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Characterization of Rational Herglotz Functions

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Abstract

In this thesis, we study rational scalar-valued Herglotz functions. Properties of the class of general Herglotz functions are discussed with examples. Rational scalar-valued Herglotz functions will then be characterised either by the their poles or by the corresponding measures in the integral representation. Lastly, the result from the scalar-valued case is used to extend the result to matrix-valued Herglotz functions.

Sammanfattning

I denna uppsats studerar vi skälarvärda Heglotz funktioner. Egenskaper hos klassen av generella Herglotz funktioner diskuteras med exempl. Rationella skälarvärda Herglotz funktioner karaktäriseras sedan antingen genom deras poler eller genom egenskaper hos motsvarande mått i integral representationen. Avslutningsvis utvidgas resultatet till matrisvärda Herlotz funktioner.

Acknowledgements

I would like to thank my supervisor Professor Annemarie Luger for suggesting the topic and the continuous support that she provided through this project. The topic related two of my favourite subjects from the masters program, measure theory and complex analysis. I was, therefore, happy to have a chance to study them deeper. Professor Annemarie's patience and deep knowledge of the topic improved my experience significantly and led me to explore the topic beyond what was initially intended.

I would also like to thank my previous teachers whose devotion to mathematics has inspired me to learn more.

Contents

1 Introduction

The class of Herglotz functions consists of analytic functions that map the closed upper half-plane into the open upper half-plane. Herglotz functions are important for many reasons, they appear in the study of the spectral theory of Schrödinger operators [14], the modeling of passive systems and circuit synthesis [10, 4]. In this thesis, we will review properties of scalar- and matrix-valued Herglotz functions and provide a characterisation of rational Herglotz functions.

A fundamental characterisation of (scalar-valued) Herglotz functions is their integral representation. Any Herglotz function admits an integral representation which depends on two scalar parameters and a positive Borel measure on $\mathbb{R}$. Similarly, matrix-valued Herglotz function have an integral representation that depends on two constant matrices and a matrix-valued measure defined on Borel sets of $\mathbb{R}$. In both cases the measure and the two constant parameters are uniquely determined by the function.

An important subclass of Herglotz functions are rational Herglotz functions. Rational Herglotz functions are important for many reasons: general Herglotz function can be approximated by rational Herglotz functions with only real poles and in applications they play a central role in passive network synthesis [4]. It is therefore natural to characterise them.

The idea of the characterisations is to divide up the poles of the function in poles on the real line and poles in the lower half-plane. The parts of the function with poles on the real line and in the lower half-plane are respectively Herglotz function. This is shown by decomposing the corresponding measure into different parts which are in turn Herglotz measures that correspond to the different part of the function with poles on the real line and in the lower half-plane. The goal then is to characterise Herglotz functions with poles only on the real line and in the lower half-plane. Herglotz functions with poles only on the real line are characterised by their poles being simple and having negative residue and Herglotz functions with poles in the lower half-plane are characterised by their corresponding measure. The same procedure is repeated verbatim for matrix-valued Herglotz functions.

2 Preliminaries

2.1 Basic Facts About Herglotz Functions

In this section, we introduce some notation and provide some classical results about Herglotz functions. The results and their proofs are based on [14, Chapter 3.4], [6, Chapter 2.], [10, Chapter 1.2.] and [2, Chapter 1].

Definition 1. *Let* $\mathbb{C}^{\pm} := \{z \in \mathbb{C} : \operatorname{Im}[z] \gtrless 0\}$ *denote the open upper and lower half-planes. A function* $h : \mathbb{C}^+ \to \mathbb{C}$ *is called a Herglotz function if* h *is analytic and* $\operatorname{Im}[h(z)] \geq 0$ *for all* $z \in \mathbb{C}^+$.

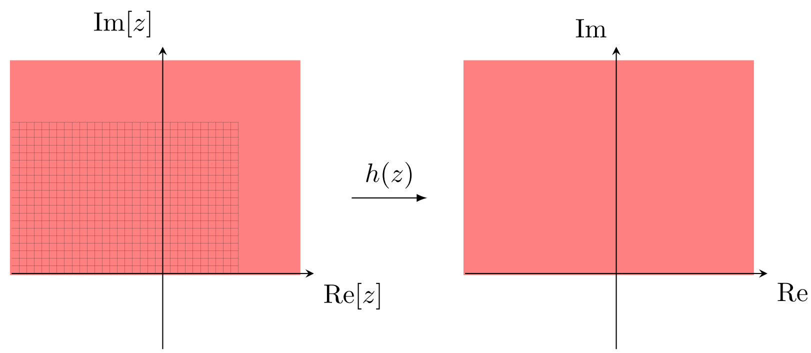

Figure 1: Herglotz Function

Example 2. *Examples of Herglotz functions are*

$$a + ib, \quad a + bz, \quad a \in \mathbb{R}, \quad b \geq 0, \tag{1}$$

$$\log(z) = \ln(|z|) + i\arg(z), \quad \arg(z) \in (-\pi, \pi], \tag{2}$$

$$z^r := e^{r\log(z)}, \quad 0 \leq r \leq 1, \tag{3}$$

$$\tan(z), \text{ since } \operatorname{Im}[\tan(x+iy)] = \frac{\sinh(2y)}{\cos(2x)+\cosh(2y)} > 0 \text{ for } y > 0, \tag{4}$$

$$\frac{1}{w-z}, \quad \operatorname{Im}[w] \leq 0. \tag{5}$$

◊

We can construct more examples by composition and linear combinations. To be more specific, any positive linear combination of Herglotz functions is a Herglotz function. In addition, if h_1 is Herglotz function that does not attain any real value, then for any other Herglotz function h_2 we have that $h_2(h_1)$ is a Herglotz function.

Example 3. *More examples of Herglotz functions*

$$-\frac{1}{\tan(z)} = -\cot(z), \tag{6}$$

$$-\frac{1}{z - \frac{3}{z} + 4i}. \tag{7}$$

A less trivial example is $\frac{\Gamma'(z)}{\Gamma(z)}$ *where* $\Gamma(z)$ *is Euler's Gamma function, see [1, Chapter 6.] for more results. To see this, we note that*

$$\Gamma(z) := e^{-\gamma z}\frac{1}{z}\prod_{n=1}^{\infty}\frac{1}{1+\frac{z}{n}}e^{\frac{z}{n}} \tag{8}$$

where $\gamma := \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \log(n)\right)$*. It can be shown that* $\Gamma(z)$ *is a non-vanishing meromorphic function with poles at the non-positive integers and no other poles. Viewing* Γ *as function on the upper half-plane, we have that* $\log(\Gamma(z))$ *is well-defined and it can be calculated*

$$\begin{aligned}\frac{d\log(\Gamma(z))}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} &= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{\frac{1}{n}}{1+\frac{z}{n}}\right) \\ &= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right)\end{aligned} \tag{9}$$

which is a Herglotz function since it analytic on $\mathbb{C}^+$ *and maps it into itself. Note that in the sum appearing when we take the logarithm, we can interchange the order of differentiation and summation since the sequence of partial sums converges uniformly on compact subsets of the upper half-plane, see [9, Theorem 11.11, page 184, Theorem 4.29].* $\lozenge$

Example 4. *In this example we mention a class of functions related to Herglotz function that are useful in passive network synthesis. To be more specific, we will consider positive-real functions.*

Definition 5. *Let* $\mathbb{C}_\pm := \{z \in \mathbb{C} : \mathrm{Re}[z] \gtrless 0\}$*. We define* $p : \mathbb{C}_+ \to \mathbb{C}$ *to be a real function if* $p(x) = \mathrm{Re}[p(x)]$ *for* $x \in \mathbb{R}_+$*.*

Definition 6. *A real function* $p(z)$ *is positive-real (PR) if* $\mathrm{Re}[p(z)] \geq 0$ *for* $\mathrm{Re}[z] > 0$ *and* p *is analytic in* $\mathbb{C}_+$*.*

One of the most significant result in circuit theory is the following

> *The driving point impedance (admittance) of an RLC circuit is a rational positive-real function.*

For a much more detailed discussion on the topic, see [16, Chapter 2, Chapter 5.2].

Remark 7. *Note that for a PR function we have that the following symmetry holds since h is analytic across the real line and fixes the real line*

$$\overline{p(z)} = p(\overline{z})$$

for $z \in \mathbb{C}_+$*. In addition, note that given a PR function* $p(z)$*, the function* $h(z) := e^{i\frac{\pi}{2}}p(e^{\frac{-i\pi}{2}}z)$ *is a Herglotz function that maps the upper part of the imaginary axis,* $\{iy : y > 0\}$*, to* $\{iy : y \geq 0\}$*. In addition, we have the following symmetry:*

$$h(-\overline{z}) = e^{i\frac{\pi}{2}}p(e^{\frac{-i\pi}{2}}(-\overline{z})) = e^{i\frac{\pi}{2}}p(e^{\frac{i\pi}{2}}\overline{z}) = e^{\frac{i\pi}{2}}p\left(\overline{e^{\frac{-i\pi}{2}}z}\right) = e^{\frac{i\pi}{2}}\overline{p(e^{\frac{-i\pi}{2}}z)} = -\overline{e^{\frac{i\pi}{2}}p(e^{\frac{-i\pi}{2}}z)} = -\overline{h(z)}.$$

$\lozenge$

A central characterisation of Herglotz functions is the classical integral representation. We present the theorem here and provide a proof in the appendix.

Theorem 8. *A function* $h : \mathbb{C}^+ \to \mathbb{C}$ *is a Herglotz function if and only if it admits a representation:*

$$h(z) = a + bz + \int_{\mathbb{R}}\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right)d\mu(\lambda) = a + bz + \int_{\mathbb{R}}\frac{1+z\lambda}{(\lambda - z)(1+\lambda^2)}d\mu(\lambda) \tag{10}$$

where μ *is a positive Borel measure that satisfies*

$$\int_{\mathbb{R}}\frac{1}{1+\lambda^2}d\mu(\lambda) < \infty \tag{11}$$

and $a \in \mathbb{R}$*,* $b \geq 0$ *are constants.*

We now provide some examples of Herglotz functions and their corresponding measure.

Example 9. *1. The function* $\frac{1}{\lambda - z}$ *for* $\lambda \in \mathbb{R}$ *is a Herglotz function with constants* $a = b = 0$ *and measure* $\mu = \delta_\lambda$ *where* δ_λ *is the point mass measure at* λ*,*

2. The function $\log(z)$ *is Herglotz function with constants* $a = b = 0$ *and measure being the Lebesgue measure restricted to the negative part of the real line, that is,*

$$\log(z) = \int_{-\infty}^{0} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\lambda$$

for $z \in \mathbb{C}^+$*.* $\Diamond$

We will now show that there exists a one-to-one correspondence between Herglotz functions and the triples (a, b, μ) satisfying Theorem 8.

Lemma 10. *Let* h *be a Herglotz function with a representation given in Theorem 8. Then we have:*

1. $a = \mathrm{Re}[h(i)]$*,*

2. $b = \lim_{z \hat{\to} \infty} \frac{h(z)}{z}$*, where* $z \hat{\to} \infty$ *denotes non-tangential limit, that is, the limit within some domain* $\{z \in \mathbb{C}^+ : \theta \leq \arg(z) \leq \pi - \theta\}$ *with* $\theta \in (0, \frac{\pi}{2}]$*, see Figure 2.*

3. The measure μ *can be reconstructed via the Stieltjes inversion formula*

$$\frac{1}{2}\mu(\{\lambda_1\}) + \frac{1}{2}\mu(\{\lambda_2\}) + \mu((\lambda_1, \lambda_2)) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \mathrm{Im}[h(\lambda + i\epsilon)] d\lambda \tag{12}$$

Proof. The first claim is clear.

