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Logarithmic Geometry and the $\mathbb{S}^1$-framed Kontsevich Operad

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Logarithmic Geometry and the $\mathbb{S}^1$-framed Kontsevich Operad

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Abstract

In this thesis, I define, for each positive integer d, an operad in the category of schemes over some base field $\Bbbk$, whose objects are the moduli spaces of stable n-pointed rooted trees of d-dimensional projective spaces, $T_{d,n}$. I then define log structures on these spaces and extend the morphisms of this operad to define an operad of log schemes without unit. Finally, I show that the Kato-Nakayama analytification of this non-unital operad is isomorphic to the operadic semidirect product $\mathcal{K}_{2d} \rtimes \mathbb{S}^1$ of the Kontsevich operad (without unit) in dimension $2d$ and the $\mathbb{S}^1$ topological group.

I detta examensarbete definierar jag, för varje positivt heltal d, en operad i kategorin av scheman över en kropp $\Bbbk$ vars objekt är moduli rummen för stabila träd av d-dimensionella projektiva rum med rot med n markerade punkter, $T_{d,n}$. Sedan definierar jag log strukturer på dessa rum och förlänger morfierna i operaden till morfier av log-scheman för att definiera en operad av log-scheman utan enhet. Slutligen visar jag att Kato-Nakayama analytifieringen av denna operad utan enhet av log-scheman är isomorf med den operadiska semidirekta produkten $\mathcal{K}_{2d} \rtimes \mathbb{S}^1$ av den topologiska Kontsevich operaden i dimension $2d$ (utan enhet) och den topologiska gruppen $\mathbb{S}^1$.

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Contents

1 Introduction

One of the many reasons to study algebraic geometry is its utility in examining the properties of complex analytic spaces. A significant and well-known example is the close relationship between various cohomology theories on a smooth scheme X over $\mathbb{C}$ and corresponding cohomology theories on the analytification of X. Due to these relationships, it is often of great interest to determine whether a topological space, or a map of topological spaces, is, up to isomorphism, the analytification of a variety or a morphism of varieties over $\mathbb{C}$. In 1999, Kato and Nakayama published the article "Log betti cohomology, log étale cohomology, and log de rham cohomology of log schemes over C" [KN99]. In this article they define an analytification of a so called "log scheme" over $\mathbb{C}$ and relate various cohomology theories for log schemes to cohomology theories on their analytifications. I will define what a log scheme is in this thesis, but for now it is sufficient to just think of it as a scheme with some extra structure. Because of the results by Kato and Nakayama, it is also interesting to know if a topological space or a map of topological spaces is, up to isomorphism, the analytification of log scheme or a morphism of schemes over $\mathbb{C}$.

In 2021, Dmitry Vaintrob published the article "Formality of little disks and algebraic geometry" [Vai21], in which he proves that the (non-unital) framed little 2 dimensional disks operad is weakly equivalent to the analytification of an operad of log-schemes and uses this result to prove some properties of the framed little disks operad. The underlying schemes in this operad are $\overline{\mathcal{M}}_{0,n+1}$, the moduli spaces of stable $(n+1)$-pointed rational curves of genus 0, which were introduced in 1983 by Knudsen [Knu83]. The goal of this thesis is to generalize this result to any even dimension $2d$. Specifically I will, for each positive integer d, define a non-unital operad of log schemes whose analytification is isomorphic to the operadic semidirect product $\mathcal{K}_{2d} \rtimes \mathbb{S}^1$ of the Kontsevich operad $\mathcal{K}_{2d}$ and the $\mathbb{S}^1$ group. It is well known that the Kontsevich operad is weakly equivalent to the little disks operad, and in dimension 2 this semidirect product is weakly equivalent to the operad of framed little disks. The underlying schemes of this operad will be $T_{d,n}$, the moduli spaces of stable n pointed rooted trees of d-dimensional projective spaces, introduced by Chen, Gibney, and Krashen in their article "Pointed trees of projective spaces" in 2006. These spaces are a natural generalization of the moduli spaces of stable pointed curves of genus 0 and Chen, Gibney, and Krashen show that $T_{1,n} \cong \overline{\mathcal{M}}_{0,n+1}$.

1.1 Notation

This section contains a list of some notation appearing in this thesis. All this notation is introduced at some point in the thesis but you can see this section for a quick remainder. Here n is a positive integer and $\mathbf{m} = (m_1, \ldots, m_n)$ is a list of positive integers.

- $[n] := \{1, 2, \ldots, n\}$
- $P(n) := \{S \mid S \subseteq [n], |S| \geq 2\}$
- B_n is (some fixed) sequence containing all elements of $P(n)$ exactly once
- $X[n]$ is the Fulton-MacPherson configuration space
- $T_{d,n}$ is the moduli space of stable n-pointed rooted trees of d-dimensional projective spaces
- $\mathrm{FM}_n(M)$ is the topological Fulton-MacPherson configuration space for a smooth manifold M
- $\mathcal{K}_{d,n}$ is the d, n Kontsevich space

- $\mathcal{K}_d$ is the Kontsevich operad in dimension d
- $p^{n,\mathbf{m}} : a \mapsto \begin{cases} 1 & 0 < a \leq m_1 \\ 2 & m_1 < a \leq m_2 \\ \vdots & \\ n & \sum_{i<n} m_i < a \leq \sum_{i \leq n} m_i \end{cases}$
- $q_r^{n,\mathbf{m}} : a \mapsto a - \sum_{i<r} m_i$

1.2 Summary of Results

In this section I list the most notable results of the thesis.

Kato-Nakayama Analytifications
The main theorem of the thesis is:

Theorem 7.35. *The analytification of the log-geometric Kontsevich operad without unit, $\mathcal{T}_d$, is isomorphic to the $\mathbb{S}^1$-framed Kontsevich Operad in dimension 2d, $\mathcal{K}_{2d} \rtimes \mathbb{S}^1$, without unit.*

In addition to this there are some other noteworthy results relating to the Kato-Nakayama analytification functor. Specifically I show the following:

Proposition 7.15. *There is an isomorphism of manifolds with corners over $(X^{\mathrm{an}})^n$*

$$(\mathsf{X}[n])^{\mathrm{KN}} \to \mathrm{FM}_n(X^{\mathrm{an}}).$$

Proposition 7.32. *Let $\mathsf{X} = (\mathbb{P}^{n-1}, (0\colon \mathcal{O} \to \mathcal{L}_1, 0\colon \mathcal{O} \to \mathcal{L}_2)$ where $\mathcal{L}_1 = \mathcal{O}_{\mathbb{P}^{n-1}}$ and $\mathcal{L}_2 = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$, and let $\mathsf{Y} = (\mathbb{P}^{n-1}, (0\colon \mathcal{O} \to \mathcal{M})$ where $\mathcal{M} = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Finally, let $f\colon \mathsf{X} \to \mathsf{Y}$ be the map given by the identity on underlying schemes and $\mathcal{M} \xrightarrow{\cong} \mathcal{L}_1 \otimes \mathcal{L}_2$. The analytification of f, $f^{\mathrm{KN}}\colon \mathbb{S}^1 \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ is the $\mathbb{S}^1$ action on $\mathbb{S}^{2n-1}$ induced by the diagonal inclusion $SO(2) \hookrightarrow SO(2n)$.*

Blow-Ups
I also prove some results relating to real oriented blow ups of smooth manifolds in sections of line bundles. It should be noted that some, if not all, of these results are likely not new but I do not know any reference for them.

Theorem 3.16. *Let Y be a smooth complete intersection in an analytic complex variety X. Let $\tilde{Y}$ be the exceptional divisor of Y in the complex blow-up $\mathrm{Bl}_Y^{\mathbb{C}} X$. There is a canonical isomorphism of blow-ups*

$$\mathrm{Bl}_{\tilde{Y}}^{\mathbb{R}} \; \mathrm{Bl}_Y^{\mathbb{C}} X \to \mathrm{Bl}_Y^{\mathbb{R}} X.$$

Furthermore, for a complex analytic subvariety $Z \subseteq X$ this diffeomorphism maps the (real) total transform of the (complex) dominant transform of Z in $\mathrm{Bl}_{\tilde{Y}}^{\mathbb{R}} \; \mathrm{Bl}_Y^{\mathbb{C}} X$ to the dominant transform of Z in $\mathrm{Bl}_Y^{\mathbb{R}} X$.

Corollary 3.15. *Let X be a complex analytic space and let Z be a closed complex analytic subspace. Let $Y_1, \ldots, Y_n$ be smooth divisors of X cut out by sections $s_n\colon X \to L_n$ of a complex line bundles on X. Additionally, assume that Z has the property that the intersection of Z and any intersection of $Y_1, \ldots, Y_{i-1}$ is either empty* *or not contained in Y_i for each i. Then the strict and total transform of Z in*

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}_n}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}_{n-1}}\ \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}_1}\ X$$

are equal where $\tilde{Y}_i$ is the total transform of Y_i under the previous blow ups.

Proposition 3.24. *Let $L_1, L_2, \dots, L_n$ be complex line bundles on a space X and let $L = \bigotimes_{i=1}^n L_i^{\otimes e_i}$ where e_i are integers and $\otimes$ is the complex tensor product. Let $\sigma_1, \dots, \sigma_n$ be sections $\sigma_i \colon X \to L_i$. Then, there is an isomorphism*

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_0}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_n}\ \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1}\ X \xrightarrow{\cong} \left(\mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_n}\ \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1}\ X\right) \times \mathbb{S}^1$$

where $\tilde{\sigma}_i$ denotes the pullback of σ_i through all previous morphisms and $\sigma_0 \colon X \to L$ is the 0 section.

Properties of $T_{d,n}$

Finally, I also show that the $T_{d,n}$ spaces defined by Chen, Gibney, Krashen in [CGK06] satisfy some interesting properties which may have applications unrelated to the operad constructed in this thesis. Specifically I prove a more general version of theorem 3.3.1 in [CGK06]

Proposition 6.21. *For any collection of positive integers $n, m_1, \dots, m_n$ there is an isomorphism*

$$T_{d,n} \times T_{d,m_1}, \times \dots \times T_{d,m_n} \cong T_{d,m}(M'_1, \dots, M'_n)$$

where $m = \sum_r m_r$ and

$$M'_r = \{1 + \sum_{i<r} m_i, \dots, m_r + \sum_{i<r} m_i\}.$$

Corollary 6.22. *For any $S \subsetneq [n]$, $|S| \geq 2$, the isomorphism of the proposition restricts to an isomorphism of closed subschemes*

$$T_{d,n}(S) \times T_{d,m_1}, \times \dots \times T_{d,m_n} \cong T_{d,m}(M'_1, \dots, M'_n, S'),$$

where $S' = (p^{n,\mathbf{m}})^{-1}(S)$ Similarly, for any $S_r \subsetneq [m_r]$, $|S_r| \geq 2$, the isomorphism of the proposition restricts to an isomorphism of closed subschemes

$$T_{d,n} \times T_{d,m_1}, \times \dots \times T_{d,m_r}(S) \times \dots \times T_{d,m_n} \cong T_{d,m}(M'_1, \dots, M'_n, S'_r),$$

where $S'_r = (q_r^{n,\mathbf{m}})^{-1}(S_r)$.

2 Operads

In this chapter I will give a brief introduction to operads and some related constructions. The purpose of this chapter is not to explain what an operad is to a reader encountering them for the first time, but rather to serve as a remainder about the precise statements of the operad axioms as well as introduce notation used in the thesis. Therefore, I will not provide any context regarding what this definition comes from or why we are interested in operads, nor will I give any examples of operads. Any readers who have not encountered operads before are thus strongly encouraged to look up some motivating examples in any standard textbook on the subject such as "Operads in algebra, topology and physics" by Markl, Shnider, and Stasheff [MSS02].

2.1 Definition

In this section I will define an operad in a symmetric monoidal category. Although some notation and some formulations deviate slightly this section is essentially a shortened version of section 1.2 in chapter 2 of [MSS02]. Note that what I refer to as an "operad" in this thesis sometimes called a "symmetric operad".

In what follows let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. An operad A in $\mathcal{C}$ is a sequence of objects in $\mathcal{C}$, $\{A(n)\}_{n\in\mathbb{N}}$ together with the following data:

- A morphism
$$\eta\colon \mathbb{1} \to A(1).$$
- For each positive integer $n \in \mathbb{N}$, a group action of the symmetric group of n elements on $A(n)$, i.e. a functor from the group category Σ_n to C sending the only object to $A(n)$.
- For each set of positive integers n and $\mathbf{m} = (m_1, \dots, m_n)$ a morphism
$$\gamma^{n,\mathbf{m}}\colon A(n) \otimes A(m_1) \otimes \cdots \otimes A(m_n) \to A(m)$$
where $m = \sum_i m_i$.

such that the following three axioms are satisfied. In what follows, let

$$A[\mathbf{v}] \coloneqq A(v_1) \otimes \cdots \otimes A(v_k),$$

for a vector of integers $\mathbf{v} = (v_1, \dots, v_k)$.

1. **Associativity**. Let n, $(m_1, \dots, m_n)$, $m = \sum_i m_i$, and $(l_1, \dots, l_m)$ be positive integers. Define $l = \sum_k l_k$, $\mathbf{m} = (m_1, \dots, m_n)$, $\mathbf{l} = (l_1, \dots, l_m)$, $l_{i,j} = l_{j+\sum_{k<i} m_i}$, $\mathbf{l}_i = (l_{i,1}, \dots, l_{i,m_i})$, $l'_i = \sum_{1\leq j\leq m_i} l_{i,j}$, and $\mathbf{l}' = (l'_1, \dots, l'_n)$. Then the following diagram commutes
$$\begin{array}{ccc}
A(n)\otimes A[\mathbf{m}]\otimes A[\mathbf{l}] & \xrightarrow{\rho} & A(n)\otimes (A_{m_1}\otimes A[\mathbf{l}_1])\otimes\cdots\otimes(A(m_n)\otimes A[\mathbf{l}_n]) \\
 & & \downarrow{\scriptstyle \mathrm{id}\otimes\gamma^{m_1,\mathbf{l}_1}\otimes\cdots\otimes\gamma^{m_n,\mathbf{l}_n}} \\
\downarrow{\scriptstyle \gamma^{n,\mathbf{m}}\otimes\mathrm{id}} & & A(n)\otimes A[\mathbf{l}'] \\
 & & \downarrow{\scriptstyle \gamma^{n,\mathbf{l}'}} \\
A(m)\otimes A[\mathbf{l}] & \xrightarrow{\gamma^{m,\mathbf{l}}} & A(l)
\end{array}$$
where ρ denotes the corresponding product permutation morphism in the symmetric category.
2. **Equivariance**. Let n, $\mathbf{m} = (m_1, \dots, m_n)$, $m = \sum_i m_i$ be positive integers. Given $\sigma \in \Sigma_n$ let
$$\sigma\mathbf{m} = (m_{\sigma^{-1}(1)}, \dots, m_{\sigma^{-1}(n)})$$
and define the block permutation $\sigma_{\mathbf{m}} \in \Sigma_m$ as the permutation which sends
$$j + \sum_{t<i} m_t \mapsto j + \sum_{t<\sigma^{-1}(i)} m_{\sigma(t)}$$
for any $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Then the following diagram commutes

$$\begin{array}{ccc} A(n) \otimes A[\mathbf{m}] & \xrightarrow{\mathrm{id} \times \overline{\sigma}} & A(n) \otimes A[\sigma \mathbf{m}] \\ & & \downarrow{\scriptstyle \gamma^{n,\sigma\mathbf{m}}} \\ \downarrow{\scriptstyle \sigma \otimes \mathrm{id}} & & A(m) \\ & & \downarrow{\scriptstyle \sigma_{\mathbf{m}}} \\ A(n) \otimes A[\mathbf{m}] & \xrightarrow{\gamma^{n,\mathbf{m}}} & A(m) \end{array}$$

where $\overline{\sigma}$ is the corresponding permutation morphism in the monoidal category.

3. **Unit**. For any positive integer n the following diagrams commute where the lower arrows are the corresponding unit object isomosphisms in the category

$$\begin{array}{ccccc} & & A(n) \otimes A(1)^{\otimes n} & & \\ & \nearrow^{\mathrm{id} \times \eta^{\otimes n}} & & \searrow^{\gamma^{1,(1,\ldots,1)}} & \\ A(n) \otimes \mathbb{1}^{\otimes n} & & \longrightarrow & & A(n) \end{array} \qquad \begin{array}{ccccc} & & A(1) \otimes A(n) & & \\ & \nearrow^{\eta \otimes \mathrm{id}} & & \searrow^{\gamma^{1,(n)}} & \\ \mathbb{1} \otimes A(n) & & \longrightarrow & & A(n) \end{array}$$

An *operad without unit* is an operad without the morphism $\eta \colon \mathbb{1} \to A(1)$ which consequently does not satisfy the unit axiom.

The collection of operads in a monoidal category $(\mathcal{C}, \otimes)$ is itself a category where a morphism between two operads $A \to B$ is a collection of morphisms $A(n) \to B(n)$ which commute with the unit, symmetry, and composition maps of the operads. It is easy to see that if a morphism of operads $A \to B$ consists of isomorphism $A(n) \to B(n)$ for each $n \in \mathbb{N}$ then it is an isomorphism. A morphism of operads without unit is defined in a similar way.

Sometimes it is also (mostly for convenience of notation) useful to introduce the "one object composition morphisms". In an operad A we define the map

$$\circ_i^{n,m} \colon A(n) \otimes A(m) \to A(n+m-1)$$

or just $\circ_i$ when n, m are clear from context, as the composition

$$\begin{array}{c} A(n) \otimes A(m) \\ \downarrow{\scriptstyle \cong} \\ A(n) \otimes \mathbb{1}^{\otimes i-1} \otimes A(m) \otimes 1^{\otimes n-i} \\ \downarrow{\scriptstyle \mathrm{id} \otimes \eta^{\otimes i-1} \otimes \mathrm{id} \otimes \eta^{\otimes n-i}} \\ A(n) \otimes A(1)^{\otimes i-1} \otimes A(m) \otimes A(1)^{\otimes n-i} \\ \downarrow{\scriptstyle \gamma} \\ A(n+m-1) \end{array}$$

2.2 Operadic Semidirect Product

In this section I will define the "operadic semidirect product". To do this we need to introduce an extra criteria and require that the symmetric monoidal category $(\mathcal{C}, \otimes)$ in which we define our operad is a category with finite products, i.e. $\mathcal{C}$ has a terminal object $\mathbb{1}$ and any two objects have a cartesian product, and that $\otimes$ is the cartesian product in the category. For example $\mathcal{C} = Top$, the category of topological spaces, is the case relevant for this thesis. In this case Salvatore and Wahl give a more in depth explanation of the operadic semidirect product in

"Framed discs operads and the equivariant recognition principle" [SW01] in which they also exemplify why this type of construction is interesting.

Definition 2.1. A group object $G \in \mathcal{C}$ *acts* on an operad A if G there is an action $a_n \colon G \times A(n) \to A(n)$ for every object in the operad such that the G and Σ_n actions commute for each n and such that the following diagram commutes.

$$
\begin{array}{ccc}
G \times A(n) \times A[\mathbf{m}] & \xrightarrow{\Delta \times \mathrm{id} \times \mathrm{id}} & G^{n+1} \times A(n) \times A[\mathbf{m}] \\
 & & \Big\downarrow \cong \\
 & & G \times A(n) \times (G \times A[m_1]) \times \cdots \times (G \times A(m_n)) \\
\Big\downarrow \mathrm{id} \times \gamma^{n,\mathbf{m}} & & \Big\downarrow \rho_n \times \rho_{m_1} \times \cdots \times \rho_{m_n} \\
 & & A(n) \times A[\mathbf{m}] \\
 & & \Big\downarrow \gamma^{n,\mathbf{m}} \\
G \times A(m) & \xrightarrow{\rho_m} & A(m)
\end{array}
$$

Here $\Delta \colon G \to G^{n+1}$ denotes the diagonal inclusion.

The goal of this section is to, given an operad A and a group object G which acts on A, define a new operad whose objects are $B(n) := G^n \times A(n)$.

The symmetry action on $B(n)$ is defined as follows. For a permutation $\sigma \in \Sigma_n$ the corresponding isomorphism $G^n \times A(n) \to G^n \times A(n)$ is defined as $\sigma_G \times \sigma_{A(n)}$ where σ_G is the permutation of factors $G^n \to G^n$ corresponding to σ and $\sigma_{A(n)}$ is the isomorphism $A(n) \to A(n)$ corresponding to σ.

Defining the composition morphisms is a bit trickier. Recall that the goal is to define morphisms

$$\gamma_G^{n,\mathbf{m}} \colon G^n \times A(n) \times (G^{m_1} \times A(m_1)) \times \cdots \times (G^{m_n} \times A(m_n)) \to G^m \times A(m)$$

where $m = \sum m_r$ such that the operad axioms are satisfied. To make it easier to see "which G is which" in the following computations I will write $G^n \times A(n) = G_1^0 \times \cdots \times G_n^0 \times A(n)$ and $G^{m_r} \times A(m_r) = G_1^r \times \cdots \times G_{m_r}^r \times A(m_r)$, i.e. G_i^r denotes the i:th G component of $G^{m_r} \times A(m_r)$. In what follows $\Delta_l \colon G \to G^l$ denotes the diagonal inclusion. First, let $g_l \colon G \times G^l \to G^l$ denote the group action given by the composition

$$G \times G_1 \times \cdots \times G_l \xrightarrow{\Delta_l \times \mathrm{id}^l} G^l \times G_1 \times \cdots \times G_l \xrightarrow{\cong} (G \times G_1) \times \cdots \times (G \times G_l) \xrightarrow{g^l} G^l$$

where $g \colon G \times G \to G$ is the group object multiplication morphism. The definition of $\gamma_G^{n,\mathbf{m}}$ is now the following composition of morphisms

$$
\begin{array}{c}
G_1^0 \times \cdots \times G_n^0 \times A(n) \times (G_1^{m_1} \times \cdots \times G_{m_1}^{m_1} \times A(m_1)) \times \cdots \times (G_1^{m_n} \times \cdots \times G_{m_n}^{m_n} \times A(m_n)) \\
\downarrow \Delta_2^n \times \mathrm{id} \\
(G_1^0)^2 \times \cdots \times (G_n^0)^2 \times A(n) \times (G_1^{m_1} \times \cdots \times G_{m_1}^{m_1} \times A(m_1)) \times \cdots \times (G_1^{m_n} \times \cdots \times G_{m_n}^{m_n} \times A(m_n)) \\
\downarrow \text{rearrange} \\
(G_1^0 \times G_1^{m_1} \times \cdots \times G_{m_1}^{m_1}) \times \cdots \times (G_n^0 \times G_1^{m_n} \times \cdots \times G_{m_n}^{m_n}) \times A(n) \times (G_1^0 \times A(m_1)) \times \cdots \times (G_n^0 \times A(m_n)) \\
\downarrow g_{m_1} \times \cdots \times g_{m_n} \times \mathrm{id} \\
G \times \cdots \times G \times A(n) \times (G_1^0 \times A(m_1)) \times \cdots \times (G_n^0 \times A(m_n)) \\
\downarrow \mathrm{id} \times a_{m_1} \times \cdots \times a_{m_n} \\
G^m \times A(n) \times A(m_1) \times \cdots \times A(m_n) \\
\downarrow \mathrm{id} \times \gamma^{n,\mathbf{m}} \\
G^m \times A(m).
\end{array}
$$

Lastly, let $e\colon \mathbb{1} \to G$ be the identity element for the group object. Then the unit map $\mathbb{1} \to G^1 \times A(1)$ is defined as $e \times \eta$ where η is the unit map for the A operad.

Definition 2.2. Let A be an operad acted on by some group object G. The *semidirect product* of A and G, denoted $A \rtimes G$ is defined as the operad with objects $G^n \times A(n)$ and with the unit, symmetry, and composition maps described above.

Remark. For this definition to make sense we must of course prove that the operad axioms are satisfied for these maps. This is true but I leave the proof as an exercise.

Example 2.3. $SO(d)$ acts on the operad of n little d-dimensional disks, D_d by rotation of the positions of the disks. The semidirect product $D_d \rtimes SO(d)$ is the operad of framed little disks. See Salvatore and Wahl [SW01] for more details.

2.3 Reduced Operad

I will introduce one additional concept relating to operads, the reduced operad. Given an operad A we define its reduced operad, denoted A^{red}, as the operad with objects

$$
A^{\text{red}}(n) := \begin{cases} \mathbb{1} & n = 1 \\ A(n) & \text{else} \end{cases}.
$$

The unit map in this operad is the identity $\mathbb{1} \to A^{\text{red}}(1)$ and the symmetry maps are the identity for $A^{\text{red}}(1)$ and the same as for $A(n)$ for all other n. To define the composition maps first let $a_n\colon A^{\text{red}}(n) \to A(n)$ be given by $a_1 = \eta$ the unit map for A and $a_n = \mathrm{id}$ for $n \geq 2$. Then we define the composition maps in A^{red}, $A^{\text{red}}(n) \otimes A^{\text{red}}[\mathbf{m}] \to A^{\text{red}}(m)$ as the identity isomorphism

$$
\mathbb{1} \otimes A(m) \to A(m)
$$

in the case $n = 1$ and the composition

$$
A^{\text{red}}(n) \otimes A^{\text{red}}(m_1) \otimes \cdots \otimes A^{\text{red}}(m_n) \xrightarrow{\mathrm{id} \otimes a_{m_1} \otimes \cdots \otimes a_{m_n}} A(n) \otimes A(m_1) \otimes \cdots \otimes A(m_n) \xrightarrow{\gamma} A(m).
$$

It is easy to verify that these maps satisfy the operad axioms.

3 Blow-Ups

In this chapter I will define the blow up of a topological space in the section of a vector bundle. In section 3.1 I will give some definitions and then section 3.2 will be dedicated to stating and proving a bunch of results about blow ups and sequences of blow ups which will be important in this thesis. I will assume here that the reader is already familiar with the scheme theoretic blow up. If you are not then this chapter might provide enough intuitive understanding for the applications relevant in this thesis but for a more thorough review see some standard textbook on algebraic geometry such as [Har13] or [Vak].

3.1 Motivation and Definition

In this section the definition of the blow-up of a topological space in a section of a vector bundle is given. The notion of a real oriented blow up is standard but notation and precise definitions, especially in "ill behaved" cases, may vary. The notation and definitions used here are taken directly from [BDPW23].

First, recall that the blow-up of a smooth variety X in a smooth complete intersection Y can be explicitly computed as follows.

Proposition 3.1. *Let X be a smooth variety and let Y be a smooth complete intersection of codimension k, cut out by equations $f = (f_1, \dots, f_n)$. Then the blow-up $\mathrm{Bl}_Y X$ is the closed subscheme of $X \times \mathbb{P}^{k-1}$ defined by equations $w_j f_i(x) = w_i f_j(x)$, where w_j is the j:th projective coordinate function. The blow-up morphism is the restriction of the projection $X \times \mathbb{P}^{k-1} \to X$ to this subscheme.*

Proof. This is a standard result. See any standard textbook on algebraic geometry, such as [Vak], for a proof (or in this case an exercise which shows you how to prove it yourself). □

Inspired by this we can define the real oriented blow-up for smooth manifolds in smooth submanifolds. In order to make the way these two concepts are related to each other clearer I will first give the following definitions.

Definition 3.2. Let X be a topological space and let $f = (f_1, f_2, \dots, f_k)$ be a continuous function $f : X \to \mathbb{C}^k$. Then the *complex blow up* of X in f, denoted $\mathrm{Bl}_f^{\mathbb{R}} X$, is defined as the space

$$\mathrm{Bl}_f^{\mathbb{C}} X = \{(x, [w_1 : \cdots : w_k]) \in X \times \mathbb{CP}^{k-1} |\ f_i(x) w_j = f_j(x) w_i \ \forall\ i, j\}$$

together with the surjective function $\rho : \mathrm{Bl}_f^{\mathbb{C}} X \to X$ given by the restriction of the projection $X \times \mathbb{S}^{k-1} \to X$.

Notice that the condition $f_i(x) w_j = f_j(x) w_i \ \forall i, j$ is equivalent to there existing an $\alpha \in \mathbb{C}$ such that $f_i(x) = \alpha w_i \ \forall i$. With this equivalent condition in mind we can define the real oriented blow up as follows.

Definition 3.3. Let X be a topological space and let $f = (f_1, \dots, f_k)$ be a continuous function $f : X \to \mathbb{R}^k$. Then the *real oriented blow up* of X in f, denoted $\mathrm{Bl}_f^{\mathbb{R}} X$, is defined as the space

$$\mathrm{Bl}_f^{\mathbb{R}} X = \{(x, w_1, \dots, w_k) \in X \times S^{k-1} |\ \exists\ \alpha \geq 0 \text{ s.t. } f_i(x) = \alpha w_i \ \forall\ 1 \leq i \leq k\}$$

together with the *blow-up map* $\rho : \mathrm{Bl}_f^{\mathbb{R}} X \to X$ given by the restriction of the projection $X \times \mathbb{S}^{k-1} \to X$.

Remark. Keep in mind that the blow-up map $\rho\colon \mathrm{Bl}^{\mathbb{R}}_f X \to X$ is part of the definition of the real oriented blow up in the same way that the blow-up map is part of the definition of the blow-up of a scheme in a closed subscheme.

You can loosely think of this as "replacing" all points in the zero locus of f with $k-1$ spheres. The topology of this new space is defined in a way such that if $f(x) = 0$ and $\{x_i\}_{i=1}^n$ is a sequence in X with $f(x_i) \neq 0$ such that $\lim_{i\to\infty} x_i = x$ then, in $\mathrm{Bl}^{\mathbb{R}}_f X$ the limit $\lim_{i\to\infty} x_i$ is the point

$$\lim_{i\to\infty}\left(\frac{f_1(x_i)}{|f(x_i)|},\ldots,\frac{f_k(x_i)}{|f(x_i)|}\right)$$

on the $k-1$-sphere we have replaced x with, provided of course that this limit exists.

Example 3.4. Let X be a topological space and let $f\colon X \to \mathbb{R}^k$ be the 0 map, $x \mapsto 0$. Then $\mathrm{Bl}^{\mathbb{R}}_f X \cong X \times S^{k-1}$.

Example 3.5. If $f\colon X \to \mathbb{R}^k$ does not send any point in X to the origin then the morphism $\mathrm{Bl}^{\mathbb{R}}_f X \to X$ is an isomorphism.

We can extend the definition of real oriented blow-ups from maps $X \to \mathbb{R}^k$ to sections of arbitrary vector bundles of X.

Definition 3.6. Let $E \to X$ be a k dimensional real vector bundle on X with a section $\sigma\colon X \to E$. Then the *real oriented blow-up* of X in σ, denoted $\mathrm{Bl}^{\mathbb{R}}_\sigma X$ and the morphism $\mathrm{Bl}^{\mathbb{R}}_\sigma X \to S$ are defined as follows. Let $U_{i i\in I}$ be an open cover of X of trivializing neighbourhoods for E. On each U_i, σ restricts to give a continuous function $\sigma_i\colon U_i \to \mathbb{R}^k$ which has a real oriented blow-up $\mathrm{Bl}^{\mathbb{R}}_{\sigma_i} U_i$ and morphism $\mathrm{Bl}^{\mathbb{R}}_{\sigma_i} U_i \to U_i$. The real oriented blow-up $\mathrm{Bl}^{\mathbb{R}}_\sigma X$ is now defined as the space we get by gluing together $\mathrm{Bl}^{\mathbb{R}}_{\sigma_i} U_i$ using the gluing maps induced by the gluing maps for the open cover $\{\sigma^{-1}(U_i)\}_{i\in I}$ of E. The morphism $\mathrm{Bl}^{\mathbb{R}}_\sigma X \to X$ is similarly defined as the morphism which restricts to $\mathrm{Bl}^{\mathbb{R}}_{\sigma_i} U_i \to U_i$ on each component of the open cover.

Remark. Notice that this construction gives a canonical embedding of the real oriented blow-up into the unit bundle of E.

There are some natural examples of this.

Example 3.7 (Unit tangent bundle). Let X be a smooth manifold, let $T \to X$ be the tangent bundle for X, and let $\sigma\colon X \to T$ be the 0 section. Then $\mathrm{Bl}^{\mathbb{R}}_\sigma X \to X$ is just the tangent unit circle bundle for X with its projection to X.

Example 3.8 (Divisors on complex varieties). Let X be an analytic complex variety and let $D \subseteq X$ be a smooth normal crossings divisor. Let $s_D\colon \mathcal{O}_X \to \mathcal{O}(D)$ be the associated (complex) line bundle with section. Then $\mathrm{Bl}^{\mathbb{R}}_{s_D} X$ is homeomorphic to the complement of a (sufficiently small) tubular neighbourhood of D in X. Furthermore, the function

$$f\colon \mathrm{Bl}^{\mathbb{R}}_{s_D} X \to X$$

restricts to an isomorphism

$$f^{-1}(X \setminus D) \xrightarrow{\cong} X \setminus D.$$

Remark. In this example I have abused notation by not distinguishing between the vector bundle sheaf with section and the associated vector bundle space with section. This will happen again.

A closely related concept is the blow-up of a smooth manifold in a closed submanifold. This can be defined in more general situations but I will stick with this somewhat simpler and sufficient definition for this thesis.

Definition 3.9. Let X be a smooth manifold and let $Y \hookrightarrow X$ be a closed submanifold such that Y has codimension k and is cut out by some smooth section σ of a k-dimensional vector bundle $E \to X$, i.e. Y is the fiber of σ of the 0-section of E. Then we define $\mathrm{Bl}_Y^{\mathbb{R}}\ X := \mathrm{Bl}_\sigma^{\mathbb{R}}\ X$ with the corresponding blow-up morphism $\rho\colon \mathrm{Bl}_Y^{\mathbb{R}}\ X \to X$. Furthermore, we define the "exceptional divisor" of the blow-up as $E_Y := \rho^{-1}(Y)$.

Remark. For this to make sense one must of course show that $\mathrm{Bl}_Y^{\mathbb{R}}\ X$ is, up to isomorphism, independent of the choice of vector bundle E with section. One way to prove this is to show that the blow-up of X in Y is diffeomorphic to a sufficiently small tubular neighbourhood of Y. Any interested readers are encouraged to try to prove this.

Example 3.10 (Blowing up the origin). Let $X = \mathbb{R}^d$ and $Y = \{0\}$, the origin. Then $\mathrm{Bl}_Y^{\mathbb{R}}\ X \cong X \setminus B(1)$, X with the (open) unit ball removed. Furthermore, $E_Y \cong S^{d-1}$. The easiest way to show this is by noting that Y is cut out by the section $x \mapsto (x, x)$ in the trivial bundle $\mathbb{R}^d \times \mathbb{R}^d$ and applying the above definition.

Finally, let us define the total, strict, and dominant transforms of a blow-up.

Definition 3.11. Let X be a smooth manifold with a subspace $Z \hookrightarrow X$. Then, for a closed submanifold Y with blow-up $\rho\colon \mathrm{Bl}_Y^{\mathbb{R}}\ X \to X$ we define

- the *total transform* of Z as $\rho^{-1}(Z)$.
- the *strict/proper* transform of Z as the closure of $\rho^{-1}(Z \setminus Y)$
- the *dominant transform* of Z as the total transform of Z if $Z \subseteq X$ and the strict transform of Z otherwise.

For the blow-up in some section σ in a vector bundle of X we define each of the above transforms in the same way but with Y replaced with the inverse image of the 0-section, $Y = \sigma^{-1}(0)$.

The methods used to extended the real oriented blow up for a map $f\colon X \to \mathbb{R}^k$, first to a the real oriented blow up of any section σ of a real vector bundle and then to the blow up of a manifold in a smooth complete intersection can be applied in the exact same way to define the complex blow up in the section of a complex vector bundle and the blow up of a complex analytic space in a complex analytic subspace that is given by a complete intersection. Furthermore, notice that by definition, if X is a smooth complex variety and $Y \hookrightarrow X$ is a smooth complete intersection, then we have $\mathrm{Bl}_{Y^{\mathrm{an}}}^{\mathbb{C}}\ X^{\mathrm{an}} \cong (\mathrm{Bl}_Y\ X)^{\mathrm{an}}$, i.e. "analytifications and complex blow ups commute".

3.2 Important Results

In this thesis many situations will arise where we want to identify two blow ups, or two sequences of blow ups, with each other. Therefore, the remainder of this section will be dedicated to stating and proving some results of this nature. In each of the following statements where I claim that there is an isomorphism of two sequences of blow ups $A \to B$ of some manifold X what I mean is that there is a diffeomorphism of spaces $A \to B$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes where the downward arrows are the corresponding blow up maps. In other words this is a diffeomorphism of manifolds over X.

Lemma 3.12. *Let Y, Z be smooth locally complete intersections in some smooth manifold X. Let $\tilde{Y}$ denote the total transform of Y in $\mathrm{Bl}^{\mathbb{R}}_Z X$ and $\tilde{Z}$ denote the total transform of Z in $\mathrm{Bl}^{\mathbb{R}}_Y X$. Then there is an isomorphism of blow ups and $\mathrm{Bl}^{\mathbb{R}}_{\tilde{Z}} \mathrm{Bl}^{\mathbb{R}}_Y X \to \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}} \mathrm{Bl}^{\mathbb{R}}_Z X$.*

Furthermore, this isomorphism maps the strict transform of the strict transform of any set $W \subseteq X$ in $\mathrm{Bl}^{\mathbb{R}}_{\tilde{Z}} \mathrm{Bl}^{\mathbb{R}}_Y X$ to the strict transform of the strict transform of W in $\mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}} \mathrm{Bl}^{\mathbb{R}}_Z X$.

Proof. I will prove this in the case where Y, Z are both cut out by equations $f = (f_1, \dots, f_k)\colon X \to \mathbb{R}^k$ and $g = (g_1, \dots, g_r)\colon X \to \mathbb{R}^r$ respectively. The general case follows from gluing these isomorphisms along a trivializing open cover for the line bundles with sections cutting out Y and Z. By definition $\mathrm{Bl}^{\mathbb{R}}_Y X = \{(x, s) \in X \times \mathbb{S}^{k-1}) | f(x) = \alpha s, \alpha \geq 0\}$. Furthermore, the total transform of Z in $\mathrm{Bl}^{\mathbb{R}}_Y X$ is cut out by

$$\tilde{g}\colon \mathrm{Bl}^{\mathbb{R}}_Y X \to \mathbb{R}^r, (x, s) \mapsto g(x).$$

Therefore

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{Z}} \mathrm{Bl}^{\mathbb{R}}_Y X = \{(x, s, t) \in X \times S^k \times S^r | f(x) = \alpha s, g(x) = \beta t,\ \alpha, \beta \geq 0\}.$$

Similarly we find that

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}} \mathrm{Bl}^{\mathbb{R}}_Z X = \{(x, t, s) \in X \times S^r \times S^k | f(x) = \alpha s, g(x) = \beta t,\ \alpha, \beta \geq 0\}.$$

Clearly these two spaces are isomorphic with isomorphism $\phi\colon (x, s, t) \mapsto (x, t, s)$.

