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An introduction to the Fourier transform in the Schwartz space and some applications to PDEs

av

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Abstract

This thesis is an introduction to the Fourier transform on functions in the Schwartz space. The definition of a function being in the Schwartz space will be stated and some properties these type of functions have, e.g. the behavior of the convolution of two functions, will be explained. Thereafter, the Fourier transform is introduced and also some important theorems, like the Fourier Inversion Formula. Finally, we will apply this theory on two famous partial differential equations, namely the wave equation and the heat equation. The wave equation will be solved in a compact form in one dimension, and the heat equation will be solved in n dimensions.

Sammanfattning

I följande uppsats ges en introduktion av fouriertransformen på funktioner i Schwartzrummet. Definitionen av funktioner i Schwartzrummet anges och även några egenskaper dessa sorters funktioner har, exempelvis beteendet vid faltning av två funktioner. Därefer introduceras fouriertransformen tillsammans med viktiga satser som inverstransformen. Slutligen, appliceras teorin på två kända partiella differential ekvationer, nämligen vågekvationen och värmeekvationen. Vågekvationen löses som en kompakt form i endimension och värmeekvationens lösning är i n-dimensioner.

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Contents

1 Introduction

In this thesis we introduce the Fourier transform as a tool for solving two interesting partial differential equations, the wave equation and the heat equation. This topic is associated with some theoretical complexity and to make this topic more understandable for me, we will reduce ourselves into the Schwartz space. Chapter 4 offers a brief explanation of the Schwartz space and the behavior of functions this space.

In this chapter, I would like to present the key figures in this thesis. The founder of the Schwartz space is Laurent Schwartz (1915–2002), most famous for his theory of distributions. Specifically that his formalization of distributions (generalized functions) always had well-defined derivatives. Schwartz was widely recognized in the mathematical community and was invited by Harald Bohr to Copenhagen in October 1947 [1]. Schwartz taught several remarkable students when he was in Nancy and in 1953, he became a professor in Paris. He continued teaching at the École Polytechnique in Paris from 1959 to 1980. Beyond his mathematical achievements, he was politically active and participated in the French elections of 1945 but failed to be elected [4].

Another key contributor to this field is Jean-Baptiste Joseph Fourier (1768-1830), who is best known for his work in mathematical physics, particularly the Fourier series. I believe many students that studies mathematics are with familiar the Fourier series, but fewer may be aware of Fourier's life. Fourier is one of the most historically famous and brilliant French mathematical scientists. He showed talents for mathematics and mechanics at a very young age and it led him to study at the Benedictine College, École Royale Militaire of Auxerre, where he became a mathematics teacher in 1790. The French Revolution took place during this time and Fourier was largely involved and it influenced his life. He desired to avoid political troubles but was still selected to join Napoleon Bonaparte's (1769–1821) Egyptian expedition in 1798. Upon his return, Fourier resumed his position as a Professor at the École Polytechnique in 1801. In 1807, he completed his most famous memoir, "On the propagation of heat in solid bodies" and submitted it to the Academy of Sciences of Paris. The paper was impressive but faced criticism for his use of trigonometric (Fourier) series expansions of functions without theoretical justification [2]. Despite this, Fourier's contributions have proved to have a large impact on mathematical physics, particularly analyzing the conduction of heat in solid bodies, the Fourier integral theorem, Fourier series and the Fourier transform, which will be examined in detail in Chapter 5 [2] [3].

It is a bit thoughtful that both Schwartz and Fourier were French mathematicians and held professorships at the École Polytechnique but in different eras, since their achievements form the foundation of this thesis. Before we start delving into the mathematical theory, I would like to explain the importance of the wave equation and the heat equation and their practical applications. Chapter 6 will focus on solving these equations with the theory from previous chapters.

The wave equation is a second-order linear partial differential equation that describes the movement of waves in different media. This equation can be applied in numerous areas such as optics e.g. X-ray diffraction optics. It is also applied in music and design of instruments, and in fluid dynamics for understanding waves in fluids, like studying the simpler case such as ripples on the surface of a lake or pond. The wave equation is also useful in predicting natural disasters like earthquakes and tsunamis [6].

The heat equation is also a partial differential equation. It explains the distribution of heat over time in various media. Heat is a fluid insider matter, capable of flowing from one position to another. The heat equation is fundamental to understanding the theory of heat conduction and is useful in studying heat transfer in solids, biological tissues, oceans or soil [8]. By exploring these equations and their solutions, this thesis aims to provide an understanding of mathematical theory and foundations with practical applications.

The main literature this thesis relies on are *An Introduction to Pseudo-Differential Operators* (3rd ed.) by M. W. Wong [9] and *The Fourier Transform and the Wave Equation* by A. Torchinsky [7] and the lecture notes *Heat Equation* from a course called "Math 220B" by Julie Levandosky at Stanford University [5].

2 Notation of differential operators

Before we start exploring the mathematical theory of the Schwartz space and the Fourier transform, we will have to introduce some notations. Firstly, this thesis will use the differential operators on $\mathbb{R}^n$ that are expressed as $\frac{\partial}{\partial x_j}$ for $j = 1, 2, ..., n$. Further, define the operator D_j given by $D_j = -i\frac{\partial}{\partial x_j}$ and $i^2 = -1$.

The most general linear partial differential operators of order m on $\mathbb{R}^n$ will be on the form

$$\sum_{\alpha_1+\alpha_2+...+\alpha_n \leq m} a_{\alpha_1,...,\alpha_n}(x) D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n},$$

where $\alpha_1, \ldots, \alpha_n$ are nonnegative integers and $a_{\alpha_1,...,\alpha_n}$ is an infinitely differentiable complex-valued function on $\mathbb{R}^n$. An n-tuple of nonnegative integers is called a multi-index and is denoted as

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n).$$

The length of α is denoted by $|\alpha| = \sum_{j=1}^{n} \alpha_j$ and we shall denote $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ and with this, rewrite the differential operator in a compact form as

$$\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha.$$

3 Preliminary

This chapter introduces some useful definitions and will present some results by applying them. Additionally, the Leibniz's rule and the Fubini's theorem will be presented and used frequently in the following chapters.

Definition 3.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is radial if and only if it can be written as $f(x) = \tilde{f}(|x|)$ for some $\tilde{f} : [0, +\infty) \to \mathbb{R}$.

Proposition 3.2. *If a function f is radial, then the following equality holds,*

$$\int_{\mathbb{R}^n} f(x)dx = c_n \int_0^{+\infty} \tilde{f}(r)r^{n-1}dr$$

where c_n is the surface area of the unit circle in $\mathbb{R}^n$.

Proof. We will prove this in two dimensions. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and it to be radial. Changing the coordinates into the polar coordinates

$$x_1 = r\cos\theta \qquad x_2 = r\sin\theta,$$

leads to

$$\begin{aligned}\int_{\mathbb{R}^2} f(x_1, x_2)dx_1dx_2 &= \int_0^{2\pi}\int_0^{+\infty} \tilde{f}(r)r\ drd\theta \\ &= 2\pi\int_0^{+\infty} \tilde{f}(r)r\ dr.\end{aligned}$$

Notice that 2π is the surface area of the unit circle in $\mathbb{R}^2$.

Similarly in three dimensions, we have $f : \mathbb{R}^3 \to \mathbb{R}$ and it to be radial with polar coordinates

$$x_1 = r\sin\theta\cos\varphi \qquad x_2 = r\sin\theta\sin\varphi \qquad x_3 = r\cos\theta.$$

Then, the integral becomes

$$\begin{aligned}\int_{\mathbb{R}^3} f(x_1, x_2, x_3)dx_1dx_2dx_3 &= \int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty} \tilde{f}(r)r^2\sin\theta\ drd\theta d\varphi \\ &= 2\pi\cdot 2\int_0^{+\infty} \tilde{f}(r)r^2\ dr \\ &= 4\pi\int_0^{+\infty} \tilde{f}(r)r^2\ dr.\end{aligned}$$

The surface area of the unit circle in $\mathbb{R}^3$ is 4π and we have shown that the proposition is true in three dimensions. □

Lemma 3.3. *The following function depending on α,*

$$\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^\alpha} dx$$

is convergent provided $\alpha > n$.

Proof. Since the function is radial, by Proposition 3.2 we get the following integral,

$$I = c_n \int_0^{+\infty} \frac{1}{(1+|r|)^\alpha}\ r^{n-1}\ dr.$$

Use the function

$$g(r) = r^{n-1-\alpha}$$

and apply the limit comparison test to get

$$\lim_{r\to+\infty} \frac{\frac{r^{n-1}}{(1+r)^\alpha}}{r^{n-1-\alpha}} = \lim_{r\to+\infty} \frac{r^\alpha}{(1+r)^\alpha} = \lim_{r\to+\infty} \frac{1}{(1+\frac{1}{r})^\alpha} = 1.$$

Since the limit exists, the integral I is convergent if and only if

$$\int_1^{+\infty} r^{n-1-\alpha} dr$$

is convergent. (This integral from 0 to 1 is convergent because the function $g(r)$ is geometric and is convergent when $|r| < 1$.) The value of this integral is

$$\lim_{N\to+\infty} \int_1^N r^{n-1-\alpha} dr = \lim_{N\to+\infty} \begin{cases} \left[\frac{r^{n-\alpha}}{n-\alpha}\right]_1^{r=N} & \text{if } n \neq \alpha, \\ [\ln r]_1^N & \text{if } n = \alpha. \end{cases}$$

Note that the only case when the limit exists is when $n - \alpha < 0$. The limit in this case is

$$\lim_{N\to+\infty} \frac{N^{n-\alpha} - 1}{n-\alpha} = \frac{1}{\alpha - n}.$$

Hence, when $\alpha > n$ the integral I in question converges. □

Lemma 3.4. *(Leibniz's rule)*
The formula

$$\begin{aligned} D^\alpha(fg) &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (D^\beta f)(D^{\alpha-\beta} g) \\ &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (-i)^\alpha (\partial^\beta f)(\partial^{\alpha-\beta} g) \end{aligned}$$

is known as Leibniz's rule.

The following theorem is a consequence of the general Fubini's theorem.

Theorem 3.5. *(Fubini's theorem)*
Let X, Y be two open sets in $\mathbb{R}^n$. Let $k : X \times Y \to \mathbb{R}$ and $k \in C(X \times Y)$ such that for all $N \in \mathbb{N}$:

$$\sup_{(x,y)\in X\times Y} (1+|x|)^N (1+|y|)^N |k(x,y)| < \infty,$$

then

$$\int_X \left(\int_Y k(x,y)\ dy \right) dx = \int_Y \left(\int_X k(x,y)\ dx \right) dy.$$

4 The Schwartz space

This chapter is an introduction to the Schwartz space. What is means for a function to be in the Schwartz space and what properties these function have will be answered in this chapter. Some important results will be presented such as two different ways to define the Schwartz space, that the product of a Schwartz function and a polynomial still is in the Schwartz space, and the convolution of two Schwartz functions belongs to the Schwartz space. The results from this chapter will be relevant in upcoming chapters.

4.1 The definition of the Schwartz space

We begin by presenting two different definitions of a function being in the Schwartz space and showing that the two definitions are equivalent.

