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The Zeta function and The Prime Number Theorem

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Abstract

The aim of this thesis is to investigate certain important properties of the Zeta function in the complex plane and use those to give a proof of The Prime Number Theorem. To do so we utilise the concept of analytic continuation, properties of the Gamma function, and briefly the Bernoulli numbers. In Chapter 1, we prove that the Gamma and Zeta functions are meromorphic in the complex plane, show different ways of representing the Zeta function, and end by investigating this function near the line $\mathrm{Re}(s) = 1$ in the complex plane. In Chapter 2, we utilise the results of Chapter 1 to look at the growth of the Zeta function near that same line, and together with Chebychev's Psi functions we prove two propositions and a theorem that together form the Prime Number Theorem. After the main chapters, we have Appendix A that covers the mathematical basis of analytic continuation and Appendix B which covers important concepts that are referred to in Chapters 1 and 2.

Sammanfattning

Målet med denna uppsats är att undersöka några viktiga egenskaper hos Zeta-funktionen i det komplexa talplanet och att sedan använda dessa för att ge ett bevis för Primtalssatsen. För att göra det kommer vi utnyttja konceptet analytisk kontinuitet samt egenskaper hos Gamma-funktionen och Bernoulli-talen kortfattat. I Kapitel 1 bevisar vi att Gamma- och Zeta-funktionerna är meromorfiska i det komplexa talplanet, sen visar vi olika sätt att representera Zeta-funktionen och avslutar med att undersöka denna funktion nära linjen $\mathrm{Re}(s) = 1$ i det komplexa talplanet. I Kapitel 2 använder vi resultaten från Kapitel 1 för att kolla på tillväxten av Zeta-funktionen nära samma linje som ovan, och tillsammans med Chebychevs Psi-funktioner bevisar vi två satser och ett teorem som tillsammans bildar Primtalssatsen. Efter huvudkapitlen har vi Appendix A som täcker den matematiska grunden till analytisk kontinuitet och Appendix B som täcker viktiga koncept som nämns i Kapitel 1 och 2.

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Contents

Introduction

Prime numbers are the natural numbers that do not have any divisors other than itself and 1. This property makes prime numbers incredibly useful in fields like cryptography, where the commonly used algorithm RSA uses very large prime numbers for secure data transmissions[6], which is important for things like online banking and e-commerce. The hard part is often finding these very large prime numbers, and a first step is to know how many there are. In this thesis it is the second question that we aim to answer: How many prime numbers are there? We can easily enough set up a prime counting function $\pi(x) = \sum_{p \leq x} 1$, where we go through all the real integers up to some large integer x, and count each prime number as we go. If $x = 10$ then we would have that $\pi(x) = 4$, and $x = 50$ would give $\pi(x) = 15$. If we go even larger, say $x = 1000$, we have $\pi(x) = 168$. Then we start being able to see a pattern for how prime numbers are distributed among the real numbers. But what happens when x becomes absurdly large? Or tends to infinity? We don't know all prime numbers, so how can we count them?

This is where the Prime Number Theorem comes in. The Prime Number Theorem is a famous theorem that goes back over a hundred years, with the first recognized proofs coming from mathematicians in 1896 [4]. This theorem states that there is a relationship between the prime counting function $\pi(x)$ and the real number x. In this thesis we prove that this relationship is the asymptotic relationship $\pi(x) \sim x/\log x$, where the symbol $\sim$ denotes the asymptotic relationship or equivalence, and $\log x$ is used for the natural logarithm throughout this thesis. This thesis aims to cover all the necessary background needed to understand the proof provided for the Prime Number Theorem. First we cover the Zeta function. This function is key to understanding multiple different results in the world of numerical analysis(including the Riemann-hypothesis which connects the Zeta function with the search for large prime numbers [5]) and we will take a closer look at it and its properties in Chapter 1. The start of Chapter 2 will serve as a bridge connecting the Zeta function to the Prime Number Theorem, and finally we cover the Prime Number Theorem itself in multiple steps. In this thesis we liberally use the concept of analytic continuation, which comes from the identity theorem, to extend a function past where it is otherwise not defined. The ability to do this is unique to the complex field and complex analysis, and it is with this concept that the Zeta function proves so strong. The statement and proof of the Identity theorem and analytic continuation can be found in Appendix A at the end of the thesis. This thesis uses the book "Complex Analysis" by Elias M. Stein and Rami Shakarchi as its primary source, with any additional sources listed in the bibliography at the end.

Chapter 1

The Zeta Function

1.1 Convergence, meromorphic continuation and representations of $\zeta(s)$

We will start off by introducing the main function of this chapter:

Definition 1.1 (The Zeta Function)**.** The Zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s = \sigma + i\tau$ is a complex number and σ, τ are real.

This definition of s, σ, τ will be applied throughout this text if nothing else is stated. We assume the reader of this text is already familiar with holomorphic and analytic functions, as well as the relation between the two. If not, please see Appendix B. In this text we will have use of two new terms to complement those two:

Definition 1.2 (Meromorphic function)**.** A function defined on an open domain is *Meromorphic* if it is holomorphic in the entire domain, except for a finite set of isolated poles.

Definition 1.3 (Entire function)**.** A function is *entire* if it is holomorphic on the whole complex plane.

In Chapter 2 the proof of the Prime Number Theorem will depend on functions that are defined and differentiable in (almost) the whole complex plane. As such we want all the functions we use to be either meromorphic or entire. This applies especially to the Zeta function.
The first step is to prove the following lemma:

Lemma 1.4 (Analyticity of $\zeta(s)$)**.** *The Zeta function is analytic in the half-plane* $\mathrm{Re}(s) > 1$.

Proof. Recall that an analytic function is one that has a convergent power series. As such we need to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

is convergent whenever $\mathrm{Re}(s) > 1$.
We will prove this convergence by using Weierstrass M-test. Let

$$M_n = \frac{1}{n^{1+\delta}}$$

where $\delta > 0, n \geq 1$ are real numbers. We will also assume that the inequality $\mathrm{Re}(s) > 1 + \delta > 1$ holds. Then we wish to prove that

$$|n^{-s}| \leq M_n.$$

To do this we will use two identities: (a)that $a^s = e^{s \log a}$ holds whenever $a \in \mathbb{R}$, and (b) $|e^{i\tau}| = 1$ for all $y \in \mathbb{R}$. Then we can write:

$$|n^{-s}| = |e^{-s \log n}| = |e^{-\mathrm{Re}(s) \log n} e^{-\mathrm{Im}(s) \log n}| = e^{-\mathrm{Re}(s) \log n} = n^{-\mathrm{Re}(s)}.$$

Since $\mathrm{Re}(s) > 1 + \delta > 1$, it is clear that $n^{-\mathrm{Re}(s)} \leq M_n$ holds true for all real $n \geq 1$.
For the final step we need to prove that $\sum_{n=1}^{\infty} M_n$ is convergent. Then by using the integral test for convergence, we wish to show that

$$\int_1^{\infty} M_x dx < \infty. \tag{1.1}$$

By evaluating the integral we find that

$$\int_1^{\infty} \frac{1}{x^{1+\delta}} dx = \lim_{N \to \infty} -\frac{1}{\delta} [x^{-\delta}]_1^N = -\frac{1}{\delta}(-1 + \lim_{N \to \infty} N^{-\delta}) = \frac{1}{\delta}.$$

Given that $\delta > 0$, we have shown that the inequality (1.1) is true, which proves by Weierstrass M-test that $\zeta(s)$ is analytic in the half-plane $\mathrm{Re}(s) > 1$. The series representing the ζ function converges both uniformly and absolutely on the half-plane $\mathrm{Re}(s) > 1$. □

Now that we have shown that $\zeta(s)$ is analytic, thus holomorphic, in the half-plane $\mathrm{Re}(s) > 1$, we want to extend $\zeta(s)$ into a meromorphic function. To do this we will use the technique called Analytic continuation to represent $\zeta(s)$ in terms of other functions that are meromorphic in $\mathbb{C}$. This will extend the domain in which $\zeta(s)$ is holomorphic from $\mathrm{Re}(s) > 1$ to the entire complex plane, except for the point $s = 1$. The validity of this method is based on the Identity Theorem, and will be discussed more in appendix A. It is strongly recommended to read appendix A first before continuing if those concepts are unfamiliar to the reader.

A large part of the proofs in this first chapter are based upon the Gamma function, so our next step is to introduce it and some of its key properties.

Definition 1.5 (The Gamma function)**.** The Gamma function is defined as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

where $\mathrm{Re}(s) > 0$.

Before we continue on to prove $\zeta(s)$ is meromorphic we need the following lemma:

Lemma 1.6. *The function $F(s) = \int_1^\infty e^{-t} t^{s-1} dt$ is entire.*

Proof. To show that $F(s)$ is entire we first aim to show that there exists a sequence of functions $F_n(s)$ which converges uniformly to $F(s)$ on some compact set Ω in the complex plane. Secondly we aim to show that $F_n(s)$ is holomorphic in Ω. Let $n > 1$, and let $F_n(s) = \int_1^n e^{-t} t^{s-1} dt$, where $F_\infty = F(s)$. Recall the integral inequality $| \int_\Omega g(x)dx| \leq \int_\Omega |g(x)|dx$. Then

$$|F_n(s) - F(s)| = \left| \int_n^\infty e^{-t} t^{s-1} dt \right| \leq \int_n^\infty |e^{-t} t^{s-1}| dt = \int_n^\infty e^{-t} t^{\sigma-1} dt.$$

We can factor the last integral so that $\int_n^\infty e^{-t/2} \cdot e^{-t/2} t^{\sigma-1} dt$, and create an upper estimate by letting $M_\Omega = \sup_\Omega e^{-t/2} t^{\sigma-1}$, and since Ω is compact M_Ω must be finite. Then

$$\int_n^\infty e^{-t} t^{\sigma-1} dt \leq M_\Omega \int_n^\infty e^{-t/2} dt \to 0$$

thus proving that $F_n(s)$ converges uniformly to $F(s)$ in Ω. Given that our choice of Ω is arbitrary, we conclude that $F_n(s)$ converges uniformly to $F(s)$ on all compact subsets of $\mathbb{C}$. Then we can conclude that $F_n(s)$ converges uniformly to $F(s)$ in $\mathbb{C}$ by Theorem 2.5.2 in [1, p.53] (See Appendix B for the statement).

