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From Abstraction to Action: Exploring Symmetry through Group and Representation Theory

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From Abstraction to Action:
Exploring Symmetry through Group and Representation Theory

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Abstract

This thesis examines the mathematical structure of symmetry found throughout group theory and representation theory. Using group theory as a foundation, it formalizes groups as abstract models of symmetry to develop key algebraic ideas. Representation theory is then introduced as a means of concretely realizing these symmetries through linear transformations on vector spaces. The final chapter synthesizes these ideas by defining symmetry as invariance under transformation and illustrating how representation theory gives form to the symmetries encoded in group theory. Throughout, the study emphasizes symmetry not only as a mathematical concept but as a unifying principle across algebraic structures.

Sammanfattning

Detta arbete undersöker den matematiska strukturen av symmetri inom gruppteori och representationsteori. Med gruppteorin som grund formaliseras grupper som abstrakta modeller av symmetri för att utveckla centrala algebraiska idéer. Representationsteori introduceras därefter för att konkretisera dessa symmetrier genom linjära transformationer på vektorrum. Det avslutande kapitlet sammanställer dessa idéer genom att definiera symmetri som invarians under transformation och visar hur representationsteori ger form åt de symmetrier som kodas i gruppteorin. Genomgående betonas symmetri, inte bara som ett matematiskt begrepp utan även som en enande princip inom algebraiska strukturer.

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1 Introduction

"For nothing is more reasonable than that physical inscription should exactly represent the geometrical, as the work, its pattern."

-*Johannes Kepler* [Kep69]

1.1 Motivation and Background

Mathematics has long been regarded as the language of the universe. In many cases, it has anticipated patterns in nature even before the emergence of empirical confirmation - such as Kepler's planetary laws - suggesting that apparent chaos may often conceal deeper symmetrical structure. Among these, *symmetry* is especially pervasive; a fundamental principle that unifies diverse mathematical and physical phenomena, continuously shaping our understanding of the natural world. Understanding this order requires moving from classical geometric intuition to abstract algebraic frameworks.

In formal mathematics, this symmetry is studied through *group theory* and *representation theory*. Group theory creates a language to describe symmetry in its most abstract form, using a rigorous algebraic framework to formalize patterns of structure-preserving transformations. These abstract groups, however, require interpretation to acquire meaning. Representation theory bridges this gap, realizing abstract groups as linear transformations on vector spaces and translating symmetry into the concrete language of linear algebra.

The integration of group theory into physics began in the early twentieth century, with physicists like Eugene Wigner and Wolfgang Pauli applying group-theoretic methods and reasoning to quantum mechanics. Their work built upon earlier developments by mathematicians such as Issai Schur and John von Neumann, who formalized what is now known as representation theory, establishing it as a central framework through which abstract algebraic structures gain physical meaning. Time and again, the physical world is revealed to be, at its most fundamental level, governed by group-theoretic symmetries [Sch06].

The inspiration for this thesis emerged from a combination of scientific curiosity and philosophical reflection. Initially motivated by an interest in Johannes Kepler's search for planetary harmonies and the way time and causality are manipulated in *The Legend of Zelda: Ocarina of Time*, I was drawn to the recurring presence of symmetry across natural sciences. This interest was deepened by Eugene Wigner's reflections in *The Unreasonable Effectiveness of Mathematics* [Wig60] regarding the role of mathematics and symmetry in shaping the universe.

These reflections, along with the scope limitations wisely (and repeatedly) suggested by my supervisor, resulted in a project focused on investigating the role of symmetry within mathematical structures. While I initially intended to explore the extension of these group-theoretic symmetries into quantum mechanics, limitations in my background in physics made this approach too ambitious given the time constraints of this thesis.

1.2 Objective and Scope

The primary goal of this thesis is to develop a foundational understanding of symmetry as a unifying thread, connecting group theory and representation theory. Rather than producing new experimental or theoretical results, the project aims to clarify core concepts and reflect on symmetry's pervasive role in mathematics. The work resonates with the spirit of Wigner's aforementioned essay, presenting symmetry as a lens through which we can interpret the mathematical structure of the universe.

This independent investigation builds primarily on two foundational texts:

- *Abstract Algebra* by David S. Dummit and Richard M. Foote, and
- *Quantum Theory, Groups and Representations: An Introduction* by Peter Woit

The thesis is structured into three main parts, each reflecting a stage in the development of my own understanding:

1. **Group Theory**: An overview of the fundamental algebra of symmetry.
2. **Representation Theory:** A translation of group theory's abstract structure into concrete, applicable form.
3. **Symmetry:** An synthesis on its central role as a bridge between group theory and its representations.

2 Group Theory: Foundational Mathematics

This chapter follows the structure and definitions outlined in *Abstract Algebra* by David S. Dummit and Richard M. Foote [DF04, Ch. 1 − 4]. This chapter presents groups as the fundamental algebraic framework capturing symmetry through composition. We begin by defining groups as sets equipped with a binary operation satisfying the group axioms, providing the foundation for understanding algebraic structure. The chapter then develops the theory systematically, exploring how groups organize their elements, how substructures arise, and how symmetries manifest through structure-preserving transformations. By examining these concepts in sequence, we build a coherent framework to analyze groups both abstractly and via their actions, setting the stage for deeper exploration of symmetry in subsequent chapters.

2.1 Introduction to Groups

In mathematics and physics, symmetries are modeled by structure-preserving transformations, such as distances in geometry or conserved quantities in physics. Group theory formalizes these structure-preserving transformations as elements of a set equipped with a binary operation satisfying specific axioms.

2.1.1 Basic Group Structure: definitions, axioms, and commutativity

At its core, a *group* is a set with a binary operation satisfying four minimal axioms: closure, associativity, the existence of an identity element, and invertibility.

Definition 2.1. A **group**[1] G is an ordered pair $(G, *)$ composed of a set G and a binary operation $*$ on that set, where $*$ is a function from $G \times G \to G$, that satisfies all group axioms:

- **A2.1** *(Closure)* $a * b \in G$ for all $a, b \in G$.
- **A2.2** *(Associativity)* $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.
- **A2.3** *(Identity)* There exists an identity element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$.

[1]From [Sec. 1.1][DF04]

- **A2.4** *(Inverse)* For every $a \in G$ there exists an inverse a^{-1}, such that

$$a * a^{-1} = a^{-1} * a = e.$$

These axioms abstract the intuitive notion of symmetry as transformations that preserve structure under composition and inversion. As we will see in *Representation Theory* (3.1), these axioms allow groups to act compatibly with matrix multiplication when represented linearly on vector spaces.

Some groups also satisfy commutativity.

Definition 2.2. If the group operation satisfies $ab = ba$ for all $a, b \in G$, the groups is called**abelian**[2], or *commutative*; otherwise, it is **non-abelian**.

The abelian property implies that the order in which elements are combined does not affect the result. This additional structure simplifies their classification and facilitates the description of element orders and subgroup structure. In contrast, many important groups in mathematics and physics are non-abelian, where the order of operations matters. This distinction has major implications, particularly in geometry and quantum physics, where non-commutativity reflects directional dependence.

2.1.2 Group order and generation

The size of a group is measured by how many elements it contains. This fundamental invariant constrains its algebraic structure, affecting properties such as the possible orders of elements, subgroup composition, and classification results like Lagrange's theorem (2.2).

Definition 2.3. The **order of a group** G, denoted $|G|$, is defined as the cardinality of the underlying set G.[3]

- If $|G|$ is finite, then G is a finite group.
- If $|G|$ is infinite, then G is an infinite group.

Notation. If $|G| = n$, where n is a positive integer, we say G has order n.

Closely related is the order of an individual group element:

[2] From [Sec. 1.1][DF04]
[3] From [Sec. 1.1][DF04]

Definition 2.4. Let G be a group with the identity element e. Let $x \in G$. The **order of the element**[4] x, denoted $|x|$, is the smallest positive integer m such that $x^m = e$.

In finite groups, repeated application of any element eventually returns to the identity, forming finite cycles. In infinite groups, some elements may still have finite order, while others have infinite order, generating infinitely many distinct powers.

Beyond measuring size, group structure can be simplified by identifying the generating set:

Definition 2.5. Let $S \subseteq G$ be a subset of group G. If every element of G can be expressed as a finite product of elements of S and their inverses, then S **generates**[5] G. In this case, we write $G = \langle S \rangle$.

For example, the symmetries of a square[6] form a finite group generated by two elements: a 90° rotation and a single reflection. The compositions of these two elements create all other symmetries in this group, illustrating how a finite group may be generated by a small subset of its elements.

2.2 Basic Groups and Their Structure

Reflecting Wigner's insight on the "unreasonable effectiveness" of mathematics [Wig60], group theory formalizes classical symmetries and provides a framework for studying less intuitive symmetry groups, such as those encountered in quantum mechanics. These fundamental symmetry patterns range from simple permutations to complex matrix transformations, establishing a connection between abstract group structures and linear algebra.

2.2.1 Symmetric Groups

Among the most fundamental types of groups for understanding symmetry are (who could have guessed) the *symmetric groups*, which consist of all possible permutations of a given set with order $n!$ [DF04, Sec. 1.3]. This rapid growth makes them central examples in understanding group structure and behavior.

[4]From [Sec. 1.1][DF04]
[5]From [Sec. 1.2][DF04]
[6]These will be explicitly shown in Example 2.8.

Definition 2.6. For a positive integer n, the **symmetric group**[7] of degree n, denoted S_n, is

$$S_n = \sigma : 1, 2, \ldots, n \to 1, 2, \ldots, n \mid \sigma \quad \text{is a permutation},$$

that is, the set of all bijections from $1, \ldots, n$ to itself.

Composition of permutations is associative, with the identity map

$$\mathrm{id}(k) = k \quad \text{for all} \quad k \in 1, \ldots, n, \tag{1}$$

acting as a neutral element and inverses σ^{-1} given by a permutation reversal

$$\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = \mathrm{id}\,.$$

Thus, S_n satisfies all group axioms (2.1) and forms a well-defined group [DF04, Sec. 1.3].

Since S_n satisfies the group axioms with explicit constructions of identity and inverses, its structure serves as a fundamental example of abstract symmetry groups. The subgroup structure and element classification provide essential tools for understanding permutation representations, which form a cornerstone in the theory of group representations developed later.

An important subclass of element in S_n are m-cycles, which cyclically permute m elements while leaving the others fixed. For $n \geq 3$, S_n is non-abelian, reflecting how the order of application affects symmetry operations. This cyclic behavior naturally leads us to the concept of *cyclic groups*, which represent the most fundamental form of symmetry and repetition.

2.2.2 Cyclic Groups

Cyclic groups are generated by a single element whose successive powers produce all group elements and satisfy a minimal positive exponent returning to the identity. This structure models ordered repetition and periodicity [DF04, Sec. 2.3].

[7] From [Sec. 1.3][DF04]

Definition 2.7. A group G is **cyclic**[8] if there exists some $x \in G$ such that

$$G = \langle x \rangle \begin{cases} x^n \mid n \in \mathbb{Z} & \text{if the group operation is multiplicative,} \\ nx \mid n \in \mathbb{Z} & \text{if the group operation is additive.} \end{cases}$$

Whether a cyclic group is finite or infinite is determined by the order of its generator (2.4). If x has finite order m, the sequence $x, x^r, \ldots, x^m = e$ repeats periodically. If x has infinite order, no such repetition occurs and all elements are distinct. For example, the finite group $\mathbb{Z}/n\mathbb{Z}$ cycles through n elements, while $\mathbb{Z}$ itself is infinite under addition.

These distinctions mirror different kinds of symmetry in both mathematics and nature. While simple rotations or rhythmic cycles map onto infinite structures, continuous progressions like time or distance relate more closely to infinite groups. Thus, cyclic groups express the core idea of symmetry through repetition, providing a natural entry point to the study of more complex group structures.

2.2.3 Dihedral Groups

Dihedral groups D_{2n} model the symmetries of regular n-sided polygons, combining n rotations and n reflections into a group of $2n$ elements. These operations follow specific algebraic rules, capturing reversible geometric symmetries. Together, these elements form a closed group containing all reversible symmetry operations of the polygon.

To illustrate how these symmetries manifest in a specific case, we examine the structure of D_{2n} in action by considering the case $n = 4$, where the group describes the symmetries of a square.

Example 2.8 (Symmetries of a square)
Consider the dihedral group D_8, where $n = 4$, representing the full symmetry group of a regular square. These symmetries combine rotations and reflections:

- 4 rotations:
$$0°, 90°, 180°, \text{ and } 270°.$$
- 4 reflections: across the x- and y-axis, and the diagonals $y = x$ and $y = -x$.

[8]From [Sec. 2.3][DF04]

As D_8 has $2n = 8$ distinct symmetries, the total number of symmetries is

$$2n = 2 \times 4 = 8.$$

These symmetries satisfy the group axioms (2.1), thus forming the dihedral group D_8.

Although the dihedral group D_8 provides a concrete and geometric example of a symmetry group, more complex or higher-dimensional symmetries cannot be fully captured by geometric intuition alone. Therefore, algebraic structures such as abstract groups and matrix groups offer a necessary framework for rigorously defining and analyzing symmetries in these broader contexts.

2.2.4 Matrix Groups

To move beyond intuitive symmetries, Dummit and Foote present matrix groups as a generalization of geometric symmetry by identifying each group element with an invertible linear transformation on a vector space [DF04, Sec. 1.4].

While dihedral groups capture planar symmetries via rotations and reflections, matrix groups extend this framework to higher-dimensional and abstract settings. This broader perspective demonstrates how matrix groups facilitate the study of symmetry in higher-dimensional and more abstract spaces, connecting mathematics with physics.

The entries of such matrices belong to a field F. Just as groups abstract symmetry, fields formalize the arithmetic underlying these spaces. Fields supply the scalars for linear actions and representations, both of which are central to understanding how groups act on spaces.[9] A field must contain at least two elements and satisfy the group axioms (2.1) under addition and multiplication, with multiplication defined on the nonzero elements, as zero has no multiplicative inverse.

The general linear group $GL_n(F)$ is the group of all invertible $n \times n$ matrices with entries from a field F, under matrix multiplication [DF04, Sec. 1.4].

Definition 2.9. For a field F, the **general linear group**[10] of degree n is

$$GL_n(F) = A \in M_{n \times n}(F) \mid \det(A) \neq 0.$$

[9]This will be discussed further in Chapter 3 (3)
[10]From [Sec. 1.4][DF04]

Here, group axioms follow naturally: associativity from matrix multiplication, identity from the identity matrix I, closure from the product of two invertible matrices, and inverses via matrix invertibility. By definition, invertibility requires $\det(A) \neq 0$, guaranteeing that every matrix in $GL_n(F)$ has a corresponding unique inverse A^{-1}.

Invertibility can also be characterized through the linear independence of the matrix's columns.

Definition 2.10. A subset S of V is a set of **linearly independent**[11] vectors if

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$$

with scalars $a_i \in F$ and vectors $v_i \in S$ implies

$$a_1 = a_2 = \dots = a_n = 0.$$

Invertibility of a matrix A is equivalent to its columns forming a basis of the vector space F^n.

This equivalence is formalized in standard linear algebra texts such as Dummit and Foote [DF04, Corollary 2, Section 1.3], linking algebraic invertibility to a geometric notion of spanning and independence.