Definition 4.1. The Schwartz space $\mathcal{S}$ consists of the set of infinitely differentiable functions on $\mathbb{R}^n$, denoted as $C^\infty(\mathbb{R}^n)$ and called as smooth funtions, satisfying that for all multi-indices α and β,

$$\sup_{x\in\mathbb{R}} \left| x^\alpha \partial^\beta f(x) \right| < +\infty.$$

Lemma 4.2. *Let $f \in C^\infty(\mathbb{R})$. The following are equivalent*

(i) for every $\alpha, \beta \in \mathbb{N}$

$$[f]_{\alpha,\beta} := \sup_{x\in\mathbb{R}} \left| x^\alpha \partial^\beta f(x) \right| < +\infty,$$

(ii) for every $\beta \in \mathbb{N}$ and for every $N \in \mathbb{N}$

$$P_{N,\beta}(f) := \sup_{x\in\mathbb{R}} (1 + |x|)^N \left| \partial^\beta f(x) \right| < +\infty.$$

Proof. Assume $P_{N,\beta}(f)$ exists and is finite. Notice that this inequality

$$|x^\alpha| \leq |x|^{|\alpha|} \leq (1 + |x|)^{|\alpha|}$$

can be used to get

$$\sup_{x\in\mathbb{R}} |x^\alpha| \left| \partial^\beta f(x) \right| \leq \sup_{x\in\mathbb{R}} (1 + |x|)^{|\alpha|} \left| \partial^\beta f(x) \right|.$$

Let the length of α be equal to N to arrive at the inequality

$$\sup_{x\in\mathbb{R}} |x^\alpha| \left|\partial^\beta f(x)\right| = \sup_{x\in\mathbb{R}} \left|x^\alpha \partial^\beta f(x)\right| \leq \sup_{x\in\mathbb{R}} (1+|x|)^N \left|\partial^\beta f(x)\right|.$$

Conversely, assume $[f]_{\alpha,\beta}$ exists and is finite. Then by the binomial expansion, for all k and for all $j = 1, .., n$ the following is true for all x,

$$(1+|x|)^{|\alpha|} = 1 + \sum_{k=1}^{|\alpha|} \binom{|\alpha|}{k} |x|^k.$$

By multiplying this with $|f(x)|$ we get

$$(1+|x|)^{|\alpha|}|f(x)| = |f(x)| + \sum_{k=1}^{|\alpha|} \binom{|\alpha|}{k} |x|^k |f(x)|.$$

Reformulate the part $|x|^k|f(x)|$ by starting with expressing

$$|x| = \sum_{j=1}^{n} |x_j| = \sqrt{\left(\sum_{j=1}^{n} |x_j|\right)^2} \leq \max_{j=1,\ldots,n} |x_j| \sqrt{\sum_{j=1}^{n} 1} = \sqrt{n} \max_{j=1,\ldots,n} |x_j|.$$

Replace this inequality in $|x|^k|f(x)|$ as

$$\begin{aligned} |x|^k|f(x)| &\leq \left(\sqrt{n} \max_{j=1,\ldots,n} |x_j|\right)^k |f(x)| \\ &= \sqrt{n}^k \max_{j=1,\ldots,n} \{|x_j|^k |f(x)|\} \\ &\leq \sqrt{n}^k \max_{j=1,\ldots,n} [f]_{ke_j,0}\,, \end{aligned}$$

where $e_j = (0, .., 0, 1, 0, ..., 0)$ and 1 is on the j-th position. This maximum exists because the maximum is chosen from a finite number of j and the function $[f]_{ke_j,0}$ is also finite by assumption. Therefore, we can furthermore write that

$$\begin{aligned} (1+|x|)^{|\alpha|}|f(x)| &= |f(x)| + \sum_{k=1}^{|\alpha|} \binom{|\alpha|}{k} |x|^k |f(x)| \\ &\leq |f(x)| + \sum_{k=1}^{|\alpha|} \binom{|\alpha|}{k} \sqrt{n}^k \max_{j=1,\ldots,n} [f]_{ke_j,0}\,. \end{aligned}$$

The supremum of the right hand side is

$$\sup_{x\in\mathbb{R}}\left(|f(x)|+\sum_{k=1}^{|\alpha|}\binom{|\alpha|}{k}\sqrt{n}^k\max_{j=1,\dots,n}[f]_{ke_j,0}\right)$$

which we can rewrite as

$$[f]_{0,0}+\sup_{x\in\mathbb{R}}\sum_{k=1}^{|\alpha|}\binom{|\alpha|}{k}\sqrt{n}^k\max_{j=1,\dots,n}[f]_{ke_j,0}\,.$$

A sum of finite terms where the maximum is finite, is also finite. We can therefore safely conclude,

$$\sup_{x\in\mathbb{R}}(1+|x|)^N\left|\partial^\beta f(x)\right|\leq\infty.$$

Hence, combing the results shows

$$\sup_{x\in\mathbb{R}}|x^\alpha|\left|\partial^\beta f(x)\right|=\sup_{x\in\mathbb{R}}(1+|x|)^N\left|\partial^\beta f(x)\right|.$$

□

4.2 Properties of functions in the Schwartz space

Functions in the Schwartz space are smooth and have its derivatives decreasing faster than a polynomial. These are pleasant properties for a function but how does a Schwartz function act on multiplication with another function or under translation and more. This section will explore these aspects and also show the Peetre's Inequality. The source of Theorem 4.6 is *An Introduction to Pseudo-Differential Operators* (3rd ed.) by M. W. Wong.

Corollary 4.3. *Prove that for all $f\in\mathcal{S}(\mathbb{R}^n)$ that*

$$\int_{\mathbb{R}^n}|f|dx<+\infty.$$

Proof. To show this, we will use that $f\in\mathcal{S}(\mathbb{R}^n)$ and the definition tell us that

$$\sup_{x\in\mathbb{R}}(1+|x|)^N\left|\partial^\beta f(x)\right|<+\infty.$$

We have

$$\begin{aligned}\int_{\mathbb{R}^n} |f| dx &= \int_{\mathbb{R}^n} \frac{(1+|x|)^N}{(1+|x|)^N} |f(x)|\, dx \\ &\leq \sup_{y\in\mathbb{R}} \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^N} (1+|x|)^N |f(x)|\, dx \\ &\leq \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^N} \sup_{y\in\mathbb{R}} (1+|x|)^N |f(x)|\, dx \\ &= P_{N,0}(f) \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^N} dx\end{aligned}$$

and the integral of bounded functions will also be bounded and therefore we have shown

$$\int_{\mathbb{R}^n} |f| dx < +\infty.$$

□

Proposition 4.4. *Let $f \in \mathcal{S}$. Then the functions*

$$(T_y f)(x) = f(x+y), \quad x \in \mathbb{R}^n,$$

and

$$(M_y f)(x) = e^{ix\cdot y} f(x), \quad x \in \mathbb{R}^n,$$

and

$$(D_a f)(x) = f(ax), \quad x \in \mathbb{R}^n$$

are in $\mathcal{S}$.

Proof. To show that $T_y f, M_y f$ and $D_\alpha f$ are in the Schwartz space we will use the definition. Firstly, we have

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^N \left|\partial^\beta (T_y f)(x)\right| = \sup_{x\in\mathbb{R}^n} (1+|x|)^N \left|\partial^\beta f(x+y)\right|$$

and by using the triangle inequality we obtain

$$\sup_{x\in\mathbb{R}^n} (1+|x+y-y|)^N \left|\partial^\beta f(x+y)\right| \leq \sup_{x\in\mathbb{R}^n} (1+|x+y|)^N (1+|y|)^N \left|\partial^\beta f(x+y)\right|.$$

The term $(1+|y|)^N$ is not dependent on x and can be considered as a constant when taking the supremum. Doing a change of variables, with $z = x+y$ allows

$$\sup_{z\in\mathbb{R}^n} (1+|z|)^N (1+|y|)^N \left|\partial^\beta f(z)\right| < \infty,$$

and since $f \in \mathcal{S}$ we have that $(T_y f)(x) \in \mathcal{S}$.

Secondly, take

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^N \left|\partial^\beta (M_y f)(x)\right| = \sup_{x\in\mathbb{R}^n}(1+|x|)^N \left|\partial^\beta e^{ix\cdot y} f(x)\right|$$

and by using Lemma 3.4, the Leibniz rule on $\partial^\beta e^{ix\cdot y} f(x)$, it is equal to

$$\begin{aligned}\sum_{k\leq\beta}\binom{\beta}{k}(-i)^\beta(\partial^k e^{ix\cdot y})(\partial^{\beta-k}f(x)) &= \sum_{k\leq\beta}\binom{\beta}{k}(-i)^\beta(iy)^k e^{ix\cdot y}\partial^{\beta-k}f(x)\\ &= C(\beta)e^{ix\cdot y}\partial^{\beta-k}f(x)\end{aligned}$$

where $C(\beta)=\sum_{k\leq\beta}\binom{\beta}{k}(-i)^\beta(iy)^k$ and is not dependent on x. Implementing this in the original equation leads to

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^N \left|C(\beta)e^{ix\cdot y}\partial^{\beta-k}f(x)\right|.$$

Since $|C(\beta)e^{ix\cdot y}|$ is bounded, we can exclude it from the supremum with respect to x and we get

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^N \left|\partial^{\beta-k}f(x)\right|.$$

Since f is in the Schwartz space, for every N and every multi-index, which in this case is $\beta-k$, this expression is finite.

Lastly, we will express $\sup_{x\in\mathbb{R}^n}(1+|x|)^N \left|\partial^\beta(D_a f)(x)\right|$ with using the variable substitution $x=x/a$ which results in

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^N \left|\partial^\beta f(ax)\right| = \sup_{x\in\mathbb{R}^n}(1+|x/a|)^N \left|\partial^\beta f(x)\right|.$$

We have two cases which are when $|a|$ is smaller or larger than 1. In the first case when $|a|>1$, we can use $|x/a|<|x|$ such that

$$\sup_{x\in\mathbb{R}^n}(1+|x/a|)^N \left|\partial^\beta f(x)\right| < \sup_{x\in\mathbb{R}^n}(1+|x|)^N \left|\partial^\beta f(x)\right| < \infty.$$

The second case when $|a|<1$, we need to apply the relation

$$1+|x/a| = \frac{|a|}{|a|}+\frac{|x|}{|a|} = \frac{|a|+|x|}{|a|} < \frac{1+|x|}{|a|}$$

on our main equation to arrive at

$$\sup_{x\in\mathbb{R}^n}(1+|x/a|)^N\left|\partial^\beta f(x)\right| < \sup_{x\in\mathbb{R}^n}\left(\frac{1+|x|}{|a|}\right)^N\left|\partial^\beta f(x)\right|.$$

Since $f \in \mathcal{S}$ and $|a|$ is a constant and does not depend on the supremum, we can write

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^N\left|\partial^\beta f(x)\right| < \infty.$$

Hence, $(D_a f)(x) \in \mathcal{S}$. □

Lemma 4.5. *If $f \in \mathcal{S}$ and P is a polynomial, then the product $f(x)P(x)$ is in the Schwartz space.*

Proof. I will restrict this proof in one dimension. To show that the product is in the Schwartz space we first have to realize that a polynomial is smooth and that f also is smooth because it is in $\mathcal{S}$. By this, the product is also smooth.

Let the polynomial be $P(x) = x^\gamma$ where γ is an integer and if we differentiate the product once, we get by Lemma 3.4 the Leibniz rule

$$\partial_x(x^\gamma f) = \gamma x^{\gamma-1}f + x^\gamma f'.$$

If we differentiate N times, it results in

$$\partial_x^N(x^\gamma f) = \sum_{j=0}^{N} c_j(x)f^{(j)}(x)$$

where $c_j(x)$ is a polynomial. Multiplying this expression with x^α will give us

$$x^\alpha\partial_x^N(x^\gamma f) = \sum_{j=0}^{N} x^\alpha c_j(x)f^{(j)}(x).$$

Every term of this expression is bounded since f is smooth and any product with f and a polynomial will be bounded per definition in Lemma 4.2. This sum will also be bounded and therefore the supremum exists. Since $\sup_{x\in\mathbb{R}}|x^\alpha\partial_x^N(x^\gamma f)| < \infty$, a product of a function in the Schwartz space and a polynomial, is in the Schwartz space. □

The following result is a corollary of Theorem 2.1 in [9].

Theorem 4.6. *Let Y be an open set in $\mathbb{R}^n$ such that*

(i) $F(x, \cdot) \in \mathcal{S}$ *for all* x *in* $\mathbb{R}^n$,

(ii) $F(\cdot, y) \in C^\infty(\mathbb{R}^n)$ *for all* y *in* Y,

(iii) $\sup_{x\in\mathbb{R}^n} \int_Y |(\partial_x^\alpha F)(x, y)|dy < \infty$ *for all multi-indices* α.

Then the integral $\int_Y F(x, y)dy$*, as a function of* x*, is in* $C^\infty(\mathbb{R}^n)$ *and*

$$\partial^\beta \int_Y F(x, y)dy = \int_Y (\partial_x^\beta F)(x, y)dy, \quad x \in \mathbb{R}^n,$$

for all multi-indices β.

