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Generating functions and their applications to counting

av

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Generating functions and their applications to counting

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Abstract

This thesis investigates generating functions and their applications in combinatorial enumeration, emphasizing how these tools simplify the process of counting complex, structured objects. It provides a clear introduction to ordinary, exponential, and Dirichlet generating functions, explaining their definitions, uses, and connections to problems in combinatorics, analysis, and number theory.

The thesis examines how algebraic operations on exponential generating functions correspond to fundamental combinatorial constructions, such as combining, partitioning, and relabeling labeled sets. It also demonstrates how generating functions can be used to solve recurrence relations, using the Catalan numbers as a key example.

A major focus is the Exponential Formula, which relates the generating functions of connected components to the generating function of the total structure they form. Through the framework of cards, decks, and hands, this result is illustrated with applications to permutations and labeled graphs. Overall, the thesis presents generating functions as powerful and unifying tools in modern combinatorics.

Sammanfattning

Denna kandidatuppsats undersöker genererande funktioner och deras tillämpningar inom kombinatorisk uppräkning, med särskild tonvikt på hur dessa verktyg förenklar processen att räkna komplexa, strukturerade objekt. Den ger en tydlig introduktion till ordinära, exponentiella och Dirichlet-genererande funktioner, där deras definitioner, användningsområden och kopplingar till problem inom kombinatorik, analys och talteori förklaras.

Uppsatsen behandlar hur algebraiska operationer på exponentiella genererande funktioner motsvarar grundläggande kombinatoriska konstruktioner, såsom att kombinera, partitionera och ometikettera märkta mängder. Den visar även hur genererande funktioner kan användas för att lösa rekursionsformler, med Catalantalen som ett centralt exempel.

Ett huvudfokus är den exponentiella formeln, som kopplar samman genererande funktioner för sammanhängande komponenter med den genererande funktionen för hela den struktur de bildar. Genom ramverket med kort, lekar och händer illustreras detta resultat med tillämpningar på permutationer och märkta grafer. Sammantaget presenterar uppsatsen genererande funktioner som kraftfulla och enhetliga verktyg inom modern kombinatorik.

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Contents

1 Introduction

This text explores the theory and powerful applications of generating functions in combinatorial enumeration, focusing on how these tools simplify the process of counting structured objects. It develops a clear understanding of ordinary generating functions (OGFs), exponential generating functions (EGFs), and Dirichlet generating functions (DGFs), and demonstrates their roles in solving both concrete counting problems and abstract structural questions.

The work begins in **chapter 2** by distinguishing between the main types of generating functions. It explains how OGFs, EGFs, and DGFs are defined, where they are most useful, and how they connect to problems in combinatorics, analysis, and number theory. This foundation provides the essential terms and understanding needed to apply generating functions effectively in a variety of contexts..

Chapter 3 investigates how basic operations on exponential generating functions correspond to combinatorial actions. Addition of EGFs represents combining sequences directly, while multiplication encodes the idea of splitting a labeled set into parts and applying separate structures to each. The derivative of an EGF corresponds to shifting the sequence in a way that reflects relabeling. A central result in this chapter shows that multiplying two EGFs captures all the ways to partition a labeled set into two disjoint subsets, a key insight for interpreting many combinatorial constructions.

In **chapter 4**, the focus shifts to using generating functions as tools for solving recurrence relations. The section illustrates this technique using the well-known Catalan numbers. By translating a recurrence into a functional equation for the generating function, a closed-form expression is derived. This bypasses recursive computation and showcases the power of generating functions in deriving elegant, explicit formulas.

Chapter 5 introduces the Exponential Formula, a central result in the theory of labeled structures. It addresses the fundamental question: if we can count connected structures of various sizes, how can we count all possible structures formed by combining these components? To answer this, a combinatorial framework involving cards, decks, and hands is developed. This model interprets complex labeled structures as collections of simpler building blocks with disjoint labels. The chapter illustrates the method with detailed examples, such as labeled graphs and permutations, showing how these familiar structures naturally arise from combining connected components. The Exponential Formula reveals that the generating function for such composite structures is the exponential of the generating function for the components. Its wide applicability makes it a cornerstone of modern enumerative combinatorics.