Corollary 2.11. *An $n \times n$ matrix A over a field F is non-singular if and only if it is invertible.*

These equivalences rely on the properties of determinants and the arithmetic structure of fields, outlined by Dummit and Foote [DF04, Sec. 14], where the general linear group $GL_n(F)$ is constructed as the group of invertible matrices over an arbitrary field F, ensuring consistent invertibility criteria.

Although checking invertibility can require direct computation, the total number of invertible $n \times n$ matrices over a finite field F is known explicitly [DF04, Sec. 1.4]:

Proposition 2.12. *For a finite field F, the following hold:*

1. *If $|F| < \infty$, there exists some prime number p and positive integer m such that*

$$|F| = p^m.$$

[11] From [Sec. 11.1][DF04]

2. *If $|F| = q < \infty$, then the order $GL_n(F)$ is*

$$|GL_n(F)| = (q^n - 1)(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}).$$

This count establishes the finite size of the general linear group, confirming that $GL_n(F)$ forms a well-defined group and establishing it as a fundamental example of matrix groups.

Example 2.13 (Elements and orders in $GL_2(\mathbb{F}_2)$)
To illustrate these ideas concretely, consider the group $GL_2(\mathbb{F}_2)$ over the field $\mathbb{F}_2$, where $n \in \mathbb{Z}^+$, and computing their orders [DF04, Sec. 1.4, Exercise 2] .

Step 1: Size of $GL_2(\mathbb{F}_2)$ By Definition 2.9 and Proposition 2.12, for the finite field $\mathbb{F}_2$ with $n = 2$ and $q = 2$, we have

$$|GL_2(\mathbb{F}_2)| = (2^2 - 1)(2^2 - 2) = 3 \cdot 2 = 6.$$

Thus, $GL_2(\mathbb{F}_2)$ contains exactly 6 elements.

Step 2: Write all elements of $GL_2(\mathbb{F}_2)$ Recall that all invertible 2×2 matrices over $\mathbb{F}_2$ are those for which $\det(A) = 1$ (2.9). By Corollary 2.11, this is equivalent to A being non-singular.

Since $\mathbb{F}_2^2$ is two-dimensional, any set of linearly independent vectors has at most two elements. It follows that any linearly independent pair must be chosen from

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

By selecting ordered pairs of linearly independent vectors in F_2^2, we define a basis which determines a unique invertible matrix. The obtained invertible matrices follow:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Step 3: Orders of elements Recall the order of an element, given in Definition (2.4).

A_1 is the identity, so

$$|A_1| = 1.$$

Continuing with A_2, the orders of the remaining matrices are determined through direct multiplication in the field F_2 (mod 2):

$$A_2^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = I \quad \Rightarrow \quad |A_2| = 2.$$

Similarly,

$$A_3^2 = I, \quad A_4^2 = I.$$

For A_5

$$A_5^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad A_5^2 \neq I,$$

$$\to A_5^2 \times A_5 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = I \quad \Rightarrow \quad A_5^3 = I.$$

So A_5 has order 3.

Finally, $A_6^2 = A_5$ implies

$$A_6^3 = A_5 \times A_6 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = I \quad \Rightarrow A_6^3 = I.$$

So A_6 also has order 3.

Solution The group $GL_2(\mathbb{F}_2)$ consists of 6 elements with the orders summarized as:

- $|A_1| = 1$
- $|A_2|, |A_3|, |A_4| = 2$
- $|A_5|, |A_6| = 3$

Transitioning to matrix groups allows us to study symmetry beyond the limits of simple geometric examples. This abstraction facilitates the study of group representations, where groups act by linear operators on vector spaces, enabling more flexible and detailed analysis of group properties and their algebraic interrelations.

2.3 Internal Structures and Classification

2.3.1 Subgroups

Studying a group's subgroups uncovers structural patterns that determine the group's overall composition [DF04, Sec. 2.1].

Definition 2.14. A nonempty subset $H \subseteq G$ is a **subgroup**[12] of G if it satisfies *closure* under the group operation, contains the *identity* element, and includes *inverses* of all its elements (A2.1, A2.3, A2.4).

A practical test for verifying subgroups is the *subgroup criterion.*

Proposition 2.15. *A subset $H \subseteq G$ is a subgroup if and only if*[13]*:*

- ***A2.5*** *(Nonempty) $H \neq \emptyset$, and*
- ***A2.6*** *(Inverse) for all $x, y \in H$, we have $xy^{-1} \in H$.*

Remark 2.16*.* In finite groups, each element has finite order, so verifying closure and identity suffices to ensure inverses exist in any finite subgroup [DF04, Sec. 2.3].

The proof for Proposition 2.15, seen in *Abstract Algebra* by Dummit and Foote [DF04, Sec. 2.1], ensures that both the identity and inverses are implicitly present in H, confirming the subgroup status of H.

Theorem 2.1 (Subgroups of a Cyclic Group)**.** *1. Every subgroup of a cyclic group is itself cyclic.*

2. In an infinite cyclic group, all subgroups are generated by x^k for some integer k.

3. In a finite cyclic group of order n, there is exactly one subgroup of order d for each divisor d of n.[14]

Recall the dihedral group from Example 2.8. In D_8, the rotations $1, r, r^2, r^3$ form a cyclic subgroup, isolating rotational behavior and simplifying the analysis of the full group's structure.

To classify these subgroups explicitly, consider the order of their generators:

Proposition 2.17. ***The Order of a Cyclic Group*** *Let $G = \langle x \rangle$ be a cyclic group generated by x.*

[12] From [Sec. 2.1][DF04]
[13] From [DF04, Sec. 2.1, Proposition 1]
[14] From [DF04, Sec. 2.3, Theorem 7]

1. *If $|x| = \infty$, then all powers of x are distinct.*
2. *If $|x| = n < \infty$, then $x^n = e$, and the elements $e, x, x^2, \dots, x^{n-1}$ are distinct.* [15]

Notation. In this context, multiplicative notation x^n is used for the group operation, regardless of the nature of the group elements.

2.3.2 Cosets and Lagrange's Theorem

Building on the structure of cyclic groups, we now examine how groups can be decomposed into equal-sized, repeating subsets, formed by translating a subgroup by a fixed group element.

Definition 2.18. Let there be a subgroup $H \leq G$. The set formed by multiplying all elements of H by any fixed element $g \in G$ is called a **coset**[16].

The **left coset**[17] of H with respect to g is the set

$$gH = gh \mid h \in H.$$

The **right coset**[18] of H with respect to g is the set

$$Hg = hg \mid h \in H.$$

As noted by Dummit and Foote, if G is abelian, commutativity implies every coset gH equals Hg [DF04, Sec. 3.1, Example 2]. In genera, left and right cosets may differ in non-abelian groups as the operation is non-commutative. Regardless, each coset has the same size as H since multiplication by g is a bijection.

Definition 2.19. The collection of all left (or right) cosets of a subgroup H in G form a **partition**[19] of G if every element $g \in G$ belongs to exactly one such coset.

In a cyclic group $G = \langle x \rangle$, the cosets of a cyclic subgroup $H = \langle x^k \rangle$ partition G into disjoint, symmetric subset, each as a translated copy of H. This structure is

[15] From [DF04, Sec. 2.3, Prop. 2]
[16] From [Sec. 3.1][DF04]
[17] From [Sec. 3.1][DF04]
[18] From [Sec. 3.1][DF04]
[19] From [Sec. 0.1][DF04]

especially clear in additive groups such as $\mathbb{Z}$.

Example 2.20 (Partitioning the group $\mathbb{Z}$)
Consider the group of integers $\mathbb{Z}$ under addition, and its subgroup

$$H = 2\mathbb{Z} = \ldots, -4, -2, 0, 2, 4, \ldots,$$

consisting of all even integers. Adding 1 to each element of H yields the coset

$$H + 1 = \ldots, -3, -1, 1, 3, 5, \ldots,$$

which is the set of all odd integers. Since $H + 1$ is not closed under addition, it is not a subgroup of $\mathbb{Z}$, but a translation of H.

Together, the cosets H and $H + 1$ form a partition of $\mathbb{Z}$ into two disjoint, equally sized cosets,

$$H \cup (H + 1) = \mathbb{Z}, \quad H \cap (H + 1) = \emptyset.$$

These coset form the quotient group

$$\mathbb{Z}/H = H, H + 1 \cong \mathbb{Z}/2\mathbb{Z},$$

reflecting the equivalence relation on $\mathbb{Z}$ defined by congruence modulo 2.

For finite groups, the number of cosets of H is the *index*, denoted $|G : H|$ [DF04, Sec. 3.2]. Since cosets partition G into subsets of size $|H|$, it follows that

$$|G| = |G : H| \times |H|. \tag{2}$$

Lagrange's Theorem follows [DF04, Sec. 1.7, Exercise 19]:

Theorem 2.2 (Lagrange's Theorem)**.** *Let $H \leq G$ be a subgroup of a finite group G. Then,*

$$|G : H| = \frac{|G|}{|H|},$$

where $|G : H|$ is the number of distinct left cosets of H in G.

Proof. Let $|H| = n$ and let $|G| = m$.

By Definition 2.18, for any fixed $g \in G$, the left coset gH is

$$gH = gh \mid h \in H = gh_1, gh_2, ..., gh_n,$$

where $H = h_1, h_2, \ldots, h_n$.

Suppose that

$$gh_i = gh_j,$$

for some $h_i, h_j \in H$. By the cancellation property of groups (2.1),

$$h_i = h_j,$$

so all elements in gH are distinct, implying

$$|gH| = |H| = n.$$

Since G is finite, the number of distinct left cosets, denoted by $p = |G : H|$, is finite.

These cosets partition G into p disjoint subsets, each of size n. Hence,

$$|G| = m = p \cdot n,$$

implying

$$p = \frac{m}{n}.$$

Therefore, $|H|$ divides $|G|$, and

$$|G| = |G : H| \times |H|.$$

□

Lagrange's Theorem connects subgroup size to the group's overall order. This result provides the necessary divisibility conditions for analyzing group structure and representation dimensions. The same principle plays a central role in representation theory, where the dimensions of irreducible representations must divide the order of the group, reflecting the same partitioning structure observed here.

Example 2.21 (Subgroups and Cosets of $\mathbb{Z}/6\mathbb{Z}$)

To illustrate the relationship between group order, subgroups and cosets, consider $\mathbb{Z}/6\mathbb{Z} = 0, 1, 2, 3, 4, 5$ under addition modulo 6.

Since $\mathbb{Z}/6\mathbb{Z}$ is cyclic of order 6, by Theorem 2.1 all subgroups are cyclic with orders dividing 6.

One such subgroup is $H = 0, 3$ of order 2, with left cosets

$$0 + H = 0, 3, \quad 1 + H = 1, 4, \quad \text{and} \quad 2 + H = 2, 5,$$

which partition the group into three disjoint subsets of size 2, satisfying

$$|G : H| = \frac{|G|}{|H|} = \frac{6}{2} = 3.$$

Similarly, the subgroup $H = 0, 2, 4$ of order 3 has left cosets

$$0 + H = 0, 2, 4, \quad \text{and} \quad 1 + H = 1, 3, 5,$$

partitioning G into two disjoint subsets of size 3, satisfying

$$|G : H| = \frac{|G|}{|H|} = \frac{6}{3} = 2.$$

In all cases, cosets partition $\mathbb{Z}/6\mathbb{Z}$ into equal-sized, disjoint subsets $|H|$, covering all elements exactly once.

Hence,

$$|\mathbb{Z}/6\mathbb{Z}| = |H| \cdot |G : H| = 6,$$

as guaranteed by Lagrange's Theorem (2.2).

Lagrange's Theorem constrains subgroup sizes without determining the group's algebraic structure or commutativity properties, which require further study.

2.3.3 Abelian vs. non-abelian groups

Recall the distinction between abelian and non-abelian groups from Definition 2.2. Commutativity influences the structure of subgroups, the behavior of cosets, and the classification of groups by constraining the possible products of elements and the relations they can satisfy.

Canonical abelian groups include the infinite group $\mathbb{Z}$ and the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$ (2.2.2), which are fully classified by their order and exhibit predictable subgroup and element structure.

Non-abelian groups such as the dihedral group D_8 (2.8) and the symmetric group S_3 (2.2.1), exhibit non-abelian behavior. For example, in D_8, the relation $rs \neq sr$ demonstrates that the composition of a reflection and a rotation depends on the order of operations.[20] Similarly, in S_3, the compositions

$$(12)(23) = (123) \quad \neq \quad (23)(12) = (132)$$

also demonstrate non-commutativity.

While D_8 illustrates non-abelian structure through geometric transformations, S_3 does so algebraically via function composition of permutations. These examples illustrate how non-commutativity arises naturally in different contexts while maintaining the same order-dependent behavior.

The failure of commutativity introduces structural complexity not present in abelian groups. In many classical systems, such as planar rotations, commutativity often holds. In contrast, non-abelian groups are essential in advanced physical theories like quantum mechanics, where the non-commutativity of operators reflects physically meaningful order-dependence.

2.3.4 Group presentations

Group presentations provide a compact algebraic description of a group in terms of generators and relations, especially useful for non-abelian groups whose structure cannot be easily enumerated.

Definition 2.22. Let there be a subset $S \subseteq G$. If G is generated by S and all relations among elements of S follow from a set of relations R, then

$$G = \langle S \mid R \rangle$$

is a **presentation**[21] of G.

[20] These relations will be discussed further as the Dihdreal Relations 1, 3, and 2.
[21] From [DF04, Sec. 1.2]

For example, the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$ is generated by a single element:

$$\mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle,$$

with all elements formed by repeated addition of 1 modulo n. Non-abelian groups typically require multiple generators to describe their structure, with nontrivial relations reflecting the importance of operation order.

The core relations that defined the structure of dihedral groups assist in understanding how these symmetries interact. When discussing the key relations of dihedral groups, we use 1 to denote the identity element of the group, following standard multiplicative notation.

The dihedral group D_{2n} is generated by two elements:

- r, a rotation by $\frac{2\pi}{n}$,
- s, a reflection across a symmetry axis.

Following the definition of *group relations* in [DF04, Sec. 1.2], the generators of D_{2n} satisfy three key relations:

1. The *rotation order* shows that applying n distinct rotations r^i returns the polygon to the original orientation, so the rotation generator r has order n:

$$r^n = 1, \text{ so } r \text{ has order } n$$

2. The *reflection order* shows that reflecting the polygon twice returns it to its original state, so the reflection generator s has order 2:

$$s^2 = 1, \text{ so } s \text{ has order } 2$$

3. The *conjugation of symmetries* inverts the direction of the polygon's rotation by conjugating a rotation by a reflection:

$$rs = sr^{-1} \equiv srs = r^{-1}$$

Together, these relations define the following group presentation:

$$D_{2n} = \langle r, s \mid r^n = 1, \quad s^2 = 1, \quad rs = sr^{-1} \rangle. \tag{3}$$

These three relations form a complete system, meaning any equation involving elements of D_{2n} can be derived from them. Each element of D_{2n} can be uniquely expressed as either r^i or sr^i for $0 \leq i < n$, making the presentation both minimal and powerful. This abstract formulation streamlines the study of dihedral symmetry and prepares us to generalize these ideas to other types of group actions and algebraic systems [DF04, Sec. 1.6].