The next lemma is called the Peetre's Inequality and we will find it useful for the following proofs.

Lemma 4.7. *(Peetre's Inequality)*
The inequality

$$(1 + |x - y|)^t \leq (1 + |x|)^t (1 + |y|)^{|t|}$$

is valid for all $t \in \mathbb{R}$ *and for all* $x, y \in \mathbb{R}^n$.

Proof. Assume that $t \geq 0$. By the triangular inequality we get

$$1 + |x - y| \leq 1 + |x| + |y| \leq (1 + |x|)(1 + |y|).$$

Hence

$$(1 + |x - y|)^t \leq (1 + |x|)^t (1 + |y|)^t.$$

Now, assume that $t < 0$, and expand the inequality as

$$1 + |x| \leq 1 + |x - y| + |y| \leq (1 + |x - y|)(1 + |y|).$$

This is equivalent to

$$\frac{1 + |x|}{1 + |x - y|} \leq 1 + |y|.$$

By taking this to the power of $|t|$ and rewriting the expression, we have the relation

$$\frac{1}{(1 + |x - y|)^{|t|}} \leq \frac{(1 + |y|)^{|t|}}{(1 + |x|)^{|t|}}.$$

Since $t < 0$ we have that $-|t| = t$ and therefore it is true to express

$$(1 + |x - y|)^t \leq (1 + |x|)^t (1 + |y|)^{|t|}.$$

□

4.3 The convolution of two functions

The convolution of two function is an important operation that will be helpful in the following chapters. We will define the convolution and show that for two function f, g in the Schwartz space, their convolution also belongs to the Schwartz space.

Definition 4.8. Let $f, g \in \mathcal{S}$. The value of the integral denoted by $(f * g)(x)$ is defined as

$$\int_{\mathbb{R}^n} f(x - y)g(y)\ dy,$$

and is called the convolution of f and g.

Lemma 4.9. *If $f, g \in \mathcal{S}(\mathbb{R}^n)$ then*

$$f * g \in C^\infty(\mathbb{R}^n)$$

and

$$\partial^\alpha (f * g)(x) = \partial^\alpha f * g(x).$$

Proof. The first expression is the convolution of f and g which is

$$f * g = \int_{\mathbb{R}^n} f(x - y)g(y)\ dy.$$

By Proposition 4.4 we know that $f(x - y) \in \mathcal{S}$. To show that the convolution is smooth, we will use Theorem 4.6. Let us argue why the conditions $(i) - (iii)$ are satisfied. The first condition is to fix y, and then the product $f(x - y)g(y)$ is only dependent on the function f and will be a translation which belongs to the Schwartz space. The second condition is to fix x, and then we have the product of f and g with respect to x to be smooth since it belongs to the Schwartz space. Lastly, we have to examine the third condition $\sup_{x \in \mathbb{R}^n} \int_Y |\partial_x^\alpha f(x - y)g(y)| dy < \infty$ for all multi-indices α. A larger term than $\partial_x^\alpha f(x - y)$ is $[f]_{0,\alpha} = \sup_{x \in \mathbb{R}} |\partial_x^\alpha f(x)|$ and it is finite since f is a Schwartz function. This means that we can with justice put this term outside of the integral as

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_x^\alpha f(x-y)g(y)|\ dy \leq \sup_{x\in\mathbb{R}^n}[f]_{0,\alpha}\int_{\mathbb{R}^n}|g(y)|\ dy.$$

The term $\int_{\mathbb{R}^n}|g(y)|\ dy$ is finite since it is independent of x so this whole expression is finite as

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_x^\alpha f(x-y)g(y)|\ dy < \infty.$$

With Theorem 4.6, we now have the result of $f(x-y)g(y)\in C^\infty(\mathbb{R}^n)$ for all $y\in\mathbb{R}^n$ and hence $f*g\in C^\infty(\mathbb{R}^n)$.

The second expression is

$$\partial^\alpha(f*g)(x) = \partial_x^\alpha\int_{\mathbb{R}^n} f(x-y)g(y)\ dy.$$

We previously explained that we can apply Theorem 4.6 on the convolution to show smoothness but we are also allowed to interchange the order of the integration and the derivation. That leads to

$$\int_{\mathbb{R}^n}\partial_x^\alpha f(x-y)g(y)\ dy = \partial^\alpha f * g(x)$$

and the proof is complete. □

Lemma 4.10. *Prove that for all functions $f,g\in\mathcal{S}(\mathbb{R}^n)$ that*

$$f*g\in\mathcal{S}(\mathbb{R}^n).$$

Proof. First, we need to prove that the convolution of f and g are smooth but that is the result of Lemma 4.9. Now, we have to show that the derivatives of the convolution are decreasing faster than a polynomial. Fix $M\geq 0$ and let $L = M+(n+1)$. We are now going to use the two different ways to prove that a function is Schwartz from Lemma 4.2. Expand the expression of the convolution and use the triangle inequality in order to get

$$P_{M,0}(f*g) = (1+|x|)^M|f*g(x)| \leq (1+|x|)^M\int_{\mathbb{R}^n}|f(x-y)g(y)|\ dy.$$

By adding and rearranging some terms we get

$$\int_{\mathbb{R}^n}\frac{(1+|x|)^M}{(1+|x-y|)^M(1+|y|)^L}(1+|x-y|)^M|f(x-y)|\cdot(1+|y|)^L|g(y)|\ dy.$$

Take the supremum with respect to x, of the integral and notice that it can be placed inside the integral since the integral is dependent on y as

$$\int_{\mathbb{R}^n} \frac{(1+|x|)^M}{(1+|x-y|)^M(1+|y|)^L} \sup_{x\in\mathbb{R}^n}(1+|x-y|)^M|f(x-y)| \cdot \sup_{x\in\mathbb{R}^n}(1+|y|)^L|g(y)| \; dy.$$

This expression has terms that is on the same form as $P_{N,\beta}(f) = \sup_{x\in\mathbb{R}}(1+|x|)^N \left|\partial^\beta f(x)\right|$ defined in Lemma 4.2 which leads to the reformulated expression

$$\int_{\mathbb{R}^n} \frac{(1+|x|)^M}{(1+|x-y|)^M(1+|y|)^L} P_{M,0}(f)P_{L,0}(g) \; dy.$$

Here again, we use the fact that this integral only depends on y such that the same terms as before, can be placed outside of the integral as

$$\left(\int_{\mathbb{R}^n} \frac{(1+|x|)^M}{(1+|x-y|)^M(1+|y|)^L} \; dy\right) P_{M,0}(f)P_{L,0}(g).$$

By using Peetre´s Inequality where $t = M$, this expression can be rewritten even further to

$$\left(\int_{\mathbb{R}^n} \frac{(1+|x|)^M}{(1+|y|)^L}(1+|x-y|)^{-M} \; dy\right) P_{M,0}(f)P_{L,0}(g) \\ \leq \left(\int_{\mathbb{R}^n} \frac{(1+|x|)^M}{(1+|x|)^M}(1+|y|)^{M-L} dy\right) P_{M,0}(f)P_{L,0}(g).$$

The expression

$$\left(\int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{L-M}} dy\right) P_{M,0}(f)P_{L,0}(g)$$

is similar to the expression in Lemma 3.3 and by using the result of it, the integral is convergent when $L - M > n$. Since we chose $L = M + (n+1)$ we have $L - M = M + (n+1) - M > n$ which is always true. We already know that $P_{M,0}(f)$ and $P_{L,0}(g)$ are convergent since both f and g are in the Schwartz space. Using this result, one obtains for some constant $C \in \mathbb{R}^n$ that

$$P_{M,0}(f*g) \leq CP_{M,0}(f)P_{L,0}(g) < \infty$$

which shows that $f*g \in \mathcal{S}(\mathbb{R}^n)$. □

5 The Fourier transform

The Fourier transform will be a fundamental tool for solving the PDEs in the following chapter. In this chapter, we will state the definition and present some main properties. After establishing these properties, we will be able to prove the Fourier inversion formula. This chapter is primarily built from chapter 4 in *An Introduction to Pseudo-Differential Operators* (3rd ed.) by M. W. Wong.

5.1 Properties of the Fourier transform

We will begin by presenting the definition and the Fourier transform of a convolution. We will then show some propositions that will be crucial for the proof of the Fourier inversion theorem.

Definition 5.1. Let $f \in \mathcal{S}$. Then

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)dx, \qquad \xi \in \mathbb{R}^n$$

is called the Fourier transform of f.

Proposition 5.2. *Let f and g be real valued functions belonging to $\mathcal{S}$. Then the Fourier transform of the convolution of f and g can be rewritten as*

$$\widehat{f * g} = (2\pi)^{n/2} \hat{f}\hat{g}.$$

Proof. By Lemma 4.10, the convolution of f and g also belongs to $\mathcal{S}$. The following expression can by definition be rewritten as

$$\begin{aligned} (2\pi)^{-n/2}\widehat{f * g}(\xi) &= (2\pi)^{-n/2} \cdot (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} (f * g)(x)dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix\xi} \left(\int_{\mathbb{R}^n} f(x-y)g(y)dy \right) dx. \end{aligned}$$

By using Fubini's theorem, the earlier expression is equal to

$$\begin{aligned} &(2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(x-y)\xi} f(x-y) \cdot e^{-iy\xi} g(y)dydx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iy\xi} g(y) \left(\int_{\mathbb{R}^n} e^{-i(x-y)\xi} f(x-y)dx \right) dy \\ &= (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} e^{-iy\xi} g(y)dy \right) \hat{f}(\xi) \\ &= \hat{g}(\xi)\hat{f}(\xi), \end{aligned}$$

which finishes the proof. □

Remark 5.3. We will use the notion "integration by parts" in the proof of the next proposition. What we mean by that is that it is possible to write that for all integers n and all multi-indices $\alpha \in \mathbb{N}^n$, that

$$\int_{\mathbb{R}^m} \partial^\alpha f(x) g(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^m} f(x) \partial^\alpha g(x) dx$$

if $f \in \mathcal{S}$ and $g \in C^\infty(\mathbb{R}^m)$ and does not grow faster than a polynomial i.e. g satisfies that for all α, there exists an integer M such that

$$\sup_{x \in \mathbb{R}^m} \frac{|\partial^\alpha g(x)|}{(1+|x|)^M} < +\infty.$$

We will show this in one dimension. The expression we will prove is

$$\int_{-\infty}^{\infty} f^{(n)}(x) g(x) dx = (-1)^n \int_{-\infty}^{\infty} f(x) g^{(n)}(x) dx.$$

Induction will be preformed to prove this statement where the base case, $n = 1$ is

$$\int_{-\infty}^{\infty} f'(x) g(x) dx = -\int_{-\infty}^{\infty} f(x) g'(x) dx.$$

This can be shown as true by using Lemma 3.4, the Leibniz rule in one dimension to get

$$\int (fg)'(x) dx = \int f'(x) g(x) dx + \int f(x) g'(x) dx$$

which is

$$\int f'(x) g(x) dx = f(x) g(x) - \int f(x) g'(x) dx.$$

The value of this integral is

$$\begin{aligned}\int_{-\infty}^{\infty} f'(x) g(x) dx &= \lim_{N \to \infty} \int_{-N}^{N} f'(x) g(x) dx \\ &= \lim_{N \to \infty} \left([f(x) g(x)]_{-N}^{N} - \int_{-N}^{N} f(x) g'(x) dx \right).\end{aligned}$$

Since the function f is in the Schwartz space and g does not grow faster than a polynomial, the product of them tends to zero as the variable x goes towards infinity. Therefore we have

$$\lim_{N\to\infty}\left([f(x)g(x)]_{-N}^{N}-\int_{-N}^{N}f(x)g'(x)dx\right)=0-\lim_{N\to\infty}\int_{-N}^{N}f(x)g'(x)dx$$
$$=-\int_{-\infty}^{\infty}f(x)g'(x)dx.$$

The induction hypothesis is

$$\int_{-\infty}^{\infty}f^{(n-1)}(x)g(x)dx=(-1)^{n-1}\int_{-\infty}^{\infty}f(x)g^{(n-1)}(x)dx.$$

The induction step is stated as

$$\int_{-\infty}^{\infty}f^{(n)}(x)g(x)dx=\int_{-\infty}^{\infty}\left(f'(x)\right)^{(n-1)}g(x)dx.$$

By using the induction hypothesis we get the previous relation to be equal to

$$(-1)^{n-1}\int_{-\infty}^{\infty}f'(x)g^{(n-1)}(x)dx$$

and the base case makes it possible to rewrite this expression as

$$(-1)^{n-1}(-1)\int_{-\infty}^{\infty}f(x)\left(g^{(n-1)}(x)\right)'dx$$

which is equal to

$$(-1)^{n}\int_{-\infty}^{\infty}f(x)g^{(n)}(x)dx.$$

Hence we have proved what we wanted to for n derivatives in the one dimensional case.

