SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Computing Stable Cohomotopy Groups of Stunted Projective Spaces

av

Nami Arabyarmohammadi

2025 - No K17

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Computing Stable Cohomotopy Groups of Stunted Projective Spaces

Nami Arabyarmohammadi

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Gregory Arone

2025

Abstract

Since infinite real projective space RP^∞ is a classifying space of $\mathbb{Z}_2$, the homotopy classes of maps $X \to RP^\infty$ corresponds bijectively to isomorphism classes of line bundles on X. This makes RP^∞ worth studying. We will thus partially compute the stable cohomotopy groups of this space and related stunted projective spaces. First we exhibit a nontrivial isomorphism of Ext groups, obtained by studying the cohomology of RP^∞ and stunted projective spaces as modules over the Steenrod algebra. This implies that the E_2 page of two Adams spectral sequences agree in an appropriate range: the one for computing the stable homotopy groups of spheres and the one for computing the stable cohomotopy of infinite stunted projective spaces, allowing us to compare the two, from which our main results will follow.

Sammanfattning

Eftersom det oändliga reella projektiva rummet RP^∞ är ett klassificerande rum for $\mathbb{Z}_2$ finns det en bijektion mellan homotopiklasser av avbildningar $X \to RP^\infty$ och isomorfiklasser av linjebuntar över X. Detta gör RP^∞ värt att studera. Därför kommer vi beräkna de stabila kohomotopigrupperna för detta rum och relaterade trunkerade projektiva rum. Först uppvisar vi en icke-trivial isomorfi av Ext grupper, erhållen genom att studera kohomologin för RP^∞ och trunkerade projektiva rum som moduler över Steenrod algebran. Detta implicerar att E_2 sidan av två Adams spektralsekvenser överensstämmer i ett lämpligt intervall: den som beräknar de stabila homotopigrupperna för sfärer och den som beräknar den stabila kohomotopin för trunkerade oändliga projektiva rum, vilket möjliggör en jämförelse mellan de två, varifrån våra huvudsakliga resultat följer.

Acknowledgments

Primarily, I wish to express my appreciation to my advisor Greg, for all the time he spent explaining heaps of mathematics that was confusing me, for reading and giving comments on this thesis, and for answering emails from morning until night. I would also like to thank my dear girlfriend Elin for all her moral support, without whom this thesis would have been completed much sooner.

Contents

1 Introduction

1.1 Motivation and background

Projective spaces are usually a mathematics students' first encounter with moduli spaces, vaguely meaning a space which parametrizes a collection of geometric objects with some common properties. In our case, projective n-space RP^n parametrizes all lines in $\mathbb{R}^{n+1}$ passing through the origin, and infinite projective space RP^∞ parametrizes all lines in $\mathbb{R}^\infty$ passing through the origin. Let X be some space. Picking a continuous map $X \to RP^\infty$ means we are associating to each point in X some line in $\mathbb{R}^\infty$ in a continuous manner. It is therefore reasonable to think that for each map $X \to RP^\infty$, we would obtain an associated line bundle $E \to X$ on X. In fact, if X is a reasonable space, like Hausdorff and paracompact, then there is a bijection between homotopy classes of maps $X \to RP^\infty$ and isomorphism classes of line bundles on X. See theorem 3.6.3 in [7] for the more general statement that the homotopy classes of maps $X \to BG$ is in one-to-one correspondence with isomorphism classes of principal G-bundles on X. Gaining homotopical information about RP^∞ would thus give us information about line bundles on spaces.

So what homotopical information will we try to get? The stable cohomotopy of RP^∞ and also the stable cohomotopy of stunted projective spaces. A *stunted projective space* is the quotient of one projective space by a smaller projective space; that is, a space of the form RP^n/RP^k for $k < n$. We also include RP^∞/RP^k to our collection of what we call stunted projective spaces. By *cohomotopy*, we mean homotopy classes of maps $X \to S^n$ into a sphere, as opposed to the homotopy groups which are maps $S^n \to X$ out of a sphere. By *stable*, we essentially mean that we are considering the homotopy classes of maps that remain after applying the suspension functor an arbitrary amount of times.

Historically, one of the main motivators for studying the stable cohomotopy of stunted projective spaces was to understand the 0:th stable cohomotopy group of $RP^\infty \cup \{*\}$ in order to prove Segal's conjecture. It states that the 0:th stable cohomotopy group of $BG \cup \{*\}$ is isomorphic to the completion of the Burnside ring $A(G)$ at its augmentation ideal, for any finite group G. As mentioned before, by theorem 3.6.3 in [7], homotopical information about BG is related to information about principal G-bundles, which is one of the reasons BG is worth studying. Segal's conjecture, if it were true, would relate homotopical information about BG to some completed ring $\widehat{A(G)}$.

The big deal about this is that $\widehat{A(G)}$ is a combinatorial object which is relatively easy to compute in many cases, see [16]. The computation of stable cohomotopy groups we discuss in this thesis was the first step in proving this conjecture because by this computation, Lin shows in theorem 1 of [19] that the conjecture holds for $G = \mathbb{Z}_2$. The last step of Segal's conjecture was proven by Carlsson in [11], and the reader is welcome to read the introductory section of that article to see how the computations of sections

4 and 5 in this thesis connect with the overarching sequence of proofs that finally led to the full proof of Segal's conjecture.

Another reason to study stunted projective spaces is due to their connection with the problem of finding the maximum number of orthogonal vector fields on a sphere. Before Adams' paper [1], it was already known that one could construct $\rho(n) - 1$ orthogonal vector fields on S^{n-1} for some function ρ, but this paper proved that there does not exist $\rho(n)$ such vector fields. He did this by proving the sufficient condition that there is no map $f : RP^{n+\rho(n)}/RP^{n-1} \to S^n$ whose restriction to $RP^n/RP^{n-1} = S^n$ is a degree 1 map of n-spheres. The problem concerning vector fields on spheres was thus reduced, in some sense, to a problem concerning the cohomotopy of stunted projective spaces.

To get information about the homotopy classes of maps $[X, S^n]$ or $[S^n, X]$ for some (stunted) projective space X is however very hard. One way to make this problem more manageable is to develop a new category where the objects are similar to topological spaces, in the sense that one has a notion of homotopy theory in this new category, and study the homotopy classes of maps $[\Sigma^\infty(X), \Sigma^\infty(S^n)]$ instead. Here $\Sigma^\infty(-)$ is a functor from topological spaces to this new category, which will be the category of spectra. We will prove that the set $[\Sigma^\infty(X), \Sigma^\infty(S^n)]$ is in some sense a simplification of $[X, S^n]$.

In this thesis, the motivation for developing spectra is because we want to restrict ourselves to only considering the homotopy classes of maps between objects which are sufficiently connected, meaning that they are n-connected for arbitrarily large n. In the case when our space X is a finite CW complex, we have that the homotopy classes of maps between spectra $[\Sigma^\infty(X), \Sigma^\infty(S^n)]$ becomes the colimit of the diagram

$$[X, S^n] \xrightarrow{\Sigma(-)} [\Sigma X, \Sigma S^n] \xrightarrow{\Sigma(-)} [\Sigma^2 X, \Sigma^2 S^n] \to \cdots ,$$

where the maps are given by applying the suspension functor. At first glance, it is reasonable to imagine that the colimit of the diagram above would be more complicated to compute than $[X, S^n]$. The work we do in the section on spectra will prove, to the contrary, that this colimit is much more manageable. It is in this sense that $[\Sigma^\infty(X), \Sigma^\infty(S^n)]$ will be shown to be a simplification of $[X, S^n]$.

1.2 Algebraic preliminaries

Let us mention some notation having to do with graded modules. In the following, fix a graded ring R and suppose all involved modules are graded R-modules. By $(M)^t$, we mean the t:th graded component of M. Unless explicitly stated otherwise, assume all maps between graded modules are graded maps.

Definition 1.1. The i:th *suspension* of a graded module M is the module $\Sigma^i M$, where $(\Sigma^i M)^t = M^{t-i}$ for all t.

Definition 1.2 (Graded Hom)**.** Given graded R-modules M, let $\mathrm{Hom}_R^t(M, N)$ denote the R-module homomorphisms lowering the degree by t.

Remark 1.3. An element in $\mathrm{Hom}_R^t(M, N)$ is the same as a graded map $\Sigma^t M \to N$.

Definition 1.4 (Graded Ext)**.** Let $P_\bullet \to M$ be a projective resolution of graded modules. Define $\mathrm{Ext}_R^{s,t}(M, N)$ as the s:th cohomology group of the cochain complex $\mathrm{Hom}_R^t(P_\bullet, N)$.

Recall that the tensor product $M \otimes_R N$ has a natural grading, given by $(M \otimes_R N)^t = \bigoplus_{i+j=t} M^i \otimes_R N^t$.

Definition 1.5 (Graded Tor)**.** Let $P_\bullet \to M$ be a projective resolution of graded modules. Define $\mathrm{Tor}^R_{s,t}(M, N)$ as the s:th homology group of the chain complex $(P_\bullet \otimes N)^t$.

Notice that Hom, Ext and Tor become graded modules under this decomposition. We finish by mentioning two basic lemmas on inverse limits that we will use readily.

Lemma 1.6. Given a short exact sequence of inverse systems $0 \to A_i \to B_i \to C_i \to 0$ of R-modules, there is a natural exact sequence

$$0 \to \varprojlim_i A_i \to \varprojlim_i B_i \to \varprojlim_i C_i \to \varprojlim_i{}^1 A_i \to \varprojlim_i{}^1 B_i \to \varprojlim_i{}^1 C_i \to 0 \ .$$

Definition 1.7. An inverse system $\{A_i\}_{i\in\mathbb{Z}}$ is *Mittag-Leffler* if for all i, there is an integer c_i such that for all $n \geq c_i$, $\mathrm{Im}\ (A_n \to A_i) = \mathrm{Im}\ (A_{c_i} \to A_i)$.

Essentially, being Mittag-Leffler means that the image into A_i stabilizes for all i. Notice that this is trivially satisfied if all modules A_i are finite.

Lemma 1.8. If $\{A_i\}_i$ is Mittag-Leffler, then $\varprojlim_i^1 A_i = 0$.

Proof. Proposition 3.5.7 in [29]. □

Remark 1.9. Lemma 1.6 implies that inverse limits are an exact functor when the leftmost inverse system in a short exact sequence is Mittag-Leffler.

2 Spectra

2.1 Motivation

The homotopy groups of spaces is something worth studying, but this is hard. For $\pi_1(-)$, we have Seifert-Van Kampen's theorem, which may be compared to the excision for (co)homology. There is however no such analog for the functors $\pi_n(-)$ when $n \geq 2$. A natural question to ask oneself is whether one can modify $\pi_n(-)$, allowing some loss of homotopical information, in order to obtain another functor from pointed topological spaces to abelian groups which is easier to compute, for example due to having an excision-type theorem. The answer is yes! It may be constructed as follows.

The suspension functor $\Sigma(-)$ induces a group homomorphism $\pi_n(X) = [S^n, X] \to \pi_{n+1}(\Sigma X) = [S^{n+1}, \Sigma X]$ given by sending a map $f : S^n \to X$ to $\Sigma f : S^{n+1} \to \Sigma X$. By the Freudenthal suspension theorem, the suspension homomorphism $\pi_{n+k}(\Sigma^k X) \to \pi_{n+k+1}(\Sigma^{k+1} X)$ is an isomorphism for sufficiently large k. Consider the sequence below, where all maps are defined by this suspension homomorphism.

$$[S^n, X] \to [S^{n+1}, \Sigma X] \to [S^{n+1}, \Sigma^2 X] \to \cdots .$$

Taking the colimit of this sequence, we obtain a group which we call the n:th *stable homotopy group of* X and denote by $\pi_n^s(X)$. Essentially, the Freudenthal suspension theorem says that after applying the suspension functor to the homotopy classes of maps $[S^n, X]$ enough times, the group stabilizes.

The intuition one should keep in mind during this section is that we want to study homotopy classes of maps $[X, Y]$, but only consider which maps remain after having applied the suspension functor an arbitrary amount of times. To be able to describe and prove theorems about such stable homotopy classes, it is natural to desire some objects $\tilde{X}$ and $\tilde{Y}$ which book-keep the data of all suspensions of X and Y. These will be our *spectra*. It is also natural to desire a map $\tilde{X} \to \tilde{Y}$ that book-keeps the data of all the maps $\Sigma^k X \to \Sigma^k Y$ for $k \geq 0$. In so doing, the hope is that one could define homotopies in this new category of spectra, and that the homotopy classes of maps $[\tilde{X}, \tilde{Y}]$ be equal to the direct limit $\varinjlim_k [\Sigma^k X, \Sigma^k Y]$ in favorable cases.

To justify this construction, we note then that the stable homotopy groups $\pi_n^s(-)$ will in fact become a generalized homology theory, meaning that we have an excision theorem, allowing us to compute the stable variant $\pi_n^s(-)$ more easily than $\pi_n(-)$. Furthermore, we will show that the homotopy classes of maps $[\tilde{X}, \tilde{Y}]$ between any two spectra are always abelian groups, allowing us to leverage its additive structure to our advantage.

The algebraic invariant we will be computing is the stable *cohomotopy* of a family of spaces. Note that the homotopy classes of maps $[X, S^k]$ is in general not a group. Once we define a suitable notion of spectrum $\tilde{S}^k$ which acts much like the topological space

S^k, not only will we have that $[\tilde{X}, \tilde{S}^k]$ is a group for any spectrum $\tilde{X}$, but that the sequence of functors $[\Sigma^\infty(-), \tilde{S}^k]$ for $k \geq 0$, will define a generalized cohomology theory! Here $\Sigma^\infty(-)$ is an (as of yet) undescribed functor which takes a space and outputs some associated spectrum. The goal of this thesis is essentially then to compute the groups obtained by evaluating this generalized cohomology theory at stunted projective spaces.

2.2 CW spectra

We note that all spaces are assumed to be pointed in the entire thesis, so by suspension we mean reduced suspension. Our two main references for section 2 is the first three chapters in part 3 of *Adams' blue book* [4] and chapter 5.2 in Hatcher's unpublished book *Spectral Sequences in Algebraic Topology* [14]. All references made to theorems concerning the homotopy theory of spaces can be found in chapter 4 of [13], whose main statements we assume the reader is vaguely familiar with. Without further ado, let us get our hands dirty with some math. We begin by defining a spectrum.

Definition 2.1. A spectrum consists of a sequence of pointed spaces $\{X_i\}_{i\geq 0}$ and maps $\sigma_i : \Sigma X_i \to X_{i+1}$ for $i \geq 0$. We call the spaces X_i the component spaces and the maps σ_i the component maps.

Definition 2.2. A CW spectrum is a spectrum where each space in the sequence is a CW complex and each map $\sigma_i : \Sigma X_i \hookrightarrow X_{i+1}$ is an inclusion of subcomplexes.

Given a spectrum consisting of CW complexes, we may force the structure maps σ_i : $\Sigma X_i \to X_{i+1}$ to be inclusions of subcomplexes by replacing our original map with a cofibration. More precisely, for all $n > 0$, replace X_n with the homotopy equivalent space $M_{\sigma_{n-1}}$, the mapping cylinder of the map $\sigma_{n-1} : \Sigma X_{n-1} \to X_n$ and let the new structure map $\tilde{\sigma}_{n-1} : \Sigma X_{n-1} \to M_{\sigma_{n-1}}$ be the inclusion map.

In this text, we will focus only on CW-spectra. From now on, we will by a spectrum always mean a CW spectrum unless explicitly stated otherwise. We will mainly focus on two types of spectra: suspension spectra and Eilenberg-Maclane spectra.

Definition 2.3 (Suspension spectra)**.** Given a CW-complex X, the suspension spectrum of X is denoted $\Sigma^\infty(X)$. Its component spaces are defined by $\Sigma^\infty(X)_i = \Sigma^i X$ and the component maps are the identity.

Example 2.4.

(1) The sphere spectrum S^k, defined as the suspension spectrum $\Sigma^\infty(S^k)$ of the k-sphere.

(2) Stunted projective spectra X_k^l, defined as $\Sigma^\infty(RP^l/RP^{k-1})$, for $0 < k \leq l$ (we will generalize these later).

To properly define Eilenberg-Maclane spectra, recall first the adjunction between suspension and the loop-space functors on pointed compactly generated weak Hausdorff spaces $[\Sigma X, Y] \cong [X, \Omega(Y)]$. From this it follows that the loop-space $\Omega K(G, n)$ has the same homotopy groups as a $K(G, n-1)$, since $[S^k, K(G, n)] \cong [S^{k-1}, \Omega K(G, n)]$. By the CW approximation theorem, there is thus a weak equivalence $K(G, n-1) \to \Omega K(G, n)$. The adjoint of this map under the adjunction above is of the form $\Sigma K(G, n-1) \to K(G, n)$. By replacing the codomain with the mapping cylinder as described above, we may assume this map is an inclusion of subcomplexes.

Definition 2.5 (Eilenberg-Maclane spectra)**.** For an abelian group G and an integer $n \geq 0$, we define the Eilenberg-Maclane spectrum $H(G, n)$ to be the spectrum with the i:th component space being the space $K(G, n+i)$ and the i:th component map being the map $\Sigma K(G, n+i) \to K(G, n+i+1)$ as described in the paragraph above.

We will now show that a lot of the theory and properties concerning CW spaces are kept intact when generalizing to the context of CW spectra. We begin by defining its cells. Given a CW spectrum X, any k cell in X_i (which is not the basepoint 0-cell) will be a $k+1$-cell in X_{i+1}, since ΣX_i is a subcomplex of X_{i+1}. We define a k-cell X_i to be equivalent to a $k+n$ cell in X_{i+n} if the latter is the iterated suspension of the former.

Definition 2.6. A cell in a CW spectrum X is the equivalence class of a cell in the CW complex X_i for some i, with the equivalence relation as described above. Taking the smallest-dimensional representative of a cell in X, we get a $k+i$-cell in X_i, for some integers k, i. The dimension of this cell in X is defined to be k.

Hopefully, this messy definition will become more intuitive with a couple examples.

Example 2.7.

(1) Let X be the sphere spectrum S^1. Then $X_0 = S^1, X_1 = S^2, X_2 = S^3$ and so on. The cell structure on the X_0 may be given by one 0-cell and one 1-cell. Consider the 1-cell in X_0. It becomes a 2-cell in X_1, a 3-cell in X_2, and so on. The equivalence class of these cells is a 1-cell in X. Furthermore, this is the only non-basepoint cell in X, because any non-basepoint cell in X_i is equivalent to the 1-cell in X_0. Therefore, X consists of one 0-cell (the basepoint) and one 1-cell, just like the actual 1-sphere.

(2) Let Y be the spectrum with $Y_i = S^1 \vee S^2 \vee S^3 \vee \dots$ for all i. Since suspensions distribute over wedge products, $\Sigma Y_i = S^2 \vee S^3 \vee \dots$, and we define σ_i to be the inclusion maps. For any integer n, we may pick a n-cell in Y_0 by taking the n-cell of $S^n \subseteq Y_0$ and this is not the suspension of any other cell in lower component spaces. The equivalence class of this cell constitutes an n-cell in Y. We thus have cells of any dimension in Y, including negative dimensions.

One can generalize the argument in the first example to show that given any CW complex C, with n_k many k-cells, its suspension spectrum $\Sigma^\infty(C)$ will have n_k many k-cells as well. The second example describes a situation which we want to avoid. To prove the convergence of the Adams spectral sequence, we must restrict ourselves to CW spectra which satisfy some conditions. Our ideal spectra have cells of dimension bounded below, finitely many cells in each dimension, or better yet finitely many cells in total.

Definition 2.8. A CW spectrum is

- *connective* if it has cells of dimension bounded below,
- of *finite type* if it has finitely many cells in each dimension,
- *finite* if it has finitely many cells.

We will finally start defining maps of spectra. One obvious way to define a map $f : X \to Y$ of spectra is as a sequence of pointed maps $f_i : X_i \to Y_i$ that is compatible with the component maps, meaning that the diagram below commutes.

$$\begin{array}{ccc} \Sigma X_i & \xrightarrow{\sigma_i^X} & X_{i+1} \\ {\scriptstyle \Sigma f_i}\downarrow & & \downarrow{\scriptstyle f_{i+1}} \\ \Sigma Y_i & \xrightarrow[\sigma_i^Y]{} & Y_{i+1} \end{array}$$

This can definitely be done, and we will call these *proper maps* of spectra. There is however a more favorable definition, which aligns more with the desire to simplify topological problems by studying what happens eventually, after suspending enough times.

Definition 2.9. A *subspectrum* $\tilde{X}$ of a CW spectrum X is a CW spectrum such that the component spaces $\tilde{X}_i$ are subcomplexes of X_i for all i, and such that the component maps σ_i of X restrict to component maps $\tilde{\sigma}_i : \Sigma\tilde{X}_i \to \tilde{X}_{i+1}$, meaning $\Sigma\tilde{X}_i \subseteq \tilde{X}_{i+1}$. The subspectrum is called *cofinal* if, given any i and any cell c in X_i, there is some integer n such that the n:th suspension of the cell $\Sigma^n c$ lies in $\tilde{X}_{i+n}$.

Being a cofinal subspectrum essentially means that each cell in X eventually lies in $\tilde{X}$. Notice that if $\Sigma^n c$ lies in $\tilde{X}_{i+n}$, then $\Sigma^N c$ lies in $\tilde{X}_{i+N}$ for all $N \geq n$. Notice that being a cofinal subspectrum is a transitive property, in the sense that if A is a cofinal in B and B is cofinal in C, then A is cofinal in C.

Definition 2.10. A *map* of CW spectra $f : X \to Y$ is a proper map $f : \tilde{X} \to Y$ of spectra for some cofinal subspectrum $\tilde{X}$ of X. We consider two maps $f, g : X \to Y$ to be equal if they agree on a common cofinal subspectrum.

Given maps of spectra $f : X \to Y$ and $g : Y \to Z$, which are proper maps $f : X' \to Y$ and $g : Y' \to Z$ from cofinal subspectra, an important question to ask is whether the composition may be properly defined. To verify this, we remark that there is a cofinal subspectrum X'' of X' which maps into Y' along f. Let the component spaces X_i'' be the union of all cells in X_i' which lie in $f^{-1}(Y_i')$. One can show that this subspectrum is cofinal. We may thus take the composition $X'' \to Y' \to Z$ as our proper map defining $g \circ f$.

We will now exhibit some basic isomorphisms of spectra, which will justify our definition of maps of CW spectra and indicate how suspending spectra compares to suspending spaces. Let us preliminarily define the *suspension* of a spectrum X by ΣX, defined by $(\Sigma X)_i = \Sigma X_i$ and the component maps $\Sigma(\sigma_i) : \Sigma^2 X_i \to \Sigma X_{i+1}$ are induced from the component maps σ_i of X by applying the suspension functor (on morphisms of spaces).

Example 2.11.

(i) Given any cofinal subspectrum $\tilde{X}$ of X, we may define a map $X \to \tilde{X}$ by taking the proper map $\mathrm{id} : \tilde{X} \to \tilde{X}$, and another map $\tilde{X} \to X$ which is the inclusion. The composition of these maps is the identity, so $\tilde{X}$ is isomorphic to X as a spectrum.

(ii) Given a spectrum X, we may define a spectrum Y with component spaces $Y_i = X_{i-1}$ for all $i > 0$, with Y_0 be a point, and define the component maps as the maps from X but shifted one degree. Then ΣY is a subspectrum of X, since $\Sigma X_{i-1} \subseteq X_i$. In fact, it is cofinal, since the suspension of any cell in X_i lies in ΣX_i, which is Y_{i+1} by definition. By the example above, for any spectrum X, there is a spectrum Y such that X is isomorphic to ΣY.

(iii) Given a spectrum X, we may define a spectrum Z with component spaces $Z_i = X_{i+1}$ for all $i \geq 0$ and define the component maps as shifted component maps from X. One can show that ΣX is a cofinal subspectrum of Z like in the example above, so Z is isomorphic to ΣX.

These examples are meant to justify our given definition of maps of spectra. The first example aligns with our initial wish to know what happens *eventually*, since considering only a cofinal part of our spectrum recovers all information. The third example shows that there are two ways to define suspensions of spectra, which are equivalent thanks to the first example. The second example exhibits the surprising property that the any spectrum is equivalent to the suspension of another spectrum, a property which does not hold in the slightest for topological spaces. This property turns out to be incredibly important, because it will imply (once we have defined it) that the homotopy classes of maps $[X, Y]$ between two arbitrary spectra is always an abelian group!

Definition 2.12. Given a spectrum X and any integer k, its k:th suspension $\Sigma^k X$ is defined by $(\Sigma^k X)_i = X_{i+k}$, where the component spaces equals a point when $i + k$ is negative, and the component maps are the shifted component maps coming from X, or constant maps when $i + k$ is negative.

This assignment is clearly functorial, since the map $\Sigma^k f : \Sigma^k X \to \Sigma^k Y$ will just be the map f shifted k steps up/down among the component spaces.

2.3 Homotopy theory

We will now introduce homotopy theory in the category of spectra, and mention without proof some theorems that are analogous to theorems from the homotopy theory of spaces, like the cellular approximation theorem, Whitehead's theorem, exactness of cofibration sequences and representability of cohomology theories.

Given a spectrum X, define $X \times I$ to be the spectrum with $(X \times I)_i = X_i \times I$, where the product means the reduced product that collapses $\{\text{basepoint}\} \times I$ to a point. Then $\Sigma(X_i \times I)$ equals $\Sigma(X_i) \times I$, so we define the component maps of $X \times I$ as $\sigma_i \times 1 : \Sigma X_i \times I \to X_{i+1} \times I$, where σ_i are the component maps of X. Notice that there are obvious maps of spectra $i_0, i_1 : X \to X \times I$ induced by the maps $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$ on each component.

Definition 2.13. A homotopy between two maps $f, g : X \to Y$ of spectra is a map $H : X \times I \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. Given two spectra X and Y, we denote by $[X, Y]$ the homotopy classes of maps.

One of the big advantages of working with spectra is that the homotopy classes of maps between any two spectra is always an abelian group! Recall that for spaces, the homotopy classes of maps between spaces is a group, if the domain equals the suspension of some space. Furthermore, it is an abelian group if the domain is the double suspension of a space. The group operation is given as follows. Given two homotopy classes of maps between spaces $f, g : \Sigma X \to Y$ one may define the sum $f + g$ as the composition

$$\Sigma X \xrightarrow{\text{pinch}} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y \ .$$

The pinch map is given by the quotient map collapsing $X \times \{0.5\} \subseteq \Sigma X$ to a point, where we consider ΣX as a quotient of $X \times I$. The map $f \vee g$ is given by mapping each point in the first or second wedge product to Y along f or g, respectively.

We can define addition of maps similarly in the case of spectra. Let $f, g : \Sigma^2 X \to Y$ be maps of spectra. Assuming f and g are proper maps by restricting to cofinal subspectra, we may define their sum on each component space by $(f + g)_i = f_i + g_i : \Sigma^2 X_i \to Y_i$. Taking for granted that $[\Sigma^2 X_i, Y_i]_{\text{spaces}}$ is an abelian group, it is easy to see that $[\Sigma^2 X, Y]_{\text{spectra}}$ is an abelian group. Since every spectrum is the double suspension of some spectrum, we conclude that for any spectra X and Y, $[X, Y]$ is an abelian group.

If $X = \Sigma^\infty C$ is the suspension spectrum of some finite CW complex C, then the homotopy classes of maps $[\Sigma^\infty C, Y]$ may be described by the direct limit of the diagram

$$[C, Y_0] \xrightarrow{\Sigma} [\Sigma C, \Sigma Y_0] \xrightarrow{(\sigma_0^Y)_*} [\Sigma C, Y_1] \xrightarrow{\Sigma} [\Sigma^2 C, \Sigma Y_1] \xrightarrow{(\sigma_0^Y)_*} [\Sigma^2 C, Y_2] \longrightarrow \cdots \ .$$

Also, considering the group $[\Sigma^k \Sigma^\infty C, Y]$ instead will give you a similar colimit but with the domains in the groups above shifted.

Proposition 2.14. If C is a finite CW complex, we have an isomorphism

$$[\Sigma^k \Sigma^\infty C, Y] \cong \varinjlim_n \, [\Sigma^{k+n} C, Y_n] \ .$$

Proof. Proposition 2.8 in [4]. □

This equivalent characterization of homotopy classes of maps is good to keep in mind, and further instills the idea that the group $[X, Y]$ only detects what happens after taking sufficient suspensions. Another proposition which reaffirms this is the following.

Proposition 2.15. The suspension map $[X, Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism for all spectra X, Y, and it is natural in X and Y.

Proof. Theorem 3.7 in [4]. □

An immediate corollary is that the inverse map $[\Sigma X, \Sigma Y] \to [X, Y]$ given by de-suspension (applying the functor $\Sigma^{-1}(-)$) is also an isomorphism. Later on in the text, we will make identifications of the form $[\Sigma^a A, \Sigma^b B] = [\Sigma^{a-b} A, B] = [A, \Sigma^{b-a} B]$ using these isomorphisms.

Example 2.16.

(i) The homotopy groups of a spectrum X are given by $\pi_n(X) = [S^n, X] = \varinjlim_i [S^{n+i}, X_i]$.

(ii) The cohomotopy groups of a spectrum X are given by $\pi^n(X) = [X, S^n]$. By the proposition above, this is isomorphic to $[\Sigma^{-n} X, S^0]$.

(iii) By proposition 2.4, it is easy to see that the homotopy group $\pi_k(H(G, n))$ equals zero if $n \neq k$ and equals G if $n = k$, for any abelian group G. This justifies calling $H(G, n)$ an Eilenberg-Maclane spectrum, due to the analogy with the defining property of Eilenberg-Maclane spaces.

Definition 2.17. The *stable homotopy groups* and *stable cohomotopy groups* of a space X are given by $\pi_n^s(X) := [S^n, \Sigma^\infty X]$ and $\pi_s^n(X) := [\Sigma^\infty X, S^n]$, respectively.

Definition 2.18. A map of spectra $f : X \to Y$ is *cellular* if for every i, the map $f : X'_i \to Y_i$ is cellular (where X' is some cofinal subspectrum).

Assuming the cellular approximation theorem for spaces, we will illustrate how one can extend the theorem to the analogous theorem concerning spectra.

Proposition 2.19. Any map $f : X \to Y$ is homotopic to a cellular map. If the map f is already cellular on a subspectrum A of X, the homotopy may be taken so that it fixes A.

Proof. Assume by restricting to a cofinal subspectrum that f is a proper map which is cellular on A. We will prove by induction that $f_n : X_n \to Y_n$ is homotopic to a cellular map relative A_n for all $n \geq 0$. We will also prove that this homotopy may be taken so that it fixes the subcomplex ΣX_{n-1}, for $n \geq 1$, so that the sequences of homotopies $H_n : X_n \times I \to Y_n$ defines a homotopy between maps of spectra $H : X \times I \to Y$.

For the base case, this means that $f_0 : X_0 \to Y_0$ is cellular on A_0. By the cellular approximation theorem for spaces, we may homotope f_0 to a cellular map on all of X_0. Assume the map $f_n : X_n \to Y_n$ is homotopic to a cellular map relative A_n. Applying the functor $\Sigma(-)$, we get that $\Sigma f_n : \Sigma X_n \to Y_{n+1}$ is cellular. By the definition of a proper map, the diagram below commutes.

$$\begin{array}{ccc} \Sigma X_i & \xrightarrow{\sigma_i^X} & X_{i+1} \\ {\scriptstyle \Sigma f_i}\downarrow & & \downarrow{\scriptstyle f_{i+1}} \\ \Sigma Y_i & \xrightarrow[\sigma_i^Y]{} & Y_{i+1} \end{array}$$

This means that Σf_n is a restriction of the map f_{n+1} to a subcomplex, since the component maps σ_i^X are by definition inclusions of subcomplexes. By the cellular approximation theorem for spaces, we can homotope f_{n+1} to a cellular map relative ΣX_n and A_{n+1}, completing the induction step. □

With homotopies between maps of spectra and homotopy groups of spectra defined, one may extend the definition of weak equivalence and homotopy equivalence to hold for spectra as well. One can then state the spectra-analog of Whitehead's theorem.

Proposition 2.20. A weak equivalence of CW spectra $X \to Y$ is a homotopy equivalence.

Proof. Theorem 3.4 in [4]. □

Corollary 2.21. Suppose the map of spectra $f : X \to Y$ induces weak equivalences $f_i : X_i \to Y_i$ on each component space. Then f is a homotopy equivalence.

Proof. The maps f_i induce isomorphisms $\pi_k(X_i) \cong \pi_k(Y_i)$ for all k. By the colimit description of $\pi_k(X)$ and $\pi_k(Y)$ from proposition 2.14 and the remark in example 2.16(i), it follows that f induces isomorphisms $\pi_k(X) \cong \pi_k(Y)$ for all k, meaning its a weak equivalence of spectra. By proposition 2.20 above, it follows that f is a homotopy equivalence. □

Given a subspectrum A of X, a natural way to define the quotient X/A would be to set $(X/A)_i = X_i/A_i$ and let the component maps $\tilde{\sigma}_i : \Sigma X_i/A_i \to X_{i+1}/A_{i+1}$ be the component maps $\sigma : \Sigma X_i \to X_{i+1}$ of X after passing to the quotient. To ensure that $\tilde{\sigma}_i$ are inclusions of subcomplexes, there is a necessary condition the subspectrum A must satisfy.

Definition 2.22. A subspectrum A of X is *closed* if given any i and any cell in X_i, if the k:th suspension of this cell lies in A_{i+k} for some k, then the cell lies in A_i from the start.

Given a closed subspectrum A of X, X/A as constructed above will thus also be a CW spectrum, since A being closed implies that the component maps of X/A are inclusions. If A is a subspectrum of X which is not closed, let $\overline{A}$ be the union of all cells in X whose suspension eventually lies in A. This may be though of as the closure of A in X. Since A is cofinal in $\overline{A}$, it follows by the example 2.11(i) that the inclusion $A \hookrightarrow \overline{A}$ is an equivalence. From now on, when taking the quotient X/A, we assume we have already identified A with $\overline{A}$.

Recall that in the category of spaces, given a subcomplex A of X, applying the functor $[-, Z]$ to the cofiber sequence $A \hookrightarrow X \twoheadrightarrow X/A \to \Sigma A \hookrightarrow \Sigma X \to \dots$ induces a natural long exact sequence

$$[A, Z] \leftarrow [X, Z] \leftarrow [X/A, Z] \leftarrow [\Sigma A, Z] \leftarrow [\Sigma X, Z] \leftarrow [\Sigma(X/A), Z] \leftarrow \cdots$$

for any CW complex Z (where exactness in the first three sets are to be considered exactness as sets). In fact, the same holds for spectra. Namely, given a closed subspectrum A of X, there is a similar cofiber sequence $A \hookrightarrow X \twoheadrightarrow X/A \to \Sigma A \hookrightarrow \Sigma X \to \dots$, and applying the contravariant functor $[-, Z]$ to it gives us an exact sequence as above, for any CW spectrum Z. In fact, since any spectrum is equivalent to the suspension of another spectrum, the sequence above may be extended indefinitely to the left.

Unlike spaces, however, applying instead the covariant functor $[Z, -]$ to the cofibration sequence of spectra also gives us an exact sequence. Again, since any spectrum is equivalent to the suspension of another spectrum, this exact sequence may be extended indefinitely to the left. We will summarize these facts as a theorem.

Proposition 2.23. Given a closed subspectrum A of X and any spectrum Z, there are natural exact sequences

$$\cdots \xleftarrow{q^*} [\Sigma^{-1}(X/A), Z] \longleftarrow [A, Z] \xleftarrow{i^*} [X, Z] \xleftarrow{q^*} [X/A, Z] \longleftarrow [\Sigma A, Z] \xleftarrow{i^*} \cdots$$

$$\cdots \xrightarrow{q_*} [Z, \Sigma^{-1}(X/A)] \longrightarrow [Z, A] \xrightarrow{i_*} [Z, X] \xrightarrow{q_*} [Z, X/A] \longrightarrow [Z, \Sigma A] \xrightarrow{i_*} \cdots ,$$

where $i : A \hookrightarrow X$ is the inclusion and $q : X \twoheadrightarrow X/A$ is the quotient map.

Proof. This is a particular case of propositions 3.9 and 3.10 in [4]. □

The proof of the proposition above is given by Adams in a more general setting, where instead of the basic cofiber sequence $A \hookrightarrow X \twoheadrightarrow X/A$, we have any map of spectra $f : X \to Y$ and a cofiber sequence $X \to Y \to C_f$, where C_f is the mapping cone of f.

This general situation reduces to ours, because in the case when X is a closed subcomplex of Y and f is the inclusion map, the mapping cone C_f will be isomorphic to Y/X. For all of this to make sense, we need to define the mapping cone of a map of spectra.

Recall that we may define the mapping cone C_f for a map of spaces $f : X \to Y$ as the colimit of the diagram below, where I is taken to have the basepoint 0.

$$\begin{array}{ccc} X & \xrightarrow{x \mapsto (x,1)} & X \wedge I \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array}$$

Another way to say this is that C_f is the adjunction space $(X \wedge I) \cup_f Y$. We may define the mapping cone C_f for a map of spectra $f : X \to Y$ similarly. Assuming f is a strict map, we define the component spaces as $(C_f)_i = C_{f_i} = (X_i \wedge I) \cup_{f_i} Y$. To define the component maps is an unnecessary technicality which will not be highly relevant for our purposes, so we will only sketch how it is done. First, we claim that ΣC_{f_i} is homeomorphic to $C_{\Sigma f_i}$. Second, we remark that by the definition of CW complexes Σf_i is the restriction of $f_{i+1} : X_{i+1} \to Y_{i+1}$ to the subcomplex ΣX_i, and from this one can conclude that there is a natural map $C_{\Sigma f_i} \to C_{f_{i+1}}$. Finally, we define the component maps as the composition $\Sigma C_{f_i} \cong C_{\Sigma f_i} \to C_{f_{i+1}}$

Furthermore, we claim that one can similarly define the mapping cylinder M_f for a map of spectra $f : X \to Y$, by setting $(M_f)_i = M_{f_i} = (X_i \times I) \cup_{f_i} Y_i$ and defining the component maps similarly as was done above. Recall that any map of CW complexes $f : X \to Y$ factors through the mapping cylinder M_f, as the composition of an inclusion of a subcomplex and a homotopy equivalence, as below.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \quad \nearrow_{\simeq} & \\ & M_f & \end{array}$$

The inclusion is given by sending X to the quotient of $X \times \{0\}$ in M_f and the homotopy equivalence is given by contracting the quotient of $X \times I$ in M_f down to $X \times \{1\}$.