To see this concept in practice, consider D_{10}, that models the rotational and reflective symmetries found in a regular pentagon.[22]

Example 2.23 (Element Orders in $D_{10}(n = 5)$)
We begin by defining the group, using the group presentation for D_{2n}:

$$D_{10} = \langle r, s \mid r^5 = s^2 = 1, \; rs = sr^{-1} \rangle.$$

For $0 \leq i < 5$, we obtain the complete list of elements for D_{10}:

$$1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4.$$

The first five elements represent rotations by multiples of $\frac{2\pi}{5}$, and the remaining five are reflections composed with each rotation.

From the relations given in Equation (3), we find that

1. By the *rotation relation* (1), $r^5 = 1$ implies $|r| = 5$. As such, its powers r, r^2, r^3, r^4 are distinct and generate a cyclic subgroup of order 5.

[22] Exercise xx from [DF04, Sec. 1.2].

2. By the *reflection relation* (2) and *conjugation relation* ((3)), we find:

$$\begin{aligned}(sr^i)^2 &= sr^i \cdot sr^i \\ &= s(r^i s)r^i \\ &= s(sr^{-1})r^i \\ &= (s^2)(r^{-i}r^i) \\ &= 1,\end{aligned}$$

so each element sr^i has order 2.

This example shows how a group presentation captures a group's algebraic structure, allowing properties like element orders and subgroup structure to be derived from a finite set of generators and relations.

We can now consider a slightly more involved example. In particular, we look at how the group relations can be used to simplify expressions within a larger dihedral group. For example, using the larger dihedral group D_{24}, representing the symmetries of a regular dodecagon, we can simplify the product $(sr^9)(sr^6)$ found in [DF04, Sec. 1.2].

Example 2.24 (Simplifying products in D_{24} using relations)
From Equation (3) for $n = 12$, we

1. Apply the *conjugation relation* (3):

$$(sr^9)(sr^6) = s(r^9 s)r^6 = s(sr^{-9})r^6.$$

2. Using $s^2 = 3$, by the *reflection relation* (2), we simplify

$$s(sr^{-9})r^6 = (s^2)r^{-9+6} = r^{-3}.$$

3. Since $r^{-3} = r^{(12-3)} = r^9$ in D_{24},

$$(sr^9)(sr^6) = r^9.$$

These computations demonstrate how a small set of defining relations, as given in a group presentation, determines not only the full element structure of a group but also enables explicit simplification of group operations. In both D_{10} and D_{24}, the relations among generators govern element orders and products, revealing the group's internal structure through purely algebraic means.

2.3.5 Uniqueness of Identity and Inverse Elements

Proposition 2.25. *For any group $(G, *)$, the following hold:*

1. ***Uniqueness of identity**: There exists a unique element $e \in G$ such that*

$$e * a = a * e = a.$$

2. ***Uniqueness of inverse:** For every $a \in G$, there exists exactly one element $a^{-1} \in G$ such that*

$$a * a^{-1} = a^{-1} * a = e.$$

Proof of uniqueness of group identity and inverses. Suppose e and e' are both identity elements in G. Applying the identity axiom (A2.3) with e to e' gives

$$e * e' = e'.$$

Since e' is also an identity element, we also have

$$e * e' = e.$$

It follows that

$$e = e',$$

establishing the uniqueness of the identity element.

Now suppose b and c are both inverses of an element $a \in G$. Then

$$a * b = b * a = e, \quad \text{and} \quad a * c = c * a = e,$$

where e is the identity element.

Using the associative axiom (A2.2) and the identity axiom (A2.3) we compute,

$$b = b * e = b * (a * c) = (b * a) * c = e * c = c.$$

Therefore,

$$b = c,$$

demonstrating the uniqueness of inverses. □

The uniqueness of the identity and inverse elements, shown in greater detail in [DF04, Sec. 1.1], shows no two distinct elements can satisfy these roles simultaneously, ensuring unambiguous group structure. This uniqueness, along with the ability to compute within a group using its presentation, is fundamental to understanding and establishing reliable algebraic behavior across all groups, reinforcing the internal coherence guaranteed by the group axioms (2.1).

This foundational uniqueness, combined with the framework of presentations, guarantees well-defined algebraic behavior across groups. Like architectural blueprints, group presentations encode the internal logic that enables visible and reliable constructions, paving the way for deeper study in representation theory.

2.4 Mappings and Group Actions

Groups can be compared and classified by examining mappings that preserve their structure. Such mappings let us identify when two groups share the same internal structure, even if their elements appear different [DF04, Sec. 1.6].

2.4.1 Group Homomorphisms and Isomorphisms

Definition 2.26. Let there be two groups G and H. A function $\varphi : G \to H$ is a **homomorphism**[23] if

$$\varphi(xy) = \varphi(x)\varphi(y), \quad \text{for all } x, y \in G.$$

Two important subsets associated with any homomorphism that play a central role in the fundamental structure of group mappings are the *kernel* and *image*.

Definition 2.27. For a homomorphism $\varphi : G \to H$, the **kernel**[24], denoted $\ker(\varphi)$,

[23] [DF04, Sec. 1.6]
[24] [DF04, Sec. 3.1]

is the subset of G defined by

$$\ker(\varphi) = g \in G \mid \varphi(g) = 1_H,$$

where 1_H is the identity of H.

Definition 2.28. For a homomorphism $\varphi : G \to H$, the **image**[25], denoted $\mathrm{im}(\varphi)$, defined by

$$\mathrm{im}(\varphi) = \varphi(g) \mid g \in G \subseteq H,$$

which forms a subgroup of H, denoted $\mathrm{im}(\varphi)$.

The kernel measures the extent to which φ fails to be injective, whereas the image reflects the extent to which φ covers H . Together, they determine the degree to which the algebraic structure of G is preserved in H.

Definition 2.29. A **group isomorphism**[26] is a bijective homomorphism. If such a map exists, the groups G and H are said to be **isomorphic**, and we write

$$G \cong H.$$

Even if their elements or operations appear different, all essential group properties are preserved under isomorphism. Isomorphisms are thus a special class of homomorphisms that preserve group structure exactly and establish an algebraic equivalence between groups.

More generally, homomorphisms reveal partial structural relationships, allowing comparison of groups beyond element-by-element correspondence. This distinction emphasizes that group theory focuses on the intrinsic pattern of element interactions rather than their individual identities or representations.

Example 2.30

Consider the group under addition modulo 4

$$\mathbb{Z}/4\mathbb{Z} = 0, 1, 2, 3,$$

[25] From [DF04, Sec. 3.1]

[26] From [DF04, Sec. 1.6]

generated by 1. Define the map

$$\varphi : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}, \quad \varphi(x) = x \pmod 2.$$

By Definition 2.26, φ is a homomorphism. Since

$$\text{im}(\varphi) = 0, 1 = \mathbb{Z}/2\mathbb{Z},$$

it is surjective. However, because

$$\ker(\varphi) = 0, 2 \neq 0,$$

it is not injective. Thus, φ fails to be an isomorphism, demonstrating a *partial* structural equivalence between $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$.

Next, consider the groups D_8 and $\mathbb{Z}/8\mathbb{Z}$, both of order 8:

- $\mathbb{Z}/8\mathbb{Z}$ is cyclic, generated by 1, and abelian.
- D_8, as described in Example 2.8, is non-abelian.

Since $\mathbb{Z}/8\mathbb{Z}$ is abelian while D_8 is not, the map

$$\psi : \mathbb{Z}/8\mathbb{Z} \to D_8$$

is not bijective. Thus

$$D_8 \ncong \mathbb{Z}/8\mathbb{Z},$$

and no isomorphism exists.

However, the rotation subgroup

$$\langle r \rangle = 1, r, r^2, r^3$$

of D_8 is cyclic of order 4. Since any two cyclic groups of the same order n are isomorphic to $\mathbb{Z}/n\mathbb{Z}$,[27] it follows that

$$\langle r \rangle \cong \mathbb{Z}/4\mathbb{Z},$$

demonstrating a full structural equivalence.

[27] See Theorem 4 in [DF04, Sec. 2.3].

This example clarifies the difference between full structural equivalence via isomorphisms and partial structural preservation via homomorphisms. While homomorphisms maintain key invariants such as subgroup orders and indices (see: Lagrange's Theorem 2.2), only isomorphisms ensure groups are algebraically indistinguishable and interchangeable in theory and application [DF04, Sec. 1.6]. Such maps are essential tools for classifying groups and understanding how complex structures relate to simpler ones

2.4.2 Group Actions and Representations

A foundational understanding of group representations allows for an expansion in perspective, leading to the analysis of how groups interact with other mathematical objects and modeling symmetry in a more generalized setting.

Definition 2.31. A **group action**[28] of a group G on a set X is a map $G \times X \to X$, denoted $(g, x) \mapsto g \cdot x$, such that for all $g, h \in G$ and $x \in X$:

- $e \cdot x = x$ where e is the identity in G,
- $g \cdot (h \cdot x) = (gh) \cdot x$.

In particular, groups act on themselves by left multiplication,

Definition 2.32. The function

$$G \times G \to G, (g, x) \mapsto gx,$$

defines a group action of G on itself, called **left multiplication**[29] .

Remark 2.33. This action is always transitive [DF04, Sec. 4.2]. That is, for any $a, b \in G$, there exists $g \in G$ such that $g \cdot a = b$.

A simple, intuitive example is the group of integers $\mathbb{Z}$ acting on itself by addition:

Example 2.34 (Group Action of $\mathbb{Z}$ on Itself by Addition)
Define a group action of a set $\mathbb{Z}$ on itself by the map

$$(n, m) \mapsto n + m \quad \text{for all } n, m \in \mathbb{Z}.$$

[28] From [DF04, Sec. 1.7]
[29] From [DF04, Sec. 4.2]

This action satisfies the group axioms:

- **A2.7** *(Identity)* For all $m \in \mathbb{Z}$,

$$0 \cdot m = m.$$

- **A2.8** *(Compatibility)* For all $n_1, n_2, m \in \mathbb{Z}$,

$$(n_1 + n_2) \cdot m = n_1 \cdot (n_2 \cdot m).$$

This corresponds to translation on $\mathbb{Z}$, and is transitive because for any $m, m' \in \mathbb{Z}$, there exists $n = m' - m \in \mathbb{Z}$ such that $n \cdot m = m'$.

This idea mirrors the internal symmetry observed in the behavior of left cosets (2.18). Although left and right cosets may differ in non-abelian groups, the theory developed through left cosets can be extended to right cosets with appropriate adjustments.

2.4.3 Cayley's Theorem

For simplicity and uniformity, we will focus on left cosets and the corresponding group action, as these provide a sufficient framework for underpinning one of the most fundamental results in group theory [DF04, Sec. 3.5]:

Theorem 2.3 (Cayley's Theorem)**.** *Every group G is isomorphic to a subgroup of S_n where $n = |G|$.*

Proof. Let G be a group. Define an action of G on itself by left multiplication (2.32):

$$G \times G \to G, \qquad (g, x) \mapsto gx.$$

By the group axioms (2.1), this is a well-defined group action, as it satisfies the conditions in Definition 2.31:

$$e \cdot x = ex = x \text{ for all } x \in G,$$

and

$$g \cdot (h \cdot x) = g \cdot (hx) = g(hx) = (gh)x = (gh) \cdot x \quad \text{for all } g, h, x \in G.$$

For each $g \in G$, define the function $\lambda_g : G \to G$ by $\lambda_g(x) = gx$. Left multiplication by g is bijective, with inverse $\lambda_{g^{-1}}$, so $\lambda_g \in S_G$, the symmetric group on the set G.

Define the map

$$\varphi : G \to S_G, \quad \varphi(g) = \lambda_g.$$

To verify that φ is a homomorphism (2.26), let $x, y, h \in G$. Then

$$\varphi(xy)(h) = (xy) \cdot h = x \cdot (yh) = (\varphi(x)\varphi(y))(h),$$

so

$$\varphi(xy) = \varphi(x)\varphi(y).$$

To show that φ is injective, assume $\varphi(x) = \varphi(y)$, so $\lambda_x = \lambda_y$. Then, $xh = yh$ for all $h \in G$. In particular, for $h = e$ we find $x = xe = ye = y$. It follows that $x = y$, confirming φ is injective.

Therefore, φ is an injective group homomorphism from G into S_G, and thus an isomorphism onto its image

$$G \cong \varphi(G) \leq S_G.$$

□

Since every group can be embedded as a subgroup of the symmetric group S_n, Cayley's Theorem establishes a concrete realization of abstract groups as permutation groups acting on finite sets. This result motivates the general strategy of studying groups via their actions on sets, a foundational approach in both representation theory and the structural analysis of groups.

2.4.4 Conjugation and Internal Symmetry

One particularly important internal action is *conjugation*, which uncovers structural features of a group.

Definition 2.35. The **conjugation action**[30] of G on itself is given by

$$g \cdot a = gag^{-1} \quad \text{for all } g, a \in G.$$

Definition 2.36. Two elements $a, b \in G$ are **conjugate**[31]

$$b = gag^{-1}$$

for some $g \in G$.

Conjugation by g, rearranges group elements in a way that respects the group operation, and extends naturally from elements to subgroups:

Definition 2.37. The **conjugate of a subgroup**[32] $H \leq G$ by g is the set

$$gHg^{-1} = ghg^{-1} \mid h \in H.$$

This construction generalizes conjugation from elements to subgroups and helps identify invariant internal structures within a group, leading to the notion of conjugacy classes:

Definition 2.38. Let there be a fixed element $a \in G$. The **conjugacy class**[33] of a is

$$Cl(a) = gag^{-1} \mid g \in G.$$

Conjugacy classes partition a group into distinct subsets of elements with the same internal symmetry. This partitioning reveals patterns in how elements relate through internal symmetry. Subgroups that remain unchanged under conjugation play a special role:

Definition 2.39. A subgroup $H \leq G$ is **normal**[34], denoted $H \trianglelefteq G$, if

$$gHg^{-1} = H \quad \text{for all } g \in G.$$

[30]From [DF04, Sec. 1.7]
[31]From [DF04, Sec. 3.1]
[32]From [DF04, Sec. 3.1, Theorem 6]
[33]From [DF04, Sec. 4.3]
[34]From [DF04, Sec. 3.1]

Any element $g \in G$ leaving H invariant under conjugation *normalizes* H. The set of all such elements form a subgroup of G

$$N_G(H) = g \in G \mid gHg^{-1} = H,$$

where $N_G(H)$ is the *normalizer* of H in G.

Normal subgroups are fundamental in group theory because their invariance under conjugation allows the construction of quotient groups. These quotient groups simplify the analysis of group structure by factoring out well-behaved internal components.

Centralizers and centers further refine this analysis:

Definition 2.40. Let there be a subset $A \subseteq G$. The **centralizer**[35] of A in G, denoted $C_G(A)$, is

$$C_G(A) = g \in G \mid gag^{-1} = a \quad \text{for all } a \in A.$$

When $A = G$, the centralizer is called the *center* of G, denoted $Z(G)$,

$$Z(G) = z \in G \mid zg = gz \quad \text{for all } g \in G.$$

Example 2.41 (Conjugation in S_3 and D_8)
The symmetric group S_3 has six elements partitioned into three conjugacy classes:

1. The identity element e,
2. The transpositions $(12), (13), (23)$,
3. The 3-cycles $(123), (132)$.

The conjugacy classes of S_n correspond exactly to elements with the same cycle structure, which is invariant under conjugation. This is formalized by Proposition 11 in [DF04, Sec. 4.4].