For the second part of the statement note that this statement is local so we may assume that Y, Z are cut out by equations $f\colon X \to \mathbb{R}^k$ and $g\colon X \to \mathbb{R}^r$. Let W_Y denote the strict transform of W in $\mathrm{Bl}^{\mathbb{R}}_Y X$. Then the strict transform of W_Y in $\mathrm{Bl}^{\mathbb{R}}_{\tilde{Z}} \mathrm{Bl}^{\mathbb{R}}_Y X$ is the set of all points $(x, s, t) \in \mathrm{Bl}^{\mathbb{R}}_{\tilde{Z}} \mathrm{Bl}^{\mathbb{R}}_Y X$ such that there exists a sequence $(x_n, s_n) \in W_Y \setminus \tilde{Y}$ such that $\lim x_n = x$, $\lim s_n = s$, and $\lim \frac{g(x_n)}{|g(x_n)|} = t$. Since $\tilde{Z}$ is closed each point (x_n, s_n) has a neighbourhood that does not intersect $\tilde{Z}$. Since $W_Y \setminus Y$ is dense in W_Y, and thus in particular on U and g is continuous we may choose this sequence such that $x_n \notin Y$. In this case we must have $s_n = \frac{f(x_n)}{|f(x_n)|}$ Hence, (x, s, t) lies in the strict transform of the strict transform of W in $\mathrm{Bl}^{\mathbb{R}}_{\tilde{Z}} \mathrm{Bl}^{\mathbb{R}}_Y X$ if and only if there is a sequence $x_n \in W \setminus (Y \cup Z)$ such that $\lim x_n = x$, $\lim \frac{f(x_n)}{|f(x_n)|} = s$, and $\lim \frac{g(x_n)}{|g(x_n)|} = t$. By symmetry the same conditions determine the points in the strict transform of the strict transform of W in $\mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}} \mathrm{Bl}^{\mathbb{R}}_Z X$. This completes the proof.

□

Remark. It may (and often does) happen that $\tilde{Y}$ or $\tilde{Z}$ is not a complete intersection of codimension k/r and thus, to be precise, we are not in general blowing up $\tilde{Y}$ or $\tilde{Z}$ here but rather we are taking the blow up of the pullback of the vector bundle with section cutting out Y to $\mathrm{Bl}^{\mathbb{R}}_Z X$ and vice versa. This perspective also extends the result of the lemma to blow ups in arbitrary line bundles with sections.

Lemma 3.13. *Let X be a smooth manifold, let $\{\sigma_i\colon X \to E_i\}_{1 \leq i \leq n}$, be a set of smooth sections of vector bundles of X, and let $Y_i \subseteq X$ be the space cut out by σ_i, i.e. zero locus of σ_i. If $Z \subseteq X$ is a closed subspace such that for each blow up $\rho_i\colon \mathrm{Bl}^{\mathbb{R}}_{\sigma_i} X$ the strict and total transform are equal for Z and for any combination of intersections between Z and the subspaces $Y_1, \dots, Y_{i-1}$ then the strict and total transforms of Z are equal in the sequence of blow ups*

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_n} \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_{n-1}} \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1} X$$

where $\tilde{\sigma}_i$ is the pullback, i.e. total transform, of the section σ_i via all previous blow ups.

Proof. Since the result can be checked locally we can assume that the line bundles E_i are trivial and thus replace σ_i with functions $f_i\colon X \to \mathbb{R}^{k_i}$. In this case we can use induction. In what follows let $\tilde{Z}_i, \tilde{Y}_{j,i}$ denote the total transforms of Z and Y_j in the blow up $\mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_i}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_{i-1}}\ \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1}\ X$, and let $\tilde{\rho}_i$ denote the blow up map $\mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_i}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_{i-1}}\ \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1}\ X \to \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_{i-1}}\ \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1}\ X$.

The base case $n = 1$ is clear by hypothesis.

For the induction step suppose the strict and total transform of Z are the same in $\mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_{i-1}}\ \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1}\ X$. By definition,

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_i}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_{i-1}}\ \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1}\ X = \{(x, s_1, \dots, s_i) \in X \times \mathbb{S}^{k_1-1} \times \dots \times \mathbb{S}^{k_i-1} |\ f_j(x) = |f_j(x)| s_j\ \forall\ 1 \leq j \leq i\},$$

and the total transform of Z is

$$\tilde{Z}_i = \{(x, s_1, \dots, s_i) \in Z \times \mathbb{S}^{k_1-1} \times \dots \times \mathbb{S}^{k_i-1} |\ f_j(x) = |f_j(x)| s_j\ \forall\ 1 \leq j \leq i\}.$$

Since the total and proper transforms of Z are equal in $\mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_i}\ \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1}\ X$ the strict transform of Z in $\mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_{i-1}}\ \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1}\ X$ is the closure of the inverse image $\tilde{\rho}_i^{-1}(\tilde{Z}_{i-1} \setminus \tilde{Y}_{i,i-1})$ by lemma 3.12.

Clearly, $\tilde{Z}_i \setminus \tilde{Y}_{i,i} = \tilde{\rho}_i^{-1}(\tilde{Z}_{i-1} \setminus \tilde{Y}_{i,i-1})$ so what we must show is that every point $(x, s_1, \dots, s_i) \in \tilde{Z}_i \cap \tilde{Y}_{i,i}$ is the limit of a sequence of points $(x^n, s_1^n, \dots, s_{i-1}^n, \frac{f_i(x)}{|f_i(x)|}) \in \tilde{\rho}_i^{-1}(\tilde{Z}_{i-1} \setminus \tilde{Y}_{i,i-1})$. To find such a sequence let I be the set of indexes $I = \{j|\ x \in Y_j \text{ and } 1 \leq j < i\}$. Since the total and proper transform of $Z \cap \bigcap_{j \in I} Y_j$ are equal in $\mathrm{Bl}^{\mathbb{R}}_{\sigma_i}\ X$ there is a sequence $x_n \in Z \cap \bigcap_{j \in I} Y_j \setminus Y_i$ such that $\lim x_n = x$ and $\lim \frac{f_i(x_n)}{|f_i(x_n)|} = s_i$. Since each Y_k is closed and $x \notin Y_k$ whenever $k \notin I$ we may without loss of generality assume $x_n \notin Y_k$ for every $k \notin I$. Now, for each x_n in this sequence the point $(x_n, s_1^n, \dots, s_{i-1}^n, \frac{f_i(x_n)}{|f_i(x_n)|})$ where $s_j^n = s_j$ for every $j \in I$ and $s_k^n = \frac{f_k(x_n)}{|f_k(x_n)|}$ lies in $\tilde{Z}_{i-1}$ by definition. Clearly,

$$\lim(x_n, s_1^n, \dots, s_{i-1}^n, \frac{f_i(x_n)}{|f_i(x_n)|}) = (x, s_1, \dots, s_i)$$

and thus we are done. □

Proposition 3.14. *Let X be a complex analytic space and let Z be a closed complex analytic subspace. Let Y be a smooth divisor of X cut out by some section $s\colon X \to L$ of a complex line bundle on X. Then the total and dominant transforms of Z under for the blow up $\rho\colon \mathrm{Bl}^{\mathbb{R}}_Y\ X \to X$ are equal.*

Proof. In the case $Z \subseteq Y$ or $Z \cap Y =$ this is clear so we assume $Z \cap Y \neq$ and $Z \not\subseteq Y$. Furthermore, the result can be proven locally so we may assume that L is trivial, i.e. Y is cut out by a single holomorphic function $f\colon X \to \mathbb{C}$. Now, let $\tilde{Z}$ denote the dominant transform of Z and let $\tilde{Y}$ denote the exceptional divisor. It is clear by definition that $\tilde{Z} \setminus \tilde{Y} = \rho^{-1}(Z) \setminus \tilde{Y}$ so we must only show that $\tilde{Z} \cap \tilde{Y} = \rho^{-1}(Z \cap Y)$. In this case note that, f is holomorphic on Z and, since $Z \not\subseteq Y$, $Z \cap Y \neq$ there are points on Z that are mapped to 0 by f and points that are not mapped to 0 by f, i.e. f is not constant. Since f is holomorphic but not constant it is an open map. Now, let $x \in Z \cap Y$, i.e. $f(x) = 0$. Since f is open, every open neighbourhood $U \subseteq Z$ of x is mapped to an open neighbourhood of the origin in $\mathbb{C}$. In particular, every open neighbourhood of x contains a point x' with $f(x')/|f(x')| = e^{i\theta}$ for every $0 \leq \theta < 2\pi$. Thus we can find a sequence in $Z \setminus Y$, such that $\lim x_n = x$

and $f(x_n)/|f(x_n)| = e^{i\theta}$. Hence, for every θ the point $(x, \theta) \in \mathrm{Bl}^{\mathbb{R}}_Y\ X$ is in the closure of $\rho^{-1}(Z \setminus Y)$. Thus $\tilde{Z} \cap \tilde{Y} = \rho^{-1}(Z \cap Y)$ and the proof is complete. □

Corollary 3.15. *Let X be a complex analytic space and let Z be a closed complex analytic subspace. Let $Y_1, \ldots, Y_n$ be smooth divisors of X cut out by sections $s_n \colon X \to L_n$ of a complex line bundles on X. Additionally, assume that Z has the property that the intersection of Z and any intersection of $Y_1, \ldots, Y_{i-1}$ is either empty or not contained in Y_i for each i. Then the strict and total transform of Z in*

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}_n}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}_{n-1}}\ \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}_1}\ X$$

are equal where $\tilde{Y}_i$ is the total transform of Y_i under the previous blow ups.

Proof. By the proposition the dominant transform of Z intersected with any combination of the Y_j divisors for $1 \leq j < i$ will have equal dominant and total transforms for the blow up $\mathrm{Bl}^{\mathbb{R}}_{Y_i}\ X \to X$ (this is trivially true in the case when the intersections are empty). Since the intersections are either empty or contained in Y_i the dominant transform is the strict transform for these subspaces and thus the corollary follows from lemma 3.13. □

Theorem 3.16. *Let Y be a smooth complete intersection in an analytic complex variety X. Let $\tilde{Y}$ be the exceptional divisor of Y in the complex blow-up $\mathrm{Bl}^{\mathbb{C}}_Y\ X$. There is a canonical isomorphism of blow-ups*

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}}\ \ \mathrm{Bl}^{\mathbb{C}}_Y\ X \to \mathrm{Bl}^{\mathbb{R}}_Y\ X.$$

Furthermore, for a complex analytic subvariety $Z \subseteq X$ this diffeomorphism maps the (real) total transform of the (complex) dominant transform of Z in $\mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}}\ \ \mathrm{Bl}^{\mathbb{C}}_Y\ X$ to the dominant transform of Z in $\mathrm{Bl}^{\mathbb{R}}_Y\ X$.

Proof. First, let us prove this in the special case where Y is a complete intersection of holomorphic functions $f_j \colon X \to \mathbb{C} \cong \mathbb{R}^2$, for $1 \leq j \leq k$. Define u_j, v_j to be the real and imaginary components of f_j (i.e. $f_j(x) = u_j(x) + iv_j(x)$). Then $\mathrm{Bl}^{\mathbb{R}}_Y\ X$ is the space

$$\{(x, (s_1, t_1, s_2, t_2, \ldots, s_k, t_k)) \in X \times \mathbb{S}^{2k-1} |\ \exists\ \alpha \geq 0 \text{ s.t } u_j(x) = \alpha s_j, v_j(x) = \alpha t_j\ \forall j\}$$

and $\mathrm{Bl}^{\mathbb{C}}_Y\ X$ is the space

$$\{(x, [w_1 \colon \ \ldots : w_k]) \in X \times \mathbb{CP}^{k-1} |\ f_l(x) w_j = f_j(x) w_l\ \forall j, l\}.$$

Now, let $U_j \subseteq \mathrm{Bl}^{\mathbb{C}}_Y\ X$ be the subspace defined by $w_j \neq 0$. On this subspace the exceptional divisor $\tilde{Y}$ of the complex blow-up is given by the single complex equation $f_j(x) = 0$, or equivalently by the two real equations $u_j(x) = 0$ and $v_j(x) = 0$. Thus, we have

$$\begin{aligned}\mathrm{Bl}^{\mathbb{R}}_{\tilde{Y} \cap U_j}\ U_j = \{(x, [w_1 \colon \ \ldots : w_k], (p_j, q_j)) \in X \times \mathbb{CP}^{k-1} \times \mathbb{S}^1 | \\ f_l(x) w_j = f_j(x) w_l\ \forall j, l \text{ and } \exists \alpha \geq 0 \colon\ u_j(x) = \alpha p_j, v_j(x) = \alpha q_j\}.\end{aligned}$$

Now we can define the function $\phi_j \colon \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y} \cap U_j} U_j \to \mathrm{Bl}^{\mathbb{R}}_Y X$ by sending $(x, [w_1 \colon \dots \colon w_k], (p_j, q_j)) \mapsto (x, (s_1, t_1, s_2, t_2, \dots, s_k, t_k))$ where

$$s_l + it_l = \lambda \frac{w_l}{w_j}(p_j + iq_j),$$

and $\lambda > 0$ is some normalization constant. It is clear that this is a diffeomorphism onto its image. Furthermore, it is also easy to see that each point in $\mathrm{Bl}^{\mathbb{R}}_Y X$ lies in the image of ϕ_j for some j. Thus, all that remains to show is that these maps ϕ_j glue together to a map $\phi \colon \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}} \mathrm{Bl}^{\mathbb{C}}_Y X \to \mathrm{Bl}^{\mathbb{R}}_Y X$. To see this, note that on the intersection $U_j \cap U_l$ we have that $g_l(x) = \frac{w_l}{w_j} g_j(x)$. Thus, the gluing map $g_{jl} \colon \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y} \cap U_j} U_j \cap \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y} \cap U_l} U_l \to \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y} \cap U_l} U_l \cap \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y} \cap U_j} U_j$ sends $(x, [w_1 \colon \dots \colon w_k], (p_j, q_j)) \mapsto (x, [w_1 \colon \dots \colon w_k], (p_l, q_l))$ where

$$p_l + iq_l = \frac{w_l/w_j}{|w_l/w_j|}(p_j + iq_j).$$

Thus, $\phi_l(g_{jl}(x, [w_1 \colon \dots \colon w_k], (p_j, q_j))) = (x, (s_1, t_1, s_2, t_2, \dots, s_k, t_k))$ where

$$s_m + it_m = \lambda \frac{w_m}{w_l} g_{ij}(p_j + iq_j) = \lambda \frac{w_m}{w_l} \frac{w_l/w_j}{|w_l/w_j|}(p_j + iq_j) = \frac{\lambda}{|w_l/w_j|} \frac{w_m}{w_j}(p_j + iq_j).$$

Thus $\phi_l \circ g_{jl} = \phi_j$ and therefore the maps glue to an isomorphism $\mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}} \ \mathrm{Bl}^{\mathbb{C}}_Y X \to \mathrm{Bl}^{\mathbb{R}}_Y X$. Commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}} \ \mathrm{Bl}^{\mathbb{C}}_Y X & \longrightarrow & \mathrm{Bl}^{\mathbb{R}}_Y X \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

also follows from construction. For the general case note that we can cover X in analytic subspaces U_i such that Y is locally cut out by functions $f_j \colon X \to \mathbb{C} \cong \mathbb{R}^2$ on each subspace U_i. To complete the proof we can glue the corresponding isomorphisms for each component U_i together. It is easy to verify that these maps "glue well" so I omit the details.

Finally, let $Z \subseteq X$ be an analytic subvariety. Clearly ϕ maps the (real) dominant transform of the (complex) dominant transform of any Z to to the dominant transform of Z. By lemma 3.14 the dominant and real transform of a complex subvariety are the same for the blow-up $\mathrm{Bl}^{\mathbb{R}}_{\tilde{Y}} \ \mathrm{Bl}^{\mathbb{C}}_Y X \to \mathrm{Bl}^{\mathbb{C}}_Y X$. From this the result follows. ☐

Remark. Note that we know from Algebraic Geometry that the exceptional divisor (by definition of the blow up) is a codimension one effective Cartier divisor and thus a complete intersection. Thus talking about the blow up in the exceptional divisor makes sense.

This result has many interesting applications. One immediate application is that this gives a simple proof for the following result.

Corollary 3.17. *Let $E \to \mathbb{CP}^{n-1}$ be the $\mathcal{O}(-1)$ complex line bundle and let $s_0 \colon \mathbb{CP}^n \to E$ be the 0 section. There is a diffeomorphism*

$$\mathrm{Bl}^{\mathbb{R}}_{s_0} \ \mathbb{CP}^n \xrightarrow{\cong} \mathbb{S}^{2n-1}.$$

Proof. Let $p \in \mathbb{A}^n_{\mathbb{C}}$ be the origin (or any point in $\mathbb{A}^n_{\mathbb{C}}$) which is clearly a codimension n, smooth complete intersection. It a standard result that the exceptional divisor $\tilde{p} \in \mathrm{Bl}^{\mathbb{C}}_p \ \mathbb{A}^n_{\mathbb{C}}$ is diffeomorphic to $\mathbb{CP}^{n-1}$ and that $\tilde{p}$

is cut out by a section of a line bundle $s\colon \mathrm{Bl}_p^{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^n \to E$ such that the restriction of E to $\tilde{p}$ is the $\mathcal{O}(-1)$ complex line bundle. Now, by the theorem

$$\mathrm{Bl}_{\tilde{p}}^{\mathbb{R}} \mathrm{Bl}_p^{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^n \cong \mathrm{Bl}_p^{\mathbb{R}} \mathbb{A}_{\mathbb{C}}^n.$$

Restricting this diffeomorphic to the real oriented blow up of the exceptional divisor $\tilde{p}$ we find that $\mathrm{Bl}_{0\in\mathcal{O}(-1)}^{\mathbb{R}} \mathbb{CP}^{n-1}$ is diffeomorphic to the exceptional divisor in $\mathrm{Bl}_p^{\mathbb{R}} \mathbb{A}_{\mathbb{C}}^n \cong \mathrm{Bl}_p^{\mathbb{R}} \mathbb{R}^{2n}$ which is of course $\mathbb{S}^{2n-1}$. □

Definition 3.18. A k-dimensional vector bundle $E \to X$ is *glued by positive scalar multiplication* if there is some trivializing open cover $\{U_i\}_{i\in I}$ of X such that for each $i, j \in I$ the gluing map of E, $\phi_{ij}\colon (U_i \cap U_j) \times \mathbb{R}^k \to (U_j \cap U_i) \times \mathbb{R}^k$ is given by $(x, \mathbf{v}) \mapsto (x, \lambda_{ij}(x)\mathbf{v}$ where $\lambda_{ij}(x)$ is a positive, real, scalar.

Lemma 3.19. *Let $g\colon E \to X$ be a k-dimensional vector bundle on X, glued by positive scalar multiplication. Then, there is an isomorphism $\mathrm{Bl}_{\sigma_0}^{\mathbb{R}} X \xrightarrow{\cong} X \times \mathbb{S}^{k-1}$, where $\sigma_0\colon X \to E$ denotes the 0-section.*

Proof. Let $\{U_i\}_{i\in I}$ be some trivializing open cover of X such that the gluing morphisms $\phi_{ij}\colon (U_i \cap U_j) \times \mathbb{R}^k \to (U_j \cap U_i) \times \mathbb{R}^k$ are all given by multiplication by a positive scalar function of x, i.e. $(x, \mathbf{v}) \mapsto (x, \lambda_{ij}(x)\mathbf{v})$ where $\lambda_{ij}(x) > 0$. Let $\rho\colon \mathrm{Bl}_{\sigma_0}^{\mathbb{R}} X \to X$ denote the blow up map and let $V_i = \rho^{-1}(U_i)$ We have $U_i \times S^{k-1} = V_i$ by definition of the real oriented blow up. The gluing morphism $g_{ij}\colon V_i \cap V_j \to V_i \cap V_j$ is given by $g_{ij}(x, s) \mapsto (x, \frac{\phi_{ij}(x,s)}{|\phi_{ij}(x,s)|})$. Since $\phi_{ij}(x, s) = \lambda_{ij}(x) \cdot s$ where $\lambda_{ij}(x) > 0$ we have $\frac{\phi_{ij}(x,s)}{|\phi_{ij}(x,s)|} = s$ and therefore g_{ij} is the identity function. Hence, the local isomorphisms $U_i \times \mathbb{S}^{k-1} \xrightarrow{\cong} V_i$ glue to give an isomorphism $X \times \mathbb{S}^{k-1} \xrightarrow{\cong} \mathrm{Bl}_{\sigma_0}^{\mathbb{R}} X$.

□

Lemma 3.20. *If $E \to X$ is a real or complex bundle glued by positive scalar multiplication then so is the dual bundle $E^\vee$.*

Proof. I will prove this in the real case. The complex is analogous. Let $\{U_i\}_{i\in I}$ be some trivializing open cover of X such that the gluing morphisms of E, $\phi_{ij}\colon (U_i \cap U_j) \times \mathbb{R}^k \to (U_j \cap U_i) \times \mathbb{R}^k$ are all given by multiplication by a positive scalar function of x, i.e. $(x, \mathbf{v}) \mapsto (x, \lambda_{ij}(x)\mathbf{v})$ where $\lambda_{ij}(x) > 0$. By construction the gluing maps of the dual bundle send

$$\phi_{ij}^\vee\colon (U_i \cap U_j) \times \hom \mathbb{R}^k, \mathbb{R} \to (U_j \cap U_i) \times \hom \mathbb{R}^k, \mathbb{R}, (x, f) \mapsto (x, f \circ \phi_{ij}^{-1}(x, -)).$$

Since $f \circ \phi_{ij}^{-1}(x, -) = \frac{1}{\lambda_{ij}(x)} f$, $E^\vee$ is also glued by positive scalar multiplication. □

Lemma 3.21. *Let $g_k\colon E_k \to X$, $1 \le k \le n$ be a collection of complex line bundles all of which are glued by positive scalar multiplication and let $e_1, \ldots, e_n$ be integers. Then the complex tensor product $\bigotimes E_k^{\otimes e_k}$ also has the property that the gluing morphisms are all given by multiplication by a real, positive, scalar function of x, i.e. $(x, z) \mapsto (x, \lambda_{ij}^k(x)z)$ where $\lambda_{ij}^k(x) \in \mathbb{R}_{>0}$.*

Proof. First note that by lemma 3.20 we may without loss of generality assume that $e_i \ge 0$. If this is true for $n = 2$, $e_1 = e_2 = 1$, then the general case follows by induction. For this case let E_1, E_2 be complex line bundles glued by positive scalar multiplication. Let $\{U_i\}_{i\in I}$ be an open cover of X trivializing both bundles such that the gluing morphisms for each E_k, $\phi_{ij}^k\colon (U_i \cap U_j) \times \mathbb{C} \to (U_j \cap U_i) \times \mathbb{C}$ are both given by multiplication by a real, positive, scalar function of x.

Then, $E_1 \otimes E_2$ will, by definition, also be trivial on U_i and the gluing maps will, again by definition of the tensor product, be given by $(x, z) \mapsto (x, \lambda^1_{ij}(x)\lambda^2_{ij}(x)z)$. Clearly $\lambda^1_{ij}(x)\lambda^2_{ij}(x) \in \mathbb{R}_{>0}$ and thus we are done. $\square$

Remark. Here $E^{\otimes(-m)}$ is taken to mean $(E^\vee)^{\otimes m}$.

Lemma 3.22. *Let $\pi\colon E \to X$ be a vector bundle glued by positive scalar multiplication and let $f\colon Y \to X$ be any continuous function. Then the pullback bundle $f^*\pi\colon f^*E \to Y$ is also glued by positive scalar multiplication.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of X such that for each $i, j \in I$ the gluing map of E, $\phi_{ij}\colon (U_i \cap U_j) \times \mathbb{R}^k \to (U_j \cap U_i) \times \mathbb{R}^k$ is given by $(x, \mathbf{v}) \mapsto (x, \lambda_{ij}(x)\mathbf{v}$ where $\lambda_{ij}(x)$ is a positive, real, scalar. Let $V_i = f^{-1}(U_i)$. By definition of the pullback, $(f^*\pi)^{-1}(V_i) \cong V_i \times \mathbb{R}^k$ and the gluing maps are given by

$$f^*\phi_{ij}\colon (y, \mathbf{v}) \mapsto (y, \phi_{ij}(f(y), \mathbf{v}) = (y, \lambda_{ij}(f(y))\mathbf{v}).$$

Since $\lambda_{ij}(f(y)) > 0$ we are done. $\square$

Lemma 3.23. *Let $\sigma\colon X \to E$ be a section of a complex line bundle on a space X. Then the pullback bundle of E on the blow-up $\rho\colon \mathrm{Bl}^{\mathbb{R}}_\sigma X \to X$, ρ^*E, is (isomorphic to a bundle) glued by positive scalar multiplication.*

Proof. Let $\{U_i\}_{i\in I}$ be a trivializing open cover of E with gluing morphisms $\phi_{ij}\colon (U_i \cap U_j) \times \mathbb{C} \to (U_j \cap U_i) \times \mathbb{C}$ and let $\sigma_i\colon U_i \to \mathbb{C}$ be the restriction of σ to U_i (composed with the projection to $\mathbb{C}$). Notice that since E is a complex line bundle ϕ_{ij} must send $(x, z) \mapsto \lambda_{ij}(x)z$ where $\lambda_{ij}(x) \in \mathbb{C}^*$. Then, if we let $V_i = \rho^{-1}(U_i)$, there are isomorphisms

$$f_i\colon V_i \to \{(x, e^{i\theta}) \in X \times \mathbb{S}^1 | \sigma_i(x) = |\sigma_i(x)|e^{i\theta}\}$$

and the gluing morphisms $g_{ij} = f_j \circ f_i^{-1}$ are given by $(x, e^{i\theta}) \mapsto (x, \frac{\lambda_{ij}(x)}{|\lambda_{ij}(x)|}e^{i\theta})$. By definition of the pullback bundle, ρ^*E is trivial on V_i and the gluing morphisms of ρ^*E are given by

$$\phi^*_{ij}\colon (V_i \cap V_j) \times \mathbb{C} \to (V_i \cap V_j) \times \mathbb{C}, (p, z) \mapsto (p, \phi_{ij}(\rho(p), z))$$

by definition of the pullback bundle. Now, we can define new trivializing homeomorphisms of ρ^*E by composing the ones we have with

$$h_i\colon V_i \times \mathbb{C} \to V_i \times \mathbb{C}, (p, z) \mapsto (p, ze^{-i\theta})$$

where θ is defined by $f_i(p) = (\rho(p), e^{i\theta})$. With these trivializing homeomorphisms the gluing maps become $h_j \circ \phi^*_{ij} \circ h_i^{-1}$. Since $g_{ij} = f_j \circ f_i^{-1}\colon (x, e^{i\theta}) \mapsto (x, \frac{\lambda_{ij}(x)}{|\lambda_{ij}(x)|}e^{i\theta})$ we have

$$h_j \circ \phi^*_{ij} \circ h_i^{-1}\colon (p, z') \mapsto (p, \lambda_i j(\rho(p))z'e^{i\theta} \cdot \frac{|\lambda_{ij}(\rho(p))|}{\lambda_{ij}(\rho(p))e^{i\theta}}) = (p, |\lambda_{ij}(\rho(p))|z').$$

Since $|\lambda_{ij}(\rho(p))| \in \mathbb{R}_{>0}$ we are done. $\square$

Proposition 3.24. *Let $L_1, L_2, \ldots, L_n$ be complex line bundles on a space X and let $L = \bigotimes_{i=1}^n L_i^{\otimes e_i}$ where e_i are integers and $\otimes$ is the complex tensor product. Let $\sigma_1, \ldots, \sigma_n$ be sections $\sigma_i\colon X \to L_i$. Then, there is an isomorphism*

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde\sigma_0} \mathrm{Bl}^{\mathbb{R}}_{\tilde\sigma_n} \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde\sigma_1} X \xrightarrow{\cong} \left(\mathrm{Bl}^{\mathbb{R}}_{\tilde\sigma_n} \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde\sigma_1} X\right) \times \mathbb{S}^1$$

where $\tilde\sigma_i$ denotes the pullback of σ_i through all previous morphisms and $\sigma_0\colon X \to L$ is the 0 section.

Proof. By lemma 3.23 and lemma 3.22 the pullbacks of the bundles L_i to $\mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_n} \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\sigma}_1}$ are glued by positive scalar multiplication. Since L is a tensor product of these bundles the pullback of L is also glued by positive scalar multiplication by lemma 3.21. Now the proposition immediately follows from lemma 3.19. □

4 Fulton-MacPherson and Kontsevich Spaces

In this chapter I will introduce the topological Fulton-MacPherson spaces, $\mathrm{FM}_n(\mathbb{R}^d)$, and the closely related Kontsevich spaces, $\mathcal{K}_{d,n}$. The Fulton-MacPherson spaces are compactifications of the configuration spaces $\mathrm{Conf}_n(\mathbb{R}^d)$ or more accurately they are spaces with a dense open embedding from $\mathrm{Conf}_n(\mathbb{R}^d)$ such that the closure of any bounded set in $\mathrm{Conf}_n(\mathbb{R}^d)$ is compact. The Kontsevich spaces compactify the quotients $\mathrm{Conf}_n(\mathbb{R}^d)/H_d$ where H_d is the group of homotheties and translations in $\mathbb{R}^d$. Although this chapter treats the manifold theoretic version of the Fulton-MacPherson spaces it is worth mentioning that this type of construction first appeared in an algebro-geometric context in the now famous article "A compactification of configuration spaces" by Fulton and MacPherson [FM94]. After defining these spaces I will define an operad with objects $\{\mathcal{K}_{d,n}\}_{n\in\mathbb{N}}$, for every d. Finally, I will define a group action by the $SO(d)$ group on this operad and, when d is even, also an action by the $\mathbb{S}^1$ group. The main goal of this thesis is to show that, in even dimensions, the semidirect product of this operad with the $\mathbb{S}^1$ topological group isomorphic to the Kato-Nakayama analytification of an operad of log schemes.

4.1 The Fulton-MacPherson Compactification

In this section I will construct the Fulton-MacPherson configuration spaces. The Kontsevich spaces will then be constructed as subspaces of these. These constructions are well known and appear in several different contexts. Much of the theory regarding these spaces was developed by Sinha in the article "Manifold-theoretic compactifications of configuration spaces" [Sin04] who in particular introduced type the pictorial notation seen in this section. In what follows I will assume d, n to be fixed positive integers so that I may omit them from the notation when convenient.

The first part of the construction is to define maps $\alpha_S\colon \mathrm{Conf}_n(\mathbb{R}^d) \to \mathbb{S}^{(|S|-1)d-1}$ for every $S \subseteq [n]$ with $|S| \geq 2$. To do this, first let $\sim$ be the equivalence relation on $(\mathbb{R}^d)^m \setminus \Delta_m$ given by $x \sim y$ if there is a $\lambda > 0$ such that $x - \lambda y \in \Delta_m$. Here Δ_m denotes the small diagonal in $(\mathbb{R}^d)^m$, i.e. the set of points $(x_1, \dots, x_m)$ such that $x_1 = x_2 = \dots = x_m$. From the definition of the N-sphere it is easy to see that the quotient $((\mathbb{R}^d)^m \setminus \Delta_m)/\sim$ is homeomorphic to $\mathbb{S}^{(m-1)d-1}$. This gives a continuous function $\alpha_m\colon \mathrm{Conf}_m(\mathbb{R}^d) \to \mathbb{S}^{(m-1)d-1}$ by composing

$$\mathrm{Conf}_m(\mathbb{R}^d) \hookrightarrow (\mathbb{R}^d)^m \setminus \Delta_m \twoheadrightarrow ((\mathbb{R}^d)^m \setminus \Delta_m)/\sim \xrightarrow{\cong} S^{(m-1)d-1}.$$

Next, for a subset $S \subseteq [n]$ let $p_S\colon X^n \to X^{|S|}$ denote the projection map onto the coordinates indexed by S, i.e. if $S = \{i_1, i_2, \dots, i_m\}$ where $i_1 < i_2 < \dots < i_m$ then $p_S(x_1, x_2, \dots, x_n) = (x_{i_1}, x_{i_2}, \dots, x_{i_m})$. Restricting to $\mathrm{Conf}_n(\mathbb{R}^d) \subseteq (\mathbb{R}^d)^n$ this gives a function $p_S\colon \mathrm{Conf}_n(\mathbb{R}^d) \to \mathrm{Conf}_{|S|}(\mathbb{R}^d)$. We define α_S as the composition $\alpha_S = \alpha_{|S|} \circ p_S$. A subtle but important thing to note here is that while S is just a set, p_S actually depends on the order of S since we are projecting to $X^{|S|}$ and not $Sym_{|S|}X$. As indicated above p_S is the projection given by the set S with elements appearing in increasing order.

Let $P(n)$, or just P when n is clear from context, denote the set of subsets of $[n]$ with at lest 2 elements, e.g. $P(3) = \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ and let $i\colon \mathrm{Conf}_n(\mathbb{R}^d) \to (\mathbb{R}^d)^n$ denote the canonical inclusion.

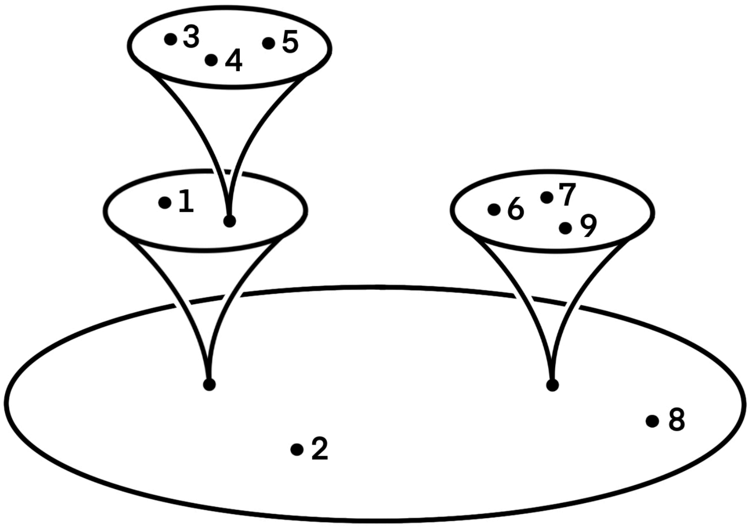


Figure 1: Configuration corresponding to a point in $\mathrm{FM}_9(\mathbb{R}^d)$

Definition 4.1. The *Fulton-MacPherson configuration space*, $\mathrm{FM}_n(\mathbb{R}^d)$ is the closure of the image of the map

$$i \times \prod_{S \in P} \alpha_S \colon \mathrm{Conf}_n(\mathbb{R}^d) \to (\mathbb{R}^d)^n \times \prod_{S \in P} \mathbb{S}^{(|S|-1)d-1}.$$

Let $\rho\colon \mathrm{FM}_n(\mathbb{R}^d) \to (\mathbb{R}^d)^n$ denote the restriction of the projection map $(\mathbb{R}^d)^n \times \prod_{S\in P} \mathbb{S}^{(|S|-1)d-1} \to (\mathbb{R}^d)^n$ and similarly let $\pi_S\colon \mathrm{FM}_n(\mathbb{R}^d) \to \mathbb{S}^{(|S|-1)d-1}$ denote the restriction of the projection to the $\mathbb{S}^{(|S|-1)d-1}$-component corresponding to the set $S \in P$. Finally, let $j\colon \mathrm{Conf}_n(\mathbb{R}^d) \hookrightarrow \mathrm{FM}_n(\mathbb{R}^d)$ denote the dense embedding of $\mathrm{Conf}_n(\mathbb{R}^d)$.

Remark. For an arbitrary manifold with an embedding $M \hookrightarrow \mathbb{R}^d$ one can define $\mathrm{FM}_n(M)$ as the closure of the image of $\mathrm{Conf}_n(M) \subseteq \mathrm{Conf}_n(\mathbb{R}^d)$. If M is compact this image will also be compact which is why this construction is often called the Fulton-MacPherson compactification. See Sinha [Sin04] for more details.

There is an oftentimes useful, intuitive, way to think of the points in this space. The picture you should have in mind is that a point in $\mathrm{FM}_n(\mathbb{R}^d)$ is a configuration of n (labeled) points in $\mathbb{R}^d$ such that two or more points are allowed to be equal but if a collection of k points "meet" then we must also give k points in $\mathbb{R}^d$, up to scaling and translation, specifying the positions of these k-points "relative to each other" such that not all of them are equal. If the relative positions of some of the k-points are also equal then we must furthermore specify their relative position up to translation and positive scaling and so on. Picture 1 illustrates an example of this picture of a point in $\mathrm{FM}_9(\mathbb{R}^d)$. In the point illustrated to this picture the points indexed by $1, 3, 4, 5$ meet in the "bottom layer" and so do the points indexed by $6, 7, 9$. Therefore, the relative positions of these groups of points are specified in a second layer of the figure. Furthermore, in the space of relative positions of the points indexed by $1, 3, 4, 5$ the points indexed by $3, 4, 5$ meet so we add an additional "layer" specifying their relative positions.

Definition 4.2. The *Kontsevich space* of n points in d dimensions, $\mathcal{K}_{d,n}$, is defined as the fiber $\rho^{-1}((0, 0, \ldots, 0)) \subseteq \mathrm{FM}_n(\mathbb{R}^d)$.

Remark. I choose the origin here for convenience but the fiber over any point on the small diagonal in $(\mathbb{R}^d)^n$

yields the same space.

Intuitively one should think of a point in $\mathcal{K}_{d,n}$ the same way we think of points in $\mathrm{FM}_n(\mathbb{R}^d)$ but this time the positions of the n points in the "bottom layer" are also only specified up to positive scaling and translation.

Even though the restriction of ρ to $\mathcal{K}_{d,n}$ is not particularly interesting the restrictions of the maps π_S are still non-trivial. I will use π_S to denote these restrictions too by abuse of notation.

The Fulton-MacPherson configuration spaces can also be expressed as a sequence of real oriented blow-ups of $(\mathbb{R}^d)^n$. Furthermore, although the order of the blow ups cannot be performed arbitrarily, there are several different orders in which the blow-ups can be performed. More details on which orders are allowed and why can be found in the 2003 article "Models for real subspace arrangements and stratified manifolds" by Gaiffi [Gai03].

Let B_n be a sequence whose elements are the subsets of $[n]$ of size ≥ 2, such that $|B_n(i)| \geq |B_n(j)|$ whenever $i \leq j$. That is B_n is any sequence which starts with $[n]$ and then the next elements in the sequence are the subsets of $[n]$ of size $n-1$, and so on, then we have.

