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Maneuvering through measure theory and Kakeya's needle problem

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Abstract

The aim of this thesis is to study Kakeya's needle problem, finding the smallest figure in the plane in which a unit segment can be rotated 180°, using the precision of measure theory. Measure theory is introduced abstractly and two measuring functions are defined, the Hausdorff measure and Lebesgue measure. The former extends the concept of dimension from the integers to the non-negative real numbers. Furthermore the equivalence between the Hausdorff measure and Lebesgue measure is shown. Finally the Kakeya conjecture, an open problem in mathematics, is briefly discussed with respect to the concepts introduced in this thesis.

Sammanfattning

Målet med denna uppsats är att studera Kakeyas nålproblem, att hitta minsta figuren i planet i vilket ett segment av enhetslängd kan rotera 180°, med måtteorins precision. Måtteori introduceras abstrakt och två mätfunktioner defininieras, Hausdorffmåttet och Lebesguemåttet. Den tidigare av dem utvidgar konceptet av dimension från heltalen till de icke-negative reella talen. Vidare visas ekvivalensen mellan Hausdorffmåttet och Lebesguemåttet. Slutligen diskuteras kort ett öppet problem inom matematiken, Kakeyas förmodan, med hänsyn till teorin introducerad i denna uppsats.

Contents

1 Introduction

In 1917 Sōichi Kakeya posed the problem of finding the smallest convex set inside which a unit segment (a needle) can be maneuvered to lie in its original position but rotated 180° without leaving the set. It was conjectured that the equilateral triangle was the smallest such set, which was proved by Gyuala Pál in 1921. Pál then reiterated the question without the convexity constraint. This became known as the Kakeya needle problem. At the same time in 1921 Abram Besicovitch succeeded in constructing a set that contained a segment in every direction and had zero area. It was realized that a simple modification to Besicovitch's set yielded a solution to the Kakeya needle problem of arbitrarily small area. [Fal85, p. 95-96]

A set (in $\mathbb{R}^2$) satisfying that a unit segment can be maneuvered inside the set to lie in its original position but rotated 180° will be denoted a "Kakeya set"(in $\mathbb{R}^2$). A set containing a unit segment in every direction will be denoted a "Besicovitch set". Section 3.1 will be dedicated to constructing a Besicovitch set in $\mathbb{R}^2$ of zero area and a Kakeya set in $\mathbb{R}^2$ with arbitrarily small area. While the constructions of these two sets can be understood at least heuristically with elementary geometry, measure theory, Section 2, gives precision. Furthermore measure theory is required to understand an open problem related to Kakeya and Besicovitch sets, the "Kakeya conjecture", briefly discussed in Section 3.2.

The theory in Sections 2 and 3 is accredited to K. J. Falconer. Most, if not all, of definitions and theorems are found in [Fal85, Chapter 1] and [Fal85, Chapter 7] with some details filled in for this thesis.

1.1 How one thinks of volume

Measure theory can be seen as an abstraction of the concept of volume. Before jumping into measure theory it may be insightful to first discuss how one thinks of volume.

A block in $\mathbb{R}^n$ has volume equal to the product of the side lengths, wether one consider this a property of the block or a definition of volume for the block has little matter. From this block one can determine the volume of other figures best illustrated by example.

We can form any rectangle into a parallelogram by moving a triangle-piece of the rectangle and vice versa ultimately showing that a parallelogram has area equal to base times height, so now we can calculate the area of parallelograms. From there any triangle can be made by splitting a parallelogram in two along the diagonal, so now we can calculate the area of triangles. Now by conjoining n congruent isosceles triangles with top-angles $360°/n$ we can form any regular n-sided polygon, enabling us to calculate the polygons area. Now take a circle and let it enclose a n-sided polygon such that the vertices of the polygon lie on the perimeter. Then letting $n \to \infty$ the gaps between the polygon and circle can be made arbitrary small, enabling us to calculate the area of the circle.

Calculating volumes of figures in this way shows that we think of the volume of any figure in reference blocks. Indeed the size of an apartment is measured in meters (or something else) *squared*, even though the floor plan may be anything but rectangular, and a soda can is measured in something *cubed*, even though a soda can is more like a cylinder.

Now say there is some figure which we can't find a method to calculate its volume as the example above. What do we do? Well, we can try covering it by a lot of blocks, which we know the volume of, and then sum up the volumes. If we have two covers of the figure, and one of them has smaller sum of volume than the other, then certainly that one is a better approximation for the volume of the figure. By this idea, taking the infimum of the sum of all block covers must be the best approximation for the volume there is, hence we may say this is the volume of the figure. This provides a general idea of volume to any figure in reference to blocks, aligning with our everyday idea of volume. What has been described is the idea behind the *Lebesgue measure*, Section 2.5.

1.2 Notation

Countable means either finite or countably infinite. Sequences will be denoted $\{a_j\}_{j=1}^{\infty}$ when countably infinite, $\{a_j\}_{j=1}^{n}$ when finite and $\{a_j\}_j$ when countable. Sums, unions and intersections will be denoted and indexed analogously.

2 Measure theory

In this section we will define two different types of measuring functions, a measure and an outer measure. The domain of these functions, Sigma fields, will be discussed with some emphasis on a special type of Sigma field, the Borel set. Furthermore two examples of outer measures, the *Hausdorff s-dimensional outer measure* and the *Lebesgue n-dimensional outer measure*, will be defined and the "equivalence" between these two outer measures will be shown when s and n are equal integers. Lastly a calculation of the Hausdorff measure and dimension will be done on a set with non-integer dimension.

2.1 Sigmafields

Definition 2.1 (σ-field)**.** Let X be any set and let Σ be a non-empty collection of subsets of X. The collection Σ is called a σ-field (Sigma field) or σ-algebra if

(i) Σ is closed under countable union,

(ii) Σ is closed under complementation.

Immediate consequences of Definition 2.1 are stated in the following theorem.

Theorem 2.2 (Consequences of Definition 2.1)**.** *If Σ is a σ-field then*

(i) Σ is closed under set difference,

(ii) Σ is closed under countable intersection,

(iii) Both $\emptyset \in \Sigma$ and $X \in \Sigma$.

Proof. By Definition 2.1(i) $E_1 \cup E_2 \in \Sigma$. By Definition 2.1(ii) $X \backslash (E_1 \cup E_2) \in \Sigma$ and also $X \backslash (E_1 \cup E_2) \cup E_1 \in \Sigma$ by Definition 2.1(i). A final use of Definition 2.1(ii) shows that

$$E_1 \backslash E_2 = X \backslash (X \backslash (E_1 \cup E_2) \cup E_2) \in \Sigma.$$

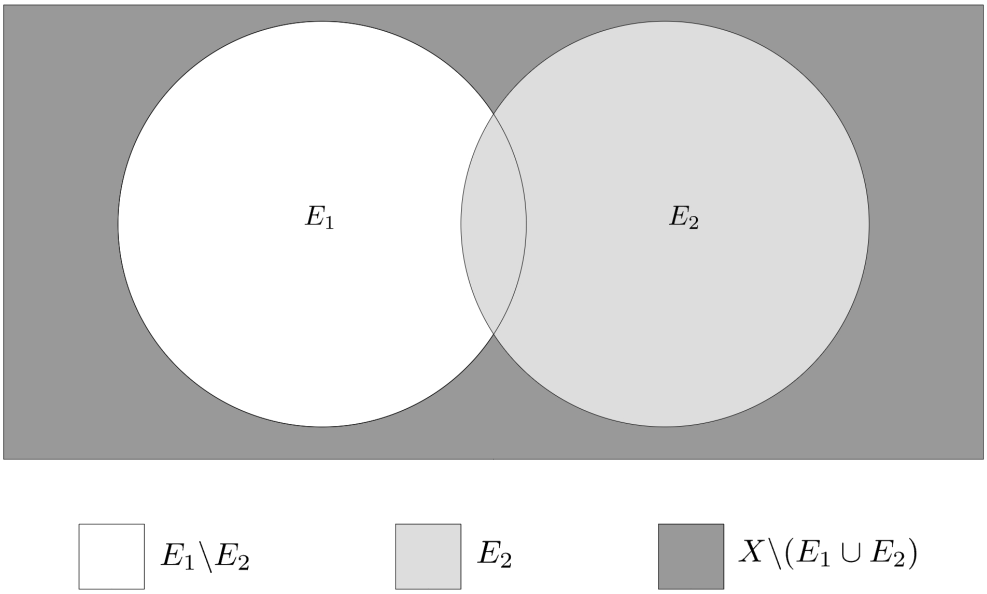

Figure 1: Visual interpretation of $E_1 \backslash E_2 = X \backslash (X \backslash (E_1 \cup E_2) \cup E_2)$.

To show (ii) we use the following equivalences

$$x \in \bigcap_j E_j \Leftrightarrow x \in E_j \text{ for } j = 1, 2, ... \Leftrightarrow x \notin X \backslash E_j \text{ for } j = 1, 2, ...$$

$$\Leftrightarrow x \notin \bigcup_j X \backslash E_j \Leftrightarrow x \in X \backslash \bigcup_j X \backslash E_j.$$

We deduce that

$$\bigcap_j E_j = X \backslash \bigcup_j X \backslash E_j$$

and conclude that the right hand side is in Σ by Definition 2.1 showing that $\bigcap_j E_j \in \Sigma$.

For (iii) since Σ is assumed to be non-empty there is some $E \subset X$ such that $E \in \Sigma$. By Definition 2.1 $X = (X \backslash E) \cup E \in \Sigma$. Since $X \in \Sigma$ it follows that $\emptyset = X \backslash X \in \Sigma$. □

Intersecting two σ-fields gives a new σ-field given the intersection is non-empty. Therefore, similarly to other branches of math, it is possible to construct a "smallest" σ-field containing a non-empty collection of subsets C of a set X by intersecting every σ-field containing each subset of C. This leads us to a definition.

Definition 2.3 (Generating a σ-field)**.** Let C be a non-empty collection of subsets of X. The σ-field generated by C is the intersection of all σ-fields containing each set of C. It is the "smallest" σ-field containing each set of C.

Definition 2.4 (Limit of a sequence of sets)**.** Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of sets. Define the lower and upper limits of $\{E_j\}_{j=1}^{\infty}$ as

$$\varliminf_{j\to\infty} E_j = \bigcup_{k=1}^{\infty}\bigcap_{j=k}^{\infty} E_j \text{ and } \varlimsup_{j\to\infty} E_j = \bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty} E_j$$

if $\varliminf_{j\to\infty} E_j = \varlimsup_{j\to\infty} E_j$ set

$$\lim_{j\to\infty} E_j = \varliminf_{j\to\infty} E_j = \varlimsup_{j\to\infty} E_j$$

as the limit of the sequence of sets $\{E_j\}_{j=1}^{\infty}$.

From Definition 2.1 and its consequences in Theorem 2.2 it is clear that if each E_j is in a σ-field Σ then $\varliminf_{j\to\infty} E_j \in \Sigma$ and $\varlimsup_{j\to\infty} E_j \in \Sigma$.

Remark 2.5. The definition implies that there is some set-relation between the lower and upper limits. Indeed $\varliminf_{j\to\infty} E_j \subset \varlimsup_{j\to\infty} E_j$ by the following argument: If $x \in \varliminf_{j\to\infty} E_j$ then $x \in \bigcap_{j=k}^{\infty} E_j$ for some k. From this $x \in E_i$ for every $i \geq k$, therefore $x \in \bigcup_{j=i}^{\infty} E_j$ for each $i \geq k$ and so $x \in \bigcap_{i=k}^{\infty}\bigcup_{j=i}^{\infty} E_j$. Furthermore since $x \in E_k$ in particular, it follows that $x \in \bigcup_{j=i}^{\infty} E_j$ for $i \leq k-1$. Therefore $x \in \bigcap_{i=1}^{k-1}\bigcup_{j=i}^{\infty} E_j$. Tying things together

$$x \in \bigcap_{i=k}^{\infty}\bigcup_{j=i}^{\infty} E_j \text{ and } x \in \bigcap_{i=1}^{k-1}\bigcup_{j=i}^{\infty} E_j$$

$$\Leftrightarrow x \in \left(\bigcap_{i=k}^{\infty}\bigcup_{j=i}^{\infty} E_j\right) \cap \left(\bigcap_{i=1}^{k-1}\bigcup_{j=i}^{\infty} E_j\right) \Leftrightarrow x \in \bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} E_j.$$

Therefore $x \in \varlimsup_{j\to\infty} E_j$.

The next theorem shows that the limit of a sequence of sets exists when we have a chain of subsets.

Theorem 2.6. *Let $\{E_j\}_{j=1}^{\infty}$ and $\{F_j\}_{j=1}^{\infty}$ be sequences of sets in a σ-field Σ.*

(i) If $\{E_j\}_{j=1}^{\infty}$ is an increasing chain of subsets, that is $E_1 \subset E_2 \subset \ldots$, then

$$\varliminf_{j\to\infty} E_j = \varlimsup_{j\to\infty} E_j = \bigcup_{j=1}^{\infty} E_j.$$

(ii) If $\{F_j\}_{j=1}^{\infty}$ is a decreasing chain of subsets, that is $F_1 \supset F_2 \supset \ldots$, then

$$\underline{\lim}_{j\to\infty} F_j = \overline{\lim}_{j\to\infty} E_j = \bigcap_{j=1}^{\infty} F_j.$$

Hence in both cases $\lim_{j\to\infty} E_j$ *and* $\lim_{j\to\infty} F_j$ *exist.*

Proof. (i) Consequences of $\{E_j\}_{j=1}^{\infty}$ being an increasing chain are

$$\bigcap_{j=k}^{\infty} E_j = E_k \text{ and } \bigcup_{j=k}^{\infty} E_j = \bigcup_{j=1}^{\infty} E_j.$$

Using Definition 2.4

$$\underline{\lim}_{j\to\infty} E_j = \bigcup_{k=1}^{\infty} E_k \text{ and } \overline{\lim}_{j\to\infty} E_j = \bigcap_{k=1}^{\infty}\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} E_j.$$

(ii) Consequences of $\{F_j\}_{j=1}^{\infty}$ being a decreasing chain are

$$\bigcap_{j=k}^{\infty} F_j = \bigcap_{j=1}^{\infty} F_j \text{ and } \bigcup_{j=k}^{\infty} F_j = F_k.$$

Again by Definition 2.4

$$\underline{\lim}_{j\to\infty} F_j = \bigcup_{k=1}^{\infty}\bigcap_{j=1}^{\infty} F_j = \bigcap_{j=1}^{\infty} F_j \text{ and } \overline{\lim}_{j\to\infty} F_j = \bigcap_{k=1}^{\infty} F_k.$$

□

2.2 Measures and Outer measures

Definition 2.7 (Measure)**.** Let Σ be a σ-field containing subsets of the set X. A function μ defined on Σ with range $[0, \infty]$ is called a *measure* if

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(\bigcup_j E_j) = \sum_j \mu(E_j)$ for every countable sequence of disjoint sets $\{E_j\}_j$ of Σ.