To prove $F_n(s)$ is entire, we will use Theorem 2.5.4 from [1, p.56] (See Appendix B for statement). We note that

$$F_n(s) = \int_1^n f(s,t) dt$$

where $s \in C$ and C is some open subset of $\mathbb{C}$. To use this theorem we will make the variable substitution $t = 1 + (n-1)u$, where $u \in [0,1]$ and $dt = (n-1)du$. Then

$$f(s,u) = e^{-(1+(n-1)u)}(1 + (n-1)u)^{s-1}(n-1)du,$$

and it is clear to see that when $n > 1$, $f(s,u)$ is holomorphic for all $u \in [0,1]$ and for each $s \in C$, and continuous on $C\times$[0,1]. Then

$$F_n(s) = \int_0^1 e^{-(1+(n-1)u)}(1 + (n-1)u)^{s-1}(n-1)du$$

is holomorphic in C. Since we chose C arbitrarily, we can choose C to be any open subset of the complex plane, including the complex plane itself. Then we have proven that $F_n(s)$ is entire, thus proving that $F(s)$ is entire.

□

Theorem 1.7. *The function $\Gamma(s)$ extends to a meromorphic function in $\mathbb{C}$ with simple poles at the negative integers $s = 0, -1, -2, \ldots$*

Proof. We begin by splitting the integral so that

$$\Gamma(s) = \int_0^1 e^{-t} t^{s-1} dt + \int_1^\infty e^{-t} t^{s-1} dt.$$

By Lemma 1.6, $\int_1^\infty e^{-t} t^{s-1} dt$ is entire. What we need to do is to show that $\int_0^1 e^{-t} t^{s-1} dt$ is at least meromorphic. Using the power series $e^{-t} = \sum_{n=0}^\infty \frac{(-t)^n}{n!}$, we find that

$$\int_0^1 e^{-t} t^{s-1} dt = \int_0^1 \sum_{n=0}^\infty \frac{(-t)^n}{n!} t^{s-1} dt.$$

When $t \in [0, 1]$ it is clear that $\sum_{n=0}^\infty \frac{(-t)^n}{n!} < \sum_{n=0}^\infty |\frac{t^n}{n!}| \leq \sum_{n=0}^\infty \frac{1}{n!}$, and since the third sum is convergent and the second is absolutely convergent, we know that the first is uniformly convergent, and we can swap the sum and integral sign. Then we can evaluate the integral and we get:

$$\sum_{n=0}^\infty \int_0^1 \frac{(-t)^n}{n!} t^{s-1} dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}.$$

Since $1/s$ is a holomorphic function in a punctured disk around $s = 0$, the sum $\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}$ describes a series that is holomorphic whenever $n \neq -s$. Since $n \geq 0$, it is clear to see that

$$\Gamma(s) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)} + \int_1^\infty e^{-t} t^{s-1} dt$$

is holomorphic when $s > 0$. The Gamma function continues being holomorphic when $s \leq 0$, except at the points where $n = -s$. These points are simple poles, occurring at $s = 0, -1, -2, \ldots$. It then follows that $\Gamma(s)$ is meromorphic in the whole complex plane with simple poles at $s = 0, -1, -2, \ldots$. □

We also need to show that the inverse of Γ is entire to prove Proposition 1.9 and Theorem1.12 up ahead. First, as shown in [1, p. 79] we have that

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin \pi a} \tag{1.2}$$

whenever $0 < a < 1$. We continue by making the change of variable $v = e^x$ in 1.2, also noting that it changes the path from $(-\infty, \infty)$ to $(0, \infty)$ as $x = \log v$. This gives us

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x} dx = \int_0^\infty \frac{v^{a-1}}{1+v} dv = \frac{\pi}{\sin \pi a}, \tag{1.2a}$$

which brings us to the following result.

Theorem 1.8. *For all $s \in \mathbb{C}$,*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Proof. First observe that since $\Gamma(s)$ has poles at $s = 0, -1, -2, ...$, $\Gamma(1-s)$ has poles at $s = 1, 2, 3...$ and the product of the two has poles at all integers, which is true for $\pi / \sin \pi s$ as well.
Next, we will prove the theorem in the region $0 < s < 1$. If the identity holds in this region, then by analytic continuation it will hold in $\mathbb{C}$. Then in this region,

$$\Gamma(1-s) = \int_0^\infty e^{-u} u^{-s} du = t \int_0^\infty e^{-vt} (vt)^{-s} dv$$

where we used the change of variables $u = vt$ with $t > 0$. Then

$$\Gamma(1-s)\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \int_0^\infty e^{-vt} t (vt)^{-s} dv$$

$$= \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-s} dv dt.$$

We are now going to change the order of integration. Doing so gives us

$$\int_0^\infty v^{-s} \left(\frac{-e^{-t(1+v)}}{1+v} \right)_{t=0}^\infty dv = \int_0^\infty \frac{v^{-s}}{1+v} dv,$$

and if we set $a = 1 - s$, we can use the identity in equation (1.2a) and we get

$$\int_0^\infty \frac{v^{-s}}{1+v} dv = \frac{\pi}{\sin \pi(1-s)} = \frac{\pi}{\sin \pi s},$$

since $\sin(\pi - \theta) = \sin(\theta)$. This concludes our proof.

$\square$

Corollary 1.8.1. *The function $\frac{1}{\Gamma(s)}$ is entire with simple zeroes at $s = 0, -1, -2, ...$*

This follows from the theorem above. Since

$$\frac{1}{\Gamma(s)} = \Gamma(1-s) \frac{\sin \pi s}{\pi},$$

$\Gamma(1-s)$ has poles at all positive integers of s and $\sin \pi s$ has zeroes at all integers of s, then all poles cancel out and we are left with zeroes at the integers $s = 0, -1, -2, ...$ Since $\Gamma(s)$ is meromorphic in the whole complex plane, $\frac{1}{\Gamma(s)}$ must then be entire.

Proposition 1.9. In the half-plane $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx, \tag{1.3}$$

where $x > 0$ is a real number.

Proof. Rewriting $\frac{1}{e^x-1}$ into a geometric series, we get

$$\frac{1}{e^x-1} = \sum_{n=1}^{\infty} e^{-nx}.$$

We can compare this series to $\sum_{n=1}^{\infty} e^{-(n-1)x}$, where $e^{-(n-1)x} \geq e^{-nx}$ for all $n \geq 1$ and $x > 0$. By means of a quick ratio test we can see that

$$|e^{-(nx)}/e^{-(n-1)x}| = |e^{(n-1)x}/(e^{nx})| = \frac{1}{e^x} < 1,$$

which confirms that the series $\sum_{n=1}^{\infty} e^{-(n-1)x}$ is convergent, and thus that $\sum_{n=1}^{\infty} e^{-nx}$ is uniformly convergent when $x > 0$. Inserting this series into the integral in equation (1.3) we get

$$\int_0^{\infty} \frac{x^{s-1}}{e^x-1} dx = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nx} x^{s-1} dx.$$

Since the sum is uniformly convergent when $x > 0$ we can interchange the sum and the integral, and we get

$$\sum_{n=1}^{\infty} \int_0^{\infty} e^{-nx} x^{s-1} dx.$$

By using the substitution $x = \frac{t}{n}$, $dx = \frac{dt}{n}$ we get a very neat expression of

$$\sum_{n=1}^{\infty} n^{-s} \int_0^{\infty} e^{-t} t^{s-1} dt = \zeta(s) \cdot \Gamma(s).$$

Inserting that back into equation(1.3) we have exactly what we wanted. ☐

Definition 1.10. The mth Bernoulli number B_m is defined by the mth degree coefficient of the series expansion

$$\frac{x}{e^x-1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m,$$

where $B_0 = 1$ and all Bernoulli numbers B_{2n+1} are zero when $n \geq 1$.

The study and computation of the Bernoulli numbers is something we will leave out of this thesis as it would become too large a tangent from the intended topic.

Lemma 1.11. *The sum* $\sum_{m=0}^{\infty} \frac{B_m}{m!} x^m$ *is a convergent power series with radius of convergence* 2π*.*

Proof. First, we make the claim that the function $\frac{z}{e^z-1}$ is analytic somewhere. Let $f(z) = z$ and $g(z) = e^z - 1$. $f(z)$ is an entire function and $g(z)$ is a periodic function with period $2\pi i$. $g(z)$ is also analytic in the disk $|z| < 2\pi$, with a zero at $z = 0$. Then $\frac{f(z)}{g(z)}$ is analytic in the annulus $0 < |z| < 2\pi$.
By Riemann's theorem on removable singularities, if

$$\lim_{z \to z_0} (z - z_0)h(z) = 0$$

then $h(z)$ is extendable to an analytic function in the neighborhood of z_0.
Let $h(z) = \frac{f(z)}{g(z)}$, then

$$\lim_{z \to 0} zh(z) = \lim_{z \to 0} \frac{z^2}{e^z - 1}.$$

Given that both $f(z)$ and $g(z)$ are continuously differentiable (since an analytic function is also holomorphic) and equal to zero at the point $z = 0$, we can apply l'Hopital's rule locally around this point, which then gives

$$\lim_{z \to 0} \frac{z^2}{e^z - 1} = \lim_{z \to 0} \frac{2z}{e^z} = 0.$$

Thus $\frac{z}{e^z-1}$ is analytic in the disk $|z| < 2\pi$. Recall that an analytic function has a convergent power series. We can find this power series from Definition 1.10 by replacing the real variable with a complex one as such

$$\sum_{m=0}^{\infty} \frac{B_m}{m!} z^m.$$

In addition we know that the radius of convergence of this series is $2\pi i$, which gives us that

$$\limsup_{m \to \infty} |B_m/m!|^{1/m} = \frac{1}{2\pi}.$$

This means that the real-valued power series is also convergent, with radius of convergence 2π, which proves the lemma.

□

We will now introduce one of the most important theorems of this chapter.

Theorem 1.12. *By analytic continuation The Zeta function can be extended into a meromorphic function over the whole complex plane by the functional equation*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^1 \frac{x^{s-1}}{e^x - 1} dx + \frac{1}{\Gamma(s)} \int_1^\infty \frac{x^{s-1}}{e^x - 1} dx$$

with only one (simple) pole at $s = 1$.

Proof. First, observe that the functional equation in this theorem is the same as the one in Proposition 1.9.