Proposition 4.3. *There is an isomorphism between* $\mathrm{FM}_n(\mathbb{R}^d)$ *and the space*

$$\mathrm{Bl}^{\mathbb{R}}_{\Delta(B_n(2^n-(n+1)))} \dots \mathrm{Bl}^{\mathbb{R}}_{\Delta(B_n(2))} \mathrm{Bl}^{\mathbb{R}}_{\Delta(B_n(1))} (\mathbb{R}^d)^n$$

where, in each blow-up $\Delta(B_n(i))$ *denotes the strict transform of the diagonal* $\Delta(B(i))$ *under all previous blow-ups. Furthermore, under this isomorphism the map* $\rho\colon \mathrm{FM}_n(\mathbb{R}^d) \to (\mathbb{R}^d)^n$ *is identified with the composition of all blow-up morphisms and furthermore the closed subspace* $\mathrm{FM}_n(\mathbb{R}^d)(S)$ *is identified with the space we get by taking the dominant transform of the* $\Delta(S)$ *diagonal in each blow-up.*

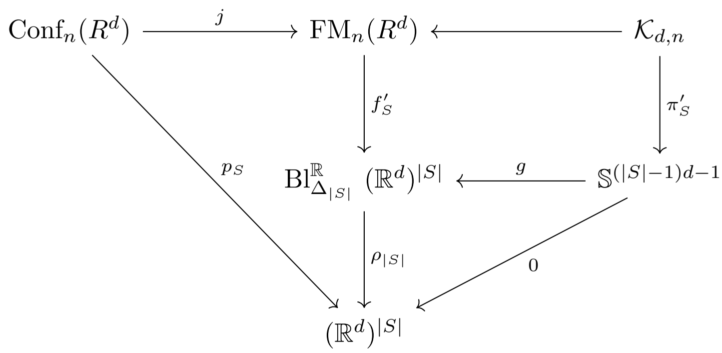
Proof. See [Gai03]. □

Before I end this section I will state and prove one last result. This will not seem particularly interesting at the moment but it will be very important in the last parts of the thesis. First, note that the blow up of $(\mathbb{R}^d)^m$ in the small diagonal Δ_m is a closed subset $\mathrm{Bl}^{\mathbb{R}}_{\Delta_m} (\mathbb{R}^d)^m \subseteq (\mathbb{R}^d)^m \times \mathbb{S}^{(m-1)d-1}$. This is clear from definition of the real oriented Blow-Up in a complete intersection. Next, note that the image of the function

$$f_S = (p_S \circ \rho) \times \pi_S \colon \mathrm{FM}_n(\mathbb{R}^d) \to (\mathbb{R}^d)^{|S|} \times \mathbb{S}^{(|S|-1)d-1}$$

is contained in $\mathrm{Bl}^{\mathbb{R}}_{\Delta_m} (\mathbb{R}^d)^m$. This is clear since this is obviously true for $f_S \circ j$ and the image of $j\colon \mathrm{Conf}_n(\mathbb{R}^d) \hookrightarrow \mathrm{FM}_n(\mathbb{R}^d)$ is dense.

Lemma 4.4. *The following diagram commutes if and only if* $f'_S = f_S$ *and* $\pi'_S = \pi_S$.



Here g is the inclusion of the fiber over the origin of the blow up $\rho_{|S|}\colon \mathrm{Bl}^{\mathbb{R}}_{\Delta_{|S|}}(\mathbb{R}^d)^{|S|} \to (\mathbb{R}^d)^{|S|}$.

Proof. It is clear by definitions that the diagram commutes for $f'_S = f_S$ and $\pi'_S = \pi_S$. Furthermore, since $\mathcal{K}_{d,n} \to \mathrm{FM}_n(\mathbb{R}^d)$ and $g\colon \mathbb{S}^{(|S|-1)d-1} \to \mathrm{Bl}^{\mathbb{R}}_{\Delta_{|S|}}(\mathbb{R}^d)^{|S|}$ are embeddings it is also clear that f'_S uniquely determines π'_S. Finally, $\rho_{|S|}$ is a homeomorphism when restricted to the inverse image of $(\mathbb{R}^d)^{|S|} \setminus \Delta_{|S|}$. Since the image of p_S is contained in $(\mathbb{R}^d)^{|S|} \setminus \Delta_{|S|}$ this means that $f'_S \circ j$ is uniquely determined, i.e. $f'_S \circ j = f_S \circ j$. Since the image of j is dense in $\mathrm{FM}_n(\mathbb{R}^d)$ it follows that $f_S = f'_S$. □

4.2 The Kontsevich Operad

For a fixed dimension d the collection $\{\mathcal{K}_{d,n}\}_{n\in\mathbb{N}}$ can be given the structure of a topological operad as follows.

First, let

$$(\{(u_S)\}_{S\in P(n)}) \in \prod_{S\in P(n)} ((\mathbb{R}^d)^S \setminus \Delta_S)/\sim$$

denote a point in $\mathcal{K}_{d,n} \subseteq \prod_{S\in P(n)}((\mathbb{R}^d)^S \setminus \Delta_S)/\sim$, where, for each $S = \{i_1, \dots, i_{|S|}\}$ ordered such that $i_1 < \cdots < i_{|S|}$, we write

$$u_S = (u^S_{i_1}, u^S_{i_2}, \dots, u^S_{i_{|S|}}) \in ((\mathbb{R}^d)^S \setminus \Delta_S)/\sim .$$

A permutation $\sigma \in \Sigma_n$ acts on $\mathrm{FM}_n(\mathbb{R}^d)$ by sending $(\{(u_S)\}_{S\in P(n)}) \mapsto (\{(v_S)\}_{S\in P(n)})$ where

$$v_S = (u^{\sigma^{-1}(S)}_{\sigma^{-1}(i_1)}, u^{\sigma^{-1}(S)}_{\sigma^{-1}(i_2)}, \dots, u^{\sigma^{-1}(S)}_{\sigma^{-1}(i_{|S|})}).$$

Note that this is not the same as $v_S = v_{\sigma^{-1}(S)}$ since the coordinates appear in a different order if σ^{-1} does not preserve the order of S. An important remark here is that this is actually the restriction of an action on $\mathrm{FM}_n(\mathbb{R}^d)$. Specifically, this is the restriction of the action where $\sigma \in \Sigma_n$ acts on $\mathrm{FM}_n(\mathbb{R}^d)$ by sending the point

$$(x_1, x_2, \dots, x_n, (\{(u_S)\}_{S\in P(n)})) \in (\mathbb{R}^d)^n \times \prod_{S\in P(n)} ((\mathbb{R}^d)^S \setminus \Delta_S)/\sim$$

to

$$(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)}, (\{(u_S)\}_{S\in P(n)})) \in (\mathbb{R}^d)^n \times \prod_{S\in P(n)} ((\mathbb{R}^d)^S \setminus \Delta_S)/\sim .$$

Furthermore, we have that the following holds.

Lemma 4.5. *For any $\sigma \in \Sigma_n$ the following diagram commutes if and only if f, g are the permutation maps described above when the leftmost arrow is the permutation of coordinates on $\mathrm{Conf}_n(\mathbb{R}^d)$ corresponding to σ.*

$$\begin{array}{ccccc} \mathrm{Conf}_n(\mathbb{R}^d) & \longrightarrow & \mathrm{FM}_n(\mathbb{R}^d) & \longleftarrow & \mathcal{K}_{d,n} \\ \downarrow & & \downarrow f & & \downarrow g \\ \mathrm{Conf}_n(\mathbb{R}^d) & \longrightarrow & \mathrm{FM}_n(\mathbb{R}^d) & \longleftarrow & \mathcal{K}_{d,n} \end{array}$$

Proof. The proof is analogous to the proof of lemma 4.4. □

Next, to define the composition maps we first define, for a positive integer n and a collection of n positive integers $\mathbf{m} = (m_1, \dots, m_n)$, the two functions of integers

$$p^{n,\mathbf{m}} \colon a \mapsto \begin{cases} 1 & 0 < a \leq m_1 \\ 2 & m_1 < a \leq m_2 \\ \vdots & \\ n & \sum_{i<n} m_i < a \leq \sum_{i \leq n} m_i \end{cases}$$

and, for $1 \leq r \leq n$

$$q_r^{n,\mathbf{m}} \colon a \mapsto a - \sum_{i<r} m_i.$$

When $n, \mathbf{m}$ are clear from context I just use p, q_r to denote these. The composition maps

$$\gamma^{n,\mathbf{m}} \colon \mathcal{K}_{d,n} \times \mathcal{K}_{d,m_1} \times \cdots \times \mathcal{K}_{d,m_n} \to \mathcal{K}_{d,m}$$

are defined by sending

$$(\{(u_S^0)\}_{S \in P(n)}), (\{(u_S^1)\}_{S \in P(m_1)}), \dots, (\{(u_S^n)\}_{S \in P(m_n)}) \mapsto (\{(v_S)\}_{S \in P(m)})$$

where, if $S = \{i_1, i_2, \dots, i_{|S|}\} \subseteq [m]$, we set

$$v_S = \begin{cases} u^r_{q_r(S)} & p(S) = \{r\} \\ (x^{p(S)}_{p(i_1)}, x^{p(S)}_{p(i_2)}, \dots, x^{p(S)}_{p(i_{|S|})}) & \text{else} \end{cases}.$$

Here $x_j^{p(S)}$ are the coordinates of $u^0_{p(S)}$. It is easy to verify that this map is well defined, i.e. maps equivalence classes to each other and has image contained in $\mathcal{K}_{d,m}$. We will later need one more lemma here in the style of 4.5 and lemma 4.4.

Lemma 4.6. *Let $S = (s_1, s_2, \dots, s_k) \subseteq [m]$ be such that $R = p^{n,\mathbf{m}}(S) = (r_1, r_2, \dots, r_l)$ has two or more elements., then there is exactly one function $g_S \colon \mathbb{S}^{(|R|-1)d-1} \to \mathbb{S}^{(|S|-1)d-1}$ making the following diagram commute.*

$$
\begin{array}{ccc}
\mathbb{S}^{(|R|-1)d-1} & \xrightarrow{\;g_S\;} & \mathbb{S}^{(|S|-1)d-1} \\
\big\downarrow{\scriptstyle i_R} & & \big\downarrow{\scriptstyle i_S} \\
\mathrm{Bl}^{\mathbb{R}}_{\Delta_{|R|}}(\mathbb{R}^d)^R & \longrightarrow & \mathrm{Bl}^{\mathbb{R}}_{\Delta_R}(\mathbb{R}^d)^{|R|} \\
\big\downarrow{\scriptstyle \rho_S} & & \big\downarrow{\scriptstyle \rho_S} \\
(\mathbb{R}^d)^R & \xrightarrow{\;d_S\;} & (\mathbb{R}^d)^S
\end{array}
$$

Where i_R, i_S are inclusions over the origin and d_S is the map given by

$$(x_{r_1}, \dots, x_{r_l}) \mapsto (x_{p^{n,\mathbf{m}}(s_1)}, x_{p^{n,\mathbf{m}}(s_2)}, \dots, x_{p^{n,\mathbf{m}}(s_k)}).$$

Furthermore, if $A_S\colon \mathcal{K}_{d,n} \times \mathcal{K}_{d,m_1} \times \cdots \times \mathcal{K}_{d,m_n} \to \mathbb{S}^{(|R|-1)d-1}$ is the function composition of the projection $\mathcal{K}_{d,n} \times \mathcal{K}_{d,m_1} \times \cdots \times \mathcal{K}_{d,m_n} \to \mathcal{K}_{d,n}$ with π_R then the above function g_S (and only this function) makes the following diagram commute.

$$
\begin{array}{ccc}
\mathcal{K}_{d,n} \times \mathcal{K}_{d,m_1} \times \cdots \times \mathcal{K}_{d,m_n} & \xrightarrow{\;\gamma^{n,\mathbf{m}}\;} & \mathcal{K}_{d,m} \\
\big\downarrow{\scriptstyle A_S} & & \big\downarrow{\scriptstyle \pi_S} \\
\mathbb{S}^{(|R|-1)d-1} & \xrightarrow{\;g_S\;} & \mathbb{S}^{(|S|-1)d-1}
\end{array}
$$

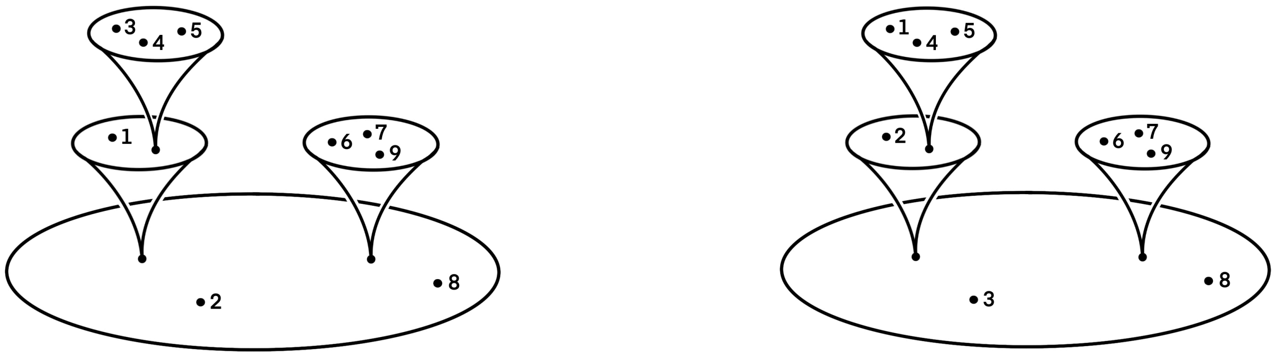
Proof. By the same argument as in the proof of lemma 4.4 there is only one g_S making the first diagram commute. It is easy to see that this also makes the second diagram commute through explicit computations. □

Finally, note that $\mathcal{K}_{d,1}$ by construction is just the one point space so there is only one possible choice for the identity map, $\eta\colon * \to \mathcal{K}_{d,1}$, namely the identity function.

Proposition/Definition 4.7. *$\{\mathcal{K}_{d,n}\}$ with the above composition, symmetry, and identity maps is an operad. I will refer to this as the Kontsevich operad, or the topological Kontsevich operad, of dimension d, denoted $\mathcal{K}_d$.*

Proof. This follows from tedious direct computations. I will leave the details as an exercise. □

There is of course also an intuitive way to think of this operad. The symmetry action maps a configuration of labeled points to the same configuration of points but with their labels permuted. Figure 2 illustrates the way a point in $\mathcal{K}_{d,9}$ is mapped by the 3-cycle $(123) \in \Sigma_9$. The composition maps $\mathcal{K}_{d,n} \times \mathcal{K}_{d,m_1} \times \cdots \times \mathcal{K}_{d,m_n} \to \mathcal{K}_{d,m}$ "attach" the collections of points in $\mathcal{K}_{d,m_r}$ to the point indexed by r in a collection of points in $\mathcal{K}_{d,n}$. Figure 3 illustrates an example of how three "collections of points" are mapped by the composition map $\mathcal{K}_{d,2} \times \mathcal{K}_{d,3} \times \mathcal{K}_{d,4} \to \mathcal{K}_{d,7}$.



(a) Point in $\mathcal{K}_{d,9}$ (b) Point in $\mathcal{K}_{d,9}$ permuted by (123)

Figure 2: Example of the symmetry action

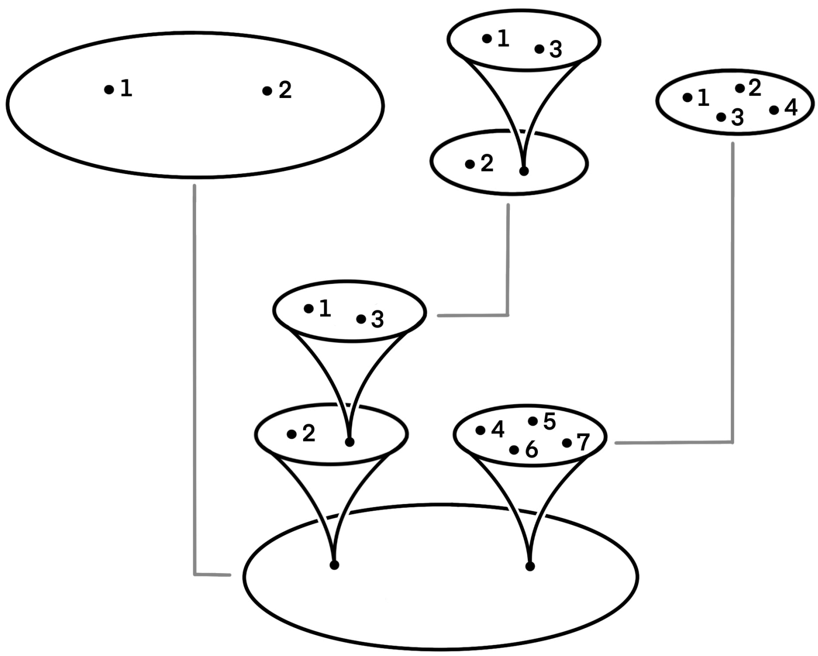


Figure 3: Example of the composition map

For some context regarding why this is an interesting operad it is well known that the Kontsevich operad is weakly equivalent to the operad of little d dimensional disks. See for example [Sal99, proposition 3.9], for a proof.

Now, let $a_m \colon SO(d) \times ((\mathbb{R}^d)^m \setminus \Delta_m)/\sim \to ((\mathbb{R}^d)^m \setminus \Delta_m)/\sim$ be the action of $SO(d)$ sending

$$(R, (x_1, x_2, \dots, x_m)) \mapsto (R(x_1), \dots, R(x_m)).$$

These actions induce an action on $\prod_{S \in P} \mathbb{S}^{(|S|-1)d-1}$ by taking the following composition

$$
\begin{array}{c}
SO(d) \times \prod_{S\in P} \mathbb{S}^{(|S|-1)d-1} \\
\big\downarrow \Delta_{|P|}\times \mathrm{id} \\
(SO(d))^{|P|} \times \prod_{S\in P} \mathbb{S}^{(|S|-1)d-1} \\
\big\downarrow \text{rearrange} \\
\prod_{S\in P} (SO(d) \times \mathbb{S}^{(|S|-1)d-1}) \\
\big\downarrow \prod_{S\in P} a_{|S|} \\
\prod_{S\in P} \mathbb{S}^{(|S|-1)d-1}
\end{array}
$$

where $\Delta_{|P|} \colon SO(d) \to SO(d)^{|P|}$ is the diagonal inclusion.

Proposition 4.8. *The above action restricts to an action on $\mathcal{K}_{d,n} \subseteq \prod_{S\in P} \mathbb{S}^{(|S|-1)d-1}$ and furthermore this gives an $SO(d)$ action on the Kontsevich operad.*

Proof. This also follows from tedious direct computations. □

While the semidirect product of the Kontsevich operad $\mathcal{K}_d$ with $SO(d)$ is arguably more interesting this is not the operad studied in this thesis. Instead we are interested in the following construction. When d is an even number $d = 2m$ there is a canonical embedding of topological groups $\mathbb{S}^1 \cong SO(2) \hookrightarrow SO(d)$ which sends

$$
R \mapsto \underbrace{R \oplus R \oplus \cdots \oplus R}_{m \text{ times}}.
$$

This induces an action of $SO(2)$ on the $\mathcal{K}_{2m}$ operad and thus we can form the semidirect product $\mathcal{K}_d \rtimes \mathbb{S}^1$.

Definition 4.9. The $\mathbb{S}^1$*-framed Kontsevich Operad in dimension* $2m$ is the semidirect product $\mathcal{K}_{2m} \rtimes \mathbb{S}^1$.

The main goal of this thesis is to construct a non-unital operad of log schemes whose Kato-Nakayama analytification is $\mathcal{K}_{2m} \rtimes \mathbb{S}^1$ for each positive integer m.

It is worth noting here that the weak equivalence between the Kontsevich operad, $\mathcal{K}_d$ and the framed little disks operad D_d constructed in [Sal99] "commutes" with the group action by $SO(d)$ on the respective operads. This implies that $\mathcal{K}_{2m} \rtimes \mathbb{S}^1$ is weakly equivalent to $D_{2m} \rtimes \mathbb{S}^1$ which means that the isomorphism in theorem 7.35 also implies that the semidirect product $D_d \rtimes \mathbb{S}^1$ (without unit) is weakly equivalent to the Kato-Nakayama analytification of the operad without unit of log schemes defined in this thesis. While commutativity with the $SO(d)$ action is straight forward to verify directly from Salvatores proof of the weak equivalence I do not know of any references for this. As such, I have decided not to include this result as a theorem in the thesis but it is still worth mentioning.

5 Logarithmic Algebraic Geometry

In this chapter I will give an introduction to logarithmic algebraic geometry and to the Kato-Nakayama analytification functor which was introduced by Kato and Nakayama in their article "Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over C"[KN99]. Those already familiar with logarithmic algebraic geometry might find this introduction unsatisfactory as I will give a simplified definition of a log-scheme is. In actuality I will be defining a so called Deligne-Faltings log scheme or a DF log scheme.

Furthermore, the definition of a DF log scheme I give here might look different from, but hopefully be equivalent to, definitions of a DF log scheme found in other sources. Large parts of this chapter are more or less a reformulation of sections 8.2 – 8.4 in the article "Hyperelliptic Curves, the Scanning Map, and Moments of Families of Quadratic L-Functions" by Bergström, Diaconu, Petersen, Westerland [BDPW23].

5.1 The Category of Log-Schemes

We begin by defining the category of log structures on a scheme X and then move on to defining the category of log-schemes.

Definition 5.1. A *log-structure* on a scheme X is a finite tuple $\mathfrak{L} = (s_i\colon \mathcal{O}_X \to \mathcal{L}_i)_{1\leq i\leq n}$ of invertible sheaves with sections. A morphism of log-structures on X,

$$(s_i\colon \mathcal{O}_X \to \mathcal{L}_i)_{1\leq i\leq n} \to (t_j\colon \mathcal{O}_X \to \mathcal{M}_j)_{1\leq j\leq m}$$

is a collection of n isomorphisms of sheaves

$$\mathcal{L}_i \xrightarrow{\cong} \bigotimes_{1\leq j\leq m} \mathcal{M}_j^{\otimes e_{ij}}$$

which also identify the sections s_i to the corresponding sections $\bigotimes_{1\leq j\leq m} t_j^{\otimes e_{ij}}$ and where $\{e\}_{ij}$ is some collection of non-negative integers.

Remark. In this definition we are using the convention that for any invertible sheaf with section $s\colon \mathcal{O}_X \to \mathcal{L}$ we have that $\mathcal{L}^{\otimes 0}$ is the structure sheaf $\mathcal{O}_X$ and the corresponding section $s^{\otimes 0}\colon \mathcal{O}_X \to \mathcal{O}_X$ is the identity section. Although this may seem a bit strange it is natural since the structure sheaf with the identity section is the identity object in the monoidal category of sheaves of modules with sections on X.

For those who prefer a categorical language there is also a different, equivalent, definition of a log-structure. Namely that a log structure is a functor between monoidal categories

$$\mathbb{N}^n \to C$$

where C is the monoidal category of invertible sheaves with sections where the product of two objects is their tensor product and where the identity is the structure sheaf with the identity section. A morphism between the log structures $\mathbb{N}^n \to C$ and $\mathbb{N}^m \to C$ is then a functor $\mathbb{N}^n \to \mathbb{N}^m$ such that the following diagram commutes up to a defined isomorphism

$$\begin{array}{ccc} \mathbb{N}^n & \longrightarrow & \mathbb{N}^m \\ & \searrow \quad \swarrow & \\ & C & \end{array}.$$

Although this is arguably a better definition I will stick to the language of the first definition.

Definition 5.2. An *inclusion* of log structures on X is a morphism of log-structures on X, $(s_i\colon \mathcal{O}_X \to \mathcal{L}_i)_{1\leq i\leq n} \to (t_j\colon \mathcal{O}_X \to \mathcal{M}_j)_{1\leq j\leq m}$ defined by sending $\mathcal{L}_i \cong \mathcal{M}_{f(i)}$ where $f\colon [n] \to [m]$ is an injective function.

Definition 5.3. The direct sum of two log structures $\mathfrak{L} = (s_i\colon \mathcal{O}_X \to \mathcal{L}_i)_{1\leq i\leq n}$ and $\mathfrak{M} = (t_j\colon \mathcal{O}_X \to \mathcal{M}_j)_{1\leq j\leq m}$

on X, denoted $\mathfrak{L} \oplus \mathfrak{M}$, is the log structure

$$((s_1 \colon \mathcal{O}_X \to \mathcal{L}_1), \dots, (s_n \colon \mathcal{O}_X \to \mathcal{L}_n), (t_1 \colon \mathcal{O}_X \to \mathcal{M}_1), \dots, (t_m \colon \mathcal{O}_X \to \mathcal{M}_m)).$$

Before we go on to define what a morphism of log-schemes is let us first give some simple examples and properties of log structures on schemes.

Example 5.4 (Log structures on fields). Let $X = \text{Spec } \Bbbk$ where k is any field. There is only one invertible sheaf on X, namely its structure sheaf $\mathcal{O}_X = \tilde{k}$. Furthermore, up to isomorphism, there are only two possible sections of this sheaf, the 0 section and the identity section. Thus any log structure on X is of the form

$$(\underbrace{\mathcal{O}_X \xrightarrow{id} \mathcal{O}_X, \dots, \mathcal{O}_X \xrightarrow{id} \mathcal{O}_X}_{n \text{ times}}, \underbrace{\mathcal{O}_X \xrightarrow{0} \mathcal{O}_X, \dots, \mathcal{O}_X \xrightarrow{0} \mathcal{O}_X}_{m \text{ times}}),$$

for some non negative integers n, m which uniquely determine the log structure. Notice that rearranging the sheaves with sections gives a canonically isomorphic log structure on X so these really are all possible log structures on X. I have one final remark to make regarding this example which the uninterested reader may safely skip. This remark is that we could have made the classification result above neater by altering the definition of what a log structure is a little bit. Specifically, the result would be a lot nicer if we could identify two log structures which are isomorphic except for the fact that one of them has a bunch of extra $\mathcal{O}_X \xrightarrow{id} \mathcal{O}_X$ elements included. This can be achieved in many ways. Out of those I could come up with within the very limited amount of time I have spent thinking about this the easiest is probably to simply redefine a log structure as an infinite tuple $(s_i \colon \mathcal{O}_X \to \mathcal{L}_i)_{i \in \mathbb{Z}}$ where all but finitely many of the invertible sheaves with sections are the structure sheaf with the identity section. This would make the category of log structures on the spectrum of a field equivalent to the category $\mathbb{N}_0$ and it would also make the classification of log structures on other schemes a little bit nicer. Alternatively we could simply not allow sections that are non vanishing in the definition of a log structure. However, these definition are, to my knowledge, not standard and since redefining concepts that are over 40 years old in a master's thesis would probably have a negative impact on readability we are stuck with this slightly uglier looking classification result.

Example 5.5 (Effective Cartier Divisors). Let X be an arbitrary scheme, let $\{D_i\}_{1 \le i \le n}$ be a set of effective Cartier divisors on X and let $D = \bigcup_{i=1}^n D_i$. Then there is a canonical morphism of log-structures

$$(s_D \colon \mathcal{O}_x \to \mathcal{O}_X(D)) \to (s_{D_i} \colon \mathcal{O}_x \to \mathcal{O}_X(D_i))_{1 \le i \le n}$$

given by the canonical isomorphism

$$\mathcal{O}_X(D) \cong \bigotimes_{i=1}^n \mathcal{O}_X(D_i).$$

The "idea" behind this example also provides some insight into morphisms of log schemes in general. For example, there can be no morphisms of log structures

$$(s_i \colon \mathcal{O}_X \to \mathcal{L}_i)_{1 \le i \le n} \to (t_j \colon \mathcal{O}_X \to \mathcal{M}_j)_{1 \le j \le m}$$

if the (scheme theoretic) union of closed subschemes cut out by the sections $(s_i)_{i=1}^n$ is not a closed subscheme of the (scheme theoretic) union of the closed subschemes cut out by the sections $(s_i)_{i=1}^n$. Similarly, there can also not be any such morphisms if there is an s_i such that the subscheme cut out by s_i is not the scheme theoretic union of subschemes cut out by powers of some of the sections t_j.

Example 5.6 (Pullback of log structures). Let $f\colon X \to Y$ be a morphism of schemes. Then a section of an invertible sheaf on Y, $s\colon \mathcal{O}_Y \to \mathcal{L}$ pulls back to a section of an invertible sheaf on X,

$$f^*s\colon \mathcal{O}_X \to f^*\mathcal{L}.$$

This means that a log structure $\mathfrak{L}$ on Y pulls back to a log structure $f^*\mathfrak{L}$ on X. Similarly a morphism of log structures on Y also pulls back to a morphism of log structures on X since pullback commutes with tensor product. Therefore f induces a functor from the category of log structures on Y to the category of log structures on X.

Motivated by example 5.4 we can introduce the following notion of equivalence of log structures.

Definition 5.7. Let $\mathfrak{L} = (s_i\colon \mathcal{O}_X \to \mathcal{L}_i)_{1\leq i\leq n}$ be a log structure on X. A map $\phi\colon \mathfrak{L} \to \mathfrak{L}$ is said to be *essentially the identity*, denoted $\phi \sim 1$, if ϕ sends each $\mathcal{L}_i$ with section either to itself via the identity map $\mathrm{id}\colon \mathcal{L}_i \to \mathcal{L}_i$ or to the "empty tensor product" $\mathcal{L}_i \to \bigotimes_{1\leq i\leq n} \mathcal{L}_i^{\otimes 0} = \mathcal{O}_X$ (where $s_i \mapsto 1$).

Remark. Note that the latter case is only possible if $\mathcal{L}_i$ is isomorphic to the trivial sheaf with the trivial section.

Definition 5.8. Two maps of log structures $f, f'\colon \mathfrak{L} \to \mathfrak{M}$ are said to be *essentially equivalent*, denoted $f \sim f'$, if there are maps $g\colon \mathfrak{L} \to \mathfrak{L}$ and $h\colon \mathfrak{M} \to \mathfrak{M}$, both of which are essentially the identity such that $h\circ f'\circ g = h\circ f\circ g$. A map of log structures $f\colon \mathfrak{L} \to \mathfrak{M}$ is said to be an *essential isomorphism* if there is a map $g\colon \mathfrak{L} \to \mathfrak{M}$ such that $g \circ f$ and $f \circ g$ are essentially the identity.

With this last example we are now ready to define a log scheme and a morphism of log schemes

Definition 5.9. A *log scheme* X is a scheme X together with a log structure $\mathfrak{L}$ on X. A morphism of log schemes $\mathsf{X} = (X, \mathfrak{L}) \to \mathsf{Y} = (Y, \mathfrak{M})$ is a morphism of schemes $f\colon X \to Y$ and a morphism of log structures on X $f^*\mathfrak{M} \to \mathfrak{L}$. A morphism of log-schemes is said to be *strict* if $f^*\mathfrak{M} \to \mathfrak{L}$ is an isomorphism of log structures and it is said to be essentially strict if $f^*\mathfrak{M} \to \mathfrak{L}$ is an essential isomorphism.

Remark. We can of course also define the category of log S-schemes where the underlying schemes are S-schemes.

Definition 5.10. A morphism of log schemes, $\mathsf{X} \to \mathsf{Y}$, is *an essential isomorphism* or *essentially the identity* if the underlying map of schemes, is an isomorphism or the identity respectively and the morphism of log structures is essentially an isomorphism or essentially the identity respectively. A morphism of log schemes is *essentially strict* if the induced map of log structures is an essential isomorphism. Finally, two maps $f, f'\colon \mathsf{X} \to \mathsf{Y}$, are *essentially equivalent*, denoted $f \sim f'$, if their underlying maps of schemes are equal and their maps of log structures are essentially equivalent.

Example 5.11. If $f\colon X \to Y$ is a function of schemes, X is any log scheme with underlying scheme X and Y is the log scheme with underlying scheme Y and no line bundles then f uniquely induces a map of log schemes $f\colon \mathsf{X} \to \mathsf{Y}$ since there are no isomorphisms of line bundles to define in this case. Note that it is not true in general that a morphism of underlying schemes always induces a morphism of log schemes and if there is a morphism of log schemes induced by a map of underlying schemes this morphism is in general not unique.

Definition 5.12. Let S be a scheme and, by abuse of notation, also a log scheme with no line bundles and let $\mathsf{X} = (X, \mathfrak{L})$ and $\mathsf{Y} = (Y, \mathfrak{M})$ be log schemes with morphisms $\mathsf{X} \to S$ and $\mathsf{Y} \to S$. Then the *fibered product*, denoted $\mathsf{X}\times_S\mathsf{Y}$, is the log scheme with underlying scheme $X\times_S Y$ and with line bundles with sections $\pi_1^*\mathfrak{L}\oplus\pi_2^*\mathfrak{M}$ where π_1,π_2 are the projections from $X\times_S Y$ to X and Y respectively.

It is easy to verify that this indeed satisfies the universal property of the fibered product. Understanding the fibered product in the category of log schemes in general is trickier but luckily we will not need this.

5.2 Kato-Nakayama Analytification

In this section I will define the so called Kato-Nakyama analytification functor from the category of DF log varieties over $\mathbb{C}$ to the category of topological spaces.

Definition 5.13. Let $\mathsf{X}=(X,(s_i\colon\mathcal{O}_X\to\mathcal{L}_i)_{1\le i\le n})$ be a log variety over $\mathbb{C}$. Then we define the *Kato-Nakayama analytification* of X, denoted X^{KN}, as the space

$$\mathrm{Bl}^{\mathbb{R}}_{s_n}\ \mathrm{Bl}^{\mathbb{R}}_{s_{n-1}}\ \ldots\mathrm{Bl}^{\mathbb{R}}_{s_1}\ X^{\mathrm{an}}$$

where X^{an} is the analytification of X and s_i is the section s_i of the vector bundle $\mathcal{L}_i$ pulled back via all previous blow-ups. We let $\rho_{\mathsf{X}}\colon\mathsf{X}^{\mathrm{KN}}\to X^{\mathrm{an}}$ denote the corresponding blow up map.

Remark. It is very important to note that since $\mathcal{L}_i$ is a one dimensional complex vector bundle it is a two dimensional real vector bundle.

Example 5.14 (Trivial Sections). The Kato-Nakayama analytification of $\mathsf{X}=(X,(0\colon\mathcal{O}_X\to\mathcal{L}))$ is the unit circle bundle of (the analytification of) $\mathcal{L}$. In particular, if $\mathcal{L}$ is the trivial line bundle this space is just $X^{\mathrm{an}}\times\mathbb{S}^1$.

Example 5.15 (Non-Vanishing Sections). Let $s\colon\mathcal{O}_X\to\mathcal{L}$ is such that the closed subscheme cut out by s, $\{x\in X|\ s(x)=0\}$, is empty. Then the Kato-Nakayama analytification of $\mathsf{X}=(X,(s\colon\mathcal{O}_X\to\mathcal{L}))$ is homeomorphic to X^{an}, i.e. the blow-up "does nothing".

Example 5.16 (Kato-Nakayama Analytification of a Blow Up). If X is a smooth variety and $Y\hookrightarrow X$ is a smooth locally complete intersection, then, by theorem 3.16 the Kato-Nakayama analytification of the log scheme given by blow up of X in Y with the associated line bundle, $(\mathrm{Bl}_Y\ X,(s_{\tilde{Y}}\colon\mathcal{O}\to\mathcal{O}(\tilde{Y})))$, is just the real oriented blow-up $\mathrm{Bl}^{\mathbb{R}}_{Y^{\mathrm{an}}}\ X^{\mathrm{an}}$.

I will not explicitly provide a definition of what the Kato-Nakayama analytification of a morphism is. This is not too difficult to do but I will not use the explicit definition of the analytification of a morphism of log schemes anywhere in this thesis and so I omit it. I will however need some important special cases. Specifically,

- Let X be a complex variety, let $\mathfrak{L}=(s_i\colon\mathcal{O}_X\to\mathcal{L}_i)_{1\le i\le n}$, let $\mathfrak{L}'=(s_i\colon\mathcal{O}_X\to\mathcal{L}_i)_{1\le i\le N}$ where $n\le N$ and let $i\colon\mathfrak{L}\hookrightarrow\mathfrak{L}'$ be the corresponding inclusion of log structures. Then, the Kato-Nakayama analytification of the map $(X,\mathfrak{L})\to(X,\mathfrak{L}')$ given by the identity $\mathrm{id}\colon X\to X$ and the inclusion $i\colon\mathrm{id}^*\mathfrak{L}=\mathfrak{L}'$ is the blow up map

$$\rho\colon(X,\mathfrak{L}')^{\mathrm{KN}}=\mathrm{Bl}^{\mathbb{R}}_{s_N}\ \mathrm{Bl}^{\mathbb{R}}_{s_{N-1}}\ \ldots\mathrm{Bl}^{\mathbb{R}}_{s_{n+1}}\ \mathrm{Bl}^{\mathbb{R}}_{s_n}\ \ldots\mathrm{Bl}^{\mathbb{R}}_{s_1}\ X^{\mathrm{an}}\to\mathrm{Bl}^{\mathbb{R}}_{s_n}\ \ldots\mathrm{Bl}^{\mathbb{R}}_{s_1}\ X^{\mathrm{an}}=(X,\mathfrak{L}')^{\mathrm{KN}}.$$

- For a strict morphism $\mathsf{X}\to\mathsf{Y}$ the following is a Cartesian diagram

$$\begin{array}{ccc} \mathsf{X}^{\mathrm{KN}} & \longrightarrow & \mathsf{Y}^{\mathrm{KN}} \\ \downarrow{\scriptstyle \rho_{\mathsf{X}}} & & \downarrow{\scriptstyle \rho_{\mathsf{Y}}} \\ X^{\mathrm{an}} & \longrightarrow & Y^{\mathrm{an}} \end{array}$$

where the top arrow is the analytification of the morphism and the bottom is the analytification of the morphism of underlying schemes.

For a precise definition of these morphisms for general log-schemes see [KN99]. See [BDPW23, 8.4.3] for an argument regarding why the construction of the Kato-Nakayama analytification of a general log scheme is equivalent to the construction by blow-ups given here.

Lemma 5.17. *The analytification of an essential identity is the identity.*

Proof. Let $\mathfrak{L} = (s_i \colon \mathcal{O}_X \to \mathcal{L}_i)_{1 \leq i \leq n}$ and let $f \colon (X, \mathfrak{L}) \to (X, \mathfrak{L})$ be an essential identity such that, without loss of generality, i gives isomorphisms

$$\mathcal{L}_i \xrightarrow{\cong} \begin{cases} \mathcal{L}_i & 1 \leq i \leq k \\ k \leq i \leq n \end{cases},$$

for some integer $1 \leq k \leq n$. Let $\mathfrak{L}' = (s_i \colon \mathcal{O}_X \to \mathcal{L}_i)_{1 \leq i \leq n}$, let $\mathsf{X}' = (X, \mathfrak{L}')$ and let $p \colon \mathsf{X} \to \mathsf{X}'$ be the map given by the identity $X \to X$ and the inclusion $\mathfrak{L}' \to \mathfrak{L}$. From definitions it is clear that the following diagram commutes.

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{\quad f \quad} & \mathsf{X} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathsf{X}' & \end{array}$$

Thus, it is enough to show that p^{KN} is an isomorphism. By definition p^{KN} is the blow-up map

$$\rho \colon \mathrm{Bl}^{\mathbb{R}}_{s_N} \mathrm{Bl}^{\mathbb{R}}_{s_{N-1}} \ldots \mathrm{Bl}^{\mathbb{R}}_{s_{n+1}} \mathrm{Bl}^{\mathbb{R}}_{s_n} \ldots \mathrm{Bl}^{\mathbb{R}}_{s_1} X^{\mathrm{an}} \to \mathrm{Bl}^{\mathbb{R}}_{s_n} \ldots \mathrm{Bl}^{\mathbb{R}}_{s_1} X^{\mathrm{an}}.$$

However, each of the line bundles with sections $s_i \colon \mathcal{O}_X \to \mathcal{L}_i$ are isomorphic to the trivial line bundle with the identity section. Thus the blow up maps in each of these line bundles are all isomorphisms and hence their composition $\rho = p^{\mathrm{KN}}$ is an isomoprhism. This completes the proof. □

Proposition 5.18. *The analytification of two essentially equivalent morphisms are equal and in particular the analytification of an essential isomorphism is an isomorphism. Furthermore, if $F \colon \mathsf{X} \to \mathsf{Y}$ with underlying map of schemes $f \colon X \to Y$ is essentially strict then the following diagram is Cartesian.*

$$\begin{array}{ccc} \mathsf{X}^{\mathrm{KN}} & \xrightarrow{F^{\mathrm{KN}}} & \mathsf{Y}^{\mathrm{KN}} \\ \downarrow{\scriptstyle \rho_{\mathsf{X}}} & & \downarrow{\scriptstyle \rho_{\mathsf{Y}}} \\ X^{\mathrm{an}} & \xrightarrow{f^{\mathrm{an}}} & Y^{\mathrm{an}} \end{array}$$

Proof. If $f, f' \colon \mathsf{X} \to \mathsf{Y}$ are essentially equivalent then, by definition, there are essential identities g, h such that $g \circ f \circ h = g \circ f' \circ h$. The analytifications of g and h are the identity by lemma 5.17 and thus the analytifications of $g \circ f \circ h$ and $g \circ f' \circ h$ are just f, f' respectively. From this the result follows.

