SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Constructing the Mean Value Theorem

av

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2025 - No K2

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Constructing the Mean Value Theorem

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Självständigt arbete i matematik 15 högskolepoäng, grundnivå

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2025

Abstract

This thesis explores the foundational framework of constructive analysis, replacing classical nonconstructive principles such as The Law of Excluded Middle (LEM) with explicit constructive methods that culminate in a proof of the Mean Value Theorem. By strengthening notions of continuity and carefully refining foundational concepts, it shows how classical results can be reestablished by discarding assumptions that fail to produce explicit constructive examples.

Sammanfattning

Denna uppsats utforskar grunden av konstruktiv analys genom att ersätta icke-konstruktiva principer så som *Lagen om det uteslutna tredje* med tydliga, konstruktiva metoder som bygger upp till att bevisa Medelvärdessatsen. Genom att förstärka begrepp om kontinuitet och förfina specifika grundläggande koncept, visar vi hur klassiska resultat inom matematisk analys kan bli återetablerade genom att undvika antaganden som inte resulterar i specifika konstruktioner.

Acknowledgements

I want to express my immense gratitude to my advisor Peter LeFanu Lumsdaine for his guidance and support throughout the creation of this thesis.

Thank you David and Cecilia for being my biggest cheerleaders.

I would also like to acknowledge the use of ChatGPT for assistance in refining grammar, formatting LaTeX code, and providing general background knowledge on the history of Constructive Mathematics, which helped me narrow down the relevant topics to cover.

Contents

1 Introduction

Constructivism is a branch of the foundations of mathematics that imposes stricter criteria on methods of proof and notions of existence compared to classical mathematics. While classical mathematics accepts that an object can be proven to exist if one can prove its nonexistence is impossible, constructive mathematics challenges the validity of those types of methods. Constructively, that interpretation of existence does not hold as constructive mathematics requires that existence claims be supported by explicit constructions. In other words, to assert that an object exists, one must provide a concrete method or algorithm to construct it.

More formally, constructive mathematics does not accept the Law of Excluded Middle or other principles that imply it, such as the Axiom of Choice.

Definition 1.1 (Law of Excluded Middle)**.** Any statement is either true or false.

This is very obvious in classical mathematics, that there is no middle ground between true and false. However, in constructive mathematics, the absence of a proof for either the truth or falsity of a statement means we cannot assert either which is the reality of many unsolved problems in present day.

Lets take a look at an example of a nonconstructive proof to the following proposition:

Proposition 1.2. *There exist irrational numbers a and b such that a^b is rational.*

Nonconstructive proof. Take the number $\sqrt{2}^{\sqrt{2}}$ and by the law of excluded middle, it is either rational or irrational.

If it is rational, then $a = b = \sqrt{2}$ and if it is not, let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.

Then

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2,$$

which is a rational number and thus concludes the proof. □

We will proceed to show how Proposition 1.2 is proven constructively.

Constructive proof. Take the number $\sqrt{2}^{\log_2 9}$, then

$$2^{\frac{1}{2}\log_2 9} = 2^{\log_2 9^{\frac{1}{2}}} = 9^{\frac{1}{2}} = \pm 3,$$

which is rational. □

In the second proof, we avoided the Law of Excluded Middle by explicitly constructing the irrational numbers a and b, thereby satisfying the constructive criteria. Of course, the irrationality of $\log_2 9$ needs to be established, but that is not harder than showing the irrationality of $\sqrt{2}$.

Our goal is to explore the theory of real analysis constructively, culminating in a constructive proof of the Mean Value Theorem, following the constructive theory developed by Errett Bishop [BB85]. Along the way, we will highlight the differences between classical and constructive approaches, particularly focusing on how certain classical results must be reformulated or proven differently to meet constructive standards. We will also examine how some classical theorems fail constructively, and how the constructive framework addresses these issues.

In this thesis, we accept all standard axioms and rules for integers and rational numbers as in classical mathematics, as classical and constructive mathematics are provable in the same way on finite discrete objects.

2 Brouwerian Counterexamples

2.1 A brief history

L.E.J. Brouwer, the founder of intuitionism criticised many classical theorems that relied on nonconstructive proofs, especially those that implied solutions to unsolved problems. He argued that such theorems could not be constructively valid because if they were, we could use them to solve these open problems. Brouwer thus developed a method that demonstrates this, the Brouwerian counterexample. One of his earliest examples of an unsolved problem that developed into such a counterexample was the question *"is there in the decimal expansion of π a decimal that in the long run occurs more often than others?"* [AS15]. In Brouwer's time, and still to this day, we do not know the truth to this statement. Therefore we can't assume the Law of Excluded Middle hold for it and is thus deemed nonconstructive.

Brouwer faced criticism his rejection of the Law of Excluded Middle, particularily from David Hilbert [Dal13]. Hilbert called out the intuitionistic limitations on mathematical concepts like irrational numbers and set theory. Brouwer, in turn, argued that LEM was an unjustified generalisation from finite systems to all mathematics, emphasising a constructivist and intuitionist perspective.

The foundational discussion soon became a conflict that was a defining moment in the history of mathematics, divided into two camps between Brouwer and Hilbert, with the latter having better talent at inspiring a measure of loyalty. Despite this setback for Brouwer, his ideas laid the groundwork for future developments in constructive mathematics and the philosophy of mathematics.