Second, we also note that $\frac{1}{\Gamma(s)}\int_1^\infty \frac{x^{s-1}}{e^x-1}dx$ is a product of two entire functions, and as such it is also entire. What we then need to show is that

$$\frac{1}{\Gamma(s)}\int_0^1 \frac{x^{s-1}}{e^x-1}dx$$

is a meromorphic function in all of $\mathbb{C}$ that only admits a pole at $s=1$. From Definition 1.10, we can rewrite the integral as:

$$\int_0^1 \frac{x^{s-1}}{e^x-1}dx = \int_0^1 x^{s-2}\sum_{m=0}^{\infty}\frac{B_m}{m!}x^m dx$$

By Lemma 1.11, the sum is absolutely (and uniformly) convergent in the interval $x \in [0,1]$, and we can interchange the sum and integral sign, leaving only x^{s+m-2} inside:

$$\int_0^1 x^{s-2}\sum_{m=0}^{\infty}\frac{B_m}{m!}x^m dx = \sum_{m=0}^{\infty}\frac{B_m}{m!}\int_0^1 x^{s+m-2}dx.$$

Evaluating this integral leaves us with

$$\sum_{m=0}^{\infty}\frac{B_m}{m!}\cdot\frac{1}{s+m-1}.$$

Whenever $s+m \neq 1$ we have that $|\frac{1}{s+m-1}| \leq 1$ for all $s \in \mathbb{C}$, $m \geq 0$, and since $\limsup_{m\to\infty}|B_m/m!|^{1/m} = \frac{1}{2\pi}$, it is easy to see by computing the root test that this sum is a convergent series in all of $\mathbb{C}$, except for the singularities at $s = 1-m$.
We now need to investigate what type of singularities these are. Recall that $B_{2m+1} = 0$ when $m \geq 1$. Then the singularities at $s = -2m+2$ have to be removable ones since poles and zeroes cancel out. The rest of the singularities are simple poles at $s = 1, 0, -1, -3, ...$

Recall that the function $\frac{1}{\Gamma(s)}$ has simple zeroes at $s = 0, -1, -2, ...$ Since zeroes and poles of the same degree cancel each other out, we are left with only a simple pole at $s=1$. This is the proof we needed to say that $\frac{1}{\Gamma(s)}\int_0^1 \frac{x^{s-1}}{e^x-1}dx$ is an analytic function in $\mathbb{C}/\{s=1\}$. As such we have shown that by the functional equation $\zeta(s) = \frac{1}{\Gamma(s)}\int_0^1 \frac{x^{s-1}}{e^x-1}dx + \frac{1}{\Gamma(s)}\int_1^\infty \frac{x^{s-1}}{e^x-1}dx$, $\zeta(s)$ extends into a meromorphic function over the whole complex plane with a simple pole at $s=1$. □

1.2 Other representations of $\zeta(s)$

While the complex-valued Zeta function has one main definition, it can be represented in different ways as shown above. In this section we will introduce three more, all of which are useful in different ways.
The first is one of great importance as we move towards our main theorem.

Proposition 1.13. There exists a sequence of functions $\{\lambda_n(s)\}_{n=1}^{\infty}$, such that

$$\sum_{1\leq n<N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1\leq n<N} \lambda_n(s)$$

where $N > 1$ is an integer and $s \in \mathbb{C}$. Furthermore, these functions satisfy the estimate $|\lambda_n(s)| \leq \frac{|s|}{n^{\sigma+1}}$.

Proof. First, we will quickly show the validity of

$$\int_1^N \frac{dx}{x^s} = \sum_{1\leq n<N} \int_n^{n+1} \frac{dx}{x^s}. \tag{1.4}$$

First, assume $s \neq 1$. Then

$$\int_1^N \frac{dx}{x^s} = -\frac{1}{(s-1)}\left(\frac{1}{N^{s-1}} - 1\right).$$

Next we evaluate the right-side integral

$$\int_n^{n+1} \frac{dx}{x^s} = -\frac{1}{s-1}\left(\frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}}\right).$$

Now if we look at the right side with the integral evaluated, we find that we are looking at a telescopic sum, which neatly computes to

$$-\frac{1}{(s-1)} \sum_{1\leq n<N} \left(\frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}}\right) = -\frac{1}{(s-1)}\left(\frac{1}{N^{s-1}} - 1\right),$$

which is the same as the left hand side, confirming our claim in (1.4). Furthermore, if $s = 1$ we can repeat this procedure and find that the equality is still valid, but the value will be different (since $\int_1^N dx/x = \log N$). As such we are justified to use this equality for all s. Now set

$$\lambda_n(s) = \int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s} dx.$$

To prove that $|\lambda_n(s)|$ is bounded by $\frac{|s|}{n^{\sigma+1}}$ we are going to use the mean-value theorem applied to $f(x) = x^{-s}$, with $n \leq x \leq n+1$:

$$|f(n) - f(x)| = \left|\int_0^1 f'(z(v))dv\right| |x-n|,$$

where $z(v) = n + v(x-n)$ and $v \in [0,1]$. This gives us that

$$|f(x) - f(n)| \leq \int_0^1 |f'(z(v))|dv$$

since $|x-n| \leq 1$ for $n \leq x \leq n+1$. By computing the derivative of $f(x)$ we get $f'(x) = \frac{-s}{x^{s+1}}$ which in turn gives

$$|f'(z(v))| = \frac{|s|}{|(n+v(x-n))^{s+1}|} = \frac{|s|}{|(n+v(x-n))|^{\sigma+1}}.$$

Since $v \in [0,1]$, we know that $n + v(x-n) \geq n$ and

$$\frac{|s|}{|(n+v(x-n))|^{\sigma+1}} \leq \frac{|s|}{n^{\sigma+1}}.$$

Then

$$|f(x) - f(n)| \leq \frac{|s|}{n^{\sigma+1}},$$

and since $\int_n^{n+1} f(n) - f(x)dx$ has length 1, we arrive at the conclusion that $|\lambda_n(s)| \leq \frac{|s|}{n^{\sigma+1}}$, which proves the proposition. □

We can arrive at an alternative estimate if we use the inequality of

$$|\lambda_n(s)| = \left| \int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s} dx \right| \leq \int_n^{n+1} \left| \frac{1}{n^s} - \frac{1}{x^s} \right| dx$$

we find that

$$\int_n^{n+1} \left| \frac{1}{n^s} - \frac{1}{x^s} \right| dx \leq \int_n^{n+1} \left| \frac{1}{n^s} \right| + \left| \frac{1}{n^s} \right| dx = \frac{2}{n^\sigma}$$

and so

$$|\lambda_n(s)| \leq \frac{2}{n^\sigma}. \tag{1.5}$$

The first estimate will be immediately useful, the second will be used in Chapter 2.

Corollary 1.13.1. *There exists a holomorphic function* $H(s) = \sum_{n=1}^{\infty} \lambda_n(s)$ *in the half-plane* $\mathrm{Re}(s) > 0$ *such that*

$$\zeta(s) - \frac{1}{s-1} = H(s)$$

Proof. This is a consequence of Proposition 1.13 if you let $N \to \infty$ and constrict the function to the half-plane $\mathrm{Re}(s) > 0$, with $x > 0$. Then the bound $|\lambda_n(s)| \leq \frac{|s|}{n^{\sigma+1}}$ gives us that the series $\sum_{n=1}^{\infty} \lambda_n(s)$ is uniformly convergent for all $\sigma > 0$ as $\sum_{n=0}^{\infty} \frac{|s|}{n^{\sigma+1}}$ is a convergent sum whenever $\sigma > 0$. This means that $\sum_{n=1}^{\infty} \lambda_n(s)$ describes a holomorphic function in the region $\mathrm{Re}(s) > 0$, and by analytic continuation we conclude that $\zeta(s)$ is also analytical in the same region.

□

The second representation of Zeta we introduce converts the Zeta function from a sum over all numbers $n \geq 1$ to a product over all prime numbers:

Definition 1.14 (Euler's Identity)**.** When $\text{Re}(s) > 1$, the Zeta function can be represented as the infinite product

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

where p is a prime number. This product is called Euler's identity.

The key to understanding the link between Euler's identity and our original definition of the zeta function requires a little bit of algebraic manipulation. This will require us to add and subtract infinite sums, which we can do since we proved $\zeta(s)$ is convergent for $\text{Re}(s) > 1$ in Lemma 1.4. What follows is a sketch of a proof that leads to this identity.
Given that $\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + ...$ we start by subtracting all multiples of $\frac{1}{2^s}$ from the right side, using $\zeta(s) - \frac{1}{2^s}\zeta(s)$:

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + ...$$

Then we subtract the terms of $\frac{1}{3^s}$, but in doing so we will subtract all terms that are multiples of $\frac{1}{2^s} \cdot \frac{1}{3^s}$, which was already done in the previous step. To account for this we simply add those terms back again

$$\left(1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{6^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + ...$$

Then the left hand side can be factorized as

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\zeta(s).$$

If we repeat this procedure for all primes up until some large real number N, where P_n is the largest prime number smaller than N, we will have

$$\prod_p^{P_n}(1 - p^{-s})\zeta(s) = 1 + \sum_{q \geq N} \frac{1}{q^s}.$$

If we continue this adding and subtracting as we let $q \to \infty$, we get

$$\zeta(s)\left(\prod_p 1 - p^{-s}\right) = 1.$$

We will isolate $\zeta(s)$ on the left hand side by claiming that the infinite product $\prod_p 1 - p^{-s}$ is convergent, and by doing so we will get Euler's identity:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \prod_p \frac{1}{1 - p^{-s}}.$$

The validity of our claim is easy to show: An infinite product is convergent if it has a non-zero finite limit. Since $0 < \frac{1}{1-p^{-s}} < 1$, for all $p > 1$ we can see that $0 < \prod_p \frac{1}{1-p^{-s}} < 1$ holds true over all prime numbers, and since $\zeta(s)$ is convergent for $\mathrm{Re}(s) > 1$ we are done.

Using Euler's identity, we can introduce and prove another lemma which will be useful in Section 1.3 and Chapter 2.

Lemma 1.15. *On the half-plane* $\mathrm{Re}(s) > 1$

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

where $c_n \geq 0$.

Proof. Using Euler's identity for $\zeta(s)$, we have

$$\log \zeta(s) = \log \left(\prod_p \frac{1}{1-p^{-s}} \right) = \sum_p \log \left(\frac{1}{1-p^{-s}} \right).$$

By using the power expansion $-\log(1-x) = \sum_{n=1}^{\infty} x^n/n$, which holds whenever $|x| < 1$, we get

$$\log \left(\frac{1}{1-p^{-s}} \right) = \sum_{m=1}^{\infty} \frac{(p^{-s})^m}{m}.$$

Putting these two steps together, we note that the sums over p and m are uniformly convergent, since $p, m \geq 1$ and $\mathrm{Re}(s) > 1$, and we get

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-sm}}{m}.$$

The final sum can be obtained if we set $c_n = \frac{1}{m}$ if $n = p^m$, or $c_n = 0$ otherwise. Then

$$\sum_{p,m} \frac{p^{-sm}}{m} = \sum_{n=1}^{\infty} \frac{c_n}{n^s}.$$

□

1.3 Abscence of zeroes on the line $\sigma = 1$

The proof of Theorem 1.12 showed us that the Zeta function admits no zeroes at $s = 1$. We now wish to show that the line $s = 1 + i\tau$, where τ is a real number, does not admit any zeroes, and as such we will provide a separate proof for that before moving onto Chapter 2 of the Thesis.