For the second part, let $\mathsf{X} = (X, \mathfrak{L})$, $\mathsf{Y} = (Y, \mathfrak{M})$ and $\mathsf{X}' = (X, f^*\mathfrak{M})$. Since $\mathsf{X} \to \mathsf{Y}$ is essentially strict this morphism factors as $\mathsf{X} \to \mathsf{X}' \to \mathsf{Y}$ where $\mathsf{X}' \to \mathsf{Y}$ is strict and $\mathsf{X} \to \mathsf{X}'$ is an essential isomorphism. Hence the left and right squares in the following diagram are Cartesian.

$$\begin{array}{ccccc} \mathsf{X}^{\mathrm{KN}} & \xrightarrow{\cong} & \mathsf{X}'^{\mathrm{KN}} & \longrightarrow & \mathsf{Y}^{\mathrm{KN}} \\ \downarrow & & \downarrow & & \downarrow \\ X^{\mathrm{an}} & \xrightarrow{\mathrm{id}} & X & \longrightarrow & Y \end{array}$$

Since both squares are Cartesian the outer square is also Cartesian. This completes the proof. □

6 Moduli Spaces of Stable n-Pointed Rooted Trees of d-Dimensional Projective Spaces

In this part of the thesis I will introduce the moduli spaces of stable n-pointed rooted trees of d-dimensional projective spaces, which we denote $T_{d,n}$. These spaces were introduced by Chen, Gibney, and Krashen in their article "Pointed Trees of Projective Spaces" [CGK06] in which they also describe many of their properties. The construction of these spaces is closely related to the Fulton-MacPherson compactification of a d-dimensional smooth variety which was introduced by Fulton and MacPherson in their now famous article "A Compactification of Configuration Spaces" [FM94]. After defining the spaces $T_{d,n}$ I will describe an operad in which they are the objects. This operad is essentially an algebro-geometric version of the Kontsevich operad introduced in section 4. In what follows all constructions are over some fixed base field $\Bbbk$ and "scheme", and "variety" will be taken to mean $\Bbbk$-scheme and $\Bbbk$-variety respectively. Many of the subsequent constructions are well defined over other base schemes as well but I will restrict my work here to only considering base fields to avoid having to specify restrictions on the base schemes when such restrictions exist.

6.1 Rooted Trees of d-Dimensional Projective Spaces

Understanding the next couple of sections is going to be very difficult without the proper intuition. Therefore I will begin this chapter by describing a way to think of the closed points in the spaces we are about to encounter. We begin with a few definitions.

Definition 6.1. A *rooted tree* (T, r) is an acyclic, connected, graph T together with a distinguished node, $r \in T$, called the root. A rooted tree has a canonical partial ordering defined by $u \leq v$ if the unique path from r to v passes through u. If u and v are neighbours and $u \leq v$ we say that u is a *parent* of v and v is a *child* of u. The set of all children of u is denoted $C(u)$. Finally, if u has no children we say that u is a *leaf*.

We will now use a tree along with some data associated with each edge and vertex to define a "tree of d-dimensional projective spaces". We proceed as follows. First, let (T, r) be a rooted tree and associate to it the following data:

- To each vertex $u \in T$ (including the root) we associate a hyperplane in d-dimensional projective space $K(v) \subseteq \mathbb{P}^d$.
- To each child $v \in C(u)$ of a node u we associate a point in projective space that does not lie in $K(u)$, $p(v) \in \mathbb{P}^d \setminus K(u)$ such that no two children of u are associated to the same point.

We will also need one more set of data but before I can describe this we will need to do some work. For every $u \in T$, we define the scheme $X(u)$ by a sequence of blow ups of $\mathbb{P}^d$ in each of the points associated to the children of u separately. Notice that this is a well defined notion since the points are all disjoint so the order in which we blow up in does not matter. For each $v \in C(u)$, let $E(v)$ denote the exceptional divisor of $p(v)$ in $X(u)$ and let $H(u)$ denote the pullback of the hyperplane $K(u)$ in $X(u)$. Notice that clearly $H(u) \cong \mathbb{P}^{d-1}$ since none of the points $p(v)$ lie in $K(u)$. With this we are ready to describe all the data associated to a rooted tree (T, r) needed to define a d-dimensional rooted tree of projective spaces.

- To each vertex $u \in T$ (including the root) we associate a hyperplane in d-dimensional projective space $K(v) \subseteq \mathbb{P}^d$.
- To each child $v \in C(u)$ of a node u we associate a point in projective space that does not lie in $K(u)$, $p(v) \in \mathbb{P}^d \setminus K(u)$ such that no two children of u are associated to the same point.
- To each child $v \in C(u)$ of a node u an isomorphism $f(v)\colon E(v) \to H(v)$ from the exceptional divisor of $p(v)$, $E(v) \subseteq X(u)$ to the hyperplane associated to v $H(v) \subseteq X(v)$.

With this data we can form a scheme $X(T, r, H, p, f)$ by gluing together the schemes $\{X(u)\}_{u \in T}$ along closed subschemes as follows. For every parent/child pair u, v glue together the schemes $X(u)$ and $X(v)$ along $E(v) \subseteq X(u)$ and $H(u) \subseteq X(v)$ via the isomorphism $f(v)$. Notice that this process is well defined without having to check any cocycle conditions since none of the closed subschemes we glue together overlap.

Remark. Even though we have not formulated it this way it is somewhat more accurate to think of p and f as functions from the set of edges in T to the respective data. That is, it would be better to write $p(uv)$ and $f(uv)$ for the point and function corresponding to the child v of u. I will ignore this.

Definition 6.2. A *rooted tree of d-dimensional projective spaces*, or d-RTPS for short, (X, H_0), is a scheme X and a closed subscheme $\mathbb{P}^{d-1} \hookrightarrow X$, which can be generated by a rooted (T, r) along with the data (H, p, f) according to the process described above such that $\mathbb{P}^{d-1} \hookrightarrow X$ identifies $\mathbb{P}^{d-1}$ with the root hyperplane $H(r)$ described above. We call (T, r) the *structure tree* of X and we call H_0 the *root hyperplane*. We let X_v denote the image of $X(v)$ in X and we call this the *branch of X associated to v*. We let H_v denote the image of $H(v) \subseteq X(v)$ in X and we call this the *hyperplane associated to v*. If v is a child of u we will say that X_v is a child branch of X_u.

Remark. Note that the singular locus of a d-RTPS is precisely the union of the (disjoint) associated hyperplanes for all non-root nodes in the structure tree. This is obvious from construction.

The spaces we will define in the next section do not parameterize rooted trees of d-dimensional projective spaces, but rather collections of n disjoint points on rooted tree of d-dimensional projective spaces for some $n \geq 1$. This motivates the following definition.

Definition 6.3. An *n-pointed rooted tree of d-dimensional projective spaces*, $(\mathbb{P}^{d-1} \hookrightarrow X, p_1, \dots, p_n)$, is a d-RTPS, $\mathbb{P}^{d-1} \hookrightarrow X$, with n disjoint marked closed points, $(p_1, \dots, p_n)$, all of which lie outside the any of the associated hyperplanes of X (including the root hyperplane). An n-pointed d-RTPS is said to be *stable* there are at least two child branches or marked points on each branch.

Remark. To clarify, an n-pointed d-RTPS is stable if the set of all child branches and marked points in a given branch contains at least two elements. In other words if a branch has one child branch and one marked point it is stable.

An n-pointed d-RTPS, $(\mathbb{P}^{d-1} \hookrightarrow X, p_1, \ldots, p_n)$, also has an associated rooted tree structure defined as follows. First let (T, r) be the rooted tree associated to $\mathbb{P}^{d-1} \hookrightarrow X$. Then, for each point p_i add a vertex i to T and add the edge ui if p_i is in the associated branch X_u.

Definition 6.4. The *tree associated to an n-pointed d-RTPS* is the tree defined according to the process described above.

It is easy, and important, to see that an n-pointed d-RTPS is stable if and only if the only leaves of this tree are the vertices added for the points $p_1, \ldots, p_n$ and each non-leaf vertex, including the root, has at least two daughters.

Definition 6.5. A morphism between n-pointed d-RTPSs

$$(\mathbb{P}^{d-1} \hookrightarrow X, p_1, \ldots, p_n) \to (\mathbb{P}^{d-1} \hookrightarrow Y, q_1, \ldots, q_n)$$

is a morphism $f: X \to Y$ such that $f(p_i) = q_i$ and such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{P}^{d-1} & \longrightarrow & X \\ & \searrow & \downarrow f \\ & & Y \end{array}$$

Such a morphism is an isomorphism if f is an isomorphism.

It is easy to check that an n-pointed d-RTPS is stable if and only if its only automorphism is the identity automorphism.

6.2 Definition of $T_{d,n}$ and its Functor of Points

In this section I will introduce the moduli spaces of stable n-pointed rooted trees of d-dimensional projective spaces, $T_{d,n}$. I will not cover the precise meaning of $T_{d,n}$ being a moduli space but for intuitions sake the space $T_{d,n}$ should be thought of as "parameterizing" the stable n-pointed d-RTPSs up to isomorphism such that each $\Bbbk$-valued point in $T_{d,n}$ corresponds to a unique isomorphism class of stable n-pointed d-RTPSs. The moduli space properties of $T_{d,n}$ will not be relevant for this article but the interested reader can see Chen, Gibney, and Krashen [CGK06] for more details on this. An easier intuitive picture to keep in mind is to think of the points in these spaces in the exact same way as we think of the points in the Kontsevich spaces in section 4 but replacing $\mathbb{R}$ with $\mathbb{C}$ and instead of identifying configurations that are identified by translation and positive scaling we identify configurations that are identified by translation and $\mathbb{C}^*$-scaling. This picture of course only works when $\Bbbk = \mathbb{C}$ and is not rigorous in any way but it is still the picture I often have in mind when trying to visualize these spaces.

An important first step in the construction of $T_{d,n}$ is to define and describe the algebraic Fulton-MacPherson compactifications of a separated scheme X which was introduced by Fulton and MacPherson in "A compactification of configuration spaces" [FM94]. I will only give a brief description of the construction and refer to other texts for proofs of important results. In order to better understand what is to come, I think it is helpful to first describe what the "goal" of defining Fulton-MacPherson compactifications is and how one should intuitively think of these spaces. This is unsurprisingly very similar to the intuition behind the topological

Fulton-MacPherson compactification. Suppose X is a separated scheme. Then we have that

$$\mathrm{Conf}_n(X) := X^n \setminus \bigcup_{1 \le i < j \le n} \Delta_{i,j}$$

is an open subscheme of X^n where Δ_{ij} denotes the i,j-diagonal in X^n. The Fulton-MacPherson compactification is a scheme $X[n]$ with an open embedding $\mathrm{Conf}_n(X) \hookrightarrow X[n]$ and a surjective morphism $\pi_n \colon X[n] \twoheadrightarrow X^n$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Conf}_n(X) & \hookrightarrow & X[n] \\ & \searrow & \downarrow \\ & & X^n \end{array}$$

Intuitively one can, for $\Bbbk = \mathbb{C}$, think of the (reduced) closed subscheme $X[n] \setminus \mathrm{Conf}_n(X)$, which I will refer to as the "boundary" of $X[n]$, as parameterizing the ways or "directions" in which points in $\mathrm{Conf}_n(X)$, i.e. sets of n disjoint points in X^n, can "approach" each other such that at least two of these points meet. This should remind you of the topological Fulton-MacPherson compactification. When constructing $X[n]$, one also defines a set of effective Cartier divisors, one for each subset $S \subseteq \{1, 2, \ldots, n\}$ with at least two elements. We will denote these divisors by $X[n](S)$. These will all lie in the boundary of $X[n]$ and their union is the entire boundary of $X[n]$. Intuitively, you should think of $X[n](S)$ as parameterizing the ways in which n disjoint points, $p_1, \ldots, p_n$, in X can approach each other such that all points corresponding to the indexes in S "meet", i.e. approach the same point in X and furthermore if any other point p_i with $i \notin S$ also approaches the same point in X then the points corresponding to indexes in S approach each other "faster" than they approach p_i.

One way to construct the Fulton-MacPherson configuration spaces of a separated scheme X is as follows. For every subset $S \subseteq [n]$ define the S-diagonal in X^n as the closed subscheme given by the relations $\Delta_S = \{(x_1, \ldots, x_n) \in X^n |\ x_i = x_j\ \forall i, j \in S\}$. Let

$$B_n = \{1, \ldots, n\}, \{1, \ldots, n-1\}, \ldots, \{n, n-3\}, \{n, n-2\}, \{n, n-1\}$$

be the sequence described in section 4.

Definition 6.6. The *Fulton-MacPherson configuration spaces for a separated scheme* X, denoted $X[n]$, are defined as the iterated blow-ups

$$\mathrm{Bl}_{\tilde{\Delta}(B_n(2^n-(n+1))} \ldots \mathrm{Bl}_{\tilde{\Delta}(B_n(2))} \mathrm{Bl}_{\tilde{\Delta}(B_n(1))} X^n$$

where, $\tilde{\Delta}(S)$ denotes the dominant transform of the diagonal $\Delta(S)$ in all previous blow-ups. The map $\pi_n \colon X[n] \to X^n$ is the composition of all blow-up maps. The divisor $X[n](S) \subseteq X[n]$ mentioned above is the diagonal $\tilde{\Delta}(S)$ in $X[n]$.

Also define the scheme $X[n,i]$ for $0 \le i \le 2^n - (n+1)$ as the iterated blow up

$$\mathrm{Bl}_{\tilde{\Delta}(B_n(i))} \ldots \mathrm{Bl}_{\tilde{\Delta}(B_n(2))} \mathrm{Bl}_{\tilde{\Delta}(B_n(1))} X^n$$

and the map $\pi_{n,i} \colon X[n,i] \to X^n$ as the composition of blow-up morphisms.

Remark. This is not the same sequence of blow-ups as the one which appeared in Fulton and MacPhersons original article [FM94]. However, for their sequence lemma 6.11 does not hold. See [Li09] for a proof of equivalence between the two sequences of blow-ups.

Note that each of the blow-up maps, by definition, is an isomorphism when restricted to $\mathrm{Conf}_n(X) \subseteq X^n$. This gives an embedding $\mathrm{Conf}_n(X) \hookrightarrow X[n]$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Conf}_n(X) & \hookrightarrow & X[n] \\ & \searrow & \downarrow{\scriptstyle \pi_n} \\ & & X^n. \end{array}$$

To simplify notation I will let $X[n](S_1, \ldots, S_r)$ denote the intersection $\bigcap_{i=1}^{r} X[n](S_i)$. The following is a summary of results by Fulton and MacPherson regarding important properties of $X[n]$.

Proposition 6.7. *The Fulton-MacPherson varieties have the following properties*

- *$X[n]$ is smooth*
- *Any set of the divisors $X[n](S)$ meet transversally (if they meet).*
- *The closed subscheme $X[n](S_1, \ldots, S_r)$ is empty if and only if any two of the sets S_i, S_j have the property $S_i \cap S_j \neq$ and $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$.*

Proof. See [FM94]. □

I will take these properties to be "obvious" for the remainder of the thesis and therefore not reference this proposition in proofs where it is important that $X[n]$ and its divisors are "nice" in the ways listed above.

Fulton and MacPherson also describe a functor of points for $X[n]$ which will prove to be of great use for us. To describe this we need a few definitions. Later in the thesis I will almost exclusively use the common notation $[n] = \{1, 2, \ldots, n\}$. However, in this part I will let $N = \{1, \ldots, n\}$ to avoid confusion with $X[n]$. For any subset $S \subseteq N$ let

$$p_S \colon X^{|N|} \to X^{|S|}$$

denote the projection morphism onto components with corresponding indexes in S. If $S_1 \subseteq S_2 \subseteq N$ let

$$p_{S_2,S_1} \colon X^{|S_2|} \to X^{|S_1|}$$

denote the projection morphism onto components corresponding to indexes in S_2, i.e. the morphism which makes the following diagram commute.

$$\begin{array}{ccc} X^{|N|} & \xrightarrow{p_{S_2}} & X^{|S_2|} \\ & \searrow{\scriptstyle p_{S_1}} & \downarrow{\scriptstyle p_{S_2,S_1}} \\ & & X^{|S_2|} \end{array}$$

Next, let $\mathcal{I}_S$ denote the closed ideal sheaf for the small diagonal in $X^{|S|}$. For $S_1 \subseteq S_2$, p_{S_1,S_2} induces a closed embedding of the corresponding small diagonals and thus induces a morphism of pullback sheaves

$$p_{S_1}^* \mathcal{I}_{S_1} \to p_{S_2}^* \mathcal{I}_{S_2}.$$

Notice that if $X = \mathbb{A}^d = \operatorname{Spec} \Bbbk[x_1, \ldots, x_d]$, and consequently $X^n \cong k[\{x_i^k\}_{i \in N, 1 \leq k, \leq d}]$, then $p_s^* \mathcal{I}_S$ is the ideal generated by all elements of the form $t_{ij}^k = x_i^k - x_j^k$ where $1 \leq k \leq d$ and $i, j \in S$ and for $S_1 \subseteq S_2$ the morphism $p_{S_1}^* \mathcal{I}_{S_1} \to p_{S_2}^* \mathcal{I}_{S_2}$ is the inclusion of ideals given by mapping $t_{ij}^k \mapsto t_{ij}^k$.

Lastly, for a map $h \colon H \to X^n$ let $h_S = p_S \circ h$ and for $S_1 \subseteq S_2$ note that the morphism $p_{S_1}^* \mathcal{I}_{S_1} \to p_{S_2}^* \mathcal{I}_{S_2}$ pulls back to a morphism

$$h_{S_1}^* \mathcal{I}_{S_1} \to h_{S_2}^* \mathcal{I}_{S_2}.$$

Definition 6.8. Given a morphism $h \colon H \to X^n$ a *screen* for h and $S \subseteq N$ is a quotient map $h_S^* \mathcal{I}_S \to \mathcal{L}$ where $\mathcal{L}$ is some invertible sheaf. A *R-collection of compatible screens* for h is a screen $\phi_S \colon h_S^* \mathcal{I}_S \to \mathcal{L}_S$ for some collection of sets R such that every $S \in P$ satisfies $S \subseteq N$, $|S| \geq 2$ and for every $S_1 \subseteq S_2$ in the collection a morphism $\mathcal{L}_{S_1} \to \mathcal{L}_{S_2}$ making the following diagram commute.

$$\begin{array}{ccc} h_{S_1}^* \mathcal{I}_{S_1} & \longrightarrow & \mathcal{L}_{S_1} \\ \downarrow & & \downarrow \\ h_{S_2}^* \mathcal{I}_{S_2} & \longrightarrow & \mathcal{L}_{S_2} \end{array}$$

Such a collection of screens is called *complete* if it contains a screen for each $S \subseteq N$ of size ≥ 2.

In what follows we let $P(n)$, or just P when n is clear from context, be the set of sets S such that $S \subseteq N$, $|S| \geq 2$. Additionally, let $P_i \subseteq P$ be the subset with the additional condition that each $S \in P_i$ appears before or at position i in the sequence B_n.

Definition 6.9. For a scheme X, the contravariant functor $\chi^X[n, i]$, or just $\chi[n, i]$ when X is clear from context, from the category of schemes to the category of sets is defined as the functor sending H to the set of pairs

$$((h \colon H \to X^n), \{\phi_S \colon h_S^* \mathcal{I}_S \to \mathcal{L}_S\}_{S \in P_i})$$

of morphisms $(h \colon H \to X^n)$ and P_i-collections of compatible screens $\{\phi_S \colon h_S^* \mathcal{I}_S \to \mathcal{L}_S\}_{S \in P_i}$ up to isomorphism of the screen data. The subfunctor $\chi^X[n, i](S_1, \ldots, S_r)$ sends H to the pairs of morphisms and compatible screens

$$((h \colon H \to X^n), \{\phi_S \colon h_S^* \mathcal{I}_S \to \mathcal{L}_S\}_{S \in P_i})$$

such that the following holds

- for any $S_i \in \{S_1, \ldots, S_r\}$ and any $j, k \in S_i$ we have $\mathrm{pr}_j \circ h = \mathrm{pr}_k \circ h$ where $\mathrm{pr}_m \colon X^n \to X$ denotes the mth projection morphism.
- for any $S_i \in \{S_1, \ldots, S_r\}$ and any $T \in P_i$ where $T \not\subseteq S_i$ and $|T \cap S_i| \in P_i$ the morphism $\mathcal{L}_{T \cap S_j} \to \mathcal{L}_T$ is trivial.

Furthermore, define $\chi^X[n] \coloneqq \chi^X[n, |B_n|]$ and $\chi^X[n](S_1, \ldots, S_r) \coloneqq \chi^X[n, |B_n|](S_1, \ldots, S_r)$

Remark. Note that for the definition of $X[n]$ we can also just replace "P_i-collection" with "complete collection" in the definition of $X[n, i]$.

One key result of Fulton and MacPherson [FM94] is that these functors are representable and their representations are the Fulton-MacPherson spaces.

Theorem 6.10. *The functor $\chi[n]$ is represented by the scheme $X[n]$ and the subfunctors $\chi[n](S_1, \ldots, S_r)$ are* *represented by the closed subschemes $X[n](S_1,\dots,S_r)$. Similarly, $\chi[n,i]$ is represented by the scheme $X[n,i]$ and the subfunctors $\chi[n,i](S_1,\dots,S_r)$ are represented by the closed subschemes $\tilde{\Delta}(S_1)\cap\cdots\cap\tilde{\Delta}(S_r)\subseteq X[n,i]$. Furthermore, the blow up map $X[n,i+1]\to X[n,i]$ induces the natural transformation of functors which sends*

$$\big((h\colon H\to X^n),\{\phi_S\colon h_S^*\mathcal{I}_S\to\mathcal{L}_S\}_{S\in P_{i+1}}\big)\in\chi[X,i+1](H)$$

to

$$((h\colon H\to X^n),\{\phi_S\colon h_S^*\mathcal{I}_S\to\mathcal{L}_S\}_{S\in P_i})\in\chi[X,i](H)$$

where $\{\phi_S\colon h_S^*\mathcal{I}_S\to\mathcal{L}_S\}_{S\in P_i}$ is the restriction of the collection of compatible P_{i+1}-screens to only those screens corresponding to sets in P_i.

Proof. See [FM94]. □

Remark. Fulton and MacPherson do not prove this statement exactly since they use a different order of blow-ups than I use in this thesis. However, the same arguments as they use can be applied to this order of blowing up too. For a more general discussion on why these two orders of blowing up are equivalent see [Li09].

Finally, we will need one more result regarding the construction of $X[n]$. This result will not be used until section 7.2.

Lemma 6.11. *For every $T\in P_i$, the dominant, strict, and total transforms of $\tilde{\Delta}(T)$ are equal for the blow-up map $X[n,i+1]=\mathrm{Bl}_{\Delta(B(i+1))}X[n,i]\to X[n,i]$.*

Proof. To avoid confusion I will let $\tilde{\Delta}(S)$ denote the (dominant transforms of) the diagonals in $X[n,i]$ and $\tilde{\Delta}'(S)$ to denote the proper transform of this, i.e. the corresponding diagonal in $X[n,i+1]$. The dominant transform is always contained in the total transform. Thus all we need to prove is that $\tilde{\Delta}'(T)$ contains the inverse image of $\tilde{\Delta}(T)$. Equivalently we can show that the natural transformation of functors in theorem 6.10 only sends elements in the image of the subfunctor $\chi[n,i+1](T)$ to elements in the image of $\chi[n,i](T)$. Let $\big((h\colon H\to X^n),\{\phi_S\colon h_S^*\mathcal{I}_S\to\mathcal{L}_S\}_{S\in P_{i+1}}\big)\in\chi[n,i+1](H)$ and suppose the natural transformation $\chi[n,i+1]\to\chi[n,i]$ sends this element to an element in $\chi[n,i](T)$. Then, for every $j,k\in T$ we have $\mathrm{pr}_j\circ h=\mathrm{pr}_k\circ h$ and for any $S\in P_i$ where $S\not\subseteq T$ and $|S\cap T|\in P_i$ the morphism $\mathcal{L}_{T\cap S_j}\to\mathcal{L}_T$ is trivial. Finally, since the size of the sets in B_n appear in decreasing order there are no sets $T\in P_i$ with $|T\cap B_n(i+1)|\in P_{i+1}$, $B_n(i+1)\not\subseteq T$ and so these are the only conditions which need to be satisfied to show that $\big((h\colon H\to X),\{\phi_S\colon h_S^*\mathcal{I}_S\to\mathcal{L}_S\}_{S\in P_{i+1}}\big)\in\chi[X,i+1](T)(H)$. □

With this result we have everything we need regarding the Fulton MacPherson compactification and we are thus ready to define the $T_{d,n}$ varieties.

Definition 6.12. Let X be a variety and let τ_n denote the composition morphism

$$X[n](N)\hookrightarrow X[n]\xrightarrow{\pi_n}X^n\xrightarrow{\mathrm{pr}_i}X.$$

Where pr_i is one of the projection morphisms. Then $T_{d,n}^{X,x}$ is defined as the fiber $\tau_n^{-1}(x)\subseteq X[n](N)$ where x is some k-valued point in X. For a collection of sets $S_i\subsetneq N$, $1\le i\le r$, we also define the closed subschemes

$T_{d,n}^{X,x}(S_1,\dots,S_r) \hookrightarrow T_{d,n}^{X,x}$ to be the fiber of x for the morphism

$$X[n](N,S_1,\dots,S_r) \hookrightarrow X[n](N) \to [\tau_n]X.$$

Equivalently $T_{d,n}^{X,x}(S_1,\dots,S_r) \hookrightarrow T_{d,n}^{X,x}$ is the pullback of $X[n](N,S_1,\dots,S_r) \hookrightarrow X[n](N)$. For both these functors the functor applied to morphisms is defined in the obvious way by composing with h and taking the pullback of the screens.

Remark. This definition is independent of which projection pr_i we choose to define τ_n. See [CGK06] for further details.

Notice that proposition 6.7 implies that the closed subscheme $T_{d,n}^{X,x}(S_1,\dots,S_r)$ is empty if any two of the sets S_i, S_j have the property $S_i \cap S_j \neq$ and $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$. This construction is essentially only interesting when X is a smooth variety of dimension d. In this case we have the following proposition.

Proposition 6.13. *For any smooth d-dimensional variety X and any point $x \in X$ there is an isomorphism $T_{d,n}^{X,x} \cong T_{d,n}^{\mathbb{A}^d,\mathbf{0}}$ and furthermore there are isomorphisms $T_{d,n}^{X,x}(S_1,\dots,S_r) \cong T_{d,n}^{\mathbb{A}^d,\mathbf{0}}(S_1,\dots,S_r)$ making the following diagram commute.*

$$\begin{array}{ccc} T_{d,n}^{X,x}(S_1,\dots,S_r) & \longrightarrow & T_{d,n}^{X,x} \\ \downarrow & & \downarrow \\ T_{d,n}^{\mathbb{A}^d,\mathbf{0}}(S_1,\dots,S_r) & \longrightarrow & T_{d,n}^{\mathbb{A}^d,\mathbf{0}} \end{array}$$

Proof. See Chen, Gibney, Krashen [CGK06]. □

Because of this we can simply let $T_{d,n}$ denote the variety $T_{d,n}^{X,x}$ for any smooth, d-dimensional, variety X and similarly $T_{d,n}(S_1,\dots,S_r)$ denotes the corresponding closed subschemes.

Definition 6.14. The *moduli space for stable n pointed rooted trees of d-dimensional projective spaces*, denoted $T_{d,n}$, is the variety described above and the closed subschemes $T_{d,n}(S_1,\dots,S_r) = \bigcap_i T_{d,n}(S_i)$ are also as above for $S_i \subsetneq N,\ |S| \geq 2$. Additionally, let $T_{d,n}(\{l\}) = T_{d,n}(N) = T_{d,n}$ for every $l \in N$.

Remark. The notation $T_{d,n}(\{l\}) = T_{d,n}(N) = T_{d,n}$ is introduced to make the notation cleaner for some results in later sections.

Chen, Gibney, and Krashen show that the spaces $T_{d,n}$ satisfy some important properties analogous to those of $X[n]$ listed in proposition 6.7.

Proposition 6.15. *The moduli spaces of stable n pointed trees of d-dimensional projective spaces have the following properties*

- *$T_{d,n}$ is smooth*
- *Any set of the divisors $T_{d,n}(S)$ meet transversally (if they meet).*
- *The closed subscheme $T_{d,n}(S_1,\dots,S_r)$ is empty if and only if any two of the sets S_i, S_j have the property $S_i \cap S_j \neq$ and $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$.*
- *$T_{d,2} \cong \mathbb{P}^{d-1}$ for every d and the restriction of the line bundle $\mathcal{O}_{X[2]}(X[2](\{1,2\}))$ to $T_{d,2}$ is $\mathcal{O}_{\mathbb{P}^{d-1}}(-1)$.*
- *$T_{1,n} \cong \overline{\mathcal{M}}_{0,n+1}$, the moduli space of stable $n+1$ pointed rational curves of genus 0.*

Proof. See [CGK06]. □

Just like the properties of $X[n]$ in proposition 6.7 I will take these properties to be "obvious" for the remainder of the thesis and therefore I will usually not reference this proposition in proofs where it is important that $T_{d,n}$ and its divisors $T_{d,n}(S)$ are "nice" in the ways listed above.

Now we are ready to describe a functor of points for $T_{d,n}$. This functor is similar to the functor of points of $X[n]$ but it is in fact much simpler.

Definition 6.16. For a scheme X, a positive integer d and a set of positive integers S, the *difference sheaf* $\mathcal{F}_S^{X,d}$ is the free, rank $(|S|-1)d$ sheaf of modules,

$$\mathcal{F}_S^{X,d} = \left(\bigoplus_{1\leq k\leq d} \bigoplus_{i,j\in S} \mathcal{O}_X t_{ij}^k \right) \Big/ \mathcal{G}_S,$$

where $\mathcal{G}_S$ is the submodule of generated by all elements of the form $t_{il}^k - t_{ij}^k - t_{jl}^k$. If X and/or d are clear from context we omit these from the notation. For any $S_1 \subseteq S_2$ the canonical injective morphism

$$i_{S_1S_2} \colon \mathcal{F}_{S_1} \to \mathcal{F}_{S_2}, t_{ij}^k \mapsto t_{ij}^k$$

is called the *difference sheaf inclusion map* corresponding to $S_1 \subseteq S_2$.

Remark. Notice that all elements of the form t_{ii}^k and $t_{ij}^k + t_{ji}^k$ lie in $\mathcal{G}_S$.

For a morphism $X \to Y$ the difference sheaves $\mathcal{F}_S^{Y,d}$ pull back to $\mathcal{F}_S^{X,d}$ and the difference sheaf inclusions $\mathcal{F}_{S_1}^{Y,d} \to \mathcal{F}_{S_2}^{Y,d}$ pull back to $\mathcal{F}_{S_1}^{X,d} \to \mathcal{F}_{S_2}^{X,d}$.

Definition 6.17. A *simple d-screen* for a set S on a variety H is a quotient map of sheaves $\mathcal{F}_S^d \twoheadrightarrow \mathcal{L}$ where s is some positive integer and $\mathcal{L}$ is an invertible sheaf. A *collection of compatible simple d-screens* is a collection of simple screens

$$\phi_S \colon \mathcal{F}_S^d \to \mathcal{L}_S,$$

for some collection of sets S and, for every inclusion of sets $S_1 \subseteq S_2$ a morphism of invertible sheaves $\mathcal{L}_{S_1} \to \mathcal{L}_{S_2}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}_{S_1}^d & \longrightarrow & \mathcal{L}_{S_1} \\ \downarrow & & \downarrow \\ \mathcal{F}_{S_2}^d & \longrightarrow & \mathcal{L}_{S_2} \end{array}$$

A collection of simple d-screens is said to be *complete* if the collection contains a screen for every $S \subseteq N$ with $|S| \geq 2$, i.e. for every $S \in P$.

To simplify the notation, a "collection of simple screens" will always be assumed to refer to a complete collection of simple screens unless I specify otherwise.

Definition 6.18. The contravariant functor $\tau_{d,n}$ from the category of schemes to the category of sets is defined as the functor sending H to the set

$$\{\phi_S \colon \mathcal{F}_S^{H,d} \to \mathcal{L}_S\}_{S\in P}$$

of complete collections of simple compatible d-screens up to isomorphism of the screen data. The subfunctor $\tau_{d,n}(S_1, \ldots, S_r)$ sends H to the set of complete collections of simple compatible d-screens,

$$\{\phi_S \colon \mathcal{F}_S^{H,d} \to \mathcal{L}_S\}_{S \in P},$$

such that for any $S_i \in \{S_1, \ldots, S_r\}$ and any $T \not\subseteq S_i$ with $|T \cap S_i| \geq 2$ the morphism $\mathcal{L}_{T \cap S_i} \to \mathcal{L}_T$ is trivial. These functors send a morphisms f to the map sending a complete collection of simple d-screens to its pullback via f.

Proposition 6.19. *The functor $\tau_{d,n}$ is represented by $T_{d,n}$ and similarly the subfunctor $\tau_{d,n}(S_1, \ldots, S_r)$ is represented by $T_{d,n}(S_1, \ldots, S_r)$.*

Proof. By proposition 6.13, $T_{d,n}$ is the fiber $\mathbb{A}^d[n](N) \times_{\mathbb{A}^d} o$ where O denotes the origin point in $(\mathbb{A}^d)^n$. Furthermore, $\mathbb{A}^d[n](N)$ represents the functor $\chi^{\mathbb{A}^d}[n](N)$. Therefore, the fiber product $\mathbb{A}^d[n](N) \times_{\mathbb{A}^d} O$ represents the functor

$$\chi[n](N) \times_{\hom(\cdot, \mathbb{A}^d)} \hom(\cdot, \mathrm{Spec}\ \Bbbk).$$

By the definition of $\chi[n](N)$ this functor sends a scheme H to the set of pairs

$$\big((h \colon H \to (\mathbb{A}^d)^n), \{\phi_S \colon h_S^* \mathcal{I}_S \to \mathcal{L}_S\}_{S \in P}\big)$$

of morphisms $(h \colon H \to X)$ and collections of compatible screens $\{\phi_S \colon h_S^* \mathcal{I}_S \to \mathcal{L}_S\}_{S \in P}$ such that $h \circ \mathrm{pr}_i = h \circ \mathrm{pr}_j$ for all indexes i, j and such that $h \circ \mathrm{pr}_i$ factors via the origin in $\mathbb{A}^d$. This is equivalent to saying that h factors via the origin in $(\mathbb{A}^d)^n$, i.e. there is a map $\tilde{h} \colon H \to \mathrm{Spec}\ \Bbbk$ making the following diagram commute.

$$\begin{array}{ccc} H & \xrightarrow{\tilde{h}} & \mathrm{Spec}\ \Bbbk \\ & \searrow^{h} & \downarrow O \\ & & (\mathbb{A}^d)^n \end{array}$$

Since Spec $\Bbbk$ is the base scheme in the category there is only one such morphism $\tilde{h}$ of $\Bbbk$-schemes. Therefore this functor sends H to the set of collections of compatible screens $\{\phi_S \colon g_S^* \mathcal{I}_S \to \mathcal{L}_S\}_{S \in P}$ where g_S denotes the composition $p_s \circ O \circ \tilde{h}$. Now, the sheaf $p_S^* \mathcal{I}_S$ is the ideal sheaf generated by all elements of the form $x_i^k - x_j^k$ for $i, j \in S$ and $1 \leq k \leq n$ where we have used the notation

$$(\mathbb{A}^d)^n = \mathrm{Spec}\ \Bbbk[x_i^k]_{i \in N, 1 \leq k \leq d}.$$

It is easy to see that the pullback of this ideal to Spec $\Bbbk$ via the origin morphism is isomorphic to the sheaf $\mathcal{F}_S^{\mathrm{Spec}\ \Bbbk, d}$ on Spec $\Bbbk$ via an isomorphism which sends the pullback of the section $x_i^k - x_j^k$ to t_{ij}^k. Furthermore, for $S_1 \subseteq S_2$ the pullback the map $p_{S_1}^* \mathcal{I}_{S_1} \to p_{S_2}^* \mathcal{I}_{S_2}$ is the map $\mathcal{F}_{S_1} \to \mathcal{F}_{S_2}$. Now, the sheaves $\mathcal{F}_S^{\mathrm{Spec}\ \Bbbk, d}$ pull back to the sheaves $\mathcal{F}_S^{H,d}$ and similarly the compatibility morphisms pull back as well. Therefore the set of collections of compatible screens $\{\phi_S \colon h_S^* \mathcal{I}_S \to \mathcal{L}_S\}_{S \in P}$ is just the set of complete collections of compatible simple d-screens over H. Furthermore, it is clear that this functor applied to a morphism of schemes f sends each complete collection of compatible simple d-screens to its pullback via f since the functor $\chi^{\mathbb{A}^d}[n](N)$ sends a map and a collection of compatible screens to their pullback via f. Thus $T_{d,n}$ represents $\tau_{d,n}$. Since

$$T_{d,n}(S_1,\dots,S_r) = \mathbb{A}^d[n](N,S_1,\dots,S_r)\times_{\mathbb{A}^d} O,$$

by proposition 6.13 this scheme represents the subfunctor of $\tau_{d,n}$ which sends H to the set of collections of compatible screens $\{\phi_S\colon g_S^*\mathcal{I}_S\to\mathcal{L}_S\}_{S\in P}$ with the property that, for any S_j and any $T\subseteq N$ with $T\not\subseteq S_j$ and $|T\cap S_j|\geq 2$ we have that $\mathcal{L}_{S_1}\to\mathcal{L}_{S_2}$ is the zero morphism. Since we have already seen that the set of compatible screens $\{\phi_S\colon g_S^*\mathcal{I}_S\to\mathcal{L}_S\}_{T\subseteq N}$ is just the set of collections of compatible simple d-screens on H this is precisely the definition of the functor $\tau_{d,n}(S_1,\dots,S_r)$. This completes the proof. □

This functor of points for of $T_{d,n}$ will prove to be extremely useful in proving a lot of the important results throughout the remainder of the thesis.