In words Definition 2.7 says that nothing has zero volume and splitting something into smaller pieces and summing the volume of the pieces should yield the same volume.

Definition 2.8 (Outer measure)**.** Let X be a set, an *outer measure*, ν, is a function with domain being the subsets of X with range $[0, \infty]$ satisfying

(i) $\nu(\emptyset) = 0$,

(ii) $\nu(A) \leq \nu(A')$ if $A \subset A'$,

(iii) $\nu(\bigcup_j A_j) \leq \sum_j \nu(A_j)$ for every countable sequence of subsets $\{A_j\}_j$.

One may ask what the point is of defining an outer measure as well as a measure. To illustrate this we first need another definition.

Definition 2.9 (ν-measurable)**.** A subset E of X is said to be *ν-measurable* if

$$\nu(A) = \nu(A \cap E) + \nu(A \backslash E) \tag{1}$$

for any set $A \subset X$.

Note that to show Equation (1) it suffices to show

$$\nu(A) \geq \nu(A \cap E) + \nu(A \backslash E) \tag{2}$$

as the inequality $\nu(A) \leq \nu(A \cap E) + \nu(A \backslash E)$ is immediate from Definition 2.8 (ii) and (iii) by taking $A_1 = A \cap E$ and $A_2 = A \backslash E$.

Say there is some set E of a σ-field Σ such that

$$\nu(A) < \nu(A \cap E) + \nu(A \backslash E)$$

for some set A. Then chopping up A into pieces of E and measuring the pieces would give a strictly larger measure than measuring A. If this is the case ν would not satisfy Definition 2.7 (ii).

In contrast if every subset E is ν-measurable then Definition 2.7 (ii) should hold for ν. Indeed Theorem 2.10 confirms this.

Theorem 2.10. *Let ν be an outer measure. The collection Σ_ν of ν-measurable sets form a σ-field, where ν is a measure.*

A detailed proof of Theorem 2.10 is found in [Fal85, Theorem 1.2, p.3].

2.3 Borel sets

Now we shall consider the theory of σ-fields, measures and outer measures applied to a metric space (X, d) leading us to a σ-field called the Borel set of X.

Definition 2.11 (Borel set)**.** Let (X, d) be a metric space, we then have the notion of closed and open sets in X. The σ-field generated by the closed sets in X is called the *Borel set* of X. A subset of X which lies in the Borel set is called a *Borel subset*.

Taking X as $\mathbb{R}^n$ with the Euclidean metric most "nice" subsets of $\mathbb{R}^n$ can be expressed by countable union, intersection or difference of closed sets. We shall see in Theorem 2.15 that these nice Borel subsets are ν-measurable if ν is something called a metric outer measure.

Definition 2.12 (Metric outer measure)**.** Let (X, d) be a metric space and define the distance between two non-empty subsets $E, F \subset X$ as

$$d(E, F) = \inf\{d(x, y) : x \in E, y \in F\}.$$

If $d(E, F) > 0$ we say that E and F are *positively separated*. An outer measure ν is a *metric outer measure* if for any two positively separated subsets $E, F \subset X$

$$\nu(E \cup F) = \nu(E) + \nu(F). \tag{3}$$

Remark 2.13. Implicitly, the assumption that two sets E and F are positively separated implies that both sets are non-empty.

Lemma 2.14. *Let ν be a metric outer measure on (X, d) and let $\{A_j\}_{j=1}^{\infty}$ be an increasing chain of subsets of X with $A = \lim_{j\to\infty} A_j$ such that $d(A_j, A \backslash A_{j+1}) > 0$ for each j. Then*

$$\lim_{j\to\infty} \nu(A_j) = \nu(A).$$

Proof. Note by Theorem 2.6 A exists and $A = \bigcup_{j=1}^{\infty} A_j$. Furthermore $\lim_{j\to\infty} \nu(A_j)$ exists but may be ∞ as by Definition 2.8 (ii) together with the fact that $\{A_j\}_{j=1}^{\infty}$ is an increasing chain the sequence $\{\nu(A_j)\}_{j=1}^{\infty}$ is monotonically increasing. It is enough to show that

$$\lim_{j\to\infty} \nu(A_j) \geq \nu(A). \tag{4}$$

This follows from the fact that since $\{A_j\}_{j=1}^{\infty}$ is a chain, $A_j \subset A$ for every j, and consequently by Definition 2.8 (ii)

$$\lim_{j\to\infty} \nu(A_j) \leq \nu(A).$$

Let $B_j = A_{j+1} \backslash A_j$, for any j we may express A as

$$A = \bigcup_{k=1}^{\infty} A_j = A_j \cup \bigcup_{k=j+1}^{\infty} B_k.$$

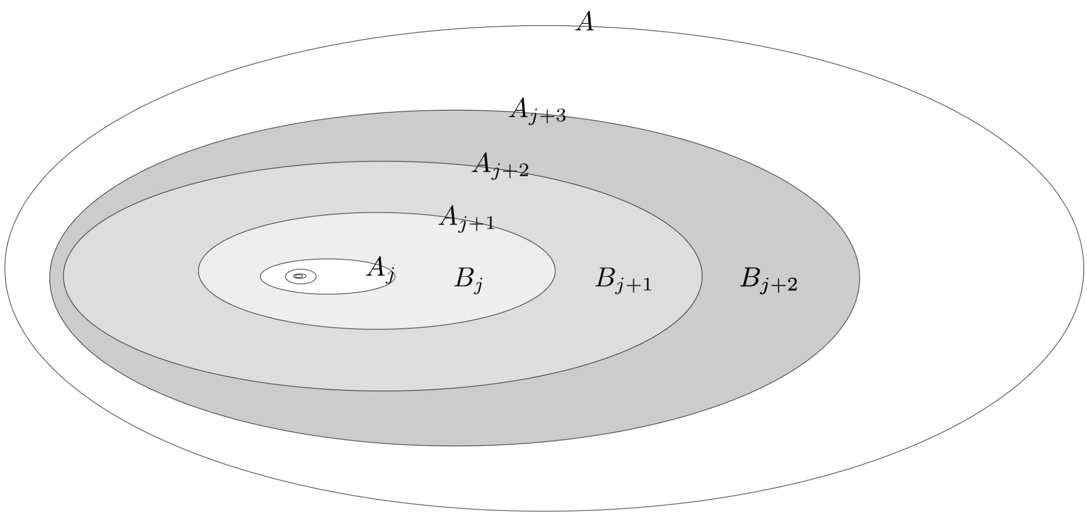

Figure 2: The B's and A's.

Note that by the condition $d(A_j, A\backslash A_{j+1}) > 0$, since $B_j \subset A_{j+1}$ and $B_{j+2} \subset A\backslash A_{j+2}$ it follows that B_i and B_j are positively separated if i and j are both even or i and j are both odd.

Using Definition 2.8 (iii)

$$\nu(A) = \nu\left(A_j \cup \bigcup_{k=j+1}^{\infty} B_k\right) \leq \nu(A_j) + \sum_{k=j+1}^{\infty} \nu(B_k). \tag{5}$$

Since the sum in the right hand side of Inequality (5) is a sum of non-negative terms it either converges or is ∞. We consider the following cases; either the sum on the right converges for all j or there exists a j such that the sum on the right is ∞.

Suppose the sum converges for every j, our aim is to show that the sum goes to 0 as $j \to \infty$. Let $\varepsilon > 0$, write

$$\sum_{k=j+1}^{\infty} \nu(B_k) = -\sum_{k=1}^{j} \nu(B_k) + \sum_{k=1}^{j} \nu(B_k) + \sum_{k=j+1}^{\infty} \nu(B_k) = -\sum_{k=1}^{j} \nu(B_k) + \sum_{k=1}^{\infty} \nu(B_k).$$

Now since the series $\sum_{k=j+1}^{\infty} \nu(B_k)$ converges for every j the sum $\sum_{k=1}^{\infty} \nu(B_k)$ converges. Consequently for every $\varepsilon > 0$ there is an N such that for all $j \geq N$

$$\left| \sum_{k=1}^{\infty} \nu(B_k) - \sum_{k=1}^{j} \nu(B_k) \right| < \varepsilon.$$

Therefore $\sum_{k=j+1}^{\infty} \nu(B_k) \to 0$ as $j \to \infty$ so taking $j \to \infty$ in Inequality (5) shows Inequality (4).

Now suppose the sum in the right hand side of Inequality (5) is ∞ for some j. Then by definition for every $2C \in \mathbb{R}$ there is some N such that for all $n \geq N$

$$\sum_{k=j+1}^{n} \nu(B_k) > 2C$$

$$\Leftrightarrow \sum_{\substack{k=j+1 \\ k \text{ odd}}}^{n} \nu(B_k) + \sum_{\substack{k=j+1 \\ k \text{ even}}}^{n} \nu(B_k) > 2C.$$

The odd B_k are positively separated and the even B_k are positively separated. Therefore since ν is a metric outer measure it follows from Definition 2.12 that

$$\sum_{\substack{k=j+1 \\ k \text{ odd}}}^{n} \nu(B_k) + \sum_{\substack{k=j+1 \\ k \text{ even}}}^{n} \nu(B_k) = \nu\left(\bigcup_{\substack{k=j+1 \\ k \text{ odd}}}^{n} B_k \right) + \nu\left(\bigcup_{\substack{k=j+1 \\ k \text{ even}}}^{n} B_k \right)$$

and finally since

$$\bigcup_{\substack{k=j+1 \\ k \text{ odd}}}^{n} B_k \subset A_{n+1} \quad \text{and} \quad \bigcup_{\substack{k=j+1 \\ k \text{ odd}}}^{n} B_k \subset A_{n+1}$$

by Definition 2.8 (ii)

$$\nu(A_{n+1}) + \nu(A_{n+1}) > 2C$$

$$\Leftrightarrow \nu(A_{n+1}) > C.$$

Since for every $2C$ there is a corresponding N we conclude that $\lim_{j\to\infty} \nu(A_j) = \infty$ and then certainly Inequality (4) holds. $\square$

Theorem 2.15 (Borel subsets are measurable)**.** *If ν is a metric outer measure on (X, d) then all Borel subsets of X are ν-measurable.*

Proof. The ν-measurable sets form a σ-field (Theorem 2.10), and the Borel subsets form the smallest σ-field containing the closed subsets of X. It is therefore sufficient to show Equation (1) when E is a closed set of X, since if each closed E is in the σ-field formed by the ν-measurable sets, the countable unions and complementations of the closed sets are in this σ-field. This is precisely the Borel set so each Borel subset is in the σ-field formed by the ν-measurable sets, that is each Borel subset is ν-measurable.

The aim is to show that Inequality (2) holds for any set A and any closed set E. If either $A = \emptyset$, $E = \emptyset$, $A\backslash E = \emptyset$ or $A \cap E = \emptyset$ the inequality holds by Definition 2.8 so moving forward assume each of these sets are non-empty. Let A_j be the points in $A\backslash E$ of distance at least $1/j$ from E. We may assume that each A_j is non-empty as if the sets are empty for finitely many j, the arguments used can be applied starting at the maximum of these j plus 1. If the sets are empty for infinitely many j then it follows that all points of $A\backslash E$ have distance 0 from E. This contradicts the fact that E^c is open hence it contradicts that E is closed.

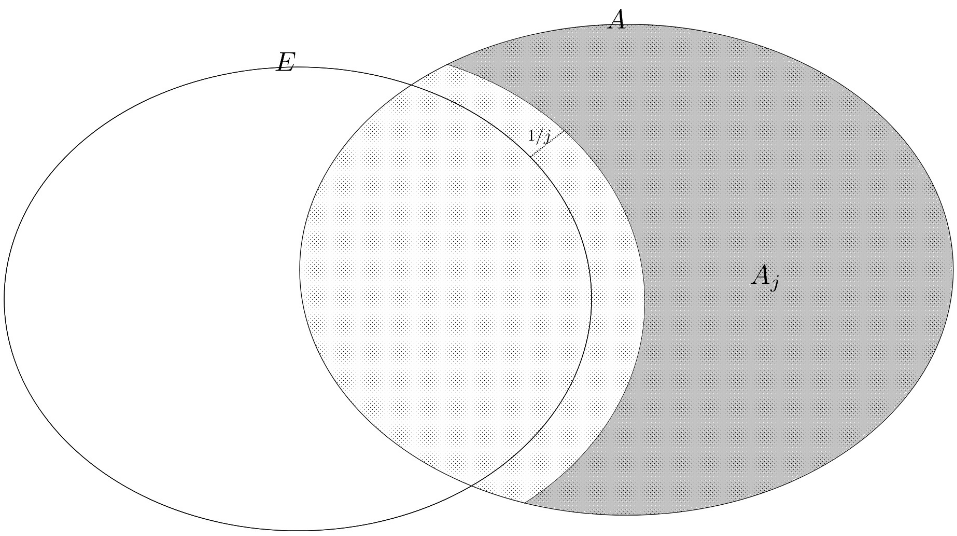

Figure 3: A and A_j.