For the second statement we note

$$\frac{h(z)}{z} = \frac{a}{z} + b + \int_{\mathbb{R}} \frac{1 + z\lambda}{z(\lambda - z)(1 + \lambda^2)} d\mu(\lambda).$$

The result then follows if we can apply the dominated convergence theorem since

$$\frac{1}{|z||\lambda - z|(1 + \lambda^2)} \leq \frac{1}{|z|| \mathrm{Im}[z]|(1 + \lambda^2)} \to 0$$

as $z \to \infty$ in any sector in the upper half plane; and similarly

$$\frac{|\lambda|}{|\lambda - z|(1 + \lambda^2)} \leq \frac{|\lambda|}{| \mathrm{Im}[z]|(1 + \lambda^2)} \to 0$$

as $z \to \infty$ in any sector in the upper half-plane. We justify applying the dominated convergence theorem by showing

$$\left| \frac{1 + z\lambda}{z(\lambda - z)(1 + \lambda^2)} \right| \leq \frac{1}{1 + \lambda^2} \frac{1 + |z|^2}{| \mathrm{Im}[z]|} \tag{13}$$

for $z \in \mathbb{C}$. To see this we maximise the function

$$\lambda \mapsto \frac{|\lambda|}{|\lambda - z|}.$$

The function attains its maximum value at $\lambda = \frac{|z|^2}{\mathrm{Re}[z]}$ (this can be found by differentiating with respect to λ). It follows that the maximum value is

$$\frac{|\lambda|}{|\lambda - z|} \leq \frac{|z|^2}{||z|^2 - z \,\mathrm{Re}[z]|} = \frac{|z|^2}{|z||\overline{z} - \mathrm{Re}[z]|} = \frac{|z|}{| \mathrm{Im}[z]|}.$$

We can now apply the dominated convergence theorem since

$$\left|\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right| = \left|\frac{1+\lambda z}{(\lambda - z)(1+\lambda^2)}\right| \leq \frac{1}{\mathrm{Im}[z](1+\lambda^2)} + \frac{|\lambda z|}{|(\lambda - z)|(1+\lambda^2)} \leq \frac{1}{1+\lambda^2}\frac{1+|z|^2}{|\,\mathrm{Im}[z]|}.$$

For proof of the last claim, let $I := [\lambda_1, \lambda_2]$. Then

$$\int_I \frac{1}{\pi}\mathrm{Im}[h(x+iy)]dx = \int_I \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(\lambda - x)^2 + y^2} d\mu(\lambda)dx. \tag{14}$$

We now use Fubini's theorem to interchange the order of integration:

$$\int_I \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(\lambda - x)^2 + y^2} d\mu(\lambda)dx = \int_{\mathbb{R}} \frac{1}{\pi}\int_I \frac{y}{(\lambda - x)^2 + y^2} dx d\mu(\lambda) = \int_{\mathbb{R}} \frac{1}{\pi}\arctan\left(\frac{x-\lambda}{y}\right)\Big|_{\lambda_1}^{\lambda_2} d\mu(\lambda). \tag{15}$$

We have that

$$\frac{1}{\pi}\left[\arctan\left(\frac{\lambda_2 - \lambda}{y}\right) - \arctan\left(\frac{\lambda_1 - \lambda}{y}\right)\right] \to \frac{1}{2}\left[\chi_{[\lambda_1,\lambda_2]}(\lambda) + \chi_{(\lambda_1,\lambda_2)}(\lambda)\right]$$

pointwise as $y \to 0^+$. To see this, we consider the different cases

- if $\lambda > \lambda_2$ then

$$\lim_{y\to 0^+}\left[\arctan\left(\frac{\lambda_2 - \lambda}{y}\right) - \arctan\left(\frac{\lambda_1 - \lambda}{y}\right)\right] = \arctan(-\infty) - \arctan(-\infty) = 0;$$

- if $\lambda < \lambda_1$ then

$$\lim_{y\to 0^+}\left[\arctan\left(\frac{\lambda_2 - \lambda}{y}\right) - \arctan\left(\frac{\lambda_1 - \lambda}{y}\right)\right] = \arctan(+\infty) - \arctan(+\infty) = 0;$$

- if $\lambda_1 < \lambda < \lambda_2$ then

$$\lim_{y\to 0^+}\frac{1}{\pi}\left[\arctan\left(\frac{\lambda_2 - \lambda}{y}\right) - \arctan\left(\frac{\lambda_1 - \lambda}{y}\right)\right] = \frac{1}{\pi}\left[\arctan(+\infty) - \arctan(-\infty)\right] = 1;$$

- if $\lambda_1 = \lambda$ then

$$\lim_{y\to 0^+}\frac{1}{\pi}\left[\arctan\left(\frac{\lambda_2 - \lambda}{y}\right) - \arctan\left(\frac{\lambda_1 - \lambda_1}{y}\right)\right] = \frac{1}{\pi}\arctan(+\infty) = \frac{1}{2};$$

- if $\lambda_2 = \lambda$ then

$$\lim_{y\to 0^+}\frac{1}{\pi}\left[\arctan\left(\frac{\lambda_2 - \lambda_2}{y}\right) - \arctan\left(\frac{\lambda - \lambda_1}{y}\right)\right] = -\frac{1}{\pi}\arctan(-\infty) = \frac{1}{2}.$$

Moreover since

$$0 \leq \frac{1}{\pi}\left[\arctan\left(\frac{\lambda_2 - \lambda}{y}\right) - \arctan\left(\frac{\lambda_1 - \lambda}{y}\right)\right] \leq 1$$

we have that that

$$\frac{1}{\pi}\left[\arctan\left(\frac{\lambda_2 - \lambda}{y}\right) - \arctan\left(\frac{\lambda_1 - \lambda}{y}\right)\right] \to 0^+$$

monotonically as $y \to 0^+$ for every $\lambda \notin [\lambda_1, \lambda_2]$ and, similarly,

$$\frac{1}{\pi}\left[\arctan\left(\frac{\lambda_2-\lambda}{y}\right) - \arctan\left(\frac{\lambda_1-\lambda}{y}\right)\right] \to \frac{1}{2}\left[\chi_{[\lambda_1,\lambda_2]}(\lambda) + \chi_{(\lambda_1,\lambda_2)}(\lambda)\right]$$

monotonically as $y \to 0^+$ for every $\lambda \in [\lambda_1, \lambda_2]$.

It follows therefore by the montone convergence theorem that

$$\lim_{y\to 0^+}\int_I \frac{1}{\pi}\operatorname{Im}[h(x+iy)]dx = \int_{\mathbb{R}} \frac{1}{2}\left[\chi_{[\lambda_1,\lambda_2]}(\lambda) + \chi_{(\lambda_1,\lambda_2)}(\lambda)\right]d\mu(\lambda) = \frac{1}{2}\mu(s\{\lambda_1\}) + \frac{1}{2}\mu(\{\lambda_2\}) + \mu((\lambda_1,\lambda_2)).$$

□

The third part of Lemma 10 is particularly useful when finding information about the corresponding measure.

Example 11. *Consider the function $h(z) = z^r$ for $0 \le r < 1$ as defined in Example 2. We note that for $z = x + iy$ and for $x > 0$ it follows that*

$$\operatorname{Im}[h(x+iy)] = \log(|x+iy|^r)\sin(r\arg(x+iy)) \to 0$$

as $y \to 0^+$ for $x > 0$. Hence, the measure vanishes on the positive part of the real axis. On the negative real axis, we have that

$$\lim_{y\to 0^+}\operatorname{Im}[h(x+iy)] = (-x)^r\sin(r\pi), \quad x<0. \tag{16}$$

It follows that for the integral representation, of Theorem 8, for h we have that $a = \cos(\frac{\pi r}{2})$ and μ is absolutely continuous with respect to the Lebesgue measure with $d\mu = \frac{1}{\pi}(-\lambda)^r\sin(\pi r)\chi_{(-\infty,0)}(r)d\lambda$ where χ is the characteristic function. ♢

We strengthen the content of Lemma 10 slightly.

Lemma 12. *Let h be a Herglotz function with integral representation given in Theorem 8. Then for any $\mathrm{C}^1(\mathbb{R})$ function $\varphi : \mathbb{R} \to \mathbb{R}$ which satisfies the inequality $|\varphi(x)| \le C\left(1+x^2\right)^{-1}$ for some $C \ge 0$ and all $x \in \mathbb{R}$, we have that*

$$\lim_{y\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}}\varphi(x)\operatorname{Im}[h(x+\mathrm{i}y)]\mathrm{d}x = \int_{\mathbb{R}}\varphi(\lambda)\mathrm{d}\mu(\lambda).$$

Proof. Assume, without loss of generality, that $b = 0$ then

$$\frac{1}{\pi}\int_{\mathbb{R}}\varphi(x)\operatorname{Im}[h(x+\mathrm{i}y)]\mathrm{d}x = \frac{1}{\pi}\int_{\mathbb{R}}\varphi(x)\left(\int_{\mathbb{R}}\frac{y}{(\lambda-x)^2+y^2}d\mu(\lambda)\right)dx \tag{17}$$

By Fubini's theorem we can interchange the integration limit to find

$$\lim_{y\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}}\varphi(x)\operatorname{Im}[h(x+\mathrm{i}y)]\mathrm{d}x = \lim_{y\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\frac{y}{(\lambda-x)^2+y^2}dxd\mu(\lambda).$$

We then apply the dominated convergence theorem to change the order of the limit and the first integral

$$\lim_{y\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\frac{y}{(\lambda-x)^2+y^2}dxd\mu(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}}\lim_{y\to 0^+}\int_{\mathbb{R}}\varphi(x)\frac{y}{(\lambda-x)^2+y^2}dxd\mu(\lambda) \tag{18}$$

Appearing in the inner integral is the Poisson kernel for the upper half-plane and by properties of the Poisson kernel it follows that

$$\lim_{y\to 0^+}\int_{\mathbb{R}}\varphi(x)\frac{1}{\pi}\frac{y}{(\lambda-x)^2+y^2}dx = \varphi(\lambda)$$

which completes the proof. □

The next result considers the case when a Herglotz function maps to an point on the real line. It turns out, such functions are trivial.

Lemma 13. *If a Herglotz function h attains a real value in the open upper half-plane, then it is a real-constant function.*

Proof. Assume that $h(z_0 = x_0 + iy_0) = \alpha$ for some $\alpha \in \mathbb{R}$ then it follows that

$$\mathrm{Im}[h(z_0)] = by_0 + \int_{\mathbb{R}} \frac{y_0}{(\lambda - x_0)^2 + y_0^2} d\mu(\lambda) = 0,$$

but since the integrand is positive, it follows that $\mu(\mathbb{R}) = 0$ and $b = 0$. In total, $h(z) \equiv \alpha$.

□

The next result deals with the non-tangential limit. It is particularly useful when considering Herglotz functions with simple poles on the real line.

Lemma 14. *The measure μ in the integral representation of Theorem 8 has a point mass at the point $\lambda_0 \in \mathbb{R}$ if and only if the limit*

$$\lim_{z \hat{\to} \lambda_0} (\lambda_0 - z)\, h(z)$$

is positive. In this case, $\mu(\{\lambda_0\}) = \lim_{z \hat{\to} \lambda_0} (\lambda_0 - z)\, h(z)$. Here, $z \hat{\to} \infty$ defines the limit within some domain $\{z \in \mathbb{C}^+ : \theta \leq \arg(z - \lambda_0) \leq \pi - \theta\}$ with $\theta \in (0, \frac{\pi}{2}]$ as indicated in Figure 2.

Proof. We assume that $a = b = 0$ in the integral representation. It follows

$$(\lambda_0 - z)h(z) = \int_{\mathbb{R}} \frac{(\lambda_0 - z)(1 + z\lambda)}{(\lambda - z)(1 + \lambda^2)} d\mu(\lambda). \tag{19}$$

By relation 13 we have

$$\left| \frac{1 + z\lambda}{(\lambda - z)(1 + \lambda^2)} \right| \leq \frac{1}{1 + \lambda^2} \frac{1 + |z|^2}{|\mathrm{Im}[z]|}$$

for each $z \in \mathbb{C}^+$ and we are then able to apply the dominated convergence theorem to find

$$(\lambda_0 - z)h(z) = \int_{\mathbb{R}} \frac{(\lambda_0 - z)(1 + z\lambda)}{(\lambda - z)(1 + \lambda^2)} d\mu(\lambda) = \int_{\mathbb{R}\setminus\{\lambda_0\}} \frac{(\lambda_0 - z)(1 + z\lambda)}{(\lambda - z)(1 + \lambda^2)} d\mu(\lambda) + \tag{20}$$

$$\int_{\{\lambda_0\}} \frac{(\lambda_0 - z)(1 + z\lambda)}{(\lambda - z)(1 + \lambda^2)} d\mu(\lambda) = \int_{\mathbb{R}\setminus\{\lambda_0\}} \frac{(\lambda_0 - z)(1 + z\lambda)}{(\lambda - z)(1 + \lambda^2)} d\mu(\lambda) + \mu(\{\lambda_0\}) \frac{1 + z\lambda_0}{1 + \lambda_0^2} \to \mu(\{\lambda_0\}) \tag{21}$$

as $z \to \lambda_0$ in any sector of the upper half plane. □

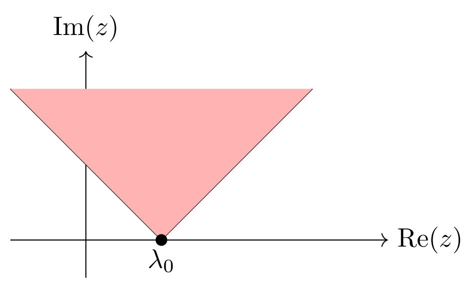

Figure 2: Non-tangential limit

Remark 15. *Note that if the limit is taken with $z = \lambda_0 + iy$ then the lemma above implies*

$$\mu(\{\lambda_0\}) = \lim_{y\to 0^+} y\,\mathrm{Im}[h(\lambda_0 + iy)]$$

and

$$0 = \lim_{y\to 0^+} y\,\mathrm{Re}[h(\lambda_0 + iy)].$$

The integral representation simplifies if we add more conditions on the function.

