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Noetherian and Artinian rings

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Abstract

Noetherian and Artinian rings are two important classes of rings. In this thesis we present the basic definitions and some of the properties of Noetherian and Artinian rings. We begin by establishing several equivalent definitions. We then prove some important theorems for Noetherian rings, including Hilberts basis theorem and that surjective homomorphisms to a different ring maintains the Noetherian property. We also explore the primary decomposition of Noetherian rings. Then we move on to Artinian rings and explore some of their properties, including their primary decomposition and the nilradical.

Sammanfattning

Noetherska och Artinska ringar är två viktiga klasser av ringar. I denna uppsats presenterar vi de grundläggande definitionerna och vissa egenskaper hos Noetherska och Artinska ringar. Vi börjar med att etablera flera ekvivalenta definitioner. Därefter bevisar vi några viktiga satser för Noetherska ringar, inklusive Hilberts bassats och att surjektiva homomorfier till en annan ring bevarar den Noetherska egenskapen. Vi undersöker också den primära dekompositionen av Noetherska ringar. Sedan går vi vidare till Artinska ringar och undersöker några av deras egenskaper, inklusive deras primära dekomposition och nilradikal.

Contents

1 Introduction

In this thesis we will outline the basic concepts and some important results relating to commutative Noetherian and Artinian rings. Noetherian and Artinian rings both have a finite ideal structure in the sense that they have put restrictions on chains of ideals, called the chain conditions. In chapter 3 we will develop these conditions further and outline three equivalent conditions for Noetherian rings and two for Artinian rings.

An important result relating to Noetherian rings is Hilbert's basis theorem, which states that if the elements in a ring of polynomials has coefficients from a Noetherian ring, then the ring of polynomials is also Noetherian. This and the corresponding theorem for rings of formal power series will be covered in chapter 4. In chapter 4 we will also cover some results regarding the primary decomposition of Noetherian rings, including that every ideal of a Noetherian ring has a primary decomposition.

Chapter 5 is devoted to Artinian rings. Topics that will be covered here include the primary decomposition and nilpotency. We will show that in Artinian rings, all prime ideals are maximal and that the number of prime ideals is finite.

The literature this thesis will rely on is *Abstract Algebra* (3rd ed.) by D. Dummit and R. Foote, which will hereafter be referred to as [DF03]. We will also be using *Introduction to Commutative Algebra* by M. F. Atiyah and I. G. Macdonald, which will be refered to as [AM69]. Finally, we will also be using *Basic Algebra* (2nd ed.) by Nathan Jacobson, which will be refered to as [Jac09].

Finally, this thesis will only cover commutative rings, so we will not be covering, for example, left/right Noetherian rings.

2 Preliminary

In this chapter, we will present some definitions and results that will be important in this thesis. This includes the definitions of rings, ideals, and quotient rings. We will also establish some relationships between ideals and the associated quotient rings. This chapter relies heavily on [DF03] as a source. We start by defining groups and rings.

2.1 Groups and rings

Definition 1. A *group* is a set G with an associated binary operation $*$ such that the following holds:

- For any two elements $x, y \in G$, it follows that $x * y \in G$.
- For all $x, y, z \in G$, it holds that $(x * y) * z = x * (y * z)$.
- An element e exists in G, with the properties that $e * x = x * e = x$ for all $x \in G$.
- For each $x \in G$ there exist $x^{-1} \in G$ with the property that $x * x^{-1} = x^{-1} * x = e$.

A group is called abelian (or commutative) if the following condition is satisfied:

- For all $x, y \in X$, it holds that $x * y = y * x$.

Definition 2. A *ring* is a set R with two associated operations $+$ and $\times$ with the properties that:

- R is an abelian group under the $+$ operation.
- For all $x, y, z \in R$, it holds that $(x \times y) \times z = x \times (y \times z)$.
- For all $x, y, z \in R$, it holds that $x \times (y + z) = (x \times y) + (x \times z)$ and $(x + y) \times z = (x \times z) + (y \times z)$.

A ring is called commutative if:

- For all $x, y \in R$, it holds that $x \times y = y \times x$.

If there exists an element $1 \in R$ such that $1 \times x = x \times 1 = x$ for all $x \in R$, then R is said to have an identity. From now on, R will be a commutative ring with a multiplicative identity.

2.2 Ideals

Now we move on to define ideals, which is a core concept of this thesis. We will also define some properties relevant to ideals that are important in this thesis. In particular, we will define prime ideals, irreducible ideals, principal ideals, primary decomposition, nilpotency, and what it means that an ideal is generated by a set of elements.

Definition 3. An *ideal* I is a subset of a ring R that satisfies the following properties:

- If two elements a and b are in I, then $x + y \in I$ and $a \times b \in I$.
- If $x \in R$ and $a \in I$, then $x \times a \in I$ and $a \times x \in I$.

If, in addition, $I \neq R$ it is called a proper ideal of R.

Definition 4. A *prime ideal* P is an ideal with the property that given two elements $a, b \in R$, if $a \times b \in P$ it follows that either $a \in P$ or $b \in P$.

Definition 5. An ideal I is irreducible if there are no ideals A and B such that

$$I \subset A, I \subset B$$

and

$$I = A \cap B.$$

Definition 6. An ideal I or a ring R is generated by a set X if every element $a_i \in I$ can be written as

$$a_i = \sum_i r_i x_i$$

for finitely many r_i such that $r_i \neq 0$ and $r_i \in R$ and $x_i \in X$. If X has only a finite number of elements, then I is said to be *finitely generated.*

Definition 7. A *Principal Ideal* is an ideal that is generated by a single element.