Similarly, we claim that any map of CW spectra $f : X \to Y$ factors through M_f in such a way, with the inclusion and homotopy equivalence of spectra being defined just as above on each component space of the spectra involved. Later on, when we construct the Adams spectral sequence, we will say that we "replace the map f by an inclusion". By this we mean that given a map of spectra $f : X \to Y$, we instead consider the inclusion $X \hookrightarrow M_f$ as constructed above. Since M_f is homotopy equivalent to Y, the groups $[X, Y]$ and $[X, M_f]$ are isomorphic. This means particular that there is a one-to-one correspondence between homotopy classes of such maps and thus that we are not losing any homotopic information after doing such a replacement.

We will illustrate a way that proposition 2.23 may be used to prove a lemma about the finite generation of $[X, Y]$ for a special case.

Lemma 2.24. If X is a finite spectrum and Y is of finite type, then $[X, Y]$ is finitely generated.

Proof. We prove this by induction on the number of non-basepoint in X. For the base case when X has one cell, X must be some sphere spectrum, and thus $[X, Y]$ is finitely generated since Y is of finite type.

For the induction step, suppose X has $n + 1$ non-basepoint cells. Let A be some non-basepoint cell in X (recall that this is an equivalence class of cells in X_i as i ranges). Then A is some sphere spectrum and X/A has n cells. By the inductive hypothesis, $[A, Y]$ and $[X/A, Y]$ are finitely generated. By the first exact sequence in proposition 2.23, we have an exact sequence

$$[A, Y] \xleftarrow{i^*} [X, Y] \xleftarrow{q^*} [X/A, Y] \ ,$$

from which we obtain the short exact sequence

$$0 \to [X/A, Y]/\ker q^* \to [X, Y] \to \text{Im } i^* \to 0 \ .$$

Since subgroups and quotients of finitely generated abelian groups are finitely generated, the left and right groups are finitely generated. If we have a short exact sequence of modules where the left and right modules are finitely generated, then the middle module must also be finitely generated. □

Let us now move on to defining the wedge product of spectra. To do this, let $\bigvee_\alpha X_\alpha$ be the spectrum with i:th component space equal to $\bigvee_\alpha (X_\alpha)_i$ and the component map equal to the composition below.

$$S^1 \wedge [\bigvee_\alpha (X_\alpha)_i] \xleftrightarrow{\cong} \bigvee_\alpha S^1 \wedge (X_\alpha)_i \xrightarrow{\bigvee_\alpha \sigma_i^{X_\alpha}} \bigvee_\alpha (X_\alpha)_{i+1}$$

Definition 2.25. The *wedge product* of a family of spectra $\{X_\alpha\}_\alpha$ is denoted by $\bigvee_\alpha X_\alpha$, with the i:th component space equal to $\bigvee_\alpha (X_\alpha)_i$ and the i:th component map equal to the composition

$$S^1 \wedge [\bigvee_\alpha (X_\alpha)_i] \xleftrightarrow{\cong} \bigvee_\alpha S^1 \wedge (X_\alpha)_i \xrightarrow{\bigvee_\alpha \sigma_i^{X_\alpha}} \bigvee_\alpha (X_\alpha)_{i+1} \ .$$

The first map is a homeomorphism, coming from the fact that smash products commute with arbitrary wedges in a convenient category of topological spaces, which is definitely true for our situation where all spaces are CW complexes. The second map is applying the component map of X_α on each wedge.

2.4 Cohomology

We have now made enough definitions and stated enough theorems to describe how one can talk about cohomology in the context of spectra. Recall that a reduced generalized cohomology theory on spaces may be defined as a sequence of contravariant functors $H^n : (\mathbf{Top}^*_{\mathrm{CW}})^{\mathrm{op}} \to \mathbf{Ab}$ from the category of pointed CW complexes to the category abelian groups satisfying the Eilenberg-Steenrod axioms (minus the dimension axiom). We may similarly define cohomology theories on spectra, as a sequence of contravariant functors from CW spectra to satisfying these axioms, because we have defined what a homotopy, suspension and wedge product means in the context of spectra.

Definition 2.26. A reduced cohomology theory of spectra is a sequence of contravariant functors H^n (for $n \in \mathbb{Z}$) from CW spectra to abelian groups satisfying the following conditions.

(i) *Homotopy invariance*: If two maps $f, g : X \to Y$ are homotopic, then $H(f) = H(g)$.

(ii) *Exactness*: If $i : A \hookrightarrow X$ is the inclusion of a closed subspectrum and $q : X \twoheadrightarrow X/A$ is the quotient map, then the induced sequence

$$H^k(X/A) \xrightarrow{q^*} H^k(X) \xrightarrow{i^*} H^k(A)$$

is exact for all $k \in \mathbb{Z}$.

(iii) *Suspension isomorphism*: There is a natural isomorphism of functors

$$H^k(-) \to H^{k+1}(\Sigma(-)) \ .$$

(iv) *Additivity*: For any family of spectra $\{X_\alpha\}_\alpha$, the natural map

$$H^*\left(\bigvee_\alpha X_\alpha\right) \to \prod_\alpha H^*(X_\alpha)$$

is an isomorphism.

For any spectrum X, we may define a sequence of functors $h^n(-) := [\Sigma^{-n}(-), X]$. This sequence of functors defines a reduced cohomology theory of spectra. Homotopy invariance is obvious. Exactness follows from our discussion about cofiber sequences. In fact, we get a long exact sequence associated to the pair (X, A) since we have natural connecting maps $H^k(A) \to H^{k+1}(X/A)$. The suspension isomorphism follows from the fact that the suspension map $[X, Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism for all spectra X, Y. Additivity follows from the observation that maps out of a wedge product may be seen as a tuple of maps out of every single wedge in the wedge product.

Furthermore, given any spectrum X, one can verify that one may define a reduced cohomology theory on *spaces* by setting $h^n(-) = [\Sigma^{-n}\Sigma^\infty(-), X]$ for all $n \in \mathbb{Z}$. If the reader was not already convinced that spectra are worth studying, hopefully this will convince you: Surprisingly, the converse statement is also true! By the Brown representability theorem given any reduced cohomology theory H^* on connected CW complexes, there is a spectrum X and an isomorphism $H^n(-) \cong [\Sigma^{-n}\Sigma^\infty(-), X]$ of functors, meaning that every cohomology theory of spaces is represented by a spectrum! We refer the reader to Brown's original proof in [8] for more on this topic.

Remark 2.27. Dually, we can define a reduced homology theory of spectra in an analogous fashion and prove that for any spectrum X, $X_n(-) = [\Sigma^n X, -]$ defines a homology theory on spectra. In particular, it follows that the homotopy groups $\pi_n(X)$ of spectra form a homology theory. This means that we have a Mayer-Vietoris long exact sequence, among other things, which one does not have for ordinary homotopy groups of spaces, making it much easier to compute!

It is known that singular (or equivalently, cellular) cohomology of spaces with coefficients in an abelian group G is represented by Eilenberg-Maclane spaces, since $H^n_{\text{sing}}(-) \cong [-, K(G,n)]$. We will now define cellular homology and cohomology of spectra with coefficients in G. One can then prove that this cohomology theory of spectra is represented by the Eilenberg-Maclane spectra, meaning that $H^n_{\text{CW}}(-) \cong [-, H(G,n)]$.

Given a spectrum X, let $C_\bullet(X_i; R)$, abbreviated to $C_\bullet(X_i)$, be the cellular chain complex of the space X_i with coefficients in some ring R. By the definition of a CW spectrum, there are inclusion $\Sigma X_i \hookrightarrow X_{i+1}$. Consequently, there are inclusions $C_\bullet(X_i) \hookrightarrow C_{\bullet+1}(X_{i+1})$, given by taking a cell in X_i, suspending it, and considering it as a cell in X_{i+1}. We define the cellular chain complex of the spectrum X as

$$C_k(X) = \varinjlim_{i\to\infty} C_{i+k}(X_i)$$

for all k, where the direct limit runs over the diagram $C_k(X_0) \hookrightarrow C_{1+k}(X_1) \hookrightarrow \cdots$ of inclusions as described above. The direct limit over i of the differentials coming from the complexes $C_\bullet(X_i)$ gives us a differential for $C_\bullet(X)$. We may thus define the cellular homology groups of X $H_k(X)$ as the k:th homology group of the complex $C_\bullet(X)$. In fact, since direct limits are an exact functor, the homology of $C_\bullet(X)$ is isomorphic to the direct limit of the homology of $C_\bullet(X_i)$. This means that the homology groups $H_k(X)$ may equivalently be defined as

$$\varinjlim_{i\to\infty} H_{i+k}(X_i) \cong H_k(X) = H_k(C_\bullet(X)) \ .$$

The cellular cohomology with coefficients in an R-module G are simply the homology groups of the cochain complex obtained by applying the functor $\text{Hom}_R(-, G)$ to the cellular chains $C_\bullet(X)$. That is, $H^k(X; G) := H_k(\text{Hom}_R(C_\bullet(X), G))$. Defining the cellular cohomology like this, one can in fact verify that the universal coefficient theorem holds in this setting.

Proposition 2.28. Given any spectrum X, $k \in \mathbb{Z}$, R-module G for R a principal ideal domain, there is a split exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{k-1}(X;R), G) \to H^k(X;G) \to \mathrm{Hom}_R(H_k(X;R), G) \to 0 \ .$$

We will now present the spectra-analog of Milnor's short exact sequence. Recall that if a CW complex X is the union $\bigcup_{i\geq 0} X_i$ of monotonically increasing subcomplexes X_i, then for any reduced cohomology theory H^*, there is a short exact sequence

$$0 \to \varprojlim_i{}^1 H^{n-1}(X_i) \to H^n(X) \to \varprojlim_i H^n(X_i) \to 0$$

for all integers n. The proof of this can be done analogously in the context of spectra.

Proposition 2.29. If the spectrum X is the union of a sequence of monotonically increasing subspectra X_i, then for any reduced cohomology theory H^* of spectra, there is for each integer n a short exact sequence

$$0 \to \varprojlim_i{}^1 H^{n-1}(X_i) \to H^n(X) \to \varprojlim_i H^n(X_i) \to 0 \ .$$

Remark 2.30. For the construction of the Adams spectral sequence, we need a lemma about a scenario when the functor $[X, -]$ commutes with *arbitrary* wedge sums of spectra. For the wedge sum of two spectra, $[X, Y_1 \vee Y_2]$ is isomorphic to $[X, Y_1] \oplus [X, Y_2]$. The isomorphism from $[X, Y_1 \vee Y_2]$ is simply given by composing a map from this group with the quotients $Y_1 \vee Y_2 \to Y_i$ for $i = 1, 2$. The inverse isomorphism from $[X, Y_1] \oplus [X, Y_2]$ is given by composing the pair of maps with the inclusions $Y_i \hookrightarrow Y_1 \vee Y_2$ and taking their sum. In fact, it is easy to prove that this isomorphism is natural in X. By induction, there is an isomorphism $[X, \bigvee_{i=1}^n Y_i] \cong \bigoplus_{i=1}^n [X, Y_i]$ natural in X, for all n. For the case when we are dealing with arbitrarily large wedge sums, we have the following lemma, whose proof will utilize plenty of the theorems we have mentioned thus far.

Lemma 2.31. Let X is a connective spectrum of finite type. Assume $\{p_i\}_{i\geq 0}$ is a sequence of integers such that $p_i \geq M$ for some M, and which repeats values only finitely many times. Then the natural map

$$\left[X, \bigvee_{i\geq 0} H(\mathbb{Z}_2, p_i)\right] \to \prod_{i\geq 0} [X, H(\mathbb{Z}_2, p_i)]$$

is an isomorphism.

Proof. Since X is a connective spectrum of finite type, the spectra X^n (consisting of all cells of dimension $\leq n$) are finite for each n, and constitute a monotonically increasing sequence of subspectra whose union is all of X. Setting $H^*(-) = [\Sigma^{-*}(-), \bigvee_i H(\mathbb{Z}_2, p_i)]$, we obtain a reduced cohomology theory. By the proposition above, we have a short exact sequence

$$0 \to \varprojlim_n{}^1 [\Sigma^1 X^n, \bigvee_i H(\mathbb{Z}_2, p_i)] \to \left[X, \bigvee_i H(\mathbb{Z}_2, p_i)\right] \to \varprojlim_n [X^n, \bigvee_i H(\mathbb{Z}_2, p_i)] \to 0 \ .$$

Consider the rightmost group. Since X^n is a finite spectrum, the group $[X^n, \bigvee_i H(\mathbb{Z}_2, p_i)]$ is equal to $[X^n, \bigvee_i^{M_n} H(\mathbb{Z}_2, p_i)]$ for some integer M_n by the cellular approximation theorem, since for $p_i > n$, the spectrum $H(\mathbb{Z}_2, p_i)$ only has cells of dimension higher than n. By the remark above, it follows that the rightmost group in the exact sequence above equals $\varprojlim_n \bigoplus_i^{M_n} [X^n, H(\mathbb{Z}_2, p_i)]$. By cellular approximation again, this group equals $\varprojlim_n \bigoplus_i [X^n, H(\mathbb{Z}_2, p_i)]$ since we are simply adding zero groups. One can confirm that this group is isomorphic to $\prod_i [X, H(\mathbb{Z}_2, p_i)]$.

To finish the proof it remains to show that the $\lim^1$ group in the short exact sequence above vanishes. First, note that $[\Sigma^1 X^n, \bigvee_i H(\mathbb{Z}_2, p_i)] \cong \bigoplus_i^{M_n} [\Sigma^1 X^n, H(\mathbb{Z}_2, p_i)]$ by the same argument as in the paragraph above. Then, notice that this group is equal to a (finite) direct sum of the p_i:th cellular cohomology group $H^{p_i}_{\mathrm{CW}}(\Sigma^1 X^n; \mathbb{Z}_2) \cong H^{p_i - 1}_{\mathrm{CW}}(X^n; \mathbb{Z}_2)$ by the representability of cohomology. Since X^n is a finite spectrum, it is easy to see that the homology groups $H_*^{\mathrm{CW}}(X^n; \mathbb{Z})$ are finitely generated in each degree, and thus by proposition 2.28 it is easy to verify that $H^{p_i}_{\mathrm{CW}}(\Sigma^1 X^n; \mathbb{Z}_2)$ is finite. From this we conclude that the inverse system $[\Sigma^1 X^n, \bigvee_i H(\mathbb{Z}_2, p_i)]$ is finite, and thus trivially Mittag-Leffler.

□

Before proving the last lemma of the section, we take a moment to recall a definition of the Steenrod algebra and explain its connection with Eilenberg-Maclane spectra. The (mod 2) Steenrod algebra, which we will denote by A, can be defined in many ways.

One way to define it is as the algebra of all stable cohomology operations $H^*(-, \mathbb{Z}_2) \to H^{*+i}(-, \mathbb{Z}_2)$ of spaces. Another way is to define it as the cohomology ring of the Eilenberg-Maclane spectrum $H(\mathbb{Z}_2, 0)$. From this definition it becomes obvious that the total cohomology of an Eilenberg-Maclane spectrum $H(\mathbb{Z}_2, n)$ is a free A-module over n, for any n. Another way is to define it as the free associative algebra over $\mathbb{Z}_2$ on the generating set $\{\mathrm{Sq}^i \mid i \geq 0\}$ modulo the Adem relations, as Adem proves in his paper [5].

Steenrod and Epstein prove in [28] that the (mod 2) Steenrod algebra A is in fact characterized by five axioms. We will thus take the following as our definition of the Steenrod algebra.

Definition 2.32. The Steenrod algebra A is generated by the collection of elements $\{\mathrm{Sq}^i \geq 0\}$ as a $\mathbb{Z}_2$-algebra, where the Steenrod squares $\mathrm{Sq}^i : H^*(-) \to H^{*+i}(-)$ satisfy the following five axioms.

(1) *Naturality*: For any space X, $\mathrm{Sq}^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$ is a homomorphism which is natural in X.

(2) $\mathrm{Sq}^0 : H^n(X) \to H^n(X)$ is the identity.

(3) If $x \in H^i(X)$, then $\mathrm{Sq}^i(x) = x \smile x$, where $\smile$ denotes the cup product.

(4) If $i > \deg(x)$, then $\mathrm{Sq}^i(x) = 0$.

(5) *Cartan formula:* $\mathrm{Sq}^n(x \smile y) = \sum_{i+j=n} \mathrm{Sq}^i(x) \smile \mathrm{Sq}^j(y)$.

From this definition of the Steenrod algebra, it is obvious that A acts on the total cohomology ring of a space, but it remains to describe how it acts on the total cohomology ring of a spectrum. We restrict ourselves to considering only spectra X of finite type. Notice that A has an obvious grading, with $|\mathrm{Sq}^i| = i$ for all i.

Let $\{X_i\}_i$ be the component spaces of X. Define as sequence of spectra K_i as follows: Let $(K_i)_n = \Sigma^{n-i}X_i$ for all $n \geq i$ and $(K_i)_n = X_n$ for $n < i$, where the component maps are the obvious ones inherited from X. It is easy to verify that the union of the K_i's equals X. Since X is a spectrum of finite type, it is easy to verify that $H^n(X) \cong \varprojlim_i H^n(K_i)$ by proposition 2.29. Furthermore, since K_i is eventually the spectrum $\Sigma^{-i}\Sigma^\infty X_i$ for all i, so it follows that $H^n(K_i) = H^n(\Sigma^{-i}\Sigma^\infty X_i) = H^{n+i}(\Sigma^\infty X_i) = H^{n+i}(X_i)$, where the last cohomology group is of a space.

Consequently, for spectra X of finite type, $H^n(X) \cong \varprojlim_i H^{n+i}(X_i)$ for all n. We may thus define the A-module structure on the total cohomology of a spectrum $H^*(X)$ by its action on $\bigoplus_n \varprojlim_i H^{n+i}(X_i)$. More explicitly, note that the inverse limit $\varprojlim_i H^{n+i}(X_i)$ is given by taking the inverse limit of the following inverse system

$$\cdots \to H^{n+2}(X_2) \xrightarrow{(\sigma_1^X)^*} H^{n+2}(\Sigma X_1) \cong H^{n+1}(X_1) \xrightarrow{(\sigma_0^X)^*} H^{n+1}(\Sigma X_0) \cong H^n(X_0)\ ,$$

where the isomorphisms are given by the suspension isomorphisms. Since the Steenrod squares are natural with respect to maps in cohomology induced by maps of spaces, and since they commute with the suspension isomorphism, it follows that we have a commutative diagram as follows for any $i \geq 0$.

$$\begin{array}{ccccccccc}
\cdots \to H^{n+2}(X_2) & \xrightarrow{(\sigma_1^X)^*} & H^{n+2}(\Sigma X_1) & \xrightarrow{\cong} & H^{n+1}(X_1) & \xrightarrow{(\sigma_0^X)^*} & H^{n+1}(\Sigma X_0) & \xrightarrow{\cong} & H^n(X_0) \\
\downarrow \mathrm{Sq}^i & & \downarrow \mathrm{Sq}^i & & \downarrow \mathrm{Sq}^i & & \downarrow \mathrm{Sq}^i & & \downarrow \mathrm{Sq}^i \\
\cdots \to H^{n+2+i}(X_2) & \xrightarrow[(\sigma_1^X)^*]{} & H^{n+2+i}(\Sigma X_1) & \xrightarrow[\cong]{} & H^{n+1+i}(X_1) & \xrightarrow[(\sigma_0^X)^*]{} & H^{n+1+i}(\Sigma X_0) & \xrightarrow[\cong]{} & H^{n+i}(X_0)
\end{array}$$

It follows that we have a natural map $\mathrm{Sq}^i : H^n(X) \to H^{n+i}(X)$ obtained from taking the inverse limits in the diagram above. This is precisely how the Steenrod algebra acts on the cohomology of spectra. From the fact that A satisfies axioms 1,2,5 from definition

2.32 on the cohomology of spaces, it is easy to see that i A, as cohomology operations on the cohomology of spectra, also satisfies 1,2,5. Notice that if we are dealing with a suspension spectrum of a space, then its cohomology as a spectrum is isomorphic as an A-module to the cohomology of the underlying space, so in this case all five axioms are fulfilled.

We will show now that the cohomology ring of $H^*(H(\mathbb{Z}_2, 0))$ is a free A-module. In [14], Hatcher proves that the map $A \to \tilde{H}^*(K(\mathbb{Z}_2, n), \mathbb{Z}_2)$ given by $\mathrm{Sq}^i \mapsto \mathrm{Sq}^i(\iota_n)$, where $\iota_n \in H^n(K(\mathbb{Z}_2, n), \mathbb{Z}_2)$ is the generator, is an isomorphism in degrees $\leq n$. Since $H(\mathbb{Z}_2, 0)$ is a spectrum of finite type, $H^n(H(\mathbb{Z}_2, 0)) \cong \varprojlim_i H^{n+i}(K(\mathbb{Z}_2, i))$, and from the isomorphism above one can verify that the induced map $A \to \bigoplus_n \varprojlim_i H^{n+i}(K(\mathbb{Z}_2, i))$ given by $\mathrm{Sq}^n \mapsto (\mathrm{Sq}^n(\iota_i))_i$ is an isomorphism in all degrees.

We will now expand on this idea and show that the cohomology of finite type spectra which are wedges of Eilenberg-Maclane spectra are also free A-modules.

Lemma 2.33. If $\{p_i\}_{i \geq 0}$ is a sequence of non-negative integers which repeats values only finitely many times, then the total cellular cohomology $H^*(\bigvee_i H(\mathbb{Z}_2, p_i); \mathbb{Z}_2)$ is a free A-module.

Proof. Notice that $\Sigma^{p_i} H(\mathbb{Z}_2, 0) = H(\mathbb{Z}_2, p_i)$, where the suspension of spectra are defined as shifting the component spaces. By $H^k(-)$ below, we mean the cellular cohomology of a spectrum with $\mathbb{Z}_2$ coefficients. By the wedge axiom and suspension isomorphism, we have that

$$\begin{aligned} H^*(\bigvee_i H(\mathbb{Z}_2, p_i)) &= \bigoplus_k H^k(\bigvee_i H(\mathbb{Z}_2, p_i)) \\ &\cong \bigoplus_k \prod_i H^k(H(\mathbb{Z}_2, p_i)) \\ &\cong \bigoplus_k \prod_i H^{k-p_i}(H(\mathbb{Z}_2, 0)) \ . \end{aligned}$$

By our assumption on $\{p_i\}_i$, $k - p_i$ is non-negative only for finitely many values of i. The direct product in the expression above may thus be taken to be a direct sum. Since direct sums commute with each other, the group above is isomorphic to

$$\bigoplus_i H^{*-p_i}(H(\mathbb{Z}_2, 0)) \ .$$

By the identification of the Steenrod algebra with the cohomology ring of $H(\mathbb{Z}_2, 0)$, we get that this group is isomorphic to

$$\bigoplus_i \Sigma^{p_i} A$$

and is thus a free A-module. □

3 Adams Spectral Sequence

3.1 General introduction to spectral sequences

In algebraic topology, one can often study a space X by relating it with other spaces, which are pieced together by some exact sequence. One example of this is the long exact sequence of homotopy groups associated to a fibration $F \hookrightarrow X \to B$. Another example is the long exact sequence in (co)homology associated to a pair (X, A). Oftentimes however, the algebraic invariant one is interested in, like the stable cohomotopy groups of X in our case, does not piece together in some satisfactory exact sequence which relates it to other groups which are more familiar. But all is not lost. We may still be able to approximate our algebraic invariant of interest by relating it to objects which are more familiar, now using sequences of chain complexes rather than just one long exact sequence. This is what spectral sequences attempt to do.

We will give an example of how spectral sequences naturally arise, and in the process prove a theorem related to spectral sequences. Suppose we want to compute the homology groups $H_*(X)$ of a space X for some homology theory $H_*(-)$. If X is complicated enough, then taking a subspace $A \subseteq X$ and studying the associated long exact sequence in homology of the pair (X, A) may not give you enough information on $H_*(X)$, because the groups $H_*(A)$ and $H_*(X, A)$ may still be too complicated to understand. But hopefully, if we take a filtration $X_1 \subseteq X_2 \subseteq \dots X$ such that $\cup_i X_i = X$, the groups $H_*(X_i)$ and $H_*(X_{i+1}, X_i)$ may be more understandable. In that case, it remains only to relate these groups to the one we are interested in. We will now show how this leads to a spectral sequence.

For each i, we have the long exact sequence associated to the pair (X_{i+1}, X_i), which we may write in the following staircase formation.

$$\begin{array}{ccccc} \dots \longrightarrow H_n(X_i) & & & & \\ \downarrow & & & & \\ H_n(X_{i+1}) & \longrightarrow & H_n(X_{i+1}, X_i) & \longrightarrow & H_{n-1}(X_i) \\ & & & & \downarrow \\ & & & & H_{n-1}(X_{i+1}) \longrightarrow \dots \end{array}$$

Writing them in this manner, we see that the staircase-shaped long exact exact sequences for each pair (X_{i+1}, X_i) fit together in the following commutative diagram, where each color represents a different staircase-shaped long exact sequence.

$$
\begin{array}{ccccccccccc}
 & & \downarrow & & & & \downarrow & & & & \downarrow & \\
\longrightarrow & & H_n(X_i) & \longrightarrow & H_n(X_i, X_{i-1}) & \longrightarrow & H_{n-1}(X_{i-1}) & \longrightarrow & H_{n-1}(X_{i-1}, X_{i-2}) & \longrightarrow & H_{n-2}(X_{i-2}) & \longrightarrow \\
 & & \downarrow & & & & \downarrow & & & & \downarrow & \\
\longrightarrow & & H_n(X_{i+1}) & \longrightarrow & H_n(X_{i+1}, X_i) & \longrightarrow & H_{n-1}(X_i) & \longrightarrow & H_{n-1}(X_i, X_{i-1}) & \longrightarrow & H_{n-2}(X_{i-1}) & \longrightarrow \\
 & & \downarrow & & & & \downarrow & & & & \downarrow & \\
\longrightarrow & & H_n(X_{i+2}) & \longrightarrow & H_n(X_{i+2}, X_{i+1}) & \longrightarrow & H_{n-1}(X_{i+1}) & \longrightarrow & H_{n-1}(X_{i+1}, X_i) & \longrightarrow & H_{n-2}(X_i) & \longrightarrow \\
 & & \downarrow & & & & \downarrow & & & & \downarrow &
\end{array}
$$

Let E be the direct sum of all the groups in the columns which are fully colored in the diagram above; that is, the groups of the form $H_m(X_j, X_{j-1})$. Let A be the direct sum of all groups in the remaining columns; that is, the groups of the form $H_m(X_j)$. Let $i : A \to A$ be the map induced by all maps of the form $H_m(X_j) \to H_m(X_{j+1})$, $j : A \to E$ the map induced by the quotient maps $H_m(X_j) \to H_m(X_j, X_{j-1})$ and $k : E \to A$ the map induced by the connecting homomorphisms $H_m(X_j, X_{j-1}) \to H_m(X_{j-1})$. The fact that the big diagram above was created by piecing together exact sequences allows us to more succinctly write it as the diagram below, which is exact at every spot.

$$
\begin{array}{ccc}
A & \xrightarrow{\;i\;} & A \\
 & {\scriptstyle k}\nwarrow \quad \swarrow{\scriptstyle j} & \\
 & E &
\end{array}
$$

An exact triangle-shaped diagram like this one with two of the groups involved being the same is called an *exact couple.* Looking back at the big colored diagram, i represents the direct sum of all vertical maps, j is the direct sum of maps with codomain of the form $H_m(X_j, X_{j-1})$, and k represents the direct sum of all maps with domain of the form $H_m(X_j, X_{j-1})$.

Once we have obtained an exact couple, we can always construct a spectral sequence, as will be shown below. Consequently, from a filtration of a space X, we obtain a spectral sequence! From now on, we may thus assume that A and E are some arbitrary abelian groups which fit into an exact couple, unless explicitly told otherwise, and show the general construction.

Given an exact couple, we obtain a chain complex $E \xrightarrow{d} E \xrightarrow{d} E$ where the boundary maps are defined by $d = jk$. Notice that $d^2 = jkjk$, which is equal to zero by exactness, so this is indeed a chain complex, and thus we may consider its homology group $E_2 = \ker d / \mathrm{Im}\ d$. Furthermore, the homology group E_2 now takes part in an exact couple of its own! Define $A_2 = \mathrm{Im}\ i$, $i_2 : A_2 \to A_2$ such that $i_2 = i|_{A_2}$, $j_2 : A_2 \to E'$ such that $j_2 = j \circ i^{-1}$ and $k_2 : E_2 \to A_2$ such that $k_2([e]) = k(e)$. By a diagram chase, one can show that j_2 and k_2 are well-defined, and also that the diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{\;i_2\;} & A_2 \\ & {\scriptstyle k_2}\nwarrow \quad \swarrow{\scriptstyle j_2} & \\ & E_2 & \end{array}$$

is exact at every corner, so this defines an exact couple! We call this new exact couple the *derived couple*. We may iterate this process of going from one exact couple to another exact couple in which the new bottom group becomes the homology of the old bottom group (with respect to the map $d_i = j_i k_i$). Doing so gives us a sequence of groups $E = E_1, E_2, E_3, \ldots$ with differentials $d = d_1, d_2, d_3, \ldots$ with the property that $E_{n+1} = \ker d_n / \mathrm{Im}\ d_n$.

Returning to our specific example, we see that the sequence of abelian groups E_i also has a natural bigrading (meaning a grading indexed over $\mathbb{Z}\times\mathbb{Z}$ rather than just over $\mathbb{Z}$) which the differentials d_i respect. We may define $E_1^{p,q} = H_{p+q}(X_p, X_{p-1})$ and $A_1^{p,q} = H_{p+q}(X_p)$. Recall that the maps i_1, j_1, k_1 were constructed by summing over maps between these bigraded components, so it is clear that the differential $d_1 = j_1 k_1 : E_1 \to E_1$ restricts to a map between these bigraded components. It follows that the groups E_n obtain a bigraded structure from E_1 since they are subquotients of it.

We will now explain in which bigraded component the map $d_n^{p,q}$ from the (p,q)-graded component lands in. By looking at the general construction of a new exact couple from an old exact couple, we see that by iterating the process n times, d_n is essentially equal to $j_1 i^{-n+1} k_1$, where i^{-n+1} denotes taking the inverse $n-1$ times. To describe what we mean by this precisely, recall that the groups E_n are subquotients of the group E_1. The map d_n may thus be regarded as the map $E_1 \to E_1$ defined by $j_1 i^{-n+1} k_1$, which is not defined on E_1 however, but it is well-defined on the subquotient E_n.

Noting that k_1 is of the form $E_1^{p,q} \to A_1^{p-1,q}$ and i_1 is of the form $A_1^{p,q} \to A_1^{p+1,q-11}$ and j_1 does not change bidegrees, it follows that $d_n^{p,q}$ is a map of the form $d_n^{p,q} : E_n^{p,q} \to E_n^{p-n,q+n-1}$. To see this pictorially, notice that $d_n^{p,q}$ is given by mapping an element $a \in E_n^{p,q}$ along k_1, taking the preimage of it $n-1$ times along i_1, then mapping it along j_1 as follows:

$$\begin{array}{ccccc} & & A_1^{p-n,q+n-1} & \xrightarrow{j_1} & E_1^{p-n,q+n-1} \ni d_n(a) \\ & & {\scriptstyle i_1}\downarrow & & \\ & & \vdots & & \\ & & {\scriptstyle i_1}\downarrow & & \\ & & A_1^{p-2,q+1} & & \\ & & {\scriptstyle i_1}\downarrow & & \\ a \in E_1^{p,q} & \xrightarrow{k_1} & A_1^{p-1,q} & & \end{array}$$

We may now define (homology) spectral sequences!

Definition 3.1. A (homology) spectral sequence consists of a sequence of bigraded abelian groups $\{E_i^{*,*}\}_i$ and homomorphisms $d_i^{p,q} : E_i^{p,q} \to E_i^{p-i,q+i-1}$ such that $d_i^2 = 0$ and $E_{i+1}^{p,q} = \ker d_i^{p,q} / \mathrm{Im}\ d_i^{p+i,q-i+1}$ for all p, q and all $i \geq 1$.

The word "homology" is written in parentheses, because the only significance this word has in relation to spectral sequences is to specify how the differentials d_i change the bidegree. To define a cohomology spectral sequence, we start precisely as above, but now the differentials point in the direction $d_i : E_i^{p,q} \to E_i^{p+i,q-i+1}$ instead.

Imposing some boundedness conditions on our given space X allows us to describe how the sequence of groups E_i eventually stabilize. For example, suppose X is a finite CW complex. Then it is D-dimensional for some integer D. Since X has a filtration $X_1, X_2, \dots$ which is bounded below (by zero), we may define $X_i = \emptyset$ for $i < 0$. Also, we have that $X_n = X$ for $n \geq D$. Using this, we see that for any fixed pair p, q, there is some large integer N such that $E_1^{p-n,q+n-1} = E_1^{p+n,q-n+1} = 0$ for all $n \geq N$. Consequently, the differential d_n exiting and entering $E_n^{p,q}$ must both be zero for any $n \geq N$. This means that $E_N^{p,q} = E_{N+1}^{p,q} = \cdots = E_{N+k}^{p,q}$ for all $k \geq 0$. We denote this stable group by $E_\infty^{p,q}$.

Summarizing what we have done in our example, we have taken a space X, filtered it into subspaces $X_1, X_2, \dots$ and pieced together long exact sequences associated to pairs of subspaces (X_i, X_{i-1}) into a spectral sequence $\{E_n\}$ and obtained for every bidegree p, q a stable group $E_\infty^{p,q}$ (assuming X is a finite complex). We will now relate these stable groups to the groups we are interested in, $H_*(X)$, and finally justify the machinery we have developed!

Consider the following exact sequence, obtained from the exact couple involving A_n and E_n.

$$E_n^{p+n-1,q-n+2} \xrightarrow{k_n} A_n^{p+n-2,q-n+2} \xrightarrow{i_n} A_n^{p+n-1,q-n+1} \xrightarrow{j_n} E_n^{p,q} \xrightarrow{k_n} A_n^{p-1,q} \xrightarrow{i_n} A_n^{p,q-1}$$

By the construction of $A_n^{p,q}$, it consists of the subgroup of $A_1^{p,q}$ which lies in the image of $i_1^{n-1} : A_1^{p-n+1,q+n-1} \to A_1^{p,q}$. Since $A_1^{p-n+1,q+n-1}$ vanishes for large enough n, (since $X_i = \emptyset$ when $i < 0$), it follows that the fifth and sixth terms vanish for large n. The leftmost term vanishes since for large n it is the subquotient of a group of the form $H_*(X, X)$, because $X_n = X$ for n large enough. Consequently, we have the following short exact sequence.

$$0 \longrightarrow A_n^{p+n-2,q-n+2} \xrightarrow{i_n} A_n^{p+n-1,q-n+1} \xrightarrow{j_n} E_n^{p,q} \longrightarrow 0$$

By the definition of $A_n^{*,*}$, we have that for n large enough, the middle group equals $\mathrm{Im}(H_{p+q}(X_p) \to H_{p+q}(X))$ and the leftmost group equals $\mathrm{Im}(H_{p+q}(X_{p-1}) \to H_{p+q}(X))$.

Consequently, by the short exact sequence above, we obtain an isomorphism

$$E_\infty^{p,q} \cong F_{p+q}^{p} / F_{p+q}^{p-1}$$

for all p, q, where we define

$$F_s^t = \mathrm{Im}(H_s(X_t) \to H_s(X)) .$$

Notice that $\{F_s^t\}_{t \geq 0}$ is a filtration for $H_s(X)$ for every s, which is the group we are interested in. In summary, we just showed that the stable groups $E_\infty^{p,q}$ are isomorphic to the successive filtration quotients of the filtered group $H_{p+q}(X)$. We will now define some terminology so that we may phrase our result as a theorem.

Definition 3.2. Let E_* be a spectral sequence. An element $c \in E_n^{s,t}$ is called a *cycle* if it lies in the kernel of d_n. It is called a *permanent cycle* if c lies in the kernel of d_i for all $i \geq n$ (where we consider c as an element in E_i). It is called a *boundary* if it lies in the image of d_n.

Definition 3.3. Let E_* be a spectral sequence. For all $n \geq 1$, call the bigraded group $E_n^{*,*}$ the E_n*-page*. Denote the group of permanent cycles in bidegree (s,t) by $E_\infty^{s,t}$. Call the bigraded group $E_\infty^{*,*}$ the E_∞*-page*.

Definition 3.4. Given a spectral sequence $E_*^{*,*}$, if for all p, q, the permanent cycles $E_\infty^{p,q}$ is isomorphic to the successive filtration quotients of some fixed filtered group G, we say that the spectral sequence *converges* to G.

Theorem 3.5. *Given a finite filtration $X_1 \subseteq X_2 \subseteq \ldots X_n = X$ on a space X such that $\cup_i X_i = X$, we obtain a spectral sequence $(E_n, d_n)_n$ with $E_1^{p,q} = H_{p+q}(X_p, X_{p-1})$ which converges to $H_*(X)$.*

This theorem allows us to relate the groups $H_p(X_q, X_{q-1})$ with the groups $H_p(X)$, in both directions. Notice that the theorem above was proven for any homology theory, but now let us specify a homology theory to give a concrete application for this theorem.

Application 3.6. Cellular homology coincides with singular homology for finite CW complexes.

Of course, this can be proven more directly, but hopefully this proof will give the reader a taste for how an easier spectral sequence argument goes.

Proof. Let X be our given finite CW-complex. Define the filtration of X such that X_i is the i-skeleton of X. Since X is a finite complex, we have for large n that $X_n = X$. By theorem 3.5, we have a spectral sequence which converges to the singular cohomology groups $H_*^{sing}(X)$.

Notice that $E_1^{p,q} = H_{p+q}^{sing}(X_p, X_{p-1})$ is isomorphic to $H_{p+q}^{sing}(X_p/X_{p-1})$. Also, X_p/X_{p-1} is a wedge of p-spheres. Assuming we are working with reduced homology, the singular homology groups of a wedge of p-spheres is concentrated in degree p. Consequently, $E_1^{p,q} = 0$ if and only if $q \neq 0$. It follows also that $E_n^{p,q} = 0$ for all $n \geq 0$ when $q \neq 0$ since these groups are subquotients of $E_1^{p,q}$.

In the case when $q = 0$, the chain complex $\{(E_1^{p,0}, d_1^{p,0})\}_{p\geq 0}$ is in fact precisely the chain complex computing the cellular cohomology of X, so $E_2^{p,0} = H_p^{CW}(X)$. Furthermore, notice that $d_2^{p,0}$ has codomain $E_2^{p-2,1}$, which is necessarily zero, and thus all differentials landing in the column $E_2^{\bullet,0}$ vanish. similarly, all differentials starting from the column $E_2^{\bullet,0}$ vanish. This implies that $E_3^{p,p} = \ker d_2^{p,p}/\mathrm{Im}\ d_2^{p+1,p+2} = E_2^{p,p}$ and thus by induction $E_\infty^{p,p} = E_2^{p,p} = H_p^{CW}(X)$.