Example 16. *Let h be a Herglotz function satisfying*

$$h(-\overline{z}) = -\overline{h(z)}. \tag{22}$$

We have that the function h is Herglotz if and only if it has an integral representation

$$h(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\mu(\lambda).$$

Since $h(i) = i\alpha$ for some $\alpha \in \mathbb{R}$, we have that $a = 0$. The symmetry of h implies

$$\mathrm{Im}[h(\overline{-(x+iy)})] = \mathrm{Im}[h(-x+iy)] = \mathrm{Im}[-\overline{h(x+iy)}] = \mathrm{Im}[h(x+iy)], \tag{23}$$

which implies by Lemma 12 that the corresponding measure μ is even. This gives

$$h(z) = bz + \lim_{R\to+\infty}\int_{-R}^{R} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\mu(\lambda) = bz + \lim_{R\to+\infty}\int_{-R}^{R} \frac{1}{\lambda - z} d\mu(\lambda) = \tag{24}$$

$$= bz - \frac{\mu(\{0\})}{z} + \lim_{R\to+\infty}\int_{0}^{R} \frac{1}{\lambda - z} - \frac{1}{\lambda+z} d\mu(\lambda) = bz - \frac{\mu(\{0\})}{z} + \int_{(0,\infty)} \frac{2z}{\lambda^2 - z^2} d\mu(\lambda). \tag{25}$$

$\Diamond$

We finish this section by briefly commenting on the extension of Herglotz functions across the real line. By the proof of Theorem 8 the integral representation which we define

$$g(z) := a + bz + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\mu(\lambda), \quad z \in \mathbb{C}\setminus\mathbb{R}$$

is analytic and well-defined on any compact subset of $\mathbb{C}\setminus\mathbb{R}$. Moreover, g satisfies the symmetry property

$$\overline{g(\overline{z})} = g(z).$$

It follows that $g(z)$ has a **symmetric extension** across the real line. Hence, if we are able to analytically extend a Herglotz function across the real line, the extension would have to be symmetric for it to coincide with the extension provided by the integral. This may not always be the case. Indeed, consider the Herglotz function $h(z) = i$ which is trivially continued across the real line. On the other hand, we have

$$\int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2+1}\right) \frac{1}{\pi} d\lambda = \begin{cases} i, & z \in \mathbb{C}^+, \\ -i, & z \in \mathbb{C}^-. \end{cases} \tag{26}$$

where the integral is calculated using residue theorem applied to the semi circle in the upper half-plane with radius R tending to ∞. Hence, the analytic extension of a Herglotz function does not always coincide with the symmetric extension given by the integral. It is then worth asking when it is possible to extend a

Herglotz function through an interval on the real line by symmetry. We note that if the measure is 0 on the interval $(\lambda_1, \lambda_2) \subset \mathbb{R}$ then the integral representation simplifies to

$$h(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\lambda = a + bz + \int_{\mathbb{R}\setminus(\lambda_1,\lambda_2)} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\lambda. \tag{27}$$

It follows that the function $h(z)$ is analytic on $\mathbb{C}^+$, well-defined and real-valued on (λ_1, λ_2) and can, therefore, be continued by symmetry

$$f(z) = \overline{f(\overline{z})}$$

across (λ_1, λ_2). The need to continue Herglotz functions turns out to be useful in the study of the complex spectra of certain non-self-adjoint operators, see [5]. A more general result is provided in [7, Chapter 1, Theorem 1.1 & 1.2].

Theorem 17. *If $h(z)$ is a Herglotz function with integral representation given by Theorem 8, then $h(z)$ can be analytically continued across $(\lambda_1, \lambda_2) \subset \mathbb{R}$ into a subset $\mathcal{D}$ of the lower half plane if and only if μ is purely absolutely continuous on (λ_1, λ_2) with associated density function μ' which is real-analytic on (λ_1, λ_2). In this case, the continuation of $h(z)$ across (λ_1, λ_2) is given by*

$$h(z) = \overline{h(\overline{z})} + 2\pi i \mu'(z), \quad z \in \mathcal{D}, \tag{28}$$

where $\mu'(z)$ denotes the complex–analytic extension of $\mu'(z)$.

Note, in particular, that an extension is provided by reflection if and only if $\mu'(\lambda) = 0$ for all $\lambda \in (\lambda_1, \lambda_2)$. The proof of the theorem is based on the inversion formula provided in Lemma 10 property 3.

2.2 Support of the Measure

As seen in the last section, property 3 from Lemma 10 was extremely useful since it shows that the measure can be recovered by studying the imaginary part of the corresponding Herglotz function along the real line. It is, therefore, relevant to study the boundary values of a Herglotz function and its relation to the measure. The purpose of this section is to glean out more information about the measure by developing the support theory for the measure. This section is based on [6, Chapter 3.], [14, Appendix A.10, Chapter 3.4] and [13, Chapter 1.1].

Let μ be a Borel measure on $\mathbb{R}$. We require in the definition of Borel measures that $\mu(K) < \infty$ on compact sets $K \subset \mathbb{R}$. The latter requirement is not an issue since the measures we will be working with will be satisfying it. We denote by

$$\mu = \mu_{ac} + \mu_s = \mu_{ac} + \mu_{sc} + \mu_{pp} \tag{29}$$

the decomposition of μ into into its absolutely continuous (ac), singularly continuous (sc), pure point (pp), and singular (s) parts with respect to Lebesgue measure on $\mathbb{R}$. We recall some necessary definitions. Given two Borel measures μ, ω on $\mathbb{R}$, we call a set S_μ a support of μ if $\mu(\mathbb{R} \setminus S_\mu) = 0$. Moreover, the support S_μ of μ is called minimal relative to ω if for any support $A \subseteq S_\mu$ we have that $\omega(S_\mu \setminus A) = 0$. We will consider minimal supports **where ω is the Lebesgue measure.**

We let $I_r(x) := (x - r, x + r)$, the interval of length $2r$ centred around x. We define

$$(D\mu)(x) := \lim_{r \to 0^+} \frac{\mu(I_r(x))}{2r} \tag{30}$$

to be the derivative of μ at $x \in \mathbb{R}$ when the limit exists. We further define the upper and lower derivatives as

$$(\overline{D}\mu)(x) := \limsup_{\epsilon \to 0^+} \frac{\mu(I_r(x))}{2r}, \tag{31}$$

$$(\underline{D}\mu)(x) := \liminf_{\epsilon \to 0^+} \frac{\mu(I_r(x))}{2r} \tag{32}$$

where

$$\limsup_{x \to a} f(x) := \lim_{\epsilon \to 0} (\sup\{f(x) : x \in I_\epsilon(a) \setminus a\}), \tag{33}$$

$$\liminf_{x \to a} f(x) := \lim_{\epsilon \to 0} (\inf\{f(x) : x \in I_\epsilon(a) \setminus a\}) \tag{34}$$

We have the following result which relates the upper derivatives to μ and the Lebesgue measure.

Lemma 18. *Let $\alpha > 0$. For every Borel set A we have*

$$\omega(\{x \in A \mid (\bar{D}\mu)(x) > \alpha\}) \leq 3^n \frac{\mu(A)}{\alpha} \tag{35}$$

and

$$\omega(\{x \in A \mid (\bar{D}\mu)(x) > 0\}) = 0, \textit{ whenever } \mu(A) = 0. \tag{36}$$

We recall Lebesgue's differentiation theorem

Theorem 19 (Lebesgue's Differentiation Theorem)**.** *Let f be locally integrable with respect to the Lebesgue measure. Then for a.e. $x \in \mathbb{R}$ it holds that*

$$\lim_{r \to 0^+} \frac{1}{2r} \int_{I_r(x)} |f(y) - f(x)| d\omega(y) = 0. \tag{37}$$

An immediate consequence of Lebesgue's differentiation theorem and Lebesgue's decomposition theorem is the following theorem.

Theorem 20. *Let μ be a Borel measure on $\mathbb{R}$. Then $D\mu$ exists a.e. with respect to the Lebesgue measure and is the Radon-Nikodym derivative of the absolutely continuous part of μ with respect to the Lebesgue measure.*

Proof. By Lebesgue's decomposition theorem we have

$$\mu = \mu_{ac} + \mu_s$$

where μ_s and the Lebesgue measure are mutually singular, that is, there exist disjoint Borel sets A and B such that

$$\mathbb{R} = A \cup B, \quad A \cap B = \emptyset,$$

and

$$\mu_s(A) = 0, \quad \omega(B) = 0.$$

By Lemma 18, we have that $\overline{D}\mu_s = 0$ for all $x \in A$. This implies that $D\mu_s = 0$ a.e. and it, therefore, suffices to show that $D\mu_{ac}$ exists a.e. with respect to the Lebesgue measure since $D\mu = D\mu_{ac} + D\mu_s$ when both values exist. We have that there exists a locally integrable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\mu_{ac}(X) = \int_X f d\omega(x)$$

for Borel sets $X \subset \mathbb{R}$. That f is locally integrable follows by the definition of Borel measures given at the start of this section. We have then by Lebesgue's Differentiation theorem

$$(D\mu_{ac})(x) = \lim_{r \to 0^+} \frac{\mu_{ac}(I_r(x))}{2r} = \lim_{r \to 0^+} \frac{1}{2r} \int_{I_r(x)} f(y) d\omega(y) = f(x) \tag{38}$$

for almost every $x \in \mathbb{R}$. Hence, $(D\mu_{ac})(x)$ exists a.e. with respect to the Lebesgue measure. □

We find supports for the absolutely continuous and singularly continuous parts using the lower and upper derivatives.

Theorem 21. *The set $M_{ac} := \{x \in \mathbb{R} : 0 < (D\mu)(x) < \infty\}$ is a support for the absolutely continuous part.*

Proof. The result follows immediately from the previous theorem since on the set $(M_{ac})^C := \mathbb{R} \setminus \{x \in \mathbb{R} : 0 < (D\mu)(x) < \infty\}$ we have

$$\mu_{ac}(D) = \int_D D\mu dx = 0.$$

□

Theorem 22. *The set $M_s := \{x \in \mathbb{R} : (\underline{D}\mu)(x) = \infty\}$ is a support for the singular part.*

Proof. See [14, Chapter A.10, Theorem A.46.] □

The following theorem strengthens the content of the last theorems and shows that the supports are minimal supports.

Lemma 23. *The set M_{ac} is a minimal support of μ_{ac}*

Proof. See [14, Chapter A.10, Lemma A.47.]. □

That M_s is a minimal support for μ_s follows immediately from the Lebesgue decomposition and the definition of a support. Indeed, any support of the absolutely singular part is minimal.
We are now ready to state the support theorem for measures corresponding to Herglotz functions. The following result shows that boundary value correspond to the Radon-Nikodym derivative of the absolutely continuous part.

Theorem 24. *Let h be a Herglotz function with associated measure μ. Then for $\lambda \in \mathbb{R}$*

$$(\underline{D}\mu)(\lambda) \leq \liminf_{\varepsilon \to 0^+} \frac{1}{\pi} \operatorname{Im}(h(\lambda + \mathrm{i}\epsilon)) \leq \limsup_{\varepsilon \to 0^+} \frac{1}{\pi} \operatorname{Im}(h(\lambda + \mathrm{i}\epsilon)) \leq (\bar{D}\mu)(\lambda). \tag{39}$$

Proof. See [13, Chapter 1.1, Theorem 1.6. (iv)] or [14, Chapter 3.4, Theorem 3.26.]. □

It follows by Theorem 20 that $\operatorname{Im}(F(\lambda + i0))$ exists a.e. and whenever it exists it is equal to the Radon-Nikodym derivative of the absolutely continuous part of the measure μ corresponding to h. In addition, by Lemma 23 and the previous discussion we have that the sets M_{ac} and M_s are minimal supports for the μ_{ac} and μ_s. We collect this information in the following theorem.

Theorem 25. *Let h be a Herglotz function with associated measure μ. Then for $\lambda \in \mathbb{R}$ the limit*

$$\operatorname{Im}[h(\lambda)] := \lim_{\epsilon \to 0^+} \operatorname{Im}[h(\lambda + \mathrm{i}\epsilon)]$$

exists a.e. with respect to both μ and Lebesgue measure (finite or infinite) and

$$(D\mu)(\lambda) = \frac{1}{\pi} \operatorname{Im}(h(\lambda))$$

whenever $(D\mu)(\lambda)$ exists. In addition, the set

$$S_{\mu_{ac}} := \left\{ \lambda \in \mathbb{R} : 0 < \lim_{y \to 0^+} \operatorname{Im}[h(\lambda + iy)] < \infty \right\} \tag{40}$$

is a minimal support for the absolutely continuous part and

$$S_{\mu_s} := \left\{ \lambda \in \mathbb{R} : \lim_{y \to 0^+} \operatorname{Im}[h(\lambda + iy)] = +\infty \right\} \tag{41}$$

is a minimal support for the singular part.

Using this result, it can also be shown that real part of the boundary value also exists almost everywhere.

Theorem 26. *Let h be a Herglotz function with associated measure μ. Then*

$$\lim_{\epsilon \to 0^+} h(\lambda + \mathrm{i}\epsilon)$$

exists a.e. with respect to both μ and Lebesgue measure.