It follows from the definition of A_j that $d(A \cap E, A_j) \geq 1/j$, so by Definition 2.12

$$\nu(A \cap E) + \nu(A_j) = \nu((A \cap E) \cup A_j) \leq \nu(A) \tag{6}$$

the last inequality coming from $(A \cap E) \cup A_j \subset A$ with Definition 2.8 (ii). Since E is closed its complement E^c is open. Therefore for every point $x \in E^c$ there is some $r > 0$ such that

$$\{y \in X : d(x, y) < r\} \subset E^c$$

$$\Leftrightarrow \{y \in X : d(x, y) < r\} \cap E = \emptyset$$

$$\Rightarrow d(x, z) \geq r \quad \forall z \in E.$$

From this we conclude that if $x \in A \backslash E$ then the distance from x to E is at least $1/j$ for some j i.e $x \in A_j$ for some j. So $A \backslash E \subset \bigcup_{j=1}^{\infty} A_j$ and by construction $\bigcup_{j=1}^{\infty} A_j \subset A \backslash E$. Therefore $A \backslash E = \bigcup_{j=1}^{\infty} A_j$.

We first consider the case when $(A \backslash E) \backslash A_{j+1}$ is non-empty for all $j + 1$, the goal is to show that $d(A_j, (A \backslash E) \backslash A_{j+1}) > 0$ for all j.

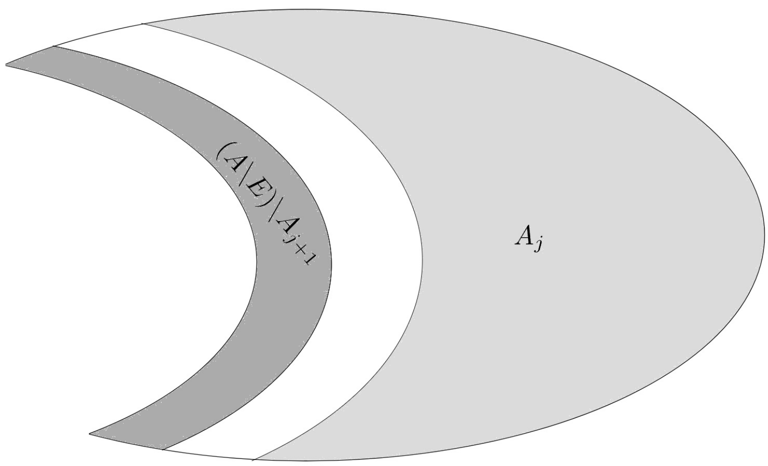

Figure 4: A_j and $(A \backslash E) \backslash A_{j+1}$.

Let $x \in (A \backslash E) \backslash A_{j+1}$, then there must exist some $z \in E$ such that $d(x, z) < 1/(j + 1)$, otherwise the distance from x to E would be greater than or equal to $1/(j + 1)$ that is $x \in A_{j+1}$. Let $y \in A_j$ then $d(y, z) \geq 1/j$ by construction. By the triangle inequality

$$d(x,y) \geq d(y,z) - d(x,z) > 1/j - 1/(j+1) > 0$$

so $d(A_j, (A\backslash E)\backslash A_{j+1}) > 0$. We are now in position to apply Lemma 2.14: $\nu(A\backslash E) = \lim_{j\to\infty} \nu(A_j)$. Plugging this into Inequality (6)

$$\nu(A \cap E) + \nu(A\backslash E) = \nu(A \cap E) + \lim_{j\to\infty} \nu(A_j) \leq \nu(A)$$

showing Inequality (2) and therefore E is ν-measurable.

Now should $(A\backslash E)\backslash A_{j+1}$ be empty for some $j+1$ then $A\backslash E \subset A_{j+1}$ and since $A_{j+1} \subset A\backslash E$ it follows that $A\backslash E = A_{j+1}$. Using this with Inequality (6)

$$\nu(A \cap E) + \nu(A\backslash E) = \nu(A \cap E) + \nu(A_{j+1}) \leq \nu(A),$$

hence Inequality (2) holds in this case. □

2.4 The Hausdorff measure

In this section we introduce a class of outer measures, the Hausdorff outer measures, measuring sets of $\mathbb{R}^n$. We shall see that these outer measures can be used to extend our notion of dimension from the positive integers to the non-negative real numbers.

Definition 2.16 (Diameter of a set)**.** Let U be a non-empty subset of $\mathbb{R}^n$ and define the *diameter* of U as

$$|U| = \sup\{|x-y| : x, y \in U\}.$$

If $E \subset \bigcup_i U_i$ where $0 < |U_i| \leq \delta$, we say that $\{U_i\}_i$ is a δ-cover of E.

Definition 2.17 (Hausdorff s-dimensional outer measure)**.** Let E be a subset of $\mathbb{R}^n$ and let $s \geq 0$. For $\delta > 0$ define

$$\mathscr{H}^s_\delta(E) = \inf \sum_i |U_i|^s,$$

the infimum taken over all countable δ-covers $\{U_i\}_i$. We define the *Hausdorff s-dimensional outer measure* of E as

$$\mathscr{H}^s(E) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(E)$$

the limit taken from positive δ to zero.

Remark 2.18. One point to make in regards to Definition 2.17 is that a subset U_j in a δ-cover $\{U_i\}_i$ of E may contain no points of E. However when calculating $\mathscr{H}^s_\delta(E)$ we may disregard such subsets as $\{U_i\}_i \backslash U_j$ is still a δ-cover and the sum in Definition 2.17 does not increase by removing a set so the infimum will be the same. Therefore at times it is convenient to only consider δ-covers of E whose sets share at least one point with E.

Remark 2.19. Another point to make regarding Definition 2.17 is that a set U_j in a δ-cover $\{U_i\}_i$ lies in a convex set of the same diameter, the convex hull of U_j. It also lies in a closed set of the same diameter, the closure of U_j. Therefore it is possible to only consider closed sets, convex sets or closed and convex sets when calculating $\mathscr{H}^s_\delta(E)$ of some set E which may be convenient at times.

Theorem 2.20. *Let $E \subset \mathbb{R}^n$. As a function of δ, $\mathscr{H}^s_\delta(E)$ is non-increasing and as a consequence $\mathscr{H}^s(E)$ always exists in $[0, \infty]$. Moreover as a function of s, $\mathscr{H}^s(E)$ is non-increasing.*

Proof. Suppose $0 < \delta_1 < \delta_2$. If $\{U_i\}_i$ is a δ_1-cover of E then it is immediate from Definition 2.16 that it is a δ_2-cover of E, as if any U_i satisifes that $0 < |U_i| \leq \delta_1$ then $0 < |U_i| \leq \delta_2$. Therefore the set of all δ_1-covers of E is a subset of the set of all δ_2 -covers of E implying that

$$\mathscr{H}^s_{\delta_1}(E) = \inf_{\delta_1\text{-covers of } E} \sum_i |U_i|^s \geq \inf_{\delta_2\text{-covers of } E} \sum_i |U_i|^s = \mathscr{H}^s_{\delta_2}(E).$$

Consequently as $\delta \to 0$, $\mathscr{H}^s_\delta(E)$ does not decrease so by monotonicity $\lim_{\delta\to 0} \mathscr{H}^s_\delta(E) = \mathscr{H}^s(E)$ exists but may be ∞.

Take $\delta \leq 1$, if $s < t$ and U_i is any set of a δ-cover then $|U_i|^s \geq |U_i|^t$. Therefore

$$\mathscr{H}^s_\delta(E) = \inf \sum_i |U_i|^s \geq \inf \sum_i |U_i|^t = \mathscr{H}^t_\delta(E)$$

as this inequality holds for every $\delta \leq 1$ it holds taking $\delta \to 0$. □

Theorem 2.21. *For any $s \geq 0$ and $\delta > 0$ $\mathscr{H}^s_\delta$ is an outer measure.*

Proof. Let s and δ be fixed, clearly the range of $\mathscr{H}^s_\delta$ is $[0, \infty]$ so we show that $\mathscr{H}^s_\delta$ defines an outer measure on $\mathbb{R}^n$ by checking conditions (i), (ii) and (iii) of Definition 2.8. Any set is a cover of the empty set, and as there for any $\varepsilon > 0$ are non-empty subsets U with $|U| = \varepsilon$, the infimum of the covers must certainly be zero showing (i) of Definition 2.8.

If $A \subset A'$ then any δ-cover of A' is a δ-cover of A. Therefore the set of δ-covers of A' is a subset of the set of δ-covers of A so

$$\mathscr{H}^s_\delta(A) = \inf_{\delta\text{-cover of } A} \sum_i |U_i|^s \leq \inf_{\delta\text{-cover of } A'} \sum_i |U_i|^s = \mathscr{H}^s_\delta(A')$$

showing (ii) of Definition 2.8.

Next consider $\bigcup_i A_i = A$. If $\sum_i \mathscr{H}^s_\delta(A_i) = \infty$

$$\mathscr{H}^s_\delta(A) \leq \infty = \sum_i \mathscr{H}^s_\delta(A_i)$$

so moving forward assume $\sum_i \mathscr{H}^s_\delta(A_i) < \infty$. If $\{V_{ij}\}_j$ is any δ-cover of A_i then $\{V_{ij}\}_{i,j}$ is a δ-cover of A implying that

$$\mathscr{H}^s_\delta(A) \leq \sum_i \sum_j |V_{ij}|^s. \tag{7}$$

Now let $\varepsilon > 0$, for every A_i there is some δ-cover $\{V_{ij}\}_j$ such that

$$\sum_j |V_{ij}|^s - \mathscr{H}^s_\delta(A_i) < 2^{-i}\epsilon \tag{8}$$

as if not then

$$\sum_j |V_{ij}|^s \geq 2^{-i}\varepsilon + \inf_{\delta\text{-covers of } A_i} \sum_j |U_j|^s$$

for all δ-covers $\{V_{ij}\}_j$ of A_i. But then the infimum on the right hand side is not the greatest lower bound hence it can't be the infimum. Consequently since Inequality (7) holds for every A_i with any δ-cover $\{V_{ij}\}_j$ using Inequality (8)

$$\mathscr{H}^s_\delta(A) < \sum_i \mathscr{H}^s_\delta(A_i) + 2^{-i}\varepsilon \leq \sum_i \mathscr{H}^s_\delta(A_i) + \varepsilon \sum_{i=1}^{\infty} 2^{-i} = \sum_i \mathscr{H}^s_\delta(A_i) + \varepsilon.$$

It is allowed to split the sum since the assumption that $\sum_i \mathscr{H}^s_\delta(A_i) < \infty$ implies that it converges by monotonicity. Since this holds for any $\varepsilon > 0$ this shows (iii) of Definition 2.8. □

Theorem 2.22. *For any $s \geq 0$ the Hausdorff s-dimensional outer measure $\mathscr{H}^s$ is a metric outer measure.*

Proof. Let s be fixed. Clearly the range of $\mathscr{H}^s$ is $[0,\infty]$ as $\mathscr{H}^s_\delta$ has range $[0,\infty]$ for any $\delta > 0$. By Theorem 2.21 for any $\delta > 0$, $\mathscr{H}^s_\delta$ is an outer measure so it satisfies (i), (ii) and (iii) of Definition 2.8:

(i) $\mathscr{H}^s_\delta(\emptyset) = 0$,

(ii) $\mathscr{H}^s_\delta(A) \leq \mathscr{H}^s_\delta(A')$ if $A \subset A'$,

(iii) $\mathscr{H}^s_\delta(\bigcup_i A_i) \leq \sum_i \mathscr{H}^s_\delta(A_i)$.

Taking $\delta \to 0$ in the equality of (i) and in the inequality of (ii) shows that $\mathscr{H}^s$ satisfies (i) and (ii) of Definition 2.8. By the monotonicity of $\mathscr{H}^s_\delta$, Theorem 2.20, $\sum_i \mathscr{H}^s_\delta(A_i) \leq \sum_i \mathscr{H}^s(A_i)$ so by inequality (iii)

$$\mathscr{H}^s_\delta\left(\bigcup_i A_i\right) \leq \sum_i \mathscr{H}^s\left(A_i\right)$$

and taking $\delta \to 0$ shows that $\mathscr{H}^s$ satisfies inequality (iii) of Definition 2.8.

To show that $\mathscr{H}^s$ is a metric outer measure we will only consider δ-covers whose sets intersect the covered set by at least one point as prompted by Remark 2.18. Assume we have two positively separated sets E and F such that $d(E,F) = r > 0$, both assumed to be non-empty, Remark 2.13 . If $0 < \delta \leq \frac{r}{4}$ any δ-cover of E is disjoint with any δ-cover of F as if U is any set in the δ-cover of E and V is any set in the δ-cover of F then U and V are positively separated; let $u \in U$ and $v \in V$ be arbitrary and let $e \in E$ and $f \in F$ be points of E and F lying in U and V respectively. By triangle inequalities

$$d(u,v) \geq d(v,e) - d(u,e) \quad \text{and} \quad d(v,e) \geq d(e,f) - d(v,f)$$

therefore

$$d(u,v) \geq d(e,f) - d(v,f) - d(u,e) \geq r - \frac{r}{4} - \frac{r}{4} = \frac{r}{2} > 0$$

so U and V are positively separated.

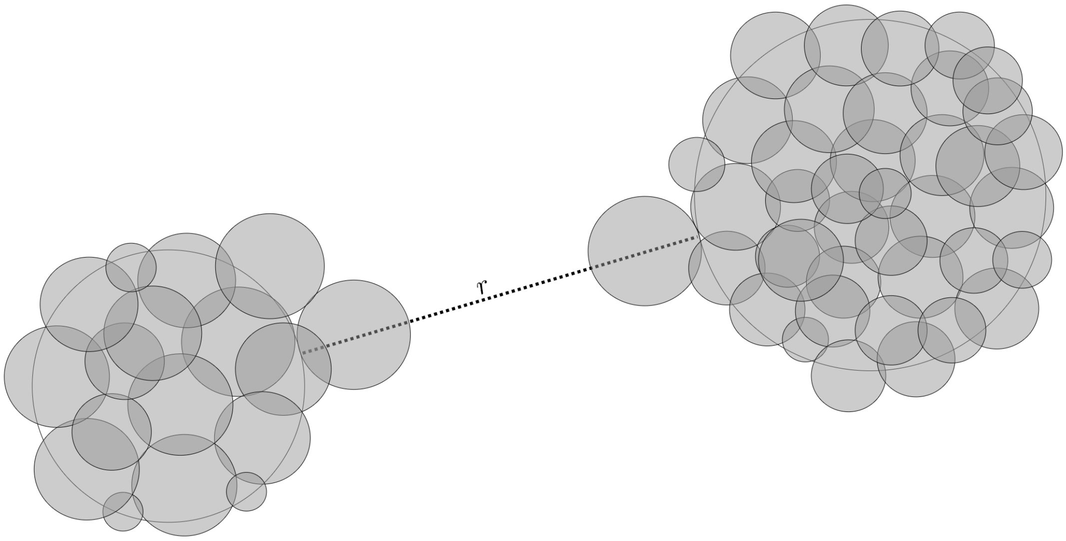

Figure 5: $\delta-$covers of E and F are disjoint.