By the convergence of the spectral sequence, it follows that $E_2^{p,0} \cong F_p^p/F_p^{p-1}$ for all p. Notice however that $H_p^{sing}(X_{p-1}) = 0$, since X_{p-1} is a $p-1$ dimensional CW-complex and thus has no singular homology groups in degree p, so $F_p^{p-1} = 0$. Furthermore, this long exact sequence in homology of the pair (X_p, X_{p-1}) shows that the map $H_p^{sing}(X_p) \to H_p^{sing}(X)$ induced by inclusion must be surjective, so $F^{p,p} = H_p(X)$. Therefore, we have that $H_p^{CW}(X) \cong H_p^{sing}(X)$. □

Now we introduce some terminology and state two lemmas, whose proofs are done by a quick diagram chase.

Definition 3.7. Let E_* and D_* be spectral sequences. A *map of spectral sequences* $f : E_* \to D_*$ is a sequence of maps $f_n : E_n^{s,t} \to D_n^{s,t}$ which commutes with the differentials of E_n and D_n for all $n \geq 1$.

Lemma 3.8. Let $f : E_* \to D_*$ be a map of spectral sequences such that for some $N \geq 1$, $f_N : E_N^{s,t} \to D_N^{s,t}$ is an isomorphism for all s, t. Then for all $n \geq N$, $f_n : E_n^{s,t} \to D_n^{s,t}$ is an isomorphism for all s, t as well.

Lemma 3.9. Let $f : E_* \to D_*$ be a map of spectral sequences such that for some $N \geq 1$, $f_N : E_N^{s,t} \to D_N^{s,t}$ (and thus for all $n \geq N$ by lemma above) is an isomorphism for all s, t. Then there is an induced map $f_\infty : E_\infty^{s,t} \to D_\infty^{s,t}$ which is an isomorphism for all s, t.

We finish the section by a discussion of filtered groups, which ends with the proof of two statements that will be necessary for the proof of our final theorem in the thesis.

Definition 3.10. A filtered group is a group G with a filtration $F^\bullet$. A map of filtered groups $f : G \to \tilde{G}$ is a group homomorphism which respects the filtrations, meaning $f(F^n) \subseteq \tilde{F}^n$.

As a remark, notice that $\varinjlim_n F^n = \bigcup_n F^n$ and that $\varprojlim_n F^n = \bigcap_n F^n$. Our convention below is that the filtration is decreasing, meaning that $F^{n+1} \subseteq F^n$ for all n. To prove the final two lemmas in this subsection, we will need the following, which is proved by an easy diagram chase.

Lemma 3.11 (The 4-lemma)**.** Given a commutative diagram of modules

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle d}\downarrow \\ \tilde{A} & \longrightarrow & \tilde{B} & \longrightarrow & \tilde{C} & \longrightarrow & \tilde{D} \end{array}$$

with exact rows such that a is surjective and d is injective, then

(1) if c is surjective then so is b,

(2) if b is injective then so is c.

Now, we can prove our lemmas.

Lemma 3.12. Let $f : G \to \tilde{G}$ be a map of filtered groups such that

- $\bigcup_n F^n = G$ and $\bigcup_n \tilde{F}^n = \tilde{G}$,
- $\bigcap_n F^n = 0$ and $\bigcap_n \tilde{F}^n = 0$,
- $\varprojlim^1_n F^n = 0$,
- f induces isomorphisms $F^n/F^{n+1} \cong \tilde{F}^n/\tilde{F}^{n+1}$ for all n.

Then f is an isomorphism of filtered groups.

Proof. Since f respects the filtrations, we have the commutative diagram below for all integers $m < n$, with exact rows and where the vertical maps are induced by f.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^n/F^{n+1} & \longrightarrow & F^m/F^{n+1} & \longrightarrow & F^m/F^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{F}^n/\tilde{F}^{n+1} & \longrightarrow & \tilde{F}^m/\tilde{F}^{n+1} & \longrightarrow & \tilde{F}^m/\tilde{F}^n & \longrightarrow & 0 \end{array} \tag{3.1}$$

The left vertical arrow is an isomorphism by assumption. If $n = m + 1$, then the right vertical arrow is also an isomorphism by assumption, and thus by the five lemma the middle vertical map is an isomorphism in the case when $n = m + 1$. By induction on n (starting at $n = m + 1$), it follows that f induces an isomorphism $F^m/F^n \cong \tilde{F}^m/\tilde{F}^n$ for all $n > m$ such that the following diagram commutes.

$$\begin{array}{ccccccc} F^m/F^n & \longrightarrow & F^{m-1}/F^n & \longrightarrow & F^{m-2}/F^n & \longrightarrow & \cdots \\ {\scriptstyle \cong}\downarrow & & {\scriptstyle \cong}\downarrow & & {\scriptstyle \cong}\downarrow & & \\ \tilde{F}^m/\tilde{F}^n & \longrightarrow & \tilde{F}^{m-1}/\tilde{F}^n & \longrightarrow & \tilde{F}^{m-2}/\tilde{F}^n & \longrightarrow & \cdots \end{array}$$

Taking the colimit as $m \to -\infty$ (which amounts to taking unions as $m \to -\infty$), gives us an isomorphism $G/F^n \cong \tilde{G}/\tilde{F}^n$ for all integers n such that a diagram similar as to the one above commutes. Taking the inverse limit over n, we obtain an isomorphism

$$\varprojlim_n \to G/F^n \cong \varprojlim_n \tilde{G}/\tilde{F}^n \ .$$

The map of short exact sequences of inverse systems

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^n & \longrightarrow & G & \longrightarrow & G/F^n & \longrightarrow & 0 \\
 & & {\scriptstyle f}\big\downarrow & & {\scriptstyle f}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \\
0 & \longrightarrow & \tilde{F}^n & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{G}/F^n & \longrightarrow & 0
\end{array}$$

induces the commutative diagram below with exact rows, by the naturality of the exact sequence in lemma 1.6.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim_n F^n & \longrightarrow & G & \longrightarrow & \varprojlim_n G/F^n & \longrightarrow & \varprojlim_n^1 F^n \\
{\scriptstyle \cong}\big\downarrow & & {\scriptstyle \cong}\big\downarrow & & \big\downarrow & & {\scriptstyle \cong}\big\downarrow & & \big\hookdownarrow \\
0 & \longrightarrow & \varprojlim_n \tilde{F}^n & \longrightarrow & \tilde{G} & \longrightarrow & \varprojlim_n \tilde{G}/F^n & \longrightarrow & \varprojlim_n^1 \tilde{F}^n
\end{array} \tag{3.2}$$

We have just shown that the fourth vertical map (from the right) is an isomorphism. The first map is trivially an isomorphism, and the second map is an isomorphism by assumption (since both groups are zero). Also, the fifth map is an injection since by assumption its domain is zero. Using lemma 3.11 on the first four columns, we see that $f : G \to \tilde{G}$ is an injection, and using this lemma on the last four columns, we see that f is a surjection.

□

Now we prove a similar theorem but regarding only surjections. The proof below will be very similar to the one above.

Lemma 3.13. Let $f : G \to \tilde{G}$ be a map of filtered groups where G is finite and the first three conditions of lemma 3.12 are satisfied, and also that f induces surjections $F^n/F^{n+1} \twoheadrightarrow \tilde{F}^n/\tilde{F}^{n+1}$ for all n. Then f is a surjection.

Proof. For $m < n$, we have the map of short exact sequences as in (3.1). The left vertical map is a surjection by assumption. The right vertical map is also a surjection by assumption, for the base case $n = m + 1$. By the five-lemma, the middle map is a surjection. By induction, the map f induces a surjection $F^m/F^n \twoheadrightarrow \tilde{F}^m/\tilde{F}^n$ for all $n > m$. Since direct limits are an exact functor, applying it to this surjection, we get a surjection $G/F^n \twoheadrightarrow \tilde{G}/\tilde{F}^n$ for all n.

In general, the inverse limit functor is not right exact, but since the kernel of $G/F^n \twoheadrightarrow \tilde{G}/\tilde{F}^n$ is a subquotient of the finite group G, the kernel must be finite for all n, and thus it is trivially Mittag-Leffler, implying by lemma 1.6 that the induced map of inverse limits

$$\varprojlim_n G/F^n \to \varprojlim_n \tilde{G}/\tilde{F}^n$$

is also a surjection. Using this, we get a diagram analogous to (3.2), but with the fourth vertical map (from the left) now being only a surjection rather than an isomorphism. Using lemma 3.11 on the last four columns of this diagram, we see that $f : G \to \tilde{G}$ is a surjection. □

3.2 Construction of Adams spectral sequence

We can finally construct the Adams spectral sequence, our main computational tool for computing the stable cohomotopy of RP^∞. Our three main references for section 3 are chapter 9 of McLeary's *User's Guide to Spectral Sequences* [21], chapter 5.2 in Hatcher's *Spectral Sequences in Algebraic Topology* [14], and the first three chapters in Bruner's *An Adams Spectral Sequence Primer* [9]. In this entire section, when we write $H^k(-)$ we mean the cellular cohomology of a spectrum with $\mathbb{Z}_2$ coefficients, unless explicitly stated otherwise.

One of the main ideas in the Adams spectral sequence is as follows. For a module over a ring M, we may study it by analyzing homological invariants, like Ext groups in our case. To compute such groups, we must first take some sort of resolution of M, which in our case will be a free resolution. The Steenrod algebra, as the algebra of all stable cohomology operations for cohomology in $\mathbb{Z}_2$, has a natural action on the cohomology groups of a spectrum. If M was the total cohomology $H^*(X)$ of some spectrum X, then it has this A-module structure. A natural question to ask is then whether one can topologically realize the resolution of $H^*(X)$ as an A-module on the level of spectra, before even passing to cohomology. More precisely, can we construct a collection of spectra $\{K_n\}_n$ and a sequence

$$X \to K_0 \to K_1 \to \cdots$$

such that applying the functor $H^*(-)$ gives us a free resolution

$$0 \leftarrow H^*(X) \leftarrow H^*(K_0) \leftarrow H^*(K_1) \leftarrow \cdot$$

of A-modules? The answer is yes! In fact, more is true. Applying $\mathrm{Hom}_A(-, H^*(Y))$ for some spectrum Y to the resolution and taking the homology gives us the groups $\mathrm{Ext}^n_A(H^*(X), H^*(Y))$, which is not inherently interesting, but what makes it interesting is that it approximates the group $[Y, X]$ in some sense. The precise connection between these is captured by the Adams spectral sequence.

Let X be a connective spectrum of finite type. Being of finite type implies that $H^k(X)$ is finitely generated for all k, and being connective implies that $H^k(X)$ vanishes for k smaller than some fixed integer. Let $\{c_\alpha\}_\alpha$ be a set of generators for $H^*(X)$ as an A-module. By the representability of cohomology, each $c_\alpha \in H^{|c_\alpha|}(X)$ corresponds to map $X \to K(\mathbb{Z}_2, |c_\alpha|)$. Notice that the values of $|c_\alpha|$ as α ranges is bounded below by some number and that it repeats values only a finite number of times. By lemma 2.31, the set $\{c_\alpha\}_\alpha$ corresponds to a map $i_0 : X \to \bigvee_\alpha K(\mathbb{Z}_2, |c_\alpha|)$. Replacing this map by an inclusion, let $X_1 = K_0/X$.

We claim that X_1 is connective of finite type. We can for all $n \geq 1$ pick models of the spaces $K(\mathbb{Z}_2, n)$ such that they have no non-basepoint cell of dimension less than n. It follows that the spectrum $H(\mathbb{Z}_2, 0)$ has no cell of dimension less than zero, and thus that

$\Sigma^n H(\mathbb{Z}_2, 0) = H(\mathbb{Z}_2, n)$ has no cell of dimension less than n. This implies that K_0 is connective of finite type, and thus that X_1 (as a quotient of K_0) is connective of finite type.

Picking a set of generators for $H^*(X)$ as an A-module gives rise to a map $X_1 \to K_1$ just as before. Replacing this map by an inclusion, we may set $X_2 = K_1/X_1$ and iterate the process. We get a sequence K_n of finite type spectra which are wedges of Eilenberg-Maclane spectra and connective spectra of finite type X_n. We also obtain the following sequence, where the dotted maps $K_n \to K_{n+1}$ are the compositions of the lower maps.

$$\begin{array}{ccccccccc} X & \xhookrightarrow[i_0]{} & K_0 & \dashrightarrow & K_1 & \dashrightarrow & K_2 & \dashrightarrow & K_3 \dashrightarrow \dots \\ & & \downarrow{\scriptstyle q_0} & \nearrow{\scriptstyle i_1} & \downarrow{\scriptstyle q_1} & \nearrow{\scriptstyle i_2} & \downarrow{\scriptstyle q_2} & \nearrow{\scriptstyle i_3} & \\ & & X_1 = K_0/X & & X_2 = K_1/X_1 & & X_3 = K_2/X_2 & & \end{array} \tag{3.3}$$

Denoting X by X_0, notice that we have cofiber sequences $X_n \hookrightarrow K_n \twoheadrightarrow X_{n+1}$ for all $i \geq 0$. Applying the total cohomology functor $H^*(-)$, we get the following diagram.

$$\begin{array}{ccccccccccc} 0 \longleftarrow & H^*(X) & \xleftarrow[(i_0)^*]{} & H^*(K_0) & \longleftarrow & H^*(K_1) & \longleftarrow & H^*(K_2) & \longleftarrow & H^*(K_3) & \longleftarrow \dots \\ & & & \uparrow{\scriptstyle (q_0)^*} & \swarrow{\scriptstyle (i_1)^*} & \uparrow{\scriptstyle (q_1)^*} & \swarrow{\scriptstyle (i_2)^*} & \uparrow{\scriptstyle (q_3)^*} & \swarrow{\scriptstyle (i_3)^*} & & \\ & & & H^*(X_1) & & H^*(X_2) & & H^*(X_3) & & & \end{array} \tag{3.4}$$

By lemma 2.33, the modules $H^*(K_n)$ are free over A. Furthermore, we claim that the top row is exact, and thus that it constitutes a free resolution of $H^*(X)$.

Lemma 3.14. The top row in (3.4) is exact.

Proof. By the exactness axiom for cohomology theories (definition 2.26(ii)), the sequences $H^*(X_{n+1}) \to H^*(K_n) \to H^*(X_n)$ are exact for all $i \geq 0$. If we can show that in (3.4) the vertical maps $(q_n)^*$ are injections and that the diagonal maps $(i_n)^*$ are surjections, the result will follow by an easy diagram chase.

We will describe how the maps i_n were constructed in slightly more detail to prove this. Pick an element c_α from the generating set for $H^*(X_n)$ as an A-module that was used in the construction of i_n. Under the isomorphism $H^{|c_\alpha|}(X_n) \cong [X_n, H(\mathbb{Z}_2, |c_\alpha|)]$, c_α corresponds to a map $f_\alpha : X_n \to H(\mathbb{Z}_2, |c_\alpha|)$ such that $f_\alpha^*(\iota) = c_\alpha$ for some particular cohomology class $\iota \in H^{|c_\alpha|}H(\mathbb{Z}_2, |c_\alpha)$ called the fundamental class. This map pulls back to some tuple $(f_\beta)_\beta$ under the surjection $\prod_\beta [X_n, K(|c_\beta|)] \to [X_n, K(|c_\alpha|)]$ which under the isomorphism in lemma 2.31 corresponds to the map $i_n : X_n \to K_n$. We describe how these elements are related to each other by the sequence below.

$$\begin{array}{ccccccc} i_n & \longmapsto & (f_\beta)_\beta & \longmapsto & f_\alpha & \longmapsto & f_\alpha^*(\iota) = c_\alpha \\ & & & & & & \\ [X_n, K_n] & \xrightarrow[\cong]{} & \prod_\beta [X_n, K(|c_\beta|)] & \longrightarrow & [X_n, K(|c_\alpha|)] & \longrightarrow & H^{|c_\alpha|}H(\mathbb{Z}_2, |c_\alpha) \end{array}$$

Notice that the first map is given by sending i_n to $(\mathrm{proj}_\beta \circ i_n)$, where $\mathrm{proj}_\beta : K_n \to H(\mathbb{Z}_2, |c_\beta|)$ is the projection map which contracts everything but the β-wedge to a point. Consequently, under the sequence above i_n maps to $c_\alpha = f_\alpha^*(\iota) = (\mathrm{proj}_\alpha \circ i_n)^*(\iota) = i_n^*(\mathrm{proj}_\alpha^*(\iota))$. In particular, this means that c_α is in the image of i_n^*, meaning that i_n^* is surjective for all n.

Taking the long exact sequence in cohomology of the pair (K_n, X_n), we get an exact sequence $\cdots \to H^{|c_\alpha|-1}(X_{n+1}) \to H^{|c_\alpha|}(K_n) \to H^{|c_\alpha|}(X_n) \to \cdots$ for all k. We just showed that the maps induced by inclusion are surjective. It follows by exactness that the maps induced by the quotient are injective. Taking the direct sum as α ranges over all possible values (and noting that direct sums are an exact functor), we obtain our result. □

We will now try to relate the homological information coming from the free resolution $H^*(K_\bullet) \to H^*(X)$ to the spectral sequence we will describe now. Recall that in the beginning of this section, we showed that given subspaces $X_0 \subseteq X_1 \subseteq \cdots$ of X, we could patch together the long exact sequences of the pairs (X_{i+1}, X_i) to obtain an exact couple, from which we obtained a spectral sequence. We will follow the same pattern now.

For each $n \geq 0$, we have cofibration sequences $X_n \hookrightarrow K_n \twoheadrightarrow X_{n+1}$. Let Y be a finite spectrum. Recall that $Y_k(-) := [\Sigma^k Y, -]$ defines a homology theory on spectra by remark 2.27. Applying this homology theory to these cofibration sequences, we get a long exact sequence in homology, which may be presented in this staircase formation.

$$
\begin{array}{ccccc}
\cdots \longrightarrow Y_{k+1}(X_{n+1}) & & & & \\
\downarrow & & & & \\
Y_k(X_n) & \longrightarrow & Y_k(K_n) & \longrightarrow & Y_k(X_{n+1}) \\
 & & & & \downarrow \\
 & & & & Y_{k-1}(X_n) \longrightarrow \cdots
\end{array}
$$

These staircase-formed exact sequences (as n ranges over all non-negative integers) can be patched together as before into the following big diagram, where each color represents a different staircase diagram.

$$
\begin{array}{ccccccccccc}
 & \downarrow & & & & \downarrow & & & & \downarrow & \\
\rightarrow & Y_{k+1}(X_{n+1}) & \rightarrow & Y_{k+1}(K_{n+1}) & \rightarrow & Y_{k+1}(X_{n+2}) & \rightarrow & Y_{k+1}(K_{n+2}) & \rightarrow & Y_{k+1}(X_{n+3}) & \rightarrow \\
 & \downarrow & & & & \downarrow & & & & \downarrow & \\
\longrightarrow & Y_k(X_n) & \longrightarrow & Y_k(K_n) & \longrightarrow & Y_k(X_{n+1}) & \longrightarrow & Y_k(K_{n+1}) & \longrightarrow & Y_k(X_{n+2}) & \longrightarrow \\
 & \downarrow & & & & \downarrow & & & & \downarrow & \\
\rightarrow & Y_{k-1}(X_{n-1}) & \rightarrow & Y_{k-1}(K_{n-1}) & \rightarrow & Y_{k-1}(X_n) & \rightarrow & Y_{k-1}(K_n) & \rightarrow & Y_{k-1}(X_{n+1}) & \rightarrow \\
 & \downarrow & & & & \downarrow & & & & \downarrow &
\end{array} \tag{3.5}
$$

Let E be the direct sum of all the columns which are fully colored; that is, the groups of the form $Y_k(K_n)$. Let A be the direct sum of all the remaining columns; that is, the groups of the form $Y_k(X_n)$. Let $i : A \to A$ the direct sum of all connecting maps, $j : A \to E$ be the map induced by inclusion, and $k : E \to A$ be the map induced by quotients. Relating these maps back to the diagram above, i takes elements down vertically, j takes elements to the right from A to E and k takes elements to the right from E to A. These three maps form the exact couple

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & A \\ & {\scriptstyle k}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & E & \end{array}$$

from which we obtain a spectral sequence E_*. This is the Adams spectral sequence! It has a natural bigrading: let $E_1^{s,t} = Y_t(K_s)$.

We will now describe the groups $E_2^{s,t}$ and provide an alternative description for the E_1 page.

Lemma 3.15. The groups $E_1^{s,t} = [\Sigma^t Y, K_s]$ are isomorphic to $\mathrm{Hom}_A^t(H^*K_s, H^*Y)$.

Proof. By lemma 2.27 we have an isomorphism $[Y, K_s] \cong \prod_{\alpha\in I}[Y, H(\mathbb{Z}_2, |c_\alpha|)]$. By the representability of cohomology, this is isomorphic to $\prod_{\alpha\in I} H^*(Y)$. This is isomorphic to the A module of all set maps $I \to H^*(Y)$, denoted by $\mathrm{Hom}_{\mathrm{Set}}(I, H^*(Y))$. Since $H^*(K_s)$ is a free A-module of rank $|I|$ by lemma 2.33, it follows by the universal property of free modules that this is isomorphic to $\mathrm{Hom}_A^0(H^*(K_s), H^*(Y))$. Composing all these isomorphisms, we obtain an isomorphism $[Y, K_s] \cong \mathrm{Hom}_A^0(H^*(K_s), H^*(Y))$ given by sending a map f to the induced map in cohomology f^*. For clarity, we describe how the composition looks in each step by the sequence below.

$$\begin{array}{ccccccccc} f & \longmapsto & (\mathrm{proj}_\alpha \circ f)_\alpha & \longmapsto & \Big(f^*\mathrm{proj}_\alpha^*(\iota)\Big)_\alpha & \mapsto & \Big[\alpha \mapsto f^*\mathrm{proj}_\alpha^*(\iota)\Big] & \mapsto & \Big[\mathrm{proj}_\alpha^*(\iota) \mapsto f^*\mathrm{proj}_\alpha^*(\iota)\Big] \\ [Y, K_s] & \longrightarrow & \prod_{\alpha\in I}[Y, H(\mathbb{Z}_2, |c_\alpha|)] & \longrightarrow & \prod_{\alpha\in I} H^*(Y) & \longrightarrow & \mathrm{Hom}_{\mathrm{Set}}(I, H^*(Y)) & \longrightarrow & \mathrm{Hom}_A^0(H^*(K_s), H^*(Y)) \end{array}$$

The image of f under this composition is indeed f^* since the collection $\{\mathrm{proj}_\alpha^*(\iota)\}_\alpha$ generate $H^*(K_s)$ as an A-module, and thus any degree-preserving A-module homomorphism $H^*K_s \to H^*Y$ is completely determined by where it maps these elements. To see that $\{\mathrm{proj}_\alpha^*(\iota)\}_\alpha$ generates H^*K_s, notice first that the natural map

$$H^*(K_s) \to \prod_\alpha H^*(H(\mathbb{Z}_2, |c_\alpha|))$$

is an isomorphism by the wedge axiom for cohomology (definition 2.26(iv)). Secondly, the fundamental class $\iota \in H^{|c_\alpha|}(K(\mathbb{Z}_2, |c_\alpha|)) \cong H^1(K(\mathbb{Z}_2, 1))$ generates $H^*K(\mathbb{Z}_2, |c_\alpha|)$ as an A-module, as one can see by our proof that the cohomology of Eilenberg-Maclane spectra are free over A just above lemma 2.33. The preimages of the generators $\iota \in H^{|c_\alpha|}(K(\mathbb{Z}_2, |c_\alpha|))$ in each factor of the direct product pull back to the collection of elements $\{\mathrm{proj}_\alpha^*(\iota)\}_\alpha$, thus proving that this collection generates $H^*(K_s)$.

We have thus shown that the map $[Y, K_s] \to \mathrm{Hom}^0_A(H^*K_s, H^*Y)$ given by $f \mapsto f^*$ is an isomorphism. Swapping Y with $\Sigma^t Y$, we get an isomorphism

$$[\Sigma^t Y, K_s] \cong \mathrm{Hom}^0_A(H^*K_s, H^*(\Sigma^t Y)) \ .$$

We have an isomorphism $H^*(\Sigma^t Y) \cong \Sigma^t H^*Y$ as graded modules, which induces the isomorphism $\mathrm{Hom}^0_A(H^*K_s, H^*(\Sigma^t Y)) \cong \mathrm{Hom}^0_A(H^*K_s, \Sigma^t H^*Y) = \mathrm{Hom}^t_A(H^*K_s, H^*Y)$. The last equality follows from the fact that a degree-preserving map of graded modules $M \to \Sigma^t N$ is precisely a degree t lowering map of graded modules $M \to N$. □

Remark 3.16. Recall that for all s, K_s is a spectrum of finite type. Also, Y is a finite spectrum. It is easy to show, just like for the analogous statement in spaces, that this implies that the cohomology of K_s is finitely generated in every degree and that the total cohomology of y is finitely generated. Since we are working with coefficients in $\mathbb{Z}_2$, finite generation is the same as being finite. Consequently, the group of degree t lowering maps $E_1^{s,t} = \mathrm{Hom}^t_A(H^*K_s, H^*Y)$ must be finite for all t. In fact, each element has order 2. Since $E_n^{s,t}$ are subquotients of $E_1^{s,t}$ for all n, it follows that in any bidegree on any page, the Adams spectral sequence is finite, given the conditions of X being connective of finite type and Y being finite.

Lemma 3.17. *The groups $E_2^{s,t}$ are isomorphic to $\mathrm{Ext}^{s,t}_A(H^*X, H^*Y)$.*

Proof. The differential on the first page is given by $d_1 = jk$. By the definition of j and k, $d_1^{s,t}$ is precisely the the map $Y_t(K_s) \to Y_t(K_{s+1})$ induced by the map $K_s \to K_{s+1}$ in (3.3). Under the isomorphism $E_1^{s,t} \cong \mathrm{Hom}^t_A(H^*K_s, H^*Y)$ given in the lemma above, the differential

$$d_1^{s,t} : \mathrm{Hom}^t_A(H^*K_s, H^*Y) \to \mathrm{Hom}^t_A(H^*K_{s+1}, H^*Y)$$

becomes precisely the map the map induced by $H^*K_{s+1} \to H^*K_s$ from the free resolution (3.4), which is in turn induced by $K_s \to K_{s+1}$ from (3.3).

Under the isomorphism from lemma 3.15, we obtain an isomorphism of chain complexes $E_1^{\bullet,\bullet} \cong \mathrm{Hom}^\bullet_A(H^*K_\bullet, H^*Y)$. Taking the homology at the bidegree (s,t) of these complexes, we thus obtain an isomorphism $E_2^{s,t} \cong \mathrm{Ext}^{s,t}_A(H^*X, H^*Y)$. □

Summarizing what we have done so far, we have constructed a sort of resolution on the level of spectra $X \to K_\bullet$ such that applying $H^*(-)$ gives us a free resolution $H^*(K_\bullet) \to H^*X$ of A-modules. The construction of the spectra-level resolution gave us cofibration sequences $X_s \hookrightarrow K_s \twoheadrightarrow X_{s+1}$, and thus long exact sequences in the homology groups $Y_k(-)$, from which we built a spectral sequence. We have identified the E_1 and E_2 pages. It remains to relate these groups to the group $[Y, X]$, which will be done by showing that the E_∞ page equals the associated graded module of $[Y, X]$ modulo odd torsion for some appropriate filtration.

We will now define some terms relating to the construction we have done thus far. Note that from (3.3), we obtain a sequence of maps $X \to X_1 \to X_2 \to \cdots$.

Definition 3.18. The sequence of maps $X \to X_\bullet$ and the cofiber sequences $X_\bullet \hookrightarrow K_\bullet \twoheadrightarrow X_{\bullet+1}$ constructed in (3.3) is called an *Adams resolution* of X.

In the colored diagram (3.5) above consisting of patched-together staircase diagrams, notice that we have columns of the form $Y_t(X_s) \to Y_{t-1}(X_{s-1}) \to \cdots \to Y_{t-s}(X)$ for all s, t. This was the A-column in the exact couple constructing the Adams spectral sequence.

Definition 3.19. For each s, t, let $F^{s,t} \subseteq Y_{t-s}(X)$ denote the image of the map $Y_t(X_s) \to Y_{t-s}(X)$ as described above. We call the sequence $\{F^{s+n,t+n}\}$ the *Adams filtration.* An element in $Y_{t-s}(X) = [\Sigma^{t-s}Y, X]$ is said to be of Adams filtration $\geq s$ if it lies in $F^{s,t}$.

Notice that the sequence $\{F^{s+n,t+n}\}_{n\in\mathbb{Z}}$ constitute a decreasing filtration of $Y_{t-s}(X)$. This filtration determines how far an element in $[\Sigma^{t-s}Y, X]$ is pulled back along the column A-column $Y_{t+*}(X_{s+*})$ in (3.5).

Looking at the cofibration sequence $\cdots \Sigma^{-1}X_{i+1} \to X_i \to K_i \to X_{i+1}$, we see that there are natural maps of the form $\Sigma^{-n-1}X_{n+1} \to \Sigma^{-n}X_n$ for all n (including the case $n = 0$ where $X_0 = X$). Applying the functor $Y_{t-s}(-)$ to the sequence $\Sigma^{-\bullet}X_\bullet \to X$, we get a sequence of maps

$$Y_{t-s}(\Sigma^{-n}X_n) \to \ldots Y_{t-s}(X)\ .$$

The groups in this sequence may be rewritten so that we obtain the sequence

$$Y_{t+n+1}(X_{s+n+1}) \to Y_{t+n}(X_{s+n}) \to \cdots Y_{t-s}(X)\ .$$

In fact, one can easily verify that the maps in this sequence are precisely equal to those in the A-column of (3.5).

Definition 3.20. The *Adams tower* of X is the sequence $X^\bullet \to X$ constructed above, where $X^n := \Sigma^{-n}X_n$ for all n.

Remark 3.21. An element in $Y_{t-s}(X)$ is thus of Adams filtration $\geq n$ precisely if it is in the image of the map $Y_{t-s}(X^n) \to Y_{t-s}(X)$, given by composition with the appropriate maps in the Adams tower.

Definition 3.22. Let $T^c \subseteq Y_c(X)$ denote the submodule of all odd-order torsion elements in $Y_c(X) = [\Sigma^c Y, X]$.

The sequence $\{F^{s+n,t+n}/(F^{s+n,t+n} \cap T^{t-s})\}$ constitutes a filtration for Y_{t-s}/T^{t-s}, which we will also call the Adams filtration. We will denote these simply by $F^{s+n,t+n}/T^{t-s}$ below, because we will show that T^{t-s} always lies in $F^{s+n,t+n}$ for reasons which will become apparent soon.

Before moving on to discussing the convergence of the Adams spectral sequence, we prove a about Adams filtrations which will come in handy later.

Lemma 3.23. If a map $f \in [\Sigma^i Y, X]$ is not of Adams filtration ≥ 1, then it does not induce the zero map in cohomology.

Proof. Applying $Y_*(-)$ cofiber sequence $X \to K_0 \to X_1$ coming from an Adams resolution of X, we obtain from the long exact sequence in particular the exact sequence

$$Y_i(\Sigma^{-1} X_1) \to Y_i(X) \to Y_i(K_0) \ .$$

The element $f \in Y_i(X)$ not being of Adams filtration ≥ 1 means precisely that it does not lie in the image of the leftmost map, and thus by exactness it maps to a nonzero element $\tilde{f} \in Y_i(K_0) = [\Sigma^i Y, K_0]$. Since the rightmost map above is defined by composition, this means that we have a commutative diagram as follows.

$$\begin{array}{ccc} \Sigma^i Y & \xrightarrow{f} & X \\ & \underset{\tilde{f}}{\searrow} & \downarrow \\ & & K_0 \end{array}$$

By lemma 3.15, the induced map in cohomology $\tilde{f}^*$ is nonzero. Applying $H^*(-)$ to the diagram above shows thus that the induced map f^* is nonzero as well. □

3.3 Convergence and naturality

Finally we can state how the E_1 and E_2 page of the spectral sequence we have constructed relates to the group $[Y, X]$. This theorem states that the Adams spectral sequence weakly converges to $[\Sigma^* Y, X]$ with respect to the Adams filtration on $[\Sigma^* Y, X]$, given some aforementioned conditions on Y and X.

Theorem 3.24. *Let X be a connective spectrum of finite type, and Y be a finite spectrum. Let $\{F^{s+n,t+n}\}_n$ be the Adams filtration on $[\Sigma^{t-s} Y, X]$ and let $E_*^{*,*}$ be the Adams spectral sequence computing $[\Sigma^* Y, X]$. Then for all s, t, we have short exact sequences*

$$0 \to F^{s+1,t+1} \to F^{s,t} \to E_\infty^{s,t} \to 0$$

for all s, t, meaning that $E_\infty^{s,t} \cong F^{s,t}/F^{s+1,t+1}$.

We will need to prove some lemmas before we can tackle the theorem above.

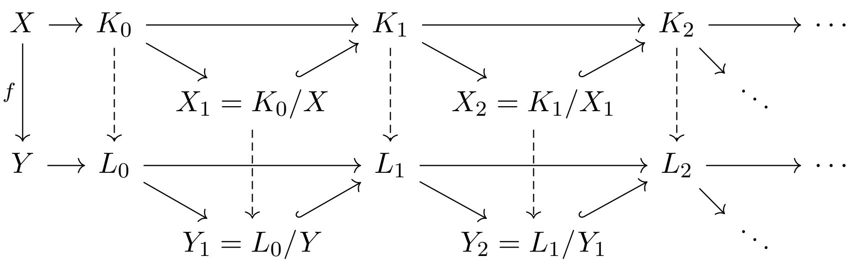
Oftentimes, we want to compare different Adams spectral sequences so that we can leverage one of the spectral sequences to get information about the other. This will not only be crucial for proving the theorem above, but is one of the main ideas in completing Lin's computation of the stable cohomotopy of infinite projective space in [19]. To this end, we prove the following lemma, helping us compare different Adams-type resolutions of spectra. It may be seen as the spectra-analog of the comparison theorem in homological algebra for projective resolutions.

Lemma 3.25. Assume we are given the filled-in arrows in the diagram below.

$$\begin{array}{ccccccccc} X & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & \dots \\ {\scriptstyle f}\downarrow & & \vdots & & \vdots & & \vdots & & \\ Y & \longrightarrow & L_0 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots \end{array}$$

Let X and Y be connective spectra of finite type. Assume that applying the functor $H^*(-)$ to the top row gives you an exact sequence $\cdots \to H^*K_1 \to H^*K_0 \to H^*X \to 0$. Assume that the L_i are wedges of finite type and are Eilenberg-Maclane spectra. Assume also that the composition of two consecutive maps in the rows are nullhomotopic. Then we can fill in the diagram above with the dashed arrows such that the diagram commutes up to homotopy.

Proof. We begin by replacing all horizontal maps with inclusions. Since the composition of two consecutive maps in the either row is always nullhomotopic, the horizontal maps factor through the quotients to give a homotopy-commutative diagram as follows (considering for now only the filled-in arrows).



By lemma 2.31 and the representability of cohomology, a map $X \to L_0$ is in one-to-one correspondence with a tuple $\{c_\alpha\}_\alpha$ of elements in $H^*(X)$. Since the induced map in cohomology $H^*K_0 \to H^*X$ is surjective by assumption, there are a tuple of classes $\{b_\alpha\}_\alpha \in \prod_\alpha H^*K_0$ that are sent to $\{f^*c_\alpha\}_\alpha$ along this map. By lemma 2.31, the tuple $\{b_\alpha\}_\alpha$ corresponds to a map $K_0 \to L_0$. By explicitly writing out the isomorphism $H^*(-) \cong [-, K(\mathbb{Z}_2, *)]$ and the isomorphism in lemma 2.31, one can confirm by a diagram chase that including this map $K_0 \to L_0$ in the diagram above gives us square involving the map $f : X \to Y$ which is homotopy-commutative.

Consequently, the map $K_0 \to L_0$ factors through the quotients to give a map $f_1 : X_1 = K_0/X \to Y_1 = L_0/Y$ which makes the square involving the map $K_0 \to L_0$ homotopy-commutative. The proof of lemma 3.14 shows in fact that the induced map in cohomology $H^*K_1 \to H^*X_1$ must be surjective and that $H^*X_1 \to H^*K_0$ is injective. Using this it follows that the diagram below satisfies the same exactness assumption in the top row, just as in our original diagram.

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & K_3 & \longrightarrow & \cdots \\ {\scriptstyle f_1}\downarrow & & & & & & & & \\ Y_1 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & L_3 & \longrightarrow & \cdots \end{array}$$

In fact, all other assumption we held for the original diagram also holds for this one. By induction, our result follows. ☐

Remark. If we set $X = Y$ and pick two different Adams resolutions of X, one can use this lemma with $f = \mathrm{id}$ to show that $E_2^{s,t}$ is independent of the choice of Adams resolutions.

Corollary 3.26. If

$$\begin{array}{ccccccccc} X \longrightarrow K_0 & & \longrightarrow & & K_1 & \longrightarrow & & K_2 & \longrightarrow \cdots \\ & \searrow & & \nearrow & & \searrow & \nearrow & & \searrow \\ & & X_1 & & & X_2 & & & \ddots \end{array}$$

is an Adams resolution for X, then

$$\begin{array}{ccccccccc} X_n \longrightarrow K_n & & \longrightarrow & & K_{n+1} & \longrightarrow & & K_{n+2} & \longrightarrow \cdots \\ & \searrow & & \nearrow & & \searrow & \nearrow & & \searrow \\ & & X_{n+1} & & & X_{n+2} & & & \ddots \end{array}$$

is an Adams resolution for X_n, for all n. Consequently, if $\cdots \to X^2 \to X^1 \to X$ is an Adams tower for X, then $\cdots \to X^{n+2} \to X^{n+1} \to X^n$ is an Adams tower for X^n, for all n.

Proof. We essentially just proved this in lemma 3.25 above for the case $n = 1$ when stating that the sequence $X_1 \to K_\bullet$ satisfies all aforementioned assumptions. The result follows by induction. ☐

Remark. From this one can conclude that the Adams filtration is independent of the choice of Adams resolution or Adams tower. It is easy to see that lemma 3.25 implies that, given any two Adams towers $X^\bullet$ and $Z^\bullet$ on X, there are sequences of maps f_i making the diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & X^2 & \longrightarrow & X^1 & \longrightarrow & X \\ & \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \cdots \longrightarrow & Z^2 & \longrightarrow & Z^1 & \longrightarrow & X \end{array}$$

commute, and similarly we can construct a similar diagram with vertical maps going the other way as well. Applying the functor $Y_t(-)$ to these two diagrams, we see that an element in $Y_t(X)$ has Adams filtration $\geq s$ with respect to the tower $X^\bullet$ if and only if it has Adams filtration $\geq s$ with respect to the tower $Z^\bullet$.