Errett Bishop revived the interest in constructive mathematics that had laid virtually stagnant by the mid-1960s with his publication Foundations of Constructive Analysis [Bis67]. He systematically developed the method of Brouwerian Counterexamples to further show a statement is nonconstructive.

For more historical context about Brouwer's influence on constructivism and his early development of Brouwerian Counterexamples, his biography L.E.J. Brouwer – Topologist, Intuitionist, Philosopher by Dirk van Dalen [Dal13] and the article Brouwerian Counterexamples by Mark Mandelkern [Man89] are warmly recommended .

2.2 Omniscience principles

Bishop systematically formulated Brouwer's nonconstructive examples into several principles of omniscience, two of which we will list in this thesis. A mathematical proof that is shown to entail one of the principles of omniscience is deemed nonconstructive and is called a Brouwerian counterexample. It is not a counterexample in the true sense of the word, but rather a demonstration that a proposition is not constructive.

The presentation of omniscience principles and the treatment of Brouwerian counterexamples in this section is influenced by authors Bridges and Vîţă with their book Techniques of Constructive Analysis [BV06] and the first chapters of Varieties of Constructive Mathematics by Bridges and Richman [BR87].

Definition 2.1 (Limited principle of omniscience)**.** If (a_n) is binary sequence of integers, then either $a_n = 0$ for all n or else there exists n such that $a_n = 1$.

The limited principle of omniscience is a special case of the law of excluded middle and thus nonconstructive. A consequence of *LPO* is the following principle, the lesser limited principle of omniscience.

Definition 2.2 (Lesser limited principle of omniscience)**.** For a binary sequence (a_n) with at most one term equal to 1 (in the sense $a_m a_n = 0$ for all $m \neq n$), either $a_{2n} = 0$ or $a_{2n+1} = 0$ for all n.

In other words, we can decide whether all even-indexed terms are zero or all odd-indexed terms are zero.

2.3 A Brouwerian counterexample

Proposition 2.3. *The statement*

$$x \geq 0 \text{ or } x \leq 0 \text{ for all } x \in \mathbb{R}$$

implies LLPO.

Proof. Consider a binary sequence (a_n) that has at most one term equal to one, and use it to define the binary expansion of a real number

$$x = \sum_{n=1}^{\infty} (-1)^n a_n 2^{-n}. \tag{2.3.1}$$

If $x = 0$ then (a_n)=0. If $x \neq 0$, then there is exactly one n such that $a_n = 1$ and

$$x = \sum_{n=1}^{\infty} (-1)^n 2^{-n} = \pm 2^{-n} \tag{2.3.2}$$

depending on whether n is even or odd.

If x is positive, then all odd indexes are zero and if x is negative, then all even indexes are zero. Therefore, proposition 2.3 implies *LLPO*.

□

Constructive real analysis

3 The Real number system

For the purposes of this thesis, we will not delve into the full construction of the real numbers. Instead, we will present only the definitions, lemmas, and propositions that are directly pertinent to proving Lemma 3.9, which plays a crucial role in our later proofs.

We construct the real numbers using sequences in $\mathbb{Q}^{\mathbb{Z}^+}$, the set of all sequences of rational numbers indexed by the positive integers.

Definition 3.1. A sequence $x \equiv (x_n)$ of rational numbers is called a *real number* if it is *regular*, that is, it satisfies

$$|x_m - x_n| \leq m^{-1} + n^{-1} \text{ for all positive integers } m \text{ and } n. \tag{3.1.1}$$

For the following definitions and lemmas, all indices m, n, and N are positive integers.

Definition 3.2. A real number $x \equiv (x_n)$ is *positive* if

$$x_n > n^{-1} \text{ for some } n. \tag{3.2.1}$$

Definition 3.3. A real number $x \equiv (x_n)$ is *nonnegative* if

$$x_n \geq -n^{-1} \text{ for all } n. \tag{3.3.1}$$

Since the conditions in these definitions for real numbers are local and pointwise, we will introduce two lemmas that offer uniform conditions. This approach will make it easier to use them in constructing proofs.

Lemma 3.4. *A real number x is positive if and only if there exists a positive integer N such that*

$$x_m \geq N^{-1} \textit{ for all } m \geq N. \tag{3.4.1}$$

Proof. ($\Rightarrow$) Assume that x is positive. Then there exists a positive integer n such that $x_n > n^{-1}$.

Let us choose N to be any sufficiently small positive integer satisfying

$$2N^{-1} \leq x_n - n^{-1}. \tag{3.4.2}$$

For all $m \geq N$, then we get

$$|x_m - x_n| \leq x_m + x_n \leq m^{-1} + n^{-1} \leq N^{-1} + n^{-1} \tag{3.4.3}$$
$$x_m \geq x_n - m^{-1} - n^{-1} \geq x_n - N^{-1} - n^{-1} \tag{3.4.4}$$
$$x_m \geq x_n - m^{-1} - n^{-1} \geq x_n - N^{-1} - n^{-1} \geq N^{-1}. \tag{3.4.5}$$

Equation (3.4.3) is due to the regularity condition (3.1.1) and reversing the inequalities gives (3.4.4). Finally, we combine this with $x_n - n^{-1} \geq 2N^{-1}$ that shows $x_m \geq N^{-1}$ for all $m \geq N$.