As a consequence if $\{W_i\}_i$ is a δ-cover of $E \cup F$ then each set W_i in the δ-cover is either part of some δ-cover of E or it is a part of some δ-cover of F, not both. Therefore counting by δ-covers $\{U_i\}_i$ and $\{V_i\}_i$ of E and F,

$$\sum_i |W_i|^s = \sum_i |U_i|^s + \sum_i |V_i|^s.$$

Now taking the infimum over all $\{W_i\}_i$ we may take the infimum over all $\{U_i\}_i$ and $\{V_i\}_i$. As the sum with $\{U_i\}_i$ does not depend on choices of $\{V_i\}_i$ and vice versa it follows that

$$\inf_{\{U_i\} \text{ and } \{V_i\}} \sum_i |U_i|^s + \sum_i |V_i|^s = \inf_{\{U_i\}} \sum_i |U_i|^s + \inf_{\{V_i\}} \sum_i |V_i|^s$$

so

$$\mathscr{H}^s_\delta(E \cup F) = \mathscr{H}^s_\delta(E) + \mathscr{H}^s_\delta(F).$$

As this holds for any δ small enough it holds when $\delta \to 0$. Therefore $\mathscr{H}^s$ satisfies Equation (3) in Definition 2.12 so it is a metric outer measure. □

Corollary 2.23. *Every Borel subset of $\mathbb{R}^n$ is measurable with respect to $\mathscr{H}^s$.*

Proof. It follows immediately from Theorem 2.22 together with Theorem 2.15. □

Lemma 2.24. *Let $E \subset \mathbb{R}^n$, suppose there is some s such that*

$$0 < \mathscr{H}^s(E) < \infty$$

then this s is unique.

Proof. If $s < t$ then for any $\delta > 0$

$$\mathscr{H}^s_\delta(E) = \inf \sum_i |U_i|^s = \delta^{s-t} \cdot \delta^{t-s} \inf \sum_i |U_i|^s = \delta^{s-t} \inf \sum_i \delta^{t-s} |U_i|^s$$

$$\geq \delta^{s-t} \inf \sum_i |U_i|^t = \delta^{s-t} \mathscr{H}^t_\delta(E),$$

assuming that $\mathscr{H}^s_\delta(E) < \infty$. The last inequality comes from the fact that if $|U_i| \leq \delta$ then $\delta^{t-s} \geq |U_i|^{t-s}$ as $t - s > 0$. Also the inequality $\mathscr{H}^s_\delta(E) \geq \delta^{s-t}\mathscr{H}^t_\delta(E)$ holds when $\mathscr{H}^s_\delta(E) = \infty$ so the inequality holds in general.

Let s be a number such that $0 < \mathscr{H}^s(E) < \infty$ and let r and t be such that $r < s < t$. By the above inequality

$$\mathscr{H}^s_\delta(E) \geq \delta^{s-t} \mathscr{H}^t_\delta(E), \tag{9}$$

$$\mathscr{H}^r_\delta(E) \geq \delta^{r-s} \mathscr{H}^s_\delta(E). \tag{10}$$

Now take $\delta \to 0$, as the Inequalities (9) and (10) hold for any $\delta > 0$ they still hold. Since $\delta^{s-t} \to \infty$ as $\delta \to 0$ it must be that $\mathscr{H}^t(E) = 0$ for (9) to hold. Similarly $\delta^{r-s} \to \infty$ and therefore $\delta^{r-s}\mathscr{H}^s_\delta(E) \to \infty$ since $\mathscr{H}^s(E) > 0$ so for (10) to hold it must be that $\mathscr{H}^r(E) = \infty$. □

Theorem 2.25 (Hausdorff dimension)**.** *Let $E \subset \mathbb{R}^n$. There is a unique value, dim E, called the Hausdorff dimension of E, such that*

$$\mathscr{H}^s(E) = \infty \text{ for all } s \text{ such that } 0 \leq s < \dim E \text{ and} \tag{11}$$

$$\mathscr{H}^s(E) = 0 \text{ for all } s \text{ such that } \dim E < s < \infty. \tag{12}$$

Proof. Suppose there is some d such that $0 < \mathscr{H}^d(E) < \infty$, then by Lemma 2.24 this d is unique, meaning for all other s, $\mathscr{H}^s(E)$ must be either 0 or ∞. By the monotonicity of $\mathscr{H}^s$, Theorem 2.20, it follows that $\mathscr{H}^s(E) = \infty$ for all $0 \leq s < d$ and $\mathscr{H}^s(E) = 0$ for all $d < s < \infty$. In this case d is the unique number satisfying

Equalities (11) and (12) so we take dim E as d.

Now suppose there is no such d, let

$$L = \{s \geq 0 : \mathscr{H}^s(E) = \infty\} \text{ and } U = \{s \geq 0 : \mathscr{H}^s(E) = 0\}.$$

By the monotonicity of $\mathscr{H}^s$ it follows that $l \leq u$ for all $l \in L$ and $u \in U$. Consider $\sup L$ and $\inf U$, it holds that $\inf U \geq l$ for every $l \in L$ otherwise it wouldn't be the greatest lower bound. We see that $\inf U$ is an upper bound of L hence it must hold that $\sup L \leq \inf U$.

We now show that $\sup L = \inf U$ by contraposition. If $\sup L < \inf U$ then there is some $\sup L < a < \inf U$. Clearly $a \notin L$ and $a \notin U$ as then $\sup L$ and $\inf U$ would not be upper and lower bounds respectively. But then $0 < \mathscr{H}^a(E) < \infty$, contradicting the assumption of no d.

Letting dim $E = \sup L = \inf U$ it is clear that it is the unique number satisfying Equalities (11) and (12). □

In Section 2.7 we attempt to connect the Hausdorff measure and dimension to our everyday interpretation of volume and dimension. This will be done through the Lebesgue measure, Section 2.5, and it will be possible to do so by the "equivalence" of the Hausdorff measure and Lebesgue measure shown in Section 2.6.

2.5 The Lebesgue measure

In the introduction, Section 1.1, a concept of calculating volume of any figure in terms of blocks, aligning with our everyday interpretation of volume, was described. The next definition will formalize this concept.

Definition 2.26 (The Lebesgue n-dimensional outer measure)**.** Let C be a coordinate block in $\mathbb{R}^n$ of the form

$$C = [a_1, b_1[\times[a_2, b_2[\times ... \times [a_n, b_n[$$

where $a_i < b_i$ for each i. Define the volume of C as

$$V(C) = (b_1 - a_1)(b_2 - a_2)...(b_n - a_n).$$

If $E \subset \mathbb{R}^n$ the *Lebesgue n-dimensional outer measure* of E is

$$\mathscr{L}^n(E) = \inf \sum_i V(C_i)$$

the infimum taken over all coverings of E by a sequence $\{C_i\}_i$ of blocks.

Theorem 2.27. *The Lebesgue n-dimensional outer measure is a metric outer measure.*

The proof that $\mathscr{L}^n$ is an outer measure can be done analogously to the proof that $\mathscr{H}^s_\delta$ is an outer measure, Theorem 2.21. Restricting oneself to δ-covers consisting only of blocks with the insight that a block of any diameter C_i can be partitioned into a sequence of blocks of diameter at most δ, $\{C_{ij}\}_{j=1}^n$, such that $V(C_i) = \sum_{j=1}^n V(C_{ij})$ enables one to apply the same arguments of Theorem 2.21 to this theorem. Also by this insight it is enough to consider covers of blocks having diameter less than δ for any $\delta > 0$, hence showing that $\mathscr{L}^n$ is a metric outer measure can be done analogously to Theorem 2.22.

2.6 $\mathscr{L}^n$ and $\mathscr{H}^n$ are "equivalent"

In this section we show that $\mathscr{L}^n$ and $\mathscr{H}^n$ are "equivalent" in the sense that if $E \subset \mathbb{R}^n$ then the Lebesgue n-dimensional outer measure of E equals the Hausdorff n-dimensional times some constant not dependent on E, Theorem 2.32.

Definition 2.28 (Vitali class)**.** A collection of sets $\mathscr{V}$ is called a *Vitali class* for E if for each $x \in E$ and each $\delta > 0$ there exists $U \in \mathscr{V}$ with $x \in U$ and $0 < |U| \leq \delta$.

An example of a Vitali class for a set E are all the closed balls of radius smaller than 1 centered at a point $x \in E$, for every $x \in E$.

Theorem 2.29. *Let E be a $\mathscr{H}^s$-measurable subset of $\mathbb{R}^n$ and let $\mathscr{V}$ be a Vitali class of closed sets of E. Then we may select a countable disjoint sequence $\{U_i\}_i$ from $\mathscr{V}$ such that either $\sum_i |U_i|^s = \infty$ or $\mathscr{H}^s(E \backslash \bigcup_i U_i) = 0$.*

Proof. We choose the sequence $\{U_i\}_i$ inductively. Suppose that $U_1, ..., U_m$ have been chosen and let d_m be the supremum of $|U|$ taken over all $U \in \mathscr{V}$ disjoint from $U_1, ..., U_m$ i.e

$$d_m = \sup \left\{ |U| : U \in \mathscr{V}, U \cap \bigcup_{i=1}^m U_m = \emptyset \right\}.$$

The supremum d_m is well-defined if the set we are taking supremum of is non-empty. If the set is non-empty then d_m is positive, if not let $d_m = 0$.

If $d_m = 0$ then $E \subset \bigcup_{i=1}^m U_i$ since suppose there is some $x \in E$ such that $x \notin \bigcup_{i=1}^m U_i$. Let r be the distance from x to $\bigcup_{i=1}^m U_i$, since $\bigcup_{i=1}^m U_i$ is closed $r > 0$. Now since $\mathscr{V}$ is a Vitali class of E there is some U with $x \in U$ and $0 < |U| \leq r/2$. Clearly U must be disjoint from $\bigcup_{i=1}^m U_i$ as every point of U has distance less than or equal to $r/2$ from x and every point of $\bigcup_{i=1}^m U_i$ has distance greater than or equal to r from x. The fact that U is disjoint from $\bigcup_{i=1}^m U_i$ contradicts the assumption that $d_m = 0$ since the set

$$\left\{U \in \mathscr{V} : U \cap \bigcup_{i=1}^{m} U_m = \emptyset\right\}$$

would not be empty. So in the case $d_m = 0$ we are done since $E \backslash \bigcup_{i=1}^m U_i = \emptyset$ and $\mathscr{H}^s(E \backslash \bigcup_{i=1}^m U_i) = \mathscr{H}^s(\emptyset) = 0$. We have found a sequence $\{U_i\}_i$ as required.

If $d_m \neq 0$ then let U_{m+1} be a set in $\mathscr{V}$ disjoint from $\bigcup_{i=1}^m U_i$ such that $|U_{m+1}| \geq \frac{1}{2} d_m$.

Suppose the process continues indefinitely and that $\sum_{i=1}^\infty |U_i|^s < \infty$. For each i let B_i be a ball with center lying in U_i and with radius $3|U_i|$. We note that $|U_i| \to 0$ since $\sum_{i=1}^\infty |U_i|^s < \infty$ and so $|B_i| \to 0$ also. Now claim for every $k > 1$

$$E \backslash \bigcup_{i=1}^{k} U_i \subset \bigcup_{i=k+1}^{\infty} B_i. \tag{13}$$

We show the claim. If $x \in E \backslash \bigcup_{i=1}^k U_i$ there exists some $U \in \mathscr{V}$ not intersecting $U_1, ..., U_k$ with $x \in U$ by an analogous argument to the one used to show $E \subset \bigcup_{i=1}^m U_i$ if $d_m = 0$. Since $|U_i| \to 0$, $|U| > 2|U_m|$ for some m. Given the method for selecting U_i, U must intersect U_i for some $k < i < m$, if not then

$$U \in \left\{V \in \mathscr{V} : V \cap \bigcup_{i=1}^{m-1} U_i = \emptyset\right\}$$

and consequently $|U| \leq d_{m-1}$. The fact that $|U_m| \geq \frac{1}{2} d_{m-1}$ together with $|U| > 2|U_m|$ leads to the contradiction. Furthermore for the U_i intersecting U we can assume that $|U| \leq 2|U_i|$ as if not then replacing m with i we can repeat the argument, and if at no point we would get a U_i intersecting U with $|U| \leq 2|U_i|$ then eventually we would be at the point where there must be a $k < i < k+1$ such that U intersects U_i and clearly there is no such integer i. Furthermore $U \subset B_i$ as if $u \in U$, y is the center of B_i, and z is a point in the intersection of U and U_i then

$$d(u,y) \le d(u,z) + d(z,y) \le |U| + |U_i| \le 2|U_i| + |U_i| = 3|U_i|$$

showing that u is within a radius of B_i of the center y, i.e $u \in B_i$. By this we conclude that $x \in B_i$. Since we can find a B_i with $i > k$ for each x Inclusion (13) follows.

Thus if $\delta > 0$, since $\mathscr{H}^s_\delta$ is an outer measure

$$\mathscr{H}^s_\delta\left(E \setminus \bigcup_{i=1}^{\infty} U_i\right) \le \mathscr{H}^s_\delta\left(E \setminus \bigcup_{i=1}^{k} U_i\right) \le \mathscr{H}^s_\delta\left(\bigcup_{i=k+1}^{\infty} B_i\right) \le \sum_{i=k+1}^{\infty} |B_i|^s = 6^s \sum_{i=k+1}^{\infty} |U_i|^s$$

provided that k is large enough to ensure that $|B_i| \le \delta$ for $i > k$ which is possible since $|B_i| \to 0$. Since $\sum_{i=1}^{\infty} |U_i|^s < \infty$ it follows that taking $k \to \infty$, $\sum_{i=k+1}^{\infty} |U_i|^s \to 0$ showing that

$$\mathscr{H}^s_\delta\left(E \setminus \bigcup_{i=1}^{\infty} U_i\right) = 0.$$

Since this holds for arbitrary $\delta > 0$ it holds taking $\delta \to 0$.