6.3 The Geometric Kontsevich Operad

In this section I define an operad with objects $T_{d,n}$ for $n\in\mathbb{N}$ and fixed dimension d. In dimension $d=1$ Chen, Gibney, and Krashen show that there are isomorphisms $T_{d,n}\cong\overline{\mathcal{M}}_{0,n+1}$ [CGK06]. In this case the operad I am about to describe is canonically isomorphic to to the "Deligne-Mumford"-operad on the moduli spaces of stable $n+1$-pointed curves of genus 0, $\overline{\mathcal{M}}_{0,n+1}$ which was defined by Ginzburg and Kapranov in section 1.4 of their article "Kozul duality for operads" [GK94]. In this section, and throughout the remainder of the thesis, I use the notation

$$[n]=\{1,2,\dots,n\}$$

for every positive integer n.

In the cartesian monoidal category of $\Bbbk$-schemes the unit object is Spec $\Bbbk$. Furthermore, it is clear, both from the explicit construction and the functor of points, that $T_{d,1}\cong$ Spec $\Bbbk$. Hence, the unit morphism of the operad

$$\eta\colon \mathbb{1}\to T_{d,1}$$

is just the identity morphism Spec $\Bbbk\to$ Spec $\Bbbk$. To define the composition morphisms we will need to state and prove proposition 6.21 which is a slight generalization of Theorem 3.3.1 (4) in [CGK06]. In the proof of this proposition as well as several other results later the notation is greatly simplified by introducing the following functions. For a positive integer n and a collection of n positive integers $\mathbf{m}=(m_1,\dots,m_n)$ define the functions of integers

$$p^{n,\mathbf{m}}\colon a\mapsto\begin{cases}1 & 0<a\leq m_1\\ 2 & m_1<a\leq m_2\\ \vdots & \\ n & \sum_{i<n}m_i<a\leq\sum_{i\leq n}m_i\end{cases}$$

and, for $1\leq r\leq n$

$$q_r^{n,\mathbf{m}}\colon a\mapsto a-\sum_{i<r}m_i.$$

These same functions also appeared in section 4.

In addition to this, we also need a set of functions of difference sheaves for some scheme X,

$$\alpha_V^{n,\mathbf{m}} \colon \mathcal{F}_V^d \to \mathcal{F}^d_{p^{n,\mathbf{m}}(V)}, t^k_{ij} \mapsto t^k_{p^{n,\mathbf{m}}(i)p^{n,\mathbf{m}}(j)}$$

and, for $1 \leq r \leq n$

$$\beta_{V,r}^{n,\mathbf{m}} \colon \mathcal{F}_V^d \to \mathcal{F}^d_{q_r^{n,\mathbf{m}}(V)}, t^k_{ij} \mapsto t^k_{q_r^{n,\mathbf{m}}(i)q_r^{n,\mathbf{m}}(j)}.$$

I will usually omit the index V here to simplify the notation since it is almost always clear from context except for in the discussion below. These functions are of course not well defined for every set V so the next step is to specify for which sets V they are well defined, or perhaps more accurately when they are well defined and interesting. In what follows we let

$$M_i' = \{1 + \sum_{j<i} m_j, 2 + \sum_{j<i} m_j \ldots, m_i + \sum_{j<i} m_j\} = (q_i^{n,\mathbf{m}})^{-1}([m_i]).$$

Notice that $p^{n,\mathbf{m}}$ is a surjective function $[m] \to [n]$ and that $q_i^{n,\mathbf{m}}$ is bijective $M_i' \to [m_i]$. Now, $\alpha_V^{n,\mathbf{m}}$ is defined for $V \subseteq [m]$ and $V \not\subseteq M_r'$ for any $r \in [m]$ and $\beta_{V,r}^{n,\mathbf{m}}$ is defined for $V \subseteq M_r'$, $|V| \geq 2$.

Lemma 6.20. *On the domains specified above, the maps $\alpha^{n,\mathbf{m}}$ and $\beta^{n,\mathbf{m}}$ commute with the difference sheaf inclusion maps. That is, if $V_1 \subseteq V_2 \subseteq M$ then*

$$i_{p^{n,\mathbf{m}}(V_1)p^{n,\mathbf{m}}(V_2)} \circ \alpha_{V_1}^{n,\mathbf{m}} = \alpha_{V_2}^{n,\mathbf{m}} \circ i_{V_1V_2}$$

and

$$i_{q_r^{n,\mathbf{m}}(V_1)q_r^{n,\mathbf{m}}(V_2)} \circ \beta_{V_1,r}^{n,\mathbf{m}} = \beta_{V_2,r}^{n,\mathbf{m}} \circ i_{V_1V_2}$$

provided that $\alpha^{n,\mathbf{m}}$ or $\beta_r^{n,\mathbf{m}}$ are well defined for both V_1 and V_2.

Proof. Using direct computation it is trivial to verify that both of these maps send $t^k_{ij} \mapsto t^k_{p^{n,\mathbf{m}}(i)p^{n,\mathbf{m}}(j)}$ or $t^k_{q_r^{n,\mathbf{m}}(i)q_r^{n,\mathbf{m}}(j)}$ respectively. Thus both equalities of maps hold. □

Proposition 6.21. *For any collection of positive integers $n, m_1, \ldots, m_n$ there is an isomorphism*

$$T_{d,n} \times T_{d,m_1}, \times \cdots \times T_{d,m_n} \cong T_{d,m}(M_1', \ldots, M_n')$$

where $m = \sum_r m_r$ and

$$M_r' = \{1 + \sum_{i<r} m_i, \ldots, m_r + \sum_{i<r} m_i\}.$$

Proof. We will prove that the functors represented by the left hand side and the right hand side are naturally isomorphic. We proceed as follows. Let

$$\text{CSS} = \{\phi_T\}_{T \in P(n)} \times \{\psi^1_{S^1}\}_{S_1 \in P(m_1)} \times \cdots \times \{\psi^n_{S^n}\}_{S_n \in P(m_n)} \in (\tau_{d,n} \times \tau_{d,m_1} \times \cdots \times \tau_{d,m_n})(H)$$

for some scheme H. We denote the image of ϕ_T by $\mathcal{L}_T^0$ and the image of $\psi^r_{S_r}$ by $\mathcal{L}^r_{S_r}$. We define a natural transformation

$$f \colon \tau_{d,n} \times \tau_{d,m_1} \times \cdots \times \tau_{d,m_n} \to \tau_{d,m}(M_1', \ldots, M_n')$$

by sending CSS to the m-collection of simple d-screens $\{\rho_V : \mathcal{F}^d_V \to \mathcal{L}_V\}_{V\in P(m)}$ defined by

$$\rho_V = \begin{cases} \psi^r_{q_r^{n,\mathbf{m}}(V)} \circ \beta_r^{n,\mathbf{m}} & V \subseteq M'_r \text{ some } r \\ \phi_{p^{n,\mathbf{m}}(V)} \circ \alpha^{n,\mathbf{m}} & \text{else} \end{cases}.$$

Note that implicit in this is that we have defined the line bundles $\mathcal{L}_V$ as

$$\mathcal{L}_V = \begin{cases} \mathcal{L}^r_{q_r^{n,\mathbf{m}}(V)} & V \subseteq M'_r \text{ some } r \\ \mathcal{L}^0_{p^{n,\mathbf{m}}(V)} & \text{else} \end{cases}.$$

Here we define the maps $\mathcal{L}_{V_1} \to \mathcal{L}_{V_2}$, for $V_1 \subseteq V_2$, in the simple screen structure as the maps

$$\begin{cases} \mathcal{L}^r_{q_r^{n,\mathbf{m}}(V_1)} \to \mathcal{L}^r_{q_r^{n,\mathbf{m}}(V_2)} & V_2 \subseteq M'_r \text{ some } r \\ \mathcal{L}^0_{p^{n,\mathbf{m}}(V_1)} \to \mathcal{L}^0_{p^{n,\mathbf{m}}(V_2)} & V_1 \not\subseteq M'_r \text{ any } r \\ 0 & \text{else} \end{cases}.$$

By lemma 6.20 it is clear that $\{\rho_V\}$ with the maps $\mathcal{L}_{V_1} \to \mathcal{L}_{V_2}$ is a compatible complete collection simple d-screen. Furthermore, we have defined $\mathcal{L}_{V_1} \to \mathcal{L}_{V_2}$ to be the zero morphism if $V_1 \subseteq M'_r$ and $V_2 \not\subseteq M'_r$. This implies that $\{\rho_V\}_{V\subseteq[m]} \in \tau_{d,m}(M'_1, \dots, M'_n)(H)$. This map

$$f : \tau_{d,n} \times \tau_{d,m_1} \times \cdots \times \tau_{d,m_n} \to \tau_{d,m}(M'_1, \dots, M'_n)$$

is clearly natural and so we have defined a natural transformation.

The next step is find an inverse g of f. Let

$$\{\rho_V : \mathcal{F}^d_V \to \mathcal{L}_V\}_{V\in P(m)} \in \tau_{d,m}(M'_1, \dots, M'_n)(H).$$

Then we define

$$g(\{\rho_V\}_{V\in P(m)}) = \{\phi_T\}_{T\in P(n)} \times \{\psi^1_{S^1}\}_{S_1\in P(m_1)} \times \cdots \times \{\psi^n_{S^n}\}_{S_n\in P(m_n)} \in (\tau_{d,n} \times \tau_{d,m_1} \times \cdots \times \tau_{d,m_n})\,(H)$$

in the following way. In what follows we again denote the image of ϕ_T by $\mathcal{L}^0_T$ and the image of $\psi^r_{S_r}$ by $\mathcal{L}^r_{S_r}$. First define

$$\phi^r_{S_r} = \rho_{(q_r^{n,\mathbf{m}})^{-1}(S_r)} \circ (\beta_r^{n,\mathbf{m}})^{-1} : \mathcal{F}_{S_r} \to \mathcal{L}^r_{S_r}$$

where $\mathcal{L}^r_{S_r} = \mathcal{L}_{\rho_{(q_r^{n,\mathbf{m}})^{-1}(S_r)}}$ and where the maps $\mathcal{L}^r_{S_{r,1}} \to \mathcal{L}^r_{S_{r,2}}$ are just the maps $\mathcal{L}^r_{(q_r^{n,\mathbf{m}})^{-1}(S_{r,1})} \to \mathcal{L}^r_{(q_r^{n,\mathbf{m}})^{-1}(S_{r,2})}$. This is clearly well defined and compatible since $q_r^{n,\mathbf{m}}$ is bijective and $\beta_r^{n,\mathbf{m}}$ is an isomorphism. Defining ϕ_T is somewhat trickier. Let $\mathcal{L}^0_T = \mathcal{L}_{(p^{n,\mathbf{m}})^{-1}(T)}$ and let the morphisms $\mathcal{L}^0_{T_1} \to \mathcal{L}^0_{T_2}$ be the morphisms $\mathcal{L}_{(p^{n,\mathbf{m}})^{-1}(T_1)} \to \mathcal{L}_{(p^{n,\mathbf{m}})^{-1}(T_2)}$. The map

$$\alpha^{n,\mathbf{m}} : \mathcal{F}_V \to \mathcal{F}_{p^{n,\mathbf{m}}(V)}$$

is a quotient map with kernel generated by all elements of the form t^k_{ij}, $i, j \in M'_r$, some r. Furthermore, since $\{\rho_V\}_{V\in P(m)} \in \tau_{d,m}(M'_1, \dots, M'_n)(H)$ we have that for any $T \subseteq [n]$, the morphism $\rho_{(p^{n,\mathbf{m}})^{-1}(T)}$ sends any element of the form t^k_{ij}, $i, j \in M'_r$ to 0. By the universal property of the quotient there is a unique morphism, which we define ϕ_T to be, making the following diagram commute.

$$\begin{array}{ccc} \mathcal{F}_{(p^{n,\mathbf{m}})^{-1}(T)} & \xrightarrow{\alpha^{n,\mathbf{m}}} & \mathcal{F}_T \\ & \searrow^{\rho_{(p^{n,\mathbf{m}})^{-1}(T)}} & \downarrow \phi_T \\ & & \mathcal{L}_T^0 = \mathcal{L}_{(p^{n,\mathbf{m}})^{-1}(T)} \end{array}$$

Using lemma 6.20 again it is clear that the maps we have defined here are again compatible. Verifying that g is the inverse of f is a simple bookkeeping exercise which is left as an exercise to the reader. Since $g = f^{-1}$ it is natural and so we have found natural isomorphisms between the functors. □

Remark. Since we defined $T_{d,n}(\{i\}) = T_{d,n}([n]) = T_{d,n}$ this statement is true even in the case $n = 1$ or $m_r = 1$.

There is a natural generalization of this statement which will be important in the next section. I will not state this in its full generality because the amount of indexes required to keep track of this is simply not worth the effort.

Corollary 6.22. *For any $S \subsetneq [n]$, $|S| \geq 2$, the isomorphism of the proposition restricts to an isomorphism of closed subschemes*

$$T_{d,n}(S) \times T_{d,m_1}, \times \cdots \times T_{d,m_n} \cong T_{d,m}(M'_1, \ldots, M'_n, S'),$$

where $S' = (p^{n,\mathbf{m}})^{-1}(S)$ Similarly, for any $S_r \subsetneq [m_r]$, $|S_r| \geq 2$, the isomorphism of the proposition restricts to an isomorphism of closed subschemes

$$T_{d,n} \times T_{d,m_1}, \times \cdots \times T_{d,m_r}(S) \times \cdots \times T_{d,m_n} \cong T_{d,m}(M'_1, \ldots, M'_n, S'_r),$$

where $S'_r = (q_r^{n,\mathbf{m}})^{-1}(S_r)$.

Proof. Let

$$f \colon \tau_{d,n} \times \tau_{d,m_1} \times \cdots \times \tau_{d,m_n} \to \tau_{d,m}(M'_1, \ldots, M'_n)$$

be as in the proof of proposition 6.21 and let

$$\{\phi_T\}_{T \in P(n)} \times \{\psi^1_{S^1}\}_{S_1 \in P(m_1)} \times \cdots \times \{\psi^n_{S^n}\}_{S_n \in P(m_n)} \in (\tau_{d,n} \times \tau_{d,m_1} \times \cdots \times \tau_{d,m_n})\,(H)$$

and

$$f\left(\{\phi_T\}_{T \in P(n)} \times \{\psi^1_{S^1}\}_{S_1 \in P(m_1)} \times \cdots \times \{\psi^n_{S^n}\}_{S_n \in P(m_n)}\right) = \{\rho_V\}_{V \in P(m)}.$$

Also, let the invertible sheaves $\mathcal{L}_V, \mathcal{L}^0_T, \mathcal{L}^r_{S_r}$ be as in the proof of 6.21.

For the first part we want to show that $\{\phi_T\}_{T \in P(n)} \in \tau_{d,n}(S)$ if and only if $\{\rho_V\}_{V \in P(m)} \in \tau_{d,m}((p^{n,\mathbf{m}})^{-1}(S))$. First suppose $\{\phi_T\}_{T \in P(n} \in \tau_{d,n}(S)$ and let $V \subseteq [m]$, such that $V \not\subseteq S'$ and $|V \cap S'| \geq 2$. Note that this implies that $V \not\subseteq M'_r$ for any r. If $V \cap S' \subseteq M'_r$ then $\mathcal{L}_{V \cap S'} \to \mathcal{L}_V$ is the 0 morphism by definition of these maps. If this is not the case then the morphism $\mathcal{L}_{V \cap S'} \to \mathcal{L}_V$ is just the map $\mathcal{L}^0_{p^{n,\mathbf{m}}(V) \cap S} \to \mathcal{L}^0_{p^{n,\mathbf{m}}(V)}$.

Since S' is the inverse image of S and V is not a subset of S' we have that $p^{n,\mathbf{m}}(V) \not\subseteq S$. Thus $\mathcal{L}^0_{p^{n,\mathbf{m}}(V) \cap S} \to \mathcal{L}^0_{p^{n,\mathbf{m}}(V)}$ is the 0 map and therefore $\{\rho_V\}_{V \subseteq [m]} \in \tau_{d,m}(S')$.

For the converse assume that $\{\rho_V\}_{V\subseteq[m]} \in \tau_{d,m}(p^{n,\mathbf{m}})^{-1}(S))$. Then let $T \subseteq [n]$, such that $T \not\subseteq S$ and $|T\cap S| \geq 2$. Then the map $\mathcal{L}^0_{S\cap T} \to \mathcal{L}^0_T$ is the map

$$\mathcal{L}_{S'\cap(p^{n,\mathbf{m}})^{-1}(T)} \to \mathcal{L}_{(p^{n,\mathbf{m}})^{-1}(T)}$$

which is the 0 morphism since $\{\rho_V\}_{V\in P(m)} \in \tau_{d,m}(S')$. This implies that $\{\phi_T\}_{T\in P(n)} \in \tau_{d,n}(S)$ and so the proof of the first statement is done. The proof of the second part is analogous. □

Proposition 6.21 allows us to define the operad composition maps

$$\gamma^{n,\mathbf{m}} \colon T_{d,n} \times T_{d,m_1}, \times \cdots \times T_{d,m_n} \to T_{d,m}$$

by composing the isomporphism $T_{d,n} \times T_{d,m_1}, \times \cdots \times T_{d,m_n} \cong T_{d,m}(M'_1,\ldots,M'_n)$ with the inclusion map $T_{d,m}(M'_1,\ldots,M'_n) \hookrightarrow T_{d,m}$.

Finally, we need to define a group action by Σ_n on $T_{d,n}$. To do this we first define permutation maps of simple screens. For each set $S \in P(n)$ a permutation $\sigma \in \Sigma_n$ induces a map

$$\sigma_S^{X,d} \colon \mathcal{F}_S^{X,d} \to \mathcal{F}_{\sigma(S)}^{X,d},$$

or just σ_S when X, d are clear from context, by sending $t^k_{ij} \mapsto t^k_{\sigma(i)\sigma(j)}$. These maps clearly commute with the inclusions of difference sheaves, i.e.

$$i_{\sigma(S_1),\sigma(S_2)} \circ \sigma_{S_1} = \sigma_{S_2} \circ i_{S_1,S_2}.$$

It is also clear that for any morphism $f \colon X \to Y$ the pullback of σ_S^Y is just σ_S^X. Lastly, it is clear from the definition that for any $\mu, \sigma \in \Sigma_n$,

$$\mu_{\sigma(S)} \circ \sigma_S = (\mu\sigma)_S.$$

For $\sigma \in \Sigma_n$ the isomorphism $\sigma \colon T_{d,n} \to T_{d,n}$ is now defined as the morphism induced by the natural transformation of functors $\sigma \colon \tau_{d,n} \to \tau_{d,n}$ which sends

$$\{\phi_S \colon F_S \twoheadrightarrow \mathcal{L}_S\}_{S\in P(n)} \in \tau_{d,n}(H) \mapsto \{\psi_S \colon \mathcal{F}_S \twoheadrightarrow \mathcal{L}'_S\}_{S\in P(n)} \in \tau_{d,n}(H),$$

where $\mathcal{L}'_S = \mathcal{L}_{\sigma^{-1}(S)}$, $\psi_S = \phi_{\sigma^{-1}(S)} \circ \sigma_S^{-1}$, and $\mathcal{L}'_{S_1} \to \mathcal{L}'_{S_2}$ is the map $\mathcal{L}_{\sigma^{-1}(S_1)} \to \mathcal{L}_{\sigma^{-1}(S_2)}$.

Since the inclusions of difference sheaves commute with σ_S^{-1} it is clear that $\{\psi_S\}_{S\in P(n)}$ is a complete collection of simple screens and since the permutation maps σ_S^Y pull back to the maps σ_S^X for any map $X \to Y$ the map $\sigma \colon \tau_{d,n} \to \tau_{d,n}$ is natural. Finally, the relationship

$$\sigma^{-1}_{\mu^{-1}(S)} \circ \mu_S^{-1} = (\mu\sigma)_S$$

implies that

$$\mu \circ \sigma \colon \tau_{d,n} \to \tau_{d,n} = \mu\sigma \colon \tau_{d,n} \to \tau_{d,n}.$$

Thus this defines an action by isomorphisms. The following property will be important in the next section.

Proposition 6.23. *For every $\sigma \in \Sigma_n$ and every $S \subsetneq [n]$, $|S| \geq 2$, the isomorphism $\sigma \colon T_{d,n} \to T_{d,n}$ restricts to* *an isomorphism of closed subschemes*

$$\sigma_n \colon T_{d,n}(S) \to T_{d,n}(\sigma(S)).$$

Proof. We have that σ maps $\{\phi_S \colon F_S \twoheadrightarrow \mathcal{L}_S\}_{S\in P(n)} \in \tau_{d,n}(H) \mapsto \{\psi_S \colon \mathcal{F}_S \twoheadrightarrow \mathcal{L}'_S\}_{S\in P(n)} \in \tau_{d,n}(H)$ where $\mathcal{L}'_S = \mathcal{L}_{\sigma^{-1}(S)}$, $\psi_S = \phi_{\sigma^{-1}(S)} \circ \sigma_S^{-1}$, and $\mathcal{L}'_{S_1} \to \mathcal{L}'_{S_2}$ is the map $\mathcal{L}_{\sigma^{-1}(S_1)} \to \mathcal{L}_{\sigma^{-1}(S_2)}$. If $\{\phi_S \colon F_S \twoheadrightarrow \mathcal{L}_S\}_{S\in P(n)} \in \tau_{d,n}(S)(H)$ then for every $T \subseteq [n]$ $T \not\subseteq S$, $|T \cap S| \geq 2$ we have that $\mathcal{L}_{T\cap S} \to \mathcal{L}_T$. This implies that whenever $T' \subseteq [n]$, $T' \not\subseteq \sigma(S)$, $|T' \cap \sigma(S)| \geq 2$ the morphism

$$\mathcal{L}_{\sigma^{-1}(T')\cap S} = \mathcal{L}'_{T'\cap\sigma(S)} \to \mathcal{L}'_{T'} = \mathcal{L}_{\sigma^{-1}(T')}$$

is trivial which implies that

$$\{\psi_S \colon \mathcal{F}_S \twoheadrightarrow \mathcal{L}'_S\}_{S\in P(n)} \in \tau_{d,n}(\sigma(S))(H).$$

The converse is similar. □

Proposition/Definition 6.24. *The schemes $T_{d,n}$ with $n \in \mathbb{N}$ form an operad, which I call the "dimension d geometric Kontsevich operad", with unit morphism $\eta \colon \mathrm{Spec}\ \Bbbk \to T_{d,1}$, composition maps $\gamma^{n,\mathbf{m}}$, and action by Σ_n as described above.*

Proof. Using the natural transformations of functors of points defining these maps verifying the operad axioms is a very tedious but straight forward bookkeeping exercise. I will omit the details to avoid spending several pages doing trivial calculations. □

7 Log Structures on $T_{d,n}$

In this chapter I will define the log schemes $\mathsf{T}_{d,n}$ whose underlying schemes are $T_{d,n}$. After defining these log varieties the composition and symmetry morphisms of the geometric Kontsevich operad from the previous section will be extended to morphisms of log varieties which still satisfy the associativity and equivariance operad axioms. Unfortunately, the unit morphism will not extend to a map of log schemes meaning that the structure I define is in fact an operad without unit. Although this operad without unit will be well defined over any base field $\Bbbk$ the main motivation behind the construction is that the $\mathbb{S}^1$-framed Kontsevich Operad in dimension $2d$, $\mathcal{K}_{2d} \rtimes \mathbb{S}^1$, without unit, which was described in section 4, is isomorphic to the Kato-Nakayama analytification of this operad without unit over $\mathbb{C}$.

7.1 The Log Geometric Kontsevich Operad

In this section I will extend the operad structure of the previous section to an operad of log-schemes. These operads of log schemes are the main structures of interest of this paper. This section will feature many expressions with a large number of indexes in various different ways. To simplify the notation somewhat I will, in this section specifically, let d be some fixed positive integer and therefore omit this from the notation when I can, writing T_n for $T_{d,n}$ for example. While using a slightly different notation for one section only may be a bit confusing I hope that this still positively affects the readability. In what follows we will make frequent use of the following result. This result is almost definitively standard and I would have preferred to reference some textbook for the proof but I could not find it in either of my two favourite books on Algebraic Geometry and therefore I have included a proof in the appendix.

Proposition 7.1. *Let $D \hookrightarrow Y$ be an effective Cartier divisor and let $s\colon \mathcal{O}_Y \to \mathcal{O}_Y(D)$ be the corresponding line bundle with section. Furthermore, let X be an irreducible scheme and let $f\colon X \to Y$ be a morphism of schemes such that there is an isomorphism of X-schemes*

$$D \times_Y X \cong D'_1 \cup \cdots \cup D'_n$$

where D'_i are all (disjoint) effective Cartier divisors with corresponding sections $s'_i\colon \mathcal{O}_X \to \mathcal{O}_X(D'_i)$. Then there is a unique isomorphism of $\mathcal{O}_X$-modules,

$$\phi\colon f^*\mathcal{O}_Y(D) \to \bigotimes_{i=1}^{n} \mathcal{O}_X(D'_i)$$

such that $\phi(f^*s) = \bigotimes_i s'_i$.

Proof. See appendix A. □

Using this we will define a DF log structure on the schemes T_n and extend the operad morphisms to morphisms of log-schemes. Recall that this just means we want to define a bunch of invertible sheaves with sections on each T_n. For every n we will add one line bundle with section for every non empty subset $S \subseteq [n]$ which will be denoted $s^n_S\colon \mathcal{O}_{T_S} \to \mathcal{O}_{T_n}(S)$ or just $s_S\colon \mathcal{O} \to \mathcal{O}(S)$ when n is clear from context. Most of these line bundles we have essentially already encountered since for $|S| \geq 2$ and $S \neq [n]$ we define

$$\mathcal{O}_{T_n}(S) := \mathcal{O}_{T_n}(T_n(S))$$

with the canonical sections s^n_S (or just s_S) which cut out the effective Cartier divisors $T_n(S)$.

The remaining line bundles are trickier to define since they do not correspond to any divisors of $T_{d,n}$. However, before we get there I think it is natural to first discuss some properties of the line bundles we have already defined. In doing this we will see that with these line bundles alone we cannot extend the operad structure of the previous section to an operad of log schemes. This then motivates why we would want to to introduce some extra line bundles with sections that do not correspond to any divisors of T_n.

Lemma 7.2. *Let n and $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ be positive integers and let*

$$\pi_0\colon T_n \times T_{m_1} \times \cdots \times T_{m_n} \to T_n$$

and

$$\pi_r\colon T_n \times T_{m_1} \times \cdots \times T_{m_n} \to T_{m_r}$$

denote the corresponding projection maps. Let $M'_r = \{\sum_{i<r} m_i + 1, \sum_{i<r} m_i + 2, \ldots, \sum_{i\leq r} m_i\}$ and let $S \subseteq [m]$ where $m = \sum_{i=1}^{n} m_i$.

- *If $S \subsetneq M'_r$ for some r then there is a unique isomorphism*

$$\gamma^*_{n,\mathbf{m}}\mathcal{O}_{T_m}(S) \cong \pi^*_r\mathcal{O}_{T_{m_r}}((q^{n,\mathbf{m}}_r)^{-1}(S))$$

which sends $(\gamma^{n,\mathbf{m}})^* s_S \mapsto \pi^*_r s_{(q^{n,\mathbf{m}}_r)^{-1}(S)}$.

- *If* $S = \bigcup_{i\in I} M'_{r_i}$ *for some set* $I \subseteq [n]$ *of at least two elements then there is a unique isomorphism*

$$(\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(S) \cong \pi_0^*\mathcal{O}_{T_n}(I) = \pi_0^*\mathcal{O}_{T_n}((p^{n,\mathbf{m}})^{-1}(S))$$

which sends $(\gamma^{n,\mathbf{m}})^* s_S \mapsto \pi_0^* s_I$.

- *If* $S \not\subseteq M_r$ *for any* r *and* S *is not of the form* $S = \bigcup_{i\in I} M'_{r_i}$, *then there is a unique isomorphism*

$$(\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(S) \cong \mathcal{O},$$

where $\mathcal{O}$ *is the structure sheaf of* $T_n \times T_{m_1} \times \cdots \times T_{m_n}$ *which sends* $(\gamma^{n,\mathbf{m}})^* s_S \mapsto 1$.

Proof. The last case, $S \not\subseteq M_r$ for any r and S is not of the form $S = \bigcup_{i\in I} M'_{r_i}$, is obvious since the image of $\gamma^{n,\mathbf{m}}$ is $T_m(M'_1, \ldots, M'_n)$ which is disjoint from the closed subscheme $T_m(S)$ which is cut out by $s_S \colon \mathcal{O}_{T_m} \to O_{T_m}(S)$. On the complement of $T_m(S)$ $\mathcal{O}_{T_m}(S)$ is trivial and the section is a unit and thus the same result for the pullback follows.

For $S \subsetneq M'_r$ note that corollary 6.22 implies that the subscheme cut out by $\pi_r^* s_{(q_r^{n,\mathbf{m}})^{-1}(S)}$ is the pullback of $T_m(S)$, the subscheme cut out by s_S and thus, by proposition 7.1 there is a unique isomorphism

$$\gamma_{n,\mathbf{m}}^*\mathcal{O}_{T_m}(S) \cong \pi_r^*\mathcal{O}_{T_{m_r}}((q_r^{n,\mathbf{m}})^{-1}(S))$$

which sends $(\gamma^{n,\mathbf{m}})^* s_S \mapsto \pi_r^* s_{(q_r^{n,\mathbf{m}})^{-1}(S)}$.

Similarly, if $S = \bigcup_{i\in I} M'_{r_i}$ for some set $I \subseteq [n]$ of at least two elements then corollary 6.22 and proposition 7.1 again imply there is a unique isomorphism

$$(\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(S) \cong \pi_0^*\mathcal{O}_{T_n}(I) = \pi_0^*\mathcal{O}_{T_n}((p^{n,\mathbf{m}})^{-1}(S))$$

which sends $(\gamma^{n,\mathbf{m}})^* s_S \mapsto \pi_0^* s_I$ by the exact same argument. □

This result is not sufficient to define a morphism of log schemes since we are yet to describe the pullbacks of the line bundles $\mathcal{O}_{T_m}(M'_r)$ with sections. In fact, with the data we have so far it is impossible to find an expression for these pullbacks with sections terms of tensor products of the line bundles we have defined so far. This is why we need to introduce more line bundles to make this operad structure well defined.

To define the last line bundles with sections in our log structure we first need to introduce a few morphisms of varieties. We first define the "add one" morphisms $a_l^n \colon T_n \to T_{n+1}$ for each $1 \le l \le n+1$. It is easier to define a_l^n from its induced natural transformation of functors $\tau_n \to \tau_{n+1}$. These functors send complete collections of simple screens

$$\tau_n(H) \ni \{\phi_S \colon \mathcal{F}_S \to \mathcal{L}_S\}_{S\in P(n)} \to \{\psi_T \colon \mathcal{F}_T \to \mathcal{L}'_T\}_{T\in P(n+1)} \in \tau_{n+1}(H).$$

When $1 \le l \le n$ we define

$$\mathcal{L}'_T := \begin{cases} \mathcal{O}_H & \text{if } T = \{l, n+1\} \\ \mathcal{L}_{s(T)} & \text{else} \end{cases}$$

where we define

$$s_l(x) = \begin{cases} x & x \neq n+1 \\ l & x = n+1 \end{cases}.$$

The quotients ψ_T are defined by

$$\psi_{\{l,n+1\}} \colon t^k_{l,n+1} \mapsto 1$$

and for $T \neq \{l, n+1\}$ we define

$$f_{T,l} \colon \mathcal{F}_T \to \mathcal{F}_{s(T)},\ t^k_{ij} \mapsto t^k_{s(i)s(j)}$$

and

$$\psi_T = \phi_{s(T)} \circ f_{T,l}.$$

When $l = n+1$ we instead define

$$\mathcal{L}'_T := \begin{cases} \mathcal{O}_H & \text{if } n+1 \in T \\ \mathcal{L}_T & \text{else} \end{cases}$$

The quotients ψ_T are defined by

$$\psi_T = \phi_T$$

if $n+1 \notin T$ and for $n+1 \in T$ we define

$$\psi_T \colon t^k_{ij} \mapsto \begin{cases} 1 & i \neq n+1, j = n+1 \\ 0 & i \neq n+1, j \neq n+1 \end{cases}.$$

Definition 7.3. For each $i \in S$, the morphism a^n_l described above is called the l:th *add one morphism* $a^n_l \colon T_n \to T_{n+1}$.

For $1 \leq l \leq n$ the add one morphisms should be intuitively thought of as sending each n-pointed d-RTPS to itself but with the i:th point replaced by a specific new branch with the i:th and the $n+1$:th point on it. For $l = n+1$ the add one morphism should be thought of as attaching the entire tree to a specific new base branch with only one other marked point on it.

There is another way to describe a^n_l which, especially for $d = 1$, helps to provide some insight in why these morphisms are useful for us to define. First let Spec $\Bbbk \to T_2 \cong \mathbb{P}^{d-1}$ be the point with homogeneous coordinates $[1 : 1 : \cdots : 1]$. Then, for $1 \leq l \leq n$, a^n_l is just the composition

$$T_n \xrightarrow{\cong} T_n \times \text{Spec } \Bbbk \to T_n \times T_2 \xrightarrow{\circ_l} T_{n+1} \xrightarrow{\sigma} T_{n+1}$$

where $\sigma \in \Sigma_{n+1}$ is the cycle $\sigma = (n+1, n, \ldots, l+2, l+1)$. Similarly, a^n_{n+1} is the composition

$$T_n \xrightarrow{\cong} \text{Spec } \Bbbk \times T_n \to T_2 \times T_n \xrightarrow{\circ_1} T_{n+1}.$$

These definitions are a bit unsatisfactory since they are not canonical in the sense that we made a choice of point Spec $\Bbbk \to T_2$. For dimension $d = 1$ things are a lot nicer since $T_{1,2} \cong$ Spec $\Bbbk$ which means that there was only one possible choice of such a point and thus the add one morphisms are canonical in this case. For $n = 2$ these sheaves can be computed explicitly. We have

Lemma 7.4. *There are isomorphisms of sheaves*

$$\mathcal{O}_{T_2}(\{1\}) \cong \mathcal{O}_{T_2}(\{2\}) \cong \mathcal{O}_{\mathbb{P}^{d-1}}(1)$$

and

$$\mathcal{O}_{T_2}(\{1,2\}) \cong \mathcal{O}_{\mathbb{P}^{d-1}}(-1).$$

Proof. This proof relies heavily on explicit computations. I will omit the actual calculations from the text here and just state the results. We have that $T_{d,3} \cong \mathrm{Bl}_{\mathcal{I}_3}\, \mathrm{Bl}_{\mathcal{I}_2}\, \mathrm{Bl}_{\mathcal{I}_1}\, \mathbb{P}^{2d-1}$ where, if we write $\mathbb{P}^{2d-1} = \mathrm{Proj}\ \Bbbk[s_1, \ldots, s_d, t_1, \ldots, t_d]$, the ideal sheaves are projectivisations of the ideals $I_1 = (s_1, \ldots, s_d)$, $I_2 = (t_1, \ldots, t_d)$, and $I_3 = (s_1 - t_1, \ldots, s_d - t_d)$. The closed subschemes Y_1, Y_2, Y_3 corresponding to the ideals I_1, I_2, I_3 are disjoint and thus the order of the blow ups do not matter and the exceptional divisor for each of the blow ups are unaffected by the other two blow ups. Each of the closed subschemes Y_1, Y_2, Y_3 is isomorphic to $\mathbb{P}^{d-1}$ and the corresponding exceptional divisors $\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3$ are all isomorphic to $\mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$ where the projective morphism $\tilde{Y}_i \to Y_i$ is given by projection onto the first component $\mathbb{P}^{d-1} \times \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$. The normal bundle for $\tilde{Y}_i$ is the $\mathcal{O}_{\mathbb{P}^{d-1}\times\mathbb{P}^{d-1}}(1,-1)$ bundle. The permutation $(3,2)$ sends $\tilde{Y}_2 \to \tilde{Y}_3$ and the restriction of the permutation morphism to $\tilde{Y}_2$ is given by the identity $\mathbb{P}^{d-1} \times \mathbb{P}^{d-1} \to \mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$.

The isomorphism

$$\mathbb{P}^{d-1} \times \mathbb{P}^{d-1} \cong T_2 \times T_2 \xrightarrow{\circ_1} T_3(\{1,2\}) \cong \mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$$

is also just the identity map on $\mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$. Thus,

$$\circ_1^* \mathcal{O}_{T_3}(\{1,2\}) \cong \circ_2^* \mathcal{O}_{T_3}(\{2,3\}) \cong \circ_1^*(2,3)^* \mathcal{O}_{T_3}(\{1,3\}) \cong \mathcal{O}_{\mathbb{P}^{d-1}\times\mathbb{P}^{d-1}}(1,-1).$$

By definition of the add one morphisms this means that, $\mathcal{O}_{T_2}(\{1\}) = a_1^* \mathcal{O}_{T_3}(\{1,3\})$ is the pullback of $\mathcal{O}_{\mathbb{P}^{d-1}\times\mathbb{P}^{d-1}}(1,-1)$ via the closed embedding $i \colon \mathbb{P}^{d-1} \hookrightarrow \mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$ given by

$$\mathbb{P}^{d-1} \cong \mathbb{P}^{d-1} \times \mathrm{Spec}\ \Bbbk \hookrightarrow \mathbb{P}^{d-1} \times \mathbb{P}^{d-1}.$$

Clearly this is $i^* \mathcal{O}_{\mathbb{P}^{d-1}\times\mathbb{P}^{d-1}}(1,-1) \cong \mathcal{O}_{\mathbb{P}^{d-1}}(1)$. By symmetry the same is true for $\mathcal{O}_{T_2}(\{2\})$.

Finally, again by definition of the add one morphisms this means that $a_3^* \mathcal{O}_{T_3}(\{1,2\})$ is the pullback of $\mathcal{O}_{\mathbb{P}^{d-1}\times\mathbb{P}^{d-1}}(1,-1)$ via the closed embedding $j \colon \mathbb{P}^{d-1} \hookrightarrow \mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$ given by

$$\mathbb{P}^{d-1} \cong \mathrm{Spec}\ \Bbbk \times \mathbb{P}^{d-1} \hookrightarrow \mathbb{P}^{d-1} \times \mathbb{P}^{d-1}.$$

Clearly this is $j^* \mathcal{O}_{\mathbb{P}^{d-1}\times\mathbb{P}^{d-1}}(1,-1) \cong \mathcal{O}_{\mathbb{P}^{d-1}}(-1)$. □

Remark. In the $d = 1$ case one can show that $\mathcal{O}_{T_{1,n}}(\{i\}) \cong O_{T_{1,n}}([n])$. Without going into too much detail this is because $T_{1,n} \cong \overline{\mathcal{M}}_{0,n+1}$ which is acted on by Σ_{n+1} and using these permutations we can show that such isomorphisms must exist. With this in mind it might seem as if a mistake has been made here since $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ are seemingly not the same line bundle. However, no mistake has been made since for $d = 1$ we have $T_{1,2} \cong \mathrm{Spec}\ \Bbbk$ and thus all line bundles on $T_{1,2}$ are isomorphic.

There are also "remove one" morphisms $r_l^n \colon T_n \to T_{n-1}$. These are defined by the functions corresponding to natural transformations

$$\tau_n(H) \ni \{\phi_S \colon \mathcal{F}_S \to \mathcal{L}_S\}_{S\in P(n)} \mapsto \{\psi_T \colon \mathcal{F}_T \to \mathcal{L}'_T\}_{T\in P(n-1)} \in \tau_{n-1}(H),$$

where $\mathcal{L}'_T := \mathcal{L}_{d_l(T)}$ and $\psi_T \colon t^k_{ij} \mapsto \phi_{d_l(T)}(t^k_{d_l(i)d_l(j)})$, where

$$d_l \colon x \mapsto \begin{cases} x & x < l \\ x+1 & x \geq l \end{cases}.$$

Notice that by this definition we have $r^n_l = r^n_n \circ \sigma$ where σ is the morphism corresponding to the permutation $(n, n-1, \ldots, i+1, i)$.