Proof. We note that $\sqrt{h(z)}$ is a Herglotz function and by definition it maps the upper half-plane into the first quadrant which implies that $i\sqrt{h(z)}$ is also a Herglotz function. We apply Theorem 25 to find that $\lim_{\epsilon \to 0^+} \operatorname{Im}[\sqrt{h(\lambda + \mathrm{i}\epsilon)}]$ and $\lim_{\epsilon \to 0^+} \operatorname{Im}[i\sqrt{h(\lambda + i\epsilon)}] = \operatorname{Re}[\sqrt{h(\lambda + \mathrm{i}\epsilon)}]$ exist almost everywhere with respect to μ and the Lebesgue measure. Finally, taking squares proves the theorem. □

Note that this naturally implies that

$$\tilde{S}_{\mu_{ac}} := \left\{ \lambda \in \mathbb{R} : \lim_{y \to 0^+} h(\lambda + iy) \text{ exists and } 0 < \lim_{y \to 0^+} \text{Im}[h(\lambda + iy)] < \infty \right\} \tag{42}$$

is a minimal support for the absolutely continuous part.
The singular part can be further decomposed using Lemma 14 where it was noted that the measure has a point mass at $\lambda_0 \in \mathbb{R}$ if and only if $\lim_{z \hat{\to} \lambda_0} (\lambda_0 - z)\, h(z) > 0$. We therefore get the following theorem.

Theorem 27. *Let h be a Herglotz function with associated measure μ. Then the set*

$$S_{\mu_{pp}} := \left\{ \lambda \in S_{\mu_s} : \lim_{z \hat{\to} \lambda_0} (\lambda_0 - z)\, h(z) > 0 \right\} \tag{43}$$

is a minimal support for the pure point part and

$$S_{\mu_{sc}} := \left\{ \lambda \in S_{\mu_s} : \lim_{z \hat{\to} \lambda_0} (\lambda_0 - z)\, h(z) = 0 \right\} \tag{44}$$

is a minimal support for the singular continuous part.

Remark 15 provides an alternative formulation of the above theorem.

Theorem 28. *Let h be a Herglotz function with associated measure μ. Then the set*

$$\left\{ \lambda \in S_{\mu_s} : \lim_{y \to 0^+} y\, \text{Im}[h(\lambda_0 + iy)] > 0 \right\} \tag{45}$$

is a minimal support for the pure point part and

$$\left\{ \lambda \in S_{\mu_s} : \lim_{y \to 0^+} y\, \text{Im}[h(\lambda_0 + iy)] = 0 \right\} \tag{46}$$

is a minimal support for the singular continuous part.

To highlight the results developed in this section, we consider the following example.

Example 29. *Consider the function $h(z) = \tan(z)$ from Example 2. We have*

$$\text{Re}[\tan(i)] = 0$$

and

$$\lim_{y \to 0^+} \frac{\tan(iy)}{iy} = \lim_{y \to 0^+} \frac{i \tanh(y)}{iy} = 0.$$

In addition we have

$$\lim_{y \to 0^+} \text{Im}[h(\lambda + iy)] = \lim_{y \to 0^+} \frac{\sinh(2y)}{\cos(2\lambda) + \cosh(2y)} = \begin{cases} \infty, & \text{if } \lambda = \frac{n}{2}\pi \text{ for odd integer,} \\ 0, & \text{else.} \end{cases} \tag{47}$$

By Theorem 25 it follows that the corresponding measure satisfies $\mu_{ac} = 0$ and the singular part is supported on $\{\frac{n}{2}\pi : n \text{ is an odd integer}\}$. Moreover, we have by Taylor expanding around $y = 0$ that for any odd integer n

$$y\, \text{Im}[h(\frac{n}{2}\pi + iy)] = 1 + y^2/3 - y^4/45 + O(y^5) \tag{48}$$

which converges to 1 *as* $y \to 0$*. Theorem 28 implies that the measure corresponding to* $\tan(z)$ *is a discrete measure with point mass at* $\frac{n}{2}\pi$ *for odd integers* n*. In total we have by Theorem 8 that*

$$\tan(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\mu(\lambda) = \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\mu(\lambda) \tag{49}$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{(k+\frac{1}{2})\pi - z} - \frac{(k+\frac{1}{2})\pi}{1+(k+\frac{1}{2})^2\pi^2} \right) \tag{50}$$

We also have that $h(z)$ *satisfies the symmetry condition* 22 *and it can be concluded by Example 22*

$$\tan(z) = \sum_{n=0}^{\infty} \frac{8z}{(2k+1)^2\pi^2 - 4z^2}. \tag{51}$$

◊

2.3 Different Representations

The integral representation in Theorem 8 is among the many characterisations of Herglotz functions. In this section we mention some other representations.

2.3.1 Exponential Herglotz representation

Assume that F is a Herglotz function. We have then that $\log(F(z))$ is a Herglotz function with representation

$$\log(F(z)) = a + bz + \int_{\mathbb{R}} \frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} d\mu(\lambda).$$

This implies

$$F(z) = \exp(a)\exp(bz)\exp(\int_{\mathbb{R}} \frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} d\mu(\lambda)).$$

We have that

$$a = \mathrm{Re}(\log(F(i))) = \ln(|F(i)|)$$

and

$$b = \lim_{z \to \infty} \frac{\log(F(z))}{z} = 0.$$

Lastly, we note that $\log(F(z))$ has bounded imaginary part which by results from the previous section implies that the corresponding measure is absolutely continuous with respect to the Lebesgue measure. The Radon-Nikodym derivative exists almost everywhere and is given by

$$\nu(x) = \lim_{y \to 0^+} \frac{1}{\pi} \mathrm{Im}(\log(F(x+iy))) = \lim_{y \to 0^+} \frac{1}{\pi} \arg(F(x+iy))$$

and bounded by $0 \leq \nu(x) \leq 1$. The exponential representation is useful in spectral theory problems. This extensive paper [2] contains many results on the topic.

2.3.2 Operator Representation

Let $\mathcal{H}$ be a Hilbert space with scalar product $(.,.)$ which is linear in the second factor. Consider a self-adjoint operator A in $\mathcal{H}$. Let $s \in \mathbb{R}$ and $u \in \mathcal{H}$, then

$$h(z) := s + (u, (A-z)^{-1}u) \tag{52}$$

is a Herglotz function. Indeed, analyticity of $h(z)$ follows by the resolvent identity for operators

$$(A-w)^{-1} - (A-z_0)^{-1} = (w-z_0)(A-w)^{-1}(A-z_0)^{-1} \tag{53}$$

and

$$\frac{h(w)-h(z_0)}{w-z_0} = \frac{(u, [(A-w)^{-1} - (A-z_0)^{-1}]u)}{w-z_0} = \tag{54}$$

$$\frac{(w-z_0)(u, (A-w)^{-1}(A-z_0)^{-1}u)}{w-z_0} \to (u, ((A-z_0)^{-1})^2 u) \tag{55}$$

as $w \to z_0 \in \mathbb{C}^+$. Lastly, we note that for self-adjoint operators we have that adjoint of $(A-z)^{-1}$ is $(A-\overline{z})^{-1}$ which implies

$$\frac{h(z)-\overline{h(z)}}{2i} = \frac{(u,(A-z)^{-1}u) - ((A-z)^{-1}u, u)}{2i} = \frac{(u, [(A-z)^{-1} - (A-\overline{z})^{-1}]u)}{2i} = \tag{56}$$

$$\frac{(u, (z-\overline{z})(A-z)^{-1}(A-\overline{z})^{-1}u)}{2i} = \mathrm{Im}[z]((A-\overline{z})^{-1}u, (A-\overline{z})^{-1}u) \geq 0. \tag{57}$$

for $z \in \mathbb{C}^+$. The result above is generalised in the following theorem

Theorem 30. *A function h is a Herglotz-Nevanlinna function if and only if there exist a Hilbert space $\mathcal{H}$, a self-adjoint linear relation A in $\mathcal{H}$, a point $z_0 \in \mathbb{C}^+$ and an element $u \in \mathcal{H}$ such that*

$$h(z) = \overline{h(z_0)} + (z - \overline{z_0}) \left(u, \left(I + (z - z_0)(A - z)^{-1}\right) u\right). \tag{58}$$

Note that the operator is replaced by a self-adjoint relation and such can be multi-valued. For more one self-adjoint relations, consider [3].

Remark 31. *We note that the form 52 does not allow for linear functions and is therefore not as general as the one in the theorem above. In regards to the uniqueness of the relation $\mathcal{A}$ and the Hilbert space $\mathcal{H}$, we have that if $\mathcal{H} = \overline{\text{span}} \left\{ \left(I + (z - z_0)(A - z)^{-1}\right) v : z \in \rho(A) \right\}$ where $\overline{span}$ denotes closed the closed linear span and ρ denotes the resolvent set then the relation A and $\mathcal{H}$ are unique up to unitary equivalence, i.e. if there exists another Hilbert space $\mathcal{H}_1$ and self-adjoint relation B such that 58 is satisfied then there exists an unitary relation $U : \mathcal{H}_1 \to \mathcal{H}$ such that $A = UBU^{-1}$.*

3 Characterisation of Rational Herglotz Functions

The goal in this section will be to characterise rational Herglotz functions. By a rational Herglotz function, we mean a Herglotz function which is a quotient of two polynomials over the field of complex numbers. Generally, given a rational function $h(z) := \frac{r(z)}{t(z)}$ where $r, t \in \mathbb{C}[z]$, for h to be a Herglotz function, it is necessary that $t(z) \neq 0$ for all $z \in \mathbb{C}^+$. But what other conditions must the polynomials r and t satisfy?

A sketch of the idea that will be used in this section

- We first characterise rational Herglotz functions with an integral representation that extends symmetrically to a rational function on $\mathbb{C} \setminus \mathbb{R}$. We note that such functions can only have poles on the real line. Examples of such functions are

$$-\frac{1}{z} \tag{59}$$

$$-\frac{1}{z-1}. \tag{60}$$

  By 26 we see that the function

$$z \mapsto i, \quad z \in \mathbb{C}^+$$

  is a non-example. The function

$$z \mapsto -\frac{1}{z+i}, \quad z \in \mathbb{C}^+ \tag{61}$$

  is also a non-example since it has a pole in the lower half-plane.

- We then characterise Herglotz functions whose integral representation when restricted to the upper half-plane gives a rational function with poles possibly only in the lower half-plane. Example of such functions is 61. The functions in 60 are non-examples.

- We characterise an arbitrary rational Herglotz function by decomposing it into two parts, a part with poles on the real line and a part one with poles in the lower half-plane.

We begin by noting that Lemma 10 property 2 is equivalent to

Corollary 32. *Let h be a Herglotz function. Then it follows that*

$$h(z) = bz + o(z)$$

as $z \to \infty$ in any sector in the upper half-plane.

We have that if $h(z)$ is a Herglotz function which does not attain 0, then $-\frac{1}{h(z)}$ is Herglotz function which by the above corollary satisfies $\frac{1}{|h(z)|} \geq C\frac{1}{|z|}$ for some constant $C \geq 0$ as $z \to \infty$ in any sector of the upper half plane. The following lemma then holds.

Lemma 33. *Let h be a rational Herglotz function with $h(z) = \frac{r(z)}{t(z)}$. It follows*

$$|\deg(r(z)) - \deg(t(z))| \leq 1. \tag{62}$$

3.1 Rational Herglotz Functions with Real Poles

In this section we focus on rational Herglotz functions with an integral representation that extends symmetrically to a rational function on $\mathbb{C} \setminus \mathbb{R}$. Herglotz functions assuming real values are dismissed since such are constant. We have the following characterisation.

Theorem 34. *An analytic function $h : \mathbb{C}^+ \to \mathbb{C}$ is a rational Herglotz functions with an integral representation that extends symmetrically to a rational function on $\mathbb{C} \setminus \mathbb{R}$ if and only if it admits a representation*

$$h(z) = a + bz + \sum_{j=1}^{n} a_j \left(\frac{1}{\tau_j - z} - \frac{\tau_j}{1 + \tau_j^2} \right), \quad z \in \mathbb{C}^+,$$

where $n \in \mathbb{N}$, $a \in \mathbb{R}$, $b \geq 0$, $a_j \geq 0$ and $\tau_j \in \mathbb{R}$ for $1 \leq j \leq n$.