Lemma 1.16. *When* $\theta \in \mathbb{R}$ *then* $3 + 4\cos\theta + \cos 2\theta \geq 0$.

Proof. The lemma follows from the simple observation that

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0$$

if θ is real. ☐

Corollary 1.16.1. *If $\sigma > 1$ and τ is real, then*

$$\log|\zeta^3(\sigma)\zeta^4(\sigma + i\tau)\zeta(\sigma + 2i\tau)| \geq 0$$

Proof. Recall that the complex logarithm is defined as

$$\log(z) = \log|z| + i\arg z$$

where $\mathrm{Re}(\log(z)) = \log|z|$. From the condition $\sigma > 1$ we note that our logarithmic branch lies in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus

$$\begin{aligned}&\log|\zeta^3(\sigma)\zeta^4(\sigma + i\tau)\zeta(\sigma + 2i\tau)|\\ &= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma + i\tau)| + \log|\zeta(\sigma + 2i\tau)|\\ &= 3\mathrm{Re}(\log\zeta(\sigma)) + 4\mathrm{Re}(\log\zeta(\sigma + i\tau)) + \mathrm{Re}(\log\zeta(\sigma + 2i\tau)).\end{aligned} \tag{1.6}$$

Then using the expression for $\log(\zeta(s))$ from Lemma 1.15, we note that $n^{-s} = e^{-s\log n}$, which gives us that $\mathrm{Re}(\zeta(s)) = \mathrm{Re}(\sum_{n=1}^{\infty} e^{-(\sigma+i\tau)\log n}) = \sum_{n=1}^{\infty} n^{-\sigma}\cos(\tau\log n)$. These two results combined gives

$$\mathrm{Re}(\log(\zeta(s)) = \sum_{n=1}^{\infty} \frac{c_n}{n^\sigma}\cos(\tau\log n).$$

Inserting it into Equation 1.6, we have

$$\sum_{n=1}^{\infty} \frac{c_n}{n^\sigma}(3 + 4\cos(\tau\log n) + \cos(2\tau\log n))$$

and given that both τ and n are real numbers, the expression in the parenthesis is positive by Lemma 1.16. Since $c_n \geq 0$, the corollary is proved. ☐

This brings us to the final theorem of this chapter:

Theorem 1.17. *The Zeta function has no zeroes on the line $s = 1 + i\tau$, where $\tau \in \mathbb{R}$.*

Proof. We will utilize the results of Corollary 1.16.1 and prove by contradiction that our theorem cannot be false. Assume for a moment that there exists points on the line $s = 1 + i\tau$ where $\zeta(s) = 0$. When $\tau = 0$, we know that $\zeta(1)$ has a simple pole, meaning as $\sigma \to 1$

$$|\zeta(\sigma)|^3 \leq C|\sigma - 1|^{-3},$$

for some constant $C > 0$. Then for some $\tau_0 \neq 0$, $\zeta(1 + i\tau_0)$ must have a zero of order 1 or greater. By inspecting the behaviour close to $\sigma = 1$, we first have that

$$|\zeta(\sigma + i\tau_0)|^4 \leq C'(\sigma - 1)^4$$

when $\sigma \to 1$ and $C' > 0$ is some constant. At the point $\sigma + 2i\tau_0$, with $\sigma \to 1$, $\zeta(\sigma + 2i\tau_0)$ is holomorphic and $|\zeta(\sigma + 2i\tau_0)| \leq B$ is bounded by some constant B.

Putting all of this together, we get

$$|\zeta(\sigma)^3 \zeta(\sigma + i\tau_0)^4 \zeta(\sigma + 2i\tau_0)| \leq |C(\sigma - 1)^{-3} C'(\sigma - 1)^4 B|.$$

Looking at the right hand side as $\sigma \to 1$, we find that

$$CC'(\sigma - 1)B \to 0.$$

Thus we conclude that the left hand side also tends to 0. However that would mean that

$$\log |\zeta(\sigma)^3 \zeta(\sigma + i\tau_0)^4 \zeta(\sigma + 2i\tau_0)| \leq 0$$

which contradicts Corollary 1.16.1, falsifying our assumption that the line $s = 1 + i\tau$ admits zeroes. As such, we have proved that $\zeta(1 + i\tau)$ does not admit any zeroes. $\square$

Chapter 2

The Prime Number Theorem

The properties of the Zeta function we explored in Chapter 1 are now going to be put to use in proving the Prime Number Theorem. The proof of the Prime Number Theorem relies on the relation between Chebyshev's ψ-function, the modified $\psi_1 = \int_1^x \psi(u)du$ and the prime counting function $\pi(x)$, as well as growth estimates of $\zeta(s)/\zeta'(s)$ near the line $\text{Re}(s) = 1$. In this section we aim to divide these parts up into small enough sections that each individual part is understandable, and then bring it all together at the end. We will start this chapter of by giving a formal mathematical definition of the prime counting function $\pi(x)$:

Definition 2.1. The prime counting function $\pi(x)$ is defined as

$$\pi(x) = \sum_{p \leq x} 1$$

whenever p is a prime number.

The author believes it may benefit some to see it written in mathematical notation to better relate to the mathematics discussed in this chapter, particularly in Section 2.2.

2.1 Estimates of the growth of $\zeta(s)$

We know from Lemma 1.4 that $\zeta(s)$ is uniformly convergent in any half-plane $\text{Re}(s) \geq 1 + \delta$, where $\delta > 0$, which guarantees a mild growth. To prove The Prime Number Theorem we will need to estimate the growth of $\zeta'(s)/\zeta(s)$ near $\text{Re}(s) = 1$, and to do that we will need to estimate the growth of first $\zeta(s)$ and then $1/\zeta(s)$.

Proposition 2.2. Let $s = \sigma + i\tau$, where $\sigma, \tau \in \mathbb{R}$. For each σ_0, $0 \leq \sigma_0 \leq 1$ and every $\epsilon > 0$ there exists a constant c_ϵ so that
(i) $|\zeta(s)| \leq c_\epsilon |\tau|^{1-\sigma_0+\epsilon}$ if $\sigma_0 \leq \sigma$ and $|\tau| \geq 1$.
(ii) $|\zeta'(s)| \leq c'_\epsilon |\tau|^\epsilon$ if $\sigma \geq 1$ and $|\tau| \geq 1$.

Proof. Recall from Proposition 1.13 the estimates $|\lambda_n(s)| \leq |s|/n^{\sigma+1}$ and $|\lambda_n(s)| \leq 2/n^{\sigma}$.
We can combine these two estimates via the observation that $y = y^{\delta}y^{1-\delta}$, with $0 < \delta < 1$ which gives us

$$|\lambda_n(s)| \leq \left(\frac{|s|}{n^{\sigma_0+1}}\right)^{\delta}\left(\frac{2}{n^{\sigma_0}}\right)^{1-\delta} \leq \frac{2|s|^{\delta}}{n^{\sigma_0+\delta}}.$$

Let's choose $\delta = 1 - \sigma_0 + \epsilon$, where $\epsilon > 0$ and $\sigma = \mathrm{Re}(s) \geq \sigma_0$, with $0 \leq \sigma_0 \leq 1$, we can apply this to the representation of $\zeta(s)$ in Corollary 1.13.1, and we get

$$|\zeta(s)| \leq \left|\frac{1}{s-1}\right| + 2|s|^{1-\sigma_0+\epsilon}\sum_{n=1}^{\infty}\frac{1}{n^{1+\epsilon}}.$$

We know that the sum is convergent, and since the fraction is bounded by $|1/(s-1)| \leq 1$, on account of $\sigma \geq \sigma_0 \geq 0$ and $|\tau| \geq 1$, we are free to choose a constant c_ϵ dependent on epsilon such that

$$\left|\frac{1}{s-1}\right| + 2|s|^{1-\sigma_0+\epsilon}\sum_{n=1}^{\infty}\frac{1}{n^{1+\epsilon}} \leq c_\epsilon|\tau|^{1-\sigma_0+\epsilon}.$$

This proves part (i).
For part (ii) we note that by the Cauchy integral formula, we have

$$\zeta'(s) = \frac{1}{2\pi r}\int_0^{2\pi}\zeta(s+re^{i\theta})e^{-i\theta}d\theta$$

where we integrate over a circle with radius r around the point s. Then

$$\begin{aligned}|\zeta'(s)| &\leq \frac{1}{2\pi r}\int_0^{2\pi}|\zeta(s+re^{i\theta})|d\theta\\ &\leq \frac{1}{2\pi r}\int_0^{2\pi}c_\epsilon|\tau|^{1-\sigma_0+\epsilon}d\theta\\ &= \frac{1}{r}c_\epsilon|\tau|^{1-\sigma_0+\epsilon}.\end{aligned}$$

If we chose $r = \epsilon$ this circle lies in the half-plane $\mathrm{Re}(s) \geq 1-\epsilon$. Then the smallest σ_0 in this circle is on the boundary, $\sigma_0 = 1-\epsilon$ and we have that

$$|\zeta'(s)| \leq c'_\epsilon|\tau|^{2\epsilon}$$

and if we replace 2ϵ by ϵ we have proven (ii).

□

Proposition 2.3. Let s be defined as above. For every $\epsilon > 0$, we have $1/|\zeta(s)| \leq c_\epsilon|\tau|^{\epsilon}$ when $\sigma \geq 1$ and $|\tau| \geq 1$.

Proof. From Corollary 1.16.1 we have that $|\zeta^3(\sigma)\zeta^4(\sigma+i\tau)\zeta(\sigma+2i\tau)| \geq 1$ whenever $\sigma \geq 1$. By using the estimate for $\zeta(s)$ from the proposition above and observing that $|\zeta(\sigma+2i\tau)| \leq c|\tau|^\epsilon$, we get

$$|\zeta^4(\sigma+i\tau)| \geq c|\zeta^{-3}(\sigma)||\tau|^{-\epsilon} \geq c'(\sigma-1)^3|\tau|^{-\epsilon}$$

for all $\sigma \geq 1$ and $|\tau| \geq 1$. Then by taking the 4th root across the expression, we end up with

$$|\zeta(\sigma+i\tau)| \geq c'(\sigma-1)^{3/4}|\tau|^{-\epsilon/4}. \tag{2.1}$$

Now consider two cases:
1. $\sigma - 1 \geq A|\tau|^{-4\epsilon}$ and
2. $\sigma - 1 < A|\tau|^{-4\epsilon}$.