Definition 8. An ideal I in R is said to be a *maximal ideal* if $I \subset R$ and there exist no other ideal J such that $I \subset J \subset R$.

Definition 9. An ideal I is *nilpotent* if there is some $k \in \mathbb{Z}^+$ such that $I^k = 0$.

Definition 10. Let R be a commutative ring and I a proper ideal of R. Then I is said to have a *primary decomposition* if there exists a finite set of primary ideals $\{P_1, ..., P_n\}$ such that

$$I = P_1 \cap P_2 \cap ... \cap P_n.$$

2.3 Integral domains and Principal ideal domains

The concepts of integral domains will be important in chapter 5. They will be defined here along with principal ideal domains which will be relevant in chapter 3.

Definition 11. An *Integral Domain* is a commutative ring R such that if $a, b \in R$ and $a, b \neq 0$, it follows that $ab \neq 0$.

Definition 12. A *Principal Ideal Domain* is an integral domain where every ideal is a principal ideal.

Proposition 1. The ring of integers $\mathbb{Z}$ is a Principal Ideal Domain.

Proof. Omitted □

2.4 Ring homomorphisms

Next, we will go over ring homomorphisms and quotient rings. These concepts will be important later in this thesis.

Definition 13. Let R and A be rings. A *ring homomorphism* is a function $f : R \longrightarrow A$ such that

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

and

$$f(x_1 x_2) = f(x_1) f(x_2)$$

for all $x_1, x_2 \in X$.

Definition 14. The *kernel* of a ring homomorphism f is those elements $x \in R$ such that $f(x) = 0 \in A$. This is denoted $ker(f)$.

Theorem 1. Let I be an ideal of a ring R and f a homomorphism $f : R \longrightarrow A$ such that $ker(f) = I$. Then A is a ring. This type of ring is called a *Quotient ring* and is denoted R/I.

Proof. See [DF03, page 240-242]. □

2.5 Quotient ring properties

Some properties of quotient rings will be relevant to our proofs and theorems. In particular the relationship between the ideal we quotient by and the quotient ring will prove important.

Lemma 1. Let R be a commutative ring and I an ideal in R. Then I is maximal if and only if R/I is a field.

Proof. See [DF03, Chapter 7, Proposition 12]. □

Lemma 2. Let R be a commutative ring and I an ideal in R. Then I is prime if and only if R/I is an integral domain.

Proof. See [DF03, Chapter 7, Proposition 13]. □

2.6 Intersection of ideals

Finally, the following results about the intersection of ideals will be important.

Proposition 2. Let R be a ring and I_i ideals of R. Then $\bigcap_{i=0}^{n} I_i$ is an ideal of R and $\bigcap_{i=0}^{n} I_i \subseteq I_i$ for all I_i.

Proof. Omitted. □

Furthermore, it is easy to see that for any ideal I

$$I \cap \left(\bigcap_{i=0}^{n} I_i \right) \subseteq \bigcap_{i=0}^{n} I_i.$$

3 Chain Conditions

Before we define Noetherian and Artinian rings we need to introduce the concept of chain conditions. In this chapter we will present the ascending chain condition, along with the maximal condition and the condition that the ring is finitely generated. We will go on to establish that these three conditions are equivalent. Beyond this we will also define the descending chain condition and the minimal condition.

This chapter will be largely based on chapter 6 in [AM69]. We start by defining ascending and descending chains of ideals.

Definition 15 (Ascending chain of ideals). Let Σ be the set of all ideals I_i of a ring R. An ascending chain of ideals is then given by a subset of Σ ordered by:

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$$

The chain is called a descending chain if it is instead ordered by $\supseteq$.

Definition 16 (Ascending chain condition (ACC)). If, for every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ in Σ, there exists an integer N such that $I_n = I_{n+1}$ for all $n \geq N$, we say that the chain stabilizes and that the ring R satisfies the *Ascending chain condition* (ACC).
If the sequence is instead ordered by $\supseteq$ this condition is called the descending chain condition (DCC).

In practice, to show that a ring satisfies the ascending chain condition, we will usually show that it is not possible to construct a strictly ascending chain of ideals; and if it does not satisfy the ACC we show that it is possible. And the same goes for the DCC. We now turn our attention to the maximal condition which we will define as the following.

Definition 17 (Maximal condition ($MaxC$)). If every non-empty subset of Σ has a maximal ideal, then we say that the ring R satisfies the *Maximal condition* ($MaxC$). If the sequence is instead ordered by $\supseteq$, the condition is called the minimal condition ($MinC$) and requires an ideal that is contained in all other ideals.

The ascending chain condition and the maximal condition establishes important concepts of finiteness in the ideal structure of a ring. There is an important relationship between these two conditions.

Proposition 3. A ring satisfies the ACC if and only if it satisfies $MaxC$.

Proof. First, we show that when ACC is satisfied, it follows that $MaxC$ is satisfied. This is equivalent to showing that when $MaxC$ is not satisfied, it follows that ACC

is not satisfied. Assume that there is a set of ideals $C \in \Sigma$ with no maximal ideal. Then, for any I_0 in C, there exists an ideal I_1 in C such that $I_0 \subset I_1$ (if I_1 does not exist, I_0 is maximal). Moreover, this means that we can find an ideal I_2 such that $I_0 \subset I_1 \subset I_2$. So for any n, you can construct a chain $I_0 \subset I_1 \subset ... \subset I_n$, that can be extended to include $I_{n+1} \supset I_n$. Hence, we have shown that when there exists no maximal ideal, it is possible to construct an infinite chain of ideals, and therefore the ACC is not satisfied. This means that when $MaxC$ is not satisfied, it follows that ACC is not satisfied, or equivalently, ACC implies $MaxC$.