Proof. That this is a sufficient condition is clear. To see that this is a necessary condition, we decompose $h(z)$ into

$$h(z) = s(z) + \frac{r_1}{t_1 - z} + \frac{r_2}{t_2 - z} + \cdots + \frac{r_n}{t_n - z} \tag{63}$$

By Lemma 33 we have that $s(z) = \tilde{a} + bz$ for some complex constants $\tilde{a}$ and b. Since $h(z)$ is Herglotz, it follows that $\tilde{a} \in \mathbb{R}$ and $b \geq 0$. If any constant $r_i \neq 0$ then it is determined by Lemma 14 to be $r_i = \mu(\{t_i\}) > 0$ for $1 \leq i \leq n$. By the uniqueness of the integral representation, it follows that the corresponding measure is a discrete measure with isolated masses of size $\mu(\{\tau_i\})$ at τ_j for $1 \leq j \leq n$. The result then follows. □

We can find more information about such functions. We have that $p(z)$ must have real coefficients. Assume now that the roots of $p(z)$ are not real then they come in pairs of the form $(a + ib, a - ib)$ which implies h maps a point of $\mathbb{C}^+$ to the real line and hence is a constant function. We may, therefore, assume the roots are real. Moreover, since $\frac{-1}{h(z)} = -\frac{q(z)}{p(z)}$ is also a Herglotz function it follows (by repeating the analysis above) that the $p(z)$ has simple roots. We gain more information on the structure of the poles and roots by considering the function $h(z = x)$ for $x \in \mathbb{R}$ where x is not a pole. Taking the derivative gives

$$h'(x) = b + \sum_{i=1}^{n} \frac{\mu(t_i)}{(t_i - x)^2}$$

and the function is strictly increasing, where it is defined. Hence, between any two zeros there must be a pole. Similarly between any two poles there must exist a zero. We state this result as a theorem.

Theorem 35. *Let h be a Herglotz function with a corresponding measure in the integral representation in Theorem 8 being a discrete measure with finitely many point masses. Then h has only first order zeros and poles on the real line which are interlacing.*

3.2 Rational Herglotz Functions with Poles in the Lower Half-Plane

In this section we will focus on rational Herglotz functions with poles only in the lower half-plane. Let $h(z)$ be a rational Herglotz function with poles in the lower half-plane. We have by partial fraction decomposition and Lemma 33 that

$$h(z) = a + bz + \sum_{i=1}^{N}\sum_{k=1}^{n_j} \frac{A_{k,j}}{(z - z_j)^k}$$

where $b \geq 0$, $A_{k,j}$ and a are constants. We note that there are no restrictions on the order of the poles and that the constants $A_{k,j}$ and a are allowed to be complex. It is not possible to characterise the function by simply considering each part in the partial fraction decomposition separately since individual parts may not be Herglotz functions. For example the function

$$-\frac{1}{z - \frac{3}{z} + 4i} = -\frac{z}{(z-(-i))\cdot(z-(-3\,i))} = -\frac{3}{2(z+3i)} + \frac{1}{2(z+i)}$$

is a Herglotz function but the terms after the second equality the are not both Herglotz functions. To be more specific, the function $\frac{-3}{2(z+3i)}$ is Herglotz, but $\frac{1}{2(z+i)}$ is not. To study the problem we will consider the density function associated with the measure corresponding to $h(z)$.

Theorem 36. *Let $h : \mathbb{C}^+ \to \mathbb{C}$ be an analytic function. Then h is a rational Herglotz function with poles only in the lower half-plane if and only if it admits a representation of the form*

$$h(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right)\nu(\lambda)d\lambda, \quad z \in \mathbb{C}^+, \tag{64}$$

where $a \in \mathbb{R}$, $b \geq 0$ are constants and $\nu(x)$ satisfies (if $\nu(x) = \frac{p(x)}{q(x)}$)

1. *$\nu(x) \geq 0$ for all $x \in \mathbb{R}$;*
2. *$0 \leq \deg(q) - \deg(p)$;*
3. *$q(x)$ has only non-real roots.*

Proof. Assume $h(z)$ is a rational Herglotz function with poles only in the lower half-plane and integral representation given in Theorem 8. Since the poles are in the lower half-plane, the corresponding measure in the integral representation given in Theorem 8 is absolutely continuous with respect to the Lebesgue measure. In this case, the density function exists and is simply given by the imaginary part of the boundary values of h. The density function is calculated

$$\lim_{y\to 0^+} \frac{1}{\pi}\operatorname{Im}(h(x+iy)) = \operatorname{Im}(h(x)) = \frac{1}{\pi}\operatorname{Im}(a) + \sum_{i=1}^{N}\sum_{k=1}^{n_j} \operatorname{Im}\left(\frac{A_{k,j}}{(x-z_j)^k}\right) = \tag{65}$$

$$\operatorname{Im}(a) + \sum_{i=1}^{N}\sum_{k=1}^{n_j}\sum_{r=1}^{k} \frac{a_r x^r}{((x-x_j)^2 + y_j^2)^k} \tag{66}$$

for some real constants a_r. The density function is also positive since h is a Herglotz function.

Conversely, assume that h given by 64. First note that h in 64 is guaranteed to be a Herglotz function since $\nu(x)$ is positive and since

$$\int_{\mathbb{R}} \frac{1}{1+x^2}\nu(x)dx < \infty.$$

We now show that the function is rational. Note that the second condition implies that after partial fraction decomposition, one should have

$$\nu(x) = a + \sum_{j=1}^{M} \sum_{k=1}^{m_j} \frac{B_{k,j}}{(x - z_j)^k} \tag{67}$$

for constants z_j, $B_{k,j}$. The third condition implies that if $z_{j_0} = x_{j_0} + iy_{j_0}$ then its conjugate is also a root, that is $z_{l_0} = x_{l_0} - iy_{l_0}$ for some $l_0 \neq j_0$ and $1 \leq l_0, j_0 \leq M$. This gives that the form in $\nu(x)$ can be rewritten

$$\nu(x) = c + \sum_{j=1}^{M_N} \sum_{k=1}^{l_j} \frac{C_{k,j}}{((x - x_j)^2 + y_j^2)^k}. \tag{68}$$

To show that it is a rational Herglotz function, we compute the integral

$$\int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) \frac{\lambda^r}{((\lambda - a)^2 + b^2)^k} d\lambda, \tag{69}$$

where $a, b \in \mathbb{R}$ with $b \neq 0$ and $0 \leq r \leq k$ are integers. We consider the case $k = r = 0$ first. If $z = i$ then the integrand becomes

$$\frac{i}{\lambda^2 + 1}$$

and

$$\int_{\mathbb{R}} \frac{i}{\lambda^2 + 1} d\lambda = i\pi.$$

We assume therefore $z \neq i$ and define

$$L(w) := \frac{(1 + wz)}{(w - z)(1 + w^2)}$$

which has simple poles at $w = z$, $w = \pm i$. We assume $z \in \mathbb{C}^+$ and use the contour from Figure 3 to evaluate the integral. By the residue theorem we have that for large enough $R > 0$

$$\int_{-R}^{R} L(w)dw + \int_{C_R} L(w)dw = 2\pi i \left[\mathrm{Res}_z(L(w)) + \mathrm{Res}_i(L(w)) \right]. \tag{70}$$

We have that

$$\mathrm{Res}_z(L(w)) = \frac{(1 + z^2)}{(1 + z^2)} = 1; \tag{71}$$

$$\mathrm{Res}_i(L(w)) = \frac{(1 + zi)}{(i - z)(2i)} = \frac{-1}{2}. \tag{72}$$

We note that the integral along C_R vanishes when $R \to \infty$ since

$$\left| \int_{C_R} L(w)dw \right| \leq \int_0^{\pi} \frac{(1 + R|z|)}{(R - |z|)(R^2 - 1)} R d\theta \to 0$$

as $R \to \infty$. Hence, we have that

$$\int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\lambda = 2\pi i \left[\mathrm{Res}_z(L(w)) + \mathrm{Res}_i(L(w)) \right] = \pi i. \tag{73}$$

Hence the theorem is satisfied if $r = k = 0$.

We note that for $k \neq 0$ the integral

$$\int_{\mathbb{R}} \frac{\lambda}{1+\lambda^2} \nu(\lambda) d\lambda, \tag{74}$$

is convergent. We focus therefore on the remaining part of the integral, that is, we consider the integral

$$\int_{\mathbb{R}} \frac{1}{\lambda - z} \cdot \frac{\lambda^r}{((\lambda-a)^2+b^2)^k} d\lambda. \tag{75}$$

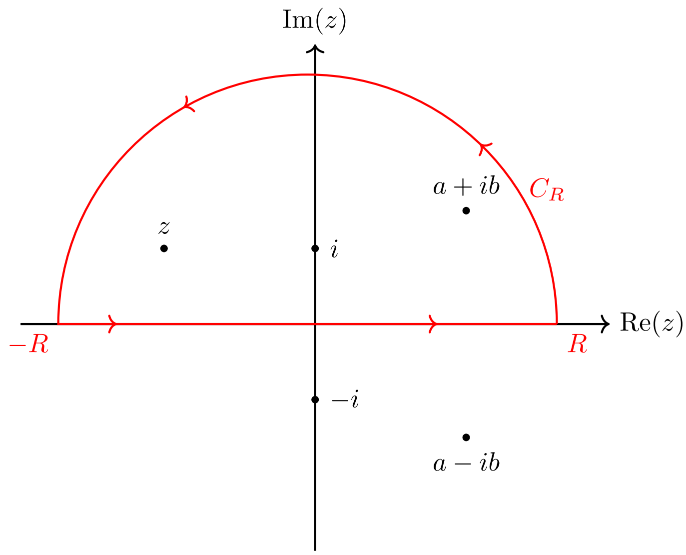

Figure 3: Semi-circle contour

By partial fraction decomposition we have

$$\frac{\lambda^r}{((\lambda-a)^2+b^2)^k} = d + \sum_{j=1}^{k} \frac{D_{1,j}}{(\lambda-(a+ib))^j} + \sum_{j=1}^{k} \frac{D_{2,j}}{(\lambda-(a-ib))^j},$$

where d and $D_{1,j}$ and $D_{2,j}$ for $\leq j \leq k$ are constants. The goal then is to calculate the integrals

$$\int_{\mathbb{R}} \frac{1}{\lambda - z} \frac{1}{((\lambda-(a+ib))^j} d\lambda; \tag{76}$$

$$\int_{\mathbb{R}} \frac{1}{\lambda - z} \frac{1}{((\lambda-(a-ib))^j} d\lambda. \tag{77}$$

For the first integral, we assume $z \neq a+bi$ and define

$$N(w) := \frac{1}{(w-z)((w-(a+ib))^j}. \tag{78}$$

which has a simple pole at $w = z$ and a pole of order j at $w = a+bi$. We consider the case when $z \in \mathbb{C}^+$ and $b > 0$ and use the contour from Figure 3 to evaluate the integral. By the residue theorem we have that

$$\int_{-R}^{R} N(w)dw + \int_{C_R} N(w)dw = 2\pi i \left[\mathrm{Res}_z(N(w)) + \mathrm{Res}_{a+bi}(N(w))\right]. \tag{79}$$

We have that

$$\text{Res}_z(N(w)) = \frac{1}{((z-(a+ib))^j}; \tag{80}$$

$$\text{Res}_{w=a+ib}(N(w)) = \frac{1}{(j-1)!}\lim_{w\to a+ib}\frac{d^{j-1}}{dw^{j-1}}\frac{1}{(w-z)}. \tag{81}$$

We note that the integral along C_R vanishes when $R \to \infty$ since

$$\left|\int_{C_R} N(w)dw\right| \leq \int_0^\pi \frac{1}{(R-|z|)(R-|a-ib|)^k} R d\theta \to 0$$

as $R \to \infty$. Hence, we have that

$$\int_{\mathbb{R}} \frac{1}{\lambda - z}\frac{1}{(\lambda-(a+ib))^j} d\lambda = 2\pi i \left[\text{Res}_z(N(w)) + \text{Res}_{a+bi}(N(w))\right] = \tag{82}$$

$$2\pi i\left[\frac{1}{((z-(a-ib))^j} + \frac{1}{(j-1)!}\lim_{w\to a+ib}\frac{d^{j-1}}{dw^{j-1}}\frac{1}{(w-z)}\right]. \tag{83}$$

Since the integral representation is well-defined and analytic on $\mathbb{C}\setminus\mathbb{R}$ it follows (we assume when solving the integral finding the integral above that $z \in \mathbb{C}^+$) that the term above has to be well-defined. Hence, there is no worry about the integrals for terms in the partial fraction decomposition where the pole is in the upper half-plane (we have also assumed that $b > 0$). Most importantly though, we have an expression which is rational in z. In the case where $z = a + bi$ we have that the integral becomes

$$\int_{\mathbb{R}} \frac{1}{((\lambda-(a-ib))^{j+1}} d\lambda = 2\pi i\left[\frac{1}{j!}\lim_{w\to a+ib}\frac{d^j}{dw^j}\frac{1}{(w-z)}\right]$$

which is rational in z.
For the second integral, we find by a similar calculation (again, we assume here that $b > 0$) that the integral becomes

$$\int_{\mathbb{R}} \frac{1}{(\lambda-z)(\lambda-(a-ib))^j} d\lambda = 2\pi i \frac{1}{(z-(a-ib))^j}$$

which is a rational function with poles in the lower half-plane. □

3.3 General Rational Herglotz Functions

We consider now a general rational Herglotz function. We let

$$h(z) = \frac{p(z)}{q(z)}$$

be an arbitrary rational Herglotz function. We then divide the function into two rational functions, one with real poles and the other with poles in the lower half-plane using partial fraction decomposition

$$h(z) = c(z) + \frac{p_1(z)}{q_1(z)} + \frac{p_2(z)}{q_2(z)}$$

where $c(z)$ is a linear function with possibly a complex constant term.