2 Definitions and Types of Generating Functions

A **generating series** is the formal power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

used to represent a sequence (a_n). It is treated purely as an algebraic object, without concern for whether the series converges for any value of x. In this context, the variable x is simply a formal symbol used to track the indices.

When the series has a **positive radius of convergence**, meaning it converges for some values of x, it can be interpreted as an actual function. In that case, it is referred to as a **generating function** in the analytic sense.

- A *generating series* refers to the formal expression.
- A *generating function* may refer either to this formal series or to the function it defines when convergence is considered.

There are different types of generating functions, each suited for specific types of problem and we are going to describe them. The following definition is inspired by

2.1 Ordinary Generating Function (OGF)

Definition 2.1.1. Let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of real numbers. The function

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

is called the **Ordinary generating function (OGF)** for the given sequence.

This definition and the following example is inspired by [Gri03].

Example 2.1.2. Let's take the sequence

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 3n - 2 & \text{if } n > 0 \end{cases}$$

and we want to find its generating series

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Now we want to find the generating function that has a positive radius of convergence. From the definition of a_n, we write:

$$G(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n = 0 + \sum_{n=1}^{\infty} (3n-2)x^n.$$

We separate the sum:

$$G(x) = 3\sum_{n=1}^{\infty} nx^n - 2\sum_{n=1}^{\infty} x^n.$$

Recall the standard generating functions:

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}.$$

Substituting, we get:

$$G(x) = 3 \cdot \frac{x}{(1-x)^2} - 2 \cdot \frac{x}{1-x}$$

$$= \frac{3x - 2x(1-x)}{(1-x)^2}.$$

$$= \frac{x + 2x^2}{(1-x)^2}.$$

Therefore,

$$G(x) = \frac{x + 2x^2}{(1-x)^2}.$$

Conclusion: The generating function for the sequence

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 3(n-1)+1 & \text{if } n > 0 \end{cases}$$

is

$$\boxed{G(x) = \frac{x + 2x^2}{(1-x)^2}}.$$

2.2 Exponential Generating Function (EGF)

Definition 2.2.1. For a sequence $a_0, a_1, a_2, a_3, \ldots$ of real numbers,

$$G_E(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

is called the **exponential generating function(EGF)** for the given sequence.

This definition follows the same book [Gri03] that we have already mentioned for previous one.

Example 2.2.2. We will show that the exponential generating function of the sequence $a_n = (n+1)!$ is $G_E(x) = \frac{1}{(1-x)^2}$.

We begin by recalling the definition of the exponential generating function (EGF) for a sequence (a_n):

$$G_E(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

Substituting $a_n = (n+1)!$, we get:

$$G_E(x) = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n = \sum_{n=0}^{\infty} (n+1) x^n.$$

From "Mathematics II - Analysis, part B - MM5011" we observe that:

$$(n+1)x^n = \frac{d}{dx}\left(x^{n+1}\right).$$

So the entire sum becomes:

$$G_E(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(x^{n+1}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^{n+1}\right).$$

We simplify the inner sum:

$$\sum_{n=0}^{\infty} x^{n+1} = x \sum_{n=0}^{\infty} x^n = \frac{x}{1-x}, \quad \text{for } |x| < 1.$$

Thus,

$$G_E(x) = \frac{d}{dx}\left(\frac{x}{1-x}\right).$$

To differentiate this, we apply the product rule and get:

$$G_E(x) = \frac{1}{1-x} + x \cdot \frac{1}{(1-x)^2} = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

Conclusion: The exponential generating function of the sequence $a_n = (n+1)!$ is

$$\boxed{G_E(x) = \frac{1}{(1-x)^2}}.$$

2.3 Dirichlet Generating Function (DGF)

Definition 2.3.1. Given a sequence $\{a_n\}_1^\infty$; we say that a formal series

$$\begin{aligned} D(s) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \\ &= a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \cdots \end{aligned}$$

is the **Dirichlet generating function (DGF)** of the sequence, and we write

$$D(s) \overset{\text{Dir}}{\longleftrightarrow} \{a_n\}_1^\infty .$$

where all $a_n \in \mathbb{C}$. Though the variable s may be real or complex, we shall only consider real values of s throughout this thesis.

We use the definition of generating functions as presented in [Wil05].