□

Theorem 2.30. *The n-dimensional volume of a n-dimensional ball of diameter d is*

$$\frac{\pi^{\frac{1}{2}n}(\frac{1}{2}d)^n}{(\frac{1}{2}n)!}$$

where $!$ is the generalized factorial, $x! = \Gamma(x+1)$.

Proof of Theorem 2.30 is found in [Gip14].

Theorem 2.31. *The n-dimensional volume of a closed convex set of diameter d is at most the volume of a ball of diameter d.*

Proof of Theorem 2.31 is found in [EG15, Theorem 2.4, p.89].

Theorem 2.32 (Equivalence of $\mathscr{L}^n$ and $\mathscr{H}^n$)**.** *If $E \subset \mathbb{R}^n$ then $\mathscr{L}^n(E) = c_n \mathscr{H}^n(E)$ where $c_n = \pi^{\frac{1}{2}n} \cdot 2^{-n}/(\frac{1}{2}n)!$.*

Proof. Given $\varepsilon > 0$ we can cover E with a collection of closed convex sets $\{U_i\}_i$ such that $\sum_i |U_i|^n < \mathscr{H}^n(E) + \varepsilon$, a consequence of Remark 2.19. By Theorem 2.31

together with Theorem 2.30

$$\mathscr{L}^n(U_i) \leq \frac{\pi^{\frac{1}{2}n}(\frac{1}{2}|U_i|)^n}{(\frac{1}{2}n)!} = c_n|U_i|^n$$

so

$$\mathscr{L}^n(E) \leq \sum_i \mathscr{L}^n(U_i) < c_n\mathscr{H}^n(E) + c_n\varepsilon,$$

the first inequality coming from the fact that $\mathscr{L}^n$ is an outer measure, $\{U_i\}_i$ is a cover of E. Since we can find such a cover $\{U_i\}_i$ for every $\varepsilon > 0$ we conclude that

$$\mathscr{L}^n(E) \leq c_n\mathscr{H}^n(E).$$

For the other inequality let $\delta > 0$ and $\{C_i\}_i$ be a collection of coordinate blocks covering E with

$$\sum_i V(C_i) < \mathscr{L}^n(E) + \varepsilon. \tag{14}$$

We can assume these blocks are open by expanding them slightly while retaining the inequality. A constructive argument for this is done for intervals in $\mathbb{R}^1$ in Section 2.8 with the intervals J_i and I_i°.

For each point of a block C_i we can have balls of radius at most δ containing that point such that the balls are contained in C_i, since C_i is open. This forms a Vitali cover of C_i. By Theorem 2.29 there exist a countable sequence of disjoint balls $\{B_{ij}\}_j$ in C_i of diameter at most δ with $\mathscr{H}^n(C_i\backslash \bigcup_j B_{ij}) = 0$ and so $\mathscr{H}^n_\delta(C_i\backslash \bigcup_j B_{ij}) = 0$ by the monotonicity of $\mathscr{H}^n_\delta$. The balls are Borel subsets and Borel subsets are $\mathscr{L}^n$-measurable, Theorem 2.15, so by Theorem 2.10 $\mathscr{L}^n$ is a measure when restricted to these balls. Since the balls are disjoint $\sum_j \mathscr{L}^n(B_{ij}) = \mathscr{L}^n(\bigcup_j B_{ij})$ per Definition 2.7. Furthermore since $\mathscr{L}^n$ is an outer measure and $\bigcup_j B_{ij} \subset C_i$ it follows from Definition 2.8 (ii) that $\mathscr{L}^n(\bigcup_j B_{ij}) \leq \mathscr{L}^n(C_i)$ so we conclude the inequality

$$\sum_j \mathscr{L}^n(B_{ij}) \leq \mathscr{L}^n(C_i). \tag{15}$$

Now for a chain of inequalities,

$$\mathscr{H}^n_\delta(E) \leq \sum_i \mathscr{H}^n_\delta(C_i) \leq \sum_i \sum_j \mathscr{H}^n_\delta(B_{ij}) + \sum_i \mathscr{H}^n_\delta\left(C_i\backslash \bigcup_j B_{ij}\right),$$

the inequalities coming from the fact that $\mathscr{H}^n_\delta$ is an outer measure. Continuing the chain

$$\sum_i \sum_j \mathscr{H}^n_\delta(B_{ij}) + \sum_i \mathscr{H}^n_\delta\left(C_i \backslash \bigcup_j B_{ij}\right) \leq \sum_i \sum_j |B_{ij}|^n,$$

since $\mathscr{H}^n_\delta(C_i \backslash \bigcup_j^\infty B_{ij}) = 0$ and $\mathscr{H}^n_\delta(B_{ij}) \leq |B_{ij}|^n$, coming from the fact that the set B_{ij} is a δ-cover of B_{ij}. Now

$$\sum_i \sum_j |B_{ij}|^n = \sum_i \sum_j c_n^{-1} \mathscr{L}^n(B_{ij})$$

as $\mathscr{L}^n(B_{ij}) = \pi^{\frac{1}{2}n}(\frac{1}{2}|B_{ij}|)^n/(\frac{1}{2}n)!$ by Theorem 2.30. Continuing

$$\sum_i \sum_j c_n^{-1} \mathscr{L}^n(B_{ij}) \leq c_n^{-1} \sum_i \mathscr{L}^n(C_i) < c_n^{-1} \mathscr{L}^n(E) + c_n^{-1} \varepsilon$$

the first inequality by Inequality (15), the strict inequality by Inequality (14). All in all $c_n \mathscr{H}^n_\delta(E) < \mathscr{L}^n(E) + \varepsilon$ for every $\varepsilon > 0$ so $c_n \mathscr{H}^n_\delta(E) \leq \mathscr{L}^n(E)$. Since $\delta > 0$ was arbitrary we conclude that

$$c_n \mathscr{H}^n(E) \leq \mathscr{L}^n(E).$$

□

2.7 Interpreting the Hausdorff measure and dimension

By Theorem 2.32 our everyday interpretation of volume, the Lebesgue measure, is “equivalent” to the Hausdorff measure. The theorem essentially shows that choosing to measure something with the Hausdorff measure rather than the Lebesgue measure is analogous to choosing to measure something in metric units rather than imperial units.

One everyday interpretation of dimension is that some set has integer dimension n if scaling the lengths of a set by a factor of $r > 0$, that is considering the sets image of the function $f(x) = rx$, its volume increases by a factor of r^n. For example stretching the side lengths of a cube in $\mathbb{R}^3$ by a factor of 2 increases its volume by a factor of 2^3, the 3 corresponding to the dimension of a cube. The next proposition shows that the Hausdorff dimension aligns with this interpretation of dimension.

Proposition 2.33. *Let $r > 0$ and*

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

$$x \mapsto rx.$$

Then $\mathscr{H}^s(f(E)) = r^s \mathscr{H}^s(E)$.

Proof. Let $\{U_i\}_i$ be a δ-cover of E, then $\{f(U_i)\}_i$ is a $r\delta$-cover of $f(E)$. Similarly if $\{V_i\}_i$ is a $r\delta$-cover of $f(E)$ then $\{f^{-1}(V_i)\}_i$ is a δ-cover of E. Therefore f is a bijection between the δ-covers of E and the $r\delta$-covers of $f(E)$. Also for any set U_i, the diameter of the image of the set is r times the diameter of the set, $|f(U_i)| = r|U_i|$. Combining these two conclusions

$$\mathscr{H}^s_{r\delta}(f(E)) = \inf \sum_i |V_i|^s = \inf \sum_i |f(U_i)|^s$$

$$= \inf \sum_i (r|U_i|)^s = r^s \inf \sum_i |U_i|^s = r^s \mathscr{H}^s_\delta(E).$$

Taking $\delta \to 0$ we are done. □

2.8 Hausdorff measure and dimension of the Cantor set

This section is dedicated to calculating the Hausdorff measure and dimension of a set with non-integer Hausdorff dimension, the Cantor set.

Example 2.34. A set known as the Cantor set is created as follows: Let $E_0 = [0, 1]$, remove the middle third of E_0 to create

$$E_1 = [0/3, 1/3] \cup [2/3, 3/3].$$

From each constituent interval of E_1 remove the middle third to create

$$E_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 9/9].$$

Iterating this process, the Cantor set is $E = \bigcap_{j=0}^{\infty} E_j$.

The Cantor set is a Borel subset meaning that it should be measurable with respect to the Hausdorff measure per Corollary 2.23.

Figure 6: The Cantor set.

The Cantor set E is compact; let $x \in [0,1]\backslash E$, then $x \in [0,1]\backslash E_j$ for some j. Since $[0,1]\backslash E_j$ is open x is an interior point of $[0,1]\backslash E_j$ and since $[0,1]\backslash E_j \subset [0,1]\backslash E$ it follows that x is an interior point of $[0,1]\backslash E$. Now if $x \in \mathbb{R}\backslash[0,1]$ it is also an interior point, so any x in the complement of E is an interior point showing that E is closed. Of course E is also bounded so E is compact.

Using the interpretation of Hausdorff dimension from Section 2.7 we can get a feel for what the Hausdorff dimension of E should be. Consider three copies of an E_j, call them E_{j1}, E_{j2}, E_{j3}. By the construction of each E_j it is possible to take each constituent interval of E_{j2} and translate them to some of the gaps of E_{j1} and E_{j3} to form E_{j-1} but scaled with a factor of 2, call it $2E_{j-1}$. Figure 7 shows the situation for $j = 2$.

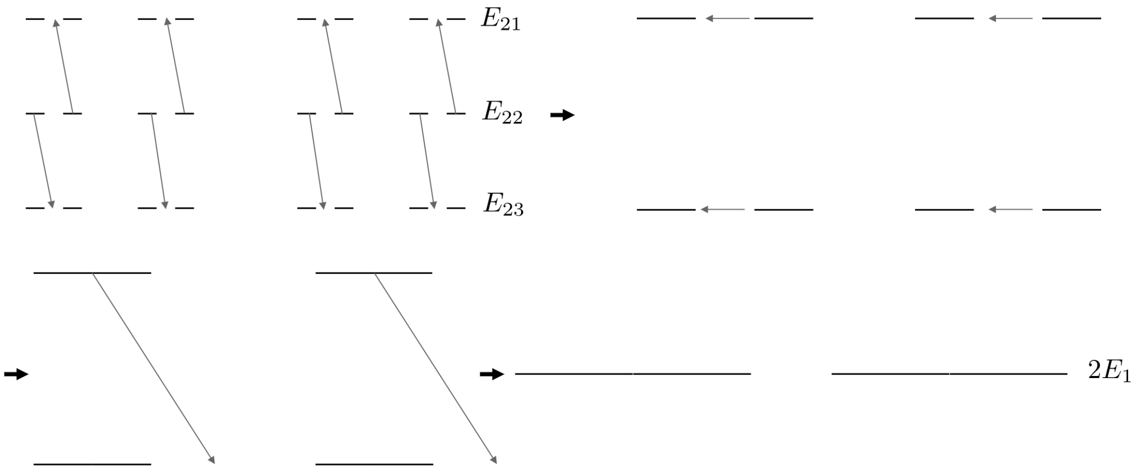

Figure 7: E_{21}, E_{22}, E_{23} and $2E_1$.

Therefore by Proposition 2.33 it seems reasonable that

$$3\mathscr{H}^s(E) = \mathscr{H}^s(2E)$$

$$\Leftrightarrow 3\mathscr{H}^s(E) = 2^s\mathscr{H}^s(E)$$
$$\Leftrightarrow s = \frac{\ln 2}{\ln 3}.$$

Let $s = \ln 2/\ln 3$, we aim to show that the Hausdorff dimension of E is s by showing that $\mathscr{H}^s(E) = 1$, as then s will be the Hausdorff dimension of E by Theorem 2.25 and its proof. Each E_j consists of 2^j constituent intervals of length 3^{-j}. Take the constituent intervals of an E_j, with $3^{-j} \leq \delta$, as a δ-cover of E, then

$$\mathscr{H}^s_\delta(E) \leq 2^j(3^{-j})^s = 2^j(3^{\ln 2/\ln 3})^{-j}.$$

By the logarithm rule $\log_a b \cdot \log_b c = \log_a c$. Taking $a = e, b = 3, c = 2$ yields that $\ln 2/\ln 3 = \log_3 2$ and therefore $3^{\ln 2/\ln 3} = 2$. This simplifies the inequality to

$$\mathscr{H}^s_\delta(E) \leq 2^j 2^{-j} = 1.$$

For every $\delta > 0$ there is an E_j such that the length of the constituent intervals of

E_j is less than δ so

$$\mathscr{H}^s(E) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(E) \leq 1$$

and one inequality is done.

For the other inequality we show that for any δ and any δ-cover $\{U_i\}_i$ of E that

$$\sum_i |U_i|^s \geq 1 - \varepsilon \tag{16}$$

for every $\varepsilon > 0$. By monotonicity the sum in Inequality (16) converges or is ∞. In the latter case Inequality (16) certainly holds so moving forward assume the sum converges. To show the inequality we ultimately replace the cover $\{U_i\}_i$ with the constituent intervals of some E_j in a way that the sum increases by at most ε.

Let $\{U_i\}_i$ be any δ-cover of E, each set U_i is contained in some closed interval $J_i = [a_i, b_i]$ with the same diameter, namely the closed convex hull of U_i, Remark 2.19. Clearly $\{J_i\}_i$ is a δ-cover of E. Replacing each U_i with these intervals J_i the sum does not increase as the diameter is the same. So $\sum_i |U_i|^s = \sum_i |J_i|^s$ and the latter sum converges as the former does.