Proposition 37. *The functions* $\frac{p_1(z)}{q_1(z)}$ *and* $\frac{p_2(z)}{q_2(z)}$ *are Herglotz functions.*

Proof. We consider the Lebesgue decomposition of the measure μ associated with the Herglotz function h.

$$\mu = \mu_{\text{ac}} + \mu_{\text{s}} = \mu_{\text{ac}} + \mu_{\text{sc}} + \mu_{\text{pp}}. \tag{84}$$

By Theorem 25

$$S_{\mu_s} = \left\{ \lambda \in \mathbb{R} \middle| \lim_{y \to 0^+} \operatorname{Im} h(\lambda + iy) = +\infty \right\} \tag{85}$$

is a support for the singular part and in particular we have by Theorem 25 that

$$S_{\mu_{sc}} = \left\{ \lambda \in S_{\mu_s} \middle| \lim_{y \to 0^+} y \operatorname{Im} h(\lambda + iy) = 0 \right\} \tag{86}$$

is a support for μ_{sc}. In the case of a rational Herglotz we have that a support for the singular part given by (85) is the set of all real poles. In particular note that if $\lambda_0 \in \mathbb{R}$ is a pole then after partial fraction decomposition, we will have that:

$$\lim_{y \to 0^+} y \operatorname{Im} \left(\frac{1}{\lambda - (\lambda_0 + iy)} \right) = \lim_{y \to 0^+} y \frac{y}{(\lambda - \lambda_0)^2 + y^2} = \begin{cases} 0, & \text{if } \lambda \neq \lambda_0 \\ 1, & \text{if } \lambda = \lambda_0 \end{cases}$$

but of course since λ_0 is a pole this means that after partial fraction decomposition the value $\lim_{y \to 0^+} \operatorname{Im}(h(\lambda_0 + iy)) = 1$, that is, the support for the singular continuous part will be empty. Hence, for a rational Herglotz function, the decomposition of the corresponding measure is

$$\mu = \mu_{ac} + \mu_s = \mu_{ac} + \mu_{pp}. \tag{87}$$

We note that poles in the lower half plane correspond to absolutely continuous measures and that poles on the real line correspond to point measures. Hence, we can conclude that both, the part with poles on the real line and the part with poles in the lower half-plane, are both Herglotz functions with corresponding measures in the integral representations being μ_{ac} and μ_{pp}. ☐

For a general Herglotz function we obtain the following characterization.

Theorem 38. *Let $h : \mathbb{C}^+ \to \mathbb{C}$ be an analytic function. Then h is a Herglotz function if and only it admits a representation as in Theorem 8 with corresponding measure μ in the integral representation. Moreover, let the Lebesgue decomposition of μ be*

$$\mu = \mu_{ac} + \mu_s = \mu_{ac} + \mu_{sc} + \mu_{pp}. \tag{88}$$

Then, it follows that h is rational if and only if $\mu_{sc} = 0$, μ_{pp} is supported on at most finitely many points and μ_{ac} has a corresponding density function $\nu(x)$ which satisfies the the conditions in Theorem 36.

The following is a characterisation of PR functions

Corollary 39. *Let $h(s) : \mathbb{C}_+ \to \mathbb{C}$ be a real-rational function. Then h is positive-real if and only if*

1. *$h(z)$ is analytic in $\operatorname{Re}[z] > 0$;*
2. *$\operatorname{Re}[h(i\omega)] \geq 0$ for all ω with $i\omega$ not a pole of $h(z)$;*
3. *Poles on the imaginary axis and infinity are simple and have non-negative residues.*

Remark 40. *The requirement that h be analytic is not needed since we have defined PR and Herglotz functions to be analytic. Analyticity does, however, follow from positive realness and there is no loss of generality in assuming h is analytic.*

Proof. We consider the corresponding rational Herglotz function f, defined as in Remark 7. The equivalent statement for f is: f is Herglotz if and only if:

1´. *$f(z)$ is analytic in $\operatorname{Im}[z] > 0$;*

2´. $\mathrm{Im}[f(x)] \geq 0$ *for all* $x \in \mathbb{R}$ *with* x *not a pole of* $f(z)$;

3´. Poles on the real line are simple and have non-positive residues and satisfy

$$\lim_{z \hat{\to} \tau_0} (\tau_0 - z) f(z) \geq 0$$

where the limit is valid in any sector of the upper half-plane (non-tangential limit), while poles at infinity are simple and have non-negative residues.

The result follows then from the Theorems 34 and 36. □

The result above does indeed characterise positive real function. It is, however, difficult to apply since one has to perform residue calculations. The article [4] provdides a modified test for positive-realness of lower order real-rational functions that avoids the need to test residue conditions.

4 Rational Matrix-valued Herglotz Functions

4.1 Definition and Basic properties

The purpose of this section is to define matrix-valued Herglotz functions and present some basic properties. The material is based on [6, Chapter 5, 6].

We briefly discuss positive definite matrices.

Definition 41. *Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices with entries in $\mathbb{C}$. We denote by $(.,.)_{\mathbb{C}^n}$ the standard Euclidean norm on $\mathbb{C}^n$ which is linear in the second factor and anti-linear in the first factor. Let $M \in M_n(\mathbb{C})$. We denote by M^* the conjugate transpose of M. We define the imaginary and real parts of M as*

$$\operatorname{Im}[M] := \frac{M - M^*}{2i}, \tag{89}$$

$$\operatorname{Re}[M] := \frac{M + M^*}{2}. \tag{90}$$

We define a Hermitian matrix $M \in M_n(\mathbb{C})$, i.e. $M = M^*$, to be nonnegative (respectively, nonpositive) if $(x, Mx) \geq 0$ for $x \in \mathbb{C}^n$ (respectively, $(x, Mx) \leq 0$)). We also call M positive (respectively, negative) if $(x, Mx) > 0$ for all $x \in \mathbb{C}^n$ (respectively, $(x, Mx) < 0$).

We mention some useful results about nonnegative matrices.

Lemma 42. *Let $A \in M_n(\mathbb{C})$. We define the principal submatrices of a matrix as the submatrices obtained from repeatedly removing out a row and the column of the same index. The determinant of principal submatrices are called principal minors. The matrix A is nonnegative, $A \geq 0$, if and only if all principal minoros are nonnegative.*

Proof. See [8, Observation 7.1.2.]. □

Corollary 43. *Let $A \in M_n(\mathbb{C})$ be any nonnegative matrix. For any fixed pairs $1 \leq j, k \leq n$*

$$|A_{j,k}| \leq A_{j,j}^{1/2} A_{k,k}^{1/2} \leq \frac{1}{2}(A_{j,j} + A_{k,k}). \tag{91}$$

Proof. Assume, without loss of generality, that $j \leq k$. Repeatedly delete the column and row of index n for $n \neq j, k$. One then ends with the following principal submatrix

$$\begin{bmatrix} A_{j,j} & A_{j,k} \\ A_{k,j} & A_{k,k} \end{bmatrix}$$

which has determinant

$$A_{j,j} A_{k,k} - |A_{j,k}|^2 \geq 0.$$

Lastly, by the previous lemma we have that $A_{j,j}, A_{k,k} \geq 0$ and the theorem follows. □

Definition 44. *A function $M : \mathbb{C}^+ \to M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is the set $n \times n$ matrices with entries in $\mathbb{C}$, is a matrix-valued Herglotz function if M is analytic on $\mathbb{C}^+$ and for any $z \in \mathbb{C}^+$, $\operatorname{Im}[M(z)]$ is non-negative.*

Remark 45. *Analyticity of the matrix here is understood as analyticity for each individual matrix element.*

Example 46. *Let $f_i(z)$ scalar-valued Herglotz functions for $1 \leq i \leq n$. The function*

$$M(z) := \begin{bmatrix} f_1(z) & 0 & 0 & \cdots & 0 \\ 0 & f_2(z) & 0 & \cdots & 0 \\ 0 & 0 & f_3(z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & f_n(z) \end{bmatrix} \tag{92}$$

is a matrix-valued Herglotz function since it is analytic and

$$\operatorname{Im}[M(z)] = \begin{bmatrix} \operatorname{Im}[f_1(z)] & 0 & 0 & \cdots & 0 \\ 0 & \operatorname{Im}[f_2(z)] & 0 & \cdots & 0 \\ 0 & 0 & \operatorname{Im}[f_3(z)] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \operatorname{Im}[f_n(z)] \end{bmatrix} \geq 0 \tag{93}$$

for $\operatorname{Im}[z] > 0$.

We note that all entries need not be scalar-valued Herglotz functions for an analytic matrix-valued function $K(z) : \mathbb{C}^+ \to M_n(\mathbb{C})$ *to be a matrix-valued Herglotz function. Consider*

$$K(z) := \begin{bmatrix} z & -i \\ i & 0 \end{bmatrix} \tag{94}$$

which is a matrix-valued Herglotz function since

$$\operatorname{Im}[K(z)] := \begin{bmatrix} \operatorname{Im}[z] & 0 \\ 0 & 0 \end{bmatrix}. \tag{95}$$

$\Diamond$

The next theorem is a useful when matrix-valued Herglotz functions.

Theorem 47. *Let* $M(z) : \mathbb{C}^+ \to M_n(\mathbb{C})$ *be a matrix-valued Herglotz function. Then it follows that* $(x, M(z)x)$ *is a scalar-valued Herglotz function for each* $x \in \mathbb{C}^n$.

Proof. By assumption $\operatorname{Im}[M(z)] = \frac{1}{2i}\left(M(z) - M(z)^*\right) \geq 0$ which is equivalent to

$$(x, \frac{1}{2i}(M(z) - M(z)^*)x) = \frac{1}{2i}((x, M(z)x) - (x, M(z)^*x)) \geq 0,$$

for any $x \in \mathbb{C}^n$. We note that $\overline{(x, M(z)x)} = (M(z)x, x) = (x, M(z)^*x)$ and therefore the Herglotz criterion is equivalent to

$$\frac{1}{2i}\left[(x, M(z)x) - (x, M(z)^*x)\right] = \frac{1}{2i}[(x, M(z)x) - \overline{(x, M(z)x)}] = \operatorname{Im}[(x, M(z)x)] \geq 0,$$

which yields the result. $\square$

We now state the main representation for matrix-valued Herglotz functions, cf. Theorem 8.

Theorem 48. *Let* $M : \mathbb{C}^+ \to M_n(\mathbb{C})$ *be an analytic matrix–valued function. Then* M *is a matrix-valued Herglotz function if and only if there exists a matrix-valued measure* Ω *on the Borel subsets of* $\mathbb{R}$ *satisfying*

$$\int_{\mathbb{R}} \frac{1}{1+\lambda^2}(x, d\Omega(\lambda)x)_{\mathbb{C}^n} < \infty \text{ for all } x \in \mathbb{C}^n$$

such that the representation

$$M(z) = C + Dz + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\Omega(\lambda), \quad z \in \mathbb{C}^+,$$
$$C = \operatorname{Re}[M(i)], \quad D = \lim_{y \to \infty^+} \left(\frac{1}{iy} M(iy)\right) \geq 0$$

holds.

Proof. See [6, Theorem 5.4.]. □

An introduction to matrix-valued measures is available at [12, 3. Matricial Integrals].

As in the scalar-valued case, the measure can be recovered via a Stieltjes inversion formula.

Lemma 49. *The measure* Ω *can be reconstructed via the Stieltjes inversion formula*

$$\frac{1}{2}\Omega(\{\lambda_1\}) + \frac{1}{2}\Omega(\{\lambda_2\}) + \Omega((\lambda_1, \lambda_2)) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \text{Im}[M(\lambda + i\epsilon)] d\lambda. \tag{96}$$

A result similar to Lemma 14 also holds.

Lemma 50. *The measure* μ *in the integral representation of Theorem 48 has a point mass at the point* $\lambda_0 \in \mathbb{R}$ *if and only if the limit*

$$\Omega(\{\lambda_0\}) = \lim_{z \hat{\to} \lambda_0} (\lambda_0 - z) M(z)$$

is non-negative.

Proof. The result follows by applying Lemma 14 to $(x, M(z)x)$ for $x \in \mathbb{C}^n$ and later using the polarisation identity. □

The result from Theorem 17 holds.

Theorem 51. *If* $M(z)$ *is a matrix-valued Herglotz function with integral representation given by Theorem 48, then* $M(z)$ *can be analytically continued across* $(a, b) \subset \mathbb{R}$ *into a subset* $\mathcal{D}$ *in the lower half plane if and only if* Ω *is purely absolutely continuous on* (a, b) *with associated matrix-valued density function* $\Omega' \geq 0$ *which is real-analytic on* (a, b)*. In this case, the continuation of* $M(z)$ *across* (a, b) *is given by*

$$M(z) = \overline{M(\overline{z})} + 2\pi i \Omega'(z), \quad z \in \mathcal{D}, \tag{97}$$

where $\Omega'(z)$ *denotes the complex–analytic extension of* $\Omega'(z)$*.*

We end the section by highlighting some properties of the matrix-valued measure in the representation theorem.