Now, we want to show that $MaxC$ implies ACC. Suppose that the $MaxC$ is satisfied. Let S be a descending chain of ideals $I_n \supseteq I_{n+1} \supseteq ....$ Then there exist a maximal ideal I_m in S. Since I_m is not contained in any ideal except itself, we have

$$I_m = I_{m+1} = I_{n+2} = ...$$

and hence, the ACC is satisfied.

Now that we have proven the implications for both directions we can conclude that the ACC is satisfied if and only if the $MaxC$ is satisfied. □

Corollary 1. A ring satisfies the DCC if and only if it satisfies the $MinC$.

Proof. This proof is the same as the proof of proposition 3 but with the ideal inclusions reversed. □

Now we have established the equivalence between the maximal condition and the ascending chain condition. There is one more condition that is equivalent to these, but first we must show the following lemma.

Lemma 3. Let $I_0 \subseteq I_1 \subseteq I_2 \subseteq ...$ be a chain of ideals. Then $\bigcup_{j=0}^{\infty} I_j$ is an ideal.

Proof. We have that

$$0 \in I_0 \subseteq \bigcup_{j=0}^{\infty} I_j.$$

Now we need to show that

$$x, y \in \bigcup_{j=0}^{\infty} I_j \Longrightarrow x + y \in \bigcup_{j=0}^{\infty} I_j.$$

If x and y are in $\bigcup_{j=0}^{\infty} I_j$, then there exists n_1 and n_2 such that

$$x \in I_{n_1},\ y \in I_{n_2}.$$

If n is equal to or greater than $\max\{n_1, n_2\}$ we have that I_{n_1} and I_{n_2} are contained in I_n, and therefore x and y are both in I_n. Because I_n is an ideal we know that

$$x + y \in I_n \subseteq \bigcup_{j=0}^{\infty} I_j.$$

Finally, we need to show that for all elements a in the ambient ring it holds that

$$ax \in \bigcup_{j=0}^{\infty} I_j,\ \forall\ x \in \bigcup_{j=0}^{\infty} I_j.$$

This follows from the fact that x is an element in some ideal I_n contained in $\bigcup_{j=0}^{\infty} I_j$ which is, by definition of ideals, closed under multiplication with elements of the ambient ring. □

Now we are ready to introduce the third condition.

Proposition 4. The ascending chain condition is satisfied if and only if every ideal the ring is finitely generated.

Proof. We start here by showing that all ideals of a ring that satisfy the ACC are finitely generated. Assume that the set of ideals of a ring R satisfies the ACC, and by extension, the $MaxC$. We now want to show that all ideals are finitely generated. First, note that if $I = R$, then $I = (1)$ is principal and by extension finitely generated. So we can assume that $I \subset R$. Now let

$$S := \{J \subseteq I \mid \textit{finitely generated ideals of } I\}.$$

As the ideals in S satisfies the $MaxC$ we know that there exists a maximal ideal M in S which is generated by some set $\{x_1, ..., x_n\}$ in R. Now, if $M = I$, then I is finitely generated. If however, $M \subset I$ we can find some element x_{n+1} that is in I but not in M such that

$$M \subset (x_1, ..., x_n, x_{n+1}) \subseteq I.$$

This contradicts our assumption that M is the maximal element in S. Hence, we can conclude that if the ACC is satisfied, all ideals are finitely generated.

To show the other direction, that a ring whose ideals are all finitely generated satisfies the ACC, we start by assuming that the ideals I_i of a ring R are finitely generated. If $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ is an ascending chain of ideals of R, then we know that $I = \bigcup_{i=1}^{\infty} I_i$ is a finitely generated ideal. Now, let $\{x_1, ..., x_n\}$ be a generating set for I. Since all I_i are contained in I, it follows that all I_i can be generated by some subset of $\{x_1, ..., x_n\}$. Hence, all I_i in the chain are finitely generated. □

Example 1. The ring of integers $\mathbb{Z}$ satisfies the ACC but not the DCC.
First we note that the ring of integers is a principal ideal domain. Let I_j be the ideal of $\mathbb{Z}$ such that

$$I_j = (m_j)$$

for some m_j in $\mathbb{Z}$. Note that, when $m_j \neq 0$, we have that $I_j \subset I_{j+1}$ if and only if $m_{j+1} \mid m_j$, which implies that if m_j is a prime number, I_j is a maximal ideal. If m_j is not prime, there exist some number m_{j+1} such that m_{j+1} divides m_j. Now, consider some m_1 that is the product of some set of prime numbers. Then we can create a chain of ideals by successively removing those prime numbers:

$$(m_1) \subset (m_2) \subset ... \subset (p)$$

where p is the last remaining prime number, and hence (p) is maximal. Therefore the ring of integers satisfies the ACC.
To show that $\mathbb{Z}$ does not satisfy the DCC, we just reverse the order of the chain and show that it becomes infinite:

$$\mathbb{Z} \supset (p_1) \supset (p_1 p_2) \supset (p_1 p_2 p_3) \supset ... \supset \left(\prod_{i=1}^{k} p_i \right) \supset ...$$

This chain can be continued indefinitely by multiplying the generator by another integer. Hence, $\mathbb{Z}$ does not satisfy the DCC.