($\Leftarrow$) Conversely, if $x_m \geq N^{-1}$ for all $m \geq N$, then $x_{N+1} \geq N^{-1} > (N+1)^{-1}$, so x is positive.

Conversely, assume that $x_m \geq N^{-1}$ for all $m \geq N$. Choose $n = N+1$. Then $n > N$, so $n^{-1} < N^{-1}$. From our assumption,

$$x_n \geq N^{-1} > n^{-1}.$$

Therefore, by definition 3.2, x is positive. □

Lemma 3.5. *A real number x is nonnegative if and only if for every positive integer n, there exists a positive integer N_n such that*

$$x_m \geq -n^{-1} \text{ for all } m \geq N_n. \tag{3.5.1}$$

Proof. ($\Rightarrow$) Assume that x is nonnegative. Then, by Definition 3.3, for all n,

$$x_m \geq -m^{-1} \geq -n^{-1}$$

where $m \geq n$.

By letting $n \equiv N_n$, equation (3.5.1) is valid.

($\Leftarrow$) Conversely, we begin by assuming (3.5.1) holds with $n \equiv N_n$ and let $k, m, n \in \mathbb{Z}^+$.

Expand $x_k \geq x_m - |x_m - x_k|$ using the regularity condition and the triangle inequality to get

$$x_k \geq x_m - |x_m - x_k| \geq -n^{-1} - m^{-1} - k^{-1}.$$

□

For the following proposition, $\mathbb{R}^*$ represents either $\mathbb{R}^+$ or $\mathbb{R}^{0+}$, where $\mathbb{R}^+$ is the set of all positive real numbers and $\mathbb{R}^{0+}$ is the set of all nonnegative real numbers.

Proposition 3.6. *If x and y are real numbers, then*

i.) $x + y, xy \in \mathbb{R}^*$ *whenever* $x, y \in \mathbb{R}^*$

ii.) $x + y \in \mathbb{R}^{0+}$ *whenever* $x \in \mathbb{R}^{0+}$ *and* $y \in \mathbb{R}^*$.

Proof.

i.) For sequences of rational numbers (x_n) and (y_n), their term-wise sum $(x_n + y_n)$ is also a sequence of rational numbers. Furthermore, if (x_n) and (y_n) are regular sequences, then $(x_n + y_n)$ is also regular since the regularity condition is preserved under addition:

$$|(x_m+y_m)-(x_n+y_n)| = |(x_m-x_n)+(y_m-y_n)| \leq |x_m-x_n|+|y_m-y_n| \leq 2(m^{-1}+n^{-1}).$$

Hence, for real numbers $x \equiv (x_n)$ and $x \equiv (y_n)$, their sum is defined as $x + y \equiv (x_n + y_n)$.

ii.) Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^{0+}$.

By Lemma 3.4, there exists a positive integer N such that $x_m \geq N^{-1}$ for all $m \geq N$.

Since y is nonnegative, for the positive integer $n > 2N$, Lemma 3.5 guarantees $y_m \geq -n^{-1}$ for all $m \geq N_n$. Remember that $-n^{-1} > (2N)$ follows from $n > 2N$.

Set $K = \max\{N, N_n\}$. Then, for all $m \geq K$, the sum $x_m + y_m$ defines as

$$x_m + y_m \geq N^{-1} - n^{-1} > N^{-1} - (2N)^{-1} = (2N)^{-1}. \tag{3.6.1}$$

Since $x_m + y_m > (2N)^{-1}$ for some integer $2N$, the sum $x + y$ defines as a positive real number. □

Definition 3.7. Let x and y be real numbers, then

$$x > y \text{ if } x - y \in \mathbb{R}^+ \tag{3.7.1}$$

$$x \geq y \text{ if } x - y \in \mathbb{R}^{0+}. \tag{3.7.2}$$

Lemma 3.8. *A real number x is nonnegative if and only if $-x$ is not positive.*

Proof. ($\Rightarrow$) Let x be a nonnegative. By definition, $x_n \geq -n^{-1}$ for all n. Assuming $-x$ is positive, then $-x_n > -n^{-1}$ for all n. We know these inequalities are contradictory since they are rationals.

($\Leftarrow$) Now, let $-x \not> 0$.

Trichotomy holds for rationals, so for each positive integer n, either $-x_n \leq n^{-1}$ or $-x_n > n^{-1}$. We can rule out the latter, since it implies that $-x$ is positive which contradicts with our assumption. Hence, $-x_n \leq n^{-1}$ for all n.

For each positive integer n, we know $-x_n \leq n^{-1}$ or $-x_n > n^{-1}$, as all these are rational. We can rule out the latter, since it implies that $-x$ is positive which contradicts with our assumption. Hence, $-x_n \leq n^{-1}$ for all n.

For each positive integer n, either $-x_n \leq n^{-1}$ or $-x_n > n^{-1}$. We can rule out the latter, since it implies that $-x$ is positive which contradicts with our assumption. Hence, $-x_n \leq n^{-1}$ for all n.