Now let

$$I_i^\circ = \left] \frac{a_i + b_i}{2} - \frac{(b_i - a_i)}{2} \cdot (1 + c), \frac{a_i + b_i}{2} + \frac{(b_i - a_i)}{2} \cdot (1 + c) \right[$$

where

$$c = \frac{\varepsilon}{\sum_i |J_i|^s},$$

$c > 0$ since $\sum_i |J_i|^s < \infty$. Purposefully I_i° is not expressed in the most simplified form. By the way it is expressed we see that the left boundary of I_i° is the midpoint of J_i, minus half the length of J_i times some factor bigger than one. Similarly the right boundary is the midpoint of J_i plus half the length of J_i times some factor bigger than one. By this construction of I_i° it is clear that $J_i \subset I_i^\circ$ so $\{I_i^\circ\}_i$ is a cover of E. Also $|I_i^\circ| = (b_i - a_i) \cdot (1 + c)$.

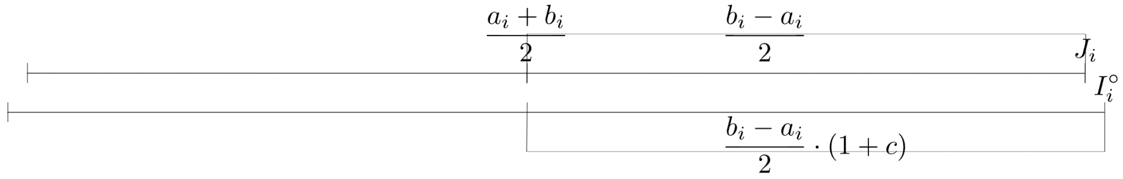

Figure 8: J_i and I_i°.

Since each I_i° is open, by the compactness of E we can extract a finite sub-cover$\{I_i^\circ\}_{i=1}^n$. Now let $I_i = \overline{I_i^\circ}$, then $\{I_i\}_{i=1}^n$ is a finite cover of E consisting of closed intervals where $|I_i| = (b_i - a_i)(1 + c)$. We compare $\sum_{i=1}^n |I_i|^s$ and $\sum_i |J_i|^s$,

$$\sum_{i=1}^n |I_i|^s = \sum_{i=1}^n (b_i - a_i)^s (1 + c)^s \le \sum_{i=1}^n (b_i - a_i)^s (1 + c)$$

$$= \sum_{i=1}^n (b_i - a_i)^s + c \sum_{i=1}^n (b_i - a_i)^s \le \sum_i (b_i - a_i)^s + c \sum_i (b_i - a_i)^s$$
$$= \sum_i |J_i|^s + c \sum_i |J_i|^s = \sum_i |J_i|^s + \varepsilon,$$

the first inequality coming from the fact that $(1 + c)^s \le (1 + c)$ since $s < 1$ and $(1 + c) > 1$. So far

$$\sum_i |U_i|^s \ge \sum_{i=1}^n |I_i| - \varepsilon.$$

Each set E_j in the construction of E consists of 2^j constituent intervals. If I is an interval in a cover of E then I covers two largest constituent intervals N and N', not necessarily from the same E_j. We may reduce each I_i in a finite cover of E consisting of closed intervals, $\{I_i\}_{i=1}^n$, such that it is a small as possible while containing some largest N and N', meaning there is some K in the complement of E such that $I_i = N \cup K \cup N'$. The situation is illustrated in Figure 9. This reduction will not increase the sum $\sum_{i=1}^n |I_i|^s$ and the collection of smallest intervals $\{I_i\}_{i=1}^n$ will still be a cover of E.

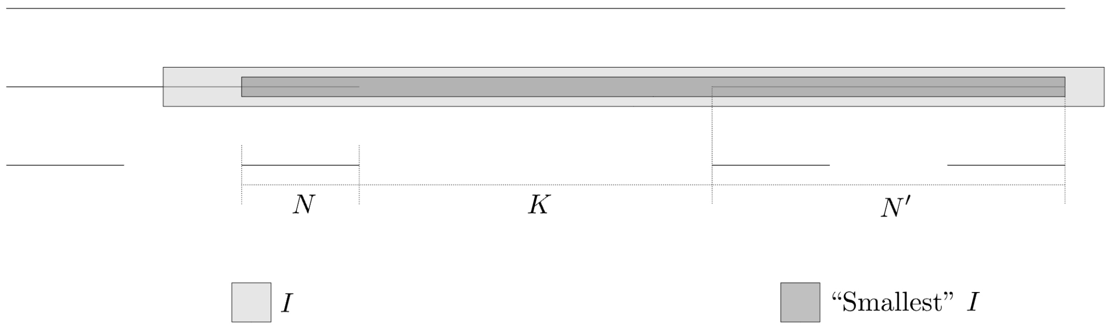

Figure 9: I, smallest I, N, K and N'.

By the construction of E this K satisfies that $|N|, |N'| \leq |K|$. Then by t^s being an increasing function with the fact that $3^s = 2$,

$$|I_i|^s = (|N| + |K| + |N'|)^s \geq \left(|N| + \frac{1}{2}|N| + \frac{1}{2}|N'| + |N'|\right)^s$$

$$= \left(\frac{3}{2}(|N| + |N|')\right)^s = 2 \cdot \left(\frac{1}{2}|N| + \frac{1}{2}|N'|\right)^s.$$

Now using the concavity of the function t^s

$$2 \cdot \left(\frac{1}{2}|N| + \frac{1}{2}|N'|\right)^s \geq 2 \cdot \frac{1}{2}(|N|^s + |N'|^s) = |N|^s + |N'|^s.$$

By this inequality we may replace each smallest interval I_i by two constituent intervals N and N' still covering E such that $\sum_{i=1}^n |I_i|^s$ does not increase. Iterate this process, but with the constituent intervals, a finite number of steps. Doing this for every interval in $\{I_i\}_{i=1}^n$ the cover $\{I_i\}_{i=1}^n$ can ultimately be replaced by all constituent intervals of an E_j, that is E_j itself, for some j, not increasing the sum. Each of the 2^j net intervals has diameter 3^{-j} so

$$\sum_{i=1}^n |I_i|^s \geq 2^j \cdot (3^{-j})^s = 1.$$

All in all

$$\sum_i |U_i|^s \geq 1 - \varepsilon$$

for every $\varepsilon > 0$ showing Inequality (16) holds. By this inequality

$$\mathscr{H}^s(E) \geq 1 - \varepsilon$$

for every $\varepsilon > 0$, together with the inequality $\mathscr{H}^s(E) \leq 1$ we conclude that $\mathscr{H}^s(E) = 1$.

3 Kakeya's needle problem

To recall a set in $\mathbb{R}^2$ in which a unit segment can be maneuvered inside the set to lie in its original position, but rotated through 180°, is denoted a Kakeya set (in $\mathbb{R}^2$). The Kakeya needle problem is to find a Kakeya set with as small area as possible. Also to recall a Besicovitch set is a set containing a unit segment in every direction. In the next section we will construct a Besicovitch set in $\mathbb{R}^2$ with zero area and a Kakeya set in $\mathbb{R}^2$ with arbitrarily small area.

3.1 Construction of Kakeya and Besicovitch sets

Lemma 3.1. *Let T_1 and T_2 be two adjacent triangles with bases on a line L of height h, both with base lengths b. These two triangles form a triangle $T_1 \cup T_2$. Take $\frac{1}{2} < \alpha < 1$. Sliding T_2 a distance $2(1-\alpha)b$ along L to overlap T_1 results in a figure S, consisting of one triangle T similar to $T_1 \cup T_2$ and two auxiliary triangles that look like ears. The area of triangle T is proportional to the area of $T_1 \cup T_2$ as follows*

$$\mathscr{L}^2(T) = \alpha^2 \mathscr{L}^2(T_1 \cup T_2),$$

and the reduction in area effected by replacing $T_1 \cup T_2$ with S is given by

$$\mathscr{L}^2(T_1 \cup T_2) - \mathscr{L}^2(S) = \mathscr{L}^2(T_1 \cup T_2)(1-\alpha)(3\alpha - 1).$$

Proof. The situation is illustrated in Figure 10.

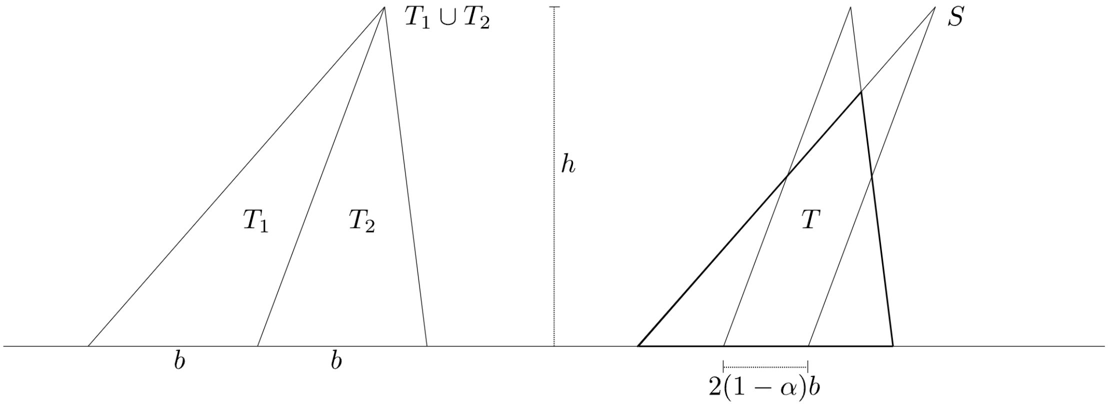

Figure 10: $T_1 \cup T_2$ and S.

The base of triangle T is

$$b + b - 2(1-\alpha)b = 2b\alpha$$

and the height, given by similarity to triangle $T_1 \cup T_2$, is

$$\frac{2b\alpha \cdot h}{2b} = \alpha h.$$

Therefore since triangle $T_1 \cup T_2$ has area $2bh/2 = bh$,

$$\mathscr{L}^2(T) = \frac{2b\alpha \cdot \alpha h}{2} = \alpha^2 \mathscr{L}^2(T_1 \cup T_2).$$

The reduction in area is precisely the area of the overlap; the leaning tower-like figure in the center of S. This tower-like figure consists of one Parallelogram P and one triangle Q, whose heights we label x and y respectively. We determine the areas of P and Q by determining x and y. Figure 11 shows P and Q along with a triangle T_2' similar to T_2 which will be used to determine x.

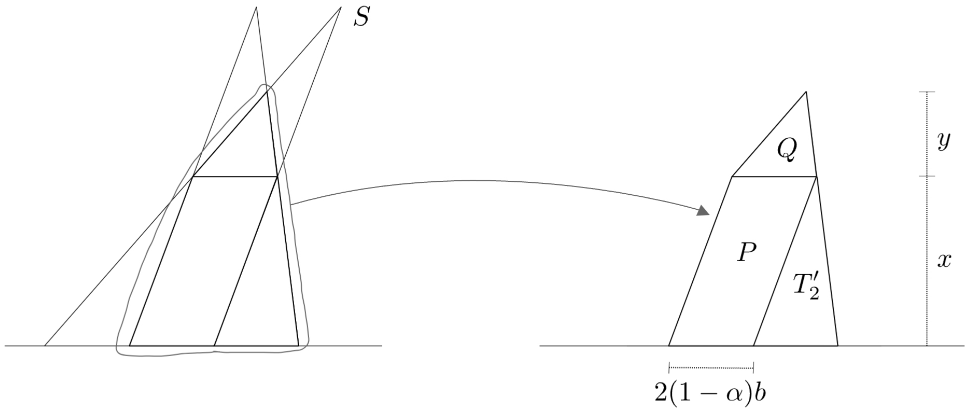

Figure 11: The leaning tower-like overlap with triangle T_2'.

Starting with x, the triangle T_2' has base $b - 2(1-\alpha)b = b(2\alpha - 1)$. By its similarity to T_2

$$x = \frac{b(2\alpha-1)h}{b} = (2\alpha - 1)h$$

yielding

$$\mathscr{L}^2(P) = 2(1-\alpha)b \cdot (2\alpha-1)h = 2bh(1-\alpha)(2\alpha-1).$$

As for y, triangle Q is similar to $T_1 \cup T_2$ since it is similar to T which is similar to $T_1 \cup T_2$. Therefore

$$y = \frac{2(1-\alpha)bh}{2b} = (1-\alpha)h$$

so

$$\mathscr{L}^2(Q) = \frac{2(1-\alpha)b \cdot (1-\alpha)h}{2} = bh(1-\alpha)^2.$$

Therefore the overlap, that is the reduction in area is,

$$\mathscr{L}^2(P) + \mathscr{L}^2(Q) = 2bh(1-\alpha)(2\alpha-1) + bh(1-\alpha)^2 = bh(1-\alpha)\Big(2(2\alpha-1) + (1-\alpha)\Big)$$

$$= bh(1-\alpha)(3\alpha-1) = \mathscr{L}^2(T_1 \cup T_2)(1-\alpha)(3\alpha-1).$$

$\square$

Lemma 3.2. *Let K be a compact set contained in an open set V. Then there is some $\varepsilon > 0$ such that translating any point of K a distance at most ε it will remain inside V after the translation.*

Proof. If V is an open set containing K then every point of K is an interior point of V. Therefore for every point $x \in K$ there is some open ball of x with positive radius $\rho(x)$, $N_{\rho(x)}(x)$, such that $N_{\rho(x)}(x) \subset V$. Also we may choose $\rho(x)$ such that $d(N_{\rho(x)}(x), V^c) > 0$. Consider the union of all these open balls $\bigcup_{x \in K} N_{\rho(x)}(x)$, this is an open cover of K which is contained in V. Since K is compact we may extract a finite sub cover, $\bigcup_{i=1}^{n} N_{\rho(x_i)}(x_i)$, which will also be contained in V. Now translating $\bigcup_{i=1}^{n} N_{\rho(x_i)}(x_i)$ a distance at most

$$\varepsilon = \min\{d(N_{\rho(x_i)}(x_i), V^c) : i = 1, 2, ..., n\}$$

its image will still be inside of V. Hence translating K a distance of at most ε it will remain inside V and so any point of K can be translated a distance at most ε while remaining inside V. □

Theorem 3.3. *Let T be a triangle with base on a line L. Divide the base of T into 2^k equal segments of length b, and join each point of division to the vertex opposite the base, forming 2^k constituent triangles $T_1, ..., T_{2^k}$, indexed from left to right. By choosing k large enough it is possible to translate these triangles along L to positions such that the area of the resulting (closed) figure S is as small as desired. Further, if V is an open set containing T, this may be achieved with $S \subset V$.*

Proof. Construction of this figure S involves repeated applications of Lemma 3.1 for a fixed value of α to be specified later.