We have now discussed three equivalent conditions on rings: the ascending chain condition, the maximal condition, and the condition that it is finitely generated. This brings us to the definitions of Noetherian and Artinian rings.

Definition 18. A ring is Noetherian if it satisfies any of the following equivalent conditions:

- The ascending chain condition.
- The maximal condition.
- All ideals are finitely generated.

Because these conditions are equivalent, one of them bieng satisfied is equivalent to all of them being satisfied.

Definition 19. A ring is Artinian if it satisfies the descending chain condition and the minimal condition.

The two upcoming chapters will cover some of the properties of Noetherian and Artinian rings.

4 Noetherian Rings

In this Section we present an overview of some properties of Noetherian rings. We will first deal ring homomorphisms from noetherian rings. Then we move on to show the Hilbert's basis theorem for rings of polynomials and the equivalent results for rings of formal power series. Finally we will explore the primary decomposition of Noetherian rings.

4.1 Ring homomorphism

Theorem 2. If R is a Noetherian ring and f is a surjective ring homomorphism to a ring A. Then A is Noetherian.

Proof. Because f is surjective there is, for every $a \in A$, an element $r \in R$ such that $f(r) = r$. Let I be an ideal of A. We want to show that I is finitely generated. The preimage of I in R is

$$f^{-1}(I) = \{r \in R \mid f(r) \in I\}.$$

The fact that f is a surjective ring homomorphism implies that $f^{-1}(I)$ is an ideal of R and that $f(f^{-1}(I)) = I$. Because $f^{-1}(I)$ is an ideal of R it is finitely generated. Let $\{r_1, ..., r_n\}$ be the set of generators of $f^{-1}(I)$. This means that since $f(f^{-1}(I)) = I$, we have that

$$I = (f(r_1), ..., f(r_n)).$$

Hence, A is Noetherian. □

4.2 Hilbert's Basis Theorem

We now move on to discussing an important result connecting the Noetherian property and rings of polynomials. First, we have the following definition.

Definition 20. Let R be a commutative ring with a multiplicative identity. and x a variable. A polynomial with coefficients in R is a formal sum

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0.$$

We say that two polynomials

$$p(x) = \sum_i a_i x^i$$

and

$$q(x) = \sum_i b_i x^i$$

are equal if $a_i = b_i$ for all i. We denote by $R[x]$ the set of all polynomials with coefficients in R. We now define addition and multiplication the same as with any polynomial addition and polynomial multiplication.

Next, we must show that the ring of polynomials is actually a ring.

Proposition 5. The set $R[x]$ with associated addition and multiplication as defined above has a ring structure.

Proof. First, we note that $R[X]$ is associative, commutative, and closed under addition. That it is closed under addition follows from the fact that when adding two polynomials $p(x)$ and $q(x)$ as defined above, the resulting coefficients will be $a_i + b_i$ for all i, and since a_i and b_i are elements of the ring R, $a_i + b_i$ must also be in R. We can also see that $R[X]$ is closed under multiplication. This can be seen by noting that when multiplying two polynomials $p(x)$ and $q(x)$, all coefficients in the resulting polynomial will be a product of coefficients in $p(x)$ and $q(x)$, and because those coefficients are elements in the ring R, the new coefficients must also be in R. Hence, $R[x]$ is a ring. □

The leading coefficients will be important when proving Hilberts basis theorem. We will will now provide a definition and prove that they form an ideal of R.

Definition 21. Consider the polynomial

$$p(x) = \sum_i a_i x^i.$$

The *leading coefficient* of $p(x)$ is $a_m \neq 0$ such that $a_j = 0$ for all $j > m$. If $p(x) = 0$, then the leading coefficient is 0.

Lemma 4. Let R be a ring, $R[x]$ the ring of polynomials with coefficients in R, and B an ideal of $R[x]$. Then the set

$$I = \{a \in R \mid a \text{ is a leading coefficient of a polynomial in } B\}$$

is an ideal of R.

Proof. Because 0 is in B it follows that 0 is in I since the leading coefficient of 0 is 0. Furthermore, consider two elements a and b in I. Then there exist elements in B such that

$$f(x) = ax^j + \textit{lower terms} \in B$$

and

$$g(x) = ax^k + \textit{lower terms} \in B.$$

If $j = k$ we have

$$f(x) + g(x) = (a+b)x^j + \textit{lower terms} \in B.$$

If $j \neq k$ we can assume that $j > k$. Then we have that

$$x^{j-k}g(x) = bx^j + \textit{lower terms} \in B$$

and then it follows that

$$f(x) + x^{j-k}g(x) = (a+b)x^j + \textit{lower terms} \in B.$$

It then follows that $a + b$ is in I. Finally, consider an element a in I and let λ be an element of R. Then there is an element of B

$$f(x) = ax^j + lower\ terms \in B$$

such that

$$\lambda f(x) = (\lambda a)x^j + lower\ terms \in B.$$

Which implies that λa is in I. Hence we can conclude that I is an ideal of R. □

Now we can move on to Hilbert's basis theorem. The original theorem as stated by Hilbert is a bit different from the one presented here. The original theorem is that if R is a field or the ring of integers, then $R[x]$ is finitely generated. Here, we present a more general version of it.

Theorem 3. *Hilbert's Basis Theorem.* If R is a Noetherian ring, then the polynomial ring $R[x]$ is also Noetherian.