Multiplying both sides of the inequality by -1, we get $x_n \geq -n^{-1}$ for all n which means that $x \geq 0$. Thus x is nonnegative. □

Lemma 3.9. *Let x and y be real numbers. Then, $x \leq y$ if and only if $x \not> y$.*

Proof. By Lemma 3.8, apply to $y - x$. □

4 Sequential convergence in the reals

Thanks to our construction of real numbers in the previous section, the concept of sequential convergence in constructive analysis closely parallels its classical counterpart. However, a careful examination of Cauchy sequences and their relationship to the real numbers becomes crucial when we move on to defining and understanding continuity of functions.

Definition 4.1. A sequence of real numbers (x_n) converges to a real number x^* if for each positive integer k, there exists a positive integer N_k such that

$$|x_n - x^*| \leq k^{-1} \text{ for all } n \geq N_k. \tag{4.1.1}$$

We call x^* the limit of sequence (x_n) which is unique. Then we say $\lim_{n\to\infty}(x_n) = x_0$. If $x_n > k$ for all $n \geq N_k$, then $\lim_{n\to\infty}(x_n) = \infty$.

Definition 4.2 (Cauchy Sequence)**.** A sequence of real numbers (x_n) is a Cauchy sequence if for each positive integer k, there exists a positive integer M_k such that

$$|x_m - x_n| \leq k^{-1} \text{ for all } n, m \geq M_k. \tag{4.2.1}$$

Theorem 4.3. *A sequence of real numbers (x_n) converges if and only if it is a Cauchy sequence.*

Proof. ($\Rightarrow$) Assume (x_n) converges to x_0 and let $M_k \equiv N_{2k}$. Then, by eq. (4.1.1)

$$|x_m - x_n| \leq |x_m - x_0| + |x_n - x_0| \leq (2k)^{-1} + (2k)^{-1} = k^{-1} \text{ for all } m, n \geq M_k. \tag{4.3.1}$$

Therefore, (x_n) is a Cauchy sequence.

($\Leftarrow$) Conversely, assume (x_n) is a Cauchy sequence and let $N_k \equiv \max\{3k, M_{2k}\}$. Then, by eq. (4.2.1)

$$|x_m - x_n| \leq 2k^{-1} \text{ for all } n, m \geq M_k. \tag{4.3.2}$$

Next, let y_k be the $2k^{th}$ rational approximation of x_{N_k}. For $m \geq n$,

$$\begin{aligned} &|y_m - y_n| \leq |y_m - x_{N_m}| + |x_{N_m} - x_{N_n}| + |x_{N_n} - y_n| \\ &\leq (2m^{-1}) + (2m)^{-1} + (2n)^{-1} + (2n)^{-1} = m^{-1} + n^{-1}. \end{aligned}$$

Thus, $y \equiv (y_n)$ is a real number.

Lastly, we will prove that (x_n) converges to y. To do so, consider $n \geq N_k$, then compute

$$\begin{aligned} &|y - x_n| \leq |y - y_n| + |y_n - x_{N_n}| + |x_{N_n} - x_n| \\ &n^{-1} + (2n)^{-1} + (2k)^{-1} \leq (3k)^{-1} + (6k)^{-1} + (2k)^{-1} = k^{-1}. \end{aligned}$$

□

5 Continuous functions

5.1 Continuity vs Uniform continuity

Definition 5.1 (Continuity)**.** A real valued function f is continuous on a compact interval $I = [a, b]$ if for all $c \in I$ and for an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in I$ with $|x - c| \leq \delta$ we have $|f(x) - f(c)| \leq \varepsilon$.The value of δ can depend either on c or ε.