In the dihedral group D_8, as previously seen in Example 2.8, conjugating a rotation

[35] From [DF04, Sec. 4.3]

r by a reflection s illustrates the internal action of conjugation, where

$$srs^{-1} = r^{-1}.$$

This *conjugation of symmetries* reveals that reflections act as inversions of rotation, demonstrating that D_8 is non-abelian.

The following proposition formalizes the characterization of these conjugacy classes within any group.

Proposition 2.42 (Characterization of conjugacy classes)**.** *A nonempty subset $C \subseteq G$ is a conjugacy class if and only if for any two elements $x, y \in C$, there exists some $g \in G$ such that*

$$y = gxg^{-1}.$$

Remark 2.43*.* This characterization follows directly from the orbit definition of the conjugation action and is supported by the orbit-stabilizer framework detailed in *Abstract Algebra* [DF04, Sec. 4.3]

Proof. Let there be a nonempty subset $C \subseteq G$.

($\Rightarrow$) Suppose $C \subseteq G$ is a conjugacy class. Then, by Definition 2.38, there exists some element $a \in G$ such that

$$C = Cl(a) = gag^{-1} \mid g \in G.$$

Let $x, y \in C$. Then there exist elements $g_1, g_2 \in G$ such that

$$x = g_1 a g_1^{-1} \quad \text{and} \quad y = g_2 a g_2^{-1}.$$

Set $g = g_2 g_1^{-1}$. Then

$$gxg^{-1} = g_2 g_1^{-1}(g_1 a g_1^{-1}) g_1 g_2^{-1} = g_2 a g_2^{-1} = y,$$

so $y = gxg^{-1}$. Hence, by Definition 2.36, any two elements $x, y \in C$ are conjugate.

($\Leftarrow$) Conversely, suppose $C \subseteq G$ is a nonempty subset such that, by Definition (2.36), any two elements $x, y \in C$ are conjugate.

Fix any $a \in C$. We claim that $C = Cl(a)$.

1. *Claim 1:* $C \subseteq Cl(a)$. Let $y \in C$. Since $a, y \in C$, and all elements in C are conjugate by assumption, there exists a $g \in G$ such that

$$y = gag^{-1}.$$

   Hence, $y \in Cl(a)$ and $C \subseteq Cl(a)$.

2. *Claim 2:* $Cl(a) \subseteq C$. Let $y \in Cl(a)$, so $y = gag^{-1}$ for some $g \in G$. Since $a \in C$ and C is closed under conjugation, it follows that $y \in C$. Therefore, $Cl(a) \subseteq C$.

Since both inclusions hold, $C = Cl(a)$, so C is a conjugacy class in G by Definition 2.38. □

As Johannes Kepler explored in *Harmonies of the World*, internal harmony often emerges through structural regularity. In group theory, conjugacy classes group together elements that are structurally equivalent under conjugation, highlighting internal symmetries within the group.

2.4.5 Automorphisms

Automorphisms are isomorphisms from a group to itself, capturing its symmetries.

Definition 2.44. An **automorphism**[36] of a group G is an isomorphism

$$\varphi : G \to G.$$

The set of all automorphisms forms a group $\mathrm{Aut}(G)$ under composition.

Automorphisms preserve group structure, offering a way to rearrange elements without violating the group's algebraic rules. Among them, some arise directly from conjugation:

Definition 2.45. Conjugation by an element $g \in G$ defines an **inner automorphism**[37].

[36] From [DF04, Sec. 4.4]
[37] From [DF04, Sec. 4.4]

The group of of inner automorphisms is denoted $\operatorname{Inn}(G)$, and forms a normal subgroup

$$\operatorname{Inn}(G) \trianglelefteq \operatorname{Aut}(G), \tag{4}$$

as shown in Corollary 15 and illustrated in Example (1) of [DF04, Sec. 4.4].

This normality naturally leads to the consideration of automorphisms that are *not* inner, implying the existence of group symmetries that cannot be realized by conjugation alone.

Definition 2.46. An **outer automorphism**[38] of a group G is a coset in the quotient group

$$\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G).$$

The group $\operatorname{Out}(G)$ measures the extent to which automorphisms of G lie outside the subgroup $\operatorname{Inn}(G)$. Each outer automorphism corresponds to a coset of $\operatorname{Inn}(G)$ and represents a structural symmetry of G not obtainable through conjugation.

When $\operatorname{Out}(G)$ is nontrivial, it indicates that G contains symmetries beyond internal transformations. This often arises when G is a normal subgroup of some larger group H, denoted $G \trianglelefteq H$, and conjugation by elements of H induces automorphisms of G that are not inner. In this sense, $\operatorname{Out}(G)$ may be interpreted as a quotient of 'external' symmetries, those arising from outside the group itself.[39]

In some cases, all automorphisms are inner, and $\operatorname{Out}(G)$ is trivial. Such groups are called complete.

Definition 2.47. A group G is **complete**[40] if and only if

1. The center is trivial:

$$Z(G) = e, \text{ and}$$

2. every automorphism of G is inner:

$$\operatorname{Aut}(G) = \operatorname{Inn}(G).$$

[38] This definition follows naturally from the definitions of automorphisms and inner automorphisms of G, as presented in [DF04, Sec. 4.4].

[39] This explanation is based on an explanation provided by my thesis supervisor, Wushi Goldring

[40] While the term *complete group* is not explicitly defined in [DF04], completing Exercise 18 in Section 4.4 shows that $\operatorname{Aut}(S_n) = \operatorname{Inn}(S_n)$ for $n \neq 6$, providing a standard example of a complete group as defined in this thesis.

Equivalently, G is complete if its center is trivial

$$Z(G) = e,$$

and every automorphism is inner

$$\mathrm{Out}(G) = e.$$

To illustrate the distinction between inner and outer automorphisms, and demonstrate the existence (or absence) of completeness in action, consider the following example from *Abstract Algebra* [DF04, Sec. 4.4, Ex. 5].

Example 2.48 (Automorphisms of D_8)
Recall the dihedral group

$$D_{2n} = \langle r, s \mid r^n = e, \quad s^2 = e, \quad rs = sr^{-1} \rangle$$

of order 8, representing the symmetries of a square from Example 2.8. Its automorphism group $\mathrm{Aut}(D_8)$, consists of all bijective homomorphisms from D_8 to itself.

Using the defining relations of D_8 from Section 2.3.4, and the basic properties of element order (2.4), we determine that any automorphism is determined by images of r and s:

- r has order 4, so its image must be r or r^{-1}.
- s has order 2 and lies outside $\langle r \rangle$, so its image must be one of s, sr, sr^2, sr^3.

This yields $2 \times 4 = 8$ possible automorphisms, all of which satisfy the defining relations. Hence,

$$|\mathrm{Aut}(D_8)| \leq 8.$$

Among these are the *inner automorphisms* $\mathrm{Inn}(D_8)$, arising from conjugation (2.35) and forming a normal subgroup of $\mathrm{Aut}(D_8)$ order 4. For example, conjugating r by s gives

$$srs = r^{-1},$$

defining the nontrivial inner automorphism

$$\varphi_s(r) = srs^{-1} = r^{-1}.$$

By Definition 2.39, D_8 is a normal subgroup of D_{16}

$$D_8 \trianglelefteq D_{16},$$

and conjugation by elements of D_{16} induces automorphisms of D_8. We conclude that these automorphisms account for all of $\mathrm{Aut}(D_8)$, establishing the isomorphism

$$\mathrm{Aut}(D_8) \cong D_8.$$

It follows that

$$\mathrm{Out}(D_8) = \mathrm{Aut}(D_8)/\,\mathrm{Inn}(D_8)$$

has order 2, confirming the existence of non-inner automorphisms.

Remark 2.49. For any abelian group G, the group of inner automorphisms is trivial. Since conjugation acts as the identity, we have

$$\mathrm{Aut}(G) = \mathrm{Out}(G).$$

For example, if $G \cong \mathbb{Z}_p$ is a cyclic group of prime order p, then

$$\mathrm{Aut}(G) \ \cong (\mathbb{Z}/p\mathbb{Z})^\times \quad \text{and} \quad \mathrm{Inn}(G) = 1,$$

so

$$\mathrm{Out}(G) \cong \mathrm{Aut}(G).$$

This analysis demonstrates that the structure of a group directly influences the nature and number of its automorphisms. The computation of $\mathrm{Aut}(D_8)$ shows that the defining presentation and non-abelian relations of the dihedral group of order 8 produce an automorphism group of size 8, isomorphic to D_8 itself. In particular, the fact that conjugation inside D_8 already accounts for half of these automorphisms (the inner subgroup of order 4) while two distinct cosets remain highlights how the interplay of rotation and reflection in D_8 both constrains and enriches its external symmetries. This concrete culmination underscores the power of group-theoretic structure and naturally paves the way for Chapter 3's shift to representation theory, where we will study how groups act as symmetries via linear transformations on vector spaces.

3 Representation Theory

Following Peter Woit's *Quantum Theory, Groups and Representations* [Woi17], this chapter extends definitions and results from group theory to study how group symmetries are realized as linear transformations via *representations*. As Woit notes, representation theory forms the standard mathematical language for symmetry in physical systems like quantum mechanics, where unitary representations preserve a complex inner product structure [Woi17, Sec. 1.3.2].

The following sections formalize fundamental concepts in representation theory and illustrate these with examples that demonstrate how representations explicitly realize group symmetries as linear transformations.

3.1 Introduction to Representations

Recall the formal definition of a group action (2.31). Representation theory specializes this concept by considering the case where the underlying set X is a vector space V, with each group element $g \in G$ acting as an invertible linear transformation:

$$g \cdot v = \rho(g)v, \tag{5}$$

where $\rho : G \to GL(V)$ is a group homomorphism. This specialization enables analysis of group symmetries via their matrix representations, enabling algebraic and geometric analysis of group structure using tools such as eigenvalues, invariants, and decompositions [Woi17, Sec. 1.3.2].

Definition 3.1 (General Representation)**.** Let G be a group. Let M be a set. A **representation**[41] of G on M is a group homomorphism (2.26):

$$\rho : G \to \mathrm{Aut}(M),$$

where $\mathrm{Aut}(M)$ is the group of bijections $M \to M$ (2.44).

When $M = V$ is a vector space, $\mathrm{Aut}(V)$ is restricted to invertible linear maps and becomes $GL(V)$, leading to the following specialization:

Definition 3.2. A **linear representation**[42] of a group G on a vector space V is a

[41] From [Woi17, Sec. 1.3.2]
[42] From [Woi17, Sec. 1.3.2]

homomorphism

$$\rho : G \to GL(V),$$

where $GL(V)$ is the group of invertible linear maps $V \to V$.

The homomorphism property $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ ensures that group multiplication is reflected in the corresponding composition of linear transformations on V, a point emphasized by Woit as the defining property of a representation [Woi17, Sec. 1.3.2]. Upon choosing a basis[43] of V, these linear operators can be represented as matrices, so that

$$GL(V) \cong GL(n, \mathbb{C}),$$

where $\dim(V) = n$ [Woi17, Sec. 4.1]. Thus, a representation becomes a homomorphism

$$G \to GL(n, \mathbb{C}),$$

assigning each group element to an $n \times n$ matrix.

Example 3.3 (Standard permutation representation of S_3)
Consider the standard permutation representation of S_3, where S_3 acts on $\mathbb{C}^3$ by permuting the standard basis vectors $\{e_1, e_2, e_3\}$.[44] By Definition 3.2, for $\sigma \in S_3$, the action

$$\rho(\sigma)(e_i) = e_{\sigma(i)}$$

defines a homomorphism $\rho : S_3 \to GL(3, \mathbb{C})$. With respect to the standard basis, the permutation matrices corresponding to $g \in S_3$ are:

$$\rho(e) = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(no change)},$$

$$\rho((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(Swaps } e_1 \text{ and } e_2\text{)},$$

[43] The ordered set of linearly independent vectors spanning the vector space V [DF04, Sec. 11.1, Def. (2)]

[44] Discussed in [Woi17, Sec. 1.3.2]

$$\rho((123)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{(Sends } e_1 \mapsto e_2, e_2 \mapsto e_3, \text{ and } e_3 \mapsto e_1\text{)}.$$

As explained by Woit [Woi17, Sec. 1.3.2, eq. 1.3], for a finite set X, the space of complex functions

$$F(X) = f : X \to \mathbb{C}$$

is a vector space of dimension $|X|$. If G acts on X, there is an induced action on $F(X)$:

$$(g \cdot f)(x) = f(g^{-1} \cdot x),$$

producing a finite-dimensional representation.

Finite sets therefore yield *finite-dimensional* representations, where group actions can be explicitly analyzed using matrix algebra [Woi17, Sec. 1.4]. Because finite-dimensional representations can be decomposed into irreducible subrepresentations, they admit a classification up to isomorphism, as formalized later in Theorem 3.1. In contrast, when X is *infinite*, the same construction produces *infinite-dimensional* representations, requiring additional tools from functional analysis to handle infinite-dimensional spaces.

Definition 3.4. For a finite dimensional complex space $V \cong \mathbb{C}^n$, a **linear representation into the complex general linear group**[45] is a homomorphism

$$\rho : G \to GL(n, \mathbb{C}).$$

In quantum mechanics, the state space of a system is modeled by $V \cong \mathbb{C}^n$ equipped with an inner product. Woit notes that group actions on such systems are realized as a unitary representation of V. This reinforces the idea that physical symmetries are best modeled as linear transformations that preserve an inner product structure [Woi17, Sec. 1.4] .

3.1.1 Unitary and orthogonal groups

In many contexts, particularly in quantum mechanics and geometry, representations are required to preserve an inner product, so that group actions are isometries

[45] From [Woi17, Sec. 2.1]

respecting the vector space's geometric structure.

Definition 3.5 (Hermitian inner-product)**.** A **Hermitian inner product**[46] on a complex vector space V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

satisfying the following properties for all $u, v, w \in V$ and scalars $\alpha \in \mathbb{C}$:

1. *Conjugate symmetry*:
$$\langle v, w \rangle = \overline{\langle w, v \rangle},$$
where the overline indicates complex conjugation.

2. *Linear in the first variable and conjugate-linear in the second*:
$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle, \quad \langle \alpha u, v \rangle = \alpha \langle u, v \rangle, \quad \langle u, \alpha v \rangle = \overline{\alpha} \langle u, v \rangle.$$

3. *Positive definiteness*:
$$\langle v, v \rangle > 0 \text{ whenever } v \neq 0, \quad \text{and} \quad \langle v, v \rangle = 0 \Leftrightarrow v = 0.$$

Remark 3.6. We use the convention linear in the *first* argument and conjugate-linear in the *second*, Following [Woi17] and an inner product on class functions in [DF04].

This inner product induces a norm via

$$\|v\|^2 = \langle v, v \rangle, \tag{6}$$

and its non-degeneracy yields a conjugate-linear isomorphism $V \to V^*$, given by $v \mapsto \ell_v$, where

$$\ell_v(w) = \langle w, v \rangle.$$

Here, ℓ_v is the functional $w \mapsto \langle w, v \rangle$. Linearity in the first argument ensures that ℓ_v is a linear functional in w, while conjugate-linearity in the second argument means that the map $v \mapsto \ell_v$ is conjugate-linear. This correspondence is an isomorphism, often denoted $V \cong V^*$.