Proof. This proof is inspired by the proof for *Hilbert's (generalized) basis theorem* in [Jac09]. Let R be a Noetherian ring and B an ideal of the polynomial ring $R[x]$. We want to show that B is finitely generated. Now, let I be the ideal of leading coefficients

$$I = \{a \in R \mid a \text{ is a leading coefficient of a polynomial in } B\}.$$

Because R is Noetherian we have that

$$I = (b_1, ..., b_i, ..., b_m)$$

where $\{b_1, ..., b_i, ..., b_m\}$ is the finite set of generators for I. Furthermore, for every $1 \leq i \leq m$ and $a_i \in R$ there is a polynomial f_i in B with

$$f_i = a_i x^{j_i} + lower\ terms.$$

Now, let

$$r = max\{J_i\}.$$

We now want to show that

$$B = (f_1, ..., f_i, ..., f_m).$$

Consider some $f = ax^k + lower\ terms$ in B. By induction on $k = deg(f)$, we will show that

$$f = \sum_i g_i f_i$$

for some $g_i \in R[x]$. If $f = 0$ or $k = 0$, this is obvious. Suppose that $k \geq r$. Note that any leading coefficient can be written as

$$a = \sum_i \lambda_i b_i$$

where $\lambda_i \in R$. Hence we have that

$$f = ax^k + lower\ terms = \left(\sum_i \lambda_i b_i\right) x^k + lower\ terms.$$

Now, if we multiply by x^{r-k} we get

$$x^{r-k} f = \left(\sum_i \lambda_i b_i\right) x^r + lower\ terms =$$

$$= \sum_i \lambda_i b_i x^{r-J_i} f_i + lower\ terms$$

which is in B because $x^{r-k}f$ has the same leading coefficient as f. Let $k_1 = deg(x^{r-k}f)$. It is clear that $k_1 < k$. If $k_1 < r$ we are done, otherwise repeat until it $k_j < r$ for some j. If $k < r$, then we can construct a polynomial h with the same leading coefficient as f where $deg(h) < k$ as follows:

$$h = f - \sum_i \lambda_i f_i$$

for some $\lambda_i \in R$. We can then repeat this process until the we get the 0 polynomial. We have now shown by induction that every $f \in B$ can be written as a finite sum $f = \sum_i g_i f_i$ for some $g_i \in R$ and $f_i \in B$. Hence B is finitely generated and $R[x]$ is Noetherian. □

4.3 Formal power series

Now that we have discussed polynomial rings we can extend the discussion to sums of powers, where the highest power with a non-zero coefficient is not finite. We start by defining the ring of formal power series, and its order.

Definition 22. Let R be a ring and x a variable. A *formal power series* is the formal sum

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + ...$$

where the coefficients a_i are elements of R. We say that two formal power series

$$p(x) = \sum_i a_i x^i$$

and

$$q(x) = \sum_i b_i x^i$$

are equal if $a_i = b_i$ for all i. We denote by $R[[x]]$ the set of all formal power series with coefficients in R. Addition and multiplication is defined as for any power series.

Definition 23. if f is a formal power series. Then the order of f is equal to the lowest power of x where the coefficient is non-zero. It is denoted $o(f)$.

Lemma 5. Let f and g be formal power series. Then we have the following:

$$o(fg) \geq o(f) + o(g)$$

and

$$o(f+g) \geq min\left(o(f), o(g)\right).$$

Proof. To prove that $o(fg) \geq o(f) + o(g)$, we note that if $a_{o(f)} \neq b_{o(g)}$, then the lowest order term in fg is

$$a_{o(f)} x^{o(f)} \cdot b_{o(g)} x^{o(g)} = a_{o(f)} b_{o(g)} x^{o(f)+o(g)},$$

which has order $o(f) + o(g)$. If $a_{o(f)} = b_{o(g)}$, then $o(fg) > o(f) + o(g)$.
That $o(f+g) \geq min\left(o(f), o(g)\right)$ is straightforward since the coefficients in the sum is equal to $a_i + b_i$ for every i. □

Lemma 6. The set $R[[x]]$ as defined above has a ring structure.

Proof. This proof is the same as the proof that $R[x]$ has a ring structure. □

Now we are ready to turn our attention to the equivalent of Hilbert's basis theorem for rings of formal power series. The following theorem establishes a relationship between Noetherian rings and formal power series.

Theorem 4. If R is a Noetherian ring, then $R[[x]]$ is also noetherian.

Proof. [Jac09, Theorem 7.11]. We use a similar strategy here as in the proof of Hilbert's basis theorem. Let R be a Noetherian ring, $R[[x]]$ the ring of polynomials with coefficients in R, and $B \subseteq R[[x]]$ an ideal. We now want to show that B is finitely generated. Here we will define the *leading coefficient* as $a_m \neq 0$ such that $a_j = 0$ for all $j < m$. As a corollary to Lemma 4 the set of leading coefficients in R is then an ideal of R. Call this ideal I. Since R is finitely generated we have that

$$I = (b_1, ..., b_i, ..., b_m)$$

for some finite set of generators $\{b_1, ..., b_i, ..., b_m\}$. For every i there is $f_i \in B$ with

$$f_i = a_i x^{J_i} + higher\ terms.$$

Let

$$r = max J_i.$$

Now, we need to show that

$$B = (f_1, ..., f_i, ..., f_m).$$

Suppose $f = ax^k + higher\ terms$, where $a \in R$. We will show that $f = \sum_i g_i f_i$ for some $g_i \in R[[x]]$ and $f_i \in B$. For some $\lambda_i \in R$ and $b_i \in I$ we have that $a = \sum_i \lambda_i b_i$. We therefore have that