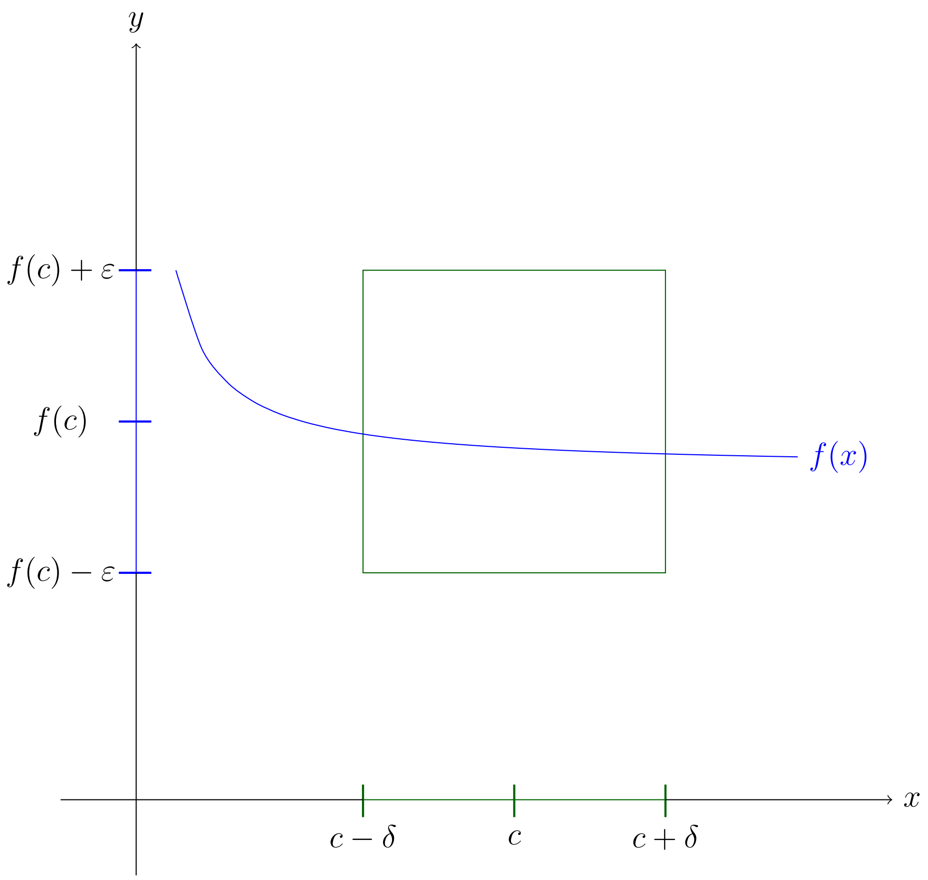

Figure 1: Graphic representation of continuity

By removing the brackets of $|x - c| \leq \delta$, we get an interval $I_\delta = [c - \delta, c + \delta]$, marked in green in Figure 1. Since $|x - c| \leq \delta$ implies $|f(x) - f(c)| \leq \varepsilon$, the interval marked in blue on the y-axis, we can visualise a rectangle that represents an area of continuity for a function $f(x)$. For a continuous function, whenever an x-value is in I_δ, the y-values have to be in the rectangle. Since δ can depend on both c and ε, the size of the rectangle may be changed to have the function pass through the rectangle in a way that does not pass through the top or bottom, thus keeping continuity.

Definition 5.2 (Uniform continuity)**.** A real valued function f is continuous on a compact interval $I = [a, b]$ if for an arbitrary $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for all $x, y \in I$ with $|x - y| \leq \delta(\varepsilon)$ we have $|f(x) - f(y)| \leq \varepsilon$.

The key difference between the two definitions is that for uniform continuity, δ (modulus of continuity) can only depend on ε. Visually this means that one rectangle will satisfy the definition for the entire function. In other words, the exact rectangle can be drawn at any x-value, and the function will pass through it from one horizontal side to the other.

The value δ is locally applicable in Definition 5.1 and globally applicable in Definition 5.2, so a uniformly continuous function is continuous but a continuous function may not necessarily be uniformly continuous.

Constructive analysis exclusively uses the definition of uniform continuity because the standard notion of continuity proves too weak for constructive purposes. Proposition 5.8 shows the constructive power of uniform continuity.

5.2 Supremum and Infimum

Definition 5.3 (Supremum and infimum)**.** A real number b is an upper bound of a nonvoid set A, if $x \leq b$ for all $x \in A$. The real number b is a *supremum* of A, or sup$A$ if it is also holds that for each $\varepsilon > 0$, there exists $x \in A$ with $x > b - \varepsilon$.

A real number b is a lower bound of a nonvoid set A, if $x \geq b$ for all $x \in A$. The real number b is an *infimum* of A, or inf$A$ if it is also applies that for each $\varepsilon > 0$, there exists $x \in A$ with $x > b + \varepsilon$.

Definition 5.4. A subset A of the real numbers $\mathbb{R}$ is located if for all $x, y \in \mathbb{R}$ with $x < y$, either y is an upper bound for A or there exists $a \in A$ such that $x < a$.

Definition 5.5. A real valued set A is called *totally bounded* if for each $\varepsilon > 0$ there exists finitely many points $y_1, y_2, \ldots y_n \in A$ such that for each $x \in A$ at least one of the numbers $|x - y_1|, |x - y_2|, \ldots, |x - y_n|$ is less than ε.

Theorem 5.6 (Least-upper-bound principle)**.** *Let A be a real valued set bounded above. Then A has a supremum if and only if A is located.*

Proof. ($\Rightarrow$) Assuming $\sup A$ exists and $x < y$, then $\sup A < y$ or $\sup A > x$.

In the case when $x < \sup A$, the inequality also implies $0 < \sup A - x$ and since $a \leq \sup A$, we can find an $a \in A$ such that

$$\sup A - (\sup A - x) < a,$$

and thus $x < a$. We have proven that the stated condition exists.

(⇐) Conversely, let's assume the stated condition holds. Let $a_1 \in A$ and let b_1 be an upper bound of A with $b_1 > a$. We construct sequences (a_n) in A and (b_n) of upper bounds of A recursively, such that for all positive integers n,

$$a_n \leq a_{n+1} < b_{n+1} \leq b_n \tag{5.6.1}$$

$$b_{n+1} - a_{n+1} \leq \frac{3}{4}(b_n - a_n). \tag{5.6.2}$$

Equation (5.6.2) ensures the distance between a_n and b_n shrinks at a controlled rate, allowing the sequences to converge.

Next, we compute two points between a_n and b_n:

$$x = a_n + \frac{1}{4}(b_n - a_n) \tag{5.6.3}$$

$$y = a_n + \frac{3}{4}(b_n - a_n). \tag{5.6.4}$$

Noting that $a_n < x < y < b_n$, we apply the stated condition to x and y.