Lemma 3.27. Let X and Y satisfy the conditions of theorem 3.24. Let $F^{s,t}$ denote the Adams filtration just as in theorem 3.24. If the groups $\pi_*(X)$ consist completely of 2-torsion elements, then the set

$$\bigcap_{n \in Z} F^{s+n,t+n}$$

of elements in $[\Sigma^{t-s} Y, X]$ of infinite Adams filtration equals zero.

Proof. By lemma 2.24 $\pi_k(X)$ is finitely generated for all k, and thus finite since it consists only of 2-torsion elements. Let n be the smallest integer such that $\pi_n(X)$ is nonzero. Let L_0 be the wedge product of $\dim_{\mathbb{Z}_2} H^n(X)$ many copies of $H(\mathbb{Z}_2, n)$. By lemma 2.31 and the representability of cohomology, a choice of $\mathbb{Z}_2$-basis for $H^n(X)$ is equivalent to a map $X \to L_0$ which induces an isomorphism when applying the functor $H^n(-)$, so we pick a basis to obtain such a map.

By the naturality of the universal coefficient theorem (proposition 2.28) and the five-lemma, it follows that the map $X \to L_0$ also induces an isomorphism when applying the functor $H_n(-)$. There is a Hurewicz theorem analog in the context of spectra which states that for $n-1$-connected spectra, the Hurewicz map $\pi_n(X) \to H_n(X)$ is an isomorphism. Applying $\pi_n(-)$ to the map $X \to L_0$ essentially gives us the map $\pi_n(X) \to \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ given by $f \mapsto f \otimes 1$ in the following sense.

By the wedge axiom, $\pi_n(L_0)$ is isomorphic to the product of $\dim_{\mathbb{Z}_2} H^n(X)$ many copies of $\mathbb{Z}_2$. By the universal coefficient theorem, $\dim_{\mathbb{Z}_2} H^n(X) = \dim_{\mathbb{Z}_2} H_n(X)$ and by the isomorphism above, $\dim_{\mathbb{Z}_2} H_n(X) = \dim_{\mathbb{Z}_2} \pi_n(X)$. Consequently, $\pi_n(L_0) \cong \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}_2$. It is easy to verify that the composition of these isomorphisms with the induced map $\pi_n(X) \to \pi_n(L)$ gives us precisely the aforementioned map $\pi_n(X) \to \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}_2$. By assumption, $\pi_n(X)$ is a group of order a power of 2, and such groups always gave an element of order 2. This element can never be in the kernel of this map, and thus the kernel is a strictly smaller group than $\pi_n(X)$. The same statement is then true for the induced map $\pi_n(X) \to \pi_n(L_0)$.

Replace the map $X \to L_0$ by an inclusion and let $Z_1 = L_0/X$. Applying the homology theory $\pi_k(-)$ to the cofibration sequence $X \hookrightarrow L_0 \twoheadrightarrow Z_1$, we get that $\pi_k(Z_1) = 0$ for $k \leq n$ and the exact sequence

$$0 \to \pi_{n+1}(Z_1) \to \pi_n(X) \to \pi_n(L_0) \ .$$

It follows that $\pi_{n+1}(Z_1)$ is the kernel of the induced map $\pi_n(X) \to \pi_n(L_0)$, which we already showed was strictly smaller than $\pi_n(X)$.

We may repeat the process. Notice that Z_1 is of finite type since it is a quotient of L_0, which itself is of finite type. Let L_1 be a wedge of $\dim_{\mathbb{Z}_2} H^{n+1}(Z_1)$ many copies of $H(\mathbb{Z}_2, n+1)$. We construct a map $Z_1 \to L_1$ which by applying $\pi_{n+1}(-)$ effectively induces the map $\pi_{n+1}(Z_1) \to \pi_{n+1}(Z_1) \otimes_{\mathbb{Z}} \mathbb{Z}_2$, guaranteeing that the kernel of the induced map $\pi_{n+1}(Z_1) \to \pi_{n+1}(L_1)$ is strictly smaller than $\pi_{n+1}(Z_1)$. Replacing $Z_1 \to L_1$ by an inclusion, we form $Z_2 = L_1/Z_1$, where $\pi_k(Z_2) = 0$ for $k \leq n+1$ and $\pi_{n+2}(Z_2)$ is isomorphic to the kernel of $\pi_{n+1}(Z_1) \to \pi_{n+1}(L_1)$, and is thus strictly smaller than $\pi_{n+1}(Z_1)$.

We continue the continue the process like this. Since $\pi_n(X)$ was finite, it follows that for some positive integer N, the process stops, and we have that Z_k is a spectrum with $\pi_{n+N}(Z_N) = 0$, and all lower homotopy groups zero as well. Now, we consider the smallest integer m such that $\pi_{n+N+m}(Z_N)$ is nonzero and continue the process of constructing spectra just as above.

Doing this, we obtain a the diagram

$$\begin{array}{ccccccccc} X & \hookrightarrow & L_0 & \dashrightarrow & L_1 & \dashrightarrow & L_2 & \longrightarrow & \cdots \\ & & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ & & Z_1 & & Z_2 & & Z_3 & & \end{array}$$

with the property that that for each i, there is an integer N such that $\pi_{i+n}(Z_n) = 0$ for all $n \geq N$. More than that, we claim that for each i, there is an integer n such that $Y_{i+n}(Z_n) = 0$ for all $n \geq N$. One can easily prove this by induction on the number of non-basepoint cells in Y (recalling that Y is a finite spectrum by assumption). One uses the first part of proposition 2.23 and the cofiber sequence $S^k \hookrightarrow Y \twoheadrightarrow Y/S^k$, where S^k is a k-cell in Y for some k.

Consider the following diagram, which satisfies the conditions in lemma 3.25.

$$\begin{array}{ccccccccc} X & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & \dots \\ {\scriptstyle \mathrm{id}}\downarrow & & & & & & & & \\ X & \longrightarrow & L_0 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots \end{array}$$

Here the spaces $K_\bullet$ are the Eilenberg-Maclane spectra from the construction of the Adams spectral sequence computing $[\Sigma^* Y, X]$. By the lemma, we may extend the diagram to obtain the homotopy-commutative diagram below.

$$\begin{array}{ccccccccccc} X & \to & K_0 & & \longrightarrow & & K_1 & & \longrightarrow & & K_2 & \longrightarrow & \cdots \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & \searrow & & \nearrow & \downarrow & \searrow & & \nearrow & \downarrow & \searrow & \\ & & & & X_1 & & & & X_2 & & & & \ddots \\ & & & & \downarrow & & & & \downarrow & & & & \\ X & \to & L_0 & & \longrightarrow & & L_1 & & \longrightarrow & & L_2 & \longrightarrow & \cdots \\ & & & \searrow & \downarrow & \nearrow & & \searrow & \downarrow & \nearrow & & \searrow & \\ & & & & Y_1 & & & & Y_2 & & & & \ddots \end{array}$$

Applying the functors $Y_t(-)$ to this diagram, we see that there is a map from the diagram (3.5) to an analogous diagram for $X \to L_\bullet$ of patched-together staircase-shaped long exact sequences coming from applying $Y_t(-)$ to the cofiber sequences $Y_i \hookrightarrow L_i \twoheadrightarrow Y_{i+1}$ making all squares commute. Picking out the relevant column $Y_\bullet(X_\bullet)$ from (3.5) and the column $Y_\bullet(Z_\bullet)$ from the analogous diagram corresponding to the sequence $X \to L_\bullet$, we have the following commutative diagram, for all integers i.

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ Y_{i+n}(X_n) & \longrightarrow & Y_{i+n}(Z_n) \\ \downarrow & & \downarrow \\ Y_{i+n-1}(X_{n-1}) & \longrightarrow & Y_{i+n-1}(Z_{n-1}) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ Y_i(X) & \xrightarrow{\mathrm{id}} & Y_i(Z) \end{array}$$

If an element in $Y_i(X)$ had infinite Adams filtration, this would mean that it pulls back to an element arbitrarily high in the leftmost column. By construction, the groups in the rightmost column vanish for large enough n. By the commutativity of the diagram, it follows for any i that any element in $Y_i(X)$ of infinite Adams filtration must be equal to zero, completing the proof. □

Lemma 3.28. Let X and Y satisfy the conditions of theorem 3.24. Let $F^{s,t}$ denote the Adams filtration just as in theorem 3.24. The subgroup of odd-order torsion T^{t-s} in $[\Sigma^{t-s}Y, X]$ is equal to

$$\bigcap_{n\in\mathbb{Z}} F^{s+n,t+n} \ ,$$

the set of elements in $[\Sigma^{t-s}Y, X]$ of infinite Adams filtration.

Proof. First, we show that $T^{t-s} \subseteq \bigcap_n F^{s+n,t+n}$. We do this by proving that all vertical maps in the column $Y_{t-*}(X_{s-*})$ of (3.5) are isomorphisms when restricted to odd-torsion elements. From this it is immediate that every element in $T^{t-s} \subseteq [\Sigma^{t-s}Y, X]$ gets pulled back arbitrarily high in this column of (3.5), from which one direction of the lemma follows.

Consider the diagram (3.5). By remark 3.16 the fully colored columns $Y_t(K_s) \cong \mathrm{Hom}_A^t(H^*K_s, H^*Y)$ are finite $\mathbb{Z}_2$-modules for all s, t. This means that all elements in this group are have order 2.

Suppose we have an element α in the kernel of $Y_{t+1}(X_{s+1}) \to Y_t(X_s)$ which is of odd torsion, meaning $k \cdot \alpha = 0$ for some odd integer k. Consider the diagram below, which is a segment of (3.5) that illustrates our short diagram chase.

$$\begin{array}{ccccc} k\cdot\beta & \longmapsto & k\cdot\alpha = 0 & & \\ \| & & \| & & \\ \beta & \longmapsto & \alpha & \longmapsto & \\ \cap & & \cap & & \downarrow \\ Y_{t+1}(K_s) & \to & Y_{t+1}(X_{s+1}) & & \\ & & \downarrow & & \\ & & Y_t(X_s) & \ni & 0 \end{array}$$

By exactness, α lies in the image of the map $Y_{t+1}(K_s) \to Y_{t+1}(X_{s+1})$. Let β be an element which maps to α. Since $\beta \mapsto \alpha$, it follows that $k \cdot \beta \mapsto k \cdot \alpha = 0$. Since $Y_{t+1}(K_s)$ only has elements of order 2, $k \cdot \beta = \beta$. Consequently, β maps to 0 and to α, which implies that $\alpha = 0$, proving injectivity. One can by a similar diagram chase prove that any odd-torsion element in $Y_t(X_s)$ pulls back to an odd-torsion element in $Y_{t+1}(X_{s+1})$ along the map $Y_{t+1}(X_{s+1}) \to Y_t(X_s)$.

It remains to show that the complement of T^k lies in the complement of $\bigcap_n F^{n,k+n}$, where $k = t - s$. Let f be an element in the complement of T^k, meaning it is 2-torsion or has infinite order. By the fundamental theorem of finitely generated abelian groups, it is easy to see that there is a positive integer i such that f does not lie in the image of the multiplication by 2^i map $2^i : Y_k(X) \to Y_k(X)$.

Let $2^i : X \to X$ be the map given by $2^i \cdot \mathrm{id}_X$, the 2^i-fold sum of the identity map with itself. The induced map in $Y_k(-)$ of $\pi_n(-)$ is precisely given by the multiplication 2^i map. Applying $\pi_n(-)$ to the cofiber sequence

$$X \xrightarrow{2^i} X \xrightarrow{q} C_{2^i} \ ,$$

where C_{2^i} denotes the mapping cone, induces a long exact sequence, which constitutes the top row in the diagram below. The two squares below commute simply due to the maps involved being group homomorphisms.

$$\begin{array}{ccccccccc}
\pi_n(X) & \xrightarrow{2^i} & \pi_n(X) & \xrightarrow{q_*} & \pi_n(C_{2^i}) & \xrightarrow{\delta} & \pi_{n-1}(X) & \xrightarrow{2^i} & \pi_{n-1}(X) \\
 & & \downarrow{\scriptstyle 2^i} & & \downarrow{\scriptstyle 2^i} & & \downarrow{\scriptstyle 2^i} & & \\
 & & \pi_n(X) & \xrightarrow{q_*} & \pi_n(C_{2^i}) & \xrightarrow{\delta} & \pi_{n-1}(X) & &
\end{array}$$

We claim $\pi_n(C_{2^i})$ consists only of 2-torsion, for any n. To see this, take any element $\alpha \in \pi_n(C_{2^i})$. Commutativity in the rightmost square and exactness imply that $\delta(2^i\alpha) = 0$. By exactness, $2^i\alpha$ lies in the image of q_*, which by commutativity in the leftmost square and exactness implies that $2^{2i}\alpha = 0$. We remark that $f \in Y_k(X)$ not lying in the image of 2^i implies by exactness of the sequence obtained by applying $Y_k(-)$ to $X \to X \to C_{2^i}$ that $q_* f$ must be a nonzero element.

Let $C_{2^i} \to (C_{2^i})_\bullet$ be an Adams resolution for C_{2^i}, and let $L_\bullet$ be the corresponding sequence of wedges of Eilenberg-Maclane spectra involved in constructing the Adams spectral sequence. Consider the following diagram, which satisfies the conditions in lemma 3.25.

$$\begin{array}{ccccccccc}
X & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & \dots \\
{\scriptstyle q}\downarrow & & & & & & & & \\
C_{2^i} & \longrightarrow & L_0 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots
\end{array}$$

Using lemma 3.25, we obtain the following commutative diagram, arguing just like in the end of lemma 3.28.

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ Y_{k+n}(X_n) & \longrightarrow & Y_{k+n}((C_{2^i})_n) \\ \downarrow & & \downarrow \\ Y_{k+n-1}(X_{n-1}) & \longrightarrow & Y_{k+n-1}((C_{2^i})_{n-1}) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ Y_k(X) & \xrightarrow{q_*} & Y_k(C_{2^i}) \end{array}$$

Lemma 3.28 shows that the only element of infinite Adams filtration in $Y_k(C_{2^i})$ is the zero map. By construction $q_*(f) \neq 0$, so $q_*(f)$ cannot be of infinite Adams filtration. The commutativity of the diagram above thus implies that f cannot be of infinite Adams filtration, because otherwise $q_*(f)$ would be so too.

□

We can finally prove the convergence of the Adams spectral sequence.

Proof of Theorem 3.24. Consider the n:th derived couple

$$\begin{array}{ccc} A_n & \xrightarrow{i_n} & A_n \\ & {}_{k_n}\nwarrow \quad \swarrow_{j_n} & \\ & E_n & \end{array}$$

of our spectral sequence. We can unroll this to obtain the following exact sequence.

$$\cdots \xrightarrow{k_n} A_n^{s-n+2,t-n+2} \xrightarrow{i_n} A_n^{s-n+1,t-n+1} \xrightarrow{j_n} E_n^{s,t} \xrightarrow{k_n} A_n^{s+1,t} \xrightarrow{i_n} A_n^{s-1,t-1} \xrightarrow{j_n} E_n^{s+n-2,t+n-2} \xrightarrow{k_n} \cdots \tag{3.6}$$

The exact sequence is written like this is so one can overlay it on the diagram (3.5) and consider the groups above as subquotients of the corresponding groups in (3.5).

Consider the map $i_n : A_n^{s,t} \to A_n^{s-1,t-1}$. We claim this is injective for large enough n. On odd-torsion elements, this follows from the fact that $i_1 : A_1^{s,t} \to A_1^{s-1,t-1}$ is injective on odd-torsion elements, as was proven in the beginning of lemma 3.28, and that i_n is simply a restriction of i_1. On infinite-order elements, this follows from exactness of the sequence above, since we know that $E_n^{*,*}$ is always a finite 2-group by remark 3.16 and thus that the image of k_n never contains infinite-order elements.

It remains to show that $i_n : A_n^{s,t} \to A_n^{s-1,t-1}$ is an injection on 2-torsion elements, which we will prove by showing that $A_n^{s,t}$ has no nonzero 2-torsion elements for large enough n. By the definition of $A_n^{s,t}$, it equals the image of the map $(i_1)^{n-1} : A_1^{s+n-1,t+n-1} \to A_1^{s,t}$. By corollary 3.26 $\cdots \to X^{s+2} \to X^{s+1} \to X^s$ is an Adams tower for X_s. Consequently, the subset $A_n^{s,t} \subseteq Y_t(X_s)$ is precisely the subset of maps of Adams filtration $\geq n-1$.

By lemma 3.27 applied on the Adams spectral sequence computing $Y_*(X_s)$, it follows that for each 2-torsion element in $A_n^{s,t} \subseteq Y_t(X_s)$ we may pick n large enough to guarantee it is zero. Since X_s is of finite type and Y is finite, $Y_t(X_s)$ is a finitely generated abelian group by lemma 2.24, and thus its subgroup of 2-torsion elements must be finite. We may thus pick a global n large enough so that $A_n^{s,t}$ contains no nontrivial 2-torsion at all, thus proving that $i_n : A_n^{s,t} \to A_n^{s-1,t-1}$ is injective.

By exactness of (3.6), it follows that k_n starting at $E_n^{s,t}$ is the zero map. Consequently, the differential $d_n = j_n \circ k_n$ starting at $E_n^{s,t}$ must be zero for large enough n. Consider now the differential d_n with codomain equal to $E_n^{s,t}$. This map has the domain $E_n^{s-n,t-n+1}$, which is zero when n is large enough, making the bidegrees negative, and thus this differential is the zero map. Therefore, for large enough n, $E_n^{s,t} = E_{n+k}^{s,t}$ for all positive integers k, and thus $E_n^{s,t} = E_\infty^{s,t}$.

Furthermore, recall that the map k_n starting at $E_n^{s,t}$ is zero. By exactness, this implies that $E_n^{s,t}$ is isomorphic to the cokernel of i_n starting at $A_n^{s-n+2,t-n+2}$. By definition, $A_n^{s-n+2,t-n+2} = i_1^{n-1}(A_1^{s+1,t+1})$. For large enough n, this is precisely $F^{s+1,t+1}$, and similarly $A_n^{s-n+1,t-n+1} = F^{s,t}$ for large enough n. Consequently, the cokernel of i_n starting at $A_n^{s-n+2,t-n+2}$ is precisely $F^{s,t}/F^{s+1,t+1}$, completing the proof. □

Remark 3.29. The isomorphism $F^{s,t}/F^{s+1,t+1} \to E_\infty^{s,t}$ is given by j_n for some large enough n.

Now that we have proved the convergence of the Adams spectral sequence given some conditions our spectra, we can state more clearly how a map of spectra induces a map of Adams spectral sequences, and (assuming both spectral sequences are convergent) how the two short exact sequences describing convergence in the respective spectral sequences may be compared. In the following, denote the Adams spectral sequence computing $[A, B]$ by $E_*(A, B)$ and $F^{s,t}(A, B)$ the corresponding Adams filtration.

Theorem 3.30 (Naturality of the Adams spectral sequence)**.**

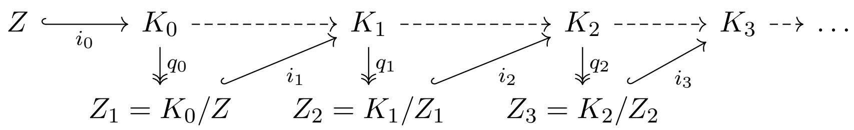
(i) *Let X, Y, Z be spectra such that X and Y are finite and Z is connective of finite type. Then any map $f : X \to Y$ induces a map of Adams spectral sequences $f_n^* : E_n^{s,t}(Y,Z) \to E_n^{s,t}(X,Z)$, and a map $f_\infty^* : E_\infty^{s,t}(Y,Z) \to E_\infty^{s,t}(X,Z)$ such that the diagram below commutes. The left and middle vertical maps are simply given by pre-composition with f.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{s+1,t+1}(Y,Z) & \longrightarrow & F^{s,t}(Y,Z) & \longrightarrow & E_\infty^{s,t}(Y,Z) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*_\infty} & & \\ 0 & \longrightarrow & F^{s+1,t+1}(X,Z) & \longrightarrow & F^{s,t}(X,Z) & \longrightarrow & E_\infty^{s,t}(X,Z) & \longrightarrow & 0 \end{array}$$

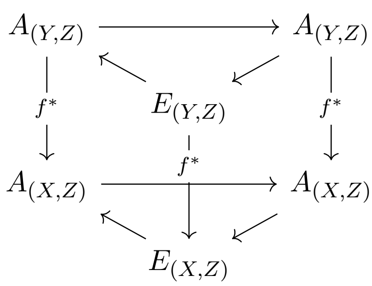
(ii) *Let X, Y, Z be spectra such that X and Y are connective of finite type and Z is finite. Then any map $f : X \to Y$ induces a map of Adams spectral sequences $f_* : E_n^{s,t}(Z,X) \to E_n^{s,t}(Z,Y)$ such that the diagram below commutes. The left and middle vertical maps are simply given by composition with f.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{s+1,t+1}(Z,X) & \longrightarrow & F^{s,t}(Z,X) & \longrightarrow & E_\infty^{s,t}(Z,X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle (f_*)_\infty} & & \\ 0 & \longrightarrow & F^{s+1,t+1}(Z,Y) & \longrightarrow & F^{s,t}(Z,Y) & \longrightarrow & E_\infty^{s,t}(Z,Y) & \longrightarrow & 0 \end{array}$$

Proof of part (i). The map $f : X \to Y$ induces a natural transformation of functors $[Y, -] \to [X, -]$, given by pre-composition with f, and thus a natural transformation $[\Sigma^t Y, -] \to [\Sigma^t X, -]$ for all t. Let the following diagram be an Adams resolution of Z.



We thus have cofiber sequences $Z_i \hookrightarrow K_i \twoheadrightarrow Z_{i+1}$ for all $i \geq 0$. Applying the homology theory $Y_t(-) = [\Sigma^t Y, -]$ or $X_t(-) = [\Sigma^t X, -]$ to the collection of cofiber sequences gives us two different patched-together staircase diagrams similar to (3.5). The natural transformations $[\Sigma^t Y, -] \to [\Sigma^t X, -]$ for all t thus gives us maps from each group in the first patched-together staircase diagram to each group in the second staircase diagram which makes all relevant diagrams commute. More concretely, this gives us the following commutative diagram between the two exact couples.



It follows that there is a similar commutative diagram involving instead the derived couples of these two exact couples. By induction, it follows that we have a map $f^* : E_n(Y,Z) \to E_n(X,Z)$ for all $n \geq 1$. By remark 3.29, the isomorphism

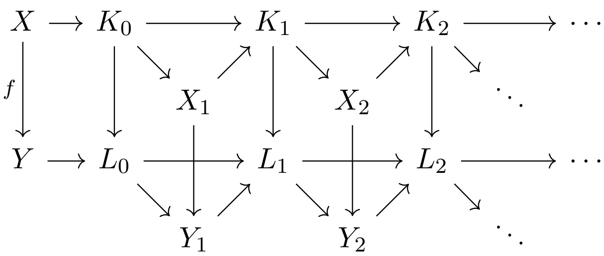
$$F^{s,t}(P,Z)/F^{s+1,t+1}(P,Z) \cong E_\infty^{s,t}(P,Z)$$

is given by j_n, for $P = X, Y$. The commutativity of the diagram above implies that the rightmost square in the diagram below commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{s+1,t+1}(Y,Z) & \longrightarrow & F^{s,t}(Y,Z) & \longrightarrow & E_\infty^{s,t}(Y,Z) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f^*_\infty} & & \\ 0 & \longrightarrow & F^{s+1,t+1}(X,Z) & \longrightarrow & F^{s,t}(X,Z) & \longrightarrow & E_\infty^{s,t}(X,Z) & \longrightarrow & 0 \end{array}$$

The commutativity of the leftmost square is obvious. □

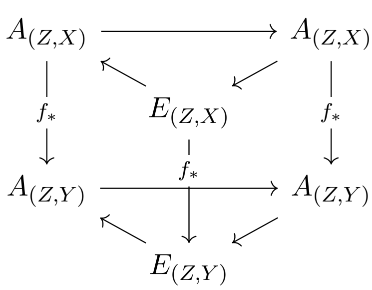
Proof of part (ii). Let $X \to X_\bullet$ be an Adams resolution, and $K_\bullet$ be the sequence of wedges of Eilenberg-Maclane spectra involved in the construction of the resolution. Similarly, let $Y \to Y_\bullet$ be an Adams resolution, with corresponding wedges of Eilenberg-Maclane spectra $L_\bullet$. By lemma 3.25, we obtain the following homotopy-commutative diagram.



The cofiber sequences $X_s \to K_s \to X_{s+1}$ and $Y_s \to L_s \to Y_{s+1}$ gives us long exact sequences in the homology theory $[\Sigma^* Z, -]$. By the diagram above, we have a map between these two long exact sequences, which means having the commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & [\Sigma^t Z, X_s] & \longrightarrow & [\Sigma^t Z, K_s] & \longrightarrow & [\Sigma^t Z, X_{s+1}] & \longrightarrow & [\Sigma^{t-1} Z, X_s] & \longrightarrow & \cdots \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ \cdots & \longrightarrow & [\Sigma^t Z, Y_s] & \longrightarrow & [\Sigma^t Z, L_s] & \longrightarrow & [\Sigma^t Z, Y_{s+1}] & \longrightarrow & [\Sigma^{t-1} Z, Y_s] & \longrightarrow & \cdots \end{array}$$

for all s, t. By taking direct sums over s, t we obtain the following commutative diagram, comparing the exact couples associated to the respective spectral sequences.



Just as before, it follows thus that f induces a map $f_* : E_n^{s,t}(Z,X) \to E_n^{s,t}(Z,Y)$ for all $n \geq 1$. Just as before, since j_n induces the convergence isomorphism for large enough n, and the diagram above commutes, it follows that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^{s+1,t+1}(Z,X) & \longrightarrow & F^{s,t}(Z,X) & \longrightarrow & E_\infty^{s,t}(Z,X) & \longrightarrow & 0 \\
 & & \big\downarrow f_* & & \big\downarrow f_* & & \big\downarrow (f_*)_\infty & & \\
0 & \longrightarrow & F^{s+1,t+1}(Z,Y) & \longrightarrow & F^{s,t}(Z,Y) & \longrightarrow & E_\infty^{s,t}(Z,Y) & \longrightarrow & 0
\end{array}$$

commutes, completing the proof. □

3.4 Multiplicative structure

We will finish off this section by showing that the Adams spectral sequence has a multiplicative structure. First, we will give some definitions of a product structure on Ext groups.

Given the graded A-modules L, M, N, there is an associative pairing

$$\mathrm{Ext}_A^{s,t}(M,N) \otimes \mathrm{Ext}_A^{s',t'}(L,M) \to \mathrm{Ext}_A^{s+s',t+t'}(L,N)$$

given as follows. Let $P_\bullet \to M$ and $P'_\bullet \to L$ be projective resolutions. Take $\alpha \in \mathrm{Ext}_A^{s,t}(M,N)$ and $\alpha' \in \mathrm{Ext}_A^{s',t'}(L,M)$, and let $f \in \mathrm{Hom}_A^t(P_s, N)$ and $f' \in \mathrm{Hom}_A^{t'}(P_{s'}, M)$ be their respective representatives. Consider the following diagram with filled-in arrows.

$$\begin{array}{ccccccccccc}
M & \longleftarrow & P_0 & \longleftarrow & P_1 & \longleftarrow & \cdots & \longleftarrow & P_s & \longleftarrow & \cdots \\
 & \nwarrow^{f'} & \uparrow f'_0 & & \uparrow f'_1 & & & & \uparrow f'_s & & \\
 & & P_{s'} & \longleftarrow & P_{s'+1} & \longleftarrow & \cdots & \longleftarrow & P_{s'+s} & \longleftarrow & \cdots
\end{array}$$

Since the rows are exact and the bottom row consists of projective modules, one can extend this to a commutative diagram including the dashed arrows $f'_\bullet$. Notice that the composition $f \circ f'_s$ lies in $\mathrm{Hom}_A^{t+t'}(P_{s+s'}, N)$.

Definition 3.31. The *Yoneda product* $\alpha\alpha'$ of α and α' given above is given by the equivalence class of $f \circ f'_s$ in $\mathrm{Ext}_A^{s+s',t+t'}(L,N)$.

One can verify that this pairing is independent of all choices of lifts, and thus that it is well-defined. One can also show that it is associative and bilinear.

Recall that given an exact sequence of modules, there is a long exact sequence in Ext. This also holds in the setting of graded modules, in which case the Ext group also inherits a grading. In fact, the connecting homomorphisms of these long exact sequences may be described as taking the Yoneda product with the short exact sequence, represented as an element in the appropriate Ext^1 group.

To make this more precise, recall that there is a bijection between set of equivalence classes of extensions $0 \to A \to ? \to B \to 0$ of R-modules and $\mathrm{Ext}^1_R(B, A)$. For a reference to this fact, see theorem 3.4.3 in [29]. If we suppose A and B above are graded R-modules, then the equivalence classes of extensions is in bijection with $\mathrm{Ext}^{1,0}_R(B, A)$. Given a short exact sequence $0 \to L \to M \to N \to 0$ of graded A-modules, there is thus a corresponding element $\alpha \in \mathrm{Ext}^{1,0}_A(N, L)$. Also, given another graded A-module K, there are two long exact sequences in Ext, given by

$$\cdots \to \mathrm{Ext}^{s,t}_A(K, L) \to \mathrm{Ext}^{s,t}_A(K, M) \to \mathrm{Ext}^{s,t}_A(K, N) \to \mathrm{Ext}^{s+1,t+1}_A(K, L) \to \cdots$$

and

$$\cdots \to \mathrm{Ext}^{s,t}_A(N, K) \to \mathrm{Ext}^{s,t}_A(M, K) \to \mathrm{Ext}^{s,t}_A(L, K) \to \mathrm{Ext}^{s+1,t+1}_A(N, K) \to \cdots$$

for all t. The bidegree-preserving maps are the obvious ones induced by the maps in the short exact sequence.

Proposition 3.32. The connecting maps

$$\mathrm{Ext}^{s,t}_A(K, N) \to \mathrm{Ext}^{s+1,t+1}_A(K, L) \quad \text{and} \quad \mathrm{Ext}^{s,t}_A(L, K) \to \mathrm{Ext}^{s+1,t+1}_A(N, K)$$

are given by left and right multiplication with $\alpha \in \mathrm{Ext}^{1,0}_A(N, L)$, respectively.

We will illustrate how exactly α is constructed from the extension $0 \to L \to M \to N \to 0$. From this, the proof of the proposition above will follow from some diagram chasing, which we will omit. Let $P_\bullet \to N$ be a projective resolution and consider the diagram below, with only the filled-in arrows.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & N & & \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Since the both rows are exact and the top row consists of projective modules (except possibly N but it does not matter), it follows that we can extend the diagram with the dashed arrows. Notice that $f \in \mathrm{Hom}^{1,0}_A(P_1, L)$. One can verify that this is a cocycle. We define α to be the equivalence of f in $\mathrm{Ext}^{1,0}(N, L)$.

We can now describe how one can multiply elements in the Adams spectral sequence, and how this relates to the Yoneda product. Let $E^{*,*}_*(X, Y)$ denote the spectral sequence computing $[\Sigma^* X, Y]$.

Proposition 3.33. For each integer $n \geq 2$ and all s, s', t, t', there is a natural bilinear and associative pairing

$$E^{s,t}_n(Y, Z) \otimes E^{s',t'}_n(X, Y) \to E^{s+s,t+t'}_n(X, Z)$$

such that the following holds.

- When $n = 2$, the pairing is the Yoneda product.
- The differentials d_n are an derivations with respect to this product, meaning $d_n(\alpha\beta) = d_n(\alpha)\beta + \alpha d_n(\beta)$.
- The product structure on E_n passes thus to a product structure on its homology groups. This product is precisely the product we have defined on E_{n+1}.

Proof. Theorem 2.1 in [25]. □

4 Ext Calculation

The sole purpose of this section is to prove that the graded map of A-modules $\phi : \mathbb{Z}_2 \to P$ given by $\lambda \mapsto \lambda x^0$ induces an isomorphism $\phi_* : \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2) \to \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P)$ for all s, t. The module P will be introduced shortly. The domain of ϕ_* consists precisely of the E_2 page of the Adams spectral sequence computing $[S^*, S^0]$, the stable (co)homotopy of spheres. The codomain of ϕ_* (in some appropriate range) will prove to be precisely the E_2 page of the Adams spectral sequence computing the stable cohomotopy of infinite projective space.

Historically, it was first conjectured by Adams in [3] that the simplest possibility is that a similar Ext group as to the one we are interested in, $\mathrm{Ext}_A^{s,t}(\Sigma^{-1}\mathbb{Z}_2, \mathbb{Z}_2)$, is isomorphic to $\mathrm{Ext}_A^{s,t}(P, \mathbb{Z}_2)$. Adams guessed that the splitting isomorphism exhibited in the sketch of the proof below would hold, and then that by passing to inverse limits, one could get such an isomorphism of Ext groups. Lin, Davis and Mahowald proved in [20] that both of Adams' conjectures hold, and from this prove that ϕ_* as mentioned above is an isomorphism! This article is our primary reference for the entire section.

This isomorphism of Ext group will thus allow us to compare these spectral sequences, and will be of utmost importance to our spectral sequence argument in the last section. In my opinion, the spectral sequence part of computing the stable cohomotopy of RP^∞ can almost be considered a straightforward corollary, once one has this isomorphism. The non-triviality of computing $\pi_s^*(RP^\infty)$ comes mainly from the material that lies in this section.

To impress upon the reader how non-trivial this isomorphism of Ext groups is, we note that computing Ext groups of modules over the Steenrod algebra, even in the case $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$, is incredibly nontrivial. Computing these groups just amounts to taking a free resolution of $\mathbb{Z}_2$ (a tiny module consisting of only two elements), applying the functor $\mathrm{Hom}_A^t(-, \mathbb{Z}_2)$ and computing the cohomology groups. Of course, if these Ext groups were easy to compute, we would have complete knowledge of the E_2 page of the Adams spectral sequence computing $[S^*, S^0]$ and thus have a lot of information $[S^*, S^0]$, so it makes sense that this should be hard.

One way $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$ is computed is by constructing a minimal free resolution $F_\bullet \to \mathbb{Z}_2$, which means that at each step, F_i is constructed by taking the minimum number of free generators in each degree. Doing this, lemma 5.49 in [14] proves that the boundary maps in the complex $\mathrm{Hom}_A^t(F_\bullet, \mathbb{Z}_2)$ are all zero, and thus that $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathrm{Hom}_A^t(F_s, \mathbb{Z}_2)$ for all s, t. The advantage of this method is that computing the Ext groups reduces to constructing a minimal free resolution within a certain (finite) range, and this is something a computer can do. See Bruner's and Rognes' preprint [10] for more on this.

The problem is, however, that we want to compare $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$ and $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P)$ in an infinite range, namely for all s, t. The main difficulty, which is what makes the proof of the isomorphism so nontrivial, is that we are exhibiting an isomorphism for all s, t between two groups, neither of which we know much about.

We will now introduce some modules and rings involved in the proof. Let A be the Steenrod algebra, as defined in definition 2.32. It is known that the subset $\{\mathrm{Sq}^0\} \cup \{\mathrm{Sq}^{2^i} \mid i \geq 0\}$ generates A as an algebra, see Corollary 1 on page 47 in [24]. With this in mind, we define A_n as the subalgebra generated by $\{\mathrm{Sq}^{2^i} \mid 0 \leq i \leq n\}$. Let P be the graded A-module $\mathbb{Z}_2[x, x^{-1}]$, where $|x| = 1$. The A-module structure on P is given by

$$\mathrm{Sq}^i x^k = \begin{cases} \binom{k}{i} x^{k+i} & \text{if } k \geq 0 \\ \binom{2^m+k}{i} x^{k+i} & \text{if } k < 0 \end{cases},$$

where the binomial coefficients are taken modulo 2, and m is some large positive integer compared to $|k|$ and i. We will later discuss why this action is well-defined, but for the moment let us take it for granted. Define $Q_{k,n}$ as the A_n-submodule of P generated by $\{x^i \mid i < k\}$.

The reason we define the A-module structure of P like this is because the A-module structure of the total cohomology of finite and infinite stunted projective spaces will coincide with submodules and subquotients of P.

Before moving on to giving a sketch of the proof, we will remark that for the entirety of this section all modules will be graded, and all maps between modules will thus be assumed to be graded maps, unless explicitly told otherwise.

4.1 Sketch of the proof

Even though we want to show that the map $\phi_* : \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2) \to \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P)$ is an isomorphism, we will for the for the majority of the section deal with Tor groups instead. Namely, we will show that the map $\gamma : P \to \Sigma^{-1}\mathbb{Z}_2$ given by $\sum_i \lambda_i x^i \mapsto \lambda_{-1}$ induces an isomorphism $\gamma_* : \mathrm{Tor}^A_{s,t}(\mathbb{Z}_2, P) \to \mathrm{Tor}^A_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2)$. After proving some results adjacent to the tensor-hom adjunction, we derive our desired isomorphism of Ext groups.