For the first step, for each $1 \leq i \leq 2^{k-1}$ slide T_{2i} to the left along L such that it overlaps T_{2i-1}, where the overlap on L is $2(1-\alpha)b$. The resulting figure S_i^1 consists of a triangle $T_i{}^1$, similar to $T_{2i-1} \cup T_{2i}$, and two auxiliary triangles that look like ears, similarly to the figure of Lemma 3.1. By Lemma 3.1

$$\mathscr{L}^2(T_i^1) = \alpha^2 \mathscr{L}^2(T_{2i-1} \cup T_{2i}) \tag{17}$$

and the resulting overlap of T_{2i-1} and T_{2i} is $(1-\alpha)(3\alpha-1)\mathscr{L}^2(T_{2i-1} \cup T_{2i})$. Note that doing this for each i each resulting triangle T_i^1 will have the same base and height. They will have the same base because the bases among all T_j are equal, and each pair T_{2i-1} and T_{2i} overlap by the same amount. Consequently if each T_i^1 has the same base length they will have the same height by their similarity to their corresponding $T_{2i-1} \cup T_{2i}$, all $T_{2i-1} \cup T_{2i}$ have the same height. Furthermore each even indexed T_{2i}^1 will have its left edge parallel to the right edge of the corresponding odd indexed T_{2i-1}^1. As both Triangles have the same height these edges must also be of equal length.

We continue the construction by working with consecutive S_i^1. For $1 \le i \le 2^{k-2}$ slide S_{2i}^1 relative to S_{2i-1}^1 such that they overlap. We will now specify how they shall overlap. Consider the triangles T_{2i}^1 and T_{2i-1}^1, as stated the left edge of T_{2i}^1 is parallel and of equal length to the right edge of T_{2i-1}^1. First slide S_{2i}^1 to the left such that T_{2i}^1 shares an edge with T_{2i-1}^1. If we disregard the auxiliary triangles of each S_{2i}^1 and S_{2i-1} we seem to be in the same situation as the first step but with a smaller base for each triangle. Therefore slide S_{2i} some more such that overlap of T_{2i-1}^1 and T_{2i}^1 is $2(1-\alpha)b_1$, where b_1 is the base length of a T_j^1. This yields a figure S_i^2 containing a triangle T_i^2 similar to $T_{2i-1}^1 \cup T_{2i}^1$, with $\mathscr{L}^2(T_i^2) = \alpha^2(\mathscr{L}^2(T_{2i-1}^1) + \mathscr{L}^2(T_{2i}^1))$ by Lemma 3.1. Furthermore, disregarding the auxiliary triangles of S_{2i-1}^1 and S_{2i}^1, the overlap yielded by the sliding is at least the overlap of T_{2i-1}^1 and T_{2i}^1 which by Lemma 3.1 is

$$(1-\alpha)(3\alpha-1)(\mathscr{L}^2(T_{2i-1}^1) + \mathscr{L}^2(T_{2i}^1))$$
$$= (1-\alpha)(3\alpha-1)\alpha^2\mathscr{L}^2(T_{4i-3} \cup T_{4i-2} \cup T_{4i-1} \cup T_{4i})$$

the last equality by Equality (17).

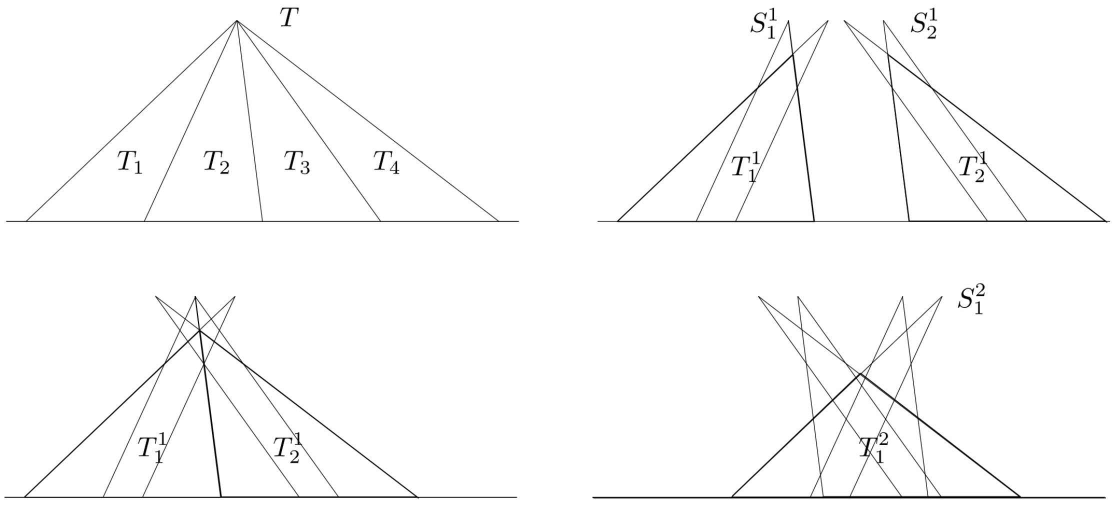

Figure 12: Construction S_1^1 and S_2^1, followed by construction of S_1^2.

Continuing in this way, at the $(r+1)$th stage we obtain S_i^{r+1} by moving S_{2i}^r relative to S_{2i-1}^r $(1 \le i \le 2^{k-r})$ so that the overlap of triangles T_{2i-1}^r and T_{2i}^r results in a reduction of area of at least $(1-\alpha)(3\alpha-1)\alpha^{2r}$ times the total area of the triangles among $T_1, T_2, ..., T_{2^k}$ used to form S_{2i-1}^r and S_{2i}^r. At the kth step we end up with a single figure S_1^k which we take as S. The reduction of area from the original triangle T to S is at least

$$(1-\alpha)(3\alpha-1)(1+\alpha^2+...+\alpha^{2(k-1)})\mathscr{L}^2(T)$$

showing that

$$\begin{aligned}\mathscr{L}^2(S) &\leq \mathscr{L}^2(T)-(1-\alpha)(3\alpha-1)(1+\alpha^2+...+\alpha^{2(k-1)})\mathscr{L}^2(T)\\ &= \left(1-(3\alpha-1)(1-\alpha)\cdot\frac{1-\alpha^{2k}}{1-\alpha^2}\right)\mathscr{L}^2(T)\\ &= \left(1-\frac{(3\alpha-1)(1-\alpha^{2k})}{1+\alpha}\right)\mathscr{L}^2(T).\end{aligned}$$

Taking α close to 1 the factor $\frac{3\alpha-1}{1+\alpha}$ is close to one. Then taking k large enough, since $1-\alpha^{2k}\to 1$ as $k\to\infty$, the term $\frac{(3\alpha-1)(1-\alpha^{2k})}{1+\alpha}$ can be made arbitrarily close to 1, showing that $\mathscr{L}^2(S)$ can be made arbitrarily small. This shows the first part of the theorem.

Now if b is the base length of triangle T, fixing T_1 and moving the other triangles relative to T_1, no triangle will be moved a distance more than b during the construction of S. Consequently dividing T into sub triangles of base lengths at most ε and then constructing an S for each of these sub triangles, the pieces involved in creating each S will need not move more than ε. Equivalently each point of T need not move more than ε in the construction of each S. By Lemma 3.2 it is possible to choose an $\varepsilon>0$ such that each point of T will remain inside of V in the construction of each of these S since T is compact (closed and bounded). By the first part of this theorem each S can be made to have arbitrarily small area so all S together can be made to have arbitrarily small area. This shows the second part of the theorem. □

Lemma 3.4. *Any simple polygon, that is a polygon whose edges do not intersect, with n vertices can be partitioned into $n-2$ triangles.*

Proof. The proof is done by induction on the number of vertices of a polygon. Clearly a polygon with three vertices can be partitioned into $3-2=1$ triangle so the base case is done.

We now show that any simple polygon with four or more vertices has a diagonal, that is a segment connecting two non-adjacent vertices which is contained in the polygon. Any simple polygon has a vertex of angle less than 180°, we say that this vertex is convex. It is easiest to convince oneself of this by trying to draw a simple polygon starting from some vertex, where every angle should be greater than 180° (there is no vertex with angle 180°). Then once one has returned to the starting vertex the figure drawn will be the exterior of a convex polygon, which is certainly not a polygon.

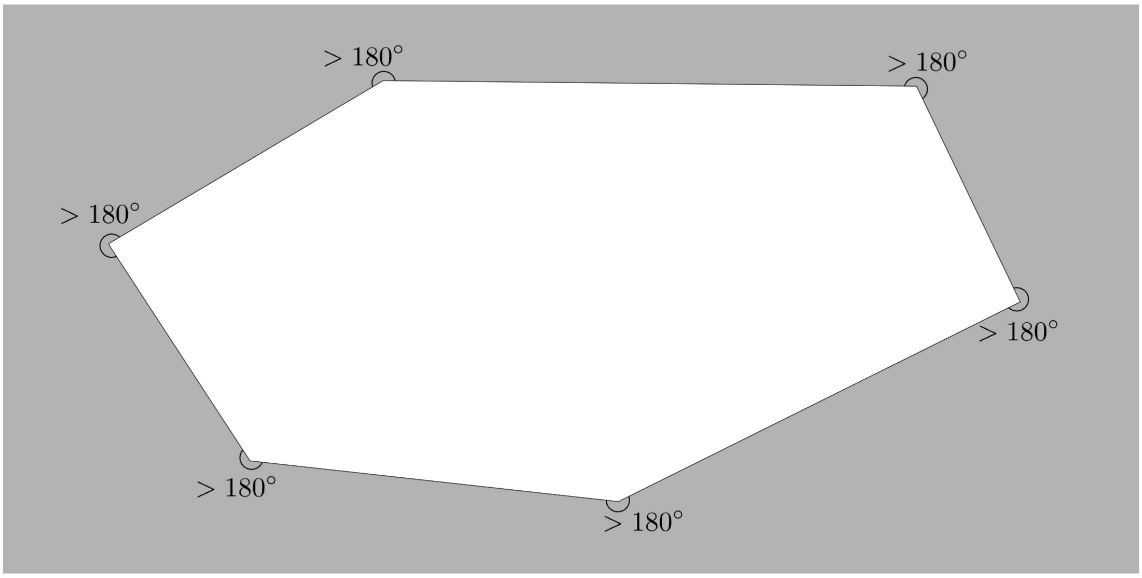

Figure 13: Trying to draw a simple polygon with every angle greater than 180°.

Let v_i be a convex vertex and let v_{i-1} and v_{i+1} be its adjacent vertices. Define a segment M with endpoints at v_{i-1} and v_i. Fix the endpoint at v_{i-1} and take the endpoint at v_i and move it along the edge between v_i and v_{i+1}. Eventually M will intersect a vertex and until then it will surely remain inside the polygon. Let w be the first vertex it intersects, we will consider the case when it intersects two or more vertices at the same time later. If $w = v_{i+1}$ then M connects v_{i-1} and v_{i+1}, M is contained inside the polygon and v_{i-1} and v_{i+1} are not adjacent, hence M is a diagonal. If $w \neq v_{i+1}$ then the segment connecting v_i and w will be contained inside the polygon since it is contained in the area traced out by M which is inside the polygon. The vertices v_i and w are not adjacent hence the segment between them is a diagonal. Now should M intersect two or more vertices at the same time, take $w \neq v_{i+1}$, the argument that the segment connecting v_i and w is a diagonal still holds.

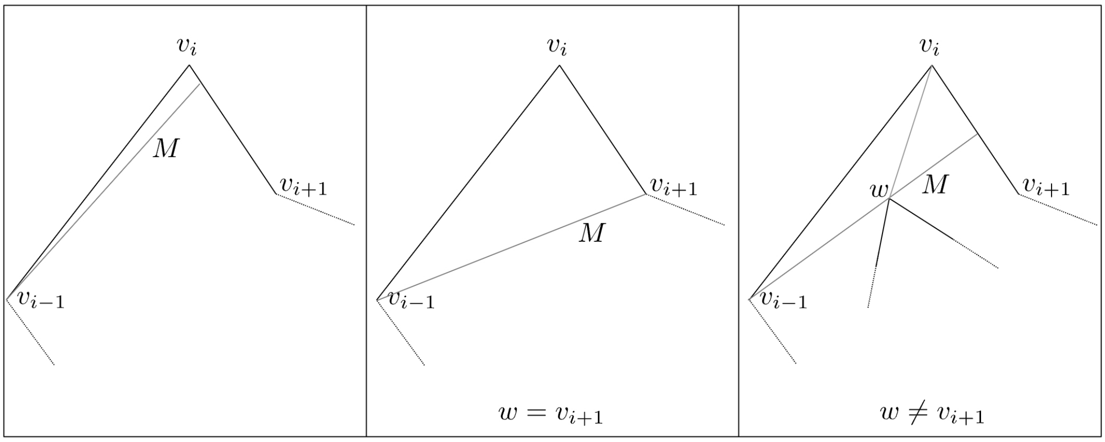

Figure 14: M and the two cases for w.

With the induction hypothesis that any simple polygon with three or more vertices and less than or equal to n vertices can be partitioned into $n-2$ triangles we prove the induction step. Take any simply polygon with $n+1$ vertices, this will be four or more vertices, and draw a diagonal. This partitions the polygon into two new polygons with k and $n+3-k$ vertices where $3 \leq k \leq n$. By the induction hypothesis these two new polygons can be partitioned into $k-2$ and $n+1-k$ triangles. Hence we can partition the original polygon into $k-2+n+1-k = n-1$ triangles, completing the induction step. $\square$

Theorem 3.5 (Construction of a Besicovitch set with zero area)**.** *There exists a set in the plane of Lebesgue measure zero which contains a unit segment in every direction.*

Proof. We construct a set F of measure zero containing unit segments in all directions of a 60° sector. The union of F with two congruent copies of F rotated 60° and 120° yields a set of the desired properties.