In the first case, if we substitute this inequality into equation (2.1) we immediately get

$$|\zeta(\sigma+i\tau)| \geq A'|\tau|^{-4\epsilon}.$$

In the second case we select a σ' such that $\sigma' > \sigma$ and $\sigma' = A|\tau|^{5\epsilon}$. Using the triangle inequality, we find

$$|\zeta(\sigma+i\tau)| \geq |\zeta(\sigma'+i\tau)| - |\zeta(\sigma'+i\tau)-\zeta(\sigma+i\tau)|$$

and by applying the same mean-value theorem as the one in Proposition 1.13, we see that

$$\begin{aligned}|\zeta(\sigma'+i\tau)-\zeta(\sigma+i\tau)| &= |\sigma'-\sigma|\left|\int_0^1 \zeta'(\sigma+r(\sigma'-\sigma)+i\tau)dr\right| \\ &\leq |\sigma'-\sigma|\int_0^1 |\zeta'(\sigma+r(\sigma'-\sigma)+i\tau)|dr,\end{aligned}$$

where $r \in [0,1]$. Now we apply the estimate of $\zeta'(s)$ from Proposition 2.2 and see that

$$\int_0^1 |\zeta'(\sigma+r(\sigma'-\sigma)+i\tau)|dr \leq \int_0^1 c'_\epsilon|\tau|^\epsilon dr = c'_\epsilon|\tau|^\epsilon.$$

This gives us

$$|\zeta(\sigma'+i\tau)-\zeta(\sigma+i\tau)| \leq c'_\epsilon|\sigma'-\sigma||\tau|^\epsilon \leq c'_\epsilon|\sigma'-1||\tau|^\epsilon.$$

This holds true as $\sigma \geq 1$ and $\sigma' > \sigma$, and since these are strictly non-negative, we can drop the absolute value signs around $\sigma' - \sigma$ and $\sigma - 1$. Now by using the estimate from equation (2.1) applied to σ' for $|\zeta(\sigma'+i\tau)|$, we get

$$|\zeta(\sigma+i\tau)| \geq c'(\sigma'-1)^{3/4}|\tau|^{-\epsilon/4} - c'_\epsilon(\sigma'-1)|\tau|^\epsilon,$$

where c' is the resulting constant A' after changing σ to σ'. Recall that we said $\sigma' - 1 = A|\tau|^{-5e}$, and set $A = (c'/(2c'_\epsilon))^4$, then we have that

$$c'(\sigma'-1)^{3/4}|\tau|^{-\epsilon/4} = 2c'_\epsilon(\sigma'-1)|\tau|^\epsilon$$

and furthermore

$$|\zeta(\sigma+i\tau)| \geq c'_\epsilon(\sigma'-1)|\tau|^\epsilon = A''|\tau|^{-4\epsilon}$$

which proves our proposition if we replace 4ϵ by ϵ. $\square$

2.2 ψ-functions

Definition 2.4 (Chebyshev's ψ-function)**.** For all integers of the form $p^m \leq x$, Chebyshev's ψ-function is defined as

$$\psi(x) = \sum_{p^m \leq x} \log p$$

where p is a prime and m is a positive integer.

An alternate form of Chebyshev's ψ-function is

$$\psi(x) = \sum_{1 \leq n \leq x} \Lambda(n), \tag{2.2}$$

where $\Lambda(n) = \log p$ whenever $n = p^m$, and zero otherwise. The reason we wish to have this alternate form is to relate the ψ-function with the ζ-function. In Lemma 1.15, we saw that $\log \zeta(s) = \sum_{p,m} \frac{p^{-sm}}{m}$ in the half-plane $\mathrm{Re}(s) > 1$. Differentiating this expression with respect to s on both sides gives

$$\log \zeta(s) = \frac{\zeta'(s)}{\zeta(s)}$$

on the left hand side and again using identity (a) from the proof of Lemma 1.4, we get

$$\left(\sum_{p,m} \frac{e^{-sm \log p}}{m} \right)' = \sum_{p,m} \frac{(-m \log p) e^{-sm \log p}}{m} = -\sum_{p,m} (\log p) p^{-sm}.$$

Finally we can rewrite this into

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1} \frac{\Lambda(n)}{n^s} \tag{2.3}$$

by recalling that $\sum_{p,m} \frac{p^{-sm}}{m} = \sum_{n=1} \frac{c_n}{n^s}$ and that $c_n = \frac{1}{m}$ whenever $n = p$, and whenever $n \neq p$ it vanishes. This final result will be used in the next proposition, but first we will introduce the ψ_1 function.

Definition 2.5. The ψ_1 function is defined as

$$\psi_1(x) = \int_1^x \psi(u) du$$

for $x > 1$.

Proposition 2.6. If $\psi_1(x) \sim \frac{x^2}{2}$ as $x \to \infty$, then $\psi(x) \sim x$ as $x \to \infty$

The notation $\sim$ indicates that there is an asymptotic relationship between two things. The expression $\psi(x) \sim x$ reads "$\psi(x)$ asymptotic (or asymptotically equal) to x". This is equivalent to stating that $\lim_{x \to \infty} \psi(x)/x = 1$.

Proof. Assume for now that the "if" statement is true (we will prove this at the end of this chapter). what we wish to show is that

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \leq 1$$

and

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq 1.$$

Given the fact that $\psi(x)$ is an increasing function, let there be two real constants α, β fulfilling the inequality $0 < \alpha < 1 < \beta$. Then we can construct the following double inequality

$$\frac{1}{(1-\alpha)x} \int_{\alpha x}^{x} \psi(u)du \leq \psi(x) \leq \frac{1}{(\beta-1)x} \int_{x}^{\beta x} \psi(u)du,$$

which holds for all values of α, β as defined above. By evaluating the integrals, we get

$$\frac{1}{(1-\alpha)x}(\psi_1(x) - \psi_1(\alpha x)) \leq \psi(x) \leq \frac{1}{(\beta-1)x}(\psi_1(\beta x) - \psi_1(x)).$$

This can be further refined by dividing throughout by x:

$$\frac{1}{(1-\alpha)}\left(\frac{\psi_1(x) - \psi_1(\alpha x)}{x^2}\right) \leq \frac{\psi(x)}{x} \leq \frac{1}{(\beta-1)}\left(\frac{\psi_1(\beta x) - \psi_1(x)}{x^2}\right).$$

If we now look only at one inequality sign at a time, starting with the left, we can further evaluate this expression. A quick rework of the left inequality gives

$$\frac{1}{(1-\alpha)}\left(\frac{\psi_1(x)}{x^2} - \frac{\psi_1(\alpha x)\alpha^2}{(\alpha x)^2}\right) \leq \frac{\psi(x)}{x},$$

which is a useful expression for the purpose of showing $\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq 1$. By letting x approach infinity we can use our claim that $\psi_1(x) \sim \frac{x^2}{2}$. If it is true, then it is also true that $\frac{\psi_1(x)}{x^2} \sim \frac{1}{2}$ when $x \to \infty$, which gives us

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq \frac{1}{2(1-\alpha)}(1-\alpha^2) = \frac{1}{2}(1+\alpha).$$

Since this holds for all $\alpha < 1$, we now simply need see that $\sup(\frac{1}{2}(1+\alpha)) \to 1$, which occurs when $\alpha \to 1$, and thus verifies that $\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq 1$.
An analogous argument can be made for the right inequality, with $\inf(\frac{1}{2}(\beta + 1)) \to 1$ when $\beta \to 1$ and our proposition is proven. ☐

Proposition 2.7. If $\psi(x) \sim x$ as $x \to \infty$, then $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$.

Proof. The proof here will be similar to that of the previous proposition, namely we wish to show that

$$\liminf_{x\to\infty} \pi(x)\frac{\log(x)}{x} \geq 1$$

and

$$\limsup_{x\to\infty} \pi(x)\frac{\log(x)}{x} \leq 1.$$

From the definition of Chebyshev's $\psi(x)$-function in Definition 2.4 and its alternate form in equation (2.2), as well as the fact that if $p^m \leq x$, then $m \leq \log x/\log p$, we can see that

$$\psi(x) = \sum_{p\leq x} \left[\frac{\log x}{\log p}\right] \log p$$

where $[\log x/\log p]$ is the greatest integer less than or equal to $\log x/\log p$. From this we get

$$\psi(x) = \sum_{p\leq x} \left[\frac{\log x}{\log p}\right] \log p \leq \sum_{p\leq x} \frac{\log x}{\log p} \log p = \pi(x)\log x.$$

Since $\pi(x) = \sum_{p\leq x} 1$. Dividing by x, we get

$$\frac{\pi(x)\log x}{x} \geq \frac{\psi(x)}{x}.$$

By letting x tend to infinity, we get

$$\liminf_{x\to\infty} \pi(x)\frac{\log x}{x} \geq 1,$$

which proves the first proposed inequality.
To prove the second we introduce the constant $0 < \alpha < 1$, and the inequalities

$$\psi(x) \geq \sum_{p\leq x} \log p \geq \sum_{x^\alpha \leq p \leq x} \log p \geq (\pi(x) - \pi(x^\alpha))\log x^\alpha. \tag{2.4}$$

The motivation for the last step is as follows: In the series $\sum_{x^\alpha<p<x} \log p$, we have by definition of $p > x^\alpha$ that $\log p > \log x^\alpha$, and by Definition 2.1

$$\pi(x) - \pi(x^\alpha) = \sum_{p\leq x} 1 - \sum_{p\leq x^\alpha} 1 = \sum_{x^\alpha<p\leq x} 1,$$

which means that

$$\sum_{x^\alpha<p\leq x} \log p > \sum_{x^\alpha<p\leq x} \log x^\alpha,$$

which proves that the inequalities in equation (2.4) are correct. Comparing the first and last inequalities, we see that

$$\psi(x) + \alpha\pi(x^\alpha)\log x \geq \alpha\pi(x)\log x.$$

If we now divide by x, we get

$$\frac{\psi}{x} + \frac{\alpha\pi(x^\alpha)\log(x)}{x} \geq \frac{\alpha\pi(x)\log x}{x}.$$

If we now let x go towards infinity, we notice that the second term on the right side will start to vanish since $\pi(x^\alpha) \leq \ x^\alpha$ and $\alpha < 1$, which means that $\frac{\alpha\pi(x^\alpha)\log x}{x} \leq \frac{x^\alpha \log x}{x} \to 0$ as $x \to \infty$. We also have the asymptotic relationship, proven in Proposition 2.6 that $\psi(x)/x \sim 1$ when $x \to \infty$, giving us

$$1 \geq \alpha \limsup_{x\to\infty} \pi(x)\frac{\log x}{x}$$

and if we let $\alpha \to 1$ (which we can do as we chose it arbitrarily), we have proven the second proposed inequality, thus proving the proposition.

□

Now what we have left to prove is that $\psi_1(x) \sim x^2/2$ as $x \to \infty$. The next two results will help us in that endeavour.