The most important and nontrivial lemma in this proof is that we have a splitting, given by the isomorphism

$$A \otimes_{A_n} P/Q_{k,n} \cong \bigoplus_{\substack{i \geq k \\ i \equiv -1 \mod 2^{n+1}}} \Sigma^i A \otimes_{A_{n-1}} \mathbb{Z}_2$$

of A-modules, for all n and k, such that the diagram

$$\begin{array}{ccc} A \otimes_{A_n} P/Q_{k,n} & \overset{\cong}{\longleftrightarrow} & \bigoplus_{\substack{i \geq k \\ i \equiv -1 \mod 2^{n+1}}} \Sigma^i A \otimes_{A_{n-1}} \mathbb{Z}_2 \\ \downarrow \mathrm{proj} & & \downarrow \theta \\ A \otimes_{A_n} P/Q_{k+1,n} & \underset{\cong}{\longleftrightarrow} & \bigoplus_{\substack{j \geq k+1 \\ j \equiv -1 \mod 2^{n+1}}} \Sigma^j A \otimes_{A_{n-1}} \mathbb{Z}_2 \end{array} \tag{4.1}$$

and the diagram below commute. Do not worry about the details of these maps for now. We will explain it all when the time comes.

$$\begin{array}{ccc} A \otimes_{A_n} P/Q_{k,n} & \overset{\cong}{\longleftrightarrow} & \bigoplus_{\substack{i \geq k \\ i \equiv -1 \mod 2^{n+1}}} \Sigma^i A \otimes_{A_{n-1}} \mathbb{Z}_2 \\ \downarrow \text{proj} & & \downarrow \psi \\ A \otimes_{A_{n+1}} P/Q_{k,n+1} & \underset{\cong}{\longleftrightarrow} & \bigoplus_{\substack{k \geq k \\ j \equiv -1 \mod 2^{n+2}}} \Sigma^j A \otimes_{A_n} \mathbb{Z}_2 \end{array} \tag{4.2}$$

Furthermore, we prove a natural change of rings isomorphism $\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, A \otimes_{A_n} M) \cong \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, M)$, where M is any left A_n-module. Applying the functor $\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, -)$ on (4.1) and using this change of rings isomorphism, we obtain an isomorphism of inverse systems. In fact, one can show that the inverse limit of the leftmost inverse system is isomorphic to $\mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P)$ and the inverse limit of the rightmost system is $\bigoplus_i \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, \Sigma^i\mathbb{Z}_2)$ such that $i \equiv -1 \mod 2^{n+1}$, with no boundedness condition on the index i anymore. Applying $\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, -)$ on (4.2) and using this isomorphism of inverse, limits, we will obtain the diagram below.

$$\begin{array}{ccc} \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P) & \overset{\cong}{\longleftrightarrow} & \bigoplus_{i \equiv -1 \mod 2^{n+1}} \mathrm{Tor}^{A_{n-1}}_{s,t}(\mathbb{Z}_2, \Sigma^i\mathbb{Z}_2) \\ \downarrow \text{proj}_* & & \downarrow \psi_* \\ \mathrm{Tor}^{A_{n+1}}_{s,t}(\mathbb{Z}_2, P) & \underset{\cong}{\longleftrightarrow} & \bigoplus_{j \equiv -1 \mod 2^{n+2}} \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, \Sigma^j\mathbb{Z}_2) \end{array}$$

The direct limit of the leftmost system is $\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, P)$. By the nature of the maps ψ_*, we will see that the direct limit of the rightmost system is $\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2)$. So we have an isomorphism between these direct limits. We will then show that $\gamma_* : \mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, P) \to \mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2)$ is the inverse to this isomorphism, implying that γ_* is an isomorphism, completing the sketch proof.

4.2 The A-module P

Let us now return to the question of why the action of A on P is well defined, in the sense that it is independent of the choice of m. To prove this, we need the lemma below, which will come in handy many times for determining how A acts on the module P.

Definition 4.1. For a non-negative integer n, we mean by its *dyadic expansion* the unique sum $n = \sum_{i \geq 0} n_i \cdot 2^i$ such that n_i is 0 or 1 for all i.

Lemma 4.2. For positive integers m and n, with $m = \sum_i m_i 2^i$ and $n = \sum_i n_i 2^i$ their respective dyadic expansion,

$$\binom{m}{n} \equiv \prod_i \binom{m_i}{n_i} \mod 2 \ .$$

Proof. Lemma 3C.6 in [13] □

The proof of the lemma below contains an explanation as to why the A-module structure of P is well defined.

Lemma 4.3 (Periodicity property of P as A_n-module)**.** For $a \in A_n$ and any integer k, $ax^k \neq 0$ if and only if $ax^{k+2^{n+1}} \neq 0$.

Proof. Since A_n is generated by the elements $\mathrm{Sq}^0, \mathrm{Sq}^1, \ldots, \mathrm{Sq}^{2^n}$, it suffices to prove the result for $a = \mathrm{Sq}^{2^i}$ for some i such that $0 \leq i \leq n$. Assume first that k is non-negative. then

$$ax^k = \binom{k}{2^i} x^{k+2^i} \ .$$

Notice that the coefficients in the dyadic expansion of 2^i are all zero, except the i:th one, which is one. By the lemma above, it follows that

$$\binom{k}{2^i} = \binom{k_i}{1} \ ,$$

where $k = \sum_i k_i 2^i$ is its dyadic expansion. Furthermore, notice that since $i < n+1$, the dyadic expansion of k and $k+2^{n+1}$ agree on the first n terms of the sum. In particular, the i:th coefficients are the same in the dyadic expansions of k and $k + 2^{n+1}$. Consequently,

$$\binom{k+2^{n+1}}{2^i} = \binom{k_i}{1}$$

by the lemma above, and thus

$$ax^{k+2^{n+1}} = \binom{k_i}{1} x^{k+2^{n+1}+2^i} \ .$$

Therefore, ax^k is nonzero if and only if $ax^{k+2^{n+1}}$ is nonzero in the case when $k \geq 0$.

Assume now that k is negative. If we define $ax^k = \mathrm{Sq}^{2^i} x^k$ to be equal to

$$\binom{2^m + k}{2^i} x^{k+2^i}$$

for m large enough with respect to $|k|$ and i, the proof will follow in the same manner. We may pick m large enough so that $2^m + k$ is positive and such that the dyadic expansion of $2^m + k$ remains unchanged in the first $m - 1$ terms. It then follows that this action of A on P is well-defined, since for any $i = 0, \dots, n$, $\binom{2^m+k}{2^i}$ will be constant for large enough m. Since $2^m + k$ is positive, it follows now from the argument above that

$$\binom{2^m + k}{2^i} = \binom{2^m + k + 2^{n+1}}{2^i} ,$$

so the coefficient in front of ax^k equals the coefficient in front of $ax^{k+2^{n+1}}$. Notice that if $k + 2^{n+1}$ is already positive, we need not add 2^m in the binomial coefficient, but either way, the i:th coefficient of the dyadic expansion of $k + 2^{n+1}$ and $2^m + k + 2^{n+1}$ will be the same, so it will not matter. Therefore, $ax^k \neq 0$ if and only if $ax^{k+2^{n+1}} \neq 0$ in the case when k is negative as well. □

Using this periodicity property, we notice another periodicity phenomenon.

Corollary 4.4 (Periodicity isomorphisms)**.** For all integers k and n, we have a isomorphisms

$$\Sigma^{2^{n+1}} P/Q_{k,n} \cong P/Q_{2^{n+1}+k,n}$$

of A_n-modules given by $x^i \mapsto x^{i+2^{n+1}}$.

Proof. The map $\Sigma^{2^{n+1}} P \to P$ given by $x^i \mapsto x^{i+2^{n+1}}$ is clearly a bijection. It is A_n-linear precisely by the periodicity property above. It descends to the quotient to give our desired isomorphism. □

Lemma 4.5. As a module over A_n, P is generated by the elements x^i such that $x \equiv -1 \mod 2^{n+1}$.

Proof. Suppose $i = 2^{n+1} - 1$ and that j is some integer such that $0 \leq j < 2^{n+1}$. Then $\mathrm{Sq}^j x^i = \binom{i}{j} x^{i+j}$. The dyadic expansions of i and j are

$$i = \sum_{\alpha=0}^{n} 1 \cdot 2^\alpha \qquad \text{and} \qquad j = \sum_{\alpha=0}^{n} j_\alpha 2^\alpha ,$$

where $j_\alpha \in \{0, 1\}$. By lemma 4.2,

$$\binom{i}{j} = \prod_{\alpha=0}^{n} \binom{1}{j_\alpha} ,$$

but since $j_\alpha \in \{0, 1\}$ for all α, all binomial coefficients on the right hand side are equal to one, and thus the expression above equals one. Consequently, $\mathrm{Sq}^j x^i = x^{i+j} = x^{2^{n+1}-1+j}$. Furthermore, one can show by induction on j and an application of the Adem relations that $\mathrm{Sq}^j \in A_n$ for all $0 \leq j < 2^{n+1}$, so $x^{2^{n+1}-1+j} \in \mathrm{span}_{A_n}(P)$. For a spelled-out proof, see proposition 4L.8 in [13]. Moving on, lemma 3.2 implies that $\mathrm{Sq}^j x^{k \cdot 2^{n+1}-1} = x^{k \cdot 2^{n+1}-1+j} \in \mathrm{span}_{A_n}(P)$ for all integers k. By the division algorithm, any integer can be written in the form $k \cdot 2^{n+1} - 1 + j$. It follows that $x^r \in \mathrm{span}_{A_n}(P)$ for all integers r. □

By our periodicity isomorphisms and the lemma above, we conclude that by studying the submodule $Q_{k,n}$ for some fixed k, we may recover all information on the whole family of submodules $Q_{k,n}$ as k varies. Notice that the proof of the lemma 4.5 shows us that taking only the powers x^i such that $i < k$ and $i \equiv -1 \mod 2^{n+1}$ suffices to generate all of $Q_{k,n}$. Consequently, $Q_{k,n} = Q_{\tilde{k}\cdot 2^{n+1}-1,n}$ for all k such that $(\tilde{k}-1)\cdot 2^{n+1}-1 < k \leq \tilde{k}\cdot 2^{n+1}-1$.

To this end, fix n, and let $Q = Q_{-1,n}$, $Q' = Q_{2^{n+1}-1,n}$ and $Q'' = Q_{2\cdot 2^{n+1}-1,n}$. All the proofs we present below that hold for Q will also hold for $Q_{k,n}$, for any integer k, modulo some obvious indexing modifications. We focus on Q and Q' to simplify notation.

As a quick remark, note that we are suspending the leftmost module below by -1 to make the map graded.

Lemma 4.6. We have an isomorphism

$$\Sigma^{-1}A_n \otimes_{A_{n-1}} \mathbb{Z}_2 \to Q'/Q$$

of A_n-modules given by $a \otimes 1 \mapsto ax^{-1}$.

Proof. To show that this map is well-defined, we need to show that the associated bilinear map $\Sigma^{-1}A_n \times \mathbb{Z}_2 \to Q'/Q$ is A_{n-1}-balanced, which reduces to showing that for any $a \in A_{n-1}$, $(a,1)$ maps to 0, except when $a = \mathrm{Sq}^0$, and then $(a,1) \mapsto x^{-1}$, because $A_{n-1}\backslash\{\mathrm{Sq}^0\}$ acts trivially on $\mathbb{Z}_2$ and Sq^0 is the identity element. It suffices to prove this on the generators of A_{n-1}, so assume $a = \mathrm{Sq}^{2^i}$ for some $0 < i \leq n-1$.

Assume m is an integer larger than n. The dyadic expansion of $2^m - 1$ is given by $\sum_{k=0}^{m-1} 1 \cdot 2^k$, so the dyadic expansion of $2^m - 1 - 2^n$ is given by a similar expression, except with the coefficient for 2^n set to zero. Notice now that in the dyadic expansion of $2^m - 1 - 2^n$, adding 2^i has the effect of flipping the coefficients in front of the terms 2^k for $k \geq i$ from one to zero until it reaches the n:th coefficient which turns from zero to one. That is, $2^m - 1 - 2^n + 2^i = \sum_{k=0} a_k 2^k$, where $a_k = 1$ for $0 \leq k < i$ and $k \geq n$ and $a_k = 0$ other otherwise. It follows by lemma 4.2 that

$$\binom{2^m + 2^i - 1 - 2^n}{2^n} = 1\ ,$$

and it clear that

$$\binom{2^m - 1}{2^i} = 1\ .$$

Consequently, $ax^{-1} = \mathrm{Sq}^{2^i}x^{-1} = \mathrm{Sq}^{2^n}x^{2^i-1-2^n}$. Since the right hand side lies in Q, it follows that $ax^{-1} = 0 \in Q'/Q$, so the map $\Sigma^{-1}A_n \otimes_{A_{n-1}} \mathbb{Z}_2 \to Q'/Q$ is well-defined.

By lemma 4.5, $\{x^i \mid i < 2^{n+1}-1,\ i \equiv -1 \mod 2^{n+1}\}$ generates Q' as an A_n-module, so $[x^{-1}] \in Q'/Q$ generates all of Q'/Q as an A_n-module, which implies that the map $\Sigma^{-1}A_n \otimes_{A_{n-1}} \mathbb{Z}_2 \to Q'/Q$ is surjective. Since this map is may also be seen as a $\mathbb{Z}_2$-linear map, if we can show that the domain and codomain have the same dimension as a $\mathbb{Z}_2$-vector space, it follows also that it is injective by linear algebra, and thus an isomorphism of A_n-modules.

In [18], Lin explains that A_n is free as a right A_{n-1}-module (module structure given by multiplication) with basis given by the set of Milnor basis elements $\mathrm{Sq}(r_1, \dots, r_{n+1})$ such that $r_i \in \{0, 2^{n+1-i}\}$ for all i. We define the Milnor basis in the start of the next subsection, so look there if you want to recall the definition, but all we need to know is that there is a set of 2^{n+1} elements in $A_n \backslash A_{n-1}$ which freely span A_n as a right A_{n-1}-module. The image of these elements in the tensor product $A_n \otimes_{A_{n-1}} \mathbb{Z}_2$ will be linearly independent over $\mathbb{Z}_2$ because the elements described above are elements which lie in A_n but not in A_{n-1}, and they also span all of the tensor product because the map $A_n \to A_n \otimes_{A_{n-1}} \mathbb{Z}_2$ is surjective. Consequently, the $\mathbb{Z}_2$-dimension of the tensor product is 2^{n+1}.

It remains to show that $\dim_{\mathbb{Z}_2} Q/Q' = 2^{n+1}$ as well. Fix an integer i such that $0 \leq i < 2^{n+1}$. Then there is a maximal integer q such that $x^{q2^{n+1}+i} \in Q'$. Otherwise, Q' would have elements of arbitrarily high degree, which is impossible because A_n is a finite algebra and Q' is the A_n-span of a set of elements which have degree bounded above, so the degree of all elements in Q' are bounded above by some number. By the maximality of q, it also follows that $x^{q2^{n+1}+i}$ does not lie in Q because if it were so, then we could have chosen q to be larger.

By the definition of Q', it follows that there is some $a \in A_n$ and x^k such that $k < 2^{n+1} - 1$ and such that $x^{q2^{n+1}+i} = ax^k$. By the periodicity property (lemma 4.3), it follows that $ax^{k-q'2^{n+1}} = x^{(q-q')2^{n+1}+i}$ as well, for any positive integer q'. Notice that this is also an element in Q'. In fact, this is an element in Q. Consequently, the only power of x equivalent to i modulo 2^{n+1} which lies in Q' but not in Q is given by $x^{q2^{n+1}+i}$. It follows that as a $\mathbb{Z}_2$-module, Q'/Q is 2^{n+1} dimensional, with one basis element for each congruence class of 2^{n+1}. This completes the proof that the map $A_n \otimes_{A_{n-1}} \mathbb{Z}_2 \to Q'/Q$ is an isomorphism.

□

We will take for granted proposition 2.3(a) in [17] which states the following.

Lemma 4.7. If an A-module P is projective, then P is as an A_n-module for all n.

Since A is a projective A-module, it follows by the lemma above that it is a projective A_n-module, and thus a flat A_n-module since projective implies flat. Using this, we prove the following.

Lemma 4.8. We have a short exact sequence

$$0 \to \Sigma^{-1} A \otimes_{A_{n-1}} \mathbb{Z}_2 \xrightarrow{\alpha} P/Q \to P/Q' \to 0$$

of A-modules.

Proof. Obviously we have a short exact sequence

$$0 \to Q'/Q \to P/Q \to P/Q' \to 0$$

of A_n-modules. Using our isomorphism from lemma 4.6, we get the short exact sequence

$$0 \to \Sigma^{-1} A_n \otimes_{A_{n-1}} \mathbb{Z}_2 \to P/Q \to P/Q' \to 0$$

of A_n-modules. Since A is flat over A_n, we may apply the exact functor $A \otimes_{A_{n-1}} (-)$ to get our desired short exact sequence of A-modules. □

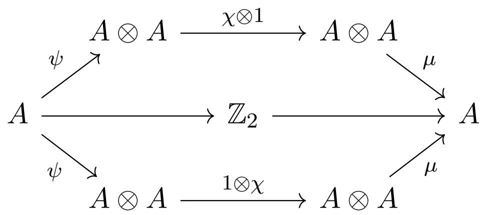
In the next section, we will prove that this short exact sequence splits and that this splitting makes the relevant diagrams, mentioned in the sketch proof, commute.

4.3 Splitting

To prove the splitting of the short exact sequence in lemma 4.8, we need to make a quick detour to speak more about the structure of the Steenrod algebra and its dual. First, we note that A is a Hopf algebra, as proven in [22]. This means that it has an associative algebra structure, with multiplication given by $\mu : A \otimes A \to A$ and a unit map $\mathbb{Z}_2 \to A$. Also, it has a coassociative coalgebra structure, with a comultiplication map $\psi : A \to A \otimes A$ determined by

$$\mathrm{Sq}^n \mapsto \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$$

and a counit $A \to \mathbb{Z}_2$ determined by the map $\mathrm{Sq}^i \mapsto 0$ if $i > 0$ and $\mathrm{Sq}^0 \mapsto 1$. Finally, it has an anti-automorphism $\chi : A \to A$ which is determined by the commutativity of the following diagram.



The commutativity of this diagram allows us to completely determine χ by induction. Notice that $\chi(\mathrm{Sq}^0) = \mathrm{Sq}^0$ by the commutativity of the diagram. For $n \geq 1$, we have by the commutativity that

$$\sum_{i+j=n} \chi(\mathrm{Sq}^i) \cdot \mathrm{Sq}^j = \sum_{i+j=n} \mathrm{Sq}^i \cdot \chi(\mathrm{Sq}^j) = 0 \ .$$

The $\mathbb{Z}_2$-dual of the Steenrod algebra $A^* := \mathrm{Hom}_{\mathbb{Z}_2}(A, \mathbb{Z}_2)$ is also a Hopf algebra. As luck would have it, A^* is a much more manageable algebra than A. In fact, Milnor proves in

[22] that A^* is a polynomial ring $\mathbb{Z}_2[\zeta_1, \zeta_2, \dots]$ with ζ_i lying in degree $2^i - 1$ for all $i \geq 0$. The dual of the multiplication map $\mu : A \otimes A \to A$ turns into the comultiplication map $A^* \to A^* \otimes A^*$ for A^*, determined by

$$\zeta_n \mapsto \sum_{i+j=n} \zeta_i^{2^j} \otimes \zeta_j \ .$$

Since A^* is a polynomial ring, it has a $\mathbb{Z}_2$-basis given by the set of all products $\prod_{i\geq 1} \zeta_i^{r_i}$ where the sequence $(r_i)_i$ ranges over all sequences of non-negative integers with finite support. The dual basis of this defines a $\mathbb{Z}_2$-basis $\{\mathrm{Sq}(r_1, r_2, \cdots,) \mid r_i \geq 0\}$ for A, which we call the *Milnor basis.* That is, $\mathrm{Sq}(r_1, r_2, \dots)$ is defined such that $\xi_1^{s_1}\xi_2^{s_2}\dots\big(\mathrm{Sq}(r_1, r_2, \dots)\big) = 1$ if $(s_1, s_2, \dots) = (r_1, r_2, \dots)$ and equal to zero otherwise. The Milnor basis relates to the subalgebras A_n as follows.

Lemma 4.9. The subalgebra A_n is a finite dimensional $\mathbb{Z}_2$-module, with a basis given by

$$\{\mathrm{Sq}(r_1, r_2, \dots, r_{n+1}) \mid r_i < 2^{n+2-i} \text{ for } i \leq n+1 \text{ and } r_i = 0 \text{ for } i > n+1\} \ .$$

Proof. Proposition 2 in [22]. □

Using the natural identification of A with its double dual $A = A^{**}$, we may thus make the identification

$$A_n = \Big(A^*/(\xi_1^{2^{n+1}}, \xi_2^{2^n}, \dots, \xi_n^{2^2}, \xi_{n+1}, \xi_{n+2}, \dots)\Big)^* \ .$$

If we take the dual of a smaller quotient of A^*, we obtain the group

$$B = \Big(A^*/(\xi_2^{2^n}, \dots, \xi_n^{2^2}, \xi_{n+1}, \xi_{n+2}, \dots)\Big)^* \ .$$

By this description of A_n, one can verify that the comultiplication map $A^* \to A^* \otimes A^*$ restricts to a map $B^* \to B^* \otimes A_{n-1}^*$, so taking the dual gives us a map $B \otimes A_{n-1} \to B$, which imbues B with a right A_{n-1}-module structure. Similarly, the comultiplication restricts to a map $B^* \to A_n \otimes B^*$, so B is also a left A_n-module. Lin explains in [20] that B is free as a right A_{n-1}-module by theorem 4.4 of [23]. Taking this for granted, we may prove the following isomorphism in the lemma below, but first, we need a definition and a lemma which we also take for granted.

Definition 4.10. Let M be a graded A-module that has finite $\mathbb{Z}_2$-dimension in each graded component and vanishes in degrees below zero. That is, M^n is zero for $n < 0$ and is a finite-dimensional $\mathbb{Z}_2$-module for $n \geq 0$. Define the *Hilbert series* $H(M)$ of M to be the power series given by

$$H(M) = \sum_{n=0}^{\infty} \dim_{\mathbb{Z}_2} M^n \cdot t^n \ .$$

Lemma 4.11. If A is a connected graded algebra over $\mathbb{Z}_2$ (like the Steenrod algebra), and B is a free graded right A-module, and C is a graded left A-module (with finite-dimensional graded components, so that $H(B)$ and $H(C)$ are well-defined), then

$$H(B \otimes_A C) = \frac{H(B)H(C)}{H(A)}$$

Proof. Lemma 7 in [6] shows that

$$\frac{H(B)H(C)}{H(A)} = \sum_{i \geq 0} H(\mathrm{Tor}_i^A(B, C)) \ .$$

Since B is free over A, it is in particular flat, and thus all higher Tor groups vanish, and $\mathrm{Tor}_0^A(B, C) = B \otimes_A C$, completing the proof. ☐

Let us now fill in the missing step to prove that the short exact sequence of lemma 4.8 splits.

Lemma 4.12. The map

$$\beta : \Sigma^{-1} B \otimes_{A_{n-1}} \mathbb{Z}_2 \to P/Q$$

given by $\beta(b \otimes 1) = bx^{-1}$ is an A_{n-1}-module isomorphism.

Proof. The map is well-defined because the associated bilinear map $\Sigma^{-1}B \times \mathbb{Z}_2 \to P/Q$ is a restriction of the previously defined map $\Sigma^{-1}A \times \mathbb{Z}_2 \to P/Q$ in the proof of lemma 4.6, which we already showed was A_{n-1}-balanced.

To prove that β is surjective, we will show that $\mathrm{Sq}^i \in B$ for all i. Notice that by the description of B above, this is equivalent to showing that evaluating any element in the ideal $(\xi_2^{2^n}, \dots, \xi_n^{2^2}, \xi_{n+1}, \xi_{n+2}, \dots) \subseteq A^*$ at Sq^i gives you zero. Proposition 8 in section 6 of [24] show that $\mathrm{Sq}(i, 0, 0, \dots)$ in fact equals Sq^i, meaning that Sq^i is dual to the element ξ_1^i. It follows that that evaluating any element in the ideal above at Sq^i will give you zero, since it does not contain ξ_1^i, and consequently, $\mathrm{Sq}^i \in B$.

Notice now that $\beta(\mathrm{Sq}^i \otimes 1) = \mathrm{Sq}^i x^{-1} = x^{i-1}$ for all i. That is, the action of Sq^i on x^{-1} is always nontrivial, from which it follows that β is surjective. This is because $\binom{2^m-1}{i}$, for some m large enough, is always equal to 1 due to lemma 4.2, because the dyadic expansion of $2^m - 1$ consists only of ones in the coefficients that matter. The argument is completely analogous to the one proving that $\mathrm{Sq}^j x^i = x^{i+j}$ in the beginning of the proof of lemma 4.5.

It remains to prove that β is injective. To do this, we will show that each graded component of $\Sigma^{-1}B \otimes \mathbb{Z}_2$ and P/F has the same dimension over $\mathbb{Z}_2$ (notice that each graded component of these modules are finite-dimensional). That is, it suffices to show that the Hilbert series of these two modules agree. Since β is in particular a surjective $\mathbb{Z}_2$-linear map between the graded components, it will then follow by linear algebra that

β induces isomorphisms between the graded components as $\mathbb{Z}_2$-modules. In particular, it will be an injection on graded components, and since β is a graded map, it follows that β must be an injection.

Notice that β restricts to the isomorphism $\Sigma^{-1}A_n \otimes_{A_{n-1}} \mathbb{Z}_2 \to Q'/Q$ of lemma 4.6. There we mentioned an explicit $\mathbb{Z}_2$-basis $\{\mathrm{Sq}(r_1, \ldots, r_{n+1}) \mid r_i = 0 \text{ or } r_i = 2^{n+1-i}\}$ for $\Sigma^{-1}A_n \otimes_{A_{n-1}} \mathbb{Z}_2$. Since the degree of $\mathrm{Sq}(r_1, \ldots, r_{n+1})$ equals $\prod_{i=1}^{n+1}(2^i - 1)r_i$, it follows that the Hilbert series $H(A_n \otimes_{A_{n-1}} \mathbb{Z}_2)$ equals

$$\prod_{i=1}^{n+1} 1 + x^{(2^i-1)2^{n+1-i}} \ ,$$

and thus that $H(Q'/Q) = H(\Sigma^{-1}A_n \otimes_{A_{n-1}} \mathbb{Z}_2) = x^{-1}H(A_n \otimes_{A_{n-1}} \mathbb{Z}_2)$.

Since all exact sequences split over a field, we have that P/Q is isomorphic to the direct sum of its filtration quotients. That is,

$$\begin{aligned} P/Q &\cong \bigoplus_{i\geq 0} Q_{(i+1)\cdot 2^{n+1}-1,n}/Q_{i\cdot 2^{n+1}-1,n} \\ &\cong \bigoplus_{i\geq 0} \Sigma^{i\cdot 2^{n+1}} Q'/Q \end{aligned}$$

as graded $\mathbb{Z}_2$-modules, where the second isomorphism comes from the periodicity isomorphism (corollary 4.4). Consequently,

$$\begin{aligned} H(P/Q) &= \sum_{i\geq 0} x^{i\cdot 2^{n+1}} H(Q'/Q) \\ &= \frac{H(Q'/Q)}{1 - x^{2^{n+1}}} \ . \end{aligned}$$

By lemma 4.11,

$$H(\Sigma^{-1}B \otimes_{A_{n-1}} \mathbb{Z}_2) = x^{-1}H(B \otimes_{A_{n-1}} \mathbb{Z}_2) = \frac{H(B)\cdot 1}{xH(A_{n-1})} \ . \tag{4.3}$$

Using the explicit algebraic description of B given in the discussion of the dual Steenrod algebra, we see that it has the additive basis

$$\{\mathrm{Sq}(r_1, \ldots, r_{n+1}) \mid r_1 \geq 0, \ r_i < 2^{n+2-i} \text{ for } i \geq 2\} \ .$$

Consequently, by combinatorics,

$$H(B) = \frac{1}{1-x} \cdot \prod_{i=2}^{r+1} \frac{1 - x^{(2^i-1)2^{r+2-i}}}{1 - x^{2^i-1}} \ ,$$

and similarly one shows that

$$H(A_{n-1}) = \prod_{i=1}^{r} \frac{1 - x^{(2^i-1)2^{r+1-i}}}{1 - x^{2^i-1}} \ .$$

From these identities, it is easy to verify that $H(B)/xH(A_{n-1}) = H(Q'/Q)/(1-x^{2^{n+1}})$, and thus one concludes by (4.3) that $H(P/Q) = H(\Sigma^{-1}B \otimes_{A_{n-1}} \mathbb{Z}_2)$, completing the proof. □

The isomorphism proven above pieces together into the following commutative diagram, where the map μ is given by multiplication and α is the leftmost map in the short exact sequence of lemma 4.8:

$$\begin{array}{ccc} \Sigma^{-1}A \otimes_{A_{n-1}} \mathbb{Z}_2 & \xrightarrow{\ \alpha\ } & A \otimes_{A_n} P/Q \\ & {}_{\mu\otimes 1}\nwarrow & \cong \uparrow {\scriptstyle 1\otimes\beta} \\ & & \Sigma^{-1}A \otimes_{A_n} B \otimes_{A_{n-1}} \mathbb{Z}_2 \end{array}$$

To see that it actually commutes, note first that by linearity it suffices to prove commutativity on simple tensors. Given any element $a \otimes b \otimes k$ in the bottom right group, $\alpha(\mu \otimes 1(a \otimes b \otimes k)) = \alpha(ab \otimes k) = ab \otimes k \cdot x^{-1}$. By the algebraic description of A_n and B given in the introduction of the dual Steenrod algebra, we see that A_n^* is a quotient of B^*, so taking duals gives us that B is a submodule of A_n. Consequently, $ab \otimes k \cdot x^{-1} = a \otimes k \cdot bx^{-1} = (1 \otimes \beta)(a \otimes b \otimes k)$, so the diagram commutes. Finally, we can prove our desired splitting lemma.

Lemma 4.13. We have a splitting

$$A \otimes_{A_n} P/Q \cong \bigoplus_{i \geq 0} \Sigma^{i \cdot 2^{n+1}-1} A \otimes_{A_{n-1}} \mathbb{Z}_2$$

as A-modules.

Proof. By the commutativity of the diagram above, it follows that $(\mu \otimes 1)(1 \otimes \beta^{-1}) \circ \alpha$ is the identity, and thus that the short exact sequence given in lemma 4.4 splits. Consequently,

$$A \otimes_{A_n} P/Q \cong \left(\Sigma^{-1}A \otimes_{A_{n-1}} \mathbb{Z}_2\right) \oplus \left(A \otimes_{A_n} P/Q'\right) \tag{4.4}$$

as graded A-modules. Since $P/Q' \cong \Sigma^{2^{n+1}} P/Q$ as A_n-modules, it follows that

$$A \otimes_{A_n} P/Q \cong \left(\Sigma^{-1}A \otimes_{A_{n-1}} \mathbb{Z}_2\right) \oplus \left(\Sigma^{2^{n+1}} A \otimes_{A_n} P/Q\right) ,$$

so by induction

$$A \otimes_{A_n} P/Q \cong \left(\Sigma^{(r+1)\cdot 2^{n+1}} A \otimes_{A_n} P/Q\right) \oplus \bigoplus_{i=0}^{r} \Sigma^{i \cdot 2^{n+1}-1} A \otimes_{A_{n-1}} \mathbb{Z}_2$$

for all r. Given any integer d, consider the degree d part of the left hand side. Depending on d, we may always pick r large enough so that the degree d part of the leftmost factor on the right hand side vanishes. This is because the smallest non-vanishing degree in A is zero, and for P/Q it is -1, so for the leftmost factor it vanishes below degree $(r+1)\cdot 2^{n+1}-1$, so making r large enough, this term must vanish at degree d. This implies that on each graded component the left hand side always equals the rightmost component on the right hand side, and thus the result follows. $\square$

To describe the splitting more explicitly, we will provide the elements which under the splitting get mapped to the generators of the respective cyclic modules $A \otimes_{A_{n-1}} \mathbb{Z}_2$. Let $\chi : A \to A$ be the canonical antiautomorphism of A and define the elements

$$y_k = \sum_{i+j=k} \chi(\mathrm{Sq}^i) \otimes x^j$$

in $A \otimes_{A_n} P/Q$. Notice that this sum is finite due to the automatic restrictions $i \geq 0$ and $j \geq -1$.

Lemma 4.14. For each $k \geq 0$, the splitting in lemma 4.6 sends $y_{k\cdot 2^{n+1}-1}$ to the generator of $\Sigma^{k\cdot 2^{n+1}-1} A \otimes_{A_{n-1}} \mathbb{Z}_2$.

Proof. Explicitly, original splitting isomorphism in (4.4) is given by

$$b \mapsto \Big((\mu \otimes 1)(1 \otimes \beta^{-1})(b) \ , \ b - \alpha(\mu \otimes 1)(1 \otimes \beta^{-1})(b) \Big) \ ,$$

where the leftmost factor in the codomain is isomorphic to the image of α and the rightmost factor is isomorphic to the kernel of $(\mu \otimes 1)(1 \otimes \beta^{-1})$. This is the splitting lemma from elementary abstract algebra. Notice that $y_{-1} = \chi(\mathrm{Sq}^0) \otimes x^{-1} = 1 \otimes x^{-1}$ and that $\beta \mathrm{Sq}^0 = x^{-1}$. From this it follows that the isomorphism above sends y_{-1} to $(1, 0)$, that is, to the generator of the leftmost factor $A \otimes \mathbb{Z}_2$.

We will now show that y_k for $k > -1$ lands in the rightmost factor under the original splitting isomorphism (4.4), meaning that $y_k \mapsto (0, y_k)$. This is equivalent to y_k lying in the kernel of $(\mu \otimes 1)(1 \otimes \beta^{-1})$. Recall that $\beta(Sq^{j+1} \otimes 1) = x^j$ for all $j \geq -1$, as explained in the proof of lemma 4.5. It follows that

$$\begin{aligned}
(\mu \otimes 1)(1 \otimes \beta^{-1}) \left(\sum_{i+j=k} \chi(\mathrm{Sq}^i) \otimes x^j \right) &= (\mu \otimes 1) \left(\sum_{i+j=k} \chi(\mathrm{Sq}^i) \otimes \mathrm{Sq}^{j+1} \otimes 1 \right) \\
&= (\mu \otimes 1) \left(\sum_{i+(j+1)=k+1} \chi(\mathrm{Sq}^i) \otimes \mathrm{Sq}^j \otimes 1 \right) \\
&= \sum_{i+(j+1)=k+1} \chi(\mathrm{Sq}^i)\mathrm{Sq}^j \otimes 1 \ ,
\end{aligned}$$

and when $k+1 \geq 1$ (which it is in this case, since $k > -1$) this is zero precisely by the definition of the antiautomorphism χ. It follows that y_k maps to $(0, y_k)$ under the splitting isomorphism in (4.4).

Furthermore, the periodicity isomorphism $A \otimes_{A_n} P/Q' \xrightarrow{\cong} \Sigma^{2^{n+1}} A \otimes_{A_n} P/Q$ sends $y_k \in A \otimes P/Q'$ to $y_{k-2^{n+1}} \in A \otimes P/Q$, because

$$\begin{aligned}\sum_{i+j=k} \chi(\mathrm{Sq}^i) \otimes x^j \mapsto \sum_{i+j=k} \chi(\mathrm{Sq}^i) \otimes x^{j-2^{n+1}} \\ = \sum_{i+j=k-2^{n+1}} \chi(\mathrm{Sq}^i) \otimes x^j \\ = y_{k-2^{n+1}} \in A \otimes_{A_n} P/Q \ .\end{aligned}$$

Under the splitting in (4.4) after we have applied the periodicity isomorphism on the rightmost factor, meaning under the isomorphism

$$A \otimes_{A_n} P/Q \cong \left(\Sigma^{-1} A \otimes_{A_{n-1}} \mathbb{Z}_2\right) \oplus \left(\Sigma^{2^{n+1}} A \otimes_{A_n} P/Q\right) \ ,$$

we see that $y_{2^{n+1}-1}$ maps to $(0, y_{-1})$ by the two paragraphs above. Applying the splitting isomorphism (4.4) again, we see that under the splitting

$$A \otimes_{A_n} P/Q \cong \left(\Sigma^{-1} A \otimes_{A_{n-1}} \mathbb{Z}_2\right) \oplus \left(\Sigma^{2^{n+1}} A \otimes_{A_n} P/Q\right) \oplus \left(\Sigma^{2 \cdot 2^{n+1}} A \otimes_{A_n} P/Q\right) \ ,$$

the element $y_{2^{n+1}-1}$ maps to $(0, 1 \otimes 1, 0)$ and $y_{2 \cdot 2^{n+1}-1}$ maps to $(0, 0, y_{-1})$. By induction, we see that the final splitting isomorphism given in the statement of lemma 4.13 sends $y_{k \cdot 2^{n+1}-1}$ to $(0, \dots, 0, 1 \otimes 1, 0, 0, \dots)$, with a $1 \otimes 1$ in the $k+1$:th factor, for all $k \geq 0$.

□

We return to considering all modules $Q_{k,n}$, as k ranges over the integers. As we've said before, all theorems proven above involving Q extend analogously to $Q_{k,n}$ for any integer k. The splitting in lemma 4.13 for example, is now a direct sum ranging over the indices i such that $i \cdot 2^{n+1} - 1 \geq k$ (notice that in the case $k = -1$, we exactly recover the indexing in lemma 4.13). Lemma 4.14 in particular tells us that $y_p \in A \otimes_{A_n} P/Q_{k,n}$ is nonzero when $p \equiv -1 \mod 2^{n+1}$ and $p \geq k$. Let us prove the converse.
We will take three identities below for granted. It relies on the fact that B is free as a left A_n-module to get the coefficients a_i from an appropriate A_n-linear combination of the Steenrod square described in the first equality in lemma 4.15 below, and then one studies the action of the $\chi(a_i)$'s on P.

Lemma 4.15. For any n, there are finitely many elements $a_i \in A_n$ with degree $i \cdot 2^{n+1} + 2^n$ so that

$$(i) \quad \mathrm{Sq}^{k\cdot 2^{n+1}+2^n} = \sum_{i+j=k} a_i \mathrm{Sq}^{j\cdot 2^{n+1}} \ ,$$

$$(ii) \quad \sum_{i+j=k} \chi(a_i) x^{j\cdot 2^{n+1}-1} = x^{k\cdot 2^{n+1}+2^n-1} \ ,$$

$$(iii) \quad \sum_{i+j=k} \chi(a_i) x^{j\cdot 2^{n+1}+2^n-1} = 0 \ .$$

Proof. Lemma 3.3 in [20]. □

Lemma 4.16. The element y_p in $A \otimes_{A_n} P/Q_{k,n}$ is zero whenever $p \not\equiv -1 \mod 2^{n+1}$.

Proof. In fact, we prove the stronger statement that $y_p \in A \otimes_{A_n} P/Q_{k,n}$ not only vanishes when $p \not\equiv -1 \mod 2^{n+1}$, but that when $p \equiv -1 \mod 2^{n+1}$, the indices in the defining sum of y_p,

$$y_p = \sum_{i+j=p} \chi(\mathrm{Sq}^i) \otimes x^j \ ,$$

ranges only over the indices such that $i \equiv 0 \mod 2^{n+1}$ and $j \equiv -1 \mod 2^{n+1}$. We prove this by induction on n.

For the base case $n = 0$, A_0 consists of only of Sq^0 and Sq^1 and their sum, since $(\mathrm{Sq}^1)^2 = 0$. For this case, the first equality in lemma 4.9 reduces to the equality $\mathrm{Sq}^{2k+1} = \mathrm{Sq}^1\mathrm{Sq}^{2k}$ by degree reasons, since for $a_i \in A_0$ to have degree $2i + 1$, we must have that $i = 0$ and thus also that $a_i = \mathrm{Sq}^1$. Also by degree reasons, notice that the antiautomorphism must send Sq^1 to Sq^1. Finally we recall that $\mathrm{Sq}^1 x^k$ is zero if and only if k is even, which can be seen directly from the the definition of the A-module structure of P.