Remark 52.
- *By the inversion formula we have that for any bounded Borel set* $X \subset \mathbb{R}$, $\Omega(X) \geq 0$*, it is a nonnegative matrix.*
- *We define*

$$w^{tr} := tr(\Omega) := \Omega_{1,1} + \Omega_{2,2} + \cdots + \Omega_{n,n} \tag{98}$$

the trace measure of Ω*. We note, again by the inversion formula and Lemma 42 that* Ω *is absolutely continuous with respect to* w^{tr}*, that is, if* $w^{tr}(X) = 0$ *for any Borel set* $X \subset \mathbb{R}$ *then* $\Omega(X) = 0$*.*

4.2 Support of the Measure

In this section we include the support theory for the measure as in section 1.2.

Theorem 53. *Let M be a matrix-valued Herglotz function with corresponding matrix-valued measure Ω. The absolutely continuous part, Ω_{ac}, of Ω with respect to the Lebesgue measure is given by*

$$d\Omega_{ac} = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \operatorname{Im}[M(\lambda + i\epsilon)] d\lambda \tag{99}$$

where the limit in the right-hand side exists almost everywhere with respect to the Lebesgue measure. In addition, we have that if

$$S_{\Omega_{ac},r} := \left\{ \lambda \in \mathbb{R} \mid \lim_{\epsilon \to 0^+} M(\lambda + i\epsilon) \text{ exists finitely, } \operatorname{rank}(\operatorname{Im}[M(\lambda + i0)]) = r \right\}, \tag{100}$$

$$S_{\Omega_{ac}} := \bigcup_{r=1}^{n} S_{\Omega_{ac},r}, \tag{101}$$

then $S_{\Omega_{ac}}$ is a minimal support of Ω_{ac}.

Proof. Let $x \in \mathbb{C}^n$ and consider the scalar-valued Herglotz function $(x, M(z)x)$. We have by Theorem 25 that the limit

$$\lim_{\epsilon \to 0^+} \operatorname{Im}[(x, M(\lambda + i\epsilon)x)]$$

exists almost everywhere with respect to the Lebesgue measure and is equal to the absolutely continuous part of the measure corresponding to $(x, M(z)x)$. Choosing $x = x_j := (x_{j,1}, x_{j,2}, \cdots, x_{j,n})^T$ such that $x_{j,l} = \delta_{j,l}$ shows that the result holds for the diagonal entries of Ω. For any general element of Ω, we apply the the polarisation identity

$$\begin{aligned}(x, M(z)y) = \frac{1}{4}\big[&((x+y), M(z)(x+y)) - ((x-y), M(z)(x-y)) + \\ &i((x-iy), M(z)(x-iy)) - i((x+iy), M(z)(x+iy))\big]\end{aligned}$$

The second part of the theorem follows by considering the trace measure of Ω_{ac}. We note note that the diagonal elements of $M(\lambda + i\epsilon)$ exist as we take $\epsilon \to 0^+$ and, in addition, if $\operatorname{rank}(\operatorname{Im}[M(\lambda + i0)]) = r$ for $r \geq 1$ then r diagonal elements of $\operatorname{Im}[M(\lambda + i0)]$ are non-zero by Theorem 47. It follows by Theorem 25 that $S_{\Omega_{ac}}$ is a minimal support for the trace measure of Ω_{ac}, which in turn implies, by Remark 52, that this is a minimal support for Ω_{ac} □

In the same spirit, the following results can be shown.

Theorem 54. *Let M be a matrix-valued Herglotz function with corresponding matrix-valued measure Ω. Define*

$$S_{\Omega_s} := \left\{ \lambda \in \mathbb{R} \mid \lim_{\varepsilon \downarrow 0} \operatorname{Im}\left(\operatorname{tr}(M(\lambda + i\varepsilon))\right) = +\infty \right\} \tag{102}$$

$$S_{\Omega_{pp},r} := \left\{ \lambda \in \mathbb{R} \mid \operatorname{rank}\left(\lim_{\varepsilon \downarrow 0} \varepsilon M(\lambda + i\varepsilon) \right) = r \right\}, \quad 1 \leq r \leq n, \tag{103}$$

$$S_{\Omega_{pp}} := \bigcup_{r=1}^{n} S_{\Omega_{pp},r}, \tag{104}$$

$$S_{\Omega_{sc}} := \left\{ \lambda \in S_{\Omega_s} \mid \lim_{\varepsilon \downarrow 0} \varepsilon \operatorname{tr}(M(\lambda + i\varepsilon)) = 0 \right\}. \tag{105}$$

Then

1. S_{Ω_s} *is a minimal support for the singular part of* Ω_s,
2. $S_{\Omega_{sc}}$ *is a minimal support for the singular continuous part of* Ω_{sc}.
3. $S_{\Omega_{pp}}$ *is the smallest support of point*
4. S_{Ω_s} *is a minimal support of* Ω.

4.3 Characterisation of Rational Herglotz Functions

We recall that the goal is to characterise the measure in the integral representation. We would like to repeat the same procedure from the scalar-valued case. We let $M(z)$ be a rational Herglotz function with corresponding measure matrix valued measure Ω. We split $M(z)$ in two parts, one in which the elements have poles (a point in $\mathbb{C}$ is a pole of a matrix valued function if it is a pole of any of the matrix entries) in $\mathbb{C}^-$ and one where the elements have poles on the real line

$$M(z) = E + Fz + M_L(z) + M_R(z),$$

where M_L consists of poles in $\mathbb{C}^-$ and M_R consists of poles on $\mathbb{R}$, $F \geq 0$ and $E \in M_n(\mathbb{C})$. Note that in the case of a rational matrix-valued Herglotz function, we have that only the poles on the real line in the diagonal elements exist in S_{Ω_s} and for such we have that

$$\lim_{\varepsilon \to 0^+} \varepsilon \mathrm{tr}(M(\lambda + i\varepsilon)) \neq 0.$$

Hence, $S_{\Omega_{sc}} = \emptyset$ and there exists only a point spectrum and an absolutely continuous part. As the point spectrum consists only of poles on the real line, we have that M_L and M_R are each matrix-valued Herglotz functions with corresponding measure being Ω_{ac}, the absolutely continuous part of Ω and Ω_{pp}, the singular part of Ω which in this case is discrete matrix valued measure. We have just proven

Proposition 55. *The matrix-valued functions M_R and M_L are Herglotz functions.*

We begin by analysing M_R. Since $(x, M(z)x)$ is scalar valued Herglotzz function for each $x \in \mathbb{C}^n$, we have that

$$\lim_{z \to \tau_0} (\tau_0 - z)(x, M(z)x) = \lim_{\epsilon \to 0^+} \varepsilon(x, \mathrm{Im}(M(\lambda + i\varepsilon))x) = (x, \Omega(\{\lambda\})x).$$

This implies that the residue of any potential pole of $(x, M_R(z), x)$ at τ_0 is non-positive and simple. For general $x, y \in \mathbb{C}^n$ we conclude by the polarisation identity that the function $(x, M_R(z)y)$ has simple poles at $\tau_0 \in \mathbb{R}$, but that the residue is not necessarily non-positive. However, we can conclude from the above identity that the residue of the entire matrix $M_R(z)$ at τ_0 (here the residue is understood as the residue at each element) is given by $-\Omega(\{\lambda\}) \leq 0$. We divide $M_R(z)$ in different parts (using partial fraction decomposition on each element)

$$M_R(z) = C + Dz + M_1(z) + \cdots + M_N(z)$$

where each part consists of elements with only a pole at τ_j. The following example is useful.

Example 56. *Letting $\Omega = (\delta_{\tau_0} = \Omega_{ij})$ be the matrix-valued measure with each entry being a point measure with point mass at $\tau_0 \in \mathbb{R}$ gives the following function in the integral representation*

$$\left(\int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\Omega \right)_{ij} = \left(\frac{1}{\tau_0 - z} - \frac{\tau_0}{1 + \tau_0^2} \right)$$

◊

We let Ω_i be the matrix-valued measure where each non-zero element is the point measure with mass equal to minus the residue at τ_i. By the uniqueness of the measure in the integral representation, we find that the measure corresponding to $M_R(z)$ is a discrete matrix-valued measure. We have thus proven the following theorem, cf. Theorem 34.

Theorem 57. *Let $H : \mathbb{C}^+ \to M_n(\mathbb{C})$ be a matrix-valued function. Then H is a rational matrix-valued Herglotz function with poles only in $\mathbb{R}$ if and only if it admits a representation*

$$H(z) = C + Dz + \sum_{j=1}^{N} A_j \left(\frac{1}{\tau_j - z} - \frac{\tau_j}{1 + \tau_j^2} \right),$$

where $\tau_j \in \mathbb{R}$, $N \in \mathbb{N}, C = C^*$, $D \geq 0$ and A_j is a non-negative matrix for $1 \leq j \leq N$.

Proof. We have shown that this is a necessary condition in the preceding theorem. To see that it is sufficient, we note that the function is analytic and

$$(x, \operatorname{Im}(H(z))x) = \frac{(x, H(z)x) - (x, H(z)^* x)}{2i} = \tag{106}$$

$$\frac{(x, Dx)(z - \overline{z})}{2i} + \frac{(x, \sum_{j=1}^{N} A_j \left(\frac{1}{\tau_j - z} - \frac{\tau_j}{1+\tau_j^2}\right) x) - (x, \sum_{j=1}^{N} A_j^* \left(\frac{1}{\tau_j - \overline{z}} - \frac{\tau_j}{1+\tau_j^2}\right) x)}{2i} = \tag{107}$$

$$(x, Dx) \operatorname{Im}(z) + \frac{(x, \left[\sum_{j=1}^{N} A_j \left(\frac{1}{\tau_j - z} - \frac{\tau_j}{1+\tau_j^2}\right) - \sum_{j=1}^{N} A_j \left(\frac{1}{\tau_j - \overline{z}} - \frac{\tau_j}{1+\tau_j^2}\right)\right] x)}{2i} = \tag{108}$$

$$(x, Dx) \operatorname{Im}(z) + (x, A_j x) \frac{\operatorname{Im}(z)}{|\tau_j - z|^2} \geq 0. \tag{109}$$

□

We attempt now to characterise matrix-valued Herglotz with poles in the lower half plane, i.e. the function M_L. We claim now the following theorem.

Theorem 58. *Let $Q(z)$ be a matrix-valued Herglotz function. Then $Q(z)$ is a rational matrix value Herglotz function with poles in the lower half-plane if and only if the corresponding measure is absolutely continuous with density matrix satisfying (if $d\Omega = P(\lambda)d\lambda$)*

1´. $P(\lambda) \geq 0, \quad$ for every $\lambda \in \mathbb{R}$;

2´. $(x, P(\lambda)x)$ is a rational function that satisfies properties (2) and (3) in Theorem 36 for every $x \in \mathbb{C}^n$.

Proof. ($\Longrightarrow$) If $Q(z)$ is a matrix-valued Herglotz function with poles only in the lower half-plane then the corresponding measure is absolutely continuous by Theorem 53 and we have that $d\Omega = P(\lambda)d\lambda$ and $P(\lambda) = \frac{1}{\pi}\operatorname{Im}(Q(\lambda + i0)) \geq 0$. In addition, since $Q(z)$ is a rational matrix-valued Herglotz function, we have that $(x, Q(z)x)$, for $x \in \mathbb{C}^n$, is a a rational Herglotz function with corresponding density function $(x, P(\lambda)x)$. Hence by Theorem 3, $(x, P(\lambda)x)$ satisfies criteria 1-3.
($\Longleftarrow$) The goal is to show that any component of $Q(z)$ is rational with poles in the lower half plane. We have that the result holds for each each function $(x, Q(z)x)$ for $x \in \mathbb{C}^n$; in fact these functions are rational Herglotz functions. We have that the result holds for $(x, Q(z)y)$ for $x, y \in \mathbb{C}^n$ by the polarisation identity:

$$(x, Q(z)y) = \tag{110}$$

$$\frac{1}{4}\left[\left((x+y), Q(z)(x+y)\right) - \left((x-y), Q(z)(x-y)\right)\right. \tag{111}$$

$$\left. +i\left((x-iy), Q(z)(x-iy)\right) - i\left((x+iy), Q(z)(x+iy)\right)\right]. \tag{112}$$

Since x and y are arbitrary the result follows for any element of $Q(z)$. □

The above theorem implies that there is no obvious difference between scalar-valued rational Herglotz functions and matrix-valued rational Herglotz functions. In fact, most of the theorems available for scalar-valued Herglotz function had an equivalent formulation for matrix-valued Herglotz functions. Indeed, the theorems described in this section and the previous one show that the it is enough to usually consider each element of the matrix or in some cases just the diagonal elements. The matrix structure was rarely taken into account. The following example shows that this is not always the case and that matrix structure has to be considered.