Proposition 5.4. *Let $\varphi\in\mathcal{S}$. Then*

(i) $(D^\alpha\varphi)^\wedge(\xi)=\xi^\alpha\hat{\varphi}(\xi)$ *for every multi-index* α*,*

(ii) $(D^\beta\hat{\varphi})(\xi)=((-x)^\beta\varphi)^\wedge(\xi)$ *for every multi-index* β*,*

(iii) $\hat{\varphi}\in\mathcal{S}$.

Proof. For the first part (iii), using the Fourier transform gives us

$$(D^\alpha\varphi)^\wedge(\xi)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-ix\xi}(D^\alpha\varphi(x))dx.$$

Since the function $\varphi \in \mathcal{S}$ and the function $e^{-ix\xi}$ can be expressed in terms of sine and cosine that are bounded functions, integrating by parts (from Remark 5.3) is feasible which generates

$$\begin{aligned}
(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} (D^\alpha \varphi(x)) dx &= (2\pi)^{-n/2} (-1)^{|\alpha|} \int_{\mathbb{R}^n} (D^\alpha e^{-ix\xi}) \varphi(x) dx \\
&= (2\pi)^{-n/2} (-1)^{|\alpha|} \int_{\mathbb{R}^n} (-i\partial_x)^\alpha (e^{-ix\xi}) \varphi(x) dx \\
&= (2\pi)^{-n/2} (-1)^{|\alpha|} \int_{\mathbb{R}^n} (-i)^{|\alpha|} (-i\xi)^\alpha e^{-ix\xi} \varphi(x) dx \\
&= (2\pi)^{-n/2} (-1)^{|\alpha|} \int_{\mathbb{R}^n} (-1)^{|\alpha|} \xi^\alpha e^{-ix\xi} \varphi(x) dx \\
&= \xi^\alpha \hat{\varphi}(\xi).
\end{aligned}$$

For the second part (ii), we will implement the Fourier transform again as

$$D^\beta \hat{\varphi}(\xi) = D^\beta \left((2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) dx \right).$$

By using Theorem 4.6 we can interchange the differential sign and the integration. We can use this theorem because when x or ξ is fixed, the product $e^{-ix\xi}\varphi(x)$ belongs to the Schwartz space. The third condition in Theorem 4.6 is valid because $e^{-ix\xi}\varphi(x)$ is in the Schwartz space and the integral of $|\partial_\xi^\alpha e^{-ix\xi}\varphi(x)|$ with respect to x is bounded by definition. Therefore, the supremum will also exist. The interchanging gives us

$$\begin{aligned}
D^\beta \left((2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) dx \right) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} D^\beta \left(e^{-ix\xi} \varphi(x) \right) dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (-x)^\beta e^{-ix\xi} \varphi(x) dx \\
&= ((-x)^\beta \varphi)^\wedge(\xi).
\end{aligned}$$

For the last part (iii), we have to show that $\hat{\varphi} \in \mathcal{S}$ which is by definition, that the function $\hat{\varphi}$ is infinitely differentiable on $\mathbb{R}^n$ satisfying that all multi-indices α and β,

$$\sup_{x \in \mathbb{R}^n} \left| \xi^\alpha D^\beta \hat{\varphi}(\xi) \right| < +\infty.$$

Firstly, the Fourier transform of φ is

$$(\varphi)^\wedge(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) dx.$$

By assumption is $\varphi \in \mathcal{S}$ and we previously explained in (ii) that this expressions satisfies the conditions for Theorem 4.6. Hence, we can say that the integral $\int_{\mathbb{R}^n} e^{-ix\xi}\varphi(x)dx$ is smooth. Secondly, let α and β be any two multi-indices and by (i) and (ii), we have

$$\left|\xi^\alpha D^\beta \hat{\varphi}(\xi)\right| = \left|\xi^\alpha ((-x)^\beta \varphi)^\wedge(\xi)\right| = \left|\{D^\alpha((-x)^\beta\varphi)\}^\wedge(\xi)\right|.$$

Since $\varphi \in \mathcal{S}$ and $(-x)^\beta$ is a polynomial, by Lemma 4.5 the product is in the Schwartz space. Furthermore, we also have that $D^\alpha((-x)^\beta\varphi) \in \mathcal{S}$ by the definition of a Schwartz space. By taking the supremum of the expression $\left|\{D^\alpha((-x)^\beta\varphi)\}^\wedge(\xi)\right|$, and applying the Fourier transform we have

$$\begin{aligned}\sup_{\xi\in\mathbb{R}^n}\left|\{D^\alpha((-x)^\beta\varphi)\}^\wedge(\xi)\right| &= \sup_{\xi\in\mathbb{R}^n}\left|(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\xi}D^\alpha((-x)^\beta\varphi(x))dx\right| \\ &\leq (2\pi)^{-n/2}\sup_{\xi\in\mathbb{R}^n}\int_{\mathbb{R}^n}\left|e^{-ix\xi}\right|\cdot\left|D^\alpha((-x)^\beta\varphi(x))\right|dx.\end{aligned}$$

We know that the integral of $e^{-ix\xi}$ is 1 over the whole real line and since we know that $D^\alpha((-x)^\beta\varphi) \in \mathcal{S}$, this integral will be finite and bounded. Therefore,

$$\sup_{\xi\in\mathbb{R}^n}\left|\xi^\alpha D^\beta\hat{\varphi}(\xi)\right| \leq (2\pi)^{-n/2}\|D^\alpha((-x)^\beta\varphi)\|_1 < \infty$$

and we have shown that $\hat{\varphi} \in \mathcal{S}$. □

Proposition 5.5. *The expressions of $T_y f, M_y f$ and $D_\alpha f$ in Propositions 4.4 are by applying the Fourier transform, equal to*

(i) $(T_y f)^\wedge(\xi) = (M_y \hat{f})(\xi), \quad \xi \in R^n,$

(ii) $(M_y f)^\wedge(\xi) = (T_{-y}\hat{f})(\xi), \quad \xi \in R^n,$

(iii) $(D_a f)^\wedge(\xi) = |a|^{-n}(D_{1/a}\hat{f})(\xi), \quad \xi \in R^n.$

Proof. By using the definition for Fourier transform on f, part *(i)* can be shown by

$$\begin{aligned}(T_y f)^\wedge(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} (T_y f)(x) dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x+y) dx \\ &= \{\textit{Change of variables: } x = x - y\} \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x-y)\xi} f(x) dx \\ &= e^{iy\xi} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx \\ &= e^{iy\xi} \hat{f}(\xi) \\ &= (M_y \hat{f})(\xi).\end{aligned}$$

The second part *(ii)*, is

$$\begin{aligned}(M_y f)^\wedge(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} (M_y f)(x) dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} e^{iyx} f(x) dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix(\xi - y)} f(x) dx \\ &= \hat{f}(\xi - y) \\ &= (T_{-y} \hat{f})(\xi).\end{aligned}$$

The last part *(iii)* can be shown by using a change of variables which is

$$\begin{aligned}(D_a f)^\wedge(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} (D_a f)(x) dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(ax) dx \\ &= \{x = \frac{x}{a}\} \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(\frac{x}{a})\xi} f(x) |a|^{-n} dx \\ &= |a|^{-n} \hat{f}\left(\frac{\xi}{a}\right) \\ &= |a|^{-n} (D_{1/a} \hat{f})(\xi).\end{aligned}$$

□

Proposition 5.6. *Let* $\varphi(x) = e^{-|x|^2/2}$*. Then* $\hat{\varphi}(\xi) = e^{-|\xi|^2/2}$*.*

Proof. First, we will restrict this proof in one dimension. The derivative of the function φ is

$$\varphi'(x) = -x \cdot e^{-|x|^2/2}.$$

Notice that it also is

$$\varphi'(x) = -x \cdot \varphi$$

and rewriting it leads to an ordinary differential equation which looks like

$$\varphi'(x) + x \cdot \varphi = 0.$$

This ODE has one initial value, $\varphi(0) = 1$. Instead of solvning this as we are used to, we are going to use the Fourier transform. The Fourier transform of this ODE is

$$(\varphi')^\wedge(\xi) + (x \cdot \varphi)^\wedge(\xi) = 0.$$

By using statement (i) and (ii) in Proposition 5.4 we can rewrite the equation as

$$\begin{aligned} -\frac{1}{i} \cdot (-i\partial\varphi)^\wedge(\xi) - (-x \cdot \varphi)^\wedge(\xi) &= -\frac{1}{i} \cdot (D\varphi)^\wedge(\xi) - (-x \cdot \varphi)^\wedge(\xi) \\ &= -\frac{1}{i}\xi\widehat{\varphi}(\xi) - D\widehat{\varphi}(\xi) \\ &= -\frac{1}{i}\xi\widehat{\varphi}(\xi) + i\partial\widehat{\varphi}(\xi) \\ &= -\frac{1}{i}\xi\widehat{\varphi}(\xi) + i\widehat{\varphi'}(\xi). \end{aligned}$$

This is equal to zero so it results in

$$\xi\widehat{\varphi}(\xi) + \widehat{\varphi'}(\xi) = 0.$$

The equation we got is on the same form as the ODE before the Fourier transform. Notice that at $\xi = 0$ the function $\widehat{\varphi}$ is

$$\widehat{\varphi}(0) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\cdot 0} \cdot e^{-|x|^2/2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-|x|^2/2} dx = 1.$$

The last expression is the density function of the normal distribution which we know is 1. The conclusion is that the ODEs of φ and $\widehat{\varphi}$ is on the same form with the same initial values which results to the unique solution of the Fourier transformed ODE to be

$$\hat{\varphi}(\xi) = e^{-|\xi|^2/2}.$$

Now we are going to show this in several dimension. The Fourier transform of $\hat{\varphi}(\xi)$ is explicilty

$$\begin{aligned}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} e^{-ix\xi} dx &= (2\pi)^{-n/2}\int_{\mathbb{R}} e^{-\frac{|x_1|^2}{2}}\cdots e^{-\frac{|x_n|^2}{2}}\cdot e^{-ix_1\xi_1}\cdots e^{-ix_n\xi_n} dx\\ &= (2\pi)^{-n/2}\prod_{j=1}^{n}\int_{\mathbb{R}} e^{-\frac{|x_j|^2}{2}} e^{-ix_j\xi_j} dx_j.\end{aligned}$$

Every term of this expression is the Fourier transform of $\varphi(x) = e^{-|x|^2/2}$ which we have shown is on the same form as itself and therefore it is possible to say that this is equal to

$$\prod_{j=1}^{n} e^{-\frac{|\xi_j|^2}{2}} = e^{-\frac{|\xi|^2}{2}}.$$

□

5.2 The Fourier inversion theorem

We are now prepared to prove the key theorem, the Fourier inversion theorem. This theorem will be applied in the final steps of solving PDEs in the following chapter. It will be used to convert solutions from their Fourier transformed expressions.