Let us now prove the base case. If $p \not\equiv -1 \mod 2$, meaning that p is even, we may split the sum defining $y_p \in A \otimes_{A_0} P/Q_{k,0}$ into two pieces as follows:

$$y_p = \sum_{\substack{i+j=p \\ i,j\equiv 0 \mod 2}} \chi(\mathrm{Sq}^i) \otimes x^j + \sum_{\substack{i+j=p \\ i,j\equiv 1 \mod 2}} \chi(\mathrm{Sq}^i) \otimes x^j \ .$$

Consider the second sum on the right hand side. Let $i = 2\tilde{i} + 1$ and $j = 2\tilde{j} + 1$. The equality $i + j = p$ is then equivalent to $(2\tilde{i}) + (2\tilde{j} + 2) = p$. We see now that second sum is equal to

$$\begin{aligned}
\sum_{(2\tilde{i})+(2\tilde{j}+2)=p} \chi(\mathrm{Sq}^{2\tilde{i}+1}) \otimes x^{2\tilde{j}+1} &= \sum_{(2\tilde{i})+(2\tilde{j}+2)=p} \chi(\mathrm{Sq}^1\mathrm{Sq}^{2\tilde{i}}) \otimes x^{2\tilde{j}+1} \\
&= \sum_{(2\tilde{i})+(2\tilde{j}+2)=p} \chi(\mathrm{Sq}^{2\tilde{i}})\mathrm{Sq}^1 \otimes x^{2\tilde{j}+1} \\
&= \sum_{(2\tilde{i})+(2\tilde{j}+2)=p} \chi(\mathrm{Sq}^{2\tilde{i}}) \otimes \mathrm{Sq}^1 x^{2\tilde{j}+1} \\
&= \sum_{(2\tilde{i})+(2\tilde{j}+2)=p} \chi(\mathrm{Sq}^{2\tilde{i}}) \otimes x^{2\tilde{j}+2} \\
&= \sum_{\substack{i+j=p \\ i,j\equiv 0 \mod 2}} \chi(\mathrm{Sq}^{i}) \otimes x^{j} \ ,
\end{aligned} \tag{4.5}$$

but this is precisely equal to the first sum of y_p. Since we are working over characteristic 2, it follows that $y_p = 0$. It remains to show that when $p \equiv -1 \mod 2$, meaning it is odd, has the previously described restriction on the indices. We may again split the sum defining y_p into two pieces as follows:

$$y_p = \sum_{\substack{i+j=p \\ i\equiv 0 \mod 2 \\ j\equiv 1 \mod 2}} \chi(\mathrm{Sq}^{i}) \otimes x^{j} + \sum_{\substack{i+j=p \\ i\equiv 1 \mod 2 \\ j\equiv 0 \mod 2}} \chi(\mathrm{Sq}^{i}) \otimes x^{j} \ .$$

We need to show that the second sum vanishes. This follows from an identical computation as in (4.5), but this time the sum vanishes since $\mathrm{Sq}^1 x^j = 0$, due to j being even, thus completing the base case.

Assume now that the (stronger) statement is proven for $n-1$. Since the projection map $A \otimes_{A_{n-1}} P/Q_{k,n-1} \to A \otimes_{A_n} P/Q_{k,n}$ is surjective, let $\overline{y}_p$ be an element which maps to $y_p \in A \otimes_{A_n} P/Q_{k,n}$ under this projection. By the inductive hypothesis, if $p \not\equiv -1 \mod 2^{n+1}$, then unless $p \equiv 2^n - 1 \mod 2^{n+1}$, it follows that $\tilde{y}_p = 0$ and thus that $y_p = 0$. To show that $p \not\equiv -1 \mod 2^{n+1}$ implies $y_p = 0$, it remains to consider the case when $p \equiv 2^n - 1 \mod 2^{n+1}$.

Assume now that $p \equiv 2^n - 1 \mod 2^{n+1}$. Note that $i \equiv 0 \mod 2^n$ is equivalent to $i \equiv 0 \mod 2^{n+1}$ or $i \equiv 2^n \mod 2^{n+1}$, and similarly $j \equiv -1 \mod 2^n$ is equivalent to $i \equiv -1 \mod 2^{n+1}$ or $i \equiv 2^n - 1 \mod 2^{n+1}$. By the inductive hypothesis, we may write $\tilde{y}_p$ as the sum on the left hand side below, and thus also the image y_p in that way. By the using the above mentioned equivalence, we may split the sum in the left hand side into four pieces by matching the congruence cases. Furthermore, note that since $p \equiv 2^n - 1 \mod 2^{n+1}$, only two of the pieces are nonempty sums, which gives us the equality with the right hand side below:

$$\sum_{\substack{i+j=p \\ i\equiv 0 \mod 2^n \\ j\equiv -1 \mod 2^n}} \chi(\mathrm{Sq}^{i}) \otimes x^{j} = \sum_{\substack{i+j=p \\ i\equiv 2^n \mod 2^{n+1} \\ j\equiv -1 \mod 2^{n+1}}} \chi(\mathrm{Sq}^{i}) \otimes x^{j} + \sum_{\substack{i+j=p \\ i\equiv 0 \mod 2^{n+1} \\ j\equiv 2^n-1 \mod 2^{n+1}}} \chi(\mathrm{Sq}^{i}) \otimes x^{j} \ .$$

If we let $p = \tilde{p} \cdot 2^{n+1} + 2^n - 1$, we may re-index the sums on the right hand side above as follows:

$$\sum_{i+j=\tilde{p}} \chi(\mathrm{Sq}^{i\cdot 2^{n+1}+2^n}) \otimes x^{j\cdot 2^{n+1}-1} + \sum_{i+j=\tilde{p}} \chi(\mathrm{Sq}^{i\cdot 2^{n+1}}) \otimes x^{j\cdot 2^{n+1}+2^n-1} .$$

Using lemma 4.9(i) we may rewrite the first sum on the right hand side above as

$$\begin{aligned}\sum_{s+t+j=\tilde{p}} \chi(\mathrm{Sq}^{t\cdot 2^{n+1}})\chi(a_s) \otimes x^{j\cdot 2^{n+1}-1} &= \sum_{s+t+j=\tilde{p}} \chi(\mathrm{Sq}^{t\cdot 2^{n+1}}) \otimes \chi(a_s)x^{j\cdot 2^{n+1}-1} \\ &= \sum_{t+r=\tilde{p}} \chi(\mathrm{Sq}^{t\cdot 2^{n+1}}) \otimes x^{r\cdot 2^{n+1}+2^n-1} ,\end{aligned}$$

but this is precisely the second sum on the right hand side, and since we are working over characteristic 2, it follows that $y_p = 0$ in this case. The first equality is due to the tensor product being over A_n, and $\chi(a_s) \in A_n$ since A_n. The second equality is due to lemma 4.9(ii).

It remains to show that when $p \equiv -1 \mod 2^{n+1}$, y_p is equal to the sum with appropriately restricted indices. The proof follows just like the case when $p \equiv 2^n - 1 \mod 2^{n+1}$. By the inductive hypothesis, we may write $\overline{y}_p$, and thus its image y_p, as a sum with restricted indices. Let $p = \tilde{p} \cdot 2^{n+1} - 1$ Using the equivalence of congruences as in the previous case, we may split it into four sums by mixing and matching the congruence cases. Since $p \equiv -1 \mod 2^{n+1}$, only two of these sums are nonempty. Consequently,

$$y_p = \sum_{i+j=\tilde{p}} \chi(\mathrm{Sq}^{i\cdot 2^{n+1}}) \otimes x^{j\cdot 2^{n+1}-1} + \sum_{i+j=\tilde{p}} \chi(\mathrm{Sq}^{i\cdot 2^{n+1}+2^n}) \otimes x^{j\cdot 2^{n+1}+2^n-1} .$$

Consider the second sum on the right hand side above; it remains to show that it vanishes. Using lemma 4.9(i) first, noting that we have tensored over A_n, and finally using 4.9(ii), we get that the second sum equals

$$\begin{aligned}\sum_{s+t+j=\tilde{p}} \chi(\mathrm{Sq}^{t\cdot 2^{n+1}})\chi(a_s) \otimes x^{j\cdot 2^{n+1}-1} &= \sum_{s+t+j=\tilde{p}} \chi(\mathrm{Sq}^{t\cdot 2^{n+1}}) \otimes \chi(a_s)x^{j\cdot 2^{n+1}-1} \\ &= \sum_{t+r=\tilde{p}} \chi(\mathrm{Sq}^{t\cdot 2^{n+1}}) \otimes 0 \\ &= 0 .\end{aligned}$$

□

4.4 Homological algebra over A-modules

In the sketch of the proof, we stated some facts in homological algebra that we will now prove. Given a right and left A-module M and N respectively, we may consider them as A_n modules for all n, and we thus have projection maps $M \otimes_{A_n} N \to M \otimes_{A_{n+1}} N$

which piece into a direct system. Given a projective resolution $P_\bullet \to L$ as A-modules, it follows by lemma 4.7 and that the forgetful functor is exact that this resolution may also be considered a projective resolution of A_n-modules for all n. Tensoring these resolutions (as n ranges) by M, we get the following commutative diagram.

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & M \otimes_{A_n} P_1 & \longrightarrow & M \otimes_{A_{n+1}} P_1 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & M \otimes_{A_n} P_0 & \longrightarrow & M \otimes_{A_{n+1}} P_0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & M \otimes_{A_n} L & \longrightarrow & M \otimes_{A_{n+1}} L & \longrightarrow & \cdots \end{array}$$

The columns are chain complexes and the rows are direct systems. A natural question to ask is whether the taking the homology first then passing to the direct limit results in a group isomorphic to first passing to the direct limit and then taking the homology. The answer is yes!

Lemma 4.17. The natural map

$$\varinjlim_n \operatorname{Tor}^{A_n}_{s,t}(M, L) \xrightarrow{\cong} \operatorname{Tor}^{A}_{s,t}(M, L)$$

is an isomorphism, for right and left A-modules M and L, respectively.

Proof. The group $\operatorname{Tor}^{A_n}_{s,t}(M, L)$ is the homology of the column $M \otimes_{A_n} P_\bullet$ in the diagram above. Since $A = \cup_n A_n$, it follows that the natural map $\varinjlim_n M \otimes_{A_n} P_k \to M \otimes_A P_k$ is an isomorphism for all k. Since direct limits are an exact functor, and exact functors commute with homology, the direct limit of $\operatorname{Tor}^{A_n}_{s,t}(M, L)$ is precisely the homology of $M \otimes_A P_\bullet$. □

For a graded $\mathbb{Z}_2$ module M, let M^* denote its graded dual $\operatorname{Hom}_{\mathbb{Z}_2}(M, \mathbb{Z}_2)$. Notice that dualizing a left A-module makes it into right A-module, and vice versa. To translate a statement about isomorphisms of Tor groups into a statement about Ext groups, we will use the tensor-hom adjunction as follows.

Lemma 4.18. We have a natural isomorphism

$$\operatorname{Hom}_{\mathbb{Z}_2}(N \otimes_A P, \mathbb{Z}_2) \cong \operatorname{Hom}_A(P, N^*) ,$$

where N and P are right and left A-modules, respectively.

Proof. Essentially, our claim is that the classical tensor-hom adjunction

$$\operatorname{Hom}_{\mathbb{Z}_2}(N \otimes_{\mathbb{Z}_2} P, \mathbb{Z}_2) \cong \operatorname{Hom}_{\mathbb{Z}_2}(P, N^*)$$

restricts to an isomorphism as described above. Recall that the adjunction is given explicitly by sending $f : N \otimes_{\mathbb{Z}_2} P \to \mathbb{Z}_2$ to $\tilde{f} : P \to N^*$ such that $p \mapsto f(- \otimes p)$ and $g : P \to N^*$ to $\tilde{g} : N \otimes_{\mathbb{Z}_2} P \to \mathbb{Z}_2$ $n \otimes p \mapsto f(p)(n)$.

We need to show that if f is A-balanced, then $\tilde{f}$ is A-linear, and that if g is A-linear, then $\tilde{g}$ is A-balanced. These statements follow from the following equalities, where $a \in A$, $n \in N$, $p \in P$.

$$\begin{aligned} \tilde{f}(a \cdot p) &= f(- \otimes a \cdot p) & \tilde{g}(n \cdot a \otimes p) &= g(p)(n \cdot a) \\ &= f(- \cdot a \otimes p) & &= \big(a \cdot g(p)\big)(n) \\ &= a \cdot f(- \otimes p) & &= g(a \cdot p)(n) \\ &= a \cdot \tilde{f}(p) & &= \tilde{g}(n \otimes a \cdot p) \end{aligned}$$

□

The isomorphism in the lemma above allows us to relate Ext and Tor groups by considering it as an isomorphism between the chain complexes computing $\mathrm{Ext}_A^{s,t}(M, N^*)$ and $\big(\mathrm{Tor}^A_{s,t}(N, M)\big)^*$, respectively, as will be shown below.

Lemma 4.19. There is an isomorphism

$$\mathrm{Ext}_A^{s,t}(M, N^*) \cong \big(\mathrm{Tor}^A_{s,t}(N, M)\big)^*$$

natural in both M and N. Here N and M are right and left A-modules which are graded $\mathbb{Z}_2$-modules.

Proof. Let $P_\bullet \to M$ be a projective resolution. Applying the functor $N \otimes_A -$ and then $(-)^*$, we get the chain complex $(N \otimes_A P_\bullet)^*$, which by lemma 4.18 is isomorphic to the complex $\mathrm{Hom}_A(P_\bullet, N^*)$. The homology of the latter complex is precisely $\mathrm{Ext}_A^{*,*}(M, N^*)$. Consider the former complex. Since $\mathbb{Z}_2$ is an injective $\mathbb{Z}_2$-module, it follows that $\mathrm{Hom}_{\mathbb{Z}_2}(-, \mathbb{Z}_2) = (-)^*$ is an exact functor, and thus commutes with homology. This implies that the homology of $(N \otimes_A P_\bullet)^*$ is isomorphic to the dual of the homology of $N \otimes_A P_\bullet$. This group is precisely $\big(\mathrm{Tor}^A_{*,*}(N, M)\big)^*$. □

Recall that A_n is a finite-dimensional $\mathbb{Z}_2$-module by lemma 4.9. This implies that A_n, as a graded $\mathbb{Z}_2$ module, is bounded above. If we let $(M)^t$ denote the t:th graded component for a graded module M, then it follows that $(A_n)^t = 0$ for $t > d$, for some integer d. If M is a bounded A-module, we have the following boundedness conditions on the corresponding Tor-group below.

Lemma 4.20. If $M^t = 0$ for all $t < u$, then $\mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, M) = 0$ for $t < u$. Also, if $M^t = 0$ for all $t > u$, then $\mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, M) = 0$ for $t > u + (s+1)d$.

Proof. It suffices to construct a free resolution of $F_\bullet \to M$ such that $(F_s)^t = 0$ for $t < u$ or $t > u + (s+1)d$, depending on the case. This is because tensoring with $\mathbb{Z}_2$ does not change the grading, and thus the grading does not change on the subquotient $\mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, M)$.

For the first case, let S_0 be the set of generators for M. Let F_0 be the free A-module on the set S_0 and let $F_0 \to M$ be the obvious map. Then $(F_0)^t = 0$ for $t < u$ since all generators $s \in S_0$ have degree at least u, and A_n cannot decrease the degree. Then let S_1 be the set of generators for $\ker(F_0 \to M)$ and let F_1 be the free module on S_1 and iterate the process.

For the second case, Let S_0 and F_0 be as above. Note then that $(F_0)^t = 0$ for $t > u + d$, because all generators $s \in S_0$ have degree at most u, and A_n increases the degree by at most d. If we let S_1 and F_1 be as before, then $(F_1)^t = 0$ for $t > u + d + d = u + 2d$, because all generators $s \in S_0$ have degree at most $u + d$, and A_n increases the degree by at most d. From induction the result follows □

In the sketch of the proof, we stated a change of rings theorem, which we will now prove.

Lemma 4.21. We have an isomorphism

$$\mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, A \otimes_{A_n} M) \cong \mathrm{Tor}^{A_n}_{*,*}(\mathbb{Z}_2, M)$$

for all n, where M is a graded A_n-module, which is natural in M.

Proof. We will show that the functor $A \otimes_{A_n} -$ is exact and takes projective A_n-modules to projective A-modules (so that it takes projective resolutions to projective resolutions). Exactness of $A \otimes_{A_n} -$ follows from A being flat over A_n, as explained below lemma 4.7. To prove that this functor takes projectives to projectives, recall that since it is an extension by scalars functor, it is left adjoint to the forgetful functor $F(-) : A\text{-Mod} \to A_n\text{-Mod}$. If P is a projective A_n module, then the functor $\mathrm{Hom}_{A_n}(P, -)$ is an exact functor of A_n-modules. Since the forgetful functor is exact, pre-composing gives us an exact functor $\mathrm{Hom}_{A_n}(P, F(-))$. By the adjunction, this functor is isomorphic to the functor $\mathrm{Hom}_A(A \otimes_{A_n} P, -)$, which must then be exact. Consequently $A \otimes_{A_n} P$ is a projective A-module.

Given a projective resolution of A_n-modules $P_\bullet \to M$, the sequence $A \otimes_{A_n} P_\bullet \to A \otimes_{A_n} M$ is a projective resolution of A-modules. We have isomorphisms of chain complexes $\mathbb{Z}_2 \otimes_A (A \otimes_{A_n} P_\bullet) \cong (\mathbb{Z}_2 \otimes_A A) \otimes_{A_n} P_\bullet \cong \mathbb{Z}_2 \otimes_{A_n} P_\bullet$. It follows that the homology of the leftmost complex is isomorphic to the homology of the rightmost complex, completing the proof. □

4.5 Passing to direct and inverse limits

Finally, we have proven enough lemmas to get to the good stuff which we promised would happen in the sketch proof. We prove that the splitting makes the diagrams mentioned in the sketch commute, that taking inverse and direct limits will give us what we promised, and finally that by a duality argument, we may go from an isomorphism of Tor groups to an isomorphism of Ext groups. To this end, consider the following two diagrams (4.6) and (4.7).

$$\begin{array}{ccc} A \otimes_{A_n} P/Q_{k,n} & \xrightarrow{\cong} & \bigoplus_{i\cdot 2^{n+1}-1\geq k} \Sigma^{i\cdot 2^{n+1}-1} A \otimes_{A_{n-1}} \mathbb{Z}_2 \\ \text{proj}\downarrow & & \downarrow\theta \\ A \otimes_{A_n} P/Q_{k',n} & \xrightarrow[\cong]{} & \bigoplus_{i\cdot 2^{n+1}-1\geq k'} \Sigma^{i\cdot 2^{n+1}-1} A \otimes_{A_{n-1}} \mathbb{Z}_2 \end{array} \tag{4.6}$$

Consider the diagram (4.6) above. The horizontal isomorphisms above are the ones from lemma 4.13. Here we are assuming that $k < k'$, and that the left vertical map "proj" is the quotient map. Finally, the right vertical map θ is the graded map which is the identity on the graded components where both groups are nonzero, and the zero map otherwise. This diagram commutes by lemma 4.14.

$$\begin{array}{ccc} A \otimes_{A_n} P/Q_{k,n} & \xrightarrow{\cong} & \bigoplus_{i\cdot 2^{n+1}-1\geq k} \Sigma^{i\cdot 2^{n+1}-1} A \otimes_{A_{n-1}} \mathbb{Z}_2 \\ \text{proj}\downarrow & & \downarrow\psi \\ A \otimes_{A_{n+1}} P/Q_{k,n+1} & \xrightarrow[\cong]{} & \bigoplus_{i\cdot 2^{n+2}-1\geq k} \Sigma^{i\cdot 2^{n+2}-1} A \otimes_{A_n} \mathbb{Z}_2 \end{array} \tag{4.7}$$

Consider now the diagram (4.7) above. The horizontal isomorphisms are the ones from lemma 4.13, but for the cases n and $n+1$ respectively. The left vertical map "proj" is the projection map $a \otimes p \mapsto a \otimes p$. Finally, the right horizontal map ψ is defined similarly to θ, being the identity on the graded components when both components are nonzero and the zero map otherwise. Notice that ψ sends half of the elements to zero, and is the identity on the other half. To see that the diagram commutes, it suffices to check commutativity on the generators, which are the y_p's by 4.14, and this is easy to verify due to lemma 4.16.

Recall that the functor $\mathrm{Tor}^A_{*,*}(\mathbb{Z}_2, -)$ commutes with arbitrary direct sums (essentially because the tensor product distributes naturally over direct sums). Using this, and lemma 4.21, we obtain the following commutative square by applying the functor $\mathrm{Tor}^A_{*,*}(\mathbb{Z}_2, -)$ on the diagram (4.6), where j and j' range over the appropriate integers as described in (4.6).

$$\begin{array}{ccc}
\mathrm{Tor}^{A_n}_{*,*}(\mathbb{Z}_2, P/Q_{k,n}) & \xrightarrow{\cong} & \bigoplus_j \mathrm{Tor}^{A_{n-1}}_{*,*}(\mathbb{Z}_2, \Sigma^j \mathbb{Z}_2) \\
\downarrow{\scriptstyle \mathrm{proj}_*} & & \downarrow{\scriptstyle \theta_*} \\
\mathrm{Tor}^{A_n}_{*,*}(\mathbb{Z}_2, P/Q_{k',n}) & \xrightarrow[\cong]{} & \bigoplus_{j'} \mathrm{Tor}^{A_{n-1}}_{*,*}(\mathbb{Z}_2, \Sigma^{j'} \mathbb{Z}_2)
\end{array} \tag{4.8}$$

We can deduce the commutativity of the diagram above by piecing together the following two commutative diagrams (4.9) and (4.10) below, because the first column in (4.9) and the last column in (4.10) create the square above. The first square in (4.9) commutes by the naturality of the change of rings isomorphism in lemma 4.21. The second square in (4.9) commutes by the functoriality of $\mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, -)$ since this diagram comes from (4.6). The first square in (4.10) commutes since the functor $\mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, -)$ commutes with direct sums. Finally, the second square in (4.10) commutes by the naturality of the change of rings isomorphism.

$$\begin{array}{ccccc}
\mathrm{Tor}^{A_n}_{*,*}(\mathbb{Z}_2, P/Q_{k,n}) & \overset{\cong}{\longleftrightarrow} & \mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, A \otimes_{A_n} P/Q_{k,n}) & \xrightarrow{\cong} & \mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, \bigoplus_j \Sigma^j A \otimes_{A_{n-1}} \mathbb{Z}_2) \\
\downarrow{\scriptstyle \mathrm{proj}_*} & & \downarrow{\scriptstyle \mathrm{proj}_*} & & \downarrow{\scriptstyle \theta_*} \\
\mathrm{Tor}^{A_n}_{*,*}(\mathbb{Z}_2, P/Q_{k',n}) & \underset{\cong}{\longleftrightarrow} & \mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, A \otimes_{A_n} P/Q_{k',n}) & \xrightarrow[\cong]{} & \mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, \bigoplus_{j'} \Sigma^{j'} A \otimes_{A_{n-1}} \mathbb{Z}_2)
\end{array} \tag{4.9}$$

$$\begin{array}{ccccc}
\mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, \bigoplus_j \Sigma^j A \otimes_{A_{n-1}} \mathbb{Z}_2) & \overset{\cong}{\longleftrightarrow} & \bigoplus_j \mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, \Sigma^j A \otimes_{A_n} \mathbb{Z}_2) & \overset{\cong}{\longleftrightarrow} & \bigoplus_j \mathrm{Tor}^{A_{n-1}}_{*,*}(\mathbb{Z}_2, \Sigma^j \mathbb{Z}_2) \\
\downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*} \\
\mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, \bigoplus_{j'} \Sigma^{j'} A \otimes_{A_{n-1}} \mathbb{Z}_2) & \underset{\cong}{\longleftrightarrow} & \bigoplus_{j}' \mathrm{Tor}^{A_{n-1}}_{*,*}(\mathbb{Z}_2, \Sigma^{j'} A \otimes_{A_n} \mathbb{Z}_2) & \underset{\cong}{\longleftrightarrow} & \bigoplus_{j}' \mathrm{Tor}^{A_{n-1}}_{*,*}(\mathbb{Z}_2, \Sigma^{j'} \mathbb{Z}_2)
\end{array} \tag{4.10}$$

One can similarly use the fact that Tor commutes with direct sums and the naturality of the isomorphism in lemma 4.21 to conclude that we have a commutative diagram as below, induced by applying the functor $\mathrm{Tor}^{A}_{*,*}(\mathbb{Z}_2, -)$ on (4.7). The indices j and j' range over the corresponding sets as described in (4.7).

$$\begin{array}{ccc}
\mathrm{Tor}^{A_n}_{*,*}(\mathbb{Z}_2, P/Q_{k,n}) & \xrightarrow{\cong} & \bigoplus_j \mathrm{Tor}^{A_{n-1}}_{*,*}(\mathbb{Z}_2, \Sigma^j \mathbb{Z}_2) \\
\downarrow{\scriptstyle \mathrm{proj}_*} & & \downarrow{\scriptstyle \psi_*} \\
\mathrm{Tor}^{A_{n+1}}_{*,*}(\mathbb{Z}_2, P/Q_{k,n+1}) & \xrightarrow[\cong]{} & \bigoplus_{j'} \mathrm{Tor}^{A_n}_{*,*}(\mathbb{Z}_2, \Sigma^{j'} \mathbb{Z}_2)
\end{array} \tag{4.11}$$

Fix a pair of integers s, t. Recall that $Q_{k,n} = \mathrm{span}_{A_n}\{x^i \mid i < k\}$. Since A_n is finite, and $Q_{k,n}$ is the A_n-linear span of a bounded above set (in terms of grading degree), it follows that $Q_{k,n}$ vanishes in grading degree greater than u, for some integer u. By lemma 4.20, it follows that $\mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, Q_{k,n}) = 0$ for small enough integers k, because small enough k

ensures that u may be picked sufficiently small. By the long exact sequence in Tor, it follows that the map induced by the quotient $\mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P) \to \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P/Q_{k,n})$ is an isomorphism for small enough k. Consequently, the map

$$\mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P) \to \varprojlim_k \left(\cdots \xrightarrow{\mathrm{proj}_*} \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P/Q_{k,n}) \xrightarrow{\mathrm{proj}_*} \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P/Q_{k+1,n}) \xrightarrow{\mathrm{proj}_*} \cdots \right)$$

induced by the quotient is an isomorphism. Notice that (4.8) gives us an isomorphism of inverse systems. Taking the inverse limit in that diagram, and using the isomorphism above, we obtain the isomorphism

$$\mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P) \xrightarrow{\cong} \bigoplus_{j\in\mathbb{Z}} \mathrm{Tor}^{A_{n-1}}_{s,t}(\mathbb{Z}_2, \Sigma^{j\cdot 2^{n+1}-1}\mathbb{Z}_2).$$

Notice there is no boundedness condition on the index of the direct sum anymore. This isomorphism takes part in the commutative diagram below, obtained by applying the functor $\varprojlim_k(-)$ to (4.11).

$$\begin{array}{ccc} \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P) & \xrightarrow{\cong} & \bigoplus_{j\in\mathbb{Z}} \mathrm{Tor}^{A_{n-1}}_{s,t}(\mathbb{Z}_2, \Sigma^{j\cdot 2^{n+1}-1}\mathbb{Z}_2) \\ \downarrow{\scriptstyle \mathrm{proj}_*} & & \downarrow{\scriptstyle \psi_*} \\ \mathrm{Tor}^{A_{n+1}}_{s,t}(\mathbb{Z}_2, P) & \xrightarrow[\cong]{} & \bigoplus_{j'\in\mathbb{Z}} \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, \Sigma^{j'\cdot 2^{n+2}-1}\mathbb{Z}_2) \end{array} \tag{4.12}$$

We can now prove the isomorphism of Tor groups promised in the sketch.

Lemma 4.22. The induced map $\gamma_* : \mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, P) \to \mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2)$ is an isomorphism.

Proof. Recall that y_{-1}, as an element in $Q_{k,n}$ for $k \leq -1$, is equal to $1 \otimes x^{-1}$. The composition

$$\Sigma^{-1} A \otimes_{A_{n-1}} \mathbb{Z}_2 \to \bigoplus_{j\cdot 2^{n+1}-1 \geq -1} \Sigma^{j\cdot 2^{n+1}-1} A \otimes_{A_{n-1}} \mathbb{Z}_2 \overset{\cong}{\leftrightarrow} A \otimes_{A_n} P/Q_{k,n} \overset{1\otimes\gamma}{\to} \Sigma^{-1} A \otimes_{A_n} \mathbb{Z}_2$$

sends $1 \otimes 1$ to $1 \otimes 1$. More precisely, it sends $1 \otimes 1$ to $(1 \otimes 1, 0, 0, \dots)$ to y_{-1} to $1 \otimes 1$. The reason that $1 \otimes \gamma(y_{-1}) = 1 \otimes 1$ is because x^{-1} is not zero in $P/Q_{k,n}$ for $k \leq -1$. Consequently, applying the functor $\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, -)$ to this composition, and using the change of rings isomorphism and that Tor commutes with direct sums, it is easy to verify that that the composition

$$\mathrm{Tor}^{A_{n-1}}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2) \longrightarrow \bigoplus_j \mathrm{Tor}^{A_{n-1}}_{s,t}(\mathbb{Z}_2, \Sigma^{j\cdot 2^{n+1}-1}\mathbb{Z}_2) \xleftrightarrow{\cong} \cdots$$

$$\cdots \xleftrightarrow{\cong} \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P/Q_{k,n}) \xrightarrow{\gamma_*} \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2)$$

is the projection map from Tor of a sub-ring to Tor of a bigger ring (where j ranges over the set $j \cdot 2^{n+1} - 1 \geq k$). Applying the functor $\varprojlim_k(-)$ on this diagram, (where the leftmost and rightmost groups may be considered as inverse systems with identity maps), we get the diagram

$$\begin{array}{c} \mathrm{Tor}^{A_{n-1}}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2) \longrightarrow \bigoplus_j \mathrm{Tor}^{A_{n-1}}_{s,t}(\mathbb{Z}_2, \Sigma^{j\cdot 2^{n+1}-1}\mathbb{Z}_2) \xleftrightarrow{\ \cong\ } \cdots \\ \cdots \xleftrightarrow{\ \cong\ } \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P) \xrightarrow{\ \gamma_*\ } \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2) \end{array}$$

where j now ranges over all integers, and where the composition is still the projection map of Tor of a sub-ring to Tor of a bigger ring.

Consider now the diagram (4.12). It presents an isomorphism between two direct systems. Taking the direct limit as $n \to \infty$, using lemma 4.17, we get an isomorphism

$$\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, P) \cong \mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2) \ .$$

This is because the only summand in the rightmost direct system of (4.12) which does not vanish, due to the definition of ψ, is the summand described in the right hand side above. Furthermore, notice that the isomorphism of direct systems (4.12) may be extended on the sides to a sequence of direct systems as follows, where the horizontal composition from the right to the left is the projection map discussed above:

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2) & \xleftarrow{\gamma_*} & \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, P) & \overset{\cong}{\leftrightarrow} & \bigoplus_{j\in\mathbb{Z}} \mathrm{Tor}^{A_{n-1}}_{s,t}(\mathbb{Z}_2, \Sigma^{j\cdot 2^{n+1}-1}\mathbb{Z}_2) & \leftarrow & \mathrm{Tor}^{A_{n-1}}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2) \\ \downarrow \mathrm{proj}_* & & \mathrm{proj}_* \downarrow & & \downarrow \psi_* & & \downarrow \mathrm{proj}_* \\ \mathrm{Tor}^{A_{n+1}}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2) & \underset{\gamma_*}{\leftarrow} & \mathrm{Tor}^{A_{n+1}}_{s,t}(\mathbb{Z}_2, P) & \underset{\cong}{\rightleftarrows} & \bigoplus_{j'\in\mathbb{Z}} \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, \Sigma^{j'\cdot 2^{n+2}-1}\mathbb{Z}_2) & \longleftarrow & \mathrm{Tor}^{A_n}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

Taking the direct limit, and using lemma 4.15, we get the diagram

$$\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2) \xleftarrow{\gamma_*} \mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, P) \cong \mathrm{Tor}^{A}_{s,t}(\mathbb{Z}_2, \Sigma^{-1}\mathbb{Z}_2)$$

whose composition (from right to left) is the identity. From this, we will conclude that γ_* is an isomorphism, completing the proof. □

We will now use our lemmas adjacent to the tensor-hom adjunction to prove that the map ϕ induces an isomorphism of Ext groups. In the proof of lemma 4.15, Lin explains that the action of $a \in A$ on x^k is nonzero if and only if the action of $\chi(a)$ on $x^{-1-k-|a|}$

is nonzero. We take this for granted. Using the anti-automorphism χ, we may imbue P with a right A-module structure, defined by $x^k \cdot a := \chi(a) \cdot x^k$. We can now see that the composition

$$P \times P \xrightarrow{\mu} P \xrightarrow{\gamma} \Sigma^{-1}\mathbb{Z}_2$$

is A-balanced, where μ is the multiplication map. Assuming $i + j + |a| = -1$ (otherwise both elements we now describe will go to zero due to the grading), this composition takes $(x^i \cdot a, x^j)$ to the coefficient of $(\chi(a)x^{-1-j|a|})x^j$ and takes $(x^i, a \cdot x^j)$ to the coefficient of $x^i(ax^j)$, and these elements are equal by what we have said before.

Since the composition $\gamma \circ \mu$ is A-balanced, it defined a map $\gamma \circ \mu : P \otimes_A P \to \Sigma^{-1}\mathbb{Z}_2$. Under the tensor-hom adjunction, this homomorphism maps to the element $\eta : P \to \mathrm{Hom}_{\mathbb{Z}_2}(P, \Sigma^{-1}\mathbb{Z}_2)$, given by $\eta(x^k) = \gamma \circ \mu(- \otimes x^k)$. In fact, as we have explained before, the fact that $\gamma \circ \mu$ is A-balanced implies that η is A-linear (as both a left and right module).

Furthermore, one can verify that η is a graded map, because it sends a degree k element $x^k \in P$ to the element $\eta(x^k)$. Consider $\eta(x^k)$ now. It takes a degree $-k-1$ element $x^{-k-1} \in P$ to the degree -1 element $1 \in \Sigma^{-1}\mathbb{Z}_2$, and for $j \neq -k-1$, $\eta(x^k)$ takes x^j to 0, which lies in $(\Sigma^{-1}\mathbb{Z}_2)^{j+k}$. Consequently, $\eta(x^k)(P^m) \subseteq (\Sigma^{-1}\mathbb{Z}_2)^{m+k}$ for any integer m, so $\eta(x^k)$ is an element of degree k, implying that the map η is graded.

It is clear that $\gamma \circ \mu$ is a non-degenerate bilinear form, meaning that for any nonzero element $p \in P$, there is an element $q \in P$ such that $\gamma \circ \mu(p \otimes q) \neq 0$. From this it follows that η is injective. Since η is an map of graded $\mathbb{Z}_2$ modules which are finite-dimensional in each graded component, it follows from linear algebra that η is a bijection. Consequently, η gives an isomorphism $P \cong \mathrm{Hom}_{\mathbb{Z}_2}(P, \Sigma^{-1}\mathbb{Z}_2)$ of graded A-modules. Therefore, $\Sigma\eta$ induces an isomorphism $\Sigma P \cong \Sigma\mathrm{Hom}_{\mathbb{Z}_2}(P, \Sigma^{-1}\mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}_2}(\Sigma^{-1}P, \Sigma^{-1}\mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}_2}(P, \mathbb{Z}_2) = P^*$.

The equality $\Sigma\mathrm{Hom}_{\mathbb{Z}_2}(P, \Sigma^{-1}\mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}_2}(\Sigma^{-1}P, \Sigma^{-1}\mathbb{Z}_2)$ above comes from the fact that degree $i-1$ maps $P \to \Sigma^{-1}\mathbb{Z}_2$ are precisely degree i maps $\Sigma^{-1}P \to \Sigma^{-1}\mathbb{Z}_2$, so

$$\left(\Sigma\mathrm{Hom}_{\mathbb{Z}_2}(P, \Sigma^{-1}\mathbb{Z}_2)\right)^i = \left(\mathrm{Hom}_{\mathbb{Z}_2}(\Sigma^{-1}P, \Sigma^{-1}\mathbb{Z}_2)\right)^i$$

for all i. Similarly, $\mathrm{Hom}_{Z_2}(\Sigma^{-1}\mathbb{Z}_2, \mathbb{Z}_2) = \Sigma\mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2, \mathbb{Z}_2)$, and this is isomorphic to $\Sigma\mathbb{Z}_2$. The isomorphism is simply given by sending $1 \in \Sigma\mathbb{Z}_2$ to the identity map.

Taking the dual of the map $\gamma : P \to \Sigma^{-1}\mathbb{Z}_2$, we obtain a map $\gamma^* : (\Sigma^{-1}\mathbb{Z}_2)^* \to P^*$. Under the isomorphism for the domain and codomain of this map, we claim the map becomes $\Sigma\phi : \Sigma\mathbb{Z}_2 \to \Sigma P$, meaning that the following diagram commutes.

$$\begin{array}{ccc} P^* & \xleftarrow{\gamma^*} & (\Sigma^{-1}\mathbb{Z}_2)^* \\ {\scriptstyle \Sigma\eta}\uparrow\cong & & \uparrow\cong \\ \Sigma P & \xleftarrow[\Sigma\phi]{} & \Sigma\mathbb{Z}_2 \end{array} \tag{4.13}$$

This amounts to showing that $\gamma^* f(-) = \Sigma\eta(x^0)(-)$, where f is the nonzero element in $(\Sigma^{-1}\mathbb{Z}_2)^*$. Given any $\sum_i e_i x^i \in P$, where $e_i \in Z_2$, we have that $\gamma^*(f)(\sum_i e_i x^i) = f(\gamma(\sum_i e_i x^i)) = f(e_{-1}) = e_{-1} = \gamma \circ \mu(\sum_i e_i x^i \otimes x^0) = \Sigma\eta(x^0)(\sum_i e_i x^i)$, from which the commutativity follows. We now have enough to prove our theorem!

Theorem 4.23. *The induced map*

$$\phi : Ext_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2) \to Ext_A^{s,t}(\mathbb{Z}_2, P)$$

is an isomorphism of A-modules.

