X$; that is, we have

$$\cdots \longrightarrow \mathcal{F}^{-r} \longrightarrow \cdots \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

with $\mathcal{F}^{-i}$ consisting of the direct sum of $\mathbb{Z}_U$. Applying the exact functor $\cdot \otimes_{\mathbb{Z}_X} K$, $K$ is flat, we get an the exact sequence

$$\cdots \longrightarrow \mathcal{F}^{-r} \otimes_{\mathbb{Z}_X} K \longrightarrow \cdots \longrightarrow \mathcal{F}^0 \otimes_{\mathbb{Z}_X} K \longrightarrow \mathcal{F} \otimes_{\mathbb{Z}_X} K \longrightarrow 0.$$

Since $\mathcal{F}^{-i} \otimes_{\mathbb{Z}_X} K$ is $f$-soft, so is $\mathcal{F} \otimes_{\mathbb{Z}_X} K$ given that we take $r$ large enough. $\square$

We now show $f^!_K \mathcal{G}$ is a sheaf when $\mathcal{G}$ is injective.

**Proposition 10.2.** *Let $K$ be a flat and $f$-soft $\mathbb{Z}_X$-module and let $\mathcal{G}$ be an injective $A_Y$-module. Then $f^!_K \mathcal{G}$ is an injective sheaf in $\mathbf{Mod}(A_X)$.*

*Proof.* Let $U$ be an open set in $X$ and let $\{U_i\}_{i \in I}$ be an open cover of $U$. Then, we have an exact sequence

$$\bigoplus_{i,j \in I} A_{U_i \cap U_j} \longrightarrow \bigoplus_{i \in I} A_{U_i} \longrightarrow A_U \longrightarrow 0,$$

and since $f_!(\cdot \otimes_{\mathbb{Z}_X} K)$ is exact by Lemma 10.1, we get another exact sequence

$$f_!(\bigoplus_{i,j \in I} A_{U_i \cap U_j} \otimes_{\mathbb{Z}_X} K) \longrightarrow f_!(\bigoplus_{i \in I} A_{U_i} \otimes_{\mathbb{Z}_X} K) \longrightarrow f_!(A_U \otimes_{\mathbb{Z}_X} K) \longrightarrow 0.$$

Since $\mathcal{G}$ is injective, $\mathrm{Hom}_{A_Y}(\cdot, \mathcal{G})$ is exact and we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{A_Y}(f_!(A_U \otimes_{\mathbb{Z}_X} K), \mathcal{G}) \longrightarrow \dots$$

$$\dots \longrightarrow \mathrm{Hom}_{A_Y}(f_!(\bigoplus_{i \in I} A_{U_i} \otimes_{\mathbb{Z}_X} K), \mathcal{G}) \longrightarrow \mathrm{Hom}_{A_Y}(f_!(\bigoplus_{i,j \in I} A_{U_i \cap U_j} \otimes_{\mathbb{Z}_X} K), \mathcal{G}).$$

This sequence is isomorphic to

$$0 \longrightarrow (f^!_K \mathcal{G})(U) \longrightarrow \prod_{i \in I} (f^!_K \mathcal{G})(U) \longrightarrow \prod_{i,j \in I} (f^!_K \mathcal{G})(U_i \cap U_j)$$

which shows $f^!_K \mathcal{G}$ is a sheaf.

To see it is injective, we showed in the proof of Lemma 10.1 that $f_!(\mathcal{F} \otimes_{\mathbb{Z}_X} K)|_{f^{-1}(y)}$ is c-soft for all $y$, hence it follows. $\square$

We need the following lemma.

**Lemma 10.3.** *Let $K$ be a flat and $f$-soft $\mathbb{Z}_X$-module and let $\mathcal{G}$ be an injective $A_Y$-module. Let $\mathcal{F}$ be an $A_X$-module. We have a canonical isomorphism*

$$\mathrm{Hom}_{A_Y}(f_!(\mathcal{F} \otimes_{\mathbb{Z}_X} K), \mathcal{G}) \cong \mathrm{Hom}_{A_Y}(\mathcal{F}, f_K^! \mathcal{G})$$

*Proof.* See Kasiwara and Schapira. □

Lastly, we need the following.