$$f = \left(\sum_i \lambda_i b_i\right) x^k + higher\ terms \in B.$$

Multiply by x^{r-k} and we get

$$x^{r-k} f = \left(\sum_i \lambda_i b_i\right) x^r + higher\ terms \in B$$

which can also be written as

$$x^{r-k}f = \sum_i \lambda_i b_i x^{r-J_i} f_i + \textit{higher terms}.$$

Now we note that

$$o(f - x^{r-k}f) > k.$$

Now, let $k_1 = r - J_i$ and lets repeat this process such that we get a sequence $k_1 < k_2 < k_3 < ....$ Then we can construct a sum

$$h = \sum_i \left(\sum_{j=1}^{\infty} \lambda_{ij} b_{ij} x^{k_j} \right) f_i + \textit{higher terms}.$$

Whose order is arbitrarily large. Note that $\sum_{j=1}^{\infty} \lambda_{ij} b_{ij} x^{k_j}$ is a formal power series in $R[[x]]$. Hence we have showed that an element $f \in B$ can be written as a sum $f = \sum g_i f_i$ for some $g_i \in R[[x]]$ and $f_i \in B$. Hence B is finitely generated by the set $\{f_1, ..., f_i, ..., f_m\}$, and therefore $R[[x]]$ is Noetherian. □

4.4 Primary decomposition

To finish of this section, we will now discuss some results relating to the primary decomposition of Noetherian rings and prove that every ideal of a Noetherian ring has a primary decomposition.

Lemma 7. Let R be a Noetherian ring. Then every ideal of R is the intersection of a finite number of irreducible ideals.

Proof. [AM69, Lemma 7.11]. By contradiction, assume that there exists a non-empty set of ideals I_j of R that are not contained in the intersection of a finite number of irreducible ideals, Call this set A. Because R is Noetherian, this set of ideals must have a maximal element I_m. The maximal element I_m is reducible, which means that

$$I_m \subset I_i \cap I_j$$

for some $i \neq j \neq m$ such that $I_m \subset I_j$ and $I_m \subset I_j$. Because I_m is maximal, this means that I_i and I_j cannot be in A, and are therefore contained in the intersection of a finite number of irreducible ideals, and by extension so is I_m. This contradicts our assumption that I_m is not the intersection of a finite number of irreducible ideals. ☐

The annihilator of a subset of a ring will an important concept here. We will now define it and explore some of properties.

Definition 24. Let R be a ring. The *annihilator* of a set $X \in R$ is the set of elements $a \in R$ such that $a \cdot x = 0$ for all $x \in X$. It is denoted $Ann(X)$

Lemma 8. The annihilator of a set $X \in R$ is an ideal of R.

Proof. Let X be a subset of a ring R. We now want to show that any linear combination of elements in $Ann(X)$ and R is an element in $Ann(X)$. Let a_i be elements in $Ann(X)$ and r_j be elements in R, then the linear combinations are given by

$$\sum_i \sum_j a_i r_j.$$

Multiply this by an element $x \in X$ and we get

$$x \sum_i \sum_j a_i r_j = \sum_i \sum_j x \cdot a_i r_j = \sum_i \sum_j (x \cdot a_i) r_j = \sum_j 0 \cdot r_j = 0.$$

Hence, we can conclude that the linear combination is in $Ann(X)$. ☐

Corollary 2. If R is a Noetherian ring and X is a subset, then $Ann(X)$ is finitely generated.

Proof. From lemma 8 we know that $Ann(X)$ is an ideal of R, and because R is Noetherian, every ideal including $Ann(X)$ must be finitely generated. ☐

Now that we have established some properties of the annihilator we can move on to the following lemma establishing a connection between irreducible ideals and primary ideals in Noetherian rings.

Lemma 9. If R is a Noetherian ring, then all irreducible ideals in R are primary ideals.

Proof. [AM69, Lemma 7.12]. Let R be a Noetherian ring and I an ideal of R. Now consider the quotient ring $A = R/I$. I is irreducible if and only if the trivial ideal $(0) \in A$ is irreducible. Assume that $(0) \in A$ is irreducible, then we want to show that $(0) \in A$ is primary. Consider the elements $x \in A$ and $y \in A$ such that $xy = 0$ and $y \neq 0$. Then we have the chain of ideals

$$Ann(x) \subseteq Ann(x^2) \subseteq ...$$

and, because of the ACC, fore some n this simplifies to

$$Ann(x^n) = Ann(x^{n+1}) = ...$$

Now, consider an element $a \in (x^n) \cap (y)$. Then, for some elements b and c we have that

$$a = bx^n = cy$$

if, and only if

$$bx^{n+1} = bx^n x = cyx = 0.$$

Which implies that

$$b \in Ann(x^{n+1}) = Ann(x^n)$$

and therefore $a = bx^n = 0$. We have thus showed that $(x^n) \cap (y) = \{0\}$. Since we have assumed that $\{0\}$ is irreducible we can conclude that $(x^n) = \{0\}$ and $(y) = \{0\}$ and by extension $x = 0$ and $y = 0$. Hence, I is primary. □

Finally, this leads us to our concluding result of this section.

Theorem 5. Let R be a Noetherian ring. Then every ideal of R has a primary decomposition.