Example 2.3.2. A fundamental example of a Dirichlet generating function is the **Riemann zeta function**, which corresponds to the sequence where $a_n = 1$ for all n:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For $\text{Re}(s) > 1$, this series converges and defines the well-known Riemann zeta function, which plays a crucial role in analytic number theory and the distribution of prime numbers.

3 Basic Operations on Generating Functions

Generating functions can be manipulated using algebraic operations that correspond to transformations of sequences.

The following lemma is adapted from [Got16].

Lemma 3.0.1. *Let* $F(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ *and* $G(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$ *be the exponential generating functions of the sequences* $(a_n)_{n\geq 0}$ *and* $(b_n)_{n\geq 0}$*, respectively. Then:*

1. $F(x) + G(x) = \sum_{n=0}^{\infty}(a_n + b_n)\dfrac{x^n}{n!}$,
2. $F(x)G(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}\right)\dfrac{x^n}{n!}$,
3. $F'(x) = \sum_{n=1}^{\infty} a_n \dfrac{x^{n-1}}{(n-1)!}$.

Proof. **(1)** By the definition of exponential generating functions, we have:

$$F(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad \text{and} \quad G(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!},$$

and these converge absolutely for $|x| < \rho$ for some $\rho > 0$.

Adding these two power series term by term gives:

$$F(x) + G(x) = \sum_{n=0}^{\infty}\left(a_n \frac{x^n}{n!} + b_n \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}(a_n + b_n)\frac{x^n}{n!}.$$

Hence, the sum of the exponential generating functions corresponds to the exponential generating function of the sequence $(a_n + b_n)_{n\geq 0}$.

(2) Their product is given by the Cauchy product of the two series:

$$F(x)G(x) = \left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty} b_m \frac{x^m}{m!}\right).$$

We use the Cauchy product formula:

$$F(x)G(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k} \cdot \frac{1}{k!} \cdot \frac{1}{(n-k)!}\right)x^n.$$

Note that:

$$\frac{1}{k!(n-k)!} = \frac{1}{n!}\binom{n}{k},$$

so we can rewrite the inner sum as:

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \cdot \frac{1}{n!}.$$

Thus, we obtain:

$$F(x)G(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{x^n}{n!},$$

which is the exponential generating function of the sequence $\left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right)_{n\geq 0}$.

(3) Differentiating term-by-term, we get:

$$F'(x) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \right) = \sum_{n=1}^{\infty} a_n \cdot \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}.$$

This is the exponential generating function of the shifted sequence $(a_1, a_2, a_3, \ldots)$.

□

The following proposition is based on the one found in [Got21] yields a useful interpretation for the coefficients of the product of two exponential generating functions. And the proof of this proposition is inspired by the argument used in [Got21].

Proposition 3.3 : For $n \in \mathbb{N}_0$, let a_n and b_n be the numbers of ways to build certain α-structure and β-structure on an n-set, respectively. Let f_n be the number of ways to partition $[n]$ into two sets S and T and then place an α-structure on S and a β structure on T. If $A(x), B(x)$, and $F(x)$ are the exponential generating functions of $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$, and $(f_n)_{n\geq 0}$, respectively, then $F(x) = A(x)B(x)$.

Proof.

Recall that the exponential generating function (EGF) of a sequence $(a_n)_{n\geq 0}$ is defined as:

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!},$$

and similarly for $B(x)$ and $F(x)$.

We want to compute the number f_n of ways to:

- Partition $[n]$ into two disjoint subsets S and T with $S \cup T = [n]$,
- Place an α-structure on S and a β-structure on T.

Let $k = |S|$. Then $|T| = n - k$. For a fixed k, the number of ways to choose $S \subseteq [n]$ of size k is $\binom{n}{k}$.

Then:

- There are a_k ways to place an α-structure on S,
- There are b_{n-k} ways to place a β-structure on T.

So for each $k = 0, 1, \ldots, n$, the number of such combinations is:

$$\binom{n}{k} a_k b_{n-k}.$$

By the rule of sum, we get:

$$f_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Now consider the product of the two EGFs:

$$A(x)B(x) = \left(\sum_{k=0}^{\infty} a_k \frac{x^k}{k!} \right) \left(\sum_{j=0}^{\infty} b_j \frac{x^j}{j!} \right).$$

The product is:

$$A(x)B(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{x^n}{n!}.$$

But this is exactly the EGF of the sequence $(f_n)_{n \geq 0}$, since:

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}, \quad \text{and} \quad f_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Therefore,

$$F(x) = A(x)B(x),$$

as claimed.