Definition 7.5. The morphism r^n_l described above is called the l:th *remove one* morphism $r^n_l \colon T_n \to T_{n-1}$.

Remark. Notice that unlike the add one morphisms the remove one morphisms are canonical for every dimension.

The remove one morphisms also has an intuitive interpretations. For $1 \leq l \leq n$ the remove one morphisms should be intuitively thought of as sending each n-pointed d-RTPS to itself but with the l:th point removed if the resulting tree is still stable. If the resulting tree is not stable there are two cases to consider. If the tree has one other marked point on the branch of the l:th point then remove this branch and add the other point to the intersection point of the removed branch and the parent branch. If the branch of the l:th point has a daughter branch then the branch of the l:th point is removed and the associated hyperplane of daughter branch is "attached" to the exceptional divisor which previously attached the now removed branch. Both of these cases can be thought of as "collapsing" the branch of the l:th point, especially in the $d = 1$ case.

The "add one" and "remove one" morphisms are related to each other in the obvious way

Lemma 7.6. *The composition*

$$r^{n+1}_{n+1} \circ a^n_l \colon T_n \to T_n$$

is the identity isomorphism.

Proof. This follows immediately from the definition of the natural transformations of functors since $d_{n+1}(x) = a_l(x) = x$ for every $x \in [n]$. ☐

With this we are ready to define the remaining line bundles $\mathcal{L}_S$. For $n > 1$ we define

$$\mathcal{O}_{T_n}(\{l\}) := (a^n_l)^* \mathcal{O}_{T_{n+1}}(T_{n+1}(\{l, n+1\}))$$

and

$$\mathcal{O}_{T_n}([n]) := (a^n_{n+1})^* \mathcal{O}_{T_{n+1}}(T_{n+1}([n])).$$

The sections $s_{\{l\}}$ and $s_{[n]}$ are the pullbacks of the corresponding canonical sections via the morphisms above. It is easy to see that these pullbacks must be identically 0 so $s_{\{l\}} = 0$ and $s_{[n]} = 0$. For $n = 1$ we define

$$\mathcal{O}_{T_1}(\{1\}) = \mathcal{O}_{T_1}$$

with section $s_{\{1\}} = 0$.

There is also one non-canonical choice we will eventually have to make in the construction of the operad of log schemes. I include it here to get it over with. For the remainder of this thesis we choose two isomorphisms

$$e_1 \colon (a_1^1)^* \mathcal{O}_{T_2}(\{1,2\}) \to \mathcal{O}_{T_1}(\{1\}),$$

and

$$e_2 \colon (a_2^1)^* \mathcal{O}_{T_2}(\{1,2\}) \to \mathcal{O}_{T_1}(\{1\}).$$

Such isomorphisms exist since all line bundles are trivial on $T_1 \cong \operatorname{Spec} \Bbbk$.

Now that all of the relevant line bundles with sections are defined we can give the following definition.

Definition 7.7. T_n denotes the log scheme

$$\mathsf{T}_n := \left(T_n, \{s_S \colon \mathcal{O}_{T_n} \to \mathcal{O}_{T_n}(S)\}_{S\subseteq[n]}\right).$$

Because it is easier, and also needed in some of the proofs to come, I will describe the symmetry group action on these log schemes before I finish defining the log scheme composition morphisms.

Proposition 7.8. *For every $S \subseteq [n]$ and every $\sigma \in \Sigma_n$ there is a canonical morphism*

$$\sigma^* \mathcal{O}_{T_n}(S) \xrightarrow{\cong} \mathcal{O}_{T_n}(\sigma^{-1}(S))$$

which sends $\sigma^* s_S mapsto s_{\sigma^{-1}(S)}$.

Proof. In the case $n = 1$ the action by Σ_1 is trivial. Thus there is nothing to prove in this case and we may assume $n \geq 2$. For $S \subsetneq [n]$ with $|S| \geq 2$ this follows immediately from proposition 6.23 and proposition 7.1.

Next, it is easy to verify that the following diagram commutes for every $i \in [n]$.

$$\begin{array}{ccc} T_n & \xrightarrow{\sigma} & T_n \\ \downarrow{\scriptstyle a_{\sigma^{-1}(i)}} & & \downarrow{\scriptstyle a_i} \\ T_{n+1} & \xrightarrow{\sigma_+} & T_{n+1} \end{array}$$

where $\sigma_+ \in \Sigma_{n+1}$ is the permutation of $n+1$ elements which fixes $n+1$ and sends all other $j \in [n+1]$ to $\sigma(j)$. Thus,

$$\sigma^* \mathcal{O}_{T_n}(\{i\}) = \sigma^* a_i^* \mathcal{O}_{T_{n+1}}(\{i, n+1\}) \cong a_{\sigma^{-1}(i)}^* \sigma_+^* \mathcal{O}_{T_{n+1}}(\{i, n+1\}).$$

Since $n + 1 \geq 3$ the first part of the proof shows us that there is a unique isomorphism of sheaves with sections

$$\mathcal{O}_{T_{n+1}}(\{\sigma^{-1}(i), n+1\}) \xrightarrow{\cong} \sigma_+^* \mathcal{O}_{T_{n+1}}(\{i, n+1\}).$$

This isomorphism pulls back to a canonical morphism of sheaves

$$\mathcal{O}_{T_n}(\sigma^{-1}(i)) = a_{\sigma^{-1}(i)}^* \mathcal{O}_{T_{n+1}}(\{\sigma^{-1}(i), n+1\}) \xrightarrow{\cong} a_{\sigma^{-1}(i)}^* \sigma_+^* \mathcal{O}_{T_{n+1}}(\{i, n+1\}) \xrightarrow{\cong} \sigma^* \mathcal{O}_{T_n}(\{i\}).$$

□

This proposition implies that the symmetry action of Σ_n on T_n extends to a symmetry action on the log-scheme T_n.

Next, we need a few more results until we are ready to go on defining the composition maps for the log schemes T_n.

Lemma 7.9. *For every $S \subsetneq [n+1]$ and $|S| \geq 2$ there is a canonical isomorphism of sheaves*

$$(a_{n+1})^* \mathcal{O}_{T_{n+1}}(S) \cong \begin{cases} \mathcal{O}_{T_n} & n+1 \in S \\ \mathcal{O}_{T_{n+1}}(S \setminus \{n+1\}) & \text{else} \end{cases}$$

which sends

$$a_{n+1}^* s_S^{n+1} \mapsto \begin{cases} 1 & n+1 \in S \text{ and } S \neq [n+1] \\ s_S^n & \text{else} \end{cases}.$$

Similarly, for $l \neq n+1$, there is a canonical isomorphism of sheaves

$$(a_l)^* \mathcal{O}_{T_{n+1}}(S) \cong \begin{cases} \mathcal{O}_{T_n} & \{i, n+1\} \not\subseteq S, \text{ and } \{i, n+1\} \cap S \neq \emptyset, \\ \mathcal{O}_{T_{n+1}}(S \setminus \{n+1\}) & \text{else} \end{cases}$$

which sends

$$a_{n+1}^* s_S^{n+1} \mapsto \begin{cases} 1 & \{i, n+1\} \not\subseteq S, \text{ and } \{i, n+1\} \cap S \neq \emptyset \\ s_S^n & \text{else} \end{cases}.$$

Proof. I will do the proof for a_{n+1}. The proof in the other case is analogous. First note that the image of a_{n+1} is contained in $T_{n+1}([n])$. By proposition 6.15 this means that im $(a_{n+1}) \cap T_{n+1}(S) = \emptyset$ for every $S \subsetneq [n+1]$ with $n+1 \in S$. Therefore, a_{n+1} factors via an open set on which $\mathcal{O}_{T_n}(S)$ is trivial with the identity section. Thus, there is an isomorphism $(a_{n+1})^* \mathcal{O}_{T_{n+1}}(S) \to \mathcal{O}_{T_{d,n}}$ sending $a_{n+1}^* s_S^{n+1} \mapsto 1$ in this case.

Next, by definition of a_{n+1} it is easy to verify that the scheme theoretic inverse image of $T_{n+1}(S)$ is $T_n(S)$ for every $S \subsetneq [n]$ and so the result follows from proposition 7.1.

Finally, if $S = [n]$ this result is the definition of $\mathcal{O}_{T_n}([n])$. □

Lemma 7.10. *For every n and $S \subseteq [n]$ there is a canonical isomorphism of sheaves*

$$(r_{n+1})^* \mathcal{O}_{T_n}(S) \cong \mathcal{O}_{T_{n+1}}(S) \otimes \mathcal{O}_{T_{n+1}}(S \cup \{n+1\})$$

which sends

$$r_{n+1}^* s_S^n \mapsto s_S^{n+1} \otimes s_{S \cup \{n+1\}}^{n+1}.$$

Proof. We will need to divide this into two cases, $n = 1$ and $n \geq 1$. First assume $n = 1$. In this case $r_2^* \mathcal{O}_{T_1}(\{1\})$ is clearly the trivial sheaf and since lemma 7.4 implies that $\mathcal{O}_{T_2}(\{1\}) \otimes \mathcal{O}_{T_2}(\{1, 2\})$ is trivial too we are done. Strictly speaking we do need to make a choice of morphism here so this is not really canonical but as long as we make one choice for any isomorphism of this form and use that every time in the thesis an isomorphism of this form comes up in the thesis we have nothing to worry about.

For $n \geq 2$ we begin with the case $S \neq [n]$ and $|S| \geq 2$. First I claim that the scheme theoretic pullback, $r_{n+1}^{-1}(T_n(S)) = T_n(S) \times_{T_n} T_{n+1}$, is the scheme theoretic union of closed subschemes $T_{n+1}(S) \cup T_{n+1}(S \cup \{n+1\})$. To show this we use the functors of points. Let

$$\{\phi_S : \mathcal{F}_S \to \mathcal{L}_S\}_{S \in P(n+1)} \in \tau_n(H)$$

and

$$r_{n+1} : \{\phi_S : \mathcal{F}_S \to \mathcal{L}_S\}_{S \in P(n+1)} \mapsto \{\psi_T : \mathcal{F}_T \to \mathcal{L}'_T\}_{T \in P(n)} \in \tau_n(H).$$

We want to show that

$$r_{n+1}^{-1}(\tau_n(S)(H)) = \tau_{n+1}(S)(H) \cup \tau_{n+1}(S \cup \{n+1\})(H).$$

Suppose

$$\{\phi_S : \mathcal{F}_S \to \mathcal{L}_S\}_{S \in P(n+1)} \in \tau_{n+1}(S)(H)$$

and let $T \subseteq [n]$ such that $|T \cap S| \geq 2$, $T \not\subseteq S$. Then, for every $i, j \in T \cap S$ we have

$$\psi_T(t_{i,j}^k) = \phi_T(t_{i,j}^k) = 0$$

since $|T \cap S| \geq 2$ and $T \not\subseteq S$ implies $\phi_T(t_{i,j}^k) = 0$. Hence, $\{\psi_T : \mathcal{F}_T \to \mathcal{L}'_T\}_{T \subseteq [n]} \in \tau_n(S)(H)$. Similarly, suppose

$$\{\phi_S : \mathcal{F}_S \to \mathcal{L}_S\}_{S \in P(n+1)} \in \tau_{n+1}(S \cup \{n+1\})(H)$$

and let $T \subseteq [n]$ such that $|T \cap S| \geq 2$, $T \not\subseteq S$. Then, for every $i, j \in T \cap S$ we have

$$\psi_T(t_{i,j}^k) = \phi_T(t_{i,j}^k) = 0$$

since $T \subseteq [n]$, $T \not\subseteq S$ implies that $T \not\subseteq S \cup \{n+1\}$ and thus, $\phi_T(t_{i,j}^k) = 0$. Again this means that $\{\psi_T : \mathcal{F}_T \to \mathcal{L}'_T\}_{T \in P(n)} \in \tau_n(S)(H)$. Thus

$$r_{n+1}^{-1}(\tau_n(S)(H)) \subseteq \tau_{n+1}(S)(H) \cup \tau_{n+1}(S \cup \{n+1\})(H).$$

For the other inclusion suppose $\{\psi_T : \mathcal{F}_T \to \mathcal{L}'_T\}_{T \subseteq [n]} \in \tau_n(S)(H)$. Then, for any $T \subseteq [n]$ with $|T \cap S| \geq 2$ and $T \not\subseteq S$ we have

$$\psi_T(t_{i,j}^k) = \psi_T(t_{i,j}^k) = 0$$

for every $i, j \in S \cap T$. Now, suppose there is some $T \subseteq [n]$, $T \not\subseteq S$, with $|S \cap T| \geq 2$ and some $i \in S \cap T$ such that $\phi_{T \cup \{n+1\}}(t_{i,n+1}^k) \neq 0$. This means that the morphism

$$\mathcal{L}_{(T \cap S) \cup \{n+1\}} \to \mathcal{L}_{T \cup \{n+1\}}$$

is non-trivial, since the kernel of a non-trivial morphism of line bundles is trivial (for irreducible schemes), we must have that

$$\phi_{V \cup \{n+1\}}(t_{i,j}^k) = 0$$

for any $V \subseteq S \cap T$ and any $i, j \in V$. Next, notice that for any $V \subseteq S$ we have that $\phi_{T \cup V}(t_{i,j}^k) = 0$ for every $i, j \in (T \cup V) \cap S$. Therefore, $\phi_{T \cup V \cup \{n+1\}}(t_{i,j}^k) = 0$ for all $i, j \in (T \cup V) \cap S$. Since, $\phi_{T \cup V \cup \{n+1\}}$ is surjective there must be some $l, m \in T \cup \{n+1\}$ such that $\phi_{T \cup V \cup \{n+1\}}(t^k_{l,m}) \neq 0$ and thus $\mathcal{L}_{T\cup\{n+1\}} \to \mathcal{L}_{T\cup V\cup\{n+1\}}$ has trivial kernel and in particular $\phi_{T\cup V\cup\{n+1\}}(t^k_{i,n+1}) \neq 0$. Repeating the argument from before we find that for any $V \subseteq S$ and any $i, j \in V$ we have that

$$\phi_{V\cup\{n+1\}}(t^k_{i,j}) = 0.$$

From this it then follows that for any $T' \subseteq [n+1]$ with $T' \not\subseteq S$ and any $i, j \in T' \cap S$ we have $\phi_{T'}(t^k_{i,j}) = 0$. Thus

$$\{\phi_S : \mathcal{F}_S \to \mathcal{L}_S\}_{S\in P(n+1)} \in \tau_{n+1}(S)(H).$$

Now, suppose no $T \subseteq [n]$, $T \not\subseteq S$, with $|S \cap T| \geq 2$ and $i \in S \cap T$ such that $\phi_{T\cup\{n+1\}}(t^k_{i,n+1}) \neq 0$ exist. Then, I claim that

$$\{\phi_S : \mathcal{F}_S \to \mathcal{L}_S\}_{S\subseteq P(n+1)} \in \tau_{n+1}(S \cup \{n+1\})(H).$$

By assumption the only thing we must verify is that if $T \cap (S \cup \{n+1\}) = \{i, n+1\}$ for some $i \in S$ then $\phi_T(t^k_{i,n+1}) = 0$. To do this notice that, for any $j \in S \setminus \{i\}$ we have that $\phi_{T\cup\{j\}}(t^k_{i,n+1}) = 0$ by assumption. By the same argument as before the morphism $\mathcal{L}_T \to \mathcal{L}_{T\cup\{j\}}$ has trivial kernel and thus $\phi_T(t^k_{i,n+1}) = 0$. Thus

$$r^{-1}_{n+1}(\tau_n(S)(H)) = \tau_{n+1}(S)(H) \cup \tau_{n+1}(S \cup \{n+1\})(H).$$

With this the result follows from proposition 7.1.

Next, for the case $S = [n]$ I claim that the following diagram commutes.

$$\begin{array}{ccc} T_{n+1} & \xrightarrow{a^{n+1}_{n+2}} & T_{n+2} \\ \downarrow{\scriptstyle r^{n+1}_{n+1}} & & \downarrow{\scriptstyle r^{n+2}_{n+1}} \\ T_n & \xrightarrow{a^n_{n+1}} & T_{n+1} \end{array}$$

This is easy to verify using the definitions. Since $\mathcal{O}_{T_n}([n]) := (a^n_{n+1})^*\mathcal{O}_{T_{n+1}}([n])$ there are canonical isomorphisms

$$(r^{n+1}_{n+1})^*\mathcal{O}_{T_n}([n]) \xrightarrow{\cong} (r^{n+1}_{n+1})^*(a^n_{n+1})^*\mathcal{O}_{T_{n+1}}([n]) \xrightarrow{\cong} (a^{n+1}_{n+2})^*(r^{n+1}_{n+2})^*\mathcal{O}_{T_{n+1}}([n]).$$

Recall that $r^{n+2}_{n+1} = r^{n+2}_{n+2} \circ \sigma$ where σ is the morphism induced by the permutation $(n+1, n+2)$. Thus, by the first part of the lemma, and proposition 7.8, there is a canonical isomorphism $(r^{n+1}_{n+2})^*\mathcal{O}_{T_{n+1}}([n]) \xrightarrow{\cong} \mathcal{O}_{T_{n+2}}([n]) \otimes \mathcal{O}_{T_{n+2}}([n+1])$. Lastly, by lemma 7.9, there is a canonical isomorphism

$$(a^{n+1}_{n+2})^* \left(\mathcal{O}_{T_{n+2}}([n]) \otimes \mathcal{O}_{T_{n+2}}([n+1])\right) \xrightarrow{\cong} \mathcal{O}_{T_{n+1}}([n]) \otimes \mathcal{O}_{T_{n+1}}([n+1]).$$

This completes the proof in this case.

Finally, the case $S = \{l\}$ follows by the same argument as in the $S = [n]$ case but with a_l instead of a_{n+1}. □

With this out of the way we can move on to finally prove the last results needed to define the composition maps for the log schemes T_n.

In the next couple of proofs there will on several occasions be a lot of indexes to keep track of. To make things a little bit more compact, and hopefully a bit clearer, I will use the following notation. Let n be a positive integer, $\mathbf{m} = (m_1, \dots, m_n)$ be a list of positive integers, and $m = \sum_i m_i$. Define

- $m'_r := \sum_{i<r} m_i$ for every $r \in [n]$.
- $M'_r := \{m'_r + 1, m'_r + 2, \dots, m'_r + m_r\}$ for every $r \in [n]$.
- $\mathbf{m}_{r+} = (m_1, \dots, m_{r-1}, m_r + 1, m_{r+1}, \dots, m_n)$ for every $r \in [n]$.
- $\mathbf{m}_* = (m_1, \dots, m_{r-1}, m_r, m_{r+1}, \dots, m_n, 1)$.
- $T[\mathbf{v}] := T_{v_1} \times \cdots \times T_{v_k}$ for every list of integers $\mathbf{v} = (v_1, \dots, v_k)$
- $A^r_i := \mathrm{id} \times \mathrm{id}^{r-1} \times a^{m_r}_i \times \mathrm{id}^{n-r} \colon T_n \times T[\mathbf{m}] \to T_n \times T[\mathbf{m}_{r+}]$ for every $r \in [n]$ and $i \in [m_r]$.
- $A^0_i \colon T_n \times T[\mathbf{m}] \to T_{n+1} \times T[\mathbf{m}_*]$ as the composition of the morphism $a^n_i \times \mathrm{id}^n \colon T_n \times T[\mathbf{m}] \to T_{n+1} \times T[\mathbf{m}]$ and the isomorphism $T_{n+1} \times T[\mathbf{m}] \cong T_{n+1} \times T[\mathbf{m}] \times \mathrm{Spec}\, k$ for each $i \in [n]$. Note that by definition $T[\mathbf{m}_*] = T[\mathbf{m}] \times T_1$ and $T_1 = \mathrm{Spec}\, k$.
- $\pi^{n,\mathbf{m}}_0 \colon T_n \times T[\mathbf{m}] \to T_n$ as the projection onto the first component.
- $\pi^{n,\mathbf{m}}_r \colon T_n \times T[\mathbf{m}] \to T_{m_r}$ as the projection onto the $(r+1)$th component.
- $\sigma_i = (i, i+1, \dots, m+1) \in \Sigma_{m+1}$ for every $i \in [m+1]$.

Lemma 7.11. *There is a canonical isomorphism of line bundles*

$$(\gamma^{n,\mathbf{m}})^* \mathcal{O}_{T_m}(M'_r) \cong \pi_0^* \mathcal{O}_{T_n}(\{r\}) \otimes \pi_r^* \mathcal{O}_{T_{m_r}}([m_r])$$

which sends $(\gamma^{n,\mathbf{m}})^* s_{M'_r} \mapsto \pi_0^* s_{\{r\}} \otimes \pi_r * s_{[m_r]}$.

Proof. First note that given the first part of the statement the mapping of sections is clear since $(\gamma^{n,\mathbf{m}})^* s_{M'_r}$, $\pi_0^* s_{\{r\}}$, and $\pi_r * s_{[m_r]}$ are all the zero sections of the respective sheaves.

Next, e will need to divide this into 3 cases. For $n = r = 1$ we have $m_r = m_1 = m$ and

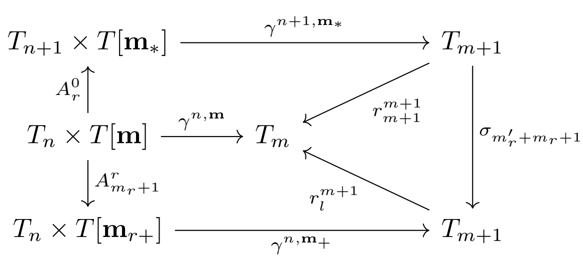
$$\gamma^{n,\mathbf{m}} \colon T_1 \times T_m \to T_m$$

is just the projection morphism $\gamma^{1,\mathbf{m}} = \pi_1$. Thus it is clear that there is a canonical isomorphism.

$$(\gamma^{1,\mathbf{m}}) * \mathcal{O}_{T_m}([m]) \cong \pi_1^* \mathcal{O}_{T_m}([m]) \cong \pi_1^* \mathcal{O}_{T_m}([m]) \otimes \pi_0 \mathcal{O}_{T_1}(\{1\}).$$

The last isomorphism here is the canonical isomorphism between a sheaf and its tensor product with the structure sheaf.

Next, let $n > 1$ and $m_r > 1$. I claim that the following is a commutative diagram.



This follows from tedious computations using the definitions of the respective maps. Proving this rigorously is a boring and not very rewarding exercise but you should be able to intuitively see why this is the case using the intuition behind each of these maps. From this diagram we conclude that there is a canonical isomorphism

$$(A_r^0)^*(\gamma^{n+1,\mathbf{m}})^*(r_{m+1}^{m+1})^*\mathcal{O}_{T_m}(M_r') \xrightarrow{\cong} (\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(M_r').$$

By lemma 7.10 there is a canonical isomorphism

$$\mathcal{O}_{T_{m+1}}(M_r') \otimes \mathcal{O}_{T_{m+1}}(M_r' \cup \{m+1\}) \cong (r_{m+1}^{m+1})^*\mathcal{O}_{T_m}(M_r').$$

Now, by lemma 7.2 there is a unique isomorphism of sheaves with sections

$$(\pi_0^{n+1,\mathbf{m}_*})^*\mathcal{O}_{T_{n+1}}(r, n+1) \xrightarrow{\cong} (\gamma^{n+1,\mathbf{m}_*})^*\mathcal{O}_{T_{m+1}}(M_r' \cup \{m+1\}).$$

Since

$$\pi_0^{n+1,\mathbf{m}_*} \circ A_r^0 = a_r^n \circ \pi_0^{n,\mathbf{m}}$$

this morphism pulls back to give a canonical isomorphism

$$(\pi_0^{n+1,\mathbf{m}})^*(a_r^n)^*\mathcal{O}_{T_{n+1}}(r, n+1) = (\pi_0^{n+1,\mathbf{m}})^*\mathcal{O}_{T_n}(r) \xrightarrow{\cong} (A_r^0)^*(\gamma^{n+1,\mathbf{m}_*})^*\mathcal{O}_{T_{m+1}}(M_r' \cup \{m+1\}).$$

Similarly, lemma 7.2, together with proposition 7.8 applied to the lower part of the diagram gives a canonical isomorphism

$$(\pi_r^{n,\mathbf{m}})^*\mathcal{O}_{T_{m_r}}([m_r]) \cong (A_{m_r+1}^r)^*(\gamma^{n,\mathbf{m}_+})^*(\sigma_{m_r'+m_r+1}^{-1})^*\mathcal{O}_{T_{m+1}}(M_r').$$

By commutativity of the diagram

$$\gamma^{n+1,\mathbf{m}_*} \circ A_r^0 = \sigma_{m_r'+m_r+1}^{-1} \circ \gamma^{n,\mathbf{m}_+} \circ A_{m_r+1}^r$$

and thus this isomorphism canonically induces an isomorphism

$$(\pi_r^{n,\mathbf{m}})^*\mathcal{O}_{T_{m_r}}([m_r]) \cong (A_r^0)^*(\gamma^{n+1,\mathbf{m}})^*\mathcal{O}_{T_{m+1}}(M_r').$$

This gives a canonical isomorphism

$$(\pi_0^{n+1,\mathbf{m}})^*\mathcal{O}_{T_n}(r) \otimes (\pi_r^{n,\mathbf{m}})^*\mathcal{O}_{T_{m_r}}([m_r]) \xrightarrow{\cong} (A_r^0)^*(\gamma^{n+1,\mathbf{m}})^* \left(\mathcal{O}_{T_{m+1}}(M_r') \otimes \mathcal{O}_{T_{m+1}}(M_r' \cup \{m+1\})\right).$$

Composed, with the isomorphism

$$\mathcal{O}_{T_{m+1}}(M_r') \otimes \mathcal{O}_{T_{m+1}}(M_r' \cup \{m+1\}) \cong (r_{m+1}^{m+1})^*\mathcal{O}_{T_m}(M_r')$$

this gives an isomorphism

$$(\pi_0^{n+1,\mathbf{m}})^*\mathcal{O}_{T_n}(r) \otimes (\pi_r^{n,\mathbf{m}})^*\mathcal{O}_{T_{m_r}}([m_r]) \xrightarrow{\cong} (A_r^0)^*(\gamma^{n+1,\mathbf{m}_*})^*(r_{m+1}^{m+1})^*\mathcal{O}_{T_m}(M_r').$$

Finally, since

$$\gamma^{n,\mathbf{m}} = r_{m+1}^{m+1} \circ \gamma^{n+1,\mathbf{m}_*} \circ A_r^0$$

this canonically induces an isomorphism

$$(\pi_0^{n+1,\mathbf{m}})^*\mathcal{O}_{T_n}(r) \otimes (\pi_r^{n,\mathbf{m}})^*\mathcal{O}_{T_{m_r}}([m_r]) \xrightarrow{\cong} (\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(M_r').$$

Finally, let $n > 1$, $m_r = 1$. For this case we will use the fact that the following diagram commutes.

$$\begin{array}{ccc} T_n \times T[\mathbf{m}] & \xrightarrow{\gamma^{n,\mathbf{m}}} & T_m \\ & & \downarrow a^m_{m_r'} \\ \downarrow A_1^r & & T_{m+1} \\ & & \downarrow \sigma_{m_r'+1} \\ T_n \times T[\mathbf{m}_{r+}] & \xrightarrow[\gamma^{n,\mathbf{m}_+}]{} & T_{m+1} \end{array}$$

Using the definition of the add one morphisms this follows directly from the fact that the composition morphisms satisfy the compatibility axioms for an operad. By definition

$$\mathcal{O}_{T_m}(M_r') = \mathcal{O}_{T_m}(\{m_r'\}) = (a^m_{m_r'})^*\mathcal{O}_{T_{m+1}}(\{m_r', m+1\}).$$

By proposition 7.8 there is a canonical isomorphism

$$\sigma^*_{m_r'+1}\mathcal{O}_{T_m+1}(m_r', m_r'+1) \xrightarrow{\cong} \mathcal{O}_{T_{m+1}}(\{m_r', m+1\}),$$

which pulls back to an isomorphism

$$(\gamma^{n,\mathbf{m}})^*(a^m_{m_r'})^*\sigma^*_{m_r'+1}\mathcal{O}_{T_m+1}(\{m_r', m_r'+1\}) \xrightarrow{\cong} (\gamma^{n,\mathbf{m}})^*(a^m_{m_r'})^*O_{T_{m+1}}(\{m_r', m+1\}).$$

Since there are canonical isomorphisms

$$(A_1^r)^*(\gamma^{n,\mathbf{m}_+})^*\mathcal{O}_{T_m+1}(\{m_r', m_r'+1\}) \cong (\gamma^{n,\mathbf{m}})^*(a^m_{m_r'})^*\sigma^*_{m_r'+1}\mathcal{O}_{T_m+1}(\{m_r', m_r'+1\})$$

and

$$(\gamma^{n,\mathbf{m}})^*(a^m_{m_r'})^*O_{T_{m+1}}(\{m_r', m+1\}) \cong (\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(\{m_r'\})$$

this induces a canonical isomorphism

$$(A_1^r)^*(\gamma^{n,\mathbf{m}_+})^*\mathcal{O}_{T_m+1}(\{m_r', m_r'+1\}) \xrightarrow{\cong} (\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(\{m_r'\}).$$

By the previous part there is a unique isomorphism of sheaves with sections

$$(\gamma^{n,\mathbf{m}_+})^*\mathcal{O}_{T_{m+1}}(\{m_r', m+1\}) \cong (\pi_0^{n,\mathbf{m}_+})^*\mathcal{O}_{T_n}(\{r\}) \otimes (\pi_r^{n,\mathbf{m}_+})^*\mathcal{O}_{T_2}(\{1,2\}).$$

By taking the pullback of this and composing with the above we get a canonical isomorphism

$$(A_r^1)^*\left((\pi_0^{n,\mathbf{m}_+})^*\mathcal{O}_{T_n}(\{r\}) \otimes (\pi_r^{n,\mathbf{m}_+})^*\mathcal{O}_{T_2}(\{1,2\})\right) \xrightarrow{\cong} (\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(\{m_r'\}),$$

which of course induces an isomorphism

$$(A_r^1)^*(\pi_0^{n,\mathbf{m}_+})^*\mathcal{O}_{T_n}(\{r\}) \otimes (A_r^1)^*(\pi_r^{n,\mathbf{m}_+})^*\mathcal{O}_{T_2}(\{1,2\}) \xrightarrow{\cong} (\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(\{m_r'\})$$

Finally, it is easy to see from definitions that

$$\pi_0^{n,\mathbf{m}_+} \circ A_r^1 = \pi_0^{n,\mathbf{m}}$$

and that

$$\pi_r^{n,\mathbf{m}_+} \circ A_r^1 = a_1 \circ \pi_r^{n,\mathbf{m}}.$$

Thus, there is a canonically induced isomorphism

$$(\pi_0^{n,\mathbf{m}})^*\mathcal{O}_{T_n}(\{r\}) \cong (A_r^1)^*(\pi_0^{n,\mathbf{m}_+})^*\mathcal{O}_{T_n}(\{r\}).$$

Furthermore, $e_1 \colon (a_1^1)^*\mathcal{O}_{T_2}(\{1,2\}) \to \mathcal{O}_{T_1}(\{1\})$ pulls back to a canonical isomorphism

$$(\pi_r^{n,\mathbf{m}})^*\mathcal{O}_{T_1}(\{1\}) \cong (A_r^1)^*(\pi_r^{n,\mathbf{m}_+})^*\mathcal{O}_{T_2}(\{1,2\}).$$

Composed with the above this finally gives us a canonical isomorphism

$$(\pi_0^{n,\mathbf{m}})^*\mathcal{O}_{T_n}(\{r\}) \otimes (\pi_r^{n,\mathbf{m}})^*\mathcal{O}_{T_1}(\{1\}) \xrightarrow{\cong} (\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(\{m_r'\}).$$

This completes the proof. □

Now, let us also show that show that the introduction of these sheaves with sections does not cause any new problems in defining our desired morphisms of log schemes.

Lemma 7.12. *For each $l \in M_r'$ there is a canonical isomorphism*

$$(\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(\{l\}) \cong \pi_r^*\mathcal{O}_{T_{m_r}}(\{q_r^{n,\mathbf{m}}(l)\})$$

if $m_r > 1$ and a canonical isomorphism

$$(\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(\{l\}) \cong \pi_0^*\mathcal{O}_{T_n}(\{r\}) \otimes \pi_r^*\mathcal{O}_{T_1}(\{1\})$$

if $m_r = 1$.

Proof. We split this into two cases. First consider $m_r > 1$. I claim that the following diagram commutes

$$\begin{array}{ccc}
T_n \times T[\mathbf{m}] & \xrightarrow{\gamma^{n,\mathbf{m}}} & T_m \\
\Big\downarrow{\scriptstyle A^r_{q_r^{n,\mathbf{m}}(l)}} & & \Big\downarrow{\scriptstyle a_l^m} \\
 & & T_{m+1} \\
 & & \Big\downarrow{\scriptstyle \sigma_{m_r'+m_r+1}} \\
T_n \times T[\mathbf{m}_{r+}] & \xrightarrow{\gamma^{n,\mathbf{m}_+}} & T_{m+1}
\end{array}$$

Through tedious computation this follows directly from the definitions of the morphisms.

From the definition of $\mathcal{O}_{T_m}(\{l\})$ it follows that there is a canonical isomorphisms

$$(\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(\{l\}) \cong (\gamma^{n,\mathbf{m}})^*(a_l^m)^*\sigma^*_{m'_r+1}\mathcal{O}_{T_{m+1}}(\{l, m'_r+m_r+1\}) \cong (A^r_{q_r^{n,\mathbf{m}}(l)})^*(\gamma^{n,\mathbf{m}_+})^*\mathcal{O}_{T_{m+1}}(\{l, m'_r+m_r+1\})$$

Lemma 7.2 implies that there is a unique isomorphism of sheaves with sections

$$(\pi_r^{n,\mathbf{m}_+})^*\mathcal{O}_{T_{m_r+1}}(\{q_r^{n,\mathbf{m}}(l), m_r+1\}) \cong (\gamma^{n,\mathbf{m}_+})^*\mathcal{O}_{T_{m+1}}(\{l, m'_r+m_r+1\}).$$

As in the proof of the previous lemma, taking the pullback of this isomorphism via $A_{q_r^{n,\mathbf{m}}(l)}$ induces a canonical isomorphism of sheaves

$$(\pi_r^{n,\mathbf{m}})^*\mathcal{O}_{T_{m_r}}(\{q_r^{n,\mathbf{m}}(l)\}) \cong (\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}(\{l\}).$$

The $m_r = 1$ case is just lemma 7.11 in the special case $m_r = 1$. □

Lemma 7.13. *There is a canonical isomorphism*

$$(\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}([m]) \cong \pi_r^*\mathcal{O}_{T_n}([n])$$

if $n > 1$ *and a canonical isomorphism*

$$(\gamma^{n,\mathbf{m}})^*\mathcal{O}_{T_m}([m]) \cong \pi_0^*\mathcal{O}_{T_1}(\{1\}) \otimes \pi_r^*\mathcal{O}_{T_{m_1}}([m_1])$$

if $n = 1$.

Proof. The proof is analogous to the proof of lemma 7.12. □

The isomorphisms of line bundles with sections from lemmas 7.2, 7.11, 7.12, and 7.13 define extensions of the composition maps $\gamma^{n,\mathbf{m}}$ of the geometric Kontsevich operad to maps of log schemes

$$\gamma^{n,\mathbf{m}} \colon \mathsf{T}_n \times \mathsf{T}_{m_1} \times \cdots \times \mathsf{T}_{m_n} \to \mathsf{T}_m.$$

Proposition/Definition 7.14. *The log schemes* T_n *with* $n \in \mathbb{N}$ *form an operad without unit with composition maps* $\gamma^{n,\mathbf{m}}$*, and action by* Σ_n *as described above. I call this the "log-geometric Kontsevich operad without unit", denoted* $\mathcal{T}_d$*.*

Proof. We already know that the underlying maps of schemes satisfy the operad axioms so we must only verify that the isomorphisms of line bundles with sections satisfy the various commutativity relations. This follows from the fact that we have made canonical choices for these isomorphisms. I will prove this in one set of cases and leave the rest as a (very) tedious exercise. Let n, $\mathbf{m} = (m_1, \dots, m_n)$, $(l_1, \dots, l_m)$ be positive integers where $m = \sum_i m_i$. Define

- $l = \sum_k l_k$.
- $\mathbf{l} = (l_1, \dots, l_m)$.
- $l_{i,j} := l_{j+\sum k<im_i}$.
- $\mathbf{l}_i := (l_{i,1}, \dots, l_{i,m_i})$.

- $l'_i = \sum_{1 \le j \le m_i} l_{i,j}$.
- $\mathbf{l}' = (l'_1, \dots, l'_n)$.
- $\mathsf{T}[\mathbf{v}] := \mathsf{T}_{v_1} \times \cdots \times \mathsf{T}_{v_k}$ for every list of integers $\mathbf{v} = (v_1, \dots, v_k)$.

Then, for the associativity axiom we want to verify that

$$\begin{array}{ccc}
\mathsf{T}_n \times \mathsf{T}[\mathbf{m}] \times \mathsf{T}[\mathbf{l}] & \xrightarrow{\rho} & \mathsf{T}_n \times (\mathsf{T}_{m_1} \times \mathsf{T}[\mathbf{l}_1]) \times \cdots \times (\mathsf{T}_{m_n} \times \mathsf{T}[\mathbf{l}_n]) \\
 & & \big\downarrow{\scriptstyle \mathrm{id} \times \gamma^{m_1,\mathbf{l}_1} \times \cdots \times \gamma^{m_n,\mathbf{l}_n}} \\
\big\downarrow{\scriptstyle \gamma^{n,\mathbf{m}} \times \mathrm{id}} & & \mathsf{T}_n \times \mathsf{T}[\mathbf{l}'] \\
 & & \big\downarrow{\scriptstyle \gamma^{n,\mathbf{l}'}} \\
\mathsf{T}_m \times \mathsf{T}[\mathbf{l}] & \xrightarrow{\gamma^{m,\mathbf{l}}} & \mathsf{T}_l
\end{array}$$

where ρ is the corresponding permutation of factors in the product. Let $S \subseteq [l]$ be such that

$$S \subsetneq L'_r = \{1 + \sum_{k<r} l_k, \dots, l_r + \sum_{k<r} l_k\}$$

and $|S| \geq 2$ for some $1 \leq r \leq m$. By definition $\gamma^{m,\mathbf{l}}$ gives the unique isomorphism of sheaves

$$(\gamma^{m,\mathbf{l}})^* \mathcal{O}_{T_l}(S) \to \pi^* \mathcal{O}_{T_{l_r}}(q^{m,\mathbf{l}}(S))$$

which sends

$$(\gamma^{m,\mathbf{l}})^* s_S \mapsto \pi^* s_{q^{m,\mathbf{l}}(S)}$$

where π is the projection map

$$T_m \times T[\mathbf{l}] \to T_{l_r}.$$

Thus, $\gamma^{m,\mathbf{l}} \circ (\gamma^{n,\mathbf{m}} \times \mathrm{id})$ gives the isomorphism of sheaves

$$\left(\gamma^{m,\mathbf{l}} \circ (\gamma^{n,\mathbf{m}} \times \mathrm{id})\right)^* \mathcal{O}_{T_l}(S) \to \pi^* \mathcal{O}_{T_{l_r}}(q^{m,\mathbf{l}}(S))$$

which sends

$$\left(\gamma^{m,\mathbf{l}} \circ (\gamma^{n,\mathbf{m}} \times \mathrm{id})\right)^* s_S \mapsto \pi^* s_{q^{m,\mathbf{l}}(S)}$$

where π is now the projection map

$$T_n \times T[\mathbf{m}] \times T[\mathbf{l}] \to T_{l_r}.$$

By similar computations

$$\gamma^{n,\mathbf{l}'} \circ (\mathrm{id} \times \gamma^{m_1,\mathbf{l}_1} \times \cdots \times \gamma^{m_n,\mathbf{l}_n}) \circ \rho$$

also gives the isomorphism of sheaves which sends

$$\left(\gamma^{n,\mathbf{l}'} \circ (\mathrm{id} \times \gamma^{m_1,\mathbf{l}_1} \times \cdots \times \gamma^{m_n,\mathbf{l}_n}) \circ \rho\right)^* \mathcal{O}_{T_l}(S) = \left(\gamma^{m,\mathbf{l}} \circ (\gamma^{n,\mathbf{m}} \times \mathrm{id})\right)^* \mathcal{O}_{T_l}(S) \to \pi^* \mathcal{O}_{T_{l_r}}(q^{m,\mathbf{l}}(S))$$

which sends

$$\left(\gamma^{m,\mathbf{l}} \circ (\gamma^{n,\mathbf{m}} \times \mathrm{id})\right)^* s_S \mapsto \pi^* s_{q^{m,\mathbf{l}}(S)}.$$

Since the section $\pi^* s_{q^{m,1}(S)}$ is non-trivial these must be the same morphism.