Lemma 2.8. *If* $c > 0$, *then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}ds = \begin{cases} 0, & \text{if } \ 0 < a \leq 1 \\ 1 - 1/a, & \text{if } \ a \geq 1 \end{cases}$$

Proof. In this proof we will use a modified version of Jordan's lemma, see Appendix B for both the statement and a proof. Let $a = e^\beta$, where $\beta = \log a$, and let $f(s) = \frac{a^s}{s(s+1)} = \frac{e^{s\beta}}{s(s+1)}$.
Now let us draw up a simple contour $T(R)$ existing of the straight line $S(R) = (c - iR, c + iR)$ and the semi-circle $C(R) = Re^{i\theta}$ oriented counter-clockwise, where $\theta \leq |\pi/2|$.

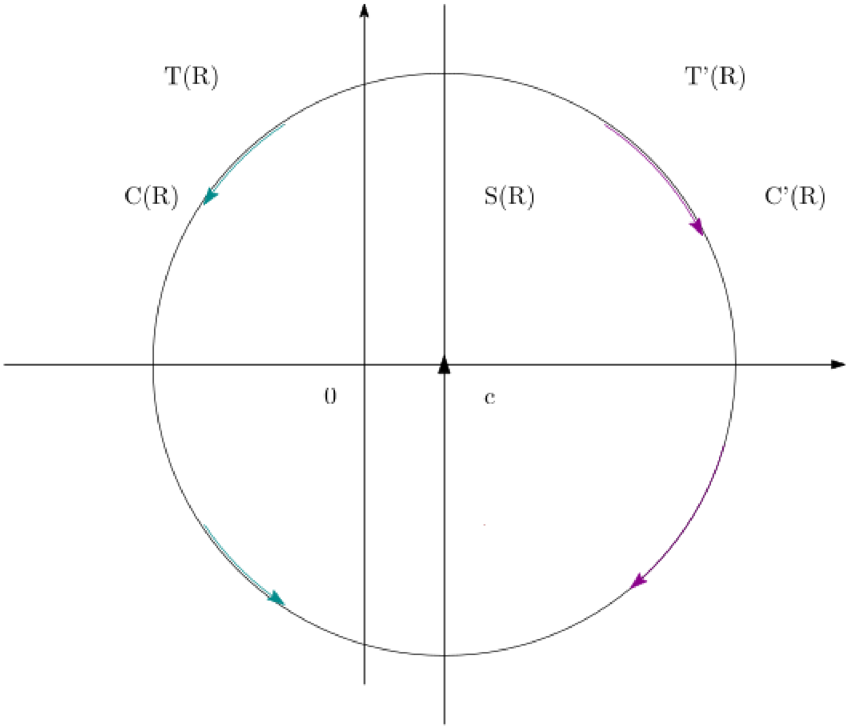

Figure 2.1: The contours T(R) (cyan and black) and T'(R)(Magenta and black)

In this region our function is holomorphic in all but the points $s = -1, 0$. By the residue theorem, $\frac{1}{2\pi i}\int_{T_R} f(s)ds = res_{s=0}f + res_{s=-1} = 1 - 1/a$.
Our aim is to show that as R grows large, the integral over $C(R)$ grows very small. In doing so we now apply Jordan's lemma to our function $f(s)$ on $C(R)$. Notice that if $s \in C(R)$, then for all large R we have

$$|s(s+1)| \geq (1/2)R^2.$$

Given our location we also know that $\text{Re}(s) < c$. That leads us to conclude that when $R \to \infty$

$$\left|\int_{C(R)} f(s)ds\right| \leq \frac{2\pi c}{R^2} \to 0$$

thus proving the lemma when $a \geq 1$.
For the second part our argument will be much the same.
Now we are operating on the right side of the line $S(R)$, with the semi circle $C'(R)$ defined just as $C(R)$, but starting at $3\pi/2$ and terminating at $\pi/2$. We will call this contour $T'(R)$.

Since $f(s)$ is holomorphic without exceptions in this region, we know by Cauchy's integral formula that $\int_{T'(R)} f(s) = 0$. By using Jordan's lemma for

$C'(R)$ in the same way we did for $C(R)$ we can see that

$$\left| \int_{C'(R)} f(s)ds \right| \leq \frac{2\pi\sigma}{R^2} \to 0$$

as $R \to \infty$. Thus we can conclude that the lemma is also correct in the case of $0 < a \leq 1$. □

Proposition 2.9. For all $c > 1$, $\psi_1(x)$ has the functional equation

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big) ds. \tag{2.5}$$

Proof. Observe that

$$\psi(u) = \sum_{n=1}^{\infty} \Lambda(n) f_n(u),$$

where $f_n(u) = 1$ if $n \leq u$ and 0 otherwise. Then

$$\begin{aligned} \psi_1(x) &= \int_1^x \psi(u)du \\ &= \sum_{n=1}^{\infty} \int_1^x \Lambda(n) f_n(u) du \\ &= \sum_{n \leq x} \Lambda(n) \int_n^x du \end{aligned}$$

which gives

$$\psi_1(x) = \sum_{n \leq x} \Lambda(n)(x - n).$$

Using equations (2.5) and (2.3) , we get

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big) ds = x \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} ds.$$

Now we can apply Lemma 2.8 (with $a = x/n$), and we get

$$\begin{aligned} x \frac{1}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n) \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s+1)} ds &= x \sum_{n \leq x} \Lambda(n) \Big(1 - \frac{n}{x} \Big) \\ &= \sum_{n \leq x} \Lambda(n)(x - n) = \psi_1(x), \end{aligned}$$

which concludes the proof. □

With all this in mind, it is time to prove the claim $\psi_1(x) \sim \frac{x^2}{2}$ when $x \to \infty$. If we do, then by Propositions 2.6 and 2.7 we have proven the Prime Number theorem.

2.3 The asymptotics of $\psi_1(x)$

Theorem 2.10. *The function*

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\big(-\frac{\zeta'(s)}{\zeta(s)}\big)ds$$

has the asymptotic relationship

$$\psi_1(x) \sim \frac{x^2}{2}$$

when $x \to \infty$.

Proof. To begin the proof, let $F(s) = \frac{x^{s+1}}{s(s+1)}(-\frac{\zeta'(s)}{\zeta(s)})$ denote the function inside the integral. Our aim here is to work as close to the line $\mathrm{Re}(s) = 1$ as we can. To do that we will start with the line $\mathrm{Re}(s) = c$, where $c > 1$. The integral along this line is the familiar

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\Big(-\frac{\zeta'(s)}{\zeta(s)}\Big).$$

We want c to be close to the line $\mathrm{Re}(s) = 1$ so that we then can deform it, moving all but a small, half-open "box" onto $\mathrm{Re}(s) = 1$. We name this contour $\gamma(T)$, where T is an appropriately large number, and note that

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} F(s)ds = \frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds.$$

This equality comes about from observing that the value of $F(s)$ in the two (infinite) rectangles formed between the two lines $\mathrm{Re}(s) = c$ and $\gamma(T)$ is bounded by $F(s) \leq A'|\tau|^{-2+\mu}$. By Proposition 2.2 and Proposition 2.3 we have that $|\zeta'(s)/\zeta(s)| \leq A|\tau|^{\mu}$ when $\sigma \geq 1,\ |\tau| \geq 1$ for some fixed $\mu > 0$. By Lemma 2.8 we also have that $|s(s+1)| \geq 1/2|\tau|^2$. Combining these two estimates gives the one above, and as $|\tau| \to \infty$, these rectangles tend to 0, which is what we need to move from one contour to the other.
Now we have a second line that mostly lies on the line we want, but not quite. This is where our third line $\gamma(T, \delta)$, where $\delta > 0$ and $1 - \delta \leq \sigma \leq 1$, comes in, but this time the half box is on the left of the point $s = 1$.

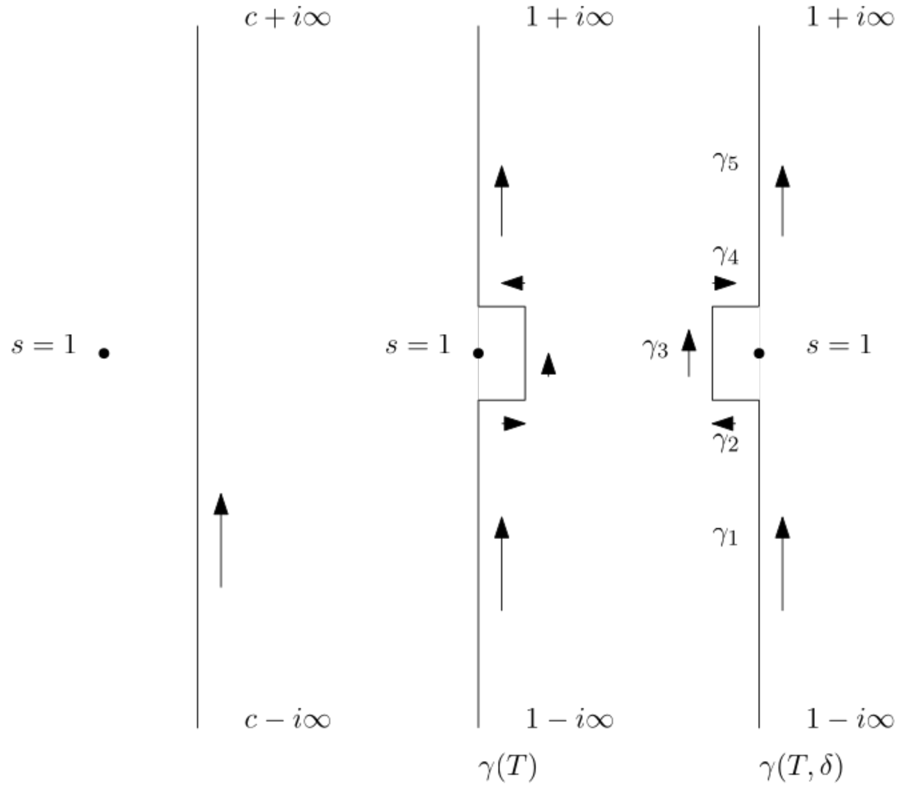

Figure 2.2: The lines $\mathrm{Re}(s) = c, \gamma(T)$ and $\gamma(T, \delta)$

Now let $|\tau| \leq T$ in the two half-boxes. We can enclose the area around $s = 1$ by combining the lines $\gamma(T)$ and $\gamma(T, \delta)$. We will call this enclosed area $\alpha(T, \delta)$. Then we can see that

$$\gamma(T) - \gamma(T, \delta) = \alpha(T, \delta).$$

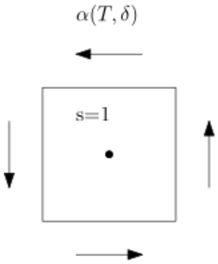

Figure 2.3: The box $\alpha(T, \delta)$

The reason why we choose to enclose this box is that we can use the residue theorem to calculate $\int_{\alpha(T,\delta)} F(s)ds$, and if we can get an upper estimate for the integral on the paths γ_{1-5} then we can also gain an estimate of $\int_{\gamma(T)} F(s)ds$, since $\int_{\gamma(T)} F(s)ds = \int_{\alpha(T,\delta)} F(s)ds + \int_{\gamma(T,\delta)} F(s)ds$.