□

4 Solving Recurrence Relations with Generating Functions

As we have seen in ”Mathematics III - Combinatorics - MM5023” generating functions provide a systematic approach to solving recurrence relations by converting them into algebraic equations.

4.1 Example: Generating Function for Catalan Numbers

The Catalan numbers are defined by the recurrence relation:

$$C_0 = 1, \quad \text{and} \quad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}, \quad \text{for } n \geq 0.$$

We define the generating function:

$$C(x) = \sum_{n=0}^{\infty} C_n x^n.$$

From the recurrence, we consider the generating function:

$$\sum_{n=0}^{\infty} C_{n+1} x^{n+1} = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^{n+1}.$$

We shift the index on the left-hand side:

$$\sum_{n=1}^{\infty} C_n x^n = x \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^n.$$

Note that the inner sum is the convolution of the sequence with itself:

$$C(x) - C_0 = xC(x)^2.$$

Since $C_0 = 1$, we get the functional equation:

$$C(x) = 1 + xC(x)^2.$$

Rewriting, we obtain a quadratic equation in $C(x)$:

$$xC(x)^2 - C(x) + 1 = 0.$$

Solving this using the quadratic formula:

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

We now determine which root corresponds to the generating function for the Catalan numbers by evaluating the behavior as $x \to 0$.

Positive root:

$$\lim_{x\to 0} \frac{1+\sqrt{1-4x}}{2x}$$

As $x \to 0$, $\sqrt{1-4x} \to 1$, so the numerator approaches 2 and the denominator approaches 0. Therefore,

$$\lim_{x\to 0} \frac{1+\sqrt{1-4x}}{2x} = \infty.$$

This diverges at the origin and cannot represent a valid generating function, since the constant term (i.e., $C(0)$) must be finite.

Negative root:

$$\lim_{x\to 0} \frac{1-\sqrt{1-4x}}{2x}$$

This is an indeterminate form $\frac{0}{0}$, so we apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{1-\sqrt{1-4x}}{2x} = \lim_{x\to 0} \frac{\frac{d}{dx}(1-\sqrt{1-4x})}{\frac{d}{dx}(2x)} = \lim_{x\to 0} \frac{2}{\sqrt{1-4x}} \cdot \frac{1}{2} = 1.$$

Thus, the negative root satisfies the initial condition $C(0) = 1$, and we conclude that the generating function is

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Our goal is to derive the closed-form expression for C_n, the coefficient of x^n in the power series expansion of $C(x)$.

The square root term can be expanded as a power series using the binomial series

$$\begin{aligned}
1-\sqrt{1-4x} &= -\sum_{n=1}^{\infty}\binom{1/2}{n}(-4x)^n = -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-3)!!}{2^n n!}(-4x)^n \\
&= -\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^{n+1}(n+1)!}(-4x)^{n+1} = \sum_{n=0}^{\infty}\frac{2^{n+1}(2n-1)!!}{(n+1)!}x^{n+1} \\
&= \sum_{n=0}^{\infty}\frac{2(2n)!}{(n+1)!n!}x^{n+1} = \sum_{n=0}^{\infty}\frac{2}{n+1}\binom{2n}{n}x^{n+1}.
\end{aligned}$$

Thus,

$$c(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n.$$

Therefore, we obtain the closed-form expression for the Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

5 Proving the Exponential Formula and Exploring Its Applications

Now we explore the Exponential Formula, a fundamental tool in combinatorial counting. It addresses how to count structures built from connected components, using labeled graphs as a motivating example. Given the number of connected graphs of each size, we seek a systematic way to count all possible graphs. The key steps involve selecting connected components and relabeling them to form the final structure. To generalize beyond graphs, we introduce a broader framework using "playing cards" and "hands." This perspective allows for diverse applications in combinatorics.

5.1 Definitions

We suppose that there is given an abstract set P of 'pictures.'

Recall that $[n]$ is the set $\{1, 2, \ldots, n\}$. We use the following definitions as presented in [Wil05].

Definition 5.1.1. A **card** $C(S, p)$ is a pair consisting of a finite set S (the *label set*) of positive integers, and a picture $p \in P$. The **weight** of C is $n = |S|$. A card of weight n is called **standard** if its label set is $[n]$.