Proof. As said before, we may imbue every left A-module with a right A-module structure, and vice versa, through the anti-automorphism χ. Doing this, it follows that the transposition map $M \otimes_A N \to N \otimes_A M$ given by $a \otimes b \mapsto b \otimes a$ is an isomorphism, and thus induces a natural isomorphism $\mathrm{Tor}^A_{s,t}(M,N) \xrightarrow{\cong} \mathrm{Tor}^A_{s,t}(N,M)$, for M and N right and left A-modules, respectively. Consequently, we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Tor}^A_{s,t}(\mathbb{Z}_2,P) & \xrightarrow[\cong]{\gamma_*} & \mathrm{Tor}^A_{s,t}(\mathbb{Z}_2,\Sigma^{-1}\mathbb{Z}_2) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathrm{Tor}^A_{s,t}(P,\mathbb{Z}_2) & \xrightarrow[\gamma_*]{\cong} & \mathrm{Tor}^A_{s,t}(\Sigma^{-1}\mathbb{Z}_2,\mathbb{Z}_2) \end{array}$$

Applying the functor $(-)^*$ to the diagram above, we get the top square in the diagram below. The commutativity of the middle square follows from the naturality of the isomorphism in lemma 4.19. Finally, the commutativity of the bottom square follows from applying the functor $\mathrm{Ext}^{s,t}_A(\mathbb{Z}_2,-)$ to (4.13).

$$\begin{array}{ccc} \mathrm{Tor}^A_{s,t}(\mathbb{Z}_2,P)^* & \xleftarrow[\cong]{(\gamma_*)^*} & \mathrm{Tor}^A_{s,t}(\mathbb{Z}_2,\Sigma^{-1}\mathbb{Z}_2)^* \\ \cong\big\uparrow & & \big\uparrow\cong \\ \mathrm{Tor}^A_{s,t}(P,\mathbb{Z}_2)^* & \xleftarrow[(\gamma_*)^*]{} & \mathrm{Tor}^A_{s,t}(\Sigma^{-1}\mathbb{Z}_2,\mathbb{Z}_2)^* \\ \cong\big\updownarrow & & \big\updownarrow\cong \\ \mathrm{Ext}^{s,t}_A(\mathbb{Z}_2,P^*) & \xleftarrow[(\gamma^*)_*]{} & \mathrm{Ext}^{s,t}_A(\mathbb{Z}_2,(\Sigma^{-1}\mathbb{Z}_2)^*) \\ (\Sigma\eta)_*\big\uparrow\cong & & \big\uparrow\cong \\ \mathrm{Ext}^{s,t}_A(\mathbb{Z}_2,\Sigma P) & \xleftarrow[(\Sigma\phi)_*]{} & \mathrm{Ext}^{s,t}_A(\mathbb{Z}_2,\Sigma\mathbb{Z}_2) \end{array}$$

By the commutativity of the diagram, it follows that the bottom map $(\Sigma\phi)_*$ is an isomorphism, and thus that the map ϕ_* we are interested in is an isomorphism.

□

5 The main theorem

Finally, we may start to compute the stable cohomotopy groups of projective spaces of the form RP^n/RP^k and RP^∞/RP^k! Our primary reference for this section is Lin's article [19], which proves our final theorem. Let us first give an outline of how the proof will go, then define the spectra we will be working with, and lastly state the theorem which we will prove.

First, we start with some preliminary computations of the stable cohomotopy groups of finite stunted projective spaces RP^n/RP^k, showing that these groups are finite 2-groups in some relevant range. This will be done with the help of the Atiyah-Hirzebruch spectral sequence.

Second, we use our isomorphism of Ext groups from the previous section to provide an isomorphism between the E_2 pages of Adams spectral sequences computing $[S^*, S^0]$ and the stable cohomotopy of RP^n/RP^k. Using our Atiyah-Hirzebruch computation, we pass to an inverse limit as $n \to \infty$ and show that the isomorphism above passes to an isomorphism between the E_2 pages of the spectral sequences computing $[S^*, S^0]$ and the stable cohomotopy of RP^∞/RP^k.

Third, we prove that the Adams spectral sequence computing the stable cohomotopy of RP^∞/RP^k is convergent. Notice that this is not guaranteed by theorem 3.24 since RP^∞/RP^k is not a finite spectrum, so some non-trivial work needs to be done to prove this. Using also the convergence of spectral sequence computing $[S^*, S^0]$, we exhibit a homomorphism of filtered groups which induces an isomorphism of successive filtration quotients, from which we conclude that the homomorphism is an isomorphism, completing the proof.

Let us now define the spectra we will be working with, and in so doing extend what we mean by the stable cohomotopy of RP^n/RP^k or RP^∞/RP^k. Define $RP^n_k := RP^n/RP^{k-1}$. Obviously, the suspension spectra of such spaces are defined for any $k-1 < n$ when k is positive. We will however also try to make define spectra which, in some sense, are suspension spectra of RP^n_k in the case when $k-1$ or n are negative. We are able to make sense of such spectra by James' periodicity equivalence.

Proposition 1.4 in [15] tells us that we have the homotopy equivalence

$$\Sigma^l(RP^{n+r}/RP^n) \simeq RP^{l+n+r}/RP^{l+n}$$

in the case where $2^{\phi(r)}$ divides l, and $\phi(r)$ is the number of integers s such that $0 < s < r$ and s is equal to $0, 1, 2$ or 4 modulo 8. This is called James' periodicity equivalence. As a quick remark, we not that the reason for such restrictions on l stems from the work on the maximal number of linearly independent vector fields of spheres! James proves that there is a homotopy equivalence described above whenever the projection $V_{n,r} \to V_{n,1} = S^{n-1}$

onto the first factor has a section, where $V_{n,r}$ is the Stiefel manifold. Having such a section is equivalent to having a collection of $r-1$ pairwise orthogonal vector fields on S^{n-1}.

Given any pair of integers n and r with $r < n$, we may pick l large enough so that $l \geq \phi(n-r+1)$ and $2^l + r - 1 > 0$. Define the spectrum X_r^n to be

$$\Sigma^{-2^l}\Sigma^\infty(RP^{2^l+n}_{2^l+r}) .$$

By James periodicity equivalence, it follows that the construction of X_r^n is independent of the chosen integer l up to a homotopy equivalence of spectra. To see this, let l be the smallest integer such that $l \geq \phi(n-r+1)$ and $2^l + r - 1 > 0$ and define the spectrum X_r^n as above. Let Y_r^n be the spectrum

$$\Sigma^{-2^{l+1}}\Sigma^\infty(RP^{2^{l+1}+n}_{2^{l+1}+r}) .$$

By the James periodicity equivalence,

$$\Sigma^{s-2^{l+1}}(RP^{2^{l+1}+n}/RP^{2^{l+1}+r}) \simeq \Sigma^{s-2^l}(RP^{2^l+n}_{2^l+r}) .$$

The left hand side is the s:th component space of Y_r^n and the right hand side is the s:th component space of X_r^n, so X_r^n and Y_r^n have homotopy equivalent component spaces. By corollary 2.21 X_r^n and Y_r^n are homotopy equivalent as spectra. In particular, it follows that X_r^n is homotopy equivalent to the suspension spectrum of RP_r^n when $0 < r \leq n$.

Below, we have a direct system of spectra where the maps below are induced by inclusions.

$$\Sigma^\infty(S^r) = X_r^r \to X_r^{r+1} \to X_r^{r+2} \to \dots$$

Define the (homotopy) direct limit of this system to be the spectrum X_r, meaning that we let X_r be the mapping telescope of this sequence. Recall that we have already discussed mapping cylinders in the context of spectra, so the mapping telescope is just a union of a sequence of mapping cylinders just as for spaces. It is easy to see that when r is positive, X_r is simply the spectrum $\Sigma^\infty(RP_r^\infty)$. In particular, $X_1 = \Sigma^\infty(RP^\infty)$.

We may now state our main theorem, whose proof is the culmination of the entire thesis. By S^c below, we mean the sphere spectrum, and by $[S^0, S^c]_2$ we mean the group $[S^0, S^c]$ modulo all odd-order torsion.

Theorem.

(i) If $k < c$ and $c > 0$, then $[X_k, S^c] = 0$.

(ii) If $k < c$ and $c = 0$, then $[X_k, S^0]$ is isomorphic to the 2-adic integers $\widehat{\mathbb{Z}}_2$ as filtered groups.

(iii) If $k < 0$ and $c < 0$, then there exists a map $f : X_k \to S^0$ such that $f^* : H^0 S^0 \to H^0 X_k$ is nonzero. Furthermore, the induced map

$$f^* : [S^0, S^c]_2 \to [X_k, S^c]$$

is an isomorphism if $k < c - 1$ and a surjection if $k = c - 1$.

We will now move on to the first step of our proof outline.

5.1 Atiyah-Hirzebruch spectral sequence computation

Our goal for this subsection is to show that the cohomotopy groups $[X^l_k, S^c]$ are finite and even finite 2-groups in some range. We start by computing the cellular homology groups of X^l_k. We use this data in conjunction with the Atiyah-Hirzebruch spectral sequence to reach our goal.

Recall that RP^n can be given a CW structure with one i-cell for each $i = 0, \dots, n$, such that for $i > 0$, the i-disk is attached to the $i-1$-skeleton of RP^n (which is RP^{i-1}) by the two-sheeted projection $\partial D^i = S^{i-1} \to RP^{i-1}$. Similarly, we can give RP^a_b a CW structure with one i-cell for each $i = 0, b, b+1, \dots, a$. The b-skeleton of RP^a_b is simply S^b. For $i > b$, the i-disk is attached to the $i-1$ skeleton of RP^a_b (which is RP^{i-1}_b) along the map $S^{i-1} \to RP^{i-1}_b$, which is just the composition of the two-sheeted projection above with the quotient $RP^{i-1} \to RP^{i-1}_b$.

Using the description of the cell structure of RP^a_b as $e^0 \cup e^b \cup \dots \cup e^a$, we may compute its cellular homology (with coefficients in $\mathbb{Z}$). The cellular chain complex is given by $C_p(RP^a_b; \mathbb{Z}) = \mathbb{Z}$ if $p = 0, b, \dots, a$ and zero otherwise. Recall that by the cellular boundary formula (the unnumbered theorem below theorem 2.35 in [13]), the boundary map of the cellular chain complex is given by $d_p(e^p) = D_p e^{p-1}$. The integer D_p is given by the degree of the composition

$$S^{p-1} \to RP^{p-1}_b \twoheadrightarrow S^{p-1} \ ,$$

where the first map is the attaching map of e^p and the second map is the map collapsing $RP^{p-1} - e^p$ to a point.

To determine the degree of the composition above, one can for example note that it is a smooth map between compact connected manifolds. Furthermore, notice that when restricting the map to any hemisphere of S^{p-1}, it is a diffeomorphism, and thus it is a local diffeomorphism. There is a theorem in differential topology that states if we have a local diffeomorphism between compact connected manifolds, then the degree of this map is given by

$$\sum_{x \in f^{-1}(y)} \deg f|_{U_x} \ ,$$

where y is a regular point and U_x is a neighborhood of x such that the restriction of the map to this open set is a diffeomorphism. If we pick the point $N = (1, 0, \dots, 0)$ from our map, its preimage is given by $\{N, -N\}$. Notice now that restricting the composition above to a neighborhood around N, we get a map homotopic to the identity, and restricting the composition above to a neighborhood around $-N$, we get a map homotopic to the antipodal map. The identity map has degree 1, and the antipodal map has degree p. Consequently, the degree of the composition is given by $1 + (-1)^p = D_p$.

We can now compute the cellular homology groups of RP^a_b, depending on the parity of a and b. If $p = 0$, then $H_p(RP^a_b) = \mathbb{Z}$ simply because R^a_b is path-connected. If $0 < p < b$ or $a < p$, then clearly $H_p(RP^a_b) = 0$. Assume $b < p < a$. If p is even, then $D_p = 2$, and thus d_p is injective, meaning that $\ker d_p = 0$, and thus that $H_p(RP^a_b) = 0$. If p is odd, then $D_p = 0$ and $D_{p-1} = 2$, implying that $H_p(RP^a_b) = \mathbb{Z}_2$. Let us now consider the edge cases: if a is even, then $\ker d_a = 0$ and thus $H_a(RP^a_b) = 0$, and if a is odd, then $H_a(RP^a_b) = \mathbb{Z}$, since $d_{a+1} = 0$. Finally, if b is even, then $D_{b+1} = 0$ and thus $H_b(RP^a_b) = \mathbb{Z}$ since d_b is necessarily zero, and if b is odd, then $D_{b+1} = 2$, and thus $H_b(RP^a_b) = \mathbb{Z}_2$. Summarizing our computations, we have that

$$H_p(RP^a_b, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0,\ p = b \text{ even},\ p = a \text{ odd} \\ \mathbb{Z}_2 & \text{if } p \text{ odd and } b < p < a, \quad \text{or if } p = b \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

The spectral sequence below is in fact constructed from the cohomological variant of the exact couple which we exhibited in the introduction in section 3. We proved convergence in that analogous case (theorem 3.5) and we argue that the proof is similar now. The computation of the E_2-page is easy to prove by some diagram chasing, so we take it for granted.

Theorem 5.1 (Atiyah-Hirzebruch cohomology spectral sequence)**.** *Let A^* be an unreduced additive generalized cohomology theory and let $\tilde{A}^*$ denote the reduced variant. Let X be a finite CW complex and X_p denote the p-skeleton of X. We have a convergent spectral sequence $\{E_n^{p,q}\}_{n\geq 1}$ such that*

- $E_2^{p,q} = H^p(X; A^q(pt))$,
- $$E_\infty^{p,q} = \frac{\ker\left(\tilde{A}^{p+q}(X) \xrightarrow{restrict} \tilde{A}^{p+q}(X_{p-1})\right)}{\ker\left(\tilde{A}^{p+q}(X) \xrightarrow{restrict} \tilde{A}^{p+q}(X_p)\right)} .$$

Define

$$F_t^s = \ker\left(\tilde{A}^t(X) \xrightarrow{\text{restrict}} \tilde{A}^t(X_s)\right) .$$

Then $F_t^\bullet$ is a decreasing filtration $\cdots \subseteq F_t^s \subseteq F_t^{s-1} \subseteq \ldots$ of $A^t(X)$ which is exhaustive (meaning $\cup_s F_t^s = A^t(X)$), and reaches both zero and the whole group $A^t(X)$ in finitely many steps. This is because we assume X is a finite CW-complex, so it is equal to its

n-skeleton for some n, implying that $F_t^s = 0$ for $s \geq n$, and also $F_t^s = A^t(X)$ for $s \leq 0$. The convergence in the theorem above may be written more succinctly now as

$$E_\infty^{p,q} \cong F_{p+q}^{p-1} / F_{p+q}^{p} \ .$$

Lemma 5.2. If $k < c < l$, then $[X_k^l, S^c]$ is a finite 2-group.

Proof. We will prove this with theorem 5.1. Notice that $X_k^L = \Sigma^{-2^\alpha}\Sigma^\infty RP_{k+2^\alpha}^{l+2^\alpha}$. For simplicity of notation, let $B_k^l = RP_{k+2^\alpha}^{l+2^\alpha}$. Recall that stable cohomotopy

$$\pi_s^*(-) = [\Sigma^\infty(-), S^*]$$

is a reduced additive cohomology theory on CW complexes. Furthermore, we may modify it so that it becomes the unreduced cohomology theory $\pi_s^*(- \sqcup pt)$. Also, note that $\pi_s^q(pt \sqcup pt) = \pi_s^q(S^0)$ which is equal to the stable homotopy group $\pi_0^s(S^q)$. The associated Atiyah-Hirzebruch spectral sequence has E_2 page $E_2^{p,q} = H^p(B_k^l; \pi_0(S^q))$ and it converges to $\pi_s^{p+q}(B_k^l) = [\Sigma^{2^\alpha} X_k^l, S^{p+q}] = [X_k^l, S^{p+q-2^\alpha}]$.

Since we are interested in computing the group $[X_k^l, S^c] = \pi_s^{2^\alpha+c}(B_k^l)$, we focus on the pairs of integers p, q such that $p + q - 2^\alpha = c$, or equivalently when $p + q = 2^\alpha + c$. By the universal coefficient theorem for cohomology,

$$E_2^{p,q} = H^p(B_k^l; \pi_0^s(S^q)) \cong \mathrm{Hom}_{\mathbb{Z}}(H_p(B_k^l), \pi_0^s(S^q)) \oplus \mathrm{Ext}_{\mathbb{Z}}^1(H_{p-1}(B_k^l), \pi_0^s(S^q)) \ . \quad (5.1)$$

When $q > 0$, $\pi_0^s(S^q)$ vanishes since the unstable homotopy groups vanish in this range, which can be seen by the cellular approximation theorem. When $q = 0$, $\pi_0^s(S^0) = \mathbb{Z}$, since it is known that $\pi_k(S^k) = \mathbb{Z}$ for all $k \geq 1$. Furthermore, Serre's finiteness theorem (theorem 1.1.8 in [26]) states that the unstable groups $\pi_{n+k}(S^n)$ are finite for $k > 0$ except in the case when $n = 2m$ and $k = 2m - 1$. Consequently, $\pi_{n+k}^s(S^n)) = \pi_n^s(S^{n-k})$ is finite for all $k > 0$, since in the colimit, we may pick a cofinal subsystem which avoids these exceptional cases. Therefore, the only case when $\pi_0^s(S^q)$ is finite is when $q = 0$.

Using (5.1) and the computation of the homology groups of B_k^l given above, it is easy to see that if $p + q = 2^\alpha + c$ and $q \leq 0$, then $E_2^{p,q}$ will always be a finite 2-group (including when it is zero also). We will prove it for one of the cases to show how it is done. The remaining cases follow from similar arguments.

If $q = 0$, then $\pi_0^s(S^q) = \mathbb{Z}$. Also, $p = 2^\alpha + c$, so $H_p(B_k^l)$ equals $\mathbb{Z}_2$ if p is odd and zero otherwise. In both cases, $\mathrm{Hom}_{\mathbb{Z}}(H_p(B_k^l, \mathbb{Z})) = 0$. If $p - 1 = 2^\alpha + k$, then $H_{p-1}(B_k^l)$ equals $\mathbb{Z}$ if $p-1$ is even and equals $\mathbb{Z}_2$ if $p-1$ is odd. It follows that $\mathrm{Ext}_{\mathbb{Z}}(H_{p-1}, \mathbb{Z})$ equals 0 or $\mathbb{Z}_2$. Finally, consider the case when $p - 1 > 2^\alpha + k$. Then $H_{p-1}(B_k^l)$ is $\mathbb{Z}_2$ or 0 depending on parity, and thus the Ext group is either $\mathbb{Z}_2$ or zero. In all of the cases, it follows by (5.1) is a direct sum of finite 2-groups.

Furthermore, notice that for $p > 2^\alpha L + 1$ $H_p(B_k^l)$ and $H_{p-1}(B_k^l)$ vanish, and thus by (5.1), $E_2^{p,q} = 0$ in this case. Consequently, for the case when $p + q = 2^\alpha + c$, the E_2 page is only non-vanishing for finitely many indices, all of which are finite 2-groups. Since the E_∞ page consists of subquotients of the E_2-page, the same holds here. The convergence of the Atiyah-Hirzebruch says that there is an isomorphism

$$E_\infty^{p,q} \cong F_{p+q}^{p-1}/F_{p+q}^{p} ,$$

where F_{p+q}^p is defined as above. Since $F_{p+q}^{2^\alpha+l} = 0$ (due to B_k^l being a CW complex of dimension $2^\alpha + l$), it follows in particular that $F_{p+q}^{2^\alpha+l-1} = F_{p+q}^{2^\alpha+l-1}/F_{p+q}^{2^\alpha+l}$ is a finite 2-group. Furthermore, the isomorphism above states that the successive filtration quotients of $\pi_s^{2^\alpha+c}(B_k^l)$ are finite 2-groups, meaning that each subgroup F_{p+q}^* has index a power of 2 in F_{p+q}^{*+1}. Consequently, F_{p+q}^s is a finite 2-group for each s. Since this filtration reaches $\pi_s^{2^\alpha+c}(B_k^l)$ in finitely many steps, it follows that$\pi_s^{2^\alpha+c}(B_k^l)$ is a finite 2-group.

□

By a completely analogous argument as above, by computing the E_2-page case-by-case using the universal coefficient theorem, we can also show the following lemma, whose proof we will omit.

Lemma 5.3. If $k \neq c$ or k is odd, then $[X_k^l, S^c]$ is finite, assuming $l > c$.

5.2 Comparing spectral sequences

When we introduced the A-module P, we claimed that the A-module structure of the total cohomology of finite and infinite stunted projective spaces will coincide with submodules and subquotients of P. It is time we proved this. First, we define the relevant submodules and subquotients of P. Let $P_k = \{x^i \mid i \geq k\}$ and for $k \leq n$, let $P_k^n = P_k/P_{n+1}$. Assume from now on that by cohomology we mean cellular homology with $\mathbb{Z}_2$ coefficients, unless explicitly stated otherwise.

After definition 2.32 of the Steenrod algebra, we discussed how the Steenrod algebra acts on the total cohomology of a spectrum. We remarked that for the suspension spectrum of a space, its cohomology as a spectrum is isomorphic as A-modules to the cohomology of its underlying space. Consequently, $H^*X_k^l \cong \Sigma^{-2^\alpha} H^*(RP_{2^\alpha k}^{2^\alpha+l})$ as graded A-modules, for some integer α.

To determine the A-module structure of $\Sigma^{-2^\alpha} H^*(RP_{2^\alpha k}^{2^\alpha+l})$, we first determine the ring structure of cohomology rings of the form $H^*(RP_b^a)$. It is easy to see that $H^*(RP_b^a) \cong P_b^a$ as graded $\mathbb{Z}_2$-modules for all $0 < b \leq a$. Recall that the cohomology ring $H^*(RP^a)$ is given by the truncated polynomial ring $\mathbb{Z}_2[x]/(x^{a+1})$, where $x \in H^1(RP^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is the nonzero element. By the long exact sequence in cohomology of the pair (RP^a, RP^b), one sees that the graded map induced by the quotient $H^*(RP_b^a) \to H^*(RP^a)$ is an injection. Consequently, the cohomology ring $H^*(RP_b^a)$ equals P_b^a as a ring.

Recalling definition 2.32 of the Steenrod algebra, we will determine now the A-module structure of $H^*(RP^a)$. Given $x \in H^*(RP^a) = \mathbb{Z}_2[x]/(x^{a+1})$. Define the total Steenrod square to be $\mathrm{Sq} := \sum_i \mathrm{Sq}^i$. Notice this sum is finite when evaluated at any element in the cohomology ring by axiom 4. By axiom 3, we have that $\mathrm{Sq}^1(x) = x^2$, and this is nonzero in $H^*(RP^a)$. Consequently, we have that $\mathrm{Sq}(x) = x + x^2 = x(1+x)$ by axiom 4. Note that the Cartan formula, axiom 5, implies that the total Steenrod square is multiplicative. Consequently, $\mathrm{Sq}(x^n) = \mathrm{Sq}(x)^n = x^n(1+x)^n = \sum_i \binom{n}{i} x^{n+i}$, where the binomial coefficients of course are taken modulo 2. By the elements in each graded component separately, we see that $\mathrm{Sq}^i(x^n) = \binom{n}{i} x^{n+i}$ for all i (where we have the convention that $\binom{n}{i} = 0$ for $n < i$. The subring $H^*(RP^a_b)$ has exactly the A-module structure as that of $H^*(RP^a)$ (in the range of elements x^n such that $b \leq n \leq a$).

We see thus that $H^*(RP^a_b)$ equals P^a_b as A-modules. It follows that $H^* X^l_k \cong \Sigma^{-2^\alpha} P^{2^\alpha + l}_{2^\alpha k} \cong P^l_k$ as graded A-modules. Furthermore, by Milnor's exact sequence (proposition 2.29), we have a short exact sequence of graded A-modules

$$0 \to \varprojlim_l{}^1 H^{*-1}(X^l_k) \to H^*(X_k) \to \varprojlim_l H^*(X^l_k) \to 0 \ .$$

The leftmost group is zero since the inverse system is in each degree composed of finite 2-groups and is thus trivially Mittag-Leffler. The rightmost group is equal to the inverse limit of P^l_k as $l \to \infty$, which is equal to P_k. Therefore, $H^*(X_k) \cong P_k$ as A-modules.

Notice that the E_2-page of the Adams spectral sequence computing the cohomotopy of X^l_k equals $\mathrm{Ext}^{s,t}_A(H^* S^0, H^* X^l_k) = \mathrm{Ext}^{s,t}_A(\mathbb{Z}_2, P_k)$. In the introduction of the previous section, we promised that $\mathrm{Ext}^{s,t}_A(\mathbb{Z}_2, P)$ equals $\mathrm{Ext}^{s,t}_A(\mathbb{Z}_2, P_k)$ in some appropriate range. It is time we proved this.

Let α be the element in $\mathrm{Ext}^{1,1-k}_A(\mathbb{Z}_2, P_k)$ associated to the extension

$$0 \to P_k \to P_{k-1} \to P^{k-1}_{k-1} = \Sigma^{k-1}\mathbb{Z}_2 \to 0 \ .$$

Lemma 5.4. The map $\mathrm{Ext}^{s,t}_A(\mathbb{Z}_2, P_k) \to \mathrm{Ext}^{s,t}_A(\mathbb{Z}_2, P)$ induced by inclusion is an isomorphism when $s - t > k$ and a surjection when $s - t = k$.

Proof. In the proof of lemma 4.20, we constructed a free resolution $F_\bullet \to \mathbb{Z}_2$ such that F_s is zero in degrees below s. We use this free resolution. The map $\mathrm{Hom}^t_A(F_s, P_k) \to \mathrm{Hom}^t_A(F_s, P)$ induced by inclusion is a surjection when $s - t \geq k$, since for every homomorphism $F_s \to P$ which lowers the degree by t, the image of this map will lie in degree at least $s - t$, and thus we may restrict the codomain to identify it as an element in $\mathrm{Hom}^t_A(F_s, P_k)$. Since F_s is by definition free, $\mathrm{Hom}_A(F_s, -)$ is an exact functor and will thus take injections to injections, implying that the map induced by inclusion will also be injective, independent of s and t. Consequently, the map $\mathrm{Hom}^t_A(F_s, P_k) \to \mathrm{Hom}^t_A(F_s, P)$ is an isomorphism when $s - t \geq k$.

By a quick diagram chase, one can prove from the isomorphism on the chain level that the induced map in homology $\mathrm{Ext}^{s,t}_A(\mathbb{Z}_2, P_k) \to \mathrm{Ext}^{s,t}_A(\mathbb{Z}_2, P)$ is an isomorphism when

$s-t>k$ and a surjection when $s-t=k$. Essentially, this follows from the fact that if we have a chain map as in the diagram below,

$$\begin{array}{ccccccccc} \dots \leftarrow & \mathrm{Hom}_A^t(F_{s+1},P_k) & \leftarrow & \mathrm{Hom}_A^t(F_s,P_k) & \leftarrow & \mathrm{Hom}_A^t(F_{s-1},P_k) & \leftarrow \dots \leftarrow & \mathrm{Hom}_A^t(F_0,P_k) \\ & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow \\ \dots \leftarrow & \mathrm{Hom}_A^t(F_{s+1},P) & \leftarrow & \mathrm{Hom}_A^t(F_s,P) & \leftarrow & \mathrm{Hom}_A^t(F_{s-1},P) & \leftarrow \dots \leftarrow & \mathrm{Hom}_A^t(F_0,P) \end{array}$$

where all vertical arrows not shown to the left are isomorphisms, then the vertical maps will always induce surjections in homology, and it will induce injections as long as the vertical map directly to the right of our given one is also an isomorphism. □

Below, we will determine the kernel of the map $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,P_k)\to\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,P)$ in the case when $s=t$. Before doing this, we will describe the diagonal $\mathrm{Ext}_A^{s,s}(\mathbb{Z}_2,\mathbb{Z}_2)$ in the E_2-page of the Adams spectral sequence computing the stable homotopy of spheres.

In the category of graded A-modules, an extension of $\Sigma^{k-1}\mathbb{Z}_2$ by $\Sigma^k\mathbb{Z}_2$ is a module with only a copy of $\mathbb{Z}_2$ in degrees k and $k-1$. By degree reasons, the A-module of such an extension is completely determined by the action of Sq^1, which will either act trivially or non-trivially. This means that there exists at most one nontrivial extension of $\Sigma^{k-1}\mathbb{Z}_2$ by $\Sigma^k\mathbb{Z}_2$. In fact, there is a nontrivial extension. Consider the short exact sequence

$$0\to\Sigma^k\mathbb{Z}_2\to P_{k-1}^k\to\Sigma^{k-1}\mathbb{Z}_2\to 0\ ,$$

where the first map is inclusion into the degree k part and the second map is projection onto the degree $k-1$ part. It is an element in $\mathrm{Ext}_A^{1,0}(\Sigma^{k-1}\mathbb{Z}_2,\Sigma^k\mathbb{Z}_2)=\mathrm{Ext}_A^{1,1}(\mathbb{Z}_2,\mathbb{Z}_2)$. We claim that this extension splits if and only if k is odd, meaning that it is nontrivial if and only if k is even. This implies that $\mathrm{Ext}_A^{1,1}(\mathbb{Z}_2,\mathbb{Z}_2)=\mathbb{Z}_2$.

For all $i\geq 2$ the elements Sq^i send all elements in P_{k-1}^k to zero by degree reasons, so the only possible nontrivial action A may have on P_{k-1}^k is by Sq^1. Notice that $\mathrm{Sq}^1(x^{k-1})=\binom{k-1}{1}x^k=(k-1)x^k$, which is zero if and only if k is odd. Consequently, when k is odd, P_{k-1}^k is a trivial A-module and thus the extension above splits, but when k is odd, then P_{k-1}^k is not isomorphic to $\Sigma^k\mathbb{Z}_2\oplus\Sigma^{k-1}\mathbb{Z}_2$ as an A-module, and thus the extension does not split. Denote the nontrivial element in $\mathrm{Ext}_A^{1,1}(\mathbb{Z}_2,\mathbb{Z}_2)$ by h_0. Furthermore, by theorem 2.3 in [12], $\mathrm{Ext}^{s,s}(Z_2,\mathbb{Z}_2)=\mathbb{Z}_2$ for all $s\geq 0$, with the generator h_0^s, where the product structure is given by the Yoneda product.

Lemma 5.5. In the case when $s-t=k$ in lemma 5.4, the map $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,P_k)\to\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,P)$ is a surjection with kernel spanned by αh_0^{s-1}. Furthermore, if k is even, then αh_0^{s-1} is nonzero for all $s\geq 1$. If k is odd, then $\alpha h_0=0$.

Proof. The short exact sequence $0\to P_k\to P_{k-1}\to\Sigma^{k-1}\mathbb{Z}_2\to 0$ induces the following long exact sequence of Ext groups:

$$\cdots\to\mathrm{Ext}_A^{s-1,t}(\mathbb{Z}_2,\Sigma^{k-1}\mathbb{Z}_2)\xrightarrow{\delta}\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,P_k)\xrightarrow{i_*}\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,P_{k-1})\to\dots$$

Since $s-t > k-1$, it follows that the map $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P_{k-1}) \to \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P)$ induced by inclusion is an isomorphism by lemma 5.4. Composing this isomorphism with the map i_* in the long exact sequence above, we get the exact sequence

$$\mathrm{Ext}_A^{s-1,t}(\mathbb{Z}_2, \Sigma^{k-1}\mathbb{Z}_2) \xrightarrow{\delta} \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P_k) \to \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P)\ ,$$

where the map to the right is the precisely the map we are interested in, namely the map induced by the inclusion $P_k \hookrightarrow P$. Furthermore, we can rewrite the domain of δ to $\mathrm{Ext}_A^{s-1,t+k-1}(\mathbb{Z}_2, \mathbb{Z}_2)$, and this is equal to $\mathrm{Ext}_A^{s-1,s-1}(\mathbb{Z}_2, \mathbb{Z}_2)$ since $s-t=k$. Consequently, we have an exact sequence

$$\mathrm{Ext}_A^{s-1,s-1}(\mathbb{Z}_2, \mathbb{Z}_2) \xrightarrow{\delta} \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P_k) \to \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P)\ .$$

The leftmost group above is isomorphic to $\mathbb{Z}_2$ with generator h_0^{s-1}. By proposition 3.32, the map δ is given by multiplying with $\alpha \in \mathrm{Ext}_A^{1,*}(\mathbb{Z}_2, P_k)$ on the left. By exactness, it follows that the kernel of the map we are interested in is spanned by the element αh_0^s.

Assume k is even. We have the following map of short exact sequences. Notice that the sequence above represents α and that the sequence below represents h_0, since k is even.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_k & \longrightarrow & P_{k-1} & \longrightarrow & \Sigma^{k-1}\mathbb{Z}_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \Sigma^k\mathbb{Z}_2 & \longrightarrow & P_{k-1}^k & \longrightarrow & \Sigma^{k-1}\mathbb{Z}_2 & \longrightarrow & 0 \end{array}$$

The map π is given by $\pi(\sum_i a_i x^i) = a_k$ in both cases. By the naturality of the long exact sequence in Ext, we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Ext}_A^{s-1,t}(\mathbb{Z}_2, \Sigma^{k-1}\mathbb{Z}_2) & \xrightarrow{\delta} & \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P_k) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi_*} \\ \mathrm{Ext}_A^{s-1,t}(\mathbb{Z}_2, \Sigma^{k-1}\mathbb{Z}_2) & \xrightarrow[\delta]{} & \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \Sigma^k\mathbb{Z}_2) \end{array}$$

By proposition 3.32, the upper horizontal map is given by multiplication with α and the lower horizontal map is given by multiplication with h_0. The upper δ thus sends the generator h_0^{s-1} to αh_0^{s-1} and the lower δ sends the generator h_0^{s-1} to h_0^s. By the commutativity of the diagram, it follows that $\pi_*(\alpha h_0^{s-1}) = h_0^s$. Since h_0^s is the generator of $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \Sigma^k\mathbb{Z}_2) = \mathrm{Ext}_A^{s,s}(\mathbb{Z}_2, \mathbb{Z}_2)$ it is nonzero, and thus αh_0^{s-1} must be nonzero in the case when k is even.

Assume now that k is odd. Then the extension $0 \to \Sigma^{k-1}\mathbb{Z}_2 \xrightarrow{i} P_{k-2}^{k-1} \to \Sigma^{k-2}\mathbb{Z}_2 \to 0$ represents h_0. From this short exact sequence we get the exact sequence

$$\mathrm{Ext}_A^{0,2-k}(\mathbb{Z}_2, \Sigma^{k-2}\mathbb{Z}_2) \xrightarrow{h_0\cdot(-)} \mathrm{Ext}_A^{1,2-k}(\mathbb{Z}_2, \Sigma^{k-1}\mathbb{Z}_2) \xrightarrow{i_*} \mathrm{Ext}_A^{1,2-k}(\mathbb{Z}_2, P_{k-2}^{k-1})\ ,$$

which is the same as having the exact sequence

$$\mathrm{Ext}_A^{0,0}(\mathbb{Z}_2,\mathbb{Z}_2) \xrightarrow{h_0\cdot(-)} \mathrm{Ext}_A^{1,1}(\mathbb{Z}_2,\mathbb{Z}_2) \xrightarrow{i_*} \mathrm{Ext}_A^{1,2-k}(\mathbb{Z}_2, P_{k-2}^{k-1})\ .$$

Since the generator h_0 lies in the image of the left map, it follows by exactness that $i_* = 0$. Below, we have a map of short exact sequences where the middle and right vertical maps are inclusions.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_k & \longrightarrow & P_{k-1} & \longrightarrow & \Sigma^{k-1}\mathbb{Z}_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & P_k & \longrightarrow & P_{k-2} & \longrightarrow & P_{k-2}^{k-1} & \longrightarrow & 0 \end{array}$$

This induces the commutative square in the diagram below by taking long exact sequences in Ext.

$$\begin{array}{ccc} \mathrm{Ext}_A^{1,2-k}(\mathbb{Z}_2,\Sigma^{k-1}\mathbb{Z}_2) & \xrightarrow{\delta} & \mathrm{Ext}_A^{2,2-k}(\mathbb{Z}_2,P_k) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Ext}_A^{1,2-k}(\mathbb{Z}_2,P_{k-2}^{k-1}) & \xrightarrow[\delta]{} & \mathrm{Ext}_A^{2,2-k}(\mathbb{Z}_2,P_k) \end{array}$$

Since $i_* = 0$, and the upper δ maps h_0 to αh_0, it follows by the commutativity of the diagram that $\alpha h_0 = 0$ in the case when k is odd. □

Notation. Let $E_*^{*,*}(l)$, $E_*^{*,*}(\infty)$ and $E_*^{*,*}(S^k)$ denote the Adams spectral sequence for computing the cohomotopy of X_k^l, X_k and the sphere spectrum S^k, respectively. Let $E_\infty^{s,t}(l)$, $E_\infty^{s,t}(\infty)$ and $E_\infty^{s,t}(S^k)$ denote the permanent cycles in the bidegree (s,t) for the respective spectral sequences. Furthermore, let $F^{*,*}(l)$ denote the filtration on the cohomotopy of X_k^l. More specifically, let $F^{s,t}(l)$ denote the subset of maps in $[X_k^l, S^{s-t}]$ of Adams filtration $\geq s$. Define $F^{*,*}(\infty)$ and $F^{*,*}(S^k)$ analogously.

In general, the inverse limit functor is not exact, which implies that it does not commute with taking homology of a chain complex. This means that the homology groups of the chain complex $\left(\varprojlim_l E_n^{*,*}(l), d_n^{*,*}\right)$ are not necessarily equal to the inverse limit of the homology of the chain complex $\left(E_n^{*,*}(l), d_n^{*,*}\right)$, which is equal to the inverse limit of $\left(E_{n+1}^{*,*}(l), d_{n+1}^{*,*}\right)$ by definition. In this case, however, all groups in the spectral sequence $E_*^{*,*}(l)$ are finite. This is because they are subquotients of $E_1^{s,t}(l) = \mathrm{Hom}_A^t(H^*K_s, H^*X_k^l)$, which is finite, due to K_s being of finite type and X_k^l being finite.

Consequently, all inverse limits involved consists of finite groups and thus trivially satisfy the Mittag-Leffler condition, implying that in this case, taking inverse limits is exact and thus that it commutes with taking homology. Therefore, taking the inverse limits of the bigraded groups $E_n^{*,*}(l)$ gives us a spectral sequence whose n:th page is equal to $\varprojlim_l E_n^{*,*}(l)$ for all $1 \leq n \leq \infty$. To prove the convergence of the spectral sequence of interest, namely the one computing $[X_k, S^*]$, we will show that it is isomorphic to the inverse limit spectral sequence given above. This will be done by induction on the pages. We start the base case on the second page.

Lemma 5.6. The map induced by the quotients

$$\operatorname{Ext}_A^{s,t}(\mathbb{Z}_2, P_k) \to \varprojlim_l \operatorname{Ext}_A^{s,t}(Z_2, P_k^l)$$

is an isomorphism.