Given that $F(s)$ is holomorphic in all but one point in the box $\alpha(T,\delta)$, we are justified in using the residue theorem. From Corollary 1.13.1, we know that $\zeta(s) = 1/(s-1) + H(s)$, where $H(s)$ is holomorphic whenever $\mathrm{Re}(s) > 0$. Set $G(s) = 1 + (s-1)H(s)$, and note that $G(s)$ defines a holomorphic function under the same conditions as $H(s)$. Then $\zeta(s) = G(s)/(s-1)$ and

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{G'(s)(s-1) - G(s)}{G(s)(s-1)} = \frac{G'(s)}{G(s)} - \frac{1}{s-1}.$$

Since $G(s)$ is holomorphic near $s = 1$, $G'(s)/G(s)$ must also be holomorphic there. We can then calculate the value of $F(s)$ in the box $\alpha(T,\delta)$ around $s = 1$. This gives:

$$\frac{1}{2\pi i}\int_{\alpha(T,\delta)} F(s)ds = \lim_{s\to 1}(s-1)\frac{x^{s+1}}{s(s+1)}\left(\frac{1}{s-1} - \frac{G'(s)}{G(s)}\right)ds = \frac{x^2}{2}$$

and subsequently

$$\frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds = \frac{x^2}{2} + \frac{1}{2\pi i}\int_{\gamma(T,\delta)} F(s)ds. \tag{2.6}$$

Now the next step is to estimate $F(s)$ on $\gamma(T,\delta)$. We will start by looking at the segment γ_1, with the argument for γ_5 being the same. Note that for $s \in \gamma_1$, $|x^{s+1}| = x^2$ and $|s(s+1)| \geq 1/2|\tau|^2$. By the integral inequality introduced in Chapter 1, we get

$$\left|\int_{\gamma_1} F(s)ds\right| \leq \int_{\gamma_1} |F(s)|ds = \int_{\gamma_1} \frac{x^2}{\tau^2}\left|\frac{-\zeta'(s)}{\zeta(s)}\right| d\tau$$

which also holds for $F(s)$ on γ_5. From Propositions 2.2 and 2.3, we have that

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq B|\tau|^{1/2},$$

whenever $\sigma \geq 1,\ |t| \geq 1$, and where $B = \frac{c'_\epsilon}{A''}$. Thus

$$\int_{\gamma_1} |F(s)ds| \leq B'x^2 \int_T^\infty \frac{|\tau|^{1/2}}{\tau^2} d\tau \leq \epsilon\frac{x^2}{2}.$$

The last step we justify by noting that the integral with respect to τ is convergent as T grows large. We can then choose a constant $\epsilon/2 > 0$ arbitrarily as long as T is sufficiently large. Since the arguments are the same, we also have

$|\int_{\gamma_5} F(s)ds| \leq \epsilon x^2/2$.

Along the paths $\gamma_2, \gamma_3, \gamma_4$ it is enough to note that $\frac{\zeta'(s)}{\zeta(s)}$ is analytic along each path as long as $\delta > 0$. Thus we can conclude that it has to be bounded by some constant that depends on δ.

On the paths γ_2, γ_4 we see that when $1-\delta \leq \sigma \leq 1$, the value of $\left|\frac{1}{s(s+1)}\right|$ can be given an upper bound C_δ, where $1/2 < C_\delta < 1$. If we start by looking at the integral over γ_2, we see that

$$\left|\int_{\gamma_2} F(s)ds\right| \leq C_\delta \int_{1-\delta}^{1} x^{1+\sigma} d\sigma.$$

Using the now familiar tricks, we can evaluate and estimate this integral to see that

$$\int_{1-\delta}^{1} x^{1+\sigma} d\sigma = e^{\log x} \int_{1-\delta}^{1} e^{\sigma \log x} d\sigma$$

$$= \frac{e^{\log x}}{\log x} e^{\sigma \log x}\Big|_{\sigma=1-\delta}^{1} = \frac{x^2(1-x^{-\delta})}{\log x} \leq \frac{x^2}{\log x}.$$

Repeating this method for the path γ_4 grants us

$$\left|\int_{\gamma_4} F(s)ds\right| \leq C'_\delta \frac{x^2(x^\delta - 1)}{\log x} \leq C'_\delta \frac{x^2}{\log x}.$$

If we combine the two constants C_δ and C'_δ into C''_δ we can see that

$$\left|\int_{\gamma_2} F(s)ds\right| + \left|\int_{\gamma_4} F(s)ds\right| \leq C''_\delta \frac{x^2}{\log x}.$$

On the path γ_3 we have

$$\left|\frac{x^{s+1}}{s(s+1)}\right| \leq C_T x^{2-\delta}$$

Where C_T is some constant that fulfills $\max_{\gamma_3} \frac{1}{|s(s+1)|} \leq C_T$. Then

$$\left|\int_{\gamma_3} F(s)ds\right| \leq C_{\delta,T} x^{2-\delta},$$

where $C_{\delta,T}$ is a constant depending on both δ and T. By combining all of this, we get that

$$\left|\int_{\gamma(T,\delta)} F(s)ds\right| \leq \epsilon x^2 + C''_\delta \frac{x^2}{\log x} + C_{\delta,T} x^{2-\delta}.$$

Combining these results with Equation (2.6) brings us to the last part of this thesis:

$$\left|\psi_1(x) - \frac{x^2}{2}\right| \leq \epsilon x^2 + C''_\delta \frac{x^2}{\log x} + C_{\delta,T} x^{2-\delta}.$$

Dividing this expression by $x^2/2$ gives

$$\left|\frac{2\psi_1(x)}{x^2} - 1\right| \leq 2\epsilon + C''_\delta \frac{2}{\log x} + C_{\delta,T}\frac{2}{x^\delta}.$$

If we now let x tend to infinity, we see that

$$\left|\frac{2\psi_1(x)}{x^2} - 1\right| \leq 4\epsilon.$$

Given the choice of $\epsilon > 0$ was arbitrary, we can see that this proves the asymptotic relationship $\psi_1(x) \sim \frac{x^2}{2}$ when $x \to \infty$, and so The Prime Number Theorem has been proven.

$\square$

Appendix A

Analytic continuation

To prove the Prime Number Theorem we have liberally used the method of Analytic continuation without ever giving any proper justification for it. In this appendix we aim to fix that. In our argument we will use the Identity theorem, which we will state in two different ways.

Theorem A.1 (The Identity theorem)**.** *Let f be a holomorphic function in an open, connected region D that vanishes on a sequence of distinct points $\{w_k\}_{k=1}^{\infty}$ that has a limit point in D. Then f is identically 0.*

The important thing to note here is that the theorem states that if the zeroes of f *accumulate* in D, then $f = 0$ everywhere.

Proof. Suppose that z_0 is a limit point for the sequence $\{w_k\}_{k=1}^{\infty}$. We start off by claiming that in a small disk around z_0 the function f, with power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)$$

is identically 0.

Now suppose our claim was false and that there exists some small integer $a_m \neq 0$ so that

$$f(z) = a_m (z - z_0)^m (1 + g(z - z_0))$$

where $g(z - z_0)$ converges to 0 as $z \to z_0$. Let $z = w_k,\ w_k \neq z_0$ and we arrive at a contradiction, since $f(w_k) = 0$ but $a_m(z - w_k)^m \neq 0$ and $1 + g(z - z_0) \neq 0$. Thus $f = 0$ in a small disk around z_0.

We now wish to expand the small disk into the region D. Let U denote the set of interior points where $f(z) = 0$, making U open and non-empty by definition. We also note that the set U has to be closed, since any limit point z_0 of z also has the property that $f(z_0) = 0$ from the previous argument, meaning if $z \in U$, then $z_0 \in U$. Now let $D = U \cup V$, where V is the complement of U in D. Since this implies that both sets are open and disjoint sets of a connected set, one of them has to be empty. Since $z \in U$, V is empty and $U = D$.

□

This gives us the following corollary, that connects the identity theorem to the rest of our thesis a bit more clearly:

Corollary A.1.1. *Let D be a region on the complex plane, and let g, h be two holomorphic functions in that region that satisfy $g(z) = h(z)$ in some open, non-empty subset $U \subset D$. Then $g = h$ in D.*

If you let $f = g - h$ in D, then it is clear that the corollary is true. The reason this theorem is so powerful is that we can extend the region D to all of $\mathbb{C}$. This is where we get the concept on analytic continuation:

Definition A.2 (Analytic Continuation)**.** If $f(z)$ is analytic in a domain D and $F(z)$ is analytic in a domain D', $D \subset D'$ and $f(z) = F(z)$ in D, then we say that $F(z)$ is the analytic continuation of $f(z)$ in D'.

This is why the concept of analytic Continuation is so strong: You only need the two functions $f(z)$ and $F(z)$ to agree on a small subset to be able to substitute $F(z)$ for $f(z)$ on a larger set. In our use of analytic Continuation in Chapter 1 we never proved that $\zeta(s)$ was equal to its continuation in the regions $\operatorname{Re}(s) \leq 1$, and this is why we did not need to.

Appendix B

Key concepts

This appendix is intended to supplement the main body of text in regards to concepts that are deemed fundamental to the text but not explained, either due to an assumption of prerequisite knowledge or to avoid breaking up the arguments of the text too much. Here they are listed so that any reader who may be unfamiliar or wish to revisit them may have an easy time accessing them. We will however assume that the reader of this text has at least attended calculus classes equivalent to Stockholm university's Matematik II - Analys B (MM5011), and preferably a class in Complex analysis. If nothing else is stated, we have chosen $\mathbb{C}$ as the (metric) space in which we operate, and U to be an open subset of $\mathbb{C}$.

Definition B.1 (Zeroes and poles)**.** A zero of a function $f(s)$ is defined as any $s_0 \in \mathbb{C}$ where $f(s_0) = 0$. A pole of a function $g(s)$ is defined as any $s_0 \in \mathbb{C}$ where $\lim_{s \to s_0} g(s) = \infty$.