Definition 5.1.2. A **hand** H is a set of cards whose label sets form a partition of $[n]$, for some n. This means that if n denotes the sum of the weights of the cards in the hand, then the label sets of the cards in H are:

- pairwise disjoint,
- nonempty, and
- their union is $[n]$.

Definition 5.1.3. The **weight** of a hand is the sum of the weights of the cards in the hand.

Definition 5.1.4. A **relabeling** of a card $C(S, p)$ with a set S' is defined if $|S| = |S'|$, and it results in the card $C(S', p)$. If $S' = [|S|]$, then we have the **standard relabeling** of the card.

Definition 5.1.5. A **deck** D is a finite set of **standard** cards whose weights are all the same and whose pictures are all different. The **weight** of the deck is the common weight of all the cards in the deck.

Definition 5.1.6. An **exponential family** $\mathcal{F}$ is a collection of decks $D_1, D_2, \ldots$ where for each $n = 1, 2, \ldots$, the deck D_n has weight n. If $\mathcal{F}$ is an exponential family, we will write d_n for the number of cards in deck D_n, and we will call

$$D(x) = \sum_{n=1}^{\infty} d_n \frac{x^n}{n!}$$

the **deck enumerator**, which is the **exponential generating function (EGF)** of the sequence $\{d_n\}_{n=1}^{\infty}$.

5.2 Examples of Exponential Families

We present some foundational examples of exponential families that frequently arise in combinatorics. And the following two examples closely follow the approach given in [Wil05].

Example 1

Consider the collection $\mathcal{F}_1$ consisting of all vertex-labeled, undirected graphs. This will serve as our first example of an exponential family.

A graph is defined as a set of vertices together with a subset of unordered pairs of these vertices, called edges. In a labeled graph, each vertex is assigned a unique positive integer as its label. A graph is said to have *standard labeling* if its vertex labels form the set $[n] = \{1, 2, \ldots, n\}$, where n is the total number of vertices.

For a graph with n vertices, there are $\binom{n}{2}$ potential edges, as each edge connects two distinct vertices. Since each possible edge may either be included or not, there are $2^{\binom{n}{2}}$ distinct labeled graphs on n vertices. For instance, with $n = 3$, we get $2^3 = 8$ such graphs.

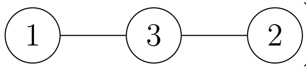
Graphs can be classified as either *connected* or *disconnected.* A graph is connected if there exists a path of edges between every pair of vertices; otherwise, it is disconnected. Among the 8 graphs on 3 labeled vertices, 4 are connected.

Let us now describe how this example fits into the framework of exponential families. A *card* in this context, denoted $\mathcal{C}(S, p)$, corresponds to a connected labeled graph. The set S represents the vertex labels used in the graph.

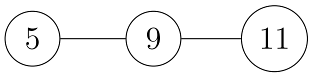
To define the *picture* p on the card, we introduce the concept of a *standard relabeling*: Given a graph G with labels in an arbitrary set S of size n, we reassign its labels to the set $[n]$ while preserving the original relative order. That is, the smallest label in S is mapped to 1, the second smallest to 2, and so on. This standard relabeling is well-defined and unique.

Suppose G is a labeled, connected graph. Then the picture p displayed on the card $\mathcal{C}(S, p)$ associated with G is the graph's standard relabeling. Thus, each card $\mathcal{C}$ presents two pieces of information: a depiction of a connected graph using standard labels, and a distinct label set of the same size.

As an example, consider a card of weight 3:

$(S, p) = \Big(\{5, 9, 11\},$  $\Big)$

This card represents the connected labeled graph:



Therefore, each card stands for a connected graph whose vertex labels need not be in standard form.

A *hand* is a multiset of cards such that the union of their label sets is exactly $[n]$, where n is the *weight* of the hand (the total number of vertices). Each hand corresponds to a not-necessarily-connected labeled graph with standard labels, whose components are represented by the cards. Though the components may have non-standard labels individually, the full graph has the label set $[n]$.