Theorem 5.7. *(The Fourier Inversion Formula)*
Let $(\hat{f})^\vee = f$ for all functions $f \in \mathcal{S}$. Here, the operation $\vee$ is defined by

$$\check{g}(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\, d\xi, \qquad g \in \mathcal{S}.$$

Proof. Applying the Fourier inversion formula on $\hat{f}$ gives us

$$(\hat{f})^\vee(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)\, d\xi.$$

The application is possible since $f \in \mathcal{S}$ and then also $\hat{f} \in \mathcal{S}$ by Proposition 5.4. Let $\epsilon > 0$ and define

$$I_\epsilon(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi-(\epsilon^2|\xi|^2)/2}\hat{f}(\xi)\, d\xi.$$

The goal with this proof is to show that $I_\epsilon(x)$ tends to f and to $(\hat{f})^\vee$ as ϵ approaches 0. Let a part of the function in the integral to be

$$g(\xi) = (2\pi)^{-n/2}\ e^{ix\cdot\xi-(\epsilon^2|\xi|^2)/2}.$$

Notice that $g(\xi)$ can be rewritten with the expressions in Proposition 5.5 as

$$g(\xi) = (M_x D_\epsilon \varphi)(\xi),$$

where

$$\varphi(\xi) = (2\pi)^{-n/2}\ e^{-|\xi|^2/2}.$$

We know that the integral of $\varphi(\xi)$ over $\mathbb{R}^n$ is 1 since it is the Gaussian function. By Propositions 5.5 we can compute the Fourier transform of $M_x D_\epsilon \varphi$ and with Proposition 5.6 we know that the Fourier transform of $\varphi(\xi)$ is,

$$\hat{g}(\eta) = (T_{-x}\epsilon^{-n} D_{1/\epsilon}\hat{\varphi})(\eta) = (2\pi)^{-n/2}\epsilon^{-n} e^{-|\eta - x|^2/(2\epsilon^2)}.$$

The expression we defined as I_ϵ can be formulated as $I_\epsilon(x) = \int_{\mathbb{R}^n} g(\xi)\hat{f}(\xi)d\xi$ and since we know the Fourier transform of g, we are going to apply it. Expanding the Fourier transform of f is

$$\int_{\mathbb{R}^n} g(\xi)\left((2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot\eta} f(\eta)d\eta\right)d\xi$$

and by Fubini's theorem, the order of intergration can be changed since the functions are in the Schwartz space. The result is

$$\int_{\mathbb{R}^n} f(\eta)\left((2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot\eta} g(\xi)d\xi\right)d\eta = \int_{\mathbb{R}^n} f(\eta)\hat{g}(\eta)d\eta.$$

Implementing the Fourier transform of $g(\eta)$ in I_ϵ is

$$(2\pi)^{-n/2}\epsilon^{-n}\int_{\mathbb{R}^n} f(\eta)e^{-|\eta - x|^2/(2\epsilon^2)}d\eta$$

and notice that this can be rewritten as a convolution like

$$I_\epsilon(x) = (f * \varphi_\epsilon)(x),$$

where $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(\frac{x}{\epsilon})$. The convolution can also be written as

$$f * \varphi_\epsilon(x) = \int_{\mathbb{R}^n} f(x-y)\varphi(y/\epsilon)\epsilon^{-n}dy.$$

Substitution of variables with $y = \epsilon z$ and $dy = \epsilon^n dz$ leads to

$$\int_{\mathbb{R}^n} f(x - \epsilon z)\varphi(z)dz.$$

Subtracting $f(x)$ from the convolution is equal to

$$f * \varphi_\epsilon(x) - f(x) = \int_{\mathbb{R}^n} f(x - y)\varphi_\epsilon(y)\ dy - f(x)$$

and since $\int_{\mathbb{R}^n} \varphi(z)dz = 1$, this integral can be multiplied as

$$\int_{\mathbb{R}^n} f(x - y)\varphi_\epsilon(y)\ dy - \int_{\mathbb{R}^n} \varphi(z)\ dz \cdot f(x).$$

By using the same substitution of variables as earlier again, we get

$$\int_{\mathbb{R}^n} f(x - \epsilon z)\epsilon^{-n}\varphi(z)\ \epsilon^n dz - \int_{\mathbb{R}^n} f(x)\varphi(z)\ dz = \int_{\mathbb{R}^n} \left(f(x - \epsilon z) - f(x)\right)\varphi(z)\ dz.$$

Notice, that the fundamental theorem of calculus on the function $g(x) = f(x - \epsilon tz)$, is

$$g(1) - g(0) = \int_0^1 g'(t)dt,$$

and then for all $x \in \mathbb{R}^n$, we have

$$f(x - \epsilon z) - f(x) = \int_0^1 \left(\frac{d}{dt} f(x - \epsilon tz)\right)\ dt.$$

Applying the chain rule on this is expressed as

$$\int_0^1 (\nabla f)(x - \epsilon tx) \cdot (-\epsilon z)dt.$$

Expressing this with a sum is

$$-\epsilon \sum_{j=1}^{n} \int_0^1 \frac{\partial f}{\partial e_j}(x - \epsilon tx)dt \cdot z_j.$$

We can now bound $f(x-\epsilon z)-f(x)$ as

$$\begin{aligned}|f(x-\epsilon z)-f(x)| &= \left|-\epsilon\sum_{j=1}^{n}\int_0^1 \frac{\partial f}{\partial e_j}(x-\epsilon tx)dt\cdot z_j\right| \\ &\leq \epsilon\sum_{j=1}^{n}\int_0^1\left|\frac{\partial f}{\partial e_j}(x-\epsilon tx)\right|dt\cdot|z_j| \\ &\leq \epsilon\sum_{j=1}^{n}P_{0,e_j}(f)|z_j| \\ &\leq \epsilon\left(\sum_{j=1}^{n}P_{0,e_j}(f)\right)|z|.\end{aligned}$$

Recall the expression with the convolution and that it now is bounded by

$$\begin{aligned}|f*\varphi_\epsilon(x)-f(x)| &= \left|\int_{\mathbb{R}^n}\left(f(x-\epsilon z)-f(x)\right)\varphi(z)\ dz\right| \\ &\leq \epsilon\sum_{j=1}^{n}P_{0,e_j}(f)\int_{\mathbb{R}^n}|\varphi(z)||z|\ dz.\end{aligned}$$

Since the function $\varphi\in\mathcal{S}$, the product of a polynomial $|z|$ also belongs to the Schwartz space by Lemma 4.5. Then the supremum of the integral of $\varphi(z)|z|$ is finite by the definition of the Schwartz space (which is Definition 4.1.)

Now, we want to show the uniform convergence of the convolution $f*\varphi_\epsilon$ to the function f. Recall, that a sequence of functions $h_n(x)$ converges uniformly to a function $h(x)$ on a set E if for every $\epsilon>0$ there exists a natural number N such that for all $n\geq N$ and for all $x\in E$,

$$|h_n(x)-h(x)|<\epsilon.$$

Also, recall that the supremum norm of a function h on a set X is defined as

$$\|h\|_\infty \coloneqq \sup\{|h(x)| : x\in X\}.$$

With our earlier deduction about the integral of $|\varphi(z)||z|$ being finite, we can take the supremum of the whole inequality and that implies

$$\|f*\varphi_\epsilon-f\|_\infty\leq\epsilon\left(\sum_{j=1}^{n}P_{0,e_j}(f)\right)C(\varphi).$$

By assumption, are both f and φ in the Schwartz space, meaning the expression $\|f * \varphi_\epsilon - f\|_\infty$ being bounded by a constant. As ϵ is tending towards zero, the whole expression is

$$\lim_{\epsilon \to 0} \|f * \varphi_\epsilon - f\|_\infty = 0.$$

Therefore the convolution $f * \varphi_\epsilon(x)$ will converge uniformly to $f(x)$. Hence, we have

$$I_\epsilon(x) = (f * \varphi_\epsilon)(x) \to f(x) = f.$$

Let us observe the behavior of the integral $I_\epsilon(x)$ in comparison to $(\hat{f})^\vee(x)$ as ϵ approaches 0. The difference is

$$\left| I_\epsilon(x) - (\hat{f})^\vee(x) \right| = \left| (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi - (\epsilon^2 |\xi|^2)/2} \hat{f}(\xi) \, d\xi - (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{f}(\xi) \, d\xi \right|$$

which we can simplify as

$$\left| (2\pi)^{-n/2} \int_{\mathbb{R}^n} (e^{-(\epsilon^2 |\xi|^2)/2} - 1) \hat{f}(\xi) \, d\xi \right|.$$

Notice, that

$$|e^{-\epsilon^2 |\xi|^2/2} - 1| \leq \epsilon^2 |\xi|^2/2$$

is true, and we can therefore write

$$\left| (2\pi)^{-n/2} \int_{\mathbb{R}^n} (e^{-(\epsilon^2 |\xi|^2)/2} - 1) \hat{f}(\xi) \, d\xi \right| \leq \left| (2\pi)^{-n/2} \frac{\epsilon^2}{2} \int_{\mathbb{R}^n} |\xi|^2 \hat{f}(\xi) \, d\xi \right|.$$

Since $f \in \mathcal{S}$, the product $|\xi|^2 |\widehat{f}(\xi)| = |\xi_1^2 + \ldots + \xi_n^2| |\widehat{f}(\xi)|$ is also in the Schwartz space. The product is therefore integrable and we can show that

$$|I_\epsilon(x) - (\hat{f})^\vee(x)| \leq C(f) \cdot \epsilon^2$$

which tends to 0 as ϵ approaches 0. Hence $I_\epsilon \to (\hat{f})^\vee(x)$ and we have therefore proved

$$(\hat{f})^\vee(x) = f(x).$$

□

6 Applications of the Fourier Transform on PDEs

This chapter discusses applications of the presented theory in this thesis. We will apply the Fourier transform to solve two PDEs, the wave equation and the heat equation. A more general solution of the wave equation will be presented in n dimensions and a simplified version will be examined in one dimension. The source of the theory corresponding to the wave equation is from *The Fourier Transform and the Wave Equation* by A. Torchinsky [7]. The solution of the heat equation will be presented in n dimensions along with a theorem related to the solution. The source of both the solution and the theorem is from the lecture notes ”Heat equation” by Julie Levandosky [5].

Before stating the PDEs, denote the Laplacian in $\mathbb{R}^n$ as

$$\nabla^2 = \partial^2_{x_1x_1} + \ldots + \partial^2_{x_nx_n}.$$

We can now formally state the wave equation.

6.1 The wave equation

The wave equation in n dimensions is

$$\partial^2_{tt}u(x,t) = \nabla^2 u(x,t) \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R},\ t > 0$$

with initial conditions

$$u(x,0) = \varphi(x) \quad \text{and} \quad \partial_t u(x,0) = \psi(x), \quad x \in \mathbb{R}^n,$$

where φ and ψ are in $\mathcal{S}(\mathbb{R}^n)$.

6.1.1 The general solution of the wave equation in n dimensions

Suppose that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Use the Fourier transform on the wave equation to obtain the Fourier solution. Consider t as a parameter and apply the Fourier transform on the second derivative of u which is expressed as

$$\widehat{\partial^2_{x_kx_k}u}(\xi,t) = i^{|2|}\xi_k^2\widehat{u}(\xi,t) = -\xi_k^2\widehat{u}(\xi,t), \quad 1 \le k \le n.$$

Here, the statement (i) in Proposition 5.4 is used in the first step. It follows that the Fourier transform of the Laplacian becomes

$$\widehat{\nabla^2 u}(\xi,t) = \sum_{k=1}^{n} \widehat{\partial^2_{x_k x_k} u}(\xi,t) = \sum_{k=1}^{n} -\xi_k^2 \widehat{u}(\xi,t) = -(\xi_1^2 + \ldots + \xi_n^2)\widehat{u}(\xi,t) = -|\xi|^2 \widehat{u}(\xi,t).$$

The Fourier transform of the wave equation is

$$\partial^2_{tt}\widehat{u}(\xi,t) + |\xi|^2 \widehat{u}(\xi,t) = 0, \quad \xi \in \mathbb{R}^n,\ t > 0.$$

with initial conditions

$$\widehat{u}(\xi,0) = \widehat{\varphi}(\xi) \quad \text{and} \quad \partial_t \widehat{u}(\xi,0) = \widehat{\psi}(\xi), \quad \xi \in \mathbb{R}^n.$$

This equation is an homogeneous ordinary differential equation in t which we know how to solve as in the next lemma.