Proof. Since P_k^l is a quotient of P_k, it follows that the $\mathbb{Z}_2$-dual $(P_k^l)^*$ is a submodule of P_k^* and it is easy to see that the natural map $\varinjlim_l (P_k^l)^* \to P_k^*$ induced by inclusions is an isomorphism. Applying the functor $\operatorname{Tor}_{s,t}^A(-, \mathbb{Z}_2)$ to this isomorphism, we obtain an isomorphism

$$\operatorname{Tor}_{s,t}^A(\varinjlim_l (P_k^l)^*, \mathbb{Z}_2) \xrightarrow{\cong} \operatorname{Tor}_{s,t}^A(P_k^*, \mathbb{Z}_2)\ .$$

Consider the group on the left hand side above. Let $F_\bullet \to \mathbb{Z}_2$ be a projective resolution of $\mathbb{Z}_2$. Then this group is equal to the homology of the chain complex $\varinjlim_l (P_k^l)^* \otimes F_\bullet$. By the tensor-hom adjunction, the tensor product functor $- \otimes F_k$ is a left adjoint functor for all k. Since left adjoint functors commute with direct limits, there is an isomorphism $\varinjlim_l (P_k^l)^* \otimes F_k \cong \varinjlim_l ((P_k^l)^* \otimes F_k)$, natural in F_k. The previously mentioned chain complex is thus isomorphic to $\varinjlim_l ((P_k^l)^* \otimes F_\bullet)$. Since the direct limit is an exact functor, and exact functors commute with homology, the homology of this chain complex is equal to $\varinjlim_l \operatorname{Tor}_{s,t}^A((P_k^l)^*, \mathbb{Z}_2)$, meaning we have an isomorphism

$$\varinjlim_l \operatorname{Tor}_{s,t}^A((P_k^l)^*, \mathbb{Z}_2) \xrightarrow{\cong} \operatorname{Tor}_{s,t}^A(P_k^*, \mathbb{Z}_2)\ .$$

Applying the functor $(-)^*$ here, and noting that the dual takes direct limits to inverse limits, we may apply the isomorphism given in lemma 4.19 to get the following diagram of isomorphisms.

$$\begin{array}{ccc}
\varinjlim_l \operatorname{Tor}_{s,t}^A((P_k^l)^*, \mathbb{Z}_2)^* & \xleftarrow{\cong} & \operatorname{Tor}_{s,t}^A(P_k^*, \mathbb{Z}_2)^* \\
\cong \updownarrow & & \updownarrow \cong \\
\varinjlim_l \operatorname{Ext}_A^{s,t}(\mathbb{Z}_2, P_k^l) & \dashleftarrow & \operatorname{Ext}_A^{s,t}(\mathbb{Z}_2, P_k)
\end{array}$$

The dotted arrow, constructed from the three other isomorphisms, gives us our desired isomorphism, which one can confirm is given by the natural map into the inverse limit induced by the projection map. □

Lemma 5.7. For every integer n, the map

$$E_n^{s,t}(\infty) \to \varprojlim_l E_n^{s,t}(l)$$

is an isomorphism.

Proof. We will prove this by induction on n. The base case is precisely the lemma above. Assume the result is true for n. By the naturality of the Adams spectral sequence (theorem 3.30), for each l the inclusion $X_k^l \to X_k$ induces a map of spectral sequences $E_*(\infty) \to E_*(l)$. This means that the map $E_n^{s,t}(\infty) \to E_n^{s,t}(l)$ induced by the inclusion commutes with the differentials in the respective spectral sequences, so that we have the following chain map, for all l.

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & E_n^{s-r,t-r+1}(\infty) & \xrightarrow{d_n} & E_n^{s,t}(\infty) & \xrightarrow{d_n} & E_n^{s+r,t+r-1}(\infty) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & E_n^{s-r,t-r+1}(L) & \xrightarrow[d_n]{} & E_n^{s,t}(L) & \xrightarrow[d_n]{} & E_n^{s+r,t+r-1}(L) & \longrightarrow & \dots
\end{array}$$

Also, since the map $E_n^{s,t}(\infty) \to E_n^{s,t}(L)$ commutes with the maps of the form $E_n^{s,t}(\tilde{L}+1) \to E_n^{s,t}(\tilde{L})$ induced by inclusion, it follows by the naturality of the Adams spectral sequence that we may take the inverse limits to obtain the commutative diagram below.

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & E_n^{s-n,t-n+1}(\infty) & \xrightarrow{d_n} & E_n^{s,t}(\infty) & \xrightarrow{d_n} & E_n^{s+n,t+n-1}(\infty) & \longrightarrow & \dots \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
\dots & \to & \varprojlim_L E_n^{s-n,t-n+1}(L) & \xrightarrow[\lim d_n]{} & \varprojlim_L E_n^{s,t}(L) & \xrightarrow[\lim d_n]{} & \varprojlim_L E_n^{s+n,t+n-1}(L) & \to & \dots
\end{array} \tag{5.2}$$

The bottom row in (5.2) is still a chain map by the functoriality of $\varprojlim(-)$. Furthermore, by the inductive hypothesis, it is an isomorphism of chain complexes.

This chain isomorphism induces an isomorphism between the homology of the top complex (which is by definition $E_{n+1}^{*,*}(\infty)$) and the homology of the bottom complex

$$H\left(\varprojlim_L E_n^{*,*}(L)\right) .$$

We will be done if we can show that the homology group above is isomorphic to

$$\varprojlim_L \left(H(E_n^{*,*}(L))\right) = \varprojlim_L \left(E_{n+1}^{*,*}(L)\right) .$$

As we discussed in remark 3.16, the spectral sequence $E_*^{*,*}(l)$ is finite in all bidegrees. The inverse system $E_n^{s,t}(l)$ is thus trivially Mittag-Leffler, meaning that the inverse limit is exact in this case, and thus commutes with homology.

□

Corollary 5.8. The map in lemma 5.7 restricts to an isomorphism

$$E_\infty^{s,t}(\infty) \xrightarrow{\cong} \varprojlim_l E_\infty^{s,t}(l)$$

on the E_∞ page .

Proof. Since the map in lemma 5.7 is a map of spectral sequences, it commutes with the differentials of the spectral sequence. It is then immediate that permanent cycles map to permanent cycles, so the codomain in the map above makes sense. Furthermore, given a permanent cycle $\{z_l\}_l$, it must for some integer N lie in $\varprojlim_l E_N^{s,t}(l)$, so by the lemma above, there is an element $z \in E_N^{s,t}(\infty)$ which maps to it. Since our map commutes with differentials, the fact that $\{z_l\}_l$ is a permanent cycle implies that z must be too, so the map is surjective. To prove injectivity, take a permanent cycle z whose image $\{z_l\}_l$ is zero. It follows then that for some integer N, $\{z_l\}_l = 0 \in \varprojlim_l E_N^{s,t}(l)$, but then $z \in E_N^{s,t}(\infty)$ is zero by the injectivity in lemma 5.7. □

5.3 Computation of stable cohomotopy groups

We will now exhibit an isomorphism between the groups these spectral sequences (as we will show later) converge to.

Definition 5.9. Given an abelian group G, we define the 2-*adic filtration* on G by $2^0 \cdot G \supseteq 2^1 \cdot G \supseteq 2^2 \cdot G \supseteq \cdots$.

Lemma 5.10. The natural map

$$[X_k, S^i] \to \varprojlim_l [X_k^l, S^i]$$

is an isomorphism for all i.

Proof. Consider Milnor's exact sequence (theorem 2.29)

$$0 \to \varprojlim_l{}^1 [X_k^l, S^{i-1}] \to [X_k, S^i] \to \varprojlim_l [X_k^l, S^i] \to 0 \ .$$

By lemma 5.3, unless $k = i-1$ and k is even, the inverse system $[X_k^l, S^{i-1}]$ consists only of finite groups, and thus is Mittag-Leffler, implying that the $\lim^1$ group above vanishes. There is only one case left to consider. Let $i - 1 = k$ and k be even.

Since we are interested in what happens in the limit, we may assume l is some integer such that $l > k+1$ and $l > k$. Applying the functor $[-, S^k]$ to the cofiber sequence $S^k \overset{j_l}{\hookrightarrow} X_k^l \twoheadrightarrow X_{k+1}^l$, we get the exact sequence

$$[X_{k+1}^l, S^k] \to [X_k^l, S^k] \xrightarrow{j_l^*} [S^k, S^k] \ .$$

The leftmost group is finite by lemma 5.3, so by exactness $\ker j_l^*$ is finite for all $l > k+1$, and thus this inverse system is Mittag-Leffler. The short exact sequence $0 \to \ker j_l^* \to [X_k^l, S^k] \to \mathrm{Im}\ j_l^* \to 0$ of inverse systems thus induces by lemma 1.6 an exact sequence

$$0 \to \varprojlim_l{}^1 [X_k^l, S^k] \to \varprojlim_l{}^1 \mathrm{Im}\ j_l^* \to 0\ ,$$

so our problem reduces to showing that the inverse system $\mathrm{Im}\ j_l^*$ is Mittag-Leffler.

Notice that $H^* X_k^l = P_k^l$ and that $H^* S^k = \Sigma^k \mathbb{Z}_2$ as graded A-modules. By the long exact sequence in cohomology associated to the cofibration sequence $S^k \hookrightarrow X_k^l \twoheadrightarrow X_{k+1}^l$, we have an exact sequence

$$H^k X_{k+1}^l \to H^k X_k^l \xrightarrow{j^*} H^k S^k\ .$$

Since the leftmost group is zero (by degree reasons), it follows that j^* is non-zero in degree k. Consequently, the map $j^* : H^* X_k^l \to H^* S^k$ is precisely the map $\pi : P_k^l \to \mathbb{Z}_2$ described in the proof of lemma 5.5 in the case when k is even. There, we showed that the induced map $(j^*)_* = \pi_* : \mathrm{Ext}_A^{s,s-k}(\mathbb{Z}_2, P_k^l) \to \mathrm{Ext}_A^{s,s}(\mathbb{Z}_2, \mathbb{Z}_2)$ sends αh_0^{s-1} to h_0^s.

Furthermore, the differential d_2 in $E_2^{*,*}(l)$ sends αh_0^{s-1} to an element in the group $\mathrm{Ext}_A^{s+2,s+1-k}(\mathbb{Z}_2, P_k^l)$. In particular, $(s+2)-(s+1-k) = k+1 > k$, so by lemma 5.4 and theorem 4.23, $\mathrm{Ext}_A^{s+2,s+1-k}(\mathbb{Z}_2, P_k^l) \cong \mathrm{Ext}_A^{s+2,s+1-k}(\mathbb{Z}_2, \mathbb{Z}_2)$. Also, since $(s+2)-(s+1-k) = k+1$, it means in particular that it is even. It is known that the group $\mathrm{Ext}_A^{\sigma,\tau}(\mathbb{Z}_2, \mathbb{Z}_2)$ vanishes for large enough σ unless $\sigma - \tau = 0$. This follows from Adams' vanishing theorem, the first theorem in [2]. Consequently, the element $d_2(\alpha h_0^{s-1})$ vanishes when $s \geq S$, for some large integer S. In general this shows that $d_r(\alpha h_0^{s-1}) \in \mathrm{Ext}_A^{s+r,s+r-1-k}(\mathbb{Z}_2, \mathbb{Z}_2)$ vanishes when $s \geq S$ and $r \geq 2$, so αh_0^{S-1} is a permanent cycle in $E_*^{*,*}(l)$.

By the convergence of the spectral sequence $E_*^{*,*}(S^k)$ for computing $[S^k, S^k]$, we have an isomorphism $E_\infty^{S,S}(\mathbb{Z}_2, \mathbb{Z}_2) \cong F^{S,S-k}(S^k)/F^{S+1,S-k+1}(S^k)$. In fact, it is known that $E_2^{\sigma,\sigma}(\mathbb{Z}_2, \mathbb{Z}_2) = E_\infty^{\sigma,\sigma}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ for all $\sigma \geq 0$; see theorem 14.7 in [27]. We claim that $h_0^S \in E_\infty^{S,S}(\mathbb{Z}_2, \mathbb{Z}_2)$ is represented by the map $2^S \cdot \mathrm{id} \in F^{S,S-k}(S^k)/F^{S+1,S-k+1}(S^k)$.

By lemma 9.19 in [21], the 2-adic filtration is contained in the Adams filtration, so $2^S \cdot [S^k, S^k] \subseteq F^{S,S-k}(S^k)$ for all S. It follows that

$$2^S \cdot [S^k, S^k]/2^{S+1} \cdot [S^k, S^k] \subseteq F^{S,S-k}(S^k)/F^{S+1,S-k+1}(S^k)\ .$$

Since $2^S \cdot \mathrm{id}$ is a nonzero element in the leftmost group, it is a (and thus the) nonzero element in the rightmost group, proving that h_0^S is represented by $2^S \cdot \mathrm{id}$.

By the convergence of the spectral sequence $E_*^{*,*}(l)$, it follows that the permanent cycle αh_0^{S-1} is represented by some map $f : X_k^l \to S^k$. By the naturality of the Adams spectral sequence, the following diagram commutes.

$$\begin{array}{ccc}
\alpha h_0^{S-1} \in E_\infty^{S,S-k}(l) & \xrightarrow{\quad j^* \quad} & E_\infty^{S,S}(S^k) \ni h_0^S \\
\cong \uparrow & & \uparrow \cong \\
f \in F^{S,S-k}(l)/F^{S+1,S-k+1}(l) & \xrightarrow[j^*]{} & F^{S,S-k}(S^k)/F^{S+1,S-k+1}(S^k) \ni 2^S \cdot \mathrm{id}
\end{array}$$

By the commutativity of the diagram, f maps to $2^S \cdot \mathrm{id}$. In particular, $2^S \cdot \mathrm{id} \in \mathrm{Im}\ j^*$, meaning that $2^S \cdot [S^k, S^k] \subseteq \mathrm{Im}\ j^*$, and thus that the order of the cokernel coker j^* is bounded above by $|[S^k, S^k]/2^S[S^k, S^k]| = 2^S$, so Im j_l^* has index at least 2^S in $[S^k, S^k]$. By the commutativity of the diagram

$$\begin{array}{ccc}
[X_k^{l+1}, S^k] & \xrightarrow{j_{l+1}^*} & [S^k, S^k] \\
\downarrow & & \| \\
[X_k^l, S^k] & \xrightarrow[j_l^*]{} & [S^k, S^k]
\end{array}$$

it follows that Im j_l^* monotonically decreases as l increases. By the boundedness of the index, it follows that Im j^* eventually becomes stationary for large enough l. Therefore, the inverse system Im j_l^* is Mittag-leffler. □

The isomorphism exhibited above respects Adams filtrations.

Lemma 5.11. $f \in [X_k, S^t]$ has Adams filtration $\geq s$ if and only if the image in the inverse limit $\{f_l\}_l \in \varprojlim_l [X_k^l, S^t]$ has Adams filtration $\geq s$:

Proof. Let $\cdots \to Y^2 \to Y^1 \to S^t$ be an Adams tower for S^t. Consider the following commutative diagram, where the horizontal maps are given by composing with the appropriate maps in the Adams tower.

$$\begin{array}{ccc}
\tilde{f} \in [X_k, Y^s] & \xrightarrow{\qquad} & [X_k, S^t] \ni f \\
\downarrow & & \downarrow \\
\{\tilde{f}_l\}_l \in \varprojlim_l [X_k^l, Y^s] & \longrightarrow & \varprojlim_l [X_k^l, S^t] \ni \{f_l\}_l
\end{array}$$

The element $f : X_k \to S^t$ lies in the upper right corner, and its image along the rightmost map is $\{f_l\}_l$. Notice that f having Adams filtration $\geq s$ means precisely that it lies in the image of the upper map. Similarly, f_l has Adams filtration $\geq s$ for all l precisely when $\{f_l\}_l$ lies in the image of the bottom map.

If f has Adams filtration $\geq s$, then there is an element $\tilde{f}$ which maps to f along the upper map. The image of $\tilde{f}$ in the inverse limit gives us a consistent sequence $\{\tilde{f}_l\}_l$ which maps to $\{f_l\}_l$, meaning that f_l has Adams filtration $\geq s$. Conversely, assume now that f_l has Adams filtration $\geq s$ for all l. Then there is an element $\{\tilde{f}_l\}_l$ which maps to $\{f_l\}_l$ along the bottom map. The leftmost map is surjective by the Milnor exact sequence. By the commutativity of the diagram, there is an element $\tilde{f}$ which maps to f. □

Corollary 5.12. The isomorphism given in lemma 5.10 restricts to an isomorphism

$$F^{s,t}(\infty) \xrightarrow{\cong} \varprojlim_l F^{s,t}(l) \ .$$

We are starting to close in on why the spectral sequence $E_*^{*,*}(\infty)$ converges. By the convergence of the spectral sequences $E_*^{*,*}(l)$, we have a short exact sequence

$$0 \to F^{s+1,t+1}(l) \to F^{s,t}(l) \to E_\infty^{s,t}(l) \to 0 \tag{5.3}$$

for all s, t, l. Taking inverse limits, and using the isomorphism from the corollary above and from lemma 5.10, we get an exact sequence

$$0 \to F^{s+1,t+1}(\infty) \to F^{s,t}(\infty) \to E_\infty^{s,t}(\infty) \ . \tag{5.4}$$

This means we have an injection from the successive filtration quotients to the permanent cycles of the spectral sequence $E_*^{*,*}(\infty)$. Notice that the rightmost map must not necessarily be surjective, since inverse limits are in general not right exact. To prove convergence, it thus remains to prove that this map is a surjection. Surprisingly, we will use a lemma from graph theory to prove this.

Lemma 5.13 (König's Lemma)**.** Given any infinite connected graph where each vertex has finitely many adjacent vertices, there is an infinitely long path that does not repeat vertices.

Proof. Take a vertex v_1. Since the graph is connected, all of the infinitely many vertices can be reached by a path starting at v_1 without repeating any vertex. Each such path must pass through one of the finitely many neighbors of v_1. By a sort of pigeonhole principle, it follows that there is some vertex, call it v_2, neighboring v_1 and through which infinitely many of these paths must pass through. Since the graph is connected and infinite all of the infinitely many vertices can be reached by a path starting at v_2 without repeating any vertex. A similar argument as above allows us to pick a new vertex v_3, extending the path $v_1 \to v_2$, and then we iterate the procedure, extending the path $v_1 \to v_2 \to v_3$. □

Theorem 5.14. *The Adams spectral sequence $E_*^{*,*}(\infty)$ converges.*

Proof. It remains to show that the rightmost map in $(*)$ is surjective. Given a permanent cycle $x \in E_\infty^{s,t}(\infty)$ its image $\{x_l\}_l$ under the map in corollary 5.8 is a permanent cycle. By the convergence of the spectral sequences $E_*^{*,*}(l)$, each $x_l \in E_\infty^{s,t}(l)$ is represented by a map $f_l : X_k^l \to S^{s-t}$ under the surjection given in (5.3). Since we care about what happens in the limit, we may assume $l > k$ and $l > s - t$. By lemma 5.2 the groups $[X_k^l, S^{s-t}]$ are all finite in this range. The naturality of the Adams spectral sequence implies that the inclusion $X_k^l \hookrightarrow X_k^{l+1}$ induces the following map of short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{s+1,t+1}(l+1) & \longrightarrow & F^{s,t}(l+1) & \longrightarrow & E_\infty^{s,t}(l+1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^{s+1,t+1}(l) & \longrightarrow & F^{s,t}(l) & \longrightarrow & E_\infty^{s,t}(l) & \longrightarrow & 0 \end{array}$$

The commutativity of the rightmost square means that given any representative f_{l+1} of the permanent cycle x_{l+1}, we can pick the restriction f_l of f_{l+1} as a representative of x_l.

Define a graph consisting of vertices all elements in $F^{s,t}(l)$ as l ranges over L_0 for some fixed integer L_0 greater than k and $s - t$. Let two vertices be adjacent if and only if one is the restriction of the other. This graph is infinite. By the last sentence in the paragraph above, this graph is connected. Lemma 5.2 shows that the groups $[X_k^l, S^{s-t}]$ are all finite, so the subgroups $F^{s,t}(l)$ are finite, and thus each vertex has finitely many neighbors.

By lemma 5.13, it follows that there is an infinitely long path in this graph that does not repeat vertices. This means precisely that there is a sequence of maps $\{f_l\}_{l \geq L_0} \in \prod_{l \geq L_0} F^{s,t}(l)$ such that the restriction of f_{l+1} to X_k^l is f_l, and this is equivalent to the sequence $\{f_l\}_l$ being an element in the inverse limit $\varprojlim_l F^{s,t}(l)$. By corollary corollary 5.12, this element pulls back to an element $f \in F^{s,t}(\infty)$, which gets mapped to our permanent cycle $z \in E_\infty^{s,t}(\infty)$ by the commutativity of the following diagram.

$$\begin{array}{ccc} f \in F^{s,t}(\infty) & \longrightarrow & E_\infty^{s,t}(\infty) \ni x \\ \downarrow & & \downarrow \\ \{f_l\}_l \in \varprojlim_l F^{s,t}(l) & \longrightarrow & \varprojlim_l E_\infty^{s,t}(l) \ni x_l \end{array}$$

This implies that the rightmost map in (5.4) is surjective, completing the proof. □

Now that we have proven convergence, we move on to proving a concrete fact about the cohomotopy groups $[X_k, S^t]$.

Lemma 5.15. If $k < t$, then maps $f : X_k \to S^t$ are of infinite Adams filtration only if $f = 0$.

Proof. Let $\cdots \to Y^2 \to Y^1 \to S^t$ be an Adams tower for S^t. For each $l > t$ we have the following commutative diagram.

$$\begin{array}{ccc} \tilde{f} \in [X_k, Y^s] & \longrightarrow & [X_k, S^t] \ni f \\ \downarrow & & \downarrow \\ \tilde{f}_l \in [X_k^l, Y^s] & \longrightarrow & [X_k^l, S^t] \ni f_l \end{array}$$

The map $f : X_k \to S^t$ being of Adams filtration $\geq s$ means precisely that there is a $\tilde{f}$ which maps into it as above. Mapping f and $\tilde{f}$ along the vertical maps, we see by the commutativity of the diagram that the restriction f_l is of Adams filtration $\geq s$. Assuming f is of infinite Adams filtration, it follows thus that f_l is of infinite filtration.

By lemma 5.2 the group $[X_k^l, S^t]$ is always a 2-group, so by lemma 3.28, it follows that $[X_k^l, S^t]$ has no elements of infinite filtration, except the constant map, so $f_l = 0$ for all $l > t$. Consequently, the map f gets mapped to 0 along the map $[X_k, S^t] \to \varprojlim_l [X_k^l, S^t]$. By the injectivity of this map, due to lemma 5.10, it follows that $f = 0$. □

We can finally prove our main theorem! By $[S^0, S^c]_2$, we mean the quotient of $[S^0, S^c]$ by all odd-order torsion.

Theorem 5.16.

(i) If $k < c$ and $c > 0$, then $[X_k, S^c] = 0$.

(ii) If $k < c$ and $c = 0$, then $[X_k, S^0]$ is isomorphic to the 2-adic integers $\widehat{\mathbb{Z}}_2$ as filtered groups.

(iii) If $k < 0$ and $c < 0$, then there exists a map $f : X_k \to S^0$ such that $f^* : H^0 S^0 \to H^0 X_k$ is nonzero. Furthermore, the induced map

$$f^* : [S^0, S^c]_2 \to [X_k, S^c]$$

is an isomorphism if $k < c - 1$ and a surjection if $k = c - 1$.

Proof of part (i). When $k < c$, it follows by lemma 5.4 and theorem 4.23 that $E_2^{s,t}(\infty) = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, P_k) \cong \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$ when $s - t = c$. Note that if $s - t = c$, then since $c > 0$, it follows in particular that $s > t$. We will show that in this case, the latter Ext group vanishes.

In the proof of lemma 4.20 we constructed a free resolution $F_\bullet \to \mathbb{Z}_2$ of $\mathbb{Z}_2$ as an A-module such that the t:th graded component $(F_s)^t$ of F_s vanishes when $t < s$. The group $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$ equals the homology at the group $\mathrm{Hom}_A^t(F_s, \mathbb{Z}_2)$. Notice that this Hom group is completely determined by where the homomorphisms send the elements in $(F_s)^t$ because the codomain of the maps are concentrated in degree zero, but in the case when $s > t$, this graded component of F_s is zero, and thus the Hom group is zero, finally implying that the homology at that group is zero.

Since $E_2^{s,t}(\infty) = 0$ for all $s - t = c$, it follows that the subquotients $E_\infty^{s,t}(\infty)$ vanish at these indices too. By the convergence of the spectral sequence, it follows that $F^{s,t}(\infty)/F^{s+1,t+1}(\infty) = 0$ for all $s - t = c$, implying that all $F^{s,t}(\infty) = F^{s+1,t+1}(\infty)$ for all $s - t = c$. Consequently, all maps in $[X_k, S^c]$ have infinite Adams filtration, and thus the group must be zero by lemma 5.15. □

Proof of part (ii). In this case, it follows by lemma 5.4 and theorem 4.23 that $\phi_* : \mathrm{Ext}_A^{s,s-i}(\mathbb{Z}_2, \mathbb{Z}_2) \to \mathrm{Ext}_A^{s,s-i}(\mathbb{Z}_2, P_k) = E_2^{s,s-i}(\infty)$ is an isomorphism for $i = 0, 1$. Recall that $\mathrm{Ext}_A^{s,s}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ for all $s \geq 0$. Also, as proven in the beginning of the proof of part (i), since $s > s-1$, it follows that $\mathrm{Ext}_A^{s,s-1}(\mathbb{Z}_2, \mathbb{Z}_2) = 0$ for all s. Consequently, the permanent cycles h_0^s map to permanent cycles $\phi_*(h_0^s)$ in $E_2^{s,s}(\infty)$, since the codomain of the differential at bidegree s, s is a subquotient of $E_2^{s-r,s+r-1}(\infty)$, which is zero, and thus necessarily $d_r(\phi_*(h_0^s)) = 0$. We will show that none of the elements $\phi_*(h_0^s)$ are boundaries.

Suppose, to the contrary, that $\phi_*(h_0^s) = d_r(x)$ for some $s \geq 0$, $r \geq 2$ and $x \in E_r^{s+r,s-r+1}(\infty)$. Since the differentials are a derivation with respect to the given product structure on the Adams spectral sequence by proposition 3.33, it follows that $d_r(x \cdot h_0^t) = d_r(x) \cdot h_0^t + x \cdot d_r(h_0^t) = \phi_*(h_0^s)h_0^t = \phi_*(h_0^{s+t})$. This element is nonzero, and thus $x \cdot h_0^t$ must be a nonzero element in $E_r^{s+r+t,s-r+1+t}(\infty)$ for all $t \geq 0$. It follows that there is a nonzero element in $E_2^{s+r+t,s-r+1+t}(\infty)$ for all t. Note that $s+r+t-(s-r+1+t) = 2r-1 > k$, so by lemma 5.4 and theorem 4.23, there is a nonzero element in $E_2^{s+r+t,s-r+1+t}(S^0)$ for all t. Since $s+r+t-(s-r+1+t) = 2r-1$ is in particular not zero, it follows that $E_2^{s+r+t,s-r+1+t}(S^0)$ vanishes for large enough t, because, as previously stated, $E_2^{\sigma,\tau}(S^0) = \mathrm{Ext}_A^{\sigma,\tau}(\mathbb{Z}_2, \mathbb{Z}_2)$ vanishes for large enough σ unless $\sigma - \tau = 0$ by Adams' vanishing theorem. This leads to a contradiction. Therefore, none of the elements $\phi_* h_0^s$ are boundaries, from which it also follows that $E_2^{s,s}(\infty) = E_\infty^{s,s}(\infty) = \mathbb{Z}_2$.

By the convergence of $E_*^{*,*}(\infty)$, it follows that the element $h_0^0 \in E_\infty^{0,0}(\infty)$ is represented by the nonzero element in $F^{0,0}(\infty)/F^{1,1}(\infty)$, call it f. By lemma 3.23 the induced map f^* in cohomology is not zero, since f is not of Adams filtration ≥ 1. Consequently, the map $f^* : H^*S^0 = \mathbb{Z}_2 \to H^*X_k = P_k$ is precisely the map $\phi : \mathbb{Z}_2 \to P_k$, which is the map that induces the isomorphism $E_2^{s,s}(S^0) \cong E_2^{s,s}(\infty)$ by lemma 5.4 and theorem 4.23. Since these groups consist of permanent cycles which are not boundaries, it follows that f induces an isomorphism $E_\infty^{s,s}(S^0) \to E_\infty^{s,s}(\infty)$. By the convergence of these spectral sequences, it follows that we have an isomorphism of filtration quotients

$$F^{s,s}(S^0)/F^{s+1,s+1}(S^0) \cong F^{s,s}(\infty)/F^{s+1,s+1}(\infty)$$

for all s. In fact, we also have this isomorphism of filtration quotients with respect to the 2-adic filtration on the group $[S^0, S^0]$, because as we showed in the proof of lemma 5.10, $2^s[S^0, S^0]/2^{s+1}[S^0, S^0]$ is equal to the filtration quotient on the left hand side above.

The isomorphism $[S^0, S^0] \to \mathbb{Z}$ given by sending the identity map to 1 makes the square below commute.

$$\begin{array}{ccc} [S^0, S^0] & \xrightarrow{g \mapsto g \circ f} & [X_k, S^0] \\ {\scriptstyle\cong}\downarrow & & \| \\ \mathbb{Z} & \xrightarrow[k \mapsto k \cdot f]{} & [X_k, S^0] \\ \downarrow & \nearrow_{k \mapsto k \cdot f} & \\ \widehat{\mathbb{Z}}_2 & & \end{array} \tag{5.5}$$

Furthermore, since $[X_k^l, S^0]$ is a finite 2-group by lemma 5.2, it has a canonical $\widehat{\mathbb{Z}}_2$-module structure given by scalar multiplication with the power series $\sum_i a_i 2^i \in \widehat{\mathbb{Z}}_2$. The inverse limit of these groups, which is $[X_k, S^0]$ by lemma 5.10, is thus also a $\widehat{\mathbb{Z}}_2$-module. The diagonal map in the diagram above (call it ψ) is given by multiplication with a 2-adic integer and it makes the triangle commute, where the vertical map in this triangle is the natural map from $\mathbb{Z}$ to its 2-adic completion.

Let the 2-adic integers $\widehat{\mathbb{Z}}_2$ be a filtered group with respect to the 2-adic filtration. Since homomorphisms distribute over addition, the map ψ in (5.5) preserves the 2-adic filtration, meaning that $\psi(2^s \cdot \widehat{\mathbb{Z}}_2) \subseteq 2^s[X_k, S^0]$. Recall that the 2-adic filtration is contained in the Adams filtration, meaning that $2^s[X_k, S^0] \subseteq F^{s,s}(\infty)$, and thus that $\psi(2^s \cdot \widehat{\mathbb{Z}}_2) \subseteq F^{s,s}(\infty)$ for all s. This means ψ is a map of filtered groups with respect to the Adams filtration on its codomain. By the same argument, it follows that the top horizontal map in (5.5) is a map of filtered groups, with respect to the 2-adic filtration on $[S^0, S^0]$ and Adams filtration on the codomain.

Consider the two vertical maps in the first column of (5.5). The first map is an isomorphism, and thus trivially induces an isomorphism of filtration quotients (with respect to the 2-adic filtrations). Furthermore, it is easy to verify that the second map does so as well, meaning that $2^s\mathbb{Z}/2^{s+1}\mathbb{Z} \cong 2^s\widehat{\mathbb{Z}}_2/2^{s+1}\widehat{\mathbb{Z}}_2$ for all s. By the commutativity of (5.5), the map ψ induces an isomorphism of filtration quotients since the top horizontal map f^* does so, as was shown above.

It is easy to see that $\bigcup_{\geq 0} 2^s\widehat{\mathbb{Z}}_2 = \widehat{\mathbb{Z}}_2$, $\bigcap_s 2^s\widehat{\mathbb{Z}}_2 = 0$ and that the Adams filtration exhausts $[X_k, S^0]$. By lemma 5.15, it follows that $\bigcap_s F^{s,s}(\infty) = 0$. Once we prove that

$$\varprojlim_s{}^1\, 2^s\widehat{\mathbb{Z}}_2 = 0 \ ,$$

all criteria for lemma 3.11 will be met, and thus we can conclude that $\psi : \widehat{\mathbb{Z}}_2 \to [X_k, S^0]$ is an isomorphism of filtered groups. We have a short exact sequence of inverse systems $0 \to 2^s\widehat{\mathbb{Z}}_2 \to \widehat{\mathbb{Z}}_2 \to \mathbb{Z}_{2^s} \to 0$ where the surjection is given by $\sum_i a_i 2^i \mapsto \sum_{i<s} a_i 2^i$ and the middle inverse system consists of identity maps. Taking inverse limits, we get by lemma 1.6 an exact sequence

$$\widehat{\mathbb{Z}}_2 \to \widehat{\mathbb{Z}}_2 \to \varprojlim_s{}^1\, 2^s\widehat{\mathbb{Z}}_2 \to 0 \ .$$

In fact, it is easy to verify that the leftmost map is an isomorphism, which is in particular surjective. By exactness, the $\lim^1$ group vanishes, completing the proof.

□

Proof of part (iii). Such a map $f : X_k \to S^0$ must exist, because there is an element in $\hat{\mathbb{Z}}_2$ which is not in $2 \cdot \hat{\mathbb{Z}}_2$, and thus by part (ii) there is an element in $[X_k, S^0]$ which is not of Adams filtration ≥ 1, and thus the map f exists due to lemma 3.23.

By assumption, the induced map $f^* : H^*S^0 = \mathbb{Z}_2 \to H^*X_k = P_k$ is precisely the map $\phi : \mathbb{Z}_2 \to P_k$. It follows thus by lemma 5.4 and theorem 4.23 that f induces an isomorphism $E_2^{s,t}(S^0) \cong E_2^{s,t}(\infty)$ when $s - t > k$. By induction on n, it follows that for all $n \geq 2$, f induces an isomorphism on the E_n-page when $s - t > k + 1$ and a surjection when $s - t = k + 1$ by a similar argument as in the proof of lemma 5.4. It follows that f induces an isomorphism on the E_∞-page when $s - t > k + 1$, and that it must induce a surjection when $s - t = k + 1$. From the convergence of these two spectral sequences, we see that f induces an isomorphism of filtration quotients $F^{s,t}(S^0)/F^{s+1,t+1}(S^0) \cong F^{s,t}(\infty)/F^{s+1,t+1}(\infty)$ when $s-t > k+1$ and a surjection when $s - t = k + 1$. We are interested in the case when $s - t = c$, so assume this from now on.

Let $T^c = \bigcap_n F^{s+n,t+n}(S^0)$. By lemma 3.28, this is precisely the odd-order torsion in $[S^0, S^c]$. By some isomorphism theorem, we have that

$$F^{s,t}(S^0)/F^{s+1,t+1}(S^0) \cong (F^{s,t}(S^0)/T^c)\Big/(F^{s+1,t+1}(S^0)/T^c) \ .$$

If we filter the group $[S^0, S^c]_2 = [S^0, S^c]/T^c$ by $F^{s+*,t+*}/T^c$, it is easy to see that this filtration exhausts the group and that $\bigcap_n F^{s+n,t+n}/T^c = 0$. Furthermore, since $[S^0, S^c]$ is finite by Serre's finiteness theorem, it is clear that

$$\varprojlim_n{}^1 F^{s+n,t+n}/T^c = 0 \ .$$

Therefore, if $c > k + 1$, then $f : [S^0, S^c]_2 \to [X_k, S^c]$ induces an isomorphism by lemma 3.12 and if $c = k + 1$, then f induces a surjection by lemma 3.13.

□

References

[1] J.F. Adams. Vector fields on spheres. *Annals of Mathematics*, 1962.

[2] J.F. Adams. A periodicity theorem in homological algebra. *Mathematical Proceedings of the Cambridge Philosophical Society*, 1965.

[3] J.F. Adams. Operations of the n:th kind in k-theory, and what we don't know about RP^∞. *Cambridge University Press*, 1974.

[4] J.F. Adams. *Stable Homotopy and Generalized Homology*. The University of Chicago Press, 1974.

[5] José Adem. The relations on steenrod powers of cohomology classes. In *Algebraic Geometry and Topology*, pages 191–238. Princeton University Press, 1957.

[6] L.L. Avramov and Ragnar-Olaf Buchweitz. Lower bounds for betti numbers. *Compositio Mathematica*, 1991.

[7] Tilman Bauer. Characteristic classes in topology, geometry and algebra, 2024. URL: `https://github.com/tilmanbauer/characteristic-classes/blob/master/characteristic-classes.pdf`.

[8] Edgar Brown. Cohomology theories. *Annals of Mathematics*, 1962.

[9] Robert Bruner. An adams spectral sequence primer, 2009. URL: `http://www.rrb.wayne.edu/papers/adams.pdf`.

[10] Robert Bruner and John Rognes. The cohomology of the mod 2 steenrod algebra, 2021. URL: `https://arxiv.org/pdf/2109.13117`.

[11] Gunnar Carlsson. Equivariant stable homotopy and segal's burnside ring conjecture. *Annals of Mathematics*, 1984.

[12] Paul Goerss. The adams-novikov spectral sequence and the homotopy groups of spheres, 2007. URL: `https://people.math.rochester.edu/faculty/doug/otherpapers/goerss-ans.pdf`.

[13] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[14] Allen Hatcher. *Spectral Sequences in Algebraic Topology*. Unpublished, 2004.

[15] I.M. James. *The Topology of Stiefel Manifolds*. Cambridge University Press, 1976.

[16] Erkki Laitinen. On the burnside ring and stable cohomotopy of a finite group. *Mathematica Scandinavica*, 1979.

[17] T.Y. Lin and H.R. Margolis. Homological aspects of modules over the steenrod algebra. *Journal of Pure and Applied Algebra*, 1974.

[18] W.H. Lin. The adams-mahowald conjecture on real projective spaces. *Mathematical Proceedings of the Cambridge Philosophical Society*, 1979.

[19] W.H. Lin. On conjectures of mahowald, segal and sullivan. *Mathematical Proceedings of the Cambridge Philosophical Society*, 1979.

[20] W.H. Lin, M.E. Mahowald, and J.F. Adams. Calculation of lin's ext groups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 1979.

[21] John McCleary. *A User's Guide to Spectral Sequences.* Cambridge University Press, 2000.

[22] John Milnor. The steenrod algebra and its dual. *Annals of Mathematics*, 1957.

[23] John Milnor and John Moore. On the structure of hopf algebras. *Annals of Mathematics*, 1965.

[24] R.E. Mosher and M.C. Tangora. *Cohomology Operations and Applications in Homotopy Theory.* Harper & Row Publishers, 1968.

[25] R.M.F. Moss. On the composition pairing of adams spectral sequences. *Journal of the Lond Mathematical Society*, 1968.

[26] Douglas Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres.* Academic Press, 1986.

[27] John Rognes. Introduction to the adams spectral sequence, 2015. URL: `https://www.mn.uio.no/math/personer/vit/rognes/kurs/mat9580v15/adams-sp-seq.010615.pdf`.

[28] N.E. Steenrod and D.B.A. Epstein. *Cohomology Operations.* Princeton University Press, 1962.

[29] Charles Weibel. *An Introduction to Homological Algebra.* Cambridge University Press, 1994.