Example 59. *Let*

$$M(z) := \begin{bmatrix} z & 1 \\ 1 & -\frac{1}{z} \end{bmatrix}$$

which is Herglotz. We also have that

$$-M(z)^{-1} = \frac{1}{2}\begin{bmatrix} -\frac{1}{z} & -1 \\ -1 & z \end{bmatrix}$$

is a matrix-valued Herglotz function. We note that for $M(z)$*, the point* $z = 0$ *is a pole and a zero- a zero of a matrix function* $M(z)$ *is defined as any pole of* $M(z)^{-1}$*. A consequence is that the interlacing property in Theorem 35 does not have an similar form for matrix-valued Herglotz functions. An interlacing property for matrix-valued meromorphic Herglotz functions is discussed in [11].*

5 Appendix- The Herglotz Representation Theorems

In this section we prove the integral representation, Theorem 8.

We will first consider a class of functions that are related to Herglotz functions. An analytic function $c : \mathbb{D} \to \overline{\mathbb{C}_+} = \{z \in \mathbb{C} : \mathrm{Re}[z] \geq 0\}$ is called a Carathéodory function. Carathéodory functions are related to Herglotz functions by the Cayley transform

$$C : z \mapsto \frac{z - i}{i + z}$$

which maps to upper half-plane conformally onto the unit disk and has inverse

$$C^{-1} : z \mapsto i\frac{1 + z}{1 - z}.$$

Indeed, given a Carathéodory function c we have that

$$f(z) := ic(C(z))$$

is a Herglotz function. Conversley, given a Herglotz function f, we have that

$$c(z) := -if(C^{-1}(z))$$

is a Carathéodory function. We define the Herglotz kernel by

$$K(w, z) := \frac{w + z}{w - z} \tag{113}$$

where $w \in \partial\mathbb{D}$ and $z \in \mathbb{D}$.

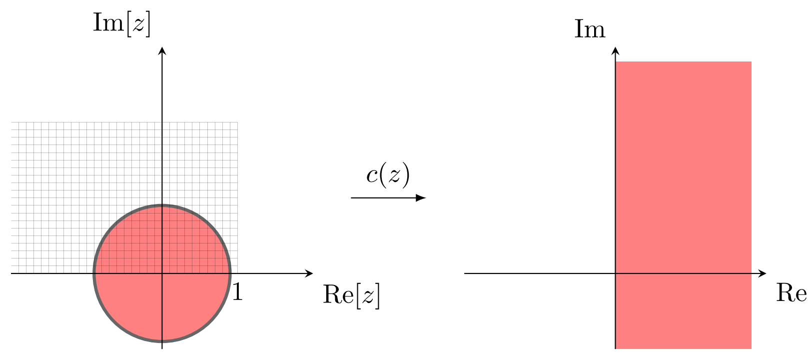

Figure 4: Carathéodory Function

The next theorem shows how to recapture an analytic function from the boundary values of its real part.

Theorem 60. *Let f be analytic in a neighbourhood of $\overline{\mathbb{D}}$. Then*

$$f(z) = i\,\mathrm{Im}[f(0)] + \frac{1}{2\pi}\int_{[0,2\pi)} K(e^{it}, z)\,\mathrm{Re}[f(e^{it})]dt. \tag{114}$$

The result holds in the more general context where $\mathrm{Re}[f(it)]$ is replaced by $u(it)$, where u is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. In this case the integral is an analytic function on $\mathbb{D}$ with real part u. For a proof, see [9, Chapter 7, Corollary 7.7]. We have the following characterisation of Carathéodory functions.

Theorem 61 (Herglotz Representation.)**.** *Every Carathéodory functions is of the form*

$$c(z) = i\,\mathrm{Im}[c(0)] + \int_{[0,2\pi)} K(e^{it}, z)d\nu \tag{115}$$

where $d\nu$ is a finite Borel measure on $[0, 2\pi)$.

Proof. We note that the right hand-side of 115 gives a Carathéodory function. Indeed, we have

$$\mathrm{Re}[i\,\mathrm{Im}[c(0)] + \int_{[0,2\pi)} K(e^{it}, z)d\nu] = \int_{[0,2\pi)} \mathrm{Re}[K(e^{it}, z)]d\nu = \int_{[0,2\pi)} \frac{1-|z|^2}{|e^{it}-z|^2}d\nu \geq 0.$$

Analyticity follows by the dominated convergence theorem since

$$\left|\frac{2e^{it}}{(-w+e^{it})(-z+e^{it})}\right| \leq \frac{2}{(1-|w|)(1-|z|)}$$

and

$$\frac{f(z)-f(w)}{z-w} = \frac{1}{z-w}\int_{[0,2\pi)} \frac{e^{it}+z}{e^{it}-z} - \frac{e^{it}+w}{e^{it}-w}d\nu = \frac{1}{z-w}\int_{[0,2\pi)} \frac{2(z-w)e^{it}}{(-w+e^{it})(-z+e^{it})}d\nu \tag{116}$$

$$= \int_{[0,2\pi)} \frac{2e^{it}}{(-w+e^{it})(-z+e^{it})}d\nu \to \int_{[0,2\pi)} \frac{2e^{it}}{(e^{it}-w)^2}d\nu \tag{117}$$

as $z \to w$, for $z, w \in \mathbb{D}$.
Conversely, suppose we are given a Carathéodory function $c(z)$. We define the functions $c_r(z) := c(rz)$ for $0 < r < 1$, which are analytic in a neighbourhood of $\overline{\mathbb{D}}$. We have by Theorem 60. that

$$c_r(z) = i\,\mathrm{Im}[c_r(0)] + \frac{1}{2\pi}\int_{[0,2\pi)} K(e^{it}, z)\,\mathrm{Re}[c_r(it)]dt.$$

Define then the family of measures $d\nu_r(t) := \mathrm{Re}[c_r(it)]\frac{dt}{2\pi}$. As $\mathrm{Re}[c_r(z)]$ is harmonic we have that

$$\nu_r([0, 2\pi)) = \frac{1}{2\pi}\int_{[0,2\pi)} \mathrm{Re}[c_r(it)]dt = c_r(0) = c(0)$$

which is bounded and independent of r. Let $r_n := 1 - \frac{1}{n}$ and consider the sequence of bounded measures (ν_{r_n}). We have by Helly's Selection Theorem (theorem below) that there exists a subsequence $(\nu_{r_{n_j}})$ that converges weakly to a positive, bounded measure ν defined on Borel sets of $[0, 2\pi)$. Hence, it follows that

$$\frac{1}{2\pi}\int_{[0,2\pi)} K(e^{it}, z)\,\mathrm{Re}[c_{r_n}(it)]dt \to \int_{[0,2\pi)} K(e^{it}, z)d\nu.$$

We note that $c_{r_n}(z) \to c(z)$ for $z \in \mathbb{D}$ and therefore

$$c(z) = i\,\mathrm{Im}[c(0)] + \int_{[0,2\pi)} K(e^{it}, z)d\nu. \tag{118}$$

□

We recall that for a locally compact metric space X, a sequence of Borel measures $\{\mu_n\}$ on X is said to converge vaguely to a Borel measure μ if

$$\int_X f d\mu_n \to \int_X f d\mu, \quad \forall f \in C_c(X).$$

Theorem 62 (Helly's Selection Theorem)**.** *Let X be a locally compact metric space. Then every bounded sequence ν_n of regular complex measures, that is $|\nu_n|(X) \leq M$, has a vaguely convergent subsequence whose limit is regular. If all ν_n are positive, every limit of a convergent subsequence is again positive.*

For a proof of the theorem, see [15, Chapter 6, Theorem 6.11]. We now turn to the proof of theorem 8.

Proof Theorem 1. We begin by showing that every function of the form

$$a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t) = a + bz + \int_{\mathbb{R}} \frac{1+zt}{(t-z)(1+t^2)} d\mu(t)$$

is Herglotz. We begin by showing that the integral is well-defined in $\mathbb{C}^+$. Let $z \in \mathbb{C}^+$, then we have the estimate

$$\left| \frac{1+tz}{t-z} \right| \leq \frac{1 + |z|(|\operatorname{Re}[\lambda - z]| + |\operatorname{Re}[z]|)}{\sqrt{\operatorname{Re}[\lambda - z]^2 + \operatorname{Im}[z]^2}} \leq |z| + \frac{1 + |z \operatorname{Re}[z]|}{|z - \lambda|} \leq |z| + \frac{1 + |z \operatorname{Re}[z]|}{\operatorname{dist}(z, \mathbb{R})}$$

which gives well-definitness. We also have that the integral is continuous since for $z \in \mathbb{C}^+$

$$\frac{1 + (z+h)t}{(t - (z+h))(1+t^2)} - \frac{1+zt}{(t-z)(1+t^2)} = \frac{1 + t(h+z)}{(t^2+1)(t-(h+z))} - \frac{1+tz}{(t^2+1)(t-z)}$$

where each part can be bounded by an integrable function as done above. Continuity then follows by the dominated convergence theorem. To see analyticity, let K be any compact subset of $\mathbb{C}^+$ and let Γ be any closed rectangle contained in K. By the above estimate we have that the function

$$\frac{1+tz}{t-z}$$

is bounded for $(t, z) \in \mathbb{R} \times K$. We can, therefore, apply Fubini's theorem to find

$$\int_{\Gamma} \int_{\mathbb{R}} \frac{1+zt}{(t-z)(1+t^2)} d\mu(t) dz = \int_{\mathbb{R}} \int_{\Gamma} \frac{1+zt}{(t-z)(1+t^2)} dz d\mu(t) = 0. \tag{119}$$

It follows by Morera's theorem that the integral is analytic in K and hence in $\mathbb{C}^+$. Lastly we note that the imaginary part of f is

$$\operatorname{Im}[f(z)] = b \operatorname{Im}[z] + \int_{\mathbb{R}} \operatorname{Im}\left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] d\mu(t) = b \operatorname{Im}[z] + \int_{\mathbb{R}} \frac{\operatorname{Im}[z]}{|t-z|^2} d\mu(t) \geq 0$$

for $z \in \mathbb{C}^+$.
Conversely, given a Herglotz function $f(z)$, we have that $c(z) := -if(C^{-1}(z))$ is Carathéodory function and Theorem 115 gives

$$c(z) = i \operatorname{Im}[c(0)] + \int_{[0,2\pi)} K(e^{it}, z) d\nu(t) \tag{120}$$

for some finite Borel measure ν on $[0, 2\pi)$. This gives

$$f(z) = ic(C(z)) = -\operatorname{Im}[c(0)] + i \int_{[0,2\pi)} K(e^{it}, C(z)) d\nu(t) \tag{121}$$

$$= -\operatorname{Im}[c(0)] + i \underbrace{K(1, C(z))}_{-iz} \nu(\{0\}) + i \int_{(0,2\pi)} K(e^{it}, C(z)) d\nu(t) = \tag{122}$$

$$-\operatorname{Im}[c(0)] + z\nu(\{0\}) + i \int_{(0,2\pi)} \frac{e^{it} + C(z)}{e^{it} - C(z)} d\nu(t) \tag{123}$$

We have that the Cayley transform is a homeomorphism from the real-line into $\partial\mathbb{D}$. In particular, we have that

$$\varphi : (0, 2\pi) \to \mathbb{R} \tag{124}$$

$$t \mapsto C^{-1}(e^{it}) = i\frac{1+e^{it}}{1-e^{it}} = -\cot\left(\frac{t}{2}\right) \tag{125}$$

is a homeomorphism. We use this map to define a a pushforward measure on $\mathbb{R}$, $\varphi_*\nu$, defined by $\varphi_*\nu(B) := \nu(\varphi^{-1}(B))$ for every Borel set in $\mathbb{R}$. Note that this is a finite Borel measure on $\mathbb{R}$. It follows that

$$\int_{\mathbb{R}} K(e^{i\varphi^{-1}(t)}, C(z))d(\varphi_*\nu) = \int_{(0,2\pi)} K(e^{it}, C(z))d\nu \tag{126}$$

We have that $\varphi^{-1}(t) = (e^{it})^{-1}(C(t)) = \arg(C(t))$ which gives

$$f(z) = -\operatorname{Im}[c(0)] + z\nu(\{0\}) + i\int_{\mathbb{R}} K(e^{i\varphi^{-1}(t)}, C(z))d(\varphi_*\nu) \tag{127}$$

$$-\operatorname{Im}[c(0)] + z\nu(\{0\}) + i\int_{\mathbb{R}} K(e^{i\arg(C(t))}, C(z))d(\varphi_*\nu) \tag{128}$$

$$-\operatorname{Im}[c(0)] + z\nu(\{0\}) + i\int_{\mathbb{R}} K(C(t), C(z))d(\varphi_*\nu) \tag{129}$$

$$-\operatorname{Im}[c(0)] + z\nu(\{0\}) + \int_{\mathbb{R}} \frac{tz+1}{t-z}d(\varphi_*\nu). \tag{130}$$

We define a measure $d\mu(t) := (1+t^2)d(\varphi_*\nu)(t)$ and find

$$f(z) = -\operatorname{Im}[c(0)] + z\nu(\{0\}) + \int_{\mathbb{R}} \frac{tz+1}{t-z}\frac{1}{1+t^2}d\mu(t) \tag{131}$$

with

$$\int_{\mathbb{R}} \frac{1}{1+t^2}d\mu(t) = \int_{\mathbb{R}} d(\varphi_*\nu) < \infty$$

□

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