Lemma 6.1. *The ordinary differential equation*

$$\partial^2_{tt}\widehat{u}(\xi,t) + |\xi|^2 \widehat{u}(\xi,t) = 0, \quad \xi \in \mathbb{R}^n,\ t > 0,$$

with initial conditions

$$\widehat{u}(\xi,0) = \widehat{\varphi}(\xi) \quad \textit{and} \quad \partial_t \widehat{u}(\xi,0) = \widehat{\psi}(\xi), \quad \xi \in \mathbb{R}^n,$$

has the solution

$$\widehat{u}(\xi,t) = \widehat{\varphi}(\xi)\cos(|\xi|t) + \frac{\widehat{\psi}(\xi)\sin(|\xi|t)}{|\xi|}.$$

Proof. The characteristic function to the ODE is

$$r^2 + |\xi|^2 = 0 \quad \Leftrightarrow \quad r = \pm\sqrt{-|\xi|^2}.$$

Putting these roots into the general form of solutions, we gain the two functions

$$\widehat{u_1}(\xi,t) = e^{i|\xi|t} \quad \text{and} \quad \widehat{u_2}(\xi,t) = e^{-i|\xi|t}$$

and by using Euler's Formula we can rewrite these solutions as

$$\widehat{u_1}(\xi,t) = \cos(|\xi|t) + i\sin(|\xi|t) \quad \text{and} \quad \widehat{u_1}(\xi,t) = \cos(|\xi|t) - i\sin(|\xi|t).$$

These solutions can be expressed as one solution by adding them as

$$\begin{aligned}\widehat{u}(\xi,t) &= c_1\widehat{u_1}(\xi,t) + c_2\widehat{u_2}(\xi,t)\\ &= (c_1+c_2)\cos(|\xi|t) + (c_1-c_2)i\sin(|\xi|t).\end{aligned}$$

The derivative of $\widehat{u}(\xi,t)$ in time is

$$\partial_t\widehat{u}(\xi,t) = -|\xi|(c_1+c_2)\sin(|\xi|t) + |\xi|(c_1-c_2)i\cos(|\xi|t).$$

To find the constants c_1 and c_2 we have to use the initial conditions and comparing them which gives us the equations system

$$\begin{cases}\widehat{u}(\xi,0) = c_1+c_2 = \widehat{\varphi}(\xi),\\ \partial_t\widehat{u}(\xi,0) = |\xi|(c_1-c_2)i = \widehat{\psi}(\xi),\end{cases} \Rightarrow \quad \begin{cases}c_1 = \frac{\widehat{\varphi}-i\widehat{\psi}/|\xi|}{2},\\ c_2 = \frac{\widehat{\varphi}+i\widehat{\psi}/|\xi|}{2}.\end{cases}$$

Hence, the final solution to the Fourier transform of the wave equation is

$$\widehat{u}(\xi,t) = \widehat{\varphi}(\xi)\cos(|\xi|t) + \frac{\widehat{\psi}(\xi)\sin(|\xi|t)}{|\xi|}.$$

□

The final solution is on a Fourier transformed version and we need to find the inversion of it by applying the inversion formula. The application is justified by arguing that the function $\widehat{u}(\xi,t)$ is in the Schwartz space, which indeed it is. The terms of $\widehat{u}(\xi,t)$ are products of $\widehat{\varphi}(\xi)\in\mathcal{S}$ and the smooth functions cosine and $\frac{\sin(|\xi|t)}{|\xi|}$ (with the value defined as 1 at $\xi=0$) and hence, $\widehat{\varphi}(\xi)$ is smooth. We also need to show that cosine and $\frac{\sin(|\xi|t)}{|\xi|}$, and all its derivatives do not grow faster than a polynomial.

Claim 6.2. *The functions* $\cos(|\xi|)$ *and* $\frac{\sin(|\xi|)}{|\xi|}$ *are smooth and also themselves and all their derivatives are bounded.*

Proof. Notice that the cosine function can be represented by its power series expressed as

$$\cos(|\xi|) = \sum(-1)^k|\xi|^{2k}/(2k!).$$

With uniform convergence on any ball of the type $|\xi|\leq R$ we have that $|\xi|^{2k} = (\xi_1^2+\ldots+\xi_n^2)^k$ is a differentiable function. The function cosine is differentiable over $\mathbb{R}^n$ but the function $|\xi|$ is not differentiable at $\xi=0$. We will examine two cases.

For $|\xi|$ larger than 1, we can use the chain rule on $\cos(|\xi|)$ as

$$\partial_j \cos(|\xi|) = \cos'(|\xi|) \cdot \partial_j |\xi|$$

where

$$\begin{aligned}\partial_j |\xi| &= \partial_j \sqrt{\xi_1^2 + \cdots + \xi_n^2} \\ &= \frac{1}{2}(\xi_1^2 + \cdots + \xi_n^2)^{-1/2} \cdot 2\xi_j \\ &= \frac{\xi_j}{|\xi|}.\end{aligned}$$

The second derivative of $\cos(|\xi|)$ is by the Leibniz rule

$$\partial_{j,k}^2 \cos(|\xi|) = \cos''(|\xi|) \cdot \partial_j |\xi| \partial_k |\xi| + \cos'(|\xi|) \cdot \partial_j^2 |\xi|$$

where

$$\begin{aligned}\partial_j^2 |\xi| &= \partial_k(\xi_j) \cdot |\xi|^{-1} + \xi_j \partial_k(|\xi|^{-1}) \\ &= \partial_k(\xi_j) \cdot |\xi|^{-1} - \xi_j |\xi|^{-2} \partial_k |\xi|.\end{aligned}$$

The expression $\partial_k(\xi_j)$ is 1 when $j = k$ and 0 when $j \neq k$. Hence

$$\begin{aligned}\left|\partial_j^2 |\xi|\right| &\leq \frac{1}{|\xi|} + \left|\frac{\xi_j}{|\xi|}\right| \frac{1}{|\xi|} \frac{\xi_k}{|\xi|} \\ &< 2,\end{aligned}$$

since $|\xi| > 1$ and the numerator of every fraction is smaller than 1. We have shown that the first and second derivative of $|\xi|$ are bounded. The higher degrees of derivatives of $|\xi|$ are functions of $\frac{\xi_j}{|\xi|}$ or $|\xi|^{-1}$ and they are also bounded. Now we have both that the derivatives of cosine and the derivatives of the function $|\xi|$ are bounded. The power series of cosine is

$$\begin{aligned}\cos(|\xi|) &= \sum_{k=0}^{\infty} \frac{(-1)^k |\xi|^{2k}}{2k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\xi_1^2 + \ldots + \xi_n^2)^k}{2k!},\end{aligned}$$

where the function

$$F(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{2k!}$$

is infinitely differentiable and $|\xi|$ is also infinitely differentiable and bounded shown earlier.

For $|\xi|$ smaller than 1, we will use that $\cos(|\xi|) \in \mathscr{C}^\infty(\mathbb{R}^n)$ and for all α we have that

$$\partial^\alpha \cos(|\xi|) \in \mathscr{C}^\infty(\mathbb{R}^n)$$

and bounded. Taking the maximum of this leads to

$$\max_{|\xi|\leq 1} |\partial^\alpha \cos(|\xi|)| \leq C_\alpha < +\infty$$

since $|\xi| \leq 1$ is a compact set and by continuity of cosine.

A similar reasoning can be applied to $\frac{\sin(|\xi|t)}{|\xi|}$. The Taylor series expansion of sine leads to

$$\frac{\sin(|\xi|t)}{|\xi|} = \frac{1}{|\xi|}\sum_{j=0}^{\infty}\frac{(-1)^j|\xi|^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty}\frac{(-1)^j|\xi|^{2j}}{(2j+1)!}.$$

Here again, the function $|\xi|^{2j}$ appeared which we have shown is infinitely differentiable and bounded and the function

$$G(z) = \sum_{j=0}^{\infty}\frac{(-1)^j z^j}{(2j+1)!}$$

is infinitely differentiable. Hence, all their derivatives are bounded. □

Applying the inversion formula on $\widehat{u}(\xi, t)$ yields the equation

$$u(x,t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\widehat{\varphi}(\xi)\cos(|\xi|t)e^{i\xi x}d\xi + \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\widehat{\psi}(\xi)\frac{\sin(|\xi|t)}{|\xi|}e^{i\xi x}d\xi.$$

This expression is one version of the solution of the wave equation. We would now like to write it on another form without mixing functions with Fourier transformed functions. In the next subchapter we will show what a solution without any Fourier transformed functions looks like in one dimension.

6.1.2 A compact solution of the wave equation in one dimension

The solution to the wave equation in one dimension is

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\varphi}(\xi)\cos(|\xi|t)e^{i\xi x}d\xi + \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\psi}(\xi)\frac{\sin(|\xi|t)}{|\xi|}e^{i\xi x}d\xi.$$

To make it more compact and readable without any Fourier transformed functions, we want to rewrite the terms in this solution as in the following lemmas.

Lemma 6.3. *The expression*

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\psi}(\xi)\frac{\sin(|\xi|t)}{|\xi|}e^{i\xi x}d\xi$$

can be rewritten as

$$\frac{1}{2}\int_{x-t}^{x+t}\psi(z)\ dz.$$

Proof. We will use the indicator function defined as

$$\mathbb{1}_A = \begin{cases} 1 & if\ x \in A, \\ 0 & if\ x \notin A. \end{cases}$$

The Fourier transform of the indicator function is

$$\begin{aligned}
\widehat{\frac{1}{2}\mathbb{1}_{[-t,t]}} &= \frac{1}{\sqrt{2\pi}}\int_{-t}^{t} e^{-ix\xi}\frac{1}{2}\cdot 1\ dx \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{-i\xi}\left[e^{-ix\xi}\right]_{-t}^{x=t} \\
&= -\frac{1}{2i\xi\sqrt{2\pi}}\cdot(-2i)\left(\frac{e^{it\xi}-e^{-it\xi}}{2i}\right) \\
&= \frac{1}{\sqrt{2\pi}}\frac{\sin t\xi}{\xi}.
\end{aligned}$$

Rewrite this relation as

$$\sqrt{2\pi}\cdot\widehat{\frac{1}{2}\mathbb{1}_{[-t,t]}} = \frac{\sin t\xi}{\xi}.$$

Since $\frac{\sin t\xi}{\xi}$ is an even function with respect to ξ, it is possible to add an absolute value on ξ without changing anything. Applying this in our wanted expression is equal to

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\psi}(\xi)\frac{\sin(|\xi|t)}{|\xi|}e^{i\xi x}d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\psi}(\xi)\sqrt{2\pi}\cdot\widehat{\frac{1}{2}\mathbb{1}_{[-t,t]}}e^{i\xi x}d\xi.$$

Evaluating the Fourier transform on the indicator functions again leads to

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\psi}(\xi)\sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\int_{-t}^{t}e^{-iy\xi}\frac{1}{2}\ dy\right)e^{i\xi x}d\xi$$

and changing the order of integration by Fubini's theorem is

$$\frac{1}{2}\int_{-t}^{t}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\psi}(\xi)e^{i\xi(x-y)}d\xi\ dy.$$

Use the Fourier inverse formula to simplify the expression above to get

$$\frac{1}{2}\int_{-t}^{t}\left(\widehat{\psi}\right)^{\vee}(x-y)dy=\frac{1}{2}\int_{-t}^{t}\psi(x-y)dy.$$

A substitution of variables where $z=x-y$ results in

$$\frac{1}{2}\int_{x+t}^{x-t}\psi(z)(-1)\ dz=\frac{1}{2}\int_{x-t}^{x+t}\psi(z)\ dz.$$

□

We have now simplified the second term of the solution to the wave equation. Notice, that the first term with cosine, is the derivative of the second term with sine with respect to t. This is presented in the following lemma.