Either y is an upper bound of A, we set $b_{n+1} \equiv a_n + \frac{3}{4}(b_n - a_n)$ and $a_{n+1} \equiv a_n$, or there exists an $a \in A$ with $a > x$. Then we set $a_{n+1} \equiv a$ and $b_{n+1} \equiv b$ which completes the recursive construction.

By Equation (5.6.1) and Equation (5.6.2), we get

$$0 \leq b_n - a_n \leq \frac{3}{4}(b_1 - a_1) \qquad (n \in \mathbb{Z}^+).$$

Constructing a modulus M_k where $k = ((\frac{3}{4})^{n-1}(b_1 - a_1))^{-1}$ results in sequences (b_n) and (a_n) being Cauchy sequences that converge to common limit l with $a_n \leq l \leq b_n$ for each positive integer n. Since every b_n is an upper bound of A, then so is l. Given $\varepsilon > 0$, there is also a certain n for $a_n \in A$ so that $l \geq a_n > l - \varepsilon$. Thus we have proven $l = \sup A$. □

A notable difference between the classical and constructive least-upper-bound theorems is that the classical principle asserts that all real and nonvoid sets that are bounded above have a supremum while the constructive theorem has to assert additional properties to set A, which is that is has to be a located set.

Corollary 5.7 (Existence of supremum and infimum)**.** *If a real-valued set A is totally bounded, then it has a supremum and an infimum.*

Proof. Firstly, a totally bounded set A is constructed by letting $x, y \in \mathbb{R}$ with $x < y$, and $\varepsilon = \frac{1}{4}(y - x)$. For each $a \in A$, choose points $a_1, \ldots a_N \in A$ such that at least one of the numbers $|a - a_1|, \ldots, |a - a_N|$ is less than ε.

For some n such that $1 \leq n \leq N$, define a_n as

$$a_n > \max\{a_1, \ldots, a_N\} - \varepsilon.$$

The properties of a_n is either $x < a_n$ or $a_n \leq x$. If $x < a_n$, the condition of Theorem 5.6 is directly satisfied. If $a_n \leq x$, then $a_n < x + 2\varepsilon$.

Consider any $a \in A$. By total boundedness, choose i with $|a - a_i| < \varepsilon$. Since $a_i < a_n + \varepsilon$, then

$$a \leq a_i + |a - a_i| < a_n + \varepsilon + \varepsilon = a_n + 2\varepsilon. \tag{5.7.1}$$

Combining Equation (5.7.1) with $a_n < x + 2\varepsilon$ the inequality becomes

$$a < a_n + 2\varepsilon < x + 4\varepsilon = y. \tag{5.7.2}$$

Value y becomes an upper bound for A and according to the least-upper-bound-property, $\sup A$ exists. It follows by symmetry that $\inf A$ exists as well. □

Proposition 5.8. *If f is a real valued continuous function on a compact interval $I = [a, b]$, then the supremum and infimum of f on I exists.*

Proof. Let $\varepsilon > 0$ and choose points $a = a_0 \leq a_1 \leq \ldots a_n = b$ such that $a_{i+1} - a_i \leq \delta(\varepsilon)$ for $0 \leq i \leq n - 1$. Since δ is a modulus of continuity for f then for each $x \in I$ with $|x - a_i| \leq \delta(\varepsilon)$, we have $|f(x) - f(a_i)| \leq \varepsilon$ for some i.

The set $\{f(x) : x \in I\}$ is totally bounded since ε is arbitrary. By Corollary 5.7, $\sup f$ and $\inf f$ exist. □

Although the standard definition of continuity allows δ to depend on both on a particular point c and ε, this notion is not strong enough for constructive proofs like the one above. Constructively, we require uniform continuity, where δ depends only on ε and not on c. This uniformity ensures that the entire domain is controlled by a single modulus of continuity, allowing the constructive existence proofs of supremum and infimum given in Proposition 5.8.

5.3 Intermediate Value Theorem

Theorem 5.9. *Let f be a continuous function on $I = [a, b]$ with $f(a) < f(b)$. Then, for each $y \in [f(a), f(b)]$ and each $\varepsilon > 0$, there is an $x \in I$ such that $|f(x) - y| < \varepsilon$.*

Proof. Since f is continuous, then $a \neq b$. Considering $y \in [f(a), f(b)]$ and $\varepsilon > 0$, let

$$m \equiv \inf\{|f(x) - y| : a \leq x \leq b\}, \tag{5.9.1}$$

which exists by Proposition 5.8.

Suppose, for contradiction that $m > 0$. Then, $f(a) - y \leq -m$ and $f(b) - y \geq m$. Divide I into points $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ and let δ be a modulus of continuity for f such that $x_{i+1} - x_i \leq \delta(m)$ for $0 \leq i \leq n-1$. It follows that

$$|(f(x_{i+1}) - y) - (f(x_i) - y)| = |f(x_{i+1}) - f(x_i)| \leq m \tag{5.9.2}$$

for some i.

Given that $|f(x) - y)| \geq m$ for all $x \in I$, the sign of $f(x_i) - y$ must remain the same for all i, either they are all positive, or all negative.