[46]From [Woi17, Sec. 4.4]

Definition 3.7 (Adjoint operator)**.** Let V be a complex vector space equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle$. For a linear operator $L : V \to V$, there exists a unique **adjoint operator** L^* such that

$$\langle Lv, w \rangle = \langle v, L^* w \rangle, \quad \text{for all} \;\; v, w \in V.$$

An operator is **self-adjoint** if $L^* = L$, and **skew-adjoint** if $L^* = -L$.

By definition, L^* is uniquely determined by the inner product structure of the space and does not depend on a choice of basis. It remains to verify that this construction yields a valid linear operator with the stated properties.

When expressed in an orthonormal basis, L^* corresponds to the conjugate transpose matrix $A^* = \overline{L^T}$ of the matrix representing L [Woi17, Sec. 4.5]. It immediately follows that applying the adjoint twice returns the original operator:

$$(L^*)^* = L.$$

In the real case, L^* reduces to the transpose as complex conjugation has no effect. This is shown explicitly in [Woi17, Sec. 4.5].

Definition 3.8 (Unitary group)**.** Let $\mathbb{C}^n$ be a complex vector space equipped with a standard Hermitian inner product $\langle v, w \rangle$. The **unitary group**[47] $U(n)$ is the group of all invertible linear transformations on $\mathbb{C}^n$ that preserve the inner product

$$\langle Lv, Lw \rangle = \langle v, w \rangle \quad \text{for all } v, w \in \mathbb{C}^n.$$

Fixing an orthonormal basis of $\mathbb{C}^n$, these operators correspond to complex matrices $A \in GL(n, \mathbb{C})$ satisfying

$$A^* A = I,$$

where A^* is the conjugate transpose.

By definition, $U(n)$ is identified with the matrix group

$$U(n) = A \in GL(n, \mathbb{C}) \mid A^* A = I,$$

which is equivalent to the condition $A^{-1} = A^*$.

[47] From [Woi17, Sec. 4.6.2]

Proposition 3.9. *A unitary group $U(n)$ is a subgroup of the general linear group $GL(n, \mathbb{C})$*[48].

Proof. By Definition 3.8,

$$U(n) = A \in GL(n, \mathbb{C}) \mid A^*A = I.$$

We verify the subgroup criterion (2.15):

1. **Nonempty**: The identity matrix I satisfies $I^*I = I$, so $I \in U(n)$.
2. **Closure**: Suppose $A, B \in U(n)$. Then

$$B^*B = I \Rightarrow B^{-1} = B^*.$$

   Using the property $(AB)^* = B^*A^*$ for the adjoint of a product, we compute

$$\begin{aligned}(AB^{-1})^*(AB^{-1}) &= (B^{-1})^*A^*AB^{-1} \\ &= BA^*AB^{-1} \\ &= BB^{-1} \\ &= I\end{aligned}$$

So $AB^{-1} \in U(n)$.

By Proposition 2.15, $U(n)$ is a subgroup of $GL(n, \mathbb{C})$. □

Definition 3.10 (Unitary representation)**.** A representation (ρ, V) of G is **unitary**[49] if

$$\langle \rho(g)v_1, \rho(g)v_2 \rangle = \langle v_1, v_2 \rangle \quad \text{for all } g \in G \text{ and } v_1, v_2 \in V.$$

In a unitary representation, group actions preserve the inner product independently of basis [Woi17, Sec. 4.4 – 4.5]. Fixing an orthonormal basis of $V \cong \mathbb{C}^n$, each $\rho(g)$ corresponds to a matrix in $GL(n, \mathbb{C})$ satisfying

$$\rho(g)^*\rho(g) = I,$$

[48] From [Woi17, Sec. 4.6.2]
[49] From [Woi17, Sec. 1.3.3]

for all g, where $\rho(g)^*$ is the adjoint matrix defined via conjugate transpose.[50] Woit notes that this condition reflects both geometric preservation of the Hermitian inner product and algebraic invariance under adjoint operations [Woi17, Sec. 4.6].

This dual geometric and algebraic perspective is fundamental in the analysis of symmetries on complex inner product spaces, providing a natural identification of the unitary group

$$O(n) = A \in GL(n, \mathbb{R}) \mid A^T A = I.$$

For real vector spaces $\mathbb{R}$, the corresponding structure is provided by a Euclidean inner product [Woi17, Sec. 4.4].

Definition 3.11 (Euclidean inner product)**.** A **Euclidean inner product**[51] on a real vector space V is a symmetric map

$$(\cdot, \cdot) : V \times V \to \mathbb{R}$$

that is non-degenerate, linear in both variables, and positive-definite.

Notation. Only positive-definite inner products are considered.[52]

Linear transformations that preserve a Euclidean inner product are called *orthogonal transformations*, as they preserve inner products, norms, and angles, thereby acting as isometries on $\mathbb{R}^n$ [Woi17, Sec. 4.6].

Definition 3.12 (Orthogonal group)**.** Let $\mathbb{R}^n$ be a real n-dimensional vector space equipped with a Euclidean inner product (v, w). The **orthogonal group**[53] $O(n)$ is the group of invertible linear transformations on $\mathbb{R}^n$ that preserve the inner product

$$(Lv, Lw) = (v, w) \quad \text{for all } v, w \in \mathbb{R}^n.$$

Relative to the standard orthonormal basis of $\mathbb{R}^n$, Woit defines $O(n)$ as the group of real invertible matrices $A \in GL(n, \mathbb{R})$ satisfying

$$A^T A = I,$$

[50] This particular case can be seen in [Woi17, Sec. 4.4].

[51] From [Woi17, Sec. 4.4]

[52] As noted by Woit [Woi17, Sec. 4.4], indefinite inner products typically arise in four-dimensional spaces, which are beyond the scope of this thesis.

[53] From [Woi17, Sec. 4.6.1]

so that $A^{-1} = A^T$.

Thus, with respect to the standard orthonormal basis, $O(n)$ is the matrix group

$$O(n) = A \in GL(n, \mathbb{R}) \mid A^T A = I.$$

Proposition 3.13. *The orthogonal group $O(n)$ is a subgroup of the general linear group $GL(n, \mathbb{R})$.*

Proof. Recall the subgroup proof for $U(n)$ (3.1.1) where closure under matrix multiplication and inversion follows from preservation of the inner product. An identical argument applies for

$$O(n) = A \in GL(n, \mathbb{R}) \mid A^T A = I,$$

with the transpose A^T replacing the adjoint A^*.

Since the identity matrix I satisfies

$$I^T I = I,$$

and for any $A, B \in O(n)$, using the property $(AB)^T = B^T A^T$, we have

$$\begin{aligned} (AB^{-1})^T (AB^{-1}) &= B^{-T} A^T A B^{-1} \\ &= B A^T A B^{-1} \\ &= B I B^{-1} \\ &= I. \end{aligned}$$

It follows from Proposition 2.15 that the orthogonal group $O(n)$ is a subgroup of the general linear group $GL(n, \mathbb{R})$. □

Definition 3.14 (Orthogonal representation)**.** Let G be a group. Let V be a finite-dimensional real vector space equipped with a Euclidean inner product $(\cdot, \cdot)$.

A representation (ρ, V) of G is **orthogonal**[54] if

$$(\rho(g)v_1, \rho(g)v_2) = (v_1, v_2) \quad \text{for all } g \in G \quad \text{and} \quad v_1, v_2 \in V.$$

Orthogonal representations are the real counterparts to unitary representations, where the underlying field is $\mathbb{R}$ and group elements are represented by orthogonal matrices relative to a chosen orthonormal basis. Together, these groups provide canonical examples of representations by isometries, preserving inner products, norms, and angles in complex and real inner product spaces, respectively [Woi17, Sec. 4.6].

3.2 Representation Decomposition

3.2.1 Irreducible Representations

Although unitary and orthogonal groups preserve inner products, they can often be decomposed into invariant subspaces. A central goal of representation theory is to classify representations (up to equivalence) by identifying those that cannot be decomposed further.

Definition 3.15 (Irreducible representation)**.** A representation (ρ, V) of a group G is **irreducible**[55] if the only G-invariant subspaces of V are 0 and V itself. If a proper non-zero subspace $W \subset V$ satisfies $\rho(g)W \subset W$ for all $g \in G$, the representation is called *reducible*.

If non-trivial invariant subspaces exist, a representation can often be decomposed as a combination of subrepresentations. Irreducible representations capture the simplest, non-trivial ways in which a group can act linearly and serve as the building blocks for all finite-dimensional representations

As an example of an irreducible representation, consider the following degree-2 real representation of the dihedral group D_{10}, adapted from [DF04, Exercise 2(a), Sec. 18.3].

Example 3.16 (Irreducibility of D_{10})

[54] Derived intuitively from the definition of unitary representation 3.10 as seen in [Woi17, Sec. 1.3.3]

[55] From [Woi17, Sec. 2.1]

Recall the dihedral group of order 10 (2.23),

$$D_{10} = \langle r, s \mid r^5 = s^2 = e, rs = sr^{-1} \rangle,$$

and define a real representation of degree-2

$$\varphi : D_{10} \to GL_2(\mathbb{R})$$

by specifying its values on the generators (3):

$$\varphi(r) = R = \begin{bmatrix} \cos\frac{2\pi}{5} & -\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{bmatrix}, \quad \varphi(s) = S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Here, R represents a rotation by $\theta = \frac{2\pi}{5}$ and S a reflection across the line $x = y$.

Step 1: Verify group relations Confirm that the matrices R and S satisfy the defining relations of D_{10}:

- $R^5 = I_2$ (1): Repeated multiplication gives

$$R^5 = \begin{bmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

- $S^2 = I_2$ (2):

$$S^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = I_2.$$

- $RS = SR^{-1}$ (3): Since R is orthogonal,

$$R^{-1} = R^T = \begin{bmatrix} \cos\frac{2\pi}{5} & \sin\frac{2\pi}{5} \\ -\sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{bmatrix}.$$

  Direct computation shows

$$RS = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix},$$

$$SR^{-1} = SR^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix},$$

so $RS = SR^{-1}$.

Hence, φ is a well-defined representation of D_{10}.

Step 2: Verify Irreducibility Any proper non-zero invariant subspace of $\mathbb{R}^2$ would be a line through the origin. Suppose $W \subset \mathbb{R}^2$ is such a one-dimensional subspace invariant under $\varphi(g)$ for all $g \in D_{10}$.

- Since the rotational angle $\theta = 72°$ does not correspond to $0°$ or $180°$, the matrix R does not preserve any one-dimensional real subspaces of $\mathbb{R}$. Hence, no nontrivial line remains invariant under R.

- The reflection S fixes the lines spanned by the vectors $(1, 1)$ and $(1, -1)$. However, these lines are not preserved by the rotation R, and thus are not invariant under the full group action.

Thus, no single one-dimensional subspace is invariant under the entire group, and by Definition 3.15, the representation φ is irreducible.

This illustrates how a matrix representation of D_{10} realizes its abstract group structure through concrete linear transformations. Establishing irreducibility isolates and identifies the fundamental components from which all representations can be constructed.

Understanding irreducible components is essential for analyzing more complex representations and characterizing the symmetries of higher-dimensional vector spaces. Since every finite-dimensional representation decomposes into irreducible subrepresentations, these components establish a direct link between the group's abstract algebraic structure and its concrete realization through linear transformations.

3.2.2 Direct Sum of Representations

While irreducible representations cannot be decomposed further, many representations can be expressed as *direct sums* of irreducible components. The *direct sum* construction formalizes this by combining representations into a larger representation whose invariant subspaces correspond exactly to the summands.

Definition 3.17 (Direct sum representation)**.** Let (ρ_1, V_1) and (ρ_2, V_2) be representations of a group G, with $\dim(V_1) = n_1$ and $\dim(V_2) = n_2$. Their **direct sum**

representation[56] , denoted $\rho_1 \oplus \rho_2$, is the representation of G on the vector space $V_1 \oplus V_2$. For each $g \in G$, the group homomorphism is defined by the block-diagonal matrix acting on the space

$$(\rho_1 \oplus \rho_2)(g) = \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}.$$

This construction extends naturally to any finite collection of representations. Each summand V_i is G-invariant, and the representation acts on $V_1 \oplus \cdots \oplus V_n$ via the direct sum of the individual actions, with

$$\dim(V) = \sum_i \dim(V_i).$$

Direct sums with at least two summands are reducible, as each summand provides a proper nonzero invariant subspace. This follows from the block-diagonal form of $\rho(g)$ matrices, where each diagonal block corresponds to the action on an invariant subspace V_i.

Example 3.18 (Direct sum decomposition of permutation representation)
To illustrate the direct sum decomposition, consider the finite abelian group

$$G = \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle a, b \mid a^2 = b^2 = e, ab = ba \rangle,$$

defined by its action on $\mathbb{C}^4$, where G permutes the standard basis vectors e_1, e_2, e_3, e_4. This defines the permutation representation

$$\rho : G \to GL(4, \mathbb{C}).$$

Define a new basis of $\mathbb{C}^4$ by

$$f_1 = e_1 + e_2, \quad f_2 = e_1 - e_2, \quad f_3 = e_3 + e_4, \quad f_4 = e_3 - e_4.$$

[56] From [Woi17, Sec. 2.1]

Let P be the change-of-basis matrix whose columns are f_1, f_2, f_3, f_4:

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Define $\rho(a)$ and $\rho(b)$ as the permutation matrices representing the generators of G:

$$\rho(a) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(swaps } e_1 \text{ and } e_2\text{)},$$

$$\rho(b) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{(swaps } e_3 \text{ and } e_4\text{)}.$$

Computing the conjugation $P^{-1}\rho(g)P$ for generators $g = a, b$, we find

$$P^{-1}\rho(a)P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P^{-1}\rho(b)P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Thus, in this basis, the representation ρ simultaneously block-diagonalizes into four one-dimensional invariant subspaces, each corresponding to a character of G.

Here, each invariant subspace corresponds to a one-dimensional representation (character) of G, demonstrating that the permutation representation decomposes as a direct sum of irreducibles.

Theorem 3.1 (Maschke's Theorem)**.** *Let G be a finite group. Let (ρ, V) be a finite-dimensional complex representation of G. Every G-invariant subspace $W \subset V$ admits a complementary G-invariant subspace. Equivalently,*

$$\rho \cong \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_m,$$

where each ρ_i is irreducible.

The theorem holds because the characteristic of $\mathbb{C}$ is zero, which does not divide the order of the finite group G, thereby guaranteeing complete reducibility of finite-dimensional complex representations [Woi17, Sec. 2.1].

Remark 3.19. Maschke's Theorem guarantees the existence of a decomposition of representations into irreducibles, unique up to isomorphism and multiplicity. The classification and multiplicities of these irreducible components will be analyzed using character theory in Section 3.3.

3.2.3 Schur's Lemma

The decomposition of representations into direct sums motivates the study of linear maps that commute with the group action on these components. Schur's Lemma characterizes the linear maps commuting with irreducible representations, showing that such maps are highly restricted and must be scalar multiples of the identity.