Lemma 6.4. *The expression*

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\varphi}(\xi)\cos(|\xi|t)e^{i\xi x}d\xi$$

is equal to

$$\frac{1}{2}\left(\varphi(x+t)+\varphi(x-t)\right).$$

Proof. The derivative with respect to t of $\frac{\sin(|\xi|t)}{|\xi|}$ is $\cos\left(|\xi|t\right)$ and we can therefore compute the derivative of the simplified expression of the sine term in Lemma 6.3 which is

$$\frac{d}{dt}\left(\frac{1}{2}\int_{x-t}^{x+t}\varphi(\tilde{x})d\tilde{x}\right).$$

Since φ is differentiable by being the Schwartz space, a primitive integral $h(\tilde{x})$ exists and we get

$$\frac{d}{dt}\left(\frac{1}{2}\left(h(x+t)-h(x-t)\right)\right).$$

Since $h(\tilde{x})$ is the primitive integral of $\varphi(\tilde{x})$, the other way around is that the derivative of $h(\tilde{x})$ is $\varphi(\tilde{x})$. Which leads us to

$$\frac{1}{2}\left(h'(x+t)-(-1)h'(x-t)\right)=\frac{1}{2}\left(\varphi(x+t)+\varphi(x-t)\right).$$

□

Finally, for the wave equation in one dimension,

$$\partial_{tt}^2 u(x,t) = \nabla^2 u(x,t) \qquad x \in \mathbb{R},\ t \in \mathbb{R},\ t > 0$$

with initial conditions

$$u(x,0) = \varphi(x) \quad \text{and} \quad \partial_t u(x,0) = \psi(x), \quad x \in \mathbb{R},$$

where φ and ψ are in $\mathcal{S}(\mathbb{R})$, the final solution in one dimension is

$$u(x,t) = \frac{1}{2}\left(\varphi(x+t) + \varphi(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \psi(z)\ dz$$

where $x \in \mathbb{R}$ and $t > 0$.

6.2 The heat equation

The heat equation in n dimensions is

$$\partial_t u(x,t) = k\nabla^2 u(x,t) \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R},\ t > 0$$

and $k > 0$ is a proportionality constant, with initial condition

$$u(x,0) = \phi(x) \quad x \in \mathbb{R}^n,$$

where ϕ is in $\mathcal{S}(\mathbb{R}^n)$.

6.2.1 The solution of the heat equation in n dimensions

We know from the last subchapter that the Fourier transform of the Laplacian is

$$\widehat{\nabla^2} u(\xi,t) = -|\xi|^2 \widehat{u}(\xi,t)$$

and by applying it in the heat equation we get

$$\partial_t \widehat{u}(\xi,t) = -k|\xi|^2 \widehat{u}(\xi,t).$$

This is an ordinary differential equation on the form

$$\partial_t \widehat{u}(\xi,t) + k|\xi|^2 \widehat{u}(\xi,t) = 0$$

which we will solve in the next lemma.

Lemma 6.5. *The ordinary differential equation*

$$\partial_t \widehat{u}(\xi,t) + k|\xi|^2 \widehat{u}(\xi,t) = 0, \quad \xi \in \mathbb{R}^n, \ t > 0,$$

with initial condition

$$\widehat{u}(\xi,0) = \widehat{\phi}(\xi) \quad \xi \in \mathbb{R}^n,$$

has the solution

$$\widehat{u}(\xi,t) = \widehat{\phi}(\xi) e^{-k|\xi|^2 t}.$$

Proof. The ODE in question has the characteristic equation

$$r + k|\xi|^2 = 0$$

with the root

$$r = -k|\xi|^2.$$

The homogenous solution to the ODE is

$$\widehat{u}(\xi,t) = Ce^{-k|\xi|^2 t}.$$

The initial condition gives us

$$\widehat{u}(\xi,0) = Ce^{-k|\xi|^2 \cdot 0} = C = \widehat{\phi}(\xi)$$

and therefore the final solution is

$$\widehat{u}(\xi,t) = \widehat{\phi}(\xi) e^{-k|\xi|^2 t}.$$

□

Since $\widehat{u}(\xi,t) \in \mathcal{S}$, applying the Fourier inversion formula on $\widehat{u}(\xi,t)$ gives us

$$u(\xi,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi x} \widehat{\phi}(\xi) e^{-k|\xi|^2 t} d\xi.$$

Expanding the Fourier transform on $\widehat{\phi}(\xi)$ yields

$$\begin{aligned}
&\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi x} \left[\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi y} \phi(y) dy \right] e^{-k|\xi|^2 t} d\xi \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \phi(y) \left[\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi(y-x)} e^{-k|\xi|^2 t} d\xi \right] dy \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \phi(y) \widehat{f}(y-x) dy
\end{aligned}$$

where $f(\xi) = e^{-k|\xi|^2 t}$. Now, we want to analyze the Fourier transform of $f(\xi)$. Notice the similarities in this expression and the expression in Proposition 5.6.

Lemma 6.6. *The Fourier transform of the function*

$$f(\xi) = e^{-k|\xi|^2 t}$$

is expressed as

$$\widehat{f}(z) = \frac{1}{(2kt)^{n/2}} e^{-|z|^2/4kt}.$$

Proof. Let the function $g(\xi)$ be

$$g(\xi) = e^{-|\xi|^2/2}$$

and in Proposition 5.6 it is stated that the Fourier transform of $g(\xi)$ is itself. Notice that the function $f(\xi)$ can be expressed in a another from by dilation with $g(\xi)$ as

$$f(\xi) = e^{-k|\xi|^2 t} = e^{2kt \ \cdot \ -|\xi|^2/2} = e^{-\left|\sqrt{2kt}\cdot\xi\right|^2/2} = g\left(\sqrt{2kt}\cdot\xi\right).$$

The Fourier transform of the dilated function $g\left(\sqrt{2kt}\cdot\xi\right)$ is by Proposition 5.5 equal to

$$(D_{\sqrt{2kt}}\ g)^\wedge(\xi) = \left|\sqrt{2kt}\right|^{-n} (D_{1/\sqrt{2kt}}\ \hat{g})(z).$$

Simplifying it results in

$$\begin{aligned}
\left|\sqrt{2kt}\right|^{-n} (D_{1/\sqrt{2kt}}\ \hat{g})(z) &= \frac{1}{(2kt)^{n/2}} e^{-\left|\frac{1}{\sqrt{2kt}}\cdot z\right|^2/2} \\
&= \frac{1}{(2kt)^{n/2}} e^{-|z|^2/4kt}.
\end{aligned}$$

Hence, the Fourier transform of $f(\xi)$ is equal to the Fourier transform of $g\left(\sqrt{2kt}\cdot\xi\right)$

which is

$$\frac{1}{(2kt)^{n/2}}e^{-|z|^2/4kt}.$$

$\square$

The result of the Fourier transform of $f(\xi) = e^{-k|\xi|^2 t}$ can be applied to the main solution of the heat equation

$$u(\xi, t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \phi(y)\widehat{f}(y-x)dy$$

which is equal to

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \phi(y)\frac{1}{(2kt)^{n/2}}e^{-|x-y|^2/4kt}dy = \frac{1}{(4k\pi t)^{n/2}}\int_{\mathbb{R}^n} \phi(y)e^{-|x-y|^2/4kt}dy.$$

We have finally reached the solution to the heat equation in n dimensions which is

$$u(\xi, t) = \frac{1}{(4k\pi t)^{n/2}}\int_{\mathbb{R}^n} \phi(y)e^{-|x-y|^2/4kt}dy$$

for $t > 0$.

6.2.2 A theorem based on the solution of the heat equation

The following theorem explains some properties the solution formula $u(x, t)$ of the heat equation have.

Theorem 6.7. *Assume $\phi \in C^\infty(\mathbb{R}^n)$ and bounded, and define*

$$u(x, t) = \frac{1}{(4k\pi t)^{n/2}}\int_{\mathbb{R}^n} \phi(y)\ e^{-|x-y|^2/4kt}dy.$$

Then the following is true,

(i) $u \in C^\infty(\mathbb{R}^n)$,

(ii) $u_t - k\nabla^2 u = 0$ *for all* $x \in \mathbb{R}^n, t > 0$,

(iii) $\lim_{(x,t)\to(x_0,0)} u(x, t) = \phi(x_0)$ *where* $x_0, x \in \mathbb{R}^n, t > 0$.

Proof. The proof will execute $(i) - (iii)$.

(i) The first part can be proved by using Theorem 4.6. Recall that ϕ is smooth and bounded which is equivalent to ϕ having the properties

$$\sup_{x\in\mathbb{R}^n} |\phi(x)| < +\infty \qquad \text{and} \qquad \phi \in C^\infty(\mathbb{R}^n).$$

The first and the second condition in Theorem 4.6 is fulfilled since for a fixed x, the product $\phi(x)e^{-|x-y|^2/4kt}$ is only the Gaussian function depending on y which we know is in the Schwartz space. For a fixed y, the product is smooth since the exponential term is the Gaussian function. The third condition is also satisfied since the derivative with respect to x of the product $\phi(x)e^{-|x-y|^2/4kt}$, is finite by the definition of the Schwartz space. We can then integrate $\phi(x)e^{-|x-y|^2/4kt}$ with respect to y and the supremum of it will also be finite. Hence, the Theorem 4.6 can be applied and therefore the function $u(x,t)$ is smooth over $\mathbb{R}^n$.

(ii) For second part, we are going to use the function $\mathcal{H}(x,t) \equiv \frac{1}{(4\pi t)^{n/2}} \ e^{-|x|^2/4t}$. This function satisfies the heat equation for the variables $\mathcal{H}(x-y,kt)$ which we will show now. By the Leibniz rule, we have the partial derivative with respect to t, to be

$$\frac{\partial}{\partial t}\mathcal{H}(x-y,kt) = \frac{d}{dt}\left((4k\pi t)^{-n/2}\right) \cdot e^{-|x-y|^2/4kt} + (4k\pi t)^{-n/2} \cdot \frac{d}{dt}\left(e^{-|x-y|^2/4kt}\right).$$

By expressing the derivatives explicitly, we have

$$-\frac{n}{2t}(4k\pi t)^{-n/2} \cdot e^{-|x-y|^2/4kt} + (4k\pi t)^{-n/2} \cdot e^{-|x-y|^2/4kt}\left(\frac{|x-y|^2}{4kt^2}\right)$$

and factorising leads to

$$\frac{\partial}{\partial t}\mathcal{H}(x-y,kt) = (4k\pi t)^{-n/2} e^{-|x-y|^2/4kt}\left(\frac{|x-y|^2}{4kt^2} - \frac{n}{2t}\right).$$

The Laplacian with respect to x is computed by first finding

$$\frac{\partial}{\partial x_j}\mathcal{H}(x-y,kt) = (4k\pi t)^{-n/2} \cdot e^{-|x-y|^2/4kt}\left(-\frac{|x-y|}{2kt}\right),$$

and then

$$\frac{\partial^2}{\partial x_j^2}\mathscr{H}(x-y,kt) = (4k\pi t)^{-n/2}\cdot e^{-|x-y|^2/4kt}\left(\frac{|x-y|^2}{4k^2t^2}-\frac{1}{2kt}\right).$$

Summing over $j=1$ to n gives us

$$\begin{aligned}\nabla^2\mathscr{H}(x-y,kt) &= \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}\mathscr{H}(x-y,kt)\\ &= \sum_{j=1}^{n}(4k\pi t)^{-n/2}\cdot e^{-|x-y|^2/4kt}\left(\frac{|x_j-y|^2}{4k^2t^2}-\frac{1}{2kt}\right)\\ &= (4k\pi t)^{-n/2}\cdot e^{-|x-y|^2/4kt}\left(\sum_{j=1}^{n}\frac{|x_j-y|^2}{4k^2t^2}-\sum_{j=1}^{n}\frac{1}{2kt}\right)\\ &= (4k\pi t)^{-n/2}\cdot e^{-|x-y|^2/4kt}\left(\frac{|x-y|^2}{4k^2t^2}-\frac{n}{2kt}\right).\end{aligned}$$

The heat equation is $\partial_t u(x,t) = k\nabla^2 u(x,t)$ and by multiplying $\nabla^2 u(x,t)$ with k we arrive at the conclusion

$$k\nabla^2 u(x,t) = k\cdot(4k\pi t)^{-n/2}\cdot e^{-|x-y|^2/4kt}\left(\frac{|x-y|^2}{4k^2t^2}-\frac{n}{2kt}\right) = \partial_t u(x,t).$$

Since the function $\mathscr{H}(x-y,kt)$ is a solution to the heat equation, we can use the previous knowledge that the solution to the heat equation is on form

$$u(x,t) = \frac{1}{(4k\pi t)^{n/2}}\int_{\mathbb{R}^n}\phi(y)e^{-|x-y|^2/4kt}dy,$$

and it becomes possible to rewrite this expression as

$$u(x,t) = \int_{\mathbb{R}^n}\phi(y)\mathscr{H}(x-y,kt)dy.$$

The differentiation in time of $u(x,t)$ is

$$u_t(x,t) = \partial_t\left(\int_{\mathbb{R}^n}\phi(y)\mathscr{H}(x-y,kt)dy\right) = \int_{\mathbb{R}^n}\partial_t\left(\phi(y)\mathscr{H}(x-y,kt)\right)dy$$

where the last step applies Theorem 4.6, passing the derivatives inside the integral, which is valid according to earlier argumentation in statement (i).