Initially, we established that $f(a) - y \leq -m$, which is negative, and $f(b) - y \geq m$, which is positive. For the chain of sign consistency to hold, we cannot start negative at x_0 and end up positive at x_n. This contradiction ensures that $m \not> 0$, therefore $m < \varepsilon$. Thus, we have established the required result. □

6 Differentiability

Definition 6.1. Let f and f' be continuous functions on a closed interval I and let δ be an operation from $\mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(x_j) - f(x_i) - f'(x_i)(x_j - x_i)| \leq \varepsilon |x_j - x_i| \tag{6.1.1}$$

whenever $\varepsilon > 0$, $x_i, x_j \in I$ and $|x_j - x_i| \leq \delta(\varepsilon)$. Then f is said to be differentiable on I, f' is the uniquely determined derivative of f on I and δ is the modulus of differentiability of f on I.

Alternatively, we interpret 6.1 as

$$\frac{f(x_j) - f(x_i)}{x_j - x_i} \to f'(x_i) \text{ as } x_j \text{ approaches } x_i \tag{6.1.2}$$

which aligns with our familiar understanding from classical analysis. This means that the difference between the actual change in f over the interval $[x_i, x_j]$ and its linear approximation by $f'(x_i)(x_j - x_i)$ remains bounded by some error, $\varepsilon > 0$, as long as $|x_j - x_i| \leq \delta(\varepsilon)$. Here, $\delta(\varepsilon)$ describes how small the interval $[x_i, x_j]$ must be to ensure that the linear approximation $f'(x)$ is accurate within a given tolerance.

Note that we are assuming that the derivative is always continuous in contrast to the classical definition.

Proposition 6.2 (Derivation rules)**.** *Let $f(x)$ and $g(x)$ be differentiable functions on I and let c be a constant. The following rules for derivation hold:*

i.) $D(f + g) = f' + g'$

ii.) $D(f \cdot g) = f'g + fg'$

iii.) $D(x) = 1$

iv.) $D(c) = 0$.

Proof. Because the definitions and properties of standard functions in constructive analysis align with their classical counterparts, and the derivation rules remain unchanged, we will omit the proofs of these differentiation formulas here. □

6.1 Rolle's theorem

Just as classically one relies on Rolle's theorem to identify a point x in the interval I where the derivative vanishes, here we cannot guarantee that the greatest lower bound lies within the closed interval, which would allow us to find an x such that $f'(x) = 0$. Nevertheless, we can establish a small $\epsilon > 0$ such that $|f'(x)| < \epsilon$ for some $x \in I$, ensuring that $f'(x) = 0$ is arbitrarily close to zero.

Lemma 6.3. *Let f be differentiable on $I = [a, b]$ and let $f(a) = f(b)$. Then, for all $\epsilon > 0$, there exists $x \in I$ such that $|f'(x)| < \epsilon$.*

Proof. Let δ be a modulus of differentiability for f on I. Since f is continuous on a closed interval, by proposition 5.8, we can define m as

$$m = \inf\{|f'(x)| : x \in I\} \tag{6.3.1}$$

and thus $|f'(x)| \geq m$ for all $x \in I$.

We claim that $m = 0$. We know that for any real number $x \geq 0$, if $x \not> 0$, then $x = 0$ (by Lemma 3.9). So, suppose for contradiction that $m > 0$. The definition of m establishes that $f'(x) \geq m$ or $f'(x) \leq -m$ at any given point $x \in I$. Since f is differentiable and thus its derivative f' is continuous, if both cases occur, the Intermediate Value Theorem implies that between values greater than or equal to m and values less than or equal to $-m$, there must be a point $\zeta \in I$ such that $|f'(\zeta)| < m$. This contradicts the definition of m as the infimum of $|f'(x)|$.

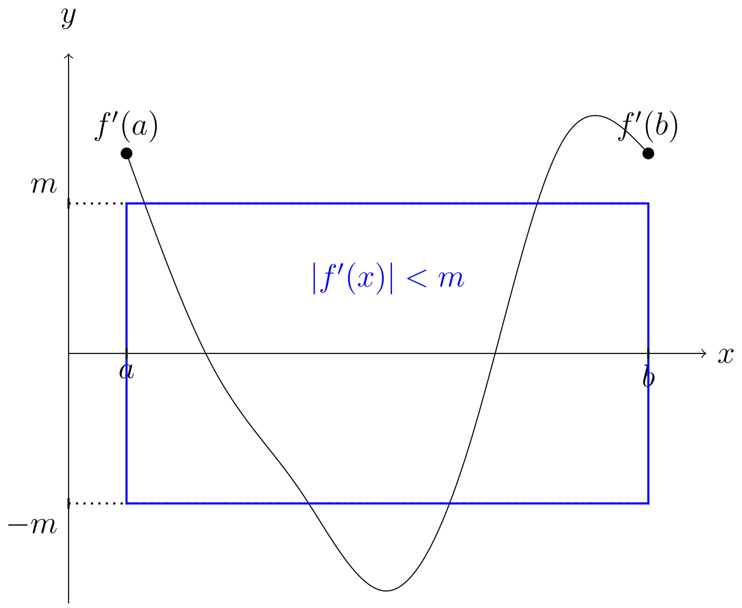

Figure 2: Example of a graphic representation of $f'(x)$ where $f'(x) \leq -m$ at some given x

Therefore, only one of the cases $f'(x) \geq m$ or $f'(x) \leq -m$ occurs. Without loss of generality, suppose $f'(x) \geq m$ for all $x \in I$.