The construction of F consists of repeated application of Theorem 3.3. Let S_1 be an equilateral triangle of unit height based on a line L contained in an open set V_1 such that $\mathscr{L}^2(\overline{V}_1) \leq 2\mathscr{L}^2(S_1)$. Consider a unit segment with an endpoint fixed at the vertex opposite of L, it is possible to trace out a 60° sector of a unit circle with this segment while staying inside of S_1. Equivalently S_1 contains a unit segment in every direction of a 60° sector.

By Theorem 3.3 it is possible to divide a triangle S_1 on a base line L into a finite number of constituent triangles, sliding them on L resulting in a closed figure S_2

such that $\mathscr{L}^2(S_2)$ is arbitrarily small and S_2 is contained in V_1. With the technique described in Theorem 3.3 we divide S_1 into a finite number of constituent triangles, sliding them on L, such that the resulting figure S_2 satisfies $\mathscr{L}^2(S_2) \leq 2^{-2}$. Furthermore, like S_1, S_2 contains a unit segment in each direction of a 60° sector since the angles at the vertices opposite of L of the constituent triangles partition a 60° sector and the height of each constituent triangle is the same as the height of S_1.

We now show that we may find an open set V_2 such that $S_2 \subset V_2 \subset V_1$ and $\mathscr{L}^2(\overline{V_2}) \leq 2\mathscr{L}^2(S_2)$. Since S_2 is compact and V_1 is open by Lemma 3.2 there is some $\varepsilon > 0$ such that translating each point of S_2 at most ε, the points will remain inside of V_1. Consequently if we consider the union of ε neighborhoods

$$A = \bigcup_{x_i \in S_2} N_\varepsilon(x_i)$$

then this set will be contained in V_1, we also note that this set is open and that it contains S_2.

By the construction of S_2 it can be considered a simple polygon. By Lemma 3.4 it is possible to partition S_2 into finitely many disjoint triangles call them $\tau_1, ..., \tau_n$. Take each τ_i and scale it such that its area increases by a factor of 2 to form new triangles $T_1, ..., T_n$ with interiors $T_1^\circ, ..., T_n^\circ$ with each T_i° having the same center as τ_i. Then $\tau_i \subset T_i^\circ$ for each i, hence $S \subset \bigcup_{i=1}^n T_i^\circ$. Now let

$$B = \bigcup_{i=1}^n T_i^\circ,$$

this set contains S_2 and

$$\mathscr{L}^2(\overline{B}) = \mathscr{L}^2(\bigcup_{i=1}^n \overline{T_i}) \leq \sum_{i=1}^n \mathscr{L}^2(\overline{T_i}) = \sum_{i=1}^n \mathscr{L}^2(T_i)$$

$$\sum_{i=1}^n 2\mathscr{L}^2(\tau_i) = 2\mathscr{L}^2(S_2).$$

This chain of equalities and inequalities holds true by closure of finite union equals finite union of the closures, $\mathscr{L}^2$ is an outer measure, the area of the closure of a triangle is the area of the triangle ($\overline{T_i} \backslash T_i$ are just points or lines with $\mathscr{L}^2$-measure 0), the relation of area between τ_i and T_i and that the τ_i are disjoint.

Now let $V_2 = A \cap B$, V_2 is open, V_2 contains S_2 since both A and B do, V_2 is a subset of V_1 since A is and V_2 satisfies that

$$\mathscr{L}^2(\overline{V_2}) = \mathscr{L}^2(\overline{A \cap B}) \leq \mathscr{L}^2(\overline{A} \cap \overline{B}) \leq \mathscr{L}^2(\overline{B}) \leq 2\mathscr{L}^2(S)$$

since $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subset \overline{B}$. We have found an open set V_2 with the desired properties.

Continuing, since S_2 consists of a finite number of triangles, we may divide each triangle into a finite number of constituent triangles moving them along L such that the resulting figure S_3 is contained in V_2 and $\mathscr{L}^2(S_3) \leq 2^{-3}$. Again S_3 contains a unit segment of each direction of a 60° sector. By the same argument for S_2 we may find a set V_3 such that $S_3 \subset V_3 \subset V_2$ and $\mathscr{L}^2(\overline{V_3}) \leq 2\mathscr{L}^2(S_3)$.

Repeating the process we obtain a sequence of figures $\{S_i\}_i$, each figure a finite union of elementary triangles based on L and of unit height, and a sequence of open sets $\{V_i\}_i$ for which

$$S_i \subset V_i \subset V_{i-1}$$

and

$$\mathscr{L}^2(\overline{V_i}) \leq 2\mathscr{L}^2(S_i) \leq 2^{-i+1}$$

for each i. Let F be the closed set $F = \bigcap_{i=1}^{\infty} \overline{V_i}$. We claim that F has the desired properties.

Certainly $\mathscr{L}^2(F) = 0$. By construction, each S_i, and thus each $\overline{V_i}$, contains a unit segment in any direction of a 60° sector. We show that this is also true for F. Let M_i be a unit segment of some normed direction vector d with $M_i \subset \overline{V_i}$ and consider the sequence $\{M_i\}_i$. We can express any point of M_i as $p_i + td$ where $t \in [0,1]$ for an end-point p_i. Considering the sequence of points $\{p_i\}_i$ we may take a subsequence of this sequence $\{p_{i(k)}\}_k$ that converges since the sequence is contained inside $\overline{V_1}$, which is compact. Assume this subsequence converges to a point p, we show that p is contained in each $\overline{V_j}$.

Since the sequence $\{V_i\}_i$ is a decreasing chain so is the sequence $\{\overline{V_i}\}_i$. By definition, since the subsequence $\{p_{i(k)}\}_k$ converges, there for every $\varepsilon > 0$ exists an N such that for all $i(k) \geq N$ it holds that $|p - p_{i(k)}| < \varepsilon$. Therefore letting $i(k) \geq \max(N, j)$, $p_{i(k)} \in \overline{V_j}$ since $p_{i(k)} \in V_j$, and $|p - p_{i(k)}| < \varepsilon$. By the latter we conclude that p is a limit point of $\overline{V_j}$ hence $p \in \overline{V_j}$. We conclude that p is contained in each $\overline{V_j}$ so $p \in F$.

Let M be the segment with endpoint p and direction d and let $p + td$ be any point of M. By the convergence of $\{p_{i(k)}\}_k$ to p we have that for every $\varepsilon > 0$ there is an N such that

$$|p - p_{i(k)}| < \varepsilon \Leftrightarrow |(p + td) - (p_{i(k)} + td)| < \varepsilon$$

for all $i(k) \geq N$. Consequently if $i(k) \geq \max(N, j)$ we see that $p+td$ is a limit point of $\overline{V_j}$ as $p_{i(k)} + td \in \overline{V_j}$. Hence $p + td \in V_j$ and since j is arbitrary $p + td \in F$. This shows that there is a unit segment of direction d in F completing the proof. □

Lemma 3.6. *Let L_1 and L_2 be parallel lines in the plane. Then, given $\varepsilon > 0$, there is a set E containing L_1 and L_2 with $\mathscr{L}^2(E) < \varepsilon$ such that a unit segment may be moved continuously from L_1 to L_2 without leaving E.*

Proof. Let x_1 and x_2 be points on L_1 and L_2 respectively. Let E be the set consisting of L_1, L_2, the segment M joining x_1 and x_2, together with the unit sector centered at x_1 lying between L_1 and M and the unit sector centered at x_2 lying between L_2 and M. Taking x_1 and x_2 sufficiently far apart the area of E can be made as small as desired. Furthermore, a unit segment may be moved from L_1 to L_2 by a rotation in the first sector, a translation along M, and a rotation in the second sector.

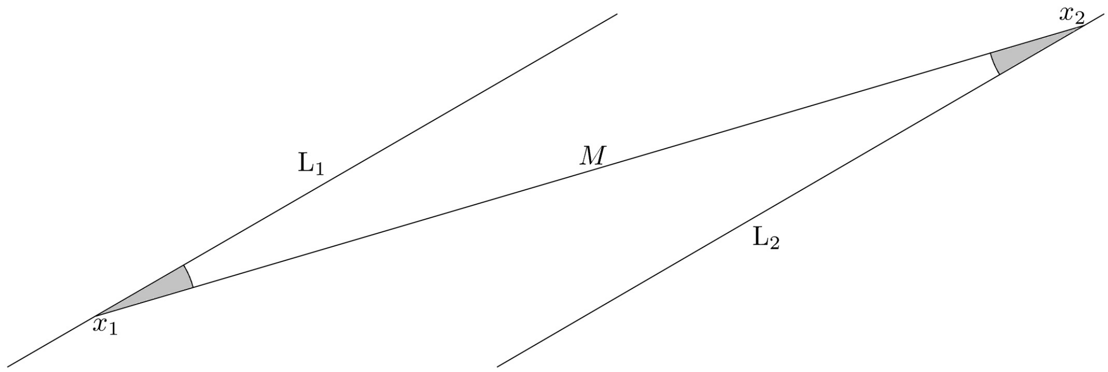

Figure 15: Figure E used to move the segment.

□

Theorem 3.7 (Construction of a Kakeya set with arbitrarily small area)**.** *Given $\varepsilon > 0$, there exists a set E in the plane with $\mathscr{L}^2(E) < \varepsilon$ with the property that a unit segment may be moved continuously to lie in its original position but rotated through 180°.*

Proof. We construct a set E_0 with $\mathscr{L}^2(E_0) < \frac{1}{4}\varepsilon$ inside which it is possible to maneuver a unit segment to a position at an angle of 60° relative its original angle.

If T is an equilateral triangle of unit height, then by Theorem 3.3 we may divide T into $m = 2^k$ constituent triangles $T_1, ..., T_m$, indexed from left to right, and sliding them along L to positions in such a way that $\mathscr{L}^2(\bigcup_{i=1}^m T_i) < \frac{1}{8}\varepsilon$. We note that for each $i \leq m-1$, the right side of T_i is parallell to the left side of T_{i+1}. Therefore by Lemma 3.6, we may for each $i \leq m-1$, add a set of measure, at most $\frac{1}{8m}\varepsilon$ allowing a unit segment to be moved from T_i to T_{i+1}. This gives a set E_0 of measure at most $\frac{1}{8}\varepsilon + (m-1)\frac{1}{8m}\varepsilon < \frac{1}{4}\varepsilon$.

Let E_1 and E_2 be congruent sets of E_0 but rotated 60° and 120° respectively. If E_0, E_1 and E_2 are placed appropriately it is possible to maneuver a unit segment continuously through the union of these sets such that it rotates 180°. Moreover the union of these sets have area at most $\frac{3}{4}\varepsilon$. Finally since the segment has been rotated 180° the line through the rotated segment is parallel to the line through the segment at its original position. Therefore by Lemma 3.6 we may add a set E_3 of area at most $\frac{1}{4}\varepsilon$ such that we can translate the rotated segment back to its original position. Taking the union of E_0, E_1, E_2 and E_3 gives the desired set E. □

3.2 Reflections on the Kakeya conjecture

An open problem, known as the Kakeya conjecture, can be stated as follows: *Let $F \subset \mathbb{R}^n$ be a Besicovitch set in $\mathbb{R}^n$, does F necessarily have Hausdorff dimension n?* The conjecture is related to Harmonic analysis and Fourier transforms, illustrated in Quanta magazine's article [Cep23].

The theory developed in Section 2 gives some understanding of the articles formulation that a Besicovitch set "can't have too much overlap". Considering each segment of a Besicovitch set taken from the unit ball, no segment is rotated, and wherever each segment lies in the Besicovitch set, which may yield more or less overlap, we still have an Hausdorff n-dimensional set. In contrast, taking each segment from the unit ball and allowing rotation, more overlap is possible than in the previous case. Allowing rotation the maximum overlap would be just one segment of one direction, something 1-dimensional.

Further should the Kakeya conjecture hold true then the Besicovitch sets of zero measure, like the one of Theorem 3.5, give an interesting example when related to the definition of Hausdorff dimension derived from Theorem 2.25. To recall the theorem states there for every $E \subset \mathbb{R}^n$ is some dim E such that $\mathscr{H}^s(E) = \infty$ for all $0 \leq s < \dim E$ and $\mathscr{H}^s(E) = 0$ for all dim $< s < \infty$ and the proof of the theorem was done by considering if $0 < \mathscr{H}^{\dim E}(E) < \infty$ or not. Naturally for integers n there are sets E for which $0 < \mathscr{H}^n(E) < \infty$, easily seen by the equivalence of the Hausdorff measure and Lebesgue measure, Theorem 2.32. These sets give examples of the first case of the theorem. Now should the Kakeya conjecture hold true then a Besicovitch set of zero measure would be an example of a set for the second case of the theorem i.e a set F for which $\mathscr{H}^s(F) = \infty$ for all $0 \leq s < n$ and $\mathscr{H}^s(F) = 0$ for all $n < s < \infty$ and also $\mathscr{H}^n(F) = 0$.

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Errata for "2025-24-Martin-Benjamin"

Benjamin Martin

2025-06-07

- p.18: "...and finally since

$$\bigcup_{\substack{k=j+1 \\ k \text{ odd}}}^{n} B_k \subset A_{n+1} \quad \text{and} \quad \bigcup_{\substack{k=j+1 \\ k \text{ odd}}}^{n} B_k \subset A_{n+1} \text{..."}$$

  Should of course be "k even"under the right union.

- p.31: "For each point of a block C_i we can have balls of radius at most δ..."

  We must have *closed* balls, which we can have, in order to apply Theorem 2.29.

- p.43: "If we disregard the auxiliary triangles of each S^1_{2i} and S_{2i-1} we seem to be in the same situation as the first step but with a smaller base for each triangle. Therefore slide S_{2i}..."

  Should of course be S^1_{2i-1} and S^1_{2i} not S_{2i-1} and S_{2i}.

- p. 47: "By the construction of S_2 it can be considered a simple polygon..."

  There is no obvious reason that S_2 can be considered a simple polygon. In fact I am unsure whether this is true or not. However, it is a *finite number of simple polygons* so S_2 can be partitioned into finitely many disjoint triangles which is the point.