Proof. This follows directly from lemma 7 and lemma 9. □

5 Artinian Rings

In this section we will explore some of the properties and ideal structure of Artinian rings. We will explore the primary decomposition of Artinian rings. From there we will move on to show that Artinian rings are a subset of the Noetherian rings. We begin with two definitions that will be important later.

Definition 25. The *Jacobson radical* of a commutative ring R is the intersection of the maximal ideals. It will be denoted as $J(R)$.

Definition 26. The *nilradical* of a commutative ring R is the intersection of the prime ideals. It will be denoted as $N(R)$.

Next, we need to establish a relationhip between an ideal of an Artinian ring and the associated quotient ring.

Lemma 10. If I is an ideal of an Artinian commutative ring R, then the quotient ring R/I is Artinian.

Proof. Consider the ideals of R/I. They have the form of a quotient ring J/I for some $I \subseteq J \subseteq R$. Because R is Artinian the set of elements $J \subseteq R$ such that $I \subseteq J$ must satisfy the DCC, and therefore, so must R/I. □

We now turn a more general result regarding quotient rings.

Lemma 11. If ρ is a prime ideal of a commutative ring R, then the quotient ring R/ρ is an integral domain.

Proof. We need to show that R/ρ has no zero divisors. Let consider two elements $a', b' \in R/\rho$. We want to show that $a'b' = 0$ if and only if $a' = 0$ or $b' = 0$. Suppose $a'b' = 0$. Then because the map from R to R/ρ is a homomorphism we have

$$a'b' = (ab)' = 0.$$

This means that $ab \in \rho$. Because ρ is a prime ideal this means that either $a \in \rho$ or $b \in \rho$, and whichever one is in ρ will be 0 in R/ρ. □

Corollary 3. Let ρ be a prime ideal of an Artinian commutative ring R. Then the quotient ring R/ρ is an Artinian integral domain.

Proof. Follows directly from Lemma 10 and 11. □

Now we are ready to establish the following theorem regarding prime ideals in Artinian rings.

Theorem 6. If a ring R is Artinian, then all prime ideals in R are maximal.

Proof. [AM69, Proposition 8.1]. Consider a prime ideal $\rho \subset R$. From corollary 3 we know that the quotient ring R/ρ is an Artinian integral domain. Now consider a non-zero element $x \in R/\rho$. We have that $(x^n) \supseteq (x^{n+1}) \supseteq ...$ and because R/ρ is Artinian we have that for some n

$$(x^n) = (x^{n+1}),$$

which implies that there exists an element y such that

$$x^n = x^{n+1}y.$$

As $x \neq 0$ and R/ρ is an integral domain this can be reduced to

$$xy = 1.$$

Therefore, y is the inverse of x and R/ρ is a field, and by extension ρ must be maximal. □

Corollary 4. If R is a commutative Artinian ring, then $J(R) = N(R)$.

Proof. Follows directly from Theorem 6 and the definitions 11 and 12. □

Next, we need to establish some important lemmas that will be important later.

Lemma 12. Consider the set of prime ideals $\{\rho_1, ..., \rho_n\} \in R$. Let m be an ideal such that m is contained in $\bigcup_{i=1}^{n} \rho_i$. Then, for some i, $m \subseteq \rho_i$.

Proof. [AM69, Proposition 1.11i]. We will prove the equivalent result that if m is not contained in any ρ_i it will not be contained in their union. If $m \not\subseteq \rho_i$ for $1 \leq i \leq n$ for some n, then

$$m \not\subseteq \bigcup_{i=1}^{n} \rho_i.$$

Now, we can use induction on n. For $n = 1$ this reduces to $m \not\subseteq \rho_i$ implies $m \not\subseteq \rho_i$, which is obvious. For $n > 1$, assuming it holds for $n-1$, we can construct a sequence of elements $x_i \in m$ with the property that when if $j \neq i$, then $x_i \notin \rho_j$. If there is an i for which there is no such x_i for which $x_i \in \rho_i$, then we are done. Otherwise, we have a sequence of elements x_i that are all in only one ideal ρ_i each. That means we can construct a new element as follows

$$y = \sum_{i=1}^{n} x_1 x_2 ... x_{i-2} x_{i-1} x_{i+1} ... x_{n-1} x_n.$$

This is a sum where each x_i is a factor of every term except one. By extension, for every i, every term in the sum is in ρ_i except one (the one where x_i is not a factor). The sum therefore consists of $n-1$ terms that, for each i, are in ρ_i and a single term is not in ρ_i. This implies that y cannot be in ρ_i for any i and by extension not in their union either. So therefore we can conclude that $m \nsubseteq \rho_i$ for all i implies $m \nsubseteq \bigcup_{i=1}^{n} \rho_i$, or equivalently that $m \subseteq \rho_i$ for some i implies

$$m \subseteq \bigcup_{i=1}^{n} \rho_i,$$

and hence we are done. □

Lemma 13. Let ρ be a prime ideal of a ring R and $\{a_1, ..., a_n\} \subseteq P(R)$ where $P(R)$ is a power set of R and a_i are ideals of R such that

$$\bigcap_{i=1}^{n} a_i \subseteq \rho.$$

Then, for some i, $a_i \subseteq \rho$. Furthermore, if $\bigcap_{i=1}^{n} a_i = \rho$, then for some i we have that $a_i = \rho$.