For the remaining $S \subseteq [l]$ the proofs are similar. In every case we end up with two isomorphisms that are equal either because they send a nontrivial section to the same nontrivial section or because they are the pullbacks (via some add one morphism) of two morphisms of sheaves of sections both of which send the same nontrivial section to the same nontrivial section. The same proof strategy can be used to verify that the equivariance axiom holds. □

Remark. A comment regarding why we need to make this an operad without unit is in order. For every d we have $\mathsf{T}_1 = (\mathrm{Spec}\ \Bbbk, 0\colon \mathcal{O} \to \mathcal{O})$ while the unit object in the category of log schemes is Spec $\Bbbk$ with no line bundles. Since there is no automorphism of $\mathcal{O}$ which identifies the 0 section with the unit section this means that we cannot define any morphism Spec $\Bbbk \to \mathsf{T}_1$ and so it is simply not possible to define a unit morphism.

7.2 Kato-Nakayama Analytifications

In this section I will show that the Kato-Nakayama analytification of the smooth log varieties $\mathsf{T}_{d,n}$ with base field $\mathbb{C}$ is diffeomorphic to $(\mathbb{S}^1)^n \times \mathcal{K}_{2d,n}$. To do this I will first compute the Kato-Nakayama analytifiction of the smooth log varieties

$$\mathsf{X}[n] := (X[n], (s_S\colon \mathcal{O}_{X[n]} \to \mathcal{O}_{X[n]}(S))_{S\in P(n)})$$

for any smooth complex variety X, where $s_S\colon \mathcal{O}_{X[n]} \to \mathcal{O}_{X[n]}(S)$ is the line bundle with section corresponding to the effective Cartier divisor $X[n](S)$.

Proposition 7.15. *There is an isomorphism of manifolds with corners over $(X^{\mathrm{an}})^n$*

$$(\mathsf{X}[n])^{\mathrm{KN}} \to \mathrm{FM}_n(X^{\mathrm{an}}).$$

Proof. I will use induction to show that

$$\mathsf{X}[n]^{\mathrm{KN}} \cong \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(2^n-(n+1))} \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(1)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i-1)} \dots \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(1)}\ (X^{\mathrm{an}})^n$$

for each $2^n-(n+1) \geq i \geq 1$. Where we in each intermediate blow up we define $\tilde{\Delta}(S)$ as the dominant transform of $\tilde{\Delta}(S)$ from the previous step and $\tilde{\Delta}(j) := \tilde{\Delta}(B_n(j))$, where in $(X^{\mathrm{an}})^n$ we let $\tilde{\Delta}(S) = \Delta(S)$, the S-diagonal. For the induction base note that this is almost clear by definition since, by definition

$$(\mathsf{X}[n])^{\mathrm{KN}} \to \mathrm{FM}_n(X^{\mathrm{an}}) = \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(2^n-(n+1))} \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(1)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(2^n-(n+1))} \dots \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(1)}\ (X^{\mathrm{an}})^n,$$

but where we define $\tilde{\Delta}(S)$ as the total and not dominant transform of $\tilde{\Delta}(S)$ from the previous blow up in each of the real oriented blow ups. However, since it is easy to verify that the conditions of corollary 3.15 are satisfied for each $X[n](S) = \tilde{\Delta}(S)$ so the total and dominant transforms are the same and thus the base case follows. For the induction step first note that in $X[n,i]^{\mathrm{an}} = \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i-1)} \dots \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(1)}\ (X^{\mathrm{an}})^n$ the subvarieties $\tilde{\Delta}(j)$ are effective Cartier divisors for each $1 \leq j \leq i$. Using the functor of points for each diagonal $\tilde{\Delta}(S)$ in this variety (see theorem 6.10) it is easy to verify that these divisors, in any order, still satisfy the conditions in corollary 3.15 for $Z = \tilde{\Delta}(j)$, $1 \leq j \leq i$. Hence, the strict and total transform of $\Delta(j)$ in $\mathrm{Bl}^{\mathbb{R}}_{\Delta(k_1)}\ \mathrm{Bl}^{\mathbb{R}}_{\Delta(k_2)\dots\mathrm{Bl}^{\mathbb{R}}_{\Delta(k_r)}}\ X[n,i]^{\mathrm{an}}$ are the same for any distinct $j, k_1, \dots, k_r \leq i$. Thus, by lemma 3.12 there is an isomorphism from

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(2^n-(n+1))} \dots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(1)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i-1)} \dots \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(1)}\ (X^{\mathrm{an}})^n$$

to the reordering of blow ups

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(2^n-(n+1)}\ \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i+1)}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i-1)}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(1)}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i-1)}\ \ldots \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(1)}\ (X^{\mathrm{an}})^n$$

where $\tilde{\Delta}(S)$ are still defined as strict transforms in each blow up. By theorem 3.16 this is isomorphic to

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(2^n-(n+1)}\ \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i+1)}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i-1)}\ \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(1)}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i-1)}\ \ldots \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(1)}\ (X^{\mathrm{an}})^n.$$

Finally, by lemma 6.11 strict, dominant, and total transforms of the diagonals $\tilde{\Delta}(1) \ldots \tilde{\Delta}(i-1)$ are equal for the blow up map $X[n,i] \to X[n,i-1]$ and thus the same is true for the blow up $\mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i)}\, X[n,i] \to X[n,i]$. Furthermore, using lemma 3.15 again the strict and total transforms of $\tilde{\Delta}(i)$ are equal for the blow up $\mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(k)}\ \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(1)}\, X[n,i-1]^{\mathrm{an}} \to X[n,i-1]^{\mathrm{an}}$ for any $1 \leq k \leq i-1$. Thus, by applying lemma 3.12 one more time there is an isomorphism from

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(2^n-(n+1)}\ \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i+1)}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i-1)}\ \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(1)}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i-1)}\ \ldots \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(1)}\ (X^{\mathrm{an}})^n$$

to

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(2^n-(n+1)}\ \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i+1)}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i)}\ \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(i-1)}\ \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(1)}\ \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(i-1)}\ \ldots \mathrm{Bl}^{\mathbb{C}}_{\tilde{\Delta}(1)}\ (X^{\mathrm{an}})^n.$$

By induction this means that $\mathsf{X}[n]^{\mathrm{KN}}$ is isomorphic to

$$\mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(2^n-(n+1))}\ \ldots \mathrm{Bl}^{\mathbb{R}}_{\tilde{\Delta}(1)}\ (X^{\mathrm{an}})^n.$$

By proposition 4.3 this is $\mathrm{FM}_n(X^{\mathrm{an}})$ and thus we are done. □

Remark. In the interest of accuracy, I should mention that proposition 4.3 only shows that the Fulton-MacPherson configuration space $\mathrm{FM}_n(M)$ is a sequence of blow ups in this way in the specific case $M = \mathbb{R}^d$. As mentioned previously, Li shows this in much more general situations in the algebraic case in [Li09] but even though this seems to be well known among experts I am unaware of any articles which show that the Fulton-MacPherson configuration spaces can be written as a sequence of blow ups in this way for an arbitrary smooth manifold M. Considering my lack of references for this statement it is of course understandable if you do not consider this proof to be adequate. If so note that the only cases which are actually of importance for the main results of this thesis are when we have $X = \mathbb{A}^d$.

Before we can apply this result to find the Kato-Nakayama analytifications of the log schemes $\mathsf{T}_{d,n}$ we will first need to do some work.

Definition 7.16. Let the "remove one" morphism $R_{n+1} \colon X[n+1] \to X[n]$ be defined as the map induced by the natural transformation of functors which sends

$$\big((h \colon H \to X^{n+1}), \{\phi_S \colon h_S^* \mathcal{I}_S \to \mathcal{L}_S\}_{S \in P(n+1)}\big) \in \chi[n+1](H)$$

to

$$\big((h_{[n]} \colon H \to X^{n}), \{\phi_S \colon h_S^* \mathcal{I}_S \to \mathcal{L}_S\}_{S \in P(n)}\big) \in \chi[n](H).$$

Lemma 7.17. *There is a unique isomorphism of line bundles*

$$R_{n+1}^* \mathcal{O}_{X[n]}(S) \xrightarrow{\cong} \mathcal{O}_{X[n+1]}(S) \otimes O_{X[n+1]}(S \cup \{n+1\})$$

sending $R_{n+1}^* s_S \mapsto s_S \otimes s_{s\cup\{n+1\}}$ *where* $s_S \colon \mathcal{O}_{X[n]} \to O_{X[n]}(S)$ *is the sheaf with section corresponding to the divisor* $X[n](S)$*.*

Proof. The results follows from the exact same arguments as used in the proof of lemma 7.10. □

Lemma 7.18. *Let* i *denote the embedding* $i \colon T_{d,n} \hookrightarrow \mathbb{A}^d[n]$*. For every* $n \geq 2$ *there is a canonical isomorphism* $i^*\mathcal{O}_{\mathbb{A}^d[n]}(\mathbb{A}^d[n])(S)) \to \mathcal{O}_{T_{d,n}}(S)$ *for every* $S \subseteq [n]$*,* $|S| \geq 2$*, sending sections* $i_n^* s_S \mapsto s_S$*.*

Proof. This is clear using proposition 7.1 whenever $S \neq [n]$. For $S = [n]$ we will use induction on n. For $n = 2$ such an isomorphism exists since proposition 6.15 and lemma 7.4 identify both of these line bundles as the $\mathcal{O}_{\mathbb{P}^{d-1}}(-1)$ line bundle. Of course there is a choice of isomorphism to be made here which cannot really be said to be canonical. However, with this choice made the rest of the isomorphisms will have canonical choices.

For the induction case we use that the following diagram commutes.

$$\begin{array}{ccc} X[n+1] & \xrightarrow{R_{n+1}} & X[n] \\ \uparrow & & \uparrow \\ T_{d,n+1} & \xrightarrow{r_{n+1}} & T_{d,n} \end{array}$$

Thus, there is a canonical isomorphism

$$r_{n+1}^* i_n^* \mathcal{O}_{\mathbb{A}^d[n]}([n]) \cong i_{n+1}^* R_{n+1}^* \mathcal{O}_{\mathbb{A}^d[n]}([n]).$$

By induction there is an isomorphism $i_n^*\mathcal{O}_{\mathbb{A}^d[n]}([n]) \to \mathcal{O}_{T_{d,n}}([n])$ so by lemma 7.10 the left hand side is canonically isomorphic to $\mathcal{O}_{T_{d,n+1}}([n]) \otimes \mathcal{O}_{T_{d,n+1}}([n+1])$. Furthermore, by lemma 7.17 the right hand side is canonically isomorphic to $i_{n+1}^*\left(\mathcal{O}_{\mathbb{A}^d[n+1]}([n+1]) \otimes \mathcal{O}_{\mathbb{A}^d[n+1]}([n])\right)$ which is isomorphic to $i_{n+1}^*\mathcal{O}_{\mathbb{A}^d[n+1]}([n+1]) \otimes \mathcal{O}_{T_{d,n+1}}([n])$ by this lemma applied to $[n] \subsetneq [n+1]$. Hence, the isomorphism $i_n^*\mathcal{O}_{\mathbb{A}^d[n]}([n]) \to \mathcal{O}_{T_{d,n}}([n])$ induces an isomorphism

$$\mathcal{O}_{T_{d,n+1}}([n+1]) \otimes \mathcal{O}_{T_{d,n+1}}([n]) \xrightarrow{\cong} i_{n+1}^*\mathcal{O}_{\mathbb{A}^d[n+1]}([n+1]) \otimes \mathcal{O}_{T_{d,n+1}}([n]).$$

By tensoring both sides with the dual of $O_{T_{d,n+1}}([n])$ we get our desired isomorphism. □

By this lemma the closed embedding $i_n \colon T_{d,n} \hookrightarrow \mathbb{A}^d[n]$ induces a strict morphism of log schemes $i_n \colon \mathsf{V}_{d,n} \to \mathsf{A}^d[n]$ where $\mathsf{A}^d[n]$ is the log scheme $(\mathbb{A}^d[n], (s_S \colon \mathcal{O}_{\mathbb{A}^d[n]} \to \mathcal{O}(D(S)))_{S \in P(n)})$ and $\mathsf{V}_{d,n}$ is the log scheme $(T_{d,n}, (s_S \colon \mathcal{O}_{T_{d,n}} \to \mathcal{O}_{T_{d,n}}(S))_{S \in P(n)})$.

Proposition 7.19. *There is a diffeomorphism,* $\mathsf{V}_{d,n}^{\mathrm{KN}} \to \mathcal{K}_{2d,n}$ *such that the following diagram commutes*

$$\begin{array}{ccc} \mathsf{V}_{d,n}^{\mathrm{KN}} & \longrightarrow & \mathsf{A}^d[n]^{\mathrm{KN}} \\ \downarrow & & \downarrow \\ \mathcal{K}_{2d,n} & \longrightarrow & \mathrm{FM}_n(X^{\mathrm{an}}) \end{array}$$

Proof. Since the closed embedding $T_{d,n} \hookrightarrow \mathbb{A}^d[n]$ induces a strict morphism of log varieties $\mathsf{V}_{d,n} \to \mathsf{A}^d[n]$ the following must be a cartesian diagram

$$\begin{array}{ccc} \mathsf{V}_{d,n}^{\mathrm{KN}} & \longrightarrow & \mathsf{A}^d[n]^{\mathrm{KN}} \\ \downarrow & & \downarrow \\ T_{d,n}^{\mathrm{an}} & \longrightarrow & X[n]^{\mathrm{an}} \end{array}$$

Since $T_{d,n}^{\mathrm{an}} \hookrightarrow X[n]^{\mathrm{an}}$ identifies $T_{d,n}$ with the fiber over the origin of the map $X[n] \to X^n$ and since the diagram

$$\begin{array}{ccc} \mathsf{A}^d[n]^{\mathrm{KN}} & \xrightarrow{\cong} & \mathrm{FM}_n(X^{\mathrm{an}}) \\ & \searrow \quad \swarrow & \\ & (X^{\mathrm{an}})^n & \end{array}$$

commutes this implies that $\mathsf{V}_{d,n}^{\mathrm{KN}} \to \mathsf{A}^d[n]^{\mathrm{KN}} \xrightarrow{\cong} \mathrm{FM}_n(\mathbb{A}_{\mathbb{C}}^d)$ identifies $\mathsf{V}_{d,n}^{\mathrm{KN}}$ with the fiber over the origin for the map $\mathrm{FM}_n(\mathbb{A}_{\mathbb{C}}^d) \to (\mathbb{A}_{\mathbb{C}}^d)^n$. Since $\mathbb{A}_{\mathbb{C}} \cong \mathbb{R}^2$ this is isomorphic to the fiber over the origin for the map $\mathrm{FM}_n(\mathbb{R}^{2d}) \to (\mathbb{R}^{2d})^n$. This is $\mathcal{K}_{2d,n}$ by definition. Commutativity of the diagram follows from construction. □

With this result we are almost ready to find the analytification of $\mathsf{T}_{d,n}$ but we need one more lemma first.

Lemma 7.20. *For every $1 \leq i \leq n$ there is a canonical isomorphism of line bundles*

$$\mathcal{O}_{T_{d,n}}(\{i\}) \xrightarrow{\cong} \bigotimes_{S \in P(n), i \in S} \mathcal{O}_{T_{d,n}}(S)^{\vee}.$$

Proof. We prove this by induction. In the case $n = 1$ this is obvious but this case will not work as a base for the induction. For $n = 2$ such isomorphisms clearly exist by lemma 7.4. As mentioned several times earlier we have made a choice that is not really canonical for an isomorphism

$$\mathcal{O}_{T_{d,2}}(\{1\}) \otimes \mathcal{O}_{T_{d,2}}(\{1,2\}) \xrightarrow{\cong} \mathcal{O}_{T_{d,2}}.$$

However, since there are canonical isomorphisms $(1,2)^*\mathcal{O}_{T_{d,2}}(\{1\}) \cong \mathcal{O}_{T_{d,2}}(\{2\})$ and $(1,2)^*\mathcal{O}_{T_{d,2}}(\{1,2\}) \cong \mathcal{O}_{T_{d,2}}(\{1,2\})$ the pullback of this morphism of line bundles via the $(1,2)$ permutation gives a morphism

$$\mathcal{O}_{T_{d,2}}(\{2\}) \otimes \mathcal{O}_{T_{d,2}}(\{1,2\}) \xrightarrow{\cong} \mathcal{O}_{T_{d,2}}$$

so at least there is only one choice to be made here.

Suppose now that there is an isomorphism

$$\mathcal{O}_{T_{d,n}}(\{i\}) \xrightarrow{\cong} \bigotimes_{S \subseteq [n], i \in S, |S| \geq 2} \mathcal{O}_{T_{d,n}}(S)^{\vee}.$$

By lemma 7.10 the pullback of this isomorphism via the remove one morphism $r_{n+1} \colon T_{d,n+1} \to T_{d,n}$ gives an isomorphism

$$\mathcal{O}_{T_{d,n+1}}(\{l\}) \otimes \mathcal{O}_{T_{d,n+1}}(\{l, n+1\}) \xrightarrow{\cong} \bigotimes_{S \in P(n), l \in S} \mathcal{O}_{T_{d,n+1}}(S)^{\vee} \otimes O_{T_{d,n+1}}(S \cup \{n+1\})^{\vee}.$$

Taking the tensor product of both sides with $\mathcal{O}_{T_{d,n+1}}(\{i,n+1\})^\vee$ gives an isomorphism

$$\mathcal{O}_{T_{d,n+1}}(\{l\}) \xrightarrow{\cong} \mathcal{O}_{T_{d,n+1}}(\{l,n+1\}) \otimes \bigotimes_{S\in P(n), l\in S} \mathcal{O}_{T_{d,n+1}}(S)^\vee \otimes O_{T_{d,n+1}}(S\cup\{n+1\})^\vee.$$

The right hand side here is isomorphic to $\bigotimes_{S\in P(n+1), l\in S} \mathcal{O}_{T_{d,n}}(S)^\vee$ via permutations of the terms. Finally, taking the pullback of this isomorphism via the permutation $(l,n+1)$ gives the desired isomorphism in the case $i=n+1$ □

Theorem 7.21. *The Kato-Nakayama analytification of* $\mathsf{T}_{d,n}$ *is diffeomorphic to* $\mathcal{K}_{2d,n}\times(\mathbb{S}^1)^n$.

Proof. By definition

$$\mathsf{T}^{\mathrm{KN}}_{d,n} \cong \mathrm{Bl}^{\mathbb{R}}_{s_{\{1\}}}\,\mathrm{Bl}^{\mathbb{R}}_{s_{\{2\}}}\,\dots\mathrm{Bl}^{\mathbb{R}}_{s_{[n]}}\,T^{\mathrm{an}}_{d,n}$$

where the blow-ups are (in any order) taken over all sections of line bundles $s_S\colon \mathcal{O}_{T_{d,n}}\to\mathcal{O}_{T_{d,n}}(S)$. Since the blowing up can be done in any order (by lemma 3.12) we can blow up in the line bundles corresponding to sets of one element, $\{l\}$, last. By lemma 7.20 $\mathcal{O}_{T_{d,n}}(\{l\})$ is a tensor product of the duals of the other line bundles. Hence, by proposition 3.24 blowing up in the zero section of each of the bundles $\mathcal{O}_{T_{d,n}}(\{l\})$ is equivalent to taking the product with $\mathbb{S}^1$ (with blow up map equal to the projection). Thus

$$\mathrm{Bl}^{\mathbb{R}}_{s_{\{1\}}}\,\mathrm{Bl}^{\mathbb{R}}_{s_{\{2\}}}\,\dots\mathrm{Bl}^{\mathbb{R}}_{s_{[n]}}\,T^{\mathrm{KN}}_{d,n} \cong (\mathbb{S}^1)^n\times\mathrm{Bl}^{\mathbb{R}}_{s_{\{n,n-1\}}}\,\dots\mathrm{Bl}^{\mathbb{R}}_{s_{[n]}}\,T^{\mathrm{an}}_{d,n}.$$

By proposition 7.19 $\mathrm{Bl}^{\mathbb{R}}_{s_{\{n,n-1\}}}\,\dots\mathrm{Bl}^{\mathbb{R}}_{s_{[n]}}\,T^{\mathrm{an}}_{d,n}\cong\mathcal{K}_{2d,n}$ and thus

$$\mathsf{T}^{\mathrm{KN}}_{d,n}\cong(\mathbb{S}^1)^n\times\mathcal{K}_{2d,n}.$$

□

7.3 Equivalence with $\mathcal{K}_{2d}\rtimes\mathbb{S}^1$

In this section I will show that the analytification of the $\mathcal{T}_d$ operad without unit over $\mathbb{C}$ is isomorphic to the $\mathbb{S}^1$-framed Kontsevich Operad in dimension $2d$, $\mathbb{S}^1\rtimes\mathcal{K}_{2d}$, without unit. However, this isomorphism is not given by the diffeomorphisms $\mathsf{T}^{\mathrm{KN}}_{d,n}\cong(\mathbb{S}^1)^n\times\mathcal{K}_{2d,n}$ seen in the previous section. Thus, the first step will be to describe the correct diffeomorphisms.

Definition 7.22. Let $\mathsf{K}_{d,n}$ denote the log scheme with base scheme $T_{d,n}$ and sheaves with sections

$$(0\colon \mathcal{O}_{T_{d,n}}\to\mathcal{L}_i)_{i\in[n]}\oplus(s_S\colon\mathcal{O}_{T_{d,n}}\to\mathcal{O}_{T_{d,n}}(S))_{S\in P(n)},$$

where $\mathcal{L}_i:=\mathcal{O}_{T_{d,n}}$.

Remark. The reason I call these line bundles $\mathcal{L}_i$ even though they are all trivial is to be able to differentiate them when defining morphisms to $\mathsf{K}_{d,n}$.

A more intuitive way to see this is that $\mathsf{K}_{d,n}$ is $\mathsf{T}_{d,n}$ but with each of the sheaves $\mathcal{O}(\{l\})$ replaced by the trivial line bundle with the 0 section. Note that by the same argument as in theorem 7.21 the analytification of $\mathsf{K}_{d,n}$ is also canonically isomorphic to $(\mathbb{S}^1)^n\times\mathcal{K}_{2d,n}$.

Lemma 7.23. *Let $k_n \colon \mathsf{T}_{d,n} \to \mathsf{K}_{d,n}$ be the map of log schemes given by the identity on the underlying map of schemes and the isomorphisms*

$$\mathcal{O}_{T_{d,n}}(S) \xrightarrow{\text{id}} \mathcal{O}_{T_{d,n}}(S)$$

and

$$\mathcal{L}_i \xrightarrow{\cong} \bigotimes_{i \in S} \mathcal{O}_{T_{d,n}}(S).$$

The Kato-Nakayama analytification of k_n is a homeomorphism.

Proof. First note that the line bundle $\bigotimes_{i\in S} \mathcal{O}_{T_{d,n}}(S)$ is canonically isomorphic to the trivial line bundle by lemma 7.20 and so k_n is a well defined morphism of log schemes. It is easy to check that the analytification is a bijection so I skip the details. Since, $\mathsf{T}_{d,n}^{\text{KN}} \cong \mathsf{K}_{d,n}^{\text{KN}} \cong (\mathbb{S}^1)^n \times \mathcal{K}_{2d,n}$ is compact this means that the function is a homoemorphism and so we are done. □

Remark. This function is in fact a diffeomorphism too. One way to prove this is to guess what the analytification of this map is, prove that the guess is correct using the methods found in the proofs of this section, and finally explicitly define a map of topological spaces that is an inverse to this map. This proof would however be significantly longer and therefore I will leave it out.

The rest of this section will be dedicated to proving that the analytification of the maps k_n (composed with the isomorphism $\mathsf{K}_{d,n}^{\text{KN}} \to (\mathbb{S}^1)^n \times \mathcal{K}_{2d,n}$) induces an isomorphism of topological operads without unit. In what follows I will abuse notation somewhat by identifying the analytifications of the log varieties with the spaces I have proven them to be homeomorphic with. For example I will say that the analytification of the diagram $\mathsf{T}_{d,n} \xrightarrow{k_n} \mathsf{K}_{d,n}$ is the diagram $(\mathbb{S}^1)^n \times \mathcal{K}_{2d,n} \xrightarrow{k_n^{\text{KN}}} (\mathbb{S}^1)^n \times \mathcal{K}_{2d,n}$. This is of course not strictly speaking true since in actuality the arrow in this diagram is the composition

$$(\mathbb{S}^1)^n \times \mathcal{K}_{2d,n} \xrightarrow{\cong} \mathsf{T}_{d,n}^{\text{KN}} \xrightarrow{k_n^{\text{KN}}} \mathsf{K}_{d,n}^{\text{KN}} \xrightarrow{\cong} (\mathbb{S}^1)^n \times \mathcal{K}_{2d,n}$$

but I will ignore this. The following lemma is the foundation for all remaining proofs.

Lemma 7.24. *There is an essentially strict map of log schemes $i \colon \mathrm{Conf}_n(X) \to \mathsf{X}[n]$, where $\mathrm{Conf}_n(X)$ is the log scheme with underlying scheme $\mathrm{Conf}_n(X)$ and no line bundles with sections. The analytification of this map is the dense inclusion $\mathrm{Conf}_n(X^{\text{an}}) \hookrightarrow \mathrm{FM}_n(X^{\text{an}})$.*

Proof. The image of the inclusion $i \colon \mathrm{Conf}_n(X) \hookrightarrow X[n]$ does not intersect any of the divisors $\mathcal{X}[n](S)$ and so there are isomorphisms of line bundles $i^*\mathcal{O}_{X[n]} \xrightarrow{\cong} \mathcal{O}_{\mathrm{Conf}_n(X)}$ which sends $i^* s_S \mapsto 1$. This isomorphisms give the desired map $i \colon \mathrm{Conf}_n(X) \to X[n]$. From definition it is clear that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Conf}_n(X) & \xrightarrow{i} & \mathsf{X}[n] \\ & \searrow & \downarrow{\scriptstyle \pi_n} \\ & & X^n. \end{array}$$

Where X^n is the scheme X^n with no line bundles and $\mathrm{Conf}_n(X) \hookrightarrow X^n$ is the canonical inclusion. The analytification of this diagram is

$$
\begin{array}{ccc}
\mathrm{Conf}_n(X^{\mathrm{an}}) & \xhookrightarrow{i^{\mathrm{KN}}} & \mathrm{FM}_n(X^{\mathrm{an}}) \\
& \searrow & \big\downarrow \rho \\
& & (X^{\mathrm{an}})^n.
\end{array}
$$

Where $\mathrm{Conf}_n(X^{\mathrm{an}}) \hookrightarrow (X^{\mathrm{an}})^n$ is the canonical inclusion. Since ρ restricts to a diffeomorphism on $\mathrm{Conf}_n(X^{\mathrm{an}}) \subseteq (X^{\mathrm{an}})^n$, i^{KN} must be the inclusion $\mathrm{Conf}_n(X^{\mathrm{an}}) \hookrightarrow \mathrm{FM}_n(X^{\mathrm{an}})$. □

Now we are ready to start proving that the maps k_n^{KN} give an isomorphism of operads without unit. We begin with showing commutativity of the symmetry maps. First note that similar to how we defined the symmetry action on $T_{d,n}$ using its functor of points and then extended it to a symmetry action on $\mathsf{T}_{d,n}$ we can define a symmetry action on $X[n]$ for any smooth variety X and then extend it to $\mathsf{X}[n]$. Since we have seen this type of construction many times in this thesis already I omit the details. Similarly we can also define an action on $\mathsf{K}_{d,n}$ in the same way.

Proposition 7.25. *Let $\sigma \in \Sigma_n$. The analytifications of the corresponding symmetry maps $\sigma_X \colon \mathsf{X}[n] \to \mathsf{X}[n]$ and $\sigma_V \colon \mathsf{V}_{d,n} \to \mathsf{V}_{d,n}$ are the corresponding symmetry maps $\mathrm{FM}_n(X^{\mathrm{an}}) \to \mathrm{FM}_n(X^{\mathrm{an}})$ and $\mathcal{K}_{2d,n} \to \mathcal{K}_{2d,n}$.*

Proof. Let $\sigma_C \colon \mathrm{Conf}_n(X) \to \mathrm{Conf}_n(X)$ be the permutation of components $X^n \to X^n$ corresponding to σ restricted to $\mathrm{Conf}_n(X)$. It is easy to verify directly from definitions that the following diagrams commutes.

$$
\begin{array}{ccccc}
\mathrm{Conf}_n(X) & \longrightarrow & \mathsf{X}[n] & \longleftarrow & \mathsf{V}_{d,n} \\
\big\downarrow \sigma_C & & \big\downarrow \sigma_X & & \big\downarrow \sigma_V \\
\mathrm{Conf}_n(X) & \longrightarrow & \mathsf{X}[n] & \longleftarrow & \mathsf{V}_{d,n}
\end{array}
$$

It is clear that the analytification of σ_C is the corresponding permutation of components $\sigma_C^{\mathrm{KN}} \colon \mathrm{Conf}_n(X^{\mathrm{an}}) \to \mathrm{Conf}_n(X^{\mathrm{an}})$. Thus the Kato-Nakayama analytification of this diagram is

$$
\begin{array}{ccccc}
\mathrm{Conf}_n(X^{\mathrm{an}}) & \longrightarrow & \mathrm{FM}_n(X^{\mathrm{an}}) & \longleftarrow & \mathcal{K}_{2d,n} \\
\big\downarrow \sigma_C^{\mathrm{an}} & & \big\downarrow \sigma_X^{\mathrm{KN}} & & \big\downarrow \sigma_V^{\mathrm{KN}} \\
\mathrm{Conf}_n(X^{\mathrm{an}}) & \longrightarrow & \mathrm{FM}_n(X^{\mathrm{an}}) & \longleftarrow & \mathcal{K}_{2d,n}
\end{array}
$$

By lemma 4.5 this diagram can only commute if σ_X^{KN} and σ_V^{KN} are the corresponding permutation maps. □

Remark. Strictly speaking lemma 4.5 only treats the case $X = \mathbb{A}^d$. The case of a general smooth variety X is a simple generalization of this but since $X = \mathbb{A}^d$ is the only case we need I will not elaborate.

Corollary 7.26. *For every $\sigma \in \Sigma_n$ the following diagram commutes*

$$
\begin{array}{ccc}
\mathsf{T}_{d,n}^{\mathrm{KN}} & \xrightarrow{\sigma_T^{\mathrm{KN}}} & \mathsf{T}_{d,n}^{\mathrm{KN}} \\
\big\downarrow k_n^{\mathrm{KN}} & & \big\downarrow k_n^{\mathrm{KN}} \\
(\mathbb{S}^1)^n \times \mathcal{K}_{2d,n} & \xrightarrow{\sigma_0} & (\mathbb{S}^1)^n \times \mathcal{K}_{2d,n}
\end{array}
$$

where σ_T is the symmetry action on $\mathsf{T}_{d,n}$ and σ_O is the symmetry action of the $\mathcal{K}_d \rtimes \mathbb{S}^1$ operad.

Proof. First I claim that σ_O is the analytification of σ_K, the symmetry morphism $\mathsf{K}_{d,n} \to \mathsf{K}_{d,n}$. To see this let P_n be the log scheme with underlying space Spec $\mathbb{C}$ and with line bundles $(0\colon \mathcal{O}_{\mathrm{Spec}\,\mathbb{C}} \to \mathcal{M}_i)_{i\in[n]}$ where, by necessity, each $\mathcal{M}_i$ is trivial. Let σ_P be the morphism $\mathsf{P} \to \mathsf{P}$ defined by $\mathcal{M}_i \xrightarrow{\cong} \mathcal{M}_{\sigma^{-1}(i)}$. Then, the following is a commutative diagram of log schemes.

$$\begin{array}{ccccc}
\mathsf{P}_n & \longleftarrow & \mathsf{K}_{d,n} & \longrightarrow & \mathsf{V}_{d,n} \\
\downarrow{\scriptstyle \sigma_P} & & \downarrow{\scriptstyle \sigma_k} & & \downarrow{\scriptstyle \sigma_V} \\
\mathsf{P}_n & \longleftarrow & \mathsf{K}_{d,n} & \longrightarrow & \mathsf{V}_{d,n}
\end{array}$$

It is easy to see that the analytification of σ_P is the permutation of factors $\sigma_F\colon (\mathbb{S}^1)^n \to (\mathbb{S}^1)^n$ and hence the analytification of this diagram is

$$\begin{array}{ccccc}
(\mathbb{S}^1)^n & \longleftarrow & (\mathbb{S}^1)^n \times \mathcal{K}_{2d,n} & \longrightarrow & \mathcal{K}_{2d,n} \\
\downarrow{\scriptstyle \sigma_F} & & \downarrow{\scriptstyle \sigma_k^{\mathrm{KN}}} & & \downarrow{\scriptstyle \sigma_V^{\mathrm{KN}}} \\
(\mathbb{S}^1)^n & \longleftarrow & (\mathbb{S}^1)^n \times \mathcal{K}_{2d,n} & \longrightarrow & \mathcal{K}_{2d,n}
\end{array}$$

By the corollary σ_V^{KN} is the permutation homeomorphism on $\mathcal{K}_{2d,n}$ and therefore this diagram also commutes if we replace the middle homeomorphism with σ_O. By the universal property of the product this means that $\sigma_K^{\mathrm{KN}} = \sigma_O$. Thus the desired commutative square is the analytification of the (clearly commutative) square

$$\begin{array}{ccc}
\mathsf{T}_{d,n} & \xrightarrow{\sigma_T} & \mathsf{T}_{d,n} \\
\downarrow{\scriptstyle k_n} & & \downarrow{\scriptstyle k_n} \\
\mathsf{K}_{d,n} & \xrightarrow{\sigma_K} & \mathsf{K}_{d,n}
\end{array}$$

This completes the proof. □

Remark. By the exact same argument the analytification of σ_T is also σ_O which is interesting but not the result we needed.

With this corollary we have shown that the homeomorphisms k_n^{KN} commute with the symmetry action. Now all that remains to show is that the maps k_n^{KN} also commute with the composition maps. This will be a bit trickier but not too difficult. As usual we first need to define some new maps. In what follows recall that, if Δ_m denotes the small diagonal in X^m and $\mathcal{I}_m$ denotes its ideal sheaf, for some smooth variety X, then $\mathrm{Bl}_{\Delta_m} X^m$ represents the functor

$$F_m\colon H \mapsto \{(f\colon H \to X^m, q\colon f^*\mathcal{I}_m \twoheadrightarrow \mathcal{L}\}$$

where $\mathcal{L}$ is a line bundle on H, up to isomorphism of the quotient $q\colon f^*\mathcal{I}_m \twoheadrightarrow \mathcal{L}$. Also recall that $\mathbb{P}^{m-1}$ represents the functor

$$P_m\colon H \mapsto \{q\colon \mathcal{O}_H^{\oplus m} \twoheadrightarrow \mathcal{L}\},$$

where $\mathcal{L}$ is a line bundle, up to isomorphism of the invertible quotient.

Definition 7.27. Let $\Pi_S\colon X[n] \to \mathrm{Bl}_{\Delta_{|S|}} X^n$ be the map defined by the natural transformation of functors $\chi[n] \to F_{|S|}$ which maps

$$((h\colon H \to X^n), \{\phi_S\colon h_S^*\mathcal{I}_S \to \mathcal{L}_S\}_{S\in P}) \in \chi[X, i+1](H)$$

to

$$(h_S \colon H \to X^{|S|}, \phi_S \colon h_S^* \mathcal{I}_S \to \mathcal{L}_S).$$

Furthermore, define $\pi_S \colon T_{d,n} \to \mathbb{P}^{(|S|-1)d-1}$ as the map induced by the natural transformation of functors $\tau_{d,n} \to P_n$

$$\tau_{d,n}(H) \ni \{\phi_S \colon \mathcal{F}_S^{H,d} \to \mathcal{L}_S\}_{S \subsetneq [n], |S| \geq 2} \mapsto (\phi_S \colon \mathcal{O}_H^{\oplus(|S|-1)d} \cong \mathcal{F}_S^{H,d} \to \mathcal{L}_S).$$

Lemma 7.28. *There are unique isomorphisms of sheaves*

$$\Pi_S^* \mathcal{O}_{\mathrm{Bl}_{\Delta_{|S|}} \ X^{|S|}}(\tilde{\Delta}) \xrightarrow{\cong} \bigotimes_{S \subseteq S'} \mathcal{O}_{X[n]}(S'),$$

sending $s_{\tilde{\Delta}} \mapsto \bigotimes_{S \subseteq S'} s_{S'}$, *where* $\tilde{\Delta}$ *is the exceptional divisor in* $\mathrm{Bl}_{\Delta_{|S|}} \ X^{|S|}$. *Similarly, there are canonical isomorphisms*

$$\pi_S^* \mathcal{O}_{\mathbb{P}^{(|S|-1)d-1}}(-1) \xrightarrow{\cong} \bigotimes_{S \subseteq S'} \mathcal{O}_{T_{d,n}}(S').$$

Proof. The first part follows from proposition 7.1 using the same argument we have seen many times in this thesis. For the second part note that if $j \colon \mathbb{P}^{(|S|-1)d-1} \hookrightarrow \mathrm{Bl}_{\Delta_{|S|}} \ X^{|S|}$ is the inclusion of the fiber over the origin for the blow up map $\mathrm{Bl}_{\Delta_{|S|}} \ X^{|S|}$ then $j^* \mathcal{O}_{\mathrm{Bl}_{\Delta_{|S|}}} \ (\tilde{\Delta})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{(|S|-1)d-1}}(-1)$. Now, it is easy to see that the following diagram commutes

$$\begin{array}{ccc} T_{d,n} & \xhookrightarrow{\quad i \quad} & X[n] \\ \big\downarrow{\scriptstyle \pi_S} & & \big\downarrow{\scriptstyle \Pi_S} \\ \mathbb{P}^{(|S|-1)d-1} & \xhookrightarrow{\ j\ } & \mathrm{Bl}_{\Delta_{|S|}} \ X^{|S|} \end{array}$$

Thus, we have canonical isomorphisms

$$\pi_S^* \mathcal{O}_{\mathbb{P}^{(|S|-1)d-1}}(-1) \cong \pi_S^* j^* \mathcal{O}_{\mathrm{Bl}_{\Delta_{|S|}} \ X^{|S|}}(\tilde{\Delta}) \cong i^* \Pi^* \mathcal{O}_{\mathrm{Bl}_{\Delta_{|S|}} \ X^{|S|}}(\tilde{\Delta}) \cong i^* \bigotimes_{S \subseteq S'} \mathcal{O}_{X[n]}(S')$$

by lemma 7.18 this is isomorphic to $\bigotimes_{S \subseteq S'} \mathcal{O}_{T_{d,n}}(S')$. □

Remark. If you do not know why $i^* \mathcal{O}_{\mathrm{Bl}_{\Delta_{|S|}}} \ (\tilde{\Delta}) \cong \mathcal{O}_{\mathbb{P}^{(|S|-1)d-1}}(-1)$ and are unwilling to take my word for it then note that it is not important that the resulting line bundle here is specifically $\mathcal{O}(-1)$.