**Lemma 10.4.** *Let $\mathbb{Z}_X$ be a sheaf. Then there $\mathbb{Z}_X$ admits a finite resolution of flat and $f$-soft $\mathbb{Z}_X$-modules; that is, we have*

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow K^0 \longrightarrow \cdots \longrightarrow K^r \longrightarrow 0$$

*where the $K^i$'s are flat and $f$-soft.*

*Proof.* See Kasiwara and Schapira [9], page 143. □

We are now ready for a variant of the Verdier duality. To state it, we include all assumptions.

**Theorem 10.5.** *Let $f : X \to Y$ be a continuous map where $X, Y$ are locally compact spaces, and $f_!$ has a finite cohomological dimension. Let $\mathcal{F}$ be an $A_X$-module and $\mathcal{G}$ an $A_Y$-module. Then there exists a functor $f^! : D^+(\mathbf{Mod}(A_Y)) \to D^+(\mathbf{Mod}(A_X))$ that is a right adjoint to $Rf_!$:*

$$\mathrm{Hom}_{D^+(\mathbf{Mod}(A_X))}(\mathcal{F}, f^! \mathcal{G}) \cong \mathrm{Hom}_{D^+(\mathbf{Mod}(A_Y))}(Rf_! \mathcal{F}, \mathcal{G}).$$

*Proof.* Let $K$ be a flat and $f$-soft resolution of $\mathbb{Z}_X$, as in Lemma 10.4. We will show that the functor $f_K^!$ satisfies this property.

Let $\mathcal{I}(X)$ and $\mathcal{I}(Y)$ be the full subcategories of $\mathbf{Mod}(A_X)$ and $\mathbf{Mod}(A_Y)$ respectively consisting of injective objects. Let $\mathcal{F}'$ and $\mathcal{G}'$ be injective resolutions of $\mathcal{F}$ and $\mathcal{G}$ respectively. By Lemma 10.3, we have an isomorphism

$$\mathrm{Hom}_{K^+(\mathbf{Mod}(A_Y))}(f_!(\mathcal{F}' \otimes_{\mathbb{Z}_X} K), \mathcal{G}') \cong \mathrm{Hom}_{K^+(\mathbf{Mod}(A_X))}(\mathcal{F}', f_K^! \mathcal{G}').$$

Further, note that $\mathcal{F}' \cong \mathcal{F}' \otimes_{\mathbb{Z}_X} \mathbb{Z}_X \to \mathcal{F}' \otimes_{\mathbb{Z}_X} K$ is a quasi-isomoprhism. Since $\mathcal{F}' \otimes_{\mathbb{Z}_X} K$ is $f$-soft, we have $R_! \mathcal{F}' \cong f_!(\mathcal{G}') \otimes_{\mathbb{Z}_X} K$ which gives the isomorphism

$$\mathrm{Hom}_{K^+(\mathbf{Mod}(A_Y))}(f_!(\mathcal{F}' \otimes_{\mathbb{Z}_X} K), \mathcal{G}') \cong \mathrm{Hom}_{K^+(\mathbf{Mod}(A_Y))}(Rf_! \mathcal{F}', \mathcal{G}').$$

Putting it together, we get the desired relationship. □

We finish this section by roughly showing that this is a generalization of Poincaré duality.

Let $X$ be an $n$-dimensional orientable manifold, $Y$ be a point space, $A = \mathbb{Q}$, $\mathcal{F} = \mathbb{Q}_X$, and $\mathcal{G} = \mathbb{Q}_{pt}$. It can be shown that $f^!\mathbb{Q}_{pt} \cong \mathbb{Q}_X[n]$, thus we have

$$\operatorname{Hom}(R\Gamma_c(X, \mathbb{Q}_X)[n], \mathbb{Q}) \cong R\Gamma(X, \mathbb{Q}_X).$$

Applying the cohomology functor, we get

$$H_c^{n-i}(X, \mathbb{Q}_X))^* \cong H^i(X, \mathbb{Q}_X)$$

for all $i \in \mathbb{N}$, where $*$ means the dual vector space over $\mathbb{Q}$, and we are done.

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