Before we proceed, we need to establish the error term of the derivative. In definition 6.1, the error term ε is defined such that $|x_{i+1} - x_i| \leq \delta(\varepsilon)$. For this proof, it is necessary to choose ε small enough so that f' provides a suitable approximation of the change in f. To avoid introducing additional terms, we set $\varepsilon = \frac{1}{2}m$. If $\varepsilon = m$

were used, the resulting error would be too large. Thus, $\varepsilon = \frac{1}{2}m$ offers a manageable error, though any value of varepsilon satisfying $\varepsilon < m$ would suffice.

Next, we apply the modulus of differentiability to select appropriate points in I. Specifically, $a = x_0 \le x_1 \le ... \le x_{n-1} \le x_n = b$ such that $x_{i+k} - x_i \le \delta(\frac{1}{2}m)$ for all i where $0 \le i \le n-1$. This ensures the derivate $f'(x)$ gives a sufficiently good approximation of the change in f over each subinterval.

Since $f(a) = f(b)$, then

$$\begin{aligned} 0 &= f(b) - f(a) \\ &= \sum_{i=0}^{n-1} f(x_{i+1}) - f(x_i) && (6.3.2) \\ &= \sum_{i=0}^{n-1} f'(x_i)(x_{i+1} - x_i) + \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i) - f'(x_i)(x_{i+1} - x_i)) && (6.3.3) \\ &\ge \sum_{i=0}^{n-1} m(x_{i+1} - x_i) - \sum_{i=0}^{n-1} \frac{1}{2}m(x_{i+1} - x_i) = \frac{1}{2}m(b-a). && (6.3.4) \end{aligned}$$

Equation (6.3.2) is a telescoping sum. In equation (6.3.3), we decompose the difference $f(x_{i+1}) - f(x_i)$ into its linear approximation $f'(x_i)(x_{i+1} - x_i)$ and the error term. By the modulus of differentiability, the error term is bounded by $\frac{1}{2}m(x_{i+1} - x_i)$. Therefore, in equation (6.3.4), we use this bound and the bound of the derivative to establish that the sum is at least $\frac{1}{2}m(b-a)$.

Since $\frac{1}{2}m(b-a) > 0$, this contradicts the fact that the expression equals to zero due to the fact $f(a) = f(b)$. Therefore, our assumption $m > 0$ is false, and we conclude $m = 0$, or rather

$$\inf\{|f'(x)| : x \in I\} = 0. \tag{6.3.5}$$

This proves the theorem. It says precisely that for every $\epsilon > 0$ there exists some $x \in I$ such that $|f'(x)| < \epsilon$. □

6.2 Mean Value Theorem

Rolle's theorem did the hard work, this is now an easy corollary.

Theorem 6.4. *Let f be differentiable on $I = [a, b]$. Then for each $\varepsilon > 0$ there is an $x \in I$ such that*

$$|f(b) - f(a) - f'(x)(b-a)| \le \varepsilon. \tag{6.4.1}$$

Proof. We begin by defining a differentiable function g on I and calculating its derivative as

$$\begin{aligned} g(x) &= (x-a)(f(b) - f(a)) - f(x)(b-a) && (6.4.2) \\ g'(x) &= f(b) - f(a) - f'(x)(b-a). && (6.4.3) \end{aligned}$$

Observe that $g(a) = -f(a)(b-a) = g(b)$. Rolle's theorem tells us that for all $\varepsilon > 0$, there exists $x \in I$ such that $|g'(x)| \leq \varepsilon$. Substituting $g'(x)$ yields (6.4.1). □

In this thesis, we explored the framework of constructive analysis and demonstrated how it differs from its classical counterpart.

Throughout our investigation, several key differences have emerged. In classical analysis, both definitions of continuity are used. Constructively however, uniform continuity is much stronger for many functions which is why uniform continuity is exclusively used. Moreover, where classical completeness principles guarantee supremum existence for every bounded set of real numbers, constructive analysis requires additional conditions such as the set being located or totally bounded. By ensuring that continuity is uniform and sets are suitably well-structured, we can provide a constructive formulation of the Mean Value Theorem.

It might seem unexpected that proof by contradiction still appear in constructive mathematics. However, the rule of proof by contradiction is different than what we might be used to in classical mathematics. It is more limited in constructive mathematics because we cannot show that positive or negative properties hold. Rather, a constructivist contradiction shows that a particular construction fails.

Additionally, Brouwerian counterexamples highlighted the logical gaps in certain classical proofs, showing where nonconstructive principles must be replaced by constructive reasoning. Can't show that positive/negative properties are holding.

By embracing these principles, constructive analysis provides a setting in which proofs reveal the computable and verifiable content of mathematical claims. In doing so, it strengthens our grasp of the underlying logic and ensures that classical results, like for the Mean Value Theorem can be reconstructed in a form that rests on more transparent, constructive foundations.

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