Theorem 3.2 (Schur's Lemma)**.** *Let (ρ, V) be a finite-dimensional, complex, irreducible representation of a group G. Suppose $M : V \to V$ is a linear map such that*

$$M\rho(g) = \rho(g)M \quad \textit{for all } g \in G.$$

Then there exists a scalar $\lambda \in \mathbb{C}$ such that

$$M = \lambda \cdot \mathrm{id}_V,$$

where id_V is the identity map on V[57]*.*

This relies crucially on $\mathbb{C}$ being algebraically closed, guaranteeing eigenvalues for all linear operators.

Proof of Schur's Lemma. Let $M : V \to V$ be a linear map such that

$$M\rho(g) = \rho(g)M \quad \text{for all } g \in G.$$

Both the image $\mathrm{im}(M)$ and kernel $\ker M$ are G-invariant subspaces, since for all $g \in G$,

$$\rho(g)(Mv) = M(\rho(g)v).$$

[57]From [Woi17, Sec. 2.1]

Since (ρ, V) is irreducible (3.15), these subspaces are either trivial 0 or the entire space V. Thus, either $M = 0$ or M is invertible.

If $M \neq 0$, then, since $\mathbb{C}$ is algebraically closed, M has at least one eigenvalue $\lambda \in \mathbb{C}$ with a corresponding nonzero eigenspace

$$W = v \in V \mid Mv = \lambda v.$$

For any $v \in W$ and $g \in G$,

$$M(\rho(g)v) = \rho(g)Mv = \lambda\rho(g)v,$$

so $\rho(g)v \in W$. Hence, W is a G-invariant subspace.

By irreducibility (3.15), W must equal V. Thus,

$$Mv = \lambda v \quad \text{for all } v \in V,$$

so

$$M = \lambda \cdot \mathrm{id}_V .$$

□

As Woit explains, its proof relies on the existence of an eigenvalue for such operators [Woi17, Sec. 2.1]. Over the real numbers $\mathbb{R}$, Schur's Lemma may fail due to the potential absence of real eigenvalues when linear operators commute with the representation.

An important consequence for abelian groups follows:

Theorem 3.3 (Commutative Group Theorem)**.** *Let G be a commutative group. Then every irreducible finite-dimensional complex representation of G is one-dimensional.*[58]

Proof. Let (ρ, V) be an irreducible finite-dimensional complex representation of an abelian group G. By Definition 2.2, for all $g, h \in G$ we have

$$\rho(g)\rho(h) = \rho(h)\rho(g),$$

[58]From [Woi17, Th. 2.2, Sec. 2.1].

meaning every operator $\rho(h)$ commutes with all $\rho(g)$.

By Theorem 3.2, each $\rho(h)$ must be a scalar multiple of the identity. Therefore, the image of ρ lies in the subgroup of scalar operators, which act by scalar multiplication on V.

Consequently, since each $\rho(h)$ acts as scalar multiplication, the representation space of V must be one-dimensional:

$$\dim(V) = 1.$$

□

This result imposes strong restrictions on the irreducible representations of abelian groups, ensuring they are one-dimensional. Consequently, the rich and complex phenomena in representation theory arise primarily from non-abelian groups, where higher-dimensional irreducible representations occur. As in Schur's Lemma (3.2), this fails over $\mathbb{R}$, where irreducible representations of abelian groups may have dimension greater than one.

Example 3.20 (Irreducibility of the standard representation of $GL(n, \mathbb{C})$)
Consider the standard representation $\rho : GL(n, \mathbb{C}) \to GL(\mathbb{C}^n)$, where $V = \mathbb{C}^n$, defined by $\rho(g) = g$.[59] The group acts on V by matrix multiplication: for all $g \in GL(n, \mathbb{C})$ and $v \in V$,

$$g \cdot v = gv.$$

Irreducibility argument Suppose $W \subseteq \mathbb{C}^n$ is a non-zero, $GL(n, \mathbb{C})$-invariant subspace:

$$g(W) \subseteq W \quad \text{for all} \quad g \in GL(n, \mathbb{C}).$$

Since $GL(n, \mathbb{C})$ acts transitively on $\mathbb{C}^n$ by invertible linear transformations, the only invariant subspaces under the standard action are 0 and $\mathbb{C}^n$. Hence, the representation is irreducible.

Application of Schur's Lemma By Definition 2.40, the center of $GL(n, \mathbb{C})$ is

$$Z(GL(n, \mathbb{C})) = A \in GL(n, \mathbb{C}) \mid AB = BA \text{ for all } B \in GL(n, \mathbb{C}).$$

Any linear map $M : V \to V$ commuting with all $\rho(g) = g$ lies in the centralizer of

[59] This example follows from the irreducibility of the standard representation of $GL(n, \mathbb{C})$ on $\mathbb{C}^{\ltimes}$, as discussed in [Woi17, Sec. 2.1-2.3].

the image of ρ (2.40). Because ρ is irreducible, Schur's Lemma (3.2) implies

$$M = \lambda \cdot \mathrm{id}_V, \quad \text{for some} \;\; \lambda \in \mathbb{C}^{\times}.$$

Thus, M acts as scalar multiplication by λ, making every vector an eigenvector with eigenvalue λ.

Therefore, the center of $GL(n, \mathbb{C})$ consists exactly of scalar matrices:

$$Z(GL(n, \mathbb{C})) = \lambda I_n \mid \lambda \in \mathbb{C}^{\times}.$$

This application demonstrates how Schur's Lemma restricts the centralizer of an irreducible representation to scalar operators, yielding a direct characterization of the center of $GL(n, \mathbb{C})$.

3.3 Character Theory

Given a representation, its character maps each group element to the trace of the corresponding linear operator, producing a function that encodes key properties of the representation in a basis-independent way.

Definition 3.21 (Character)**.** Let G be a group and (ρ, V) a representation of G over $\mathbb{C}$. The **character**[60] $\chi_V : G \to \mathbb{C}$ is defined by

$$\chi_V(g) = \mathrm{tr}(\rho(g)) \quad \text{for all } g \in G,$$

where tr denotes the sum of diagonal entries, or *trace*, of the linear operator $\rho(g)$.

Consequently, equivalent representations have identical characters, since conjugate matrices have equal trace[61]. This invariance follows from the following basic matrix identity, adapted from [DF04, Exercise 1, Sec. 18.3][62]:

Example 3.22 (Trace identity)
Let A and B be two arbitrary $n \times n$ complex matrices.

[60]From [Woi17, Sec. 9.4.2]
[61]From [DF04, (18.1), Sec. 18.3]
[62]This example is adjusted in order to account for the exclusion of *ring theory* in this thesis.

By definition of the trace,

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}.$$

Interchanging the summations yields

$$\operatorname{tr}(AB) = \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik}.$$

Recognizing that

$$\sum_{i=1}^{n} b_{ki} a_{ik} = (BA)_{kk},$$

it follows that

$$\operatorname{tr}(AB) = \sum_{k=1}^{n} (BA)_{kk} = \operatorname{tr}(BA).$$

Thus, for any such matrices,

$$\operatorname{tr}(AB) = \operatorname{tr}(BA),$$

even though AB and BA may not be equal.

Applying this to $\rho(hgh^{-1}) = \rho(h)\rho(g)\rho(h)^{-1}$, it follows that characters are invariant under conjugation and thereby constant on conjugacy classes.[63]

Proposition 3.23 (Conjugation invariance of characters)**.** *Let (ρ, V) be a representation of a group G. Then,*

$$\operatorname{tr}(\rho(hgh^{-1})) = \operatorname{tr}(\rho(g)), \quad \textit{for all } g, h \in G.$$

Therefore, the character χ_V satisfies

$$\chi_V(g^{-1}xg) = \chi_V(x), \tag{7}$$

making it a class function.

[63] From [DF04, (18.2), Sec. 18.3]

3.3.1 Class Functions

Definition 3.24 (Class function)**.** A **class function**[64] on a group G is any function $f : G \to F$ satisfying

$$f(g^{-1}xg) = f(x) \quad \text{for all } g, x \in G.$$

The space $\mathcal{C}(G)$ of complex-valued class functions forms a finite-dimensional complex inner product space. Recalling the general notion of a Hermitian inner product from Definition 3.5, we now specialize this structure to class functions and definite it as an average over group elements, aligning naturally with the discrete structure of G, and fascilitating the study of characters.

Definition 3.25 (Hermitian inner product on class functions)**.** Let G be a finite group. Let $\mathcal{C}(G)$ denote the space of complex-valued class functions on G. This space carries a natural Hermitian inner product[65] defined by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)} \quad \text{for all } \phi, \psi \in \mathcal{C}(G).$$

Two fundamental properties of characters concerns their value at the identity and their behavior under direct sums:[66]

Proposition 3.26 (Value of the character at the identity)**.** *For a representation* (ρ, V)*,*

$$\chi_V(e) = \operatorname{tr}(\rho(e)) = \operatorname{tr}(I_V) = \dim(V),$$

since $\rho(e) = I_V$*.*

This establishes a baseline value for characters, as the trace at the identity equals the dimension of the representation.

Proposition 3.27 (Character of a direct sum is the sum of characters)**.** *If* $\rho = \rho_1 \oplus \rho_2$*, is a direct sum of representations of* G*, then*

$$\chi_\rho = \chi_{\rho_1} + \chi_{\rho_2}.$$

[64] From [DF04, Sec. 18.3].
[65] From [DF04, Sec. 18.3].
[66] From [DF04, (18.3), Sec. 18.3]

This follows from the linearity of trace and the block-diagonal form of $\rho(g)$ relative to the decomposition $V = V_1 \oplus V_2$ [Woi17, 9.4.2].

By Maschke's Theorem 3.1, every finite-dimensional complex representation decomposes *uniquely* as a direct sum of irreducible representations:

Theorem 3.4 (Complete reducibility with multiplicities)**.** *Let (ρ, V) be a finite-dimensional complex representation of a finite group G. Then there exist irreducible subrepresentations $(\rho_1, V_1), \ldots, (\rho_k, V_k)$ of G, and nonnegative integers $m_1, \ldots, m_k$, such that*

$$(\rho, V) \cong m_1(\rho_1, V_1) \oplus m_2(\rho_2, V_2) \oplus \cdots \oplus m_k(\rho_k, V_k),$$

where each m_i denotes the multiplicity of ρ_i [DF04, Eq. (18.6), Sec. 18.3].

Consequently, by Proposition 3.27, the character χ_ρ decomposes as a sum of irreducible characters weighted by their multiplicities [DF04, Eq. (18.8), Sec. 18.3]:

$$\chi_\rho = m_1\chi_{\rho_1} + m_2\chi_{\rho_2} + \cdots + m_k\chi_{\rho_k}. \tag{8}$$

This bijection holds because irreducible characters form a linearly independent basis of the space of class functions, ensuring that characters uniquely determine the isomorphism class and decomposition multiplicities of representations, thereby underpinning character theory as an effective tool for analyzing and classifying representations [DF04, Eq. (18.9), Sec. 18.3].

Accordingly, $\mathcal{C}(G)$ is a k-dimensional vector space, and is isomorphic to $\mathbb{C}^k$. The irreducible characters $\chi_1 \ldots, \chi_k$ form an orthonormal basis of this space.

Theorem 3.5 (Orthonormality of irreducible characters)**.** *Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G, where k is the number of all conjugacy classes in G. Then, for all $1 \leq i, j \leq k$,*

$$\langle \chi_i, \chi_j \rangle = \delta_{ij},$$

where δ_{ij} is the Kronecker delta[67] *[DF04, Eq. (18.10), Sec. 18.3].*

It follows that any class function can be uniquely expressed as a linear combination of these irreducible characters. A class function corresponds to the character of a representation if and only if its expansion in the irreducible character basis has non-negative integer coefficients.

[67] The *Kronecker delta* is defined in [DF04, Sec. 18.3].

Theorem 3.6 (Character criterion for class functions)**.** *Let G be a finite group, and let $f : G \to \mathbb{C}$ be a class function. Then f is the character of some finite-dimensional complex representation of G if and only if*

$$f = \sum_{i=1}^{k} m_i \chi_i,$$

where each $m_i \in \mathbb{Z}_{\geq 0}$[68] *and $\chi_1, \ldots, \chi_k$ are the irreducible characters of G.*

Proof. Let $\chi_1 \ldots, \chi_k$ be the irreducible characters forming an orthonormal basis of $\mathcal{C}(G)$.

(⇒) Suppose f is the character of a finite-dimensional complex representation (ρ, V). By Theorem 3.4, ρ decomposes uniquely as a direct sum of irreducibles:

$$(\rho, V) \cong m_1(\rho_1, V_1) \oplus \cdots \oplus m_k(\rho_k, V_k),$$

with $m_i \in \mathbb{Z}_{\geq 0}$. By Proposition 3.27, the character χ_V decomposes as

$$f = \chi_V = \sum_{i=1}^{k} m_i \chi_i.$$

This character decomposition is uniquely determined by the representation, since irreducible decomposition is unique up to isomorphism and ordering of summands.

(⇐) Conversely, given

$$f = \sum_{i=1}^{k} m_i \chi_i,$$

with $m_i \in \mathbb{Z}_{\geq 0}$, define

$$V = \underbrace{V_1 \oplus \cdots \oplus V_1}_{m_1 \text{ times}} \oplus \underbrace{V_2 \oplus \cdots \oplus V_2}_{m_2 \text{ times}} \oplus \cdots \oplus \underbrace{V_k \oplus \cdots \oplus V_k}_{m_k \text{ times}},$$

and let ρ act on each copy of V_i via ρ_i. By Proposition 3.27, the character of (ρ, V) is

$$\chi_V = \sum_{i=1}^{k} m_i \chi_i = f.$$

Thus, f is a character of G. □

[68] Derived from Exercise 5 in [DF04, 18.3]

This theorem highlights the completeness of irreducible characters by showing they not only form a basis of $\mathcal{C}(G)$, but also help us determine which class functions arise as characters of representations. The coefficients in such linear combinations correspond precisely to the multiplicities of irreducible subrepresentations in the decomposition.

3.4 Orthogonality Relations

The inner product on characters induces orthogonality relations that distinguish irreducible representations and characterize the decomposition of group representations into irreducible characters. These relations offer two complementary perspectives: algebraically, irreducible characters form an orthonormal basis of the class function space; analytically, unitary representations act by inner product-preserving transformations on vector spaces. Together, these views link the group's algebraic structure with the geometry of its representation spaces.

3.4.1 Algebraic perspective

The first orthogonality relation extends Theorem 3.5, showing that every class function has a unique expansion as a linear combination of irreducible characters, corresponding to the decomposition of representations into irreducibles.

Theorem 3.7 (The first orthogonality relation for group characters)**.** *Let G be a finite group. Let $\chi_1, \ldots, \chi_r$ be the distinct irreducible complex characters of G. Then, with respect to the Hermitian inner product on class functions $\mathcal{C}(G)$ (3.25),*

$$\langle \chi_i, \chi_j \rangle = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. In particular, the irreducible characters form an orthonormal basis for the space $\mathcal{C}(G)$[69].