To summarize:

- Cards correspond to connected labeled graphs (with arbitrary labels),
- Deck $\mathcal{D}_n$ consists of all connected labeled graphs with standard labeling on n vertices,
- A hand of weight n is a labeled graph (not necessarily connected) on vertex set $[n]$,
- The number of cards in $\mathcal{D}_n$, denoted d_n, is the number of connected labeled graphs on n vertices,
- The number of hands of weight n consisting of k cards, denoted $h(n, k)$, counts the number of labeled graphs with k connected components.

Thus, the question of how to relate connected and arbitrary labeled graphs translates to understanding the relationship between the counts d_n and the total number of labeled graphs.

Example 2

In this case, we explore how the collection of all permutations forms an exponential family structure.

To begin, let's define the notion of a *card* in this context. Each card visually displays a number of labeled points arranged in a circular configuration. Suppose there are n points labeled with elements from the set $[n]$, ordered in some way around the circle. The circular layout includes arrows pointing clockwise to indicate the direction of traversal.

Alongside the circular diagram, every card also features a set S consisting of n distinct positive integers.

Each card, therefore, represents a cyclic permutation of the elements in S; that is, a permutation forming a single cycle. For example, the card shown in Figure 1 with label set $S = \{2, 4, 7, 9, 10\}$ corresponds to the cyclic permutation:

$$2 \to 7 \to 4 \to 10 \to 9 \to 2$$

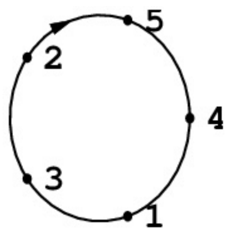


Figure 1: A visual representation of a cyclic permutation.

A *deck* in this family consists of all such standard cards for a fixed size n. That is, deck $\mathcal{D}_n$ includes exactly one card for each distinct cyclic permutation of the standard set $[n]$. Since there are exactly $(n-1)!$ distinct cyclic permutations of $[n]$, we have $d_n = (n-1)!$.

Now consider a *hand* as a collection of such cards. Each card in a hand contains a cyclic permutation and an associated label set. The label sets of the cards in a hand are disjoint and together cover the entire set $\{1, 2, \ldots, n\}$, where n is the total weight of the hand (the sum of the sizes of the label sets).

The union of these cards gives a complete permutation of n elements, where each cycle in the permutation comes from one of the cards. To construct the actual permutation, each cycle (originally drawn with labels from $[k]$) is relabeled using the corresponding set on the card, preserving relative order.

Since every permutation can be uniquely written as a product of disjoint cycles, we conclude that hands of weight n are in bijection with all permutations of n elements.

Therefore, the collection of all permutations forms an exponential family, which we denote by $\mathcal{F}_2$. In this setup:

- The number of cards in $\mathcal{D}_n$ is $d_n = (n-1)!$.
- A hand of weight n and size k corresponds to a permutation of n elements that decomposes into exactly k cycles.

5.3 The Main Counting Theorems

The exponential formula allows us to count complex combinatorial structures by building them from simpler, connected components. This section focuses on how to *merge two exponential families*—collections of labeled combinatorial structures—into a new family.

Given two disjoint exponential families $\mathcal{F}'$ and $\mathcal{F}''$, the merged family $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ is formed by combining the decks (i.e., sets of structures) of equal weight from each family, ensuring that all structures remain distinct. This operation preserves the exponential generating function structure and leads to powerful results for counting labeled objects.

The Fundamental Lemma of Labeled Counting. Let $\mathcal{F}', \mathcal{F}''$ be two disjoint exponential families, and let $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ be their merger. Further, let $\mathcal{H}'(x, y), \mathcal{H}''(x, y), \mathcal{H}(x, y)$ be the respective 2 -variables hand enumerators of these families. Then

$$\mathcal{H}(x, y) = \mathcal{H}'(x, y)\mathcal{H}''(x, y).$$

We follow a similar method to that of [Wil05] in proving this result.

Proof. Every hand in the merged family $\mathcal{F}$ consists of two disjoint subhands: one from $\mathcal{F}'$ and one from $\mathcal{F}''$. Suppose we are constructing a hand of total weight n and size k in $\mathcal{F}$. To do this, we choose a subset $S \subseteq [n]$ of size n', and assign it to the subhand from $\mathcal{F}'$. The remaining $n - n'$ labels go to the subhand from $\mathcal{F}''$.