The same reasoning is applicable for $\nabla^2 u(x,t)$ which is expressed as

$$\nabla^2 u(x,t) = \int_{\mathbb{R}^n} \nabla^2 \left(\phi(y)\mathcal{H}(x-y,kt)\right) dy.$$

Hence, the expression $u_t - k\nabla^2 u$ is equal to

$$\int_{\mathbb{R}^n} \partial_t \left(\phi(y)\mathcal{H}(x-y,kt)\right) dy - \int_{\mathbb{R}^n} k\nabla^2 \left(\phi(y)\mathcal{H}(x-y,kt)\right) dy$$

and $\phi(y)$ is not dependent on t or x, which means it can be thought of as a constant. Then the expression is simplified as

$$\int_{\mathbb{R}^n} \phi(y) \left(\mathcal{H}_t - k\nabla^2\mathcal{H}\right)(x-y,kt)dy.$$

In the beginning of the proof, it was shown that $\mathcal{H}(x-y,kt)$ is a solution to the heat equation which implies the expression $(\mathcal{H}_t - k\nabla^2\mathcal{H})(x-y,kt)$ being zero. Hence, we have shown that

$$u_t - k\nabla^2 u = \int_{\mathbb{R}^n} \phi(y) \left(\mathcal{H}_t - \nabla^2\mathcal{H}\right)(x-y,kt)dy = 0$$

for all $x \in \mathbb{R}^n$ and $t > 0$.

(iii) For the last part, we will use some known definitions. Since $\phi \in C^\infty(\mathbb{R}^n)$, recall that the definition for pointwise continuity is, for all $\epsilon > 0$, there exists a $\gamma > 0$ such that $|y - x_0| < \gamma$ implies $|\phi(y) - \phi(x_0)| < \epsilon$.

By using the fact that the density function of the normal distribution can be expressed as

$$\int_{\mathbb{R}^n} \frac{1}{(4k\pi t)^{n/2}} e^{-|x-y|^2/4kt} dy = 1$$

and multiplying this with $\phi(x_0)$ implies

$$\phi(x_0) = \frac{1}{(4k\pi t)^{n/2}} \int_{\mathbb{R}^n} \phi(x_0) e^{-|x-y|^2/4kt} dy.$$

The expression $\lim_{(x,t)\to(x_0,0)} u(x,t) = \phi(x_0)$ is proved by showing

$$\lim_{(x,t)\to(x_0,0)} |\phi(x_0) - u(x,t)| = 0.$$

Using the density function leads to

$$|\phi(x_0) - u(x,t)| = \left| \frac{1}{(4k\pi t)^{n/2}} \int_{\mathbb{R}^n} \phi(x_0) e^{-|x-y|^2/4kt} dy \right.$$
$$\left. - \frac{1}{(4k\pi t)^{n/2}} \int_{\mathbb{R}^n} \phi(y) e^{-|x-y|^2/4kt} dy \right|.$$

The terms can be written as a single integral since they have the same boundary and that is expressed as

$$\begin{aligned} |\phi(x_0) - u(x,t)| &= \left| \frac{1}{(4k\pi t)^{n/2}} \int_{\mathbb{R}^n} (\phi(x_0) - \phi(y)) e^{-|x-y|^2/4kt} dy \right| \\ &= \left| \int_{\mathbb{R}^n} (\phi(x_0) - \phi(y)) \mathscr{H}(x-y, kt) dy \right|. \end{aligned}$$

Let γ be the one from the definition of pointwise continuity of the function ϕ. Let the space $\mathbb{R}^n$ be equal to the two unions $\mathbb{R}^n = B(x_0,\gamma) \cup \mathbb{R}^n \setminus B(x_0,\gamma)$. The integral can be formulated as two integrals with this new boundary, that is

$$\left| \int_{B(x_0,\gamma)} (\phi(x_0) - \phi(y)) \mathscr{H}(x-y, kt) dy \right| + \left| \int_{\mathbb{R}^n \setminus B(x_0,\gamma)} (\phi(x_0) - \phi(y)) \mathscr{H}(x-y, kt) dy \right|.$$

By placing the absolute values inside the integral, our expression becomes bounded by

$$\int_{B(x_0,\gamma)} |\phi(x_0) - \phi(y)| \mathscr{H}(x-y, kt) dy + \int_{\mathbb{R}^n \setminus B(x_0,\gamma)} |\phi(x_0) - \phi(y)| \mathscr{H}(x-y, kt) dy$$

and the function $\mathscr{H}(x-y, kt)$ is non-negative. Our goal is to show that this expression tends to zero as (x,t) tends to $(x_0, 0)$. Let us look closer at each term.

Part *I*: The first part is

$$I = \int_{B(x_0,\gamma)} |\phi(x_0) - \phi(y)| \mathscr{H}(x-y, kt) dy.$$

The ball $B(x_0,\gamma)$ is equivalent to a point $y \in \mathbb{R}^n$ fulfilling $|x_0 - y| < \gamma$. With this, the integral I with respect to y is

$$I \leq \sup_{|x_0-y|<\gamma} |\phi(x_0) - \phi(y)| \int_{B(x_0,\gamma)} \mathcal{H}(x-y,kt)dy.$$

For the ϵ from the definition for pointwise continuity, the supremum part $\sup_{|x_0-y|<\gamma} |\phi(x_0) - \phi(y)|$ is smaller than ϵ since ϕ is continuous by assumption and $|x_0 - y|$ is smaller than γ. The integral $\int_{B(x_0,\gamma)} \mathcal{H}(x-y,kt)dy$ is bounded by the integral $\int_{\mathbb{R}^n} \mathcal{H}(x-y,kt)dy$ since $B(x_0,\gamma) \subset \mathbb{R}^n$ and the integral of it is the density function of the normal distribution which is equal to one. Therefore, the integral with boundary of the ball is bounded. Hence, the whole expression I goes to zero.

Part *II*: The second part is

$$II = \int_{\mathbb{R}^n \setminus B(x_0,\gamma)} |\phi(x_0) - \phi(y)| \, \mathcal{H}(x-y,kt)dy.$$

Notice,

$$|\phi(x_0) - \phi(y)| \leq |\phi(x_0)| + |\phi(y)| \leq 2 \sup_{z\in\mathbb{R}^n} |\phi(z)| = 2 \|\phi\|_\infty .$$

Then integral II is bounded by

$$\int_{\mathbb{R}^n \setminus B(x_0,\gamma)} |\phi(x_0) - \phi(y)| \, \mathcal{H}(x-y,kt)dy \leq 2 \|\phi\|_\infty \int_{\mathbb{R}^n \setminus B(x_0,\gamma)} \mathcal{H}(x-y,kt)dy.$$

Since $\|\phi\|_\infty$ is bounded by assumption and ϕ is not dependent on y, we need to show that $\int_{\mathbb{R}^n \setminus B(x_0,\gamma)} \mathcal{H}(x-y,kt)$ tends to zero as (x,t) approaches $(x_0,0)$. The set $\mathbb{R}^n \setminus B(x_0,\gamma)$ is everything in $\mathbb{R}^n$ except the ball with radius γ centered at x_0. If we take a point $y \in \mathbb{R}^n$, then $|y - x_0| > \gamma$ has to be fulfilled, which means that the point y is not in the ball. We will continue to expand this inequality $|y - x_0|$ to eventually reach at an interesting expression. With the triangle inequality we get

$$|y - x_0| \leq |y - x| + |x - x_0| \leq |y - x| + \frac{\gamma}{2}$$

where x lies inside the ball and we assume it is closer to the center x_0 than $\frac{\gamma}{2}$.

With $|y - x_0| > \gamma$ the inequality is

$$|y - x_0| \leq |y - x| + \frac{|y - x_0|}{2}.$$

Rearranging it creates

$$\frac{|y - x_0|}{2} \leq |y - x|$$

which is equivalent to

$$-\frac{|y - x_0|^2}{4} \geq -|y - x|^2$$

and then by dividing with $4kt$ we get

$$-\frac{|y - x_0|}{16kt} \geq -\frac{|y - x|}{4kt}.$$

Taking this expressions to the exponent and the is

$$e^{-\frac{|y-x|^2}{4kt}} \leq e^{-\frac{|y-x_0|^2}{16kt}}.$$

Let us go back to the integral that we wanted to show tends to zero which is

$$\int_{\mathbb{R}^n \setminus B(x_0,\gamma)} \mathcal{H}(x - y, kt)\ dy = \int_{\mathbb{R}^n \setminus B(x_0,\gamma)} \frac{1}{(4k\pi t)^{n/2}}\ e^{-|x-y|^2/4kt}\ dy.$$

The boundary of this integral can also be written as $|y - x_0| > \gamma$ as explained earlier, and by applying the inequality we just derived, we get

$$\int_{\mathbb{R}^n \setminus B(x_0,\gamma)} \frac{1}{(4k\pi t)^{n/2}}\ e^{-|x-y|^2/4kt}\ dy \leq \int_{|y-x_0|>\gamma} \frac{1}{(4k\pi t)^{n/2}}\ e^{-|y-x_0|^2/16kt}\ dy.$$

There exists a $\delta > 0$ and by assumption of t, take $0 < t < \delta$, then the square root of t will also be non-negative. We can therefore use the variable substitution, $z = (y - x_0)/\sqrt{t}$ inside the absolute value. With the substitution the expression is

$$\int_{|z|>\gamma/\sqrt{t}} \frac{1}{(4k\pi t)^{n/2}}\ e^{-|z|^2/16k}(\sqrt{t})^n\ dz.$$

By canceling the variable t in the integral and taking out the constant $C =$

$1/(4k\pi)^{n/2}$ from the integral, we now have

$$C \int_{|z|>\gamma/\sqrt{t}} e^{-|z|^2/16k}\ dz$$

and this the integral of a Gaussian distribution. As t goes to zero, $|z|$ goes to infinity, meaning that the integral is over the double tail of the Gaussian distribution. The integral is therefore approaching zero.

We have now proved that $I + II$ tends to zero as (x, t) tends to $(x_0, 0)$. Hence, $\lim_{(x,t)\to(x_0,0)} u(x, t) = \phi(x_0)$ where $x_0, x \in \mathbb{R}^n, t > 0$.

□

7 Conclusion

In this thesis, we have used the Fourier transform as a powerful mathematical tool to solve the wave equation and the heat equation for functions in the Schwartz space. We started with defining the Schwartz space and showed that the convolution of two Schwartz functions also belongs to this space. Utilizing the convolution of two functions, we were able to prove the Fourier inversion formula. Lastly, we could solve the PDEs by using Fourier transformed expressions and rewriting them as ODEs which are more straightforward to solve. However, the resulting solutions remained a bit tricky until we simplified some terms by clever mathematical techniques. We also noticed that the solution of the wave equation in one dimension takes a surprisingly simple expression.

By working in the Schwartz space, we avoided to concern us about discontinuities or unbounded behavior. However, functions in the real life are not always that pleasant. While the introduction of this thesis explained some practical applications of the wave and the heat equation, it is important to understand that the solutions presented here only exists in the Schwartz space, which is unusual in real world scenarios.

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