The second orthogonality relation further illuminates how these patterns interact, by connecting character values across conjugacy classes:

Theorem 3.8 (The Second Orthogonality Relation for group characters)**.** *Use the*

[69] From [DF04, Sec. 18.3]

same notation as in the first orthogonality relation (3.7)*. Then, for all* $x, y \in G$*,*

$$\sum_{i=1}^{r} \chi_i(x)\overline{\chi_i(y)} = \begin{cases} \frac{|G|}{|C_G(x)|} & \text{if } x \text{ is conjugate to } y, \\ 0 & \text{otherwise,} \end{cases}$$

where $C_G(x)$ *denotes the centralizer of* x *in* G[70]*.*

This algebraic framework shows that irreducible characters serve as coordinate functions for class functions, offering a complete basis for describing how conjugacy-invariant information is modeled algebraically. Theorem 3.7 captures their orthonormal basis structure, while Theorem 3.8 reveals how these characters reflect the group's internal symmetries. These results create a clear connection between conjugacy, class functions, and representation theory, providing a natural transition to a complementary analytical perspective.

3.4.2 Analytical perspective

Unitary representations provide an analytical counterpart, realizing symmetry via linear transformations that preserve inner products on complex vector spaces. Orthogonality appears here between vectors in the representation space rather than between functions on the group.

Woit analyzes how the unitary condition imposes orthonormality on matrix rows, leading to algebraic constraints among entries. Although his treatment focuses on the special unitary group $SU(2)$, these conditions similarly apply to the unitary group $U(2)$ [Woi17, Sec. 3.1.2]. We extend this analysis to explicitly parameterize $U(2)$ as matrices preserving the standard Hermitian inner product on $\mathbb{C}^2$.

Example 3.28 (Unitary orthonormality)
Let $U(2)$ be a unitary matrix

$$U = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in M_2(\mathbb{C})$$

such that

$$U^*U = I.$$

[70]From [DF04, Sec. 18.3]

Writing this condition explicitly gives

$$U^* = \begin{bmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{bmatrix} \quad \Rightarrow \quad U^*U = \begin{bmatrix} |\alpha|^2 + |\gamma|^2 & \alpha\overline{\beta} + \gamma\overline{\delta} \\ \overline{\beta}\alpha + \overline{\delta}\gamma & |\beta|^2 + |\delta|^2 \end{bmatrix} = I.$$

From this, we obtain the orthonormality conditions on the rows and columns:

- **Row orthonormality:**

$$|\alpha|^2 + |\beta|^2 = 1, \quad |\gamma|^2 + |\delta|^2 = 1, \quad \alpha\overline{\gamma} + \beta\overline{\delta} = 0$$

- **Column orthonormality**

$$|\alpha|^2 + |\gamma|^2 = 1, \quad |\beta|^2 + |\delta|^2 = 1, \quad \overline{\alpha}\beta + \overline{\gamma}\delta = 0.$$

This ensures that U preserves a standard Hermitian inner product on $\mathbb{C}^2$, embodying the notion of symmetry as a structure-preserving transformation (3.10).

An explicit construction of such a unitary matrix is

$$U = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}, \quad \text{where } |\alpha|^2 + |\beta|^2 = 1.$$

Unitarity is verified by computing

$$U^*U = \begin{bmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} = \begin{bmatrix} |\alpha|^2 + |\beta|^2 & 0 \\ 0 & |\alpha|^2 + |\beta|^2 \end{bmatrix} = I.$$

Hence, $U \in U(2)$.

This construction illustrates that the rows and columns of a unitary matrix are unit vectors in $\mathbb{C}^2$, preserving vector norms. Thus, as noted by Woit, Thus, unitary transformations act as isometries of the unit sphere in $\mathbb{C}^2$, emphasizing the geometric structure of the representation space over the discrete group elements [Woi17, Sec. 7.4].

The algebraic and analytical perspectives provide complementary descriptions of group symmetry. Algebraically, orthogonality relations establish irreducible characters as an orthonormal basis of class functions, reflecting conjugacy-class structure.

Analytically, unitarity corresponds to invariance under inner product-preserving transformations, encoding geometric symmetry. Together, these frameworks unify discrete group structure and continuous geometric invariance, foundational for further study of symmetry as a central organizing concept in various areas of mathematics.

4 Symmetry: A Unifying Principle

Symmetry, understood as a principle of invariance, manifests across mathematics, natural sciences, and physical systems. The group axioms (2.1) provide the framework to formalize intuitive symmetry patterns as algebraic structures.

Wigner described abstract groups as fundamental structures underlying physical laws, with representations providing their realization in physical systems [Wig60]. As established in Chapter 3, when a group acts on an inner product space via unitary representations, it preserves the inner product structure. Thus, symmetry emerges as invariance under the group action. More generally, a group action on a set X partitions it into equivalence classes called orbits where elements are related by the action of group elements.[71] In this setting, symmetry is realized by the group acting on X through structure-preserving transformations.

We now formalize symmetry as invariance under a group action.

Definition 4.1. Let G be a group acting on a non-empty set X[72]. A ***symmetry of*** x is an element $g \in G$ such that

$$g \cdot x = x.$$

Two elements $x, y \in X$ are ***related* by symmetry** under G if there exists a $g \in G$ such that

$$g \cdot x = y.$$

Definition 4.2. The set of all symmetries of x under this action is the *stabilizer* of x, denoted $\text{stab}_G(x)$.

[71] For a more detailed exploration of orbits, see [DF04, Sec. 4.1].
[72] From [DF04, Sec. 1.7]

When X is an inner product space, group actions that preserve the inner product yield geometric interpretations of symmetry; a shown by the orthogonality relations in Section 3.4. These relations transform abstract invariance into geometric structure. In this way, these relations and group action framework reveal the intersection of algebraic and geometric aspects of symmetry, fundamental to the theory of representations.

4.1 Symmetry through geometry

Representing group elements as linear transformations imposes strong constraints on character values. These constraints often reveal direct connections between the group's algebraic structure and the arithmetic properties of its characters.

Two dihedral groups, based on [DF04, Exercise 26, Sec. 18.3], provide a clear illustration of these connections. In the case of D_8, every irreducible character takes values in $\mathbb{Q}$, reflecting a relatively simple symmetry structure.

Example 4.3 (Character values of D_8)
Recall the dihedral group D_8 of order 8 from Equation (3)

$$D_8 = \langle r, s \mid r^4 = s^2 = e, srs = r^{-1} \rangle.$$

Step 1: Group Structure

Since characters are constant on conjugacy classes, any character $\chi : D_8 \to \mathbb{C}$ is determined by its values on conjugacy classes (2.38).

$$C_1 = e, \quad C_2 = r^2, \quad C_3 = r, r^3,$$

$$C_4 = s, sr^2, \quad C_5 = sr, sr^3.$$

Thus, D_8 has 5 conjugacy classes.

Step 2: Irreducible characters

By Maschke's Theorem (3.1), D_8 admits five irreducible representations with degrees satisfying

$$\sum_{i=1}^{5} \dim(V_i)^2 = |D_8| = 8.$$

The only positive integer solution is

$$1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8,$$

so D_8 has four one-dimensional one two-dimensional irreducible representations.

Step 3: Constructing characters

Let $\chi_1, \chi_2, \chi_3, \chi_4$ denote the four one-dimensional characters. Since these are group homomorphisms into $\mathbb{C}^\times$, their values lie in $1, -1 \subset \mathbb{Q}$.

The two-dimensional irreducible representation ρ_5 is defined by

$$\rho_5(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \rho_5(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By Definition 3.21, its character values are

$$\chi_5(e) = \mathrm{tr}(I_2) = 2, \quad \chi_5(r) = \mathrm{tr}(\rho_5(r)) = 0, \quad \chi_5(r^2) = \mathrm{tr}(\rho_5(r)^2) = -2,$$

$$\chi_5(r^3) = \mathrm{tr}(\rho_5(r^3)) = 0, \quad \chi_5(s) = \mathrm{tr}(\rho_5(s)) = 0, \quad \chi_5(sr^k) = 0 \text{ for } k = 1, 2, 3.$$

Thus, all irreducible characters of D_8 are rational-valued.

In contrast, the group D_{10} exhibits a representation whose character values involve irrational values. Stemming from the geometry of a regular pentagon, these values appear through the term $\omega + \omega^{-1}$, where $\omega = e^{2\pi i/5}$, reflecting a deeper arithmetic complexity.

Example 4.4 (An irrational-valued irreducible character of D_{10})
Step 1: Group Structure

The group $D_{10} = \langle r, s \mid r^5 = s^2 = e, srs = r^{-1} \rangle$ has order 10 and five conjugacy classes (2.38):

$$C_1 = e, \quad C_2 = r, r^4, \quad C_3 = r^2, r^3$$

$$C_4 = s, sr^2, \quad C_5 = sr, sr^3.$$

By similar reasoning as in Example 4.3, D_{10} has five irreducible representations. Their degrees satisfy

$$1^2 + 1^2 + 2^2 + 2^2 = 10,$$

so D_{10} has two one-dimensional and three two-dimensional irreducible representations.

Step 2: Irrational trace of a representation

Define a representation $\rho : D_{10} \to GL(2, \mathbb{C})$ by

$$\rho(r) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}, \quad \rho(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $\omega = e^{2\pi i/5}$. This satisfies the group relations:

$$\rho(r)^5 = I, \quad \rho(s)^2 = I, \rho(s)\rho(r)\rho(s) = \rho(r)^{-1},$$

so ρ is a well-defined irreducible representation of D_{10}.

The corresponding character χ_ρ is:

$$\chi_\rho(e) = 2, \tag{9}$$
$$\chi_\rho(r) = \omega + \omega^{-1} = 2\cos(2\pi/5), \tag{10}$$
$$\chi_\rho(r^2) = \omega^2 + \omega^{-1} = 2\cos(4\pi/5), \tag{11}$$
$$\chi_\rho(s) = \text{ tr}(\rho(s)) = 0, \tag{12}$$
$$\chi_\rho(sr^k) = 0 \text{ for } 1 \leq k \leq 4. \tag{13}$$

Step 3: Irrationality We know that

$$\cos(2\pi/5) = \frac{\sqrt{5} - 1}{4} \quad \text{and} \quad \cos(4\pi/5) = -\frac{\sqrt{5} + 1}{4},$$

which are irrational numbers. Hence, χ_ρ is not rational-valued, showing that D_{10} has irreducible characters with irrational values.

Together, these examples illustrate that character values extend beyond integers. Rational-valued characters, as in D_8, correspond to simpler symmetry structures, whereas irrational values in D_{10} reflect deeper geometric and arithmetic complexity. Character theory thus connects algebraic symmetry with arithmetic and geometric properties.

4.2 Symmetry through representation

Representation theory analyzes group actions by decomposing them into invariant subspaces, thereby making the internal structure of the symmetry explicit. While character values encode information about a representation's trace behavior, the full representation reveals its structure via invariant subspaces and corresponding block-diagonal matrix forms.

Example 4.5
Let S_3 be standard permutation representation $\rho : S_3 \to GL(3, \mathbb{C})$ (seen in Example 3.3) by the standard basis e_1, e_2, e_3 of $\mathbb{C}^3$, given by[73]

$$\rho(g)(e_i) = e_{g(i)}, \quad \text{for each } g \in S_3.$$

By Definition 3.2, this defines a homomorphism, so $(\rho, \mathbb{C}^3)$ is a linear representation of S_3.

Step 1: Identify the trivial representation Define the vector

$$u = e_1 + e_2 + e_3 \in \mathbb{C}^3.$$

By definition of the permutation representation ρ,

$$\rho(g)u = \rho(g)(e_1 + e_2 + e_3) = e_{g(1)} + e_{g(2)} + e_{g(3)} = u.$$

leaving v fixed under the action of every $g \in S_3$. Consequently, the subset

$$W = \text{span}(v) \in \mathbb{C}^3$$

is invariant under ρ. The restriction of ρ to W is isomorphic to the trivial representation ρ_{triv}, so

$$W \cong \rho_{\text{triv}}.$$

Since S_3 is a finite group and $\mathbb{C}$ is of characteristic zero, Maschke's Theorem 3.1 guarantees that every invariant subspace has an invariant complement. Therefore there exists a subspace $V \subset \mathbb{C}^3$ such that

$$\mathbb{C}^3 = \rho_{\text{triv}} \oplus V,$$

[73] Adapted from Problem 1 and Group Representations in (**??**)Sec. B.1 and 1.3.2]Woit.

where V is the two-dimensional complementary subspace whose structure will be determined in Example 4.8.

Step 2: Block-diagonalization Without defining or choosing a basis for V, we can write the matrix of any $g \in G$ under ρ in block-diagonal form abstractly as

$$\rho(g) \cong \begin{bmatrix} \rho_{\text{triv}}(g) & 0 \\ 0 & \rho_V(g) \end{bmatrix} \quad \text{for all } g \in S_3.$$

As the trivial representation of ρ is the action on a 1-dimensional space, $\rho_{\text{triv}}(g)$ is identified with the 1×1 identity matrix

$$\rho_{\text{triv}}(g) = I_1 = [1] \in GL(1, \mathbb{C}).$$

This results in the block-matrix form

$$\rho(g) \sim \begin{bmatrix} 1 & 0 \\ 0 & \rho_V(g) \end{bmatrix}, \quad \text{for all } g \in S_3,$$

where $\rho_V(g)$ defines the unknown irreducible V.

This block-diagonalization explicitly displays the decomposition of the representation into invariant subspaces, illustrating how the group acts independently on each component. The decomposition expresses the abstract group action in terms of explicit linear transformations on invariant subspaces, turning abstract symmetry into a concrete algebraic form.

4.3 Character Tables and symmetry classification

While decomposing a single representation reveals its internal structure, it does not capture the full landscape of a group's irreducible representations. Character tables summarize the group's irreducible representations by tabulating their characters across conjugacy classes. This matrix provides a complete numerical fingerprint of the group's internal symmetry.

Because characters are additive over direct sums, the character of any representation is the sum of the characters of its irreducible components. This additivity forms the basis for the orthogonality relations and classification theorems that follow, enabling systematic construction and analysis of character tables [DF04, Exercise 10, Sec.

11.2].

For a finite group G, the character table is an $n \times n$ matrix, where n is the number of conjugacy classes. In this matrix, columns correspond to *conjugacy classes* (2.38), and rows correspond to *irreducible representations* (3.15).

Definition 4.6 (Character table)**.** Let G be a finite group with n conjugacy classes and irreducible characters $\chi_1, \chi_2, ..., \chi_n$. The **character table** of G is the $n \times n$ matrix:

$$\begin{bmatrix} \chi_1(g_1) & \chi_1(g_2) & \cdots & \chi_1(g_n) \\ \chi_2(g_1) & \chi_2(g_2) & \cdots & \chi_2(g_n) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_r(g_1) & \chi_r(g_2) & \cdots & \chi_r(g_n) \end{bmatrix},$$

where $g_1, \ldots, g_n$ are representatives of the conjugacy classes of G.

Each entry $\chi_i(g_j)$ reflects how the i-th irreducible representation responds to the symmetry represented by the conjugacy class g_j. In this way, the character table serves as a symbolic fingerprint of the group's internal symmetry.

The symmetric group S_3 from Section 2.2 provides a classic example as it organizes its elements according to symmetry type. This structure is reflected in its character table.

Example 4.7 (Character Table of S_3)
Recall the conjugacy classes of the symmetric group S_3 from Example 2.41.