Let k' be the number of cards in the subhand from $\mathcal{F}'$, so the subhand from $\mathcal{F}''$ must have $k - k'$ cards. The number of ways to choose the label set S is $\binom{n}{n'}$, and the number of ways to choose and relabel the two subhands is given by $h'(n', k')$ and $h''(n - n', k - k')$ respectively, where $h'(n', k')$ denotes the number of hands in $\mathcal{F}'$ of weight n' and size k', and similarly for h''.

Therefore, summing over all possible ways to split n and k between the two subhands, we get:

$$h(n, k) = \sum_{n'=0}^{n} \sum_{k'=0}^{k} \binom{n}{n'} h'(n', k') \cdot h''(n - n', k - k').$$

This convolution formula corresponds precisely to the coefficient of $\frac{x^n}{n!} y^k$ in the product of the generating functions:

$$\mathcal{H}(x, y) = \mathcal{H}'(x, y) \cdot \mathcal{H}''(x, y), \tag{5.3.1}$$

as desired. □

The central insight is that the operation of merging exponential families corresponds precisely to multiplying their exponential generating functions. In equation (5.3.1), the appearance of the binomial coefficient alongside the weight variable n' but not with the card count variable k', reflects the hybrid nature of the generating function: the variable x behaves like an egf (exponential generating function), while y operates more like an ordinary generating function (ogs). Thanks to the Fundamental Lemma of Labeled Counting, we can derive the relationship between deck and hand enumerators step by step. The process begins with the most elementary case: an exponential family where only one deck contains a single card. Here, the hand enumerator is straightforward to determine. We then expand to families with multiple cards in a single deck, and eventually to the general case. At each stage, the merging of families is guided by the Fundamental Lemma, allowing us to incrementally build up the exponential formula, transforming a simple initial setup into a more complex and powerful result.

Theorem 5.3.1 (Exponential Formula). *Let $\mathcal{F}$ be an exponential family, with* *deck enumerator* $\mathcal{D}(x)$ *and hand enumerator* $\mathcal{H}(x, y)$*. Then*

$$\mathcal{H}(x, y) = e^{y\mathcal{D}(x)}.$$

In particular, the number of hands of weight n *containing exactly* k *cards is given by*

$$h(n, k) = \left[\frac{x^n}{n!}\right] \left\{\frac{\mathcal{D}(x)^k}{k!}\right\}. \tag{5.3.2}$$

To prove this theorem, we extend the method in [Wil05] to a broader class of structures.

Proof. In an exponential family, a *hand* is defined as a multiset of *cards*, where each card is a labeled structure with a positive weight, and the total weight of a hand is the sum of the weights of its constituent cards.

Suppose we want to count the number of hands of total weight n and containing exactly k cards. Each such hand is constructed by:

- Selecting k cards (with repetition allowed) from the available decks $\mathcal{D}_1, \mathcal{D}_2, \ldots$, such that the total weight of the cards is n,
- Assigning distinct labels from the set $[n]$ to the n elements of the hand, in such a way that the labeling is consistent with the labeling rules of the cards.

The exponential generating function $\mathcal{D}(x)$ encodes the count of individual labeled cards, where each card of weight m contributes a term $\frac{d_m x^m}{m!}$. Since the cards are labeled and distinguishable by their labelings, the exponential formula tells us that the generating function for forming hands with exactly k such cards (and relabeling all the elements in the union of those cards with a set of size n) is:

$$\frac{\mathcal{D}(x)^k}{k!}.$$

Summing over all $k \geq 0$, we obtain the full exponential generating function for all hands (of any number of cards):

$$\mathcal{H}(x, y) = \sum_{k=0}^{\infty} \frac{\mathcal{D}(x)^k}{k!} y^k = e^{y\mathcal{D}(x)}.$$

The coefficient $h(n, k)$ is the number of such hands of weight n and size k, which corresponds to the coefficient of $\frac{x^n}{n!}$ in $\frac{\mathcal{D}(x)^k}{k!}$:

$$h(n,k) = \left[\frac{x^n}{n!}\right]\left\{\frac{\mathcal{D}(x)^k}{k!}\right\}.$$

This completes the proof. □

By summing (5.3.2) over all k we obtain the following See details in [Wil05]:

Corollary 5.3.2. *Let $\mathcal{F}$ be an exponential family, let $\mathcal{D}(x)$ be the egf of the sequence $\{d_n\}_1^\infty$ of sizes of the decks, and let $\mathcal{H}(x) \xrightarrow{egf} \{h_n\}_0^\infty$, where h_n is the number of hands of weight n. Then*

$$\mathcal{H}(x) = e^{\mathcal{D}(x)}.$$

By summing (5.3.2) over just those k that lie in a given set T, we obtain

Corollary 5.3.3 (The exponential formula with numbers of cards restricted). *Let T be a set of positive integers, let $e_T(x) = \sum_{n\in T} x^n/n!$, and let $h_n(T)$ be the number of hands whose weight is n and whose number of cards belongs to the allowable set T. Then*

$$\{h_n(T)\}_0^\infty \overset{egf}{\longleftrightarrow} e_T(\mathcal{D}(x)).$$

5.4 Application: Counting Labeled 2-Regular Graphs

A *2-regular graph* is an undirected graph in which every vertex has degree exactly 2. For a graph on n labeled vertices, this means that every vertex belongs to a cycle. Therefore, a 2-regular graph is simply a **disjoint union of cycles** that cover all n vertices. This structural insight provides the foundation for applying the *exponential formula* to enumerate such graphs.

We aim to count the number $g(n)$ of labeled 2-regular graphs on n vertices, where the label set is $[n] = \{1, 2, \ldots, n\}$.

Constructing the Exponential Family

In the framework of exponential families, each **cycle** of size $n \geq 3$ on a labeled vertex set corresponds to a *card* of weight n. Since we are considering *undirected* cycles, we count the number of distinct undirected circular arrangements of n labeled vertices.

Let d_n denote the number of such undirected cycles. For $n \geq 3$, we have:

$$d_n = \frac{(n-1)!}{2},$$

where the division by 2 accounts for the two possible orientations of a cycle, and $(n-1)!$ counts circular permutations modulo rotation.

For $n = 1$ or 2, no such cycles exist, so we define $d_1 = d_2 = 0$.

Deck Enumerator

We compute the *deck enumerator* $\mathcal{D}(x)$ according to [Wil05, p. 83] by following:

$$\begin{aligned}
\mathcal{D}(x) &= \sum_{n\geq 3} \frac{d_n}{n!} x^n = \sum_{n\geq 3} \frac{(n-1)!}{2n!} x^n \\
&= \frac{1}{2} \sum_{n\geq 3} \frac{x^n}{n} \\
&= \frac{1}{2} \left(\log \frac{1}{1-x} - x - \frac{x^2}{2} \right),
\end{aligned}$$

where we used the expansion $\log \frac{1}{1-x} = \sum_{n\geq 1} \frac{x^n}{n}$ and excluded the terms for $n = 1$ and $n = 2$.

Applying the Exponential Formula

The exponential formula states that the *hand enumerator* $\mathcal{H}(x, y)$ for an exponential family with deck enumerator $\mathcal{D}(x)$ is given by:

$$\mathcal{H}(x, y) = \exp \left(y \cdot \mathcal{D}(x) \right).$$

To count all labeled 2-regular graphs regardless of the number of components (i.e., cycles), we set $y = 1$, yielding the exponential generating function for $g(n)$:

$$\begin{aligned}
\sum_{n\geq 0} g(n) \frac{x^n}{n!} &= \exp \left(\frac{1}{2} \log \frac{1}{1-x} - \frac{x}{2} - \frac{x^2}{4} \right) \\
&= \frac{e^{-\frac{1}{2}x - \frac{1}{4}x^2}}{\sqrt{1-x}}.
\end{aligned}$$

Interpretation and Significance

This compact formula elegantly encodes the enumeration of labeled 2-regular graphs. Each coefficient $g(n)$ gives the number of such graphs on n labeled vertices. Importantly, the method circumvents the direct combinatorial challenge of counting cycle decompositions by harnessing the structure of exponential families and the algebra of generating functions.

What makes this application remarkable is that the decomposition of 2-regular graphs into cycles — a highly nontrivial combinatorial object — becomes tractable when each cycle is viewed as a separate card in an exponential family. The exponential formula then automatically accounts for all valid partitions and relabelings.

This result stands as a shining example of the power of generating functions and symbolic methods in enumerative combinatorics: a deep counting problem is resolved through an elegant and efficient algebraic identity.

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