Proof. [AM69, Proposition 1.11ii]. We will prove the equivalent statement that if for all i, $a_i \nsubseteq \rho$, then $\bigcap_{i=1}^{n} a_i \nsubseteq \rho$. Assume that for all i we had $a_i \nsubseteq \rho$. Then, for every i, there must be an element $x_i \in a_i$ such that $x_i \notin \rho$. This means that

$$\prod_{i=1}^{n} x_i \in \bigcap_{i=1}^{n} a_i.$$

However, because ρ is a prime ideal, we have that

$$\prod_{i=1}^{n} x_i \notin \rho,$$

and by extension

$$\bigcap_{i=1}^{n} a_i \nsubseteq \rho.$$

Hence we can conclude that if $a_i \subseteq \rho$ for at least one i, then $\bigcap_{i=1}^{n} a_i = \rho$. Lastly, we note that if $\rho = \bigcap_{i=1}^{n} a_i$, then $\rho \subseteq a_i$ for all i. Adding this to our previous results we get that if $\rho = \bigcap_{i=1}^{n} a_i$, then $a_i = \rho$ for some i. □

Now we are ready to prove the following theorem about Artinian rings.

Theorem 7. If a ring R is Artinian, then R has a finite number of maximal ideals.

Proof. [AM69, Proposition 8.3]. Let m_i denote maximal ideals in an Artinian ring R, and M the set containing every intersection $m_1 \cap m_2 \cap \dots$. Since all intersections of ideals are also ideals (Proposition 2), we know that every element in M is an ideal of R. Now to show that M has a minimal element, suppose there is no minimal element in M, then we can construct an infinite chain

$$m_1 \supset m_1 \cap m_2 \supset \dots \supset \bigcap_{i=1}^{n} m_i \supset \dots$$

This would contradict the DCC, and hence we know that this sequence stabilizes at some minimal element, say $\bigcap_{i=1}^{n} m_i$. This implies that if m is any maximal ideal in R, we get

$$m \cap \left(\bigcap_{i=1}^{n} m_i \right) = \bigcap_{i=1}^{n} m_i$$

and by extension

$$\bigcap_{i=1}^{n} m_i \subseteq m.$$

From this follows, as a result of lemma 13, that there exists an i such that $m_i \subseteq m$, and because m_i is maximal it follows that $m_i = m$. □

Finally, we will show a result regarding the nilradical of Artinian rings. But first we must provide a definition and a lemma.

Definition 27. Let I and J be two ideals. The product IJ is defined as the ideal generated by the set

$$\{ij \mid i \in I, j \in J\}.$$

As a corollary, we have that I^n is the ideal generated by the set

$$\{\prod_{i=1}^{n} x_i \mid x_i \in I\}$$

Lemma 14. If R is a ring and I an ideal of R, then $I^{n+1} \subseteq I^n$ for all $n \geq 1$.

Proof. First, we show that this holds for $n = 1$, i.e. $I^2 \subseteq I$. Every element in I^2 is a sum of products of two elements in I. Because I is an ideal it is closed under multiplication and addition, which means that any element in I^2 must also be an element in I, and hence $I^2 \subseteq I$. Now we proceed by induction. Suppose that $I^n \subseteq I^{n-1}$ holds for some $n \geq 2$. We now want to show that $I^{n+1} \subseteq I^n$. Let a be an element in I^{n+1}. It can then be written as a sum of products of elements in I^n. Again, because I^n is an ideal it is closed under multiplication and addition. This means that $a \in I^{n+1}$ implies that $a \in I^n$, and therefore $I^{n+1} \subseteq I^n$. By induction, we have therefore shown that $I^{n+1} \subseteq I^n$ for all $n \geq 1$. □

Proposition 6. If a ring R is Artinian, the nilradical $N(R)$ is nilpotent.

Proof. [AM69, Proposition 8.4]. Let R be an Artinian ring and $\mathfrak{R} = N(R)$. Because R is Artinian and $\mathfrak{R}$ is an ideal of R, the *DCC* and Lemma 14 implies that there exist som $k \in \mathbb{Z}^+$ such that

$$\mathfrak{R}^k = \mathfrak{R}^{k+1} = ... = a.$$

Note also that this means that $a^n = a$ for all $n \in \mathbb{Z}^+$. Assume that a is non-zero and define a set

$$S = \{b \in R \mid b \text{ is ideal such that } ab \neq 0\}.$$

We know that a is in S, so it is not empty. Because R is Artinian we know that S has a minimal element, which we will call c. Let x be an element in c such that $xa \neq 0$ (which also means $x \neq 0$). Then it is clear that $(x) \subseteq c$, and because c is minimal, we can therefore conclude that $(x) = c$. We also have that $xa \subseteq (x)$, and that $(xa)a = xa^2 = xa \neq 0$. Because c is minimal this implies that $xa = (x) = c$. Since $x \in (x)$ there must exist some element $y \in a$ such that $x = xy$ and by extension

$$x = xy = xy^2 = \dots$$

However, since y is in a power of the nilradical, y must be nilpotent which means that $x = xy^n = 0$ for all $n \in \mathbb{Z}^+$. Since we picked x to be non-zero this is a contradiction. Hence $a = \mathfrak{R}^k = 0$. □

References

[AM69] Michael F. Atiyah and Ian G. Macdonald. *Introduction To Commutative Algebra*. Addison-Wesley Publishing Company, 1969.

[DF03] David S. Dummit and Richard M. Foote. *Abstract Algebra*. John Wiley & Sons Inc., third edition, 2003.

[Jac09] Nathan Jacobson. *Basic Algebra II*. Dover Publications Inc., second edition, 2009.