The corresponding irreducible characters are:

1. χ_1: Trivial representation

$$\chi_1(e) = 1, \quad \chi_1((12)) = 1, \quad \chi_1((123)) = 1$$

2. χ_2: Sign representation

$$\chi_2(e) = 1, \quad \chi_2((12)) = -1, \quad \chi_2((123)) = 1$$

3. χ_3: Standard 2-dimensional representation

$$\chi_3(e) = 2, \quad \chi_3((12)) = 0, \quad \chi_3((123)) = -1$$

Together, these form the character table of S_3[74]:

Classes	1	(12)	(123)
Sizes	1	3	2
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

This table captures structural information about S_3. Each row shows how the group action decomposes into irreducible components, while orthogonality among rows and columns reflects the group's underlying symmetry. The face that the number of irreducible characters aligns with the number of conjugacy classes is formalized in:

Theorem 4.1. *Let G be a finite group. Then the number of non-isomorphic irreducible complex representations of G is equal to the number of conjugacy classes of G*[75].

From this perspective, representation theory transforms abstract group structure into explicit, computable patterns.

Revisiting the permutation action of S_3 illustrates how symmetry appears not only as invariance, but also as decomposition [Woi17, B.1, Problem 1]. By constructing explicit matrix representations, the group's internal structure appears as a direct sum of fundamental symmetry types.

Example 4.8 (Decomposing a permutation representations of S_3)
From Example 4.5, recall the direct sum decomposition

$$\mathbb{C}^3 = \rho_{\text{triv}} \oplus V,$$

of $\rho : S_3 \to GL(3, \mathbb{C})$, where ρ_{triv} is the trivial representation of $\mathbb{C}^3$, and V is the complementary two-dimensional subspace.

Step 1: Identify the Standard Representation

Define the standard representation of $\mathbb{C}3$ as

$$V = \{x \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 0\}.$$

[74]Given by [DF04, Sec. 19.1]
[75]From [DF04, Sec. 18.3]

The invariance of V is verified under the standard permutation representation ρ_{std}, where

$$\rho(g)(x_i) = x_{g^{-1}(i)}, \quad \text{for all } g \in S_3,$$

as for any $x \in V$, we have

$$x_1 + x_2 + x_3 = 0.$$

Then, the transformed vector $\rho(g)(x)$ satisfies

$$\rho(g)(x_1, x_2, x_3) = x_{g^{-1}(1)} + x_{g^{-1}(2)} + x_{g^{-1}(3)} = x_1 + x_2 + x_3 = 0,$$

making V ρ-invariant.

Step 2: Choose a Basis for V

We choose a basis for V

$$v_1 = e_1 - e_2, \quad v_2 = e_1 - e_3$$

satisfying

$$(v_1)_1 + (v_1)_2 + (v_1)_3 = 1 - 1 + 0 = 0, \quad (v_2)_1 + (v_2)_2 + (v_2)_3 = 1 + 0 - 1 = 0.$$

For two linearly independent vectors v_1, v_2 Definition 2.10,

$$V = \text{span}\{v_1, v_2\}.$$

The restriction of ρ to V is isomorphic to the two-dimensional standard representation, denoted ρ_{std}

$$V \cong \rho_{\text{std}}.$$

Step 3: Final block-diagonal form

From ρ_{triv} and ρ_{std} we obtain the ordered basis

$$\mathcal{B} = \{u, v_1, v_2\} = \{(1, 1, 1), (1, -1, 0), (1, 0, -1)\}.$$

With respect to this new basis, we compute the images of v_1, and v_2 under $\rho(g)$, expressing the result as linear combinations of v_1 and v_2 take block-diagonal form.

1. The identity $g = e$:

$$\rho(e)(v_1) = v_1, \quad \rho(e)(v_2) = v_2.$$

So

$$\rho_{\text{std}}(e) = I_2 \quad \Longrightarrow \quad \rho(e) \cong \begin{bmatrix} 1 & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

2. The transpose $g = (12)$:

$$\rho((12))(v_1) = \rho((12))(e_1 - e_2) = e_2 - e_1 = -v_1,$$

$$\rho((12))(v_2) = \rho((12))(e_1 - e_3) = e_2 - e_3 = (e_2 - e_1) + (e_1 - e_3) = -v_1 + v_2$$

So

$$\rho_{\text{std}}((12)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \Longrightarrow \quad \rho((12)) \cong \begin{bmatrix} 1 & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \end{bmatrix}.$$

3. The 3-cycle $g = (123)$:

$$\rho((123))(v_1) = \rho((123))(e_1 - e_2) = e_2 - e_3 = (-v_1 + v_2),$$

$$\rho((123))(v_2) = \rho((123))(e_1 - e_3) = e_2 - e_1 = -v_1.$$

So

$$\rho_{\text{std}}((123)) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \quad \Longrightarrow \quad \rho((123)) \cong \begin{bmatrix} 1 & 0 \\ 0 & \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}.$$

Each matrix exhibits a $1 \oplus 2$ block structure, with the top-left scalar entry corresponding to the trivial representation, and the lower 2×2 block giving the standard representation.

Thus, the block-diagonalization of the permutation representation decomposes as

$$\rho \cong \rho_{\text{triv}} \oplus \rho_{\text{std}},$$

with matrix representation

$$\rho(g) \sim \begin{bmatrix} 1 & 0 \\ 0 & \rho_{\text{std}}(g) \end{bmatrix}$$

This decomposition reveals that a simple permutation action *categorizes* the symmetries of S_3 into irreducible building blocks. The trivial subspace corresponds to vectors fixed under all permutations, while the complementary subspace captures the essential nontrivial action of the group. These decompositions not only help classify representations but also clarify the internal symmetries of the spaces on which the group acts.

Building on these decompositions, the next step is to study representations preserving additional geometric structure. Inner product preservation and orthogonality enriches symmetry analysis, linking algebraic decompositions to the geometric invariants arising from inner product spaces.

4.4 Preservation through orthogonal representations

Problem 4 in [Woi17, B.2], which characterizes orthogonal matrices via orthonormality of their rows, illustrates a key characterization of orthogonal matrices by showing their defining property implies that both their rows and columns are orthonormal sets.

Example 4.9 (Characterization of orthogonal matrices via row orthonormality) Let $M \in M_n(\mathbb{R})$. Then $M \in O(n)$ if and only if the rows of M form an orthonormal set with respect to a standard Euclidean inner product on $\mathbb{R}^n$ (3.11).

Let

$$M = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

where each $v_i \in \mathbb{R}^n$ is the i-th row vector of M. Recall the standard Euclidean inner product on $\mathbb{R}^n$ is given by

$$(x, y) = x^T y,$$

for any $x, y \in \mathbb{R}^n$ (3.11).

Then, the (i, j)-th entry of the matrix product $MM^T \in M_n(\mathbb{R})$ is

$$(MM^T)_{ij} = v_i \cdot v_j = (v_i, v_j).$$

Hence,

$$MM^T = I \quad \Leftrightarrow \quad (v_i, v_j) = \delta_{ij} \quad \text{for all} \;\; i, j = 1, \ldots, n.$$

That is, the rows of M are mutually orthonormal vectors.

Since $M \in O(n)$ implies both $M^T M = I$ and $MM^T = I$, the columns of M also form an orthonormal set under a Euclidean inner product.

This characterization of orthogonal matrices in terms of orthonormal rows follows from the preservation of the Euclidean inner product by orthogonal transformations [Woi17, Sec. 4.1 and 4.4]. Because of this preservation, they define isometries of $\mathbb{R}^n$, exemplifying how algebraic conditions translate to geometric invariances via representations.

Extending this to complex vector spaces, unitary matrices preserve the Hermitian inner product, providing a natural generalization [Woi17, Problem 4, B.2].

Example 4.10 (Characterization of unitary matrices via column orthonormality) Let $A \in M_n(\mathbb{C})$. Then $A \in U(n)$ if and only if the columns of A form an orthonormal set with respect to a standard Hermitian inner product on $\mathbb{C}^n$. The same holds for the rows.

Let $A = [v_1 \mid \cdots \mid v_n]$, where $v_j \in \mathbb{C}^n$ is the j-th column of A. Recall that a standard Hermitian inner product on $\mathbb{C}^n$ is given by

$$\langle x, y \rangle = x^* y,$$

where x^* denotes the conjugate transpose of x. Then the (i, j)-th entry of $A^* A$ is

$$(A^* A)_{ij})v_i^* v_j = \langle v_i, v_j \rangle.$$

Thus,

$$A^* A = I \Leftrightarrow \langle v_i, v_j \rangle = \delta_{ij}.$$

That is, the columns $v_1, \ldots, v_n$ form an orthonormal set in $\mathbb{C}^n$ with respect to the Hermitian inner product.

Since $A \in U(n)$ if and only if $A^*A = I$, the result follows. A similar argument applied to $A^* \in U(n)$ shows that the rows of A are also orthonormal.

Orthogonal and unitary representations thus preserve the geometric structure of the vector space, including inner products, lengths, and angles. Through these structure-preserving transformations, representation theory links abstract symmetry to the concrete geometry of invariant lengths and angles.[76].

[76]See [Woi17, Sec. 3.1.2 and Ch. 4] for a detailed treatment of structure-preserving representations.

5 Conclusion

5.1 Symmetry as a spectrum

Symmetry lies at the intersection of algebraic abstraction and geometric realization, connecting formal structure with spatial intuition. Defining symmetry as invariance under transformation and realizing groups as linear actions on vector spaces trace a path from intuitive patterns to a rigorous framework. Through this lens, symmetry serves as both a structural invariant and a dynamic tool for characterizing transformations.

Examples from previous sections highlight this connection between abstract and concrete descriptions. For instance, the symmetric group S_3, introduced in Group Theory (**??**), is defined as the group of all bijections on a three-element set. Abstractly, its structure encodes the ways elements can be permuted. In Chapter 3 (3.1), this abstraction is made concrete through the linear transformations given by its standard representation. This example demonstrates how a combinatorial group action induces linear operators on a vector space, making the abstract group structure accessible to computation and geometry.

A similar connection appears in the analysis of the dihedral group, introduced in Group Theory Example 4.3 and revisited via its representations in Chapter 3. Algebraically, D_8 is defined by generators and relations. In geometric terms, these correspond to the isometries of a square. The representation of D_8 acting on $\mathbb{R}^2$ realizes it as a group of real matrices, illustrating how group operations preserve spatial structure. This representation aligns algebraic operations with geometric transformations, showing how group symmetries preserve spatial configuration.

These examples illustrate that symmetry is not a binary property but a unifying spectrum in which geometry, algebra, and analysis describe the same phenomena from different perspectives. The formalization of this spectrum via linear transformations is a theme emphasized in both *Abstract Algebra* by Dummit and Foote and *Quantum theory, Groups and Representations* by Woit.

This spectrum-oriented perspective also contextualizes the broader significance of symmetry. In physics, for instance, Noether's theorem identifies a direct correspondence between continuous symmetries and conserved quantities, such as momentum arising from translational invariance [Woi17, Sec. 35.2]. Although the thesis is restricted to finite groups, its concluding reflections point toward the extension of these ideas to continuous symmetries. Representation theory connects algebraic groups with differential geometry and quantum physics. The thesis's examples foreshadow how this algebra-to-geometry bridge becomes central to understanding structure and transformation, exemplifying why mathematical abstraction can so effectively capture the patterns observed in nature.

5.2 Further areas of study

The study of symmetry extends far beyond the foundations of the finite groups and representations that appear in this thesis.

Within the scope of the primary references on which this thesis is based, natural directions for further study include:

- **Continuous symmetries (Lie groups and Lie algebras):** The thesis primarily addressed finite and discrete symmetry groups, leaving the theory of continuous (Lie) groups as a natural next step. Future work could explore the classification of irreducible representations of groups such as $SO(3)$ and $SU(2)$, and examine how continuous symmetries in geometry and quantum theory correspond to algebraic structures. Woit discusses how the representation theory of Lie groups connects algebra with differential geometry and underlies the structure of physical laws [Woi17, Sec. 8.3].

- **Quantum Groups and Physica:** The thesis initially aimed to include elementary physical applications [Woi17, Sec. 35.2], such as deriving conservation laws from translation invariance via Noether's theorem and showing how group-theoretic arguments have historically predicted elementary particles. However, constraints of time and background made this infeasible. A natural extension would be to investigate how continuous-symmetry principles manifest in modern physics, particularly in classification problems beyond the finite simple-group theorem. Building on the idea of "linearizing" symmetry through group representations [Woi17], future directions include studying representations of algebraic and quantum groups, which connect representation theory to topology and number theory.

- **Galois Theory and Field Extentions:** Beyond geometry, symmetry also arises in field extensions, where Galois groups capture the structure of polynomial solutions. This reflects a distinct but parallel perspective on symmetry, rooted in the same group-theoretic principles [DF04, Ch. 13-14].[77]

- **Harmonic and Orbital Symmetry:** Beyond formal mathematics, symmetry inspires creative interdisciplinary projects. One example is the *SYSTEM Sounds*[78] project, which transforms the orbital resonances[79] of planetary systems into harmonic musical compositions. This interplay between celestial mechanics and sound revives Johannes Kepler's vision of a mathematically harmonious cosmos. Though speculative in his time, Kepler's ideas now find formal expression through tools such as harmonic analysis and representation theory, allowing his musical intuition to gain new scientific relevance.

5.3 Final thoughts

As Wigner famously observed, the ability of abstract mathematics to describe physical reality is remarkable.Symmetry, a unifying principle at the heart of these structures, offers one possible explanation: it encodes the invariances underlying the laws of nature. In this light, the study of symmetry is not only mathematical but also philosophical, raising questions about the nature of structure and reality.

This thesis has explored how the abstract language of group theory and its concrete realizations via representation theory reinforce each other within a coherent mathematical framework. As one source succinctly puts it, group theory is "the study of symmetry," while representation theory reduces such problems to linear algebra [Woi17]. In practice, these ideas manifest through examples ranging from permutation groups to matrix actions, illustrating how the same underlying symmetry can operate on diverse mathematical structures. By articulating both perspectives, the thesis clarifies how structural insights from abstract groups and concrete realizations through representations interact to form a coherent mathematical picture.

Across these domains, symmetry offers both conceptual clarity and structural depth as a unifying mathematical theme. Whether expressed analytically, geometrically, or computationally, symmetry reveals a unified framework for understanding the patterns underlying space, time, and transformation. In this light, Woit's remark

[77] Woit also emphasizes this point via discussion on the quantum behavior of fundamental particles, linking abstract algebraic structures to quantum behavior [Woi17].
[78] This project can be viewed at https://www.system-sounds.com/nasa/
[79] These are periodic ratios between planetary orbital periods.

that the framework of quantum mechanics appears "perfectly natural" through the lens of representation theory acquires particular relevance. Abstract constructions often gain concreteness when interpreted as symmetries acting on physical or geometric systems. Representation theory thus stands, in both principle and practice, as a central theme in modern mathematics.

Reflecting on the project as a whole, I find that it bridges foundational theory with meaningful application in a way that is both coherent and informative. The progression was structured to ensure that each step built logically upon the previous. Engaging with these concepts has deepened my understanding of symmetry as a guiding principle and clarified its role as a bridge between abstraction and application.

Further study promises to deepen both the mathematical framework of symmetry and our understanding of the structures it illuminates. From ancient celestial harmonies to their modern reinterpretations, symmetry continues to connect intuition with formal structure. It remains, in mathematics and beyond, a bridge linking the tangible and the abstract, guiding our understanding of the language of the universe.

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