Equivalently stated, if $g(s) = \frac{1}{f(s)}$, then any zeroes of $f(s)$ are poles of $g(s)$. In addition we often speak about the *order* of the zero/pole, which refers to how many zeroes/poles exists at a given point, e.g. The function $f(s) = s^3$ has a zero of the third order at $s = 0$, and the function $g(s) = s/f(s)$ has a pole of the second order at the same point.

Definition B.2 (Holomorphic Function)**.** Given an open set U and a function defined on $f : U \to \mathbb{C}$, if the derivative $f'(z)$ exists for all $z \in \mathbb{C}$, then f is a holomorphic function in U.

Definition B.3 (Analytic function)**.** With f, U as above, f is said to be an analytic function in U if for all $z_0 \in U$ f has a convergent power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)$.

By Theorem 1.2.6 and Theorem 2.4.4 in [1, p.16 and p. 49], a Holomorphic function is analytic, and an analytic function is holomorphic, which means that the two terms can be used interchangeably.

Definition B.4 (Convergence)**.** A sequence p_n is said to be convergent if, for every $\epsilon > 0$ there exists an integer N such that $n \geq N$, which further implies that

$$|p_n - p| < \epsilon.$$

When we say a function is convergent we also mean that it is pointwise convergent, as opposed to the stronger uniform convergent:

Definition B.5 (Uniform Convergence)**.** Let $\{f_n\}$ be a sequence of functions in U, where $n = 1, 2, 3, \ldots.$ If there exists $\epsilon > 0$ and a (large) integer N such that $n \geq N$, implies

$$|f_n(x) - f(x)| \leq \epsilon$$

we say that $\{f_n\}$ converges uniformly to the function f in U.

What Definition B.4 and B.5 both describe is the idea that if an infinite sequence can be summed up into something finite, then terms of the larger indices have to be very small, and the "difference" (or more correctly, the distance) between them has to be very small. The main difference then between (pointwise) convergence and uniform convergence is the choice of ϵ. If a function is convergent, then different values of f_n may lead to different values of ϵ, so that every value of f_n lies close to f. If a function is uniformly convergent, there must exist a ϵ so that every single f_n lies at most within ϵ-distance of f.

Theorem B.6 (Weierstrass M-test)**.** *Suppose $\{f_n\}$ is a sequence of functions defined on U, and suppose*

$$|f_n(x)| \leq M_n$$

where $x \in U$ and $n = 1, 2, 3 \ldots.$ Then $\sum f_n$ converges uniformly in U if $\sum M_n$ converges.

For the proof, see Theorem 7.10 in [2, p. 148].

Theorem B.7 (Convergence for power series/radius of convergence)**.** *Given a power series of the form $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)$, we define the radius of convergence as the number $r = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{a_n}}$.*

If $r < 1$ the series converges, if $r > 1$ it diverges. For proof, see either [1, p.15] or [2, p.69].

Theorem B.8 (Theorem 2.5.2)**.** *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of U, then f is holomorphic in U.*

Theorem B.9 (Theorem 2.5.4)**.** *Let $F(z, s)$ be defined for $(z, s) \in U \times [0, 1]$, where U is an open set in $\mathbb{C}$. Suppose F satisfies the following properties:*
(i) $F(z, s)$ is holomorphic in z for each s
(ii) F is continuous on $U \times [0, 1]$.
Then the function f defined on U by

$$f(z) = \int_0^1 F(z, s)ds$$

is holomorphic.

The proof for these last two theorems can be found in [1, p. 53 and p.56].

Theorem B.10 (Riemann's theorem on removable singularities)**.** *Suppose f is a holomorphic function in U in all points except maybe z_0. If f is bounded in U/z_0, then z_0 is a removable singularity.*

We will give a rough sketch of the proof.
Let $g(z_0) = 0$, and whenever $z \neq z_0$ we have

$$g(z) = (z - z_0)^2 f(z),$$

and since $f(z)$ is holomorphic, so is $g(z)$. Given that $f(z)$ is bounded, we also have that $g(z) \to 0$ as $z \to z_0$. Then by the definition of a derivative we can confirm that $g'(z_0)$ exists, since

$$\frac{g(z) - g(z_0)}{z - z_0} = (z - z_0) f(z_0) \to 0.$$

This is the part we used in the proof of Lemma 1.11. Since g has a derivative at z_0, it has to be holomorphic and we can evaluate it as a power series around this point, with the first two terms being 0. If you then express f in terms of g, so that $f(z) = g(z)(z - z_0)^{-2}$, you can combine that with the power series of g and find a finite value for $f(z_0)$, which concludes that z_0 is a removable singularity and f is holomorphic in that neighborhood.

Theorem B.11 (Residue Theorem)**.** *Let f be a function that is holomorphic on a disk D inside U, except for a finite number of points z_k inside D, where $k = 0, 1, ..., n$. Then*

$$\int_D f(z)dz = 2\pi i \sum_{k=0}^{n} res_{z_k} f$$

where $res_{z_k} f = \lim_{z \to z_k} (z - z_0) f(z)$.

See proof in [1, p. 77].

Definition B.12 (Limsup)**.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, and let $A = \sup\{a_n\}_{n=N}^{\infty}$ for some large $N > n$. Then the sequence $\{A_n\}_{n=1}^{\infty}$ denotes all the upper bounds of the sequence $\{a_n\}_{n=1}^{\infty}$. We can then define the Limit Superior ("limsup") of the sequence $\{a_n\}_{n=1}^{\infty}$ as

$$\limsup_{n \to \infty} a_n = \inf\{A_n\}_{n=1}^{\infty}.$$

In other words, the limsup of a_n is the smallest upper bound of the sequence $\{a_n\}_{n=1}^{\infty}$.

Lemma B.13 (Jordan's lemma)**.** *Let $f(z)$ be a continuous, complex valued function of the form $f(z) = e^{iaz} g(z)$ and defined on a semicircular contour*

$C_R = \{Re^{i\theta} \mid \theta \in [0, \pi]\}$, *where the radius* R *and the constant* a *are positive. Then the contour integral* $\int_{C_R} f(z)dz$ *has an upper bound*

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{\pi}{a} M_R$$

where $M_R = max_{\theta \in [0,\theta]} |g(Re^{i\theta})|$ *is the maximum value of* $g(z)$ *on the contour.*

A quick remark before we begin the proof is that if f is continuous on the semicircular contour C_R for all large R, then

$$\lim_{R \to \infty} M_R = 0.$$

Proof. By the definition of a complex line integral,

$$\int_{C_R} f(z)dz = \int_0^{\pi} g(Re^{i\theta}) e^{iaR(\cos\theta + i \sin\theta)} iRe^{i\theta} d\theta,$$

the now familiar inequality $| \int_a^b f(x)dx| \leq \int_b^a |f(x)|dx$ yields

$$I_R := \left| \int_{C_R} f(z)dz \right| \leq R \int_0^{\pi} \left| g(Re^{i\theta}) e^{aR(i\cos\theta - \sin\theta)} e^{i\theta} \right| = R \int_0^{\pi} |g(Re^{i\theta})| e^{-aR\sin\theta} d\theta.$$

By utilizing the symmetry $\sin\theta = \sin(\pi - \theta)$, and using M_R as stated in the above remark, we get

$$I_R \leq RM_R \int_0^{\pi} e^{-aR\sin\theta} d\theta = 2RM_R \int_0^{\pi/2} e^{-aR\sin\theta} d\theta.$$

In the interval $\theta \in [0, \pi/2]$ the graph of $\sin\theta$ lies above the straight line connecting its endpoints, meaning that $\sin\theta \geq 2\theta/\pi$ in that region. This further implies that

$$I_R \leq 2RM_R \int_0^{\pi/2} e^{-2aR\theta/\pi} d\theta = \frac{\pi}{a}(1 - e^{-aR}) M_R \leq \frac{\pi}{a} M_R.$$

□

Corollary B.13.1 (Jordan's Rotated lemma)**.** *Jordan's lemma also holds for functions of the form* $h(z) = e^{az} g(z)$ *if the semi-circular contour is rotated by* 90° *counter-clockwise.*

Proof. The proof is mostly the same as for Jordan's lemma, except for the initial integral. The first few steps are near identical as in Lemma B.13 so we will skip them. Let $S_R = \{Re^{i\theta} | \theta \leq |\pi/2|\}$ and let $N_R := max_{\theta \leq |\pi/2|}$. Then

$$J_R := \left| \int_{S_R} h(z)dz \right| \leq \int_{-\pi/2}^{\pi/2} |g(Re^{i\theta}) e^{aR(\cos\theta + i\sin\theta)} iRe^{i\theta}| d\theta = R \int_{-\pi/2}^{\pi/2} |g(Re^{i\theta})| e^{aR\cos(\theta)} d\theta.$$

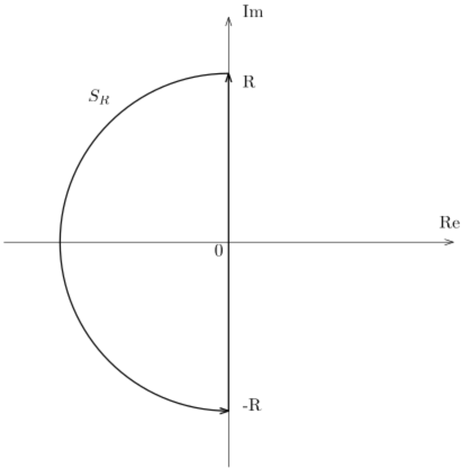

Figure B.1: The contour S_R

We now intend to show that it is possible to rotate this integral by 90°, thus proving that we are justified in applying Jordan's lemma on a semi-circle different from C_R. Since $\cos\theta = -\sin(\pi/2+\theta)$ and $d(\theta+\pi/2) = d(\theta)$, we get

$$\int_{-\pi/2}^{\pi/2} e^{aR\cos(\theta)}d\theta = \int_0^{\pi} e^{-aR\sin\theta}d\theta.$$

We also have that

$$e^{i(\theta+\pi/2)} = ie^{i\theta},$$

and since multiplication with i induces a 90° rotation but does not change the value, we get $N_R = M_R$, which supports our claim. Thus we now have enough proof to claim that

$$RN_R \int_{-\pi/2}^{\pi/2} e^{aR\cos(\theta)}d\theta = RM_R \int_0^{\pi} e^{-aR\sin\theta}d\theta$$

and the rest was proven in Lemma B.13. □

The above proof can be repeated to show that Jordan's lemma works in clockwise rotation too, you just need to repeat the proof with the function $h(z) = e^{-az}g(z)$, and use $\cos\theta = \sin(\theta-\pi/2)$.

AI Statement

AI has not been used in any way in the making of this thesis. AI has not been intentionally used in the research for this thesis, though the author is aware that it can at times be difficult to tell what on the internet is written by a real person and what has been generated by AI.

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