By this lemma the morphisms Π_S and π_S induce maps of log schemes

$$\Pi_S \colon \mathsf{X}[n] \to (\mathrm{Bl}_{\Delta_{|S|}} \ X^{|S|}, \mathcal{O}(\tilde{\Delta}))$$

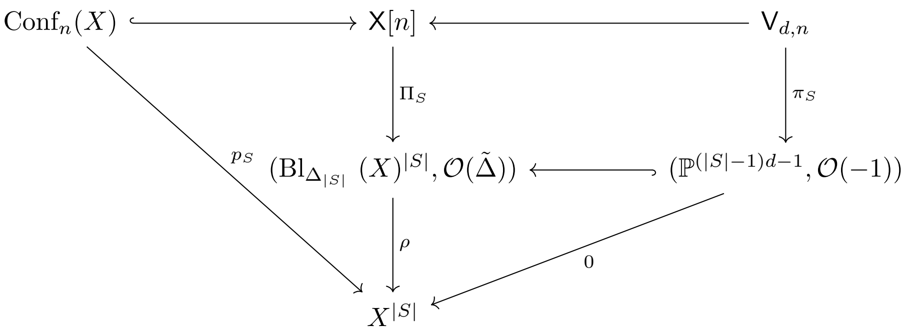
and

$$\pi_s \colon \mathsf{V}_{d,n} \to (\mathbb{P}^{(|S|-1)d-1}, \mathcal{O}(-1))$$

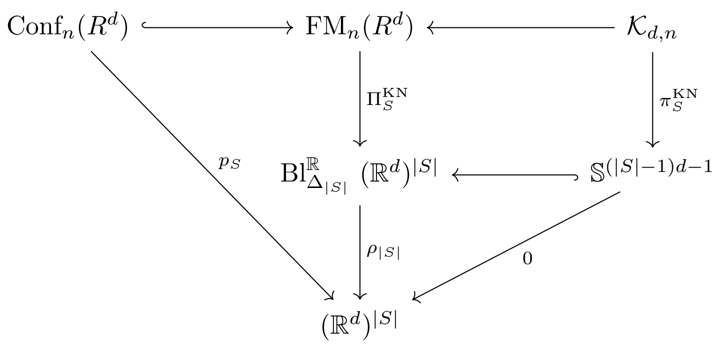
via the isomorphisms of sheaves from the lemma. For the next lemma, recall that by theorem 3.16 the Kato-Nakayama analytification of $(\mathrm{Bl}_{\Delta_{|S|}} \ X^{|S|}, \mathcal{O}(\tilde{\Delta}))$ is $\mathrm{Bl}^{\mathbb{R}}_{\Delta_{|S|}} \ (X^{\mathrm{an}})^{|S|}$ and the analytification of $(\mathbb{P}^{(|S|-1)d-1}, \mathcal{O}(-1))$ is $\mathbb{S}^{(|S|-1)2d-1}$.

Lemma 7.29. *The Kato-Nakayama analytification of* $\pi_S \colon \mathsf{V}_{d,n} \to (\mathbb{P}^{(|S|-1)d-1}, \mathcal{O}(-1))$ *is the projection map* $\pi_S \colon \mathcal{K}_{2d,n} \to \mathbb{S}^{(|S|-1)2d-1}$ *defined in section 4.*

Proof. First note that it is easy to verify that the following diagram commutes



where p_S is the inclusion $\mathrm{Conf}_n(X) \hookrightarrow X$ composed with the projection onto coordinates corresponding to elements in S $X^n \to X^{|S|}$. For $X = \mathbb{A}^d_{\mathbb{C}}$, the analytification of this diagram is



By lemma 4.4 the commutativity of this diagram implies that π_S^{KN} is π_S from section 4 (and furthermore that $\Pi_S^{\mathrm{KN}} = f_S$). ∎

With this we are ready to prove that the maps k_n^{KN} commute with the composition maps. To avoid making the proof too long I have divided it into 3 lemmas. Before stating them I will introduce some notation. In what follows let n be a positive integer, $\mathbf{m} = (m_1, \ldots, m_n)$ be a list of positive integers, and $m = \sum_i m_i$. Define

- $A(n) \coloneqq (\mathbb{S}^1)^n \times \mathcal{K}_{2d,n}$
- $A[\mathbf{m}] \coloneqq A(m_1) \times A(m_2) \times \cdots \times A(m_n)$
- $\delta^{n,\mathbf{m}} \colon A(n) \times A[\mathbf{m}] \to A(m)$ to be the composition map in the $\mathcal{K}_{2d} \rtimes \mathbb{S}^1$ operad
- $\mathsf{T}(n) \coloneqq \mathsf{T}_{d,n}$
- $\mathsf{T}[\mathbf{m}] \coloneqq \mathsf{T}(m_1) \times \cdots \times \mathsf{T}(m_n)$

- $\gamma^{n,\mathbf{m}}\colon \mathsf{T}(n) \times \mathsf{T}[\mathbf{m}] \to \mathsf{T}(m)$ to be the composition maps for the $\mathsf{T}_{d,n}$
- $\mathsf{K}(n) := \mathsf{K}_{d,n}$
- $\mathsf{K}[\mathbf{m}] := \mathsf{K}(m_1) \times \cdots \times \mathsf{K}(m_n)$
- $k_{\mathbf{m}} := k_{m_1} \times \cdots \times k_{m_n} \colon \mathsf{T}[\mathbf{m}] \to \mathsf{K}[\mathbf{m}]$
- $\mathsf{V}(n) := \mathsf{V}_{d,n}$
- $\mathsf{V}[\mathbf{m}] := \mathsf{V}(m_1) \times \cdots \times \mathsf{V}(m_n)$
- $\mathsf{P}_n := (\mathbb{P}^{(n-1)d-1}, (0\colon \mathcal{O} \to \mathcal{O}(-1)))$
- $\mathsf{C} := (\operatorname{Spec} \mathbb{C}, (0\colon \mathcal{O} \to \mathcal{O}))$
- $h_S\colon \mathsf{K}(n) \to \mathsf{P}_{|S|}$ to be the composition $\mathsf{K}(n) \to \mathsf{V}(n) \xrightarrow{\pi_S} \mathsf{P}_{|S|}$ for $S \subseteq [n]$, $|S| \geq 2$
- $h_i\colon \mathsf{K}(n) \to \mathsf{C}$ to be the map given by (the only possible) $f\colon T_{d,n} \to \operatorname{Spec} \mathbb{C}$ and $f^*\mathcal{O} \xrightarrow{\cong} \mathcal{L}_i$.
- $H_S\colon \mathsf{K}(n) \times \mathsf{K}[\mathbf{m}] \to \mathsf{P}_{|p^{n,\mathbf{m}}(S)|}$ for $S \subseteq [m]$, $|p^{n,\mathbf{m}}(S)| \geq 2$ as the composition

$$\mathsf{K}(n) \times \mathsf{K}[\mathbf{m}] \to \mathsf{K}(n) \xrightarrow{a_{p^{n,\mathbf{m}}(S)}} \mathsf{P}_{|p^{n,\mathbf{m}}(S)|}.$$

- $H_S\colon \mathsf{K}(n) \times \mathsf{K}[\mathbf{m}] \to \mathsf{P}_{|q_r^{n,\mathbf{m}}(S)|} \times \mathsf{C}$ for $S \subseteq [m]$, $p^{n,\mathbf{m}}(S) = \{r\}$, $|S| \geq 2$ as the composition

$$\mathsf{K}(n) \times \mathsf{K}[\mathbf{m}] \to \mathsf{K}(n) \times K(m_r) \xrightarrow{a_r \times a_{q_r^{n,\mathbf{m}}(s)}} \mathsf{C} \times \mathsf{P}_{|p^{n,\mathbf{m}}(S)|}.$$

- $H_i\colon \mathsf{K}(n) \times \mathsf{K}[\mathbf{m}] \to \mathsf{C} \times \mathsf{C}$ for $i \in [m]$, $p^{n,\mathbf{m}}(i) = r$, as the composition

$$\mathsf{K}(n) \times \mathsf{K}[\mathbf{m}] \to \mathsf{K}(n) \times K(m_r) \xrightarrow{a_{\{r\}} \times a_{\{q_r^{n,\mathbf{m}}(i)\}}} \mathsf{C} \times \mathsf{C}.$$

Lemma 7.30. *For every $i \in [m]$ here is a morphism $g\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathsf{T}(n) \times \mathsf{T}[\mathbf{m}] & \xrightarrow{\gamma^{n,\mathbf{m}}} & \mathsf{T}(m) \\ \downarrow{\scriptstyle k_n \times k_{\mathbf{m}}} & & \downarrow{\scriptstyle k_{\mathbf{m}}} \\ \mathsf{K}(n) \times \mathsf{K}[\mathbf{m}] & & \mathsf{K}(m) \\ \downarrow{\scriptstyle H_i} & & \downarrow{\scriptstyle h_i} \\ \mathsf{C} \times \mathsf{C} & \xrightarrow{g} & \mathsf{C} \end{array}$$

Furthermore, the analytification of the bottom half of this diagram commutes with $\delta^{n,\mathbf{m}}$, i.e. the following diagram commutes

$$\begin{array}{ccc} A(n) \times A[\mathbf{m}] & \xrightarrow{\delta^{n,\mathbf{m}}} & A(m) \\ \downarrow{\scriptstyle H_i} & & \downarrow{\scriptstyle h_i} \\ \mathbb{S}^1 \times \mathbb{S}^1 & \xrightarrow{g^{\mathrm{KN}}} & \mathbb{S}^1 \end{array}$$

Proof. First note that $\mathsf{C} \times \mathsf{C} \cong (\operatorname{Spec} C, (0\colon \mathcal{O} \to \mathcal{L}_1, 0\colon \mathcal{O} \to \mathcal{L}_2))$. By applying lemma 7.20 is easy to verify that the diagram commutes when g is the identity on the underlying map of schemes and the isomorphism of line bundles is $\mathcal{O} \xrightarrow{\cong} \mathcal{L}_1 \otimes \mathcal{L}_2$. From the definition of the analytification of a map of log schemes the induced map $g^{\mathrm{KN}} \colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ is the product morphism for the topological group $\mathbb{S}^1$. The analytification of A_i is the product of the projection from $A(n) \times A[\mathbf{m}]$ to the rth $\mathbb{S}^1$ component of $A(n)$ and the $q_r^{n,\mathbf{m}}(i)$th $\mathbb{S}^1$ component of $A(m_r)$ and the analytification of a_i is the projection to the ith $\mathbb{S}^1$ component of $A(m)$. By definition or the composition maps $\delta^{n,\mathbf{m}}$ the diagram

$$\begin{array}{ccc} A(n) \times A[\mathbf{m}] & \xrightarrow{\delta^{n,\mathbf{m}}} & A(m) \\ \big\downarrow{\scriptstyle H_i} & & \big\downarrow{\scriptstyle h_i} \\ \mathbb{S}^1 \times \mathbb{S}^1 & \xrightarrow{g^{\mathrm{KN}}} & \mathbb{S}^1 \end{array}$$

commutes if (and only if) g^{KN} is the multiplication map which, as stated previously, it is. □

Lemma 7.31. *For every $S \subseteq [m]$ with $p^{n,\mathbf{m}}(S) \geq 2$ there is a morphism $b_S \colon \mathsf{P}_{|p^{n,\mathbf{m}}(S)|} \to P_{|S|}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathsf{T}(n) \times \mathsf{T}[\mathbf{m}] & \xrightarrow{\gamma^{n,\mathbf{m}}} & \mathsf{T}(m) \\ \big\downarrow{\scriptstyle k_n \times k_{\mathbf{m}}} & & \big\downarrow{\scriptstyle k_{\mathbf{m}}} \\ \mathsf{K}(n) \times \mathsf{K}[\mathbf{m}] & & \mathsf{K}(m) \\ \big\downarrow{\scriptstyle H_S} & & \big\downarrow{\scriptstyle h_S} \\ \mathsf{P}_{|p^{n,\mathbf{m}}(S)|} & \xrightarrow{b_S} & \mathsf{P}_{|S|} \end{array}$$

Furthermore, the analytification of the bottom half of this diagram commutes with $\delta^{n,\mathbf{m}}$, i.e. the following diagram commutes

$$\begin{array}{ccc} A(n) \times A[\mathbf{m}] & \xrightarrow{\delta^{n,\mathbf{m}}} & A(m) \\ \big\downarrow{\scriptstyle H_S} & & \big\downarrow{\scriptstyle h_S} \\ \mathbb{S}^{(|p^{n,\mathbf{m}}(S)|-1)d-1} & \xrightarrow{b_S^{\mathrm{KN}}} & \mathbb{S}^{(|S|-1)d-1} \end{array}$$

Proof. To make the notation a little bit simpler let $R = p^{n,\mathbf{m}}(S)$. We start by defining the map b_S for the underlying schemes $\mathbb{P}^{(|R|-1)d-1} \to \mathbb{P}^{(|S|-1)d-1}$. Since $\mathcal{F}_S^{H,d}$ is naturally isomorphic to $\mathcal{O}_H^{\oplus(|S|-1)d}$ for any scheme H, we have that $\mathbb{P}^{(|R|-1)d-1}$ represents the functor

$$H \to \{\phi \colon \mathcal{F}_R^{H,d} \twoheadrightarrow \mathcal{L}\}$$

and $\mathbb{P}^{(|S|-1)d-1}$ represents the functor

$$H \to \{\Phi \colon \mathcal{F}_S^{H,d} \twoheadrightarrow \mathcal{L}\}.$$

The desired morphism of schemes is the one induced by the natural transformation

$$(\phi \colon \mathcal{F}_R^{H,d} \twoheadrightarrow \mathcal{L}) \mapsto (\phi \circ \alpha^{n,\mathbf{m}} \colon \mathcal{F}_R^{H,d} \twoheadrightarrow \mathcal{L})$$

where $\alpha^{n,\mathbf{m}}$ is as in section 6.3. This is the restriction to fibers over the origin of a map $B_S \colon \mathrm{Bl}_{\Delta_{|R|}} \mathbb{A}^{|R|} \to \mathrm{Bl}_{\Delta_{|R|}} \mathbb{A}^{|R|}$ which is defined in the obvious way. Since this type of construction has appeared many times I will now omit the details. The inverse image of the exceptional divisor in $\mathrm{Bl}_{\Delta_{|R|}} \mathbb{A}^{|R|}$ via this map is the exceptional divisor in $\mathrm{Bl}_{\Delta_{|R|}} \mathbb{A}^{|R|}$ so by applying proposition 7.1 and restricting to $\mathbb{P}^{(|S|-1)d-1}$ we find that there is a canonical isomorphism $b_S^*\mathcal{O}(-1) \cong O(-1)$. This gives a morphism of log schemes $b_S\colon \mathsf{P}_{|R|} \to \mathsf{P}_{|S|}$. It is a trivial exercise to verify that the resulting diagram of morphisms of log schemes commutes. Finally, it is also easy to use this construction to make a commutative diagram

$$\begin{array}{ccc} \mathsf{P}_{|R|} & \xrightarrow{b_S} & \mathsf{P}_{|S|} \\ \big\downarrow{\scriptstyle i_R} & & \big\downarrow{\scriptstyle i_S} \\ (\mathrm{Bl}_{\Delta_{|R|}}\ (\mathbb{A}^d)^{|R|}, (s\colon \mathcal{O} \to \mathcal{O}(\tilde{\Delta}_{|R|})) & \longrightarrow & (\mathrm{Bl}_{\Delta_{|S|}}\ (\mathbb{A}^d)^{|S|}, (s\colon \mathcal{O} \to \mathcal{O}(\tilde{\Delta}_{|S|})) \\ \big\downarrow{\scriptstyle \rho_R} & & \big\downarrow{\scriptstyle \rho_S} \\ (\mathbb{A}^d)^{|R|} & \xrightarrow{d_S} & (\mathbb{A}^d)^{|S|} \end{array}$$

where the analytification of d_S is the map d_S in lemma4.6. Thus, by lemma 4.6 the diagram

$$\begin{array}{ccc} A(n) \times A[\mathbf{m}] & \xrightarrow{\delta^{n,\mathbf{m}}} & A(m) \\ \big\downarrow{\scriptstyle H_S} & & \big\downarrow{\scriptstyle h_S} \\ \mathbb{S}^{(|p^{n,\mathbf{m}}(S)|-1)d-1} & \xrightarrow{b_S^{\mathrm{KN}}} & \mathbb{S}^{(|S|-1)d-1} \end{array}$$

commutes. □

The last result we need follows from a result that is interesting on its own. Therefore I will state it as a proposition.

Proposition 7.32. *Let $\mathsf{X} = (\mathbb{P}^{n-1}, (0\colon \mathcal{O} \to \mathcal{L}_1, 0\colon \mathcal{O} \to \mathcal{L}_2)$ where $\mathcal{L}_1 = \mathcal{O}_{\mathbb{P}^{n-1}}$ and $\mathcal{L}_2 = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$, and let $\mathsf{Y} = (\mathbb{P}^{n-1}, (0\colon \mathcal{O} \to \mathcal{M})$ where $\mathcal{M} = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Finally, let $f\colon \mathsf{X} \to \mathsf{Y}$ be the map given by the identity on underlying schemes and $\mathcal{M} \xrightarrow{\cong} \mathcal{L}_1 \otimes \mathcal{L}_2$. The analytification of f, $f^{\mathrm{KN}}\colon \mathbb{S}^1 \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ is the $\mathbb{S}^1$ action on $\mathbb{S}^{2n-1}$ induced by the diagonal inclusion $SO(2) \hookrightarrow SO(2n)$.*

Proof. First, let $g\colon \mathbb{A}^1 \times \mathbb{A}^n \to \mathbb{A}^n$ be the map $(z, (x_1, \ldots, x_n)) \mapsto (zx_1, \ldots, zx_n)$. By the universal property of the blow up this induces a map $\tilde{g}\colon \mathbb{A}^1 \times \mathrm{Bl}_p\ \mathbb{A}^n \to \mathrm{Bl}_p\ \mathbb{A}^n$, where p denotes the origin in $\mathbb{A}^n$. Now, let $\tilde{p}$ be the exceptional divisor in $\mathrm{Bl}_p\ \mathbb{A}^n$, let $s\colon \mathcal{O}_{\mathrm{Bl}_p\ \mathbb{A}^n} \to \mathcal{M}$ be the corresponding line bundle with section and let $t\colon \mathcal{O}_{\mathbb{A}^1} \to \mathcal{L}$ be the line bundle with section corresponding to the origin in $\mathbb{A}^1$. By abuse of notation denote the pullbacks of these two sheaves with sections to the product $\mathbb{A}^1 \times \mathrm{Bl}_p\ \mathbb{A}^n$ in the same way. It is clear that $g^{-1}(\tilde{p})$ is the scheme theoretic union $\mathbb{A}^1 \times \tilde{p} \cup o \times \mathrm{Bl}_p\ \mathbb{A}^n$, where o denotes the origin in $\mathbb{A}^1$. Hence by proposition 7.1 there is a unique isomorphism of line bundles

$$\tilde{g}^*\mathcal{M} \xrightarrow{\cong} \mathcal{L} \otimes \mathcal{M}$$

sending $\tilde{g}^*s \mapsto s \otimes t$. This extends $\tilde{g}$ to a morphism of log schemes

$$(\mathbb{A}^1 \times \mathrm{Bl}_p\ \mathbb{A}^n, (s\colon \mathcal{O} \to \mathcal{L}, t\colon \mathcal{O} \to \mathcal{M})) \to (\mathrm{Bl}_p\ \mathbb{A}^n, t\colon \mathcal{O} \to \mathcal{M}).$$

It is easy to verify that the following diagram of log schemes commutes.

$$\begin{array}{ccc}
\mathsf{X} & \xrightarrow{\quad f \quad} & \mathsf{Y} \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\
(\mathbb{A}^1 \times \mathrm{Bl}_p\ \mathbb{A}^n, (s\colon \mathcal{O} \to \mathcal{L}, t\colon \mathcal{O} \to \mathcal{M})) & \xrightarrow{\quad \tilde{g} \quad} & (\mathrm{Bl}_p\ \mathbb{A}^n, t\colon \mathcal{O} \to \mathcal{M}) \\
\downarrow{\scriptstyle \mathrm{id}\times\rho} & & \downarrow{\scriptstyle \rho} \\
\mathbb{A}^1 \times \mathbb{A}^n & \xrightarrow{\quad g \quad} & \mathbb{A}^n
\end{array}$$

Where i, j are the strict morphisms induced by the inclusions of fibers over the origin for the underlying schemes, and $\mathbb{A}^1 \times \mathbb{A}^n$ and $\mathbb{A}^n$ are the corresponding schemes with no line bundles. The analytification of this diagram is the following,

$$\begin{array}{ccc}
\mathbb{S}^1 \times \mathbb{S}^{2n-1} & \xrightarrow{\quad f' \quad} & \mathbb{S}^{2n-1} \\
\downarrow{\scriptstyle i^{\mathrm{KN}}} & & \downarrow{\scriptstyle j^{\mathrm{KN}}} \\
\mathrm{Bl}_o^{\mathbb{R}}\ \mathbb{R}^2 \times \mathrm{Bl}_p^{\mathbb{R}}\ \mathbb{R}^{2n} & \xrightarrow{\quad \tilde{g}' \quad} & \mathrm{Bl}_p^{\mathbb{R}}\ \mathbb{R}^{2n} \\
\downarrow{\scriptstyle \mathrm{id}\times\rho^{\mathrm{KN}}} & & \downarrow{\scriptstyle \rho^{\mathrm{KN}}} \\
\mathbb{R}^2 \times \mathbb{R}^{2n} & \xrightarrow{\quad g^{\mathrm{an}} \quad} & \mathbb{R}^{2n}
\end{array}$$

where $\tilde{g}' = \tilde{g}^{\mathrm{KN}}$ and $f' = f^{\mathrm{KN}}$. Since $(a+bi)(c+di) = (ac-bd) + (ad+bc)$ the analytification g^{an} is given by

$$((a,b),(x_1,y_1,\ldots,x_n,y_n)) \mapsto (R_{a,b}(x_1,y_1),\ldots,R_{a,b}(x_n,y_n))$$

where $R_{a,b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation

$$R_{a,b} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Now, it is easy to construct a function $\tilde{g}'$ such that the diagram commutes with f' equal to the group action morphism. Since ρ is a homeomorphism on a dense subset there is only one $\tilde{g}'$ that can make the diagram commute and since $i^{\mathrm{KN}}, j^{\mathrm{KN}}$ the function f' is uniquely determined by $\tilde{g}'$. Hence the group action function is the only possible function which can make the diagram commute and so f^{KN} must be this function. □

Lemma 7.33. *For every $S \subseteq [m]$ with $p^{n,\mathbf{m}}(S) = \{r\}$ and $|S| \geq 2$ there is a morphism $b_S\colon \mathsf{C} \times \mathsf{P}_{|q_r^{n,\mathbf{m}}(S)|} \to P_{|S|}$ such that the following diagram commutes*

$$\begin{array}{ccc}
\mathsf{T}(n) \times \mathsf{T}[\mathbf{m}] & \xrightarrow{\quad \gamma^{n,\mathbf{m}} \quad} & \mathsf{T}(m) \\
\downarrow{\scriptstyle k_n \times k_{\mathbf{m}}} & & \downarrow{\scriptstyle k_{\mathbf{m}}} \\
\mathsf{K}(n) \times \mathsf{K}[\mathbf{m}] & & \mathsf{K}(m) \\
\downarrow{\scriptstyle H_S} & & \downarrow{\scriptstyle h_S} \\
\mathsf{C} \times \mathsf{P}_{|q_r^{n,\mathbf{m}}(S)|} & \xrightarrow{\quad b_S \quad} & \mathsf{P}_{|S|}
\end{array}$$

Furthermore, the analytification of the bottom half of this diagram commutes with $\delta^{n,\mathbf{m}}$, i.e. the following diagram commutes

$$\begin{array}{ccc} A(n) \times A[\mathbf{m}] & \xrightarrow{\delta^{n,\mathbf{m}}} & A(m) \\ \downarrow H_S & & \downarrow h_S \\ \mathbb{S}^1 \times \mathbb{S}^{(|q_r^{n,\mathbf{m}}(S)|-1)d-1} & \xrightarrow{b_S^{\mathrm{KN}}} & \mathbb{S}^{(|S|-1)d-1} \end{array}$$

Proof. $\mathsf{C} \times \mathsf{P}_{|q_r^{n,\mathbf{m}}(S)|}$ is isomorphic to the log scheme $(\mathbb{P}^{(|S|-1)d-1}, (0\colon \mathcal{O} \to \mathcal{L}_1, 0\colon \mathcal{O} \to \mathcal{L}_2))$ where $\mathcal{L}_1 = \mathcal{O}$ and $\mathcal{L}_2 = \mathcal{O}(-1)$, and $\mathsf{P}_{|S|}$ is the log variety $(\mathbb{P}^{(|S|-1)d-1}, (0\colon \mathcal{O} \to \mathcal{M}))$ where $\mathcal{M} = \mathcal{O}(-1)$. Now, it is easy to verify that the first diagram is commutative when b_S is the map of log schemes given by the identity on the underlying schemes and the isomorphism of line bundles $\mathcal{M} \cong \mathcal{L}_1 \otimes \mathcal{L}_2$. By propositon 7.32 the analytification of this map is the group action function $\mathbb{S}^1 \times \mathbb{S}^{(|S-1|)2d-1} \to \mathbb{S}^{(|S-1|)2d-1}$. By definition of the composition functions $\delta^{n,\mathbf{m}}$ it is clear that the second diagram commutes for this map. □

Lemma 7.34. *The following diagram commutes for every $n, \mathbf{m}$.*

$$\begin{array}{ccc} \mathsf{T}(n)^{\mathrm{KN}} \times \mathsf{T}(m_1)^{\mathrm{KN}} \times \cdots \times \mathsf{T}(m_n)^{\mathrm{KN}} & \xrightarrow{(\gamma^{n,\mathbf{m}})^{\mathrm{KN}}} & \mathsf{T}(m)^{\mathrm{KN}} \\ \downarrow k_n^{\mathrm{KN}} \times k_{m_1}^{\mathrm{KN}} \times \cdots \times k_{m_n}^{\mathrm{KN}} & & \downarrow k_m^{\mathrm{KN}} \\ A(n) \times A(m_1) \times \cdots \times A(m_n) & \xrightarrow{\delta^{n,\mathbf{m}}} & A(m) \end{array}$$

Proof. From lemmas 7.30, 7.31, and 7.33 it is evident that

$$a_S \circ k_m^{\mathrm{KN}} \circ (\gamma^{n,\mathbf{m}})^{\mathrm{KN}} = a_S \circ \delta^{n,\mathbf{m}} \circ k_n^{\mathrm{KN}} \times k_{\mathbf{m}}^{\mathrm{KN}}$$

for every $S \subseteq [m]$. Since the maps a_S are the inclusion

$$A(m) \hookrightarrow (\mathbb{S}^1)^m \times \prod_{S \in P(m)} \mathbb{S}^{(|S|-1)d-1}$$

composed with the projection maps to each of the components the result immediately follows from the universal property of the product. □

Theorem 7.35. *The analytification of the log-geometric Kontsevich operad without unit, $\mathcal{T}_d$, is isomorphic to the $\mathbb{S}^1$-framed Kontsevich Operad in dimension $2d$, $\mathcal{K}_{2d} \rtimes \mathbb{S}^1$, without unit.*

Proof. Since the maps k_n^{KN} are homeomorphisms by lemma 7.23 they give the desired isomorphism of operads without unit by corollary 7.26 and lemma 7.34. □

Remark. While we cannot extend this to an isomorphism of operads with units there is a somewhat stronger version of this result. Namely, even though there is no unit in our operad of log schemes the "one object composition morphisms" $\circ_i\colon \mathsf{T}_{d,n} \times \mathsf{T}_{d,m} \to \mathsf{T}_{d,n+m-1}$ can still be defined in the obvious way and their analytifications are identified with the "one object composition morphisms" of $\mathcal{K}_d \rtimes \mathbb{S}^1$ by the isomorphisms k_n.

As a final remark to end this thesis I will suggest two solutions to the problem of defining a unit morphism for this operad. These "solutions" should be seen as suggestions for further research rather than theories I have actually fully developed.

1. The perhaps most obvious way to get around this problem is by redefining $\mathsf{T}_{d,1}$ as Spec $\Bbbk$ without any line bundles. We can still isomorphisms of line bundles in the same way as we did in section 7.1 by just

removing $\otimes\mathcal{O}_{T_1}(\{1\})$ from any of the tensor products in which this appears. This operad of log schemes has a unit and the analytification of this operad is the reduced $\mathbb{S}^1$-framed Kontsevich Operad in dimension $2d$.

2. Another thing we could do is to change categories to the category of log schemes over $(\operatorname{Spec} \Bbbk, 0\colon \mathcal{O} \to \mathcal{O})$. In this category $\mathsf{T}_{d,1}$ is the unit object so the unit morphism would certainly be well defined. For this to work we would have to define morphisms $\mathsf{T}_{d,n} \to (\operatorname{Spec} \Bbbk, 0\colon \mathcal{O} \to \mathcal{O})$ making $\mathsf{T}_{d,n}$ log schemes over $(\operatorname{Spec} \Bbbk, 0\colon \mathcal{O} \to \mathcal{O})$. There is only one candidate for the map of underlying schemes so this is equivalent to expressing $\mathcal{O}_{T_{d,n}}$ as a tensor product of the other sheaves with sections. The most natural candidate for this is
$$\mathcal{O} \cong \bigotimes_{S\subseteq[n]} \mathcal{O}_{T_{d,n}}^{\otimes|S|}.$$
There are some details to work out here since the Cartesian product in this category is not the same as that in the category of log schemes over Spec $\Bbbk$. However, I think (with emphasis on think) that one will still be able to define the composition maps in this category.

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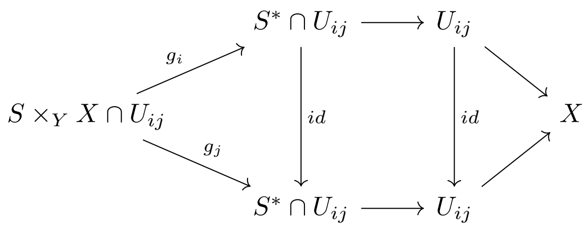
A Pullbacks of Effective Cartier Divisors

This appendix is dedicated to giving a proof of proposition 7.1.

Lemma A.1. *Let $s\colon \mathcal{O}_Y \to \mathcal{L}$ be a section of a line bundle on a scheme Y and let $f\colon X \to Y$ be a morphism of schemes. Furthermore, let $S \hookrightarrow Y$ denote the closed subscheme cut out by s and let $S^* \hookrightarrow X$ denote the subscheme cut out by the pullback of s, f^*s. Then there is an isomorphism of X-schemes,*

$$S \times_Y X \cong S^*.$$

Proof. First notice that if we can find an open cover of $\bigcup_i U_i = X$ such that for each U_i there is an isomorphism of U_i-schemes $g_i\colon S \times_Y X \cap U_i \to S^* \cap U_i$ then these maps glue to an isomorphism of X-schemes, $g\colon S \times_Y X \to S^*$. This follows from the fact that since g_i, g_j are X-morphisms the outer cycle of the following diagram must commute. Since $S^* \cap U_{ij} \to X$ is a monomorphism this implies that the inner cycle is commutative too, i.e. g_i, g_j satisfy the cocycle condition.



This statement reduces the proof to the case where X, Y are affine and $\mathcal{L}$ is the trivial line bundle since we can choose an affine open cover of Y which trivializes $\mathcal{L}$, $\bigcup_i V_i = Y$, and an affine open cover of X, $\bigcup_{i,j} U_{ij}$, $U_{ij} = \text{Spec } B_{ij}$ such that $U_{ij} \subseteq f^{-1}(V_i)$.

In the case $X = \text{Spec } B$, $Y = \text{Spec } A$ and $s \in \Gamma(Y, \mathcal{O}_Y) = A$ we have that $S = \text{Spec } A/(s) \hookrightarrow \text{Spec } A = Y$. Furthermore, if $f\colon X \to Y$ is induced by $\phi\colon A \to B$ we have that $f^*s \in \Gamma(X, \mathcal{O}_X) = B$ is the section, $\phi(s)$ and thus $S^* = \text{Spec } B/(\phi(s)) \hookrightarrow \text{Spec } B = X$. Hence, the desired result is, in the affine, trivial $\mathcal{L}$, case, equivalent to claiming that there is a B-algebra isomorphism $\sigma\colon A/(s) \otimes_A B \to B/(\phi(s))$. This is a standard result from commutative algebra and thus we are done. □

Lemma A.2. *Let $s\colon \mathcal{O}_X \to \mathcal{L}$ and $t\colon \mathcal{O}_X \to \mathcal{M}$ be non zero sections of line bundles on an irreducible scheme X such that s, t both cut out the same closed subscheme of X. Then there is a unique isomorphism of $\mathcal{O}_X$-modules*

$$\phi\colon \mathcal{L} \to \mathcal{M}$$

sending $s \mapsto t$.

Proof. Let $\bigcup_i U_i$ be a cover of trivializing open affines, $U_i = \text{Spec } A_i$, for both $\mathcal{L}$ and $\mathcal{M}$ and let $\mu_i\colon \mathcal{O}_{U_i} \to \mathcal{L}|_{U_i}$ and $\sigma_i\colon \mathcal{O}_{U_i} \to \mathcal{M}|_{U_i}$ be the corresponding trivializing isomorphisms. Furthermore, define $a_i, b_i \in A_i$ to be the elements such that $\mu_i(a_i) = s|_{U_i}$ and $\sigma_i(b_i) = t|_{U_i}$. Finally, let $l_{i,j}$ and $m_{i,j}$ denote the $U_i \cap U_j$ connecting homomorphisms for $\mathcal{L}$ and $\mathcal{M}$ respectively. Now, since a_i, b_i are non-zero divisors which cut out isomorphic closed subschemes of Spec A_i there is a unique automorphism, $\rho_i\colon A \to A$ (seen as an A-module) sending $a_i \mapsto b_i$

given by $a \mapsto r_i a$. This implies that there is a unique isomorphism $\phi_i \colon \mathcal{L}|_{U_i} \to \mathcal{M}|_{U_i}$ of $\mathcal{O}_{U_i}$-modules sending $s|_{U_i} \to t|_{U_i}$; namely,

$$\phi_i = \sigma_i \circ \rho_i \circ \mu_i^{-1}.$$

Lastly notice that

$$\phi_j \circ l_{i,j}(s_i) = \phi_j(s_j) = t_j = m_{i,j}(t_i) = m_{i,j}(\phi_i(s_i)).$$

Since a_i is not a zero divisor and $\mu_i(a_i) = s_i$ this implies that $\phi_j \circ l_{i,j} = m_{i,j} \circ \phi_i$. Since the isomorphisms ϕ_i commute with the structure morphisms for the respective line bundles they glue to give an isomorphism of sheaves $\phi \colon \mathcal{L} \to \mathcal{M}$ sending $s \mapsto t$. This isomorphism is unique since the restriction of ϕ to each U_i was unique. □

Remark. We could have allowed X to be reducible and instead required that s, t do not restrict to 0 on any irreducible component of X but this would have made the proof slightly more tedious so I decided against including this.

Finally, we are now ready for the proof of proposition 7.1. I will restate it for clarity.

Proposition. *Let $D \hookrightarrow Y$ be an effective Cartier divisor and let $s \colon \mathcal{O}_Y \to \mathcal{O}_Y(D)$ be the corresponding line bundle with section. Furthermore, let X be an irreducible scheme and let $f \colon X \to Y$ be a morphism of schemes such that there is an isomorphism of X-schemes*

$$D \times_Y X \cong D_1' \cup \cdots \cup D_n'$$

where D_i' are all (distinct) effective Cartier divisors with corresponding sections $s_i' \colon \mathcal{O}_X \to \mathcal{O}_X(D_i')$. Then there is a unique isomorphism of $\mathcal{O}_X$-modules,

$$\phi \colon f^*\mathcal{O}_Y(D) \to \bigotimes_{i=1}^{n} \mathcal{O}_X(D_i')$$

such that $\phi(s^*) = \bigotimes_i s_i'$.

Proof. By lemmas A.1 and A.2 we must only check that the zero locus of $\bigotimes_i s_i'$ is the scheme theoretic union $D_1' \cup \cdots \cup D_n'$. Using a gluing argument similar to that in the proof of lemma A.1 we can reduce to the case when X is affine, $X = \operatorname{Spec} A$, and the sheaves $\mathcal{O}_X(D_i')$ are trivial with sections $s_i' \in A$ respectively. Here we abuse notation somewhat by considering s_i' to be elements of A rather than global sections of a quasi coherent sheaf. In this case the ideal for the closed subscheme corresponding to the scheme theoretic union of closed subschemes $\bigcup_i \operatorname{Spec} A/(s_i')$ is the intersection of ideals $\bigcap_i (s_i')$. Furthermore, the section

$$s_1' \otimes \cdots \otimes s_n' \in A \otimes \cdots \otimes A \cong A$$

is just the element $s_1 s_2 \ldots s_n$ and therefore The closed subscheme cut out by $\bigotimes_i s_i'$ corresponds to the ideal $(s_1 s_2 \ldots s_n)$. Finally, since D_i' is irreducible for every i the ideals (s_i') are disjoint prime ideals which implies that

$$\bigcap_i (s_i) = (\prod_i s_i).$$

Thus, the two closed subschemes we have defined are the same and the proof is complete. □

Remark. Notice that we could have made this statement more general by only requiring D to be cut out by a single equation in some line bundle $\mathcal{L}$ and similarly only requiring each D_i' to be cut out by a single equation in $\mathcal{L}_i'$ such that none of them share any irreducible components. However, since we do not need this in full generality I decided to only include this weaker result.