SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

The Gelfand duality

av

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2025 - No K6

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Självständigt arbete i matematik 15 högskolepoäng, grundnivå

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2025

Abstract

In this text, I lay out the Gelfand representation. I start by building up the necessary background terminology, along with some useful theorems, assuming relatively little knowledge of the discussed topics. I then construct the Gelfand representation and show how it is and anti-equivalence of the categories of locally compact Hausdorff spaces Haus and the category of C*-algebras C^*-Alg.

Sammanfattning

I den här texten beskriver jag gelfandrepresentationen. Jag börjar med att bygga upp nödvändig bakgrundsterminologi, samt en del användbara satser och antar relativt låg förkunskap inom de områden jag tar upp. Jag konstruerar sedan Gelfandrepresentationen och visar hur den är en antiekvivalens mellan katogrierna av lokalt kompakta Hausdorffrum Haus och kategorin av C*-algebror C^*-Alg

Acknowledgments

Thank you to my mother, for providing support while I worked. Thank you to Håkan and Joel, who do not know it, but they helped me along the way. Special thanks to my friend Cielo without whom I might not have gotten as far as I did.

A very big thank you to my advisor Peter who put up with me and provided help and support and encouragement in all the ways I could have possibly asked for.

Contents

1 Introduction

The Gelfand duality is a very well celebrated theorem, or at least so claims the text about it that I have read. Judging by how easy it is to fin such texts though, I believe it is fair to say they probably know what they are talking about. When I first laid eyes on the theorem, it looked very intimidating. I hope that I have managed to provide detailed enough background knowledge to make that leap easier for the uninformed reader, even though the background subjects are highly interesting in and of themselves and I simply do not have enough space here to come even close to do them justice. As always, any reader wishing to know more about any subject may look towards the bibliography given at the end. Most of the resources I used while writing this are available online although not everything may be free.

2 Background knowledge

2.1 Category theory

We will not delve deeper into this subject than is necessary to establish the concepts needed in this thesis. For a more complete introduction, the reader may find [Rie16] to be useful. This section will mostly be an excerpt from chapter one of her book.

Definition 2.1.1 (Category)**.** A *Category* $\mathcal{C}$, consists of a collection of *objects* and a collection of *morphisms* such that the following axioms are upheld

(1) A collection of *objects* $ob(\mathcal{C})$. This may be a set or a proper class.

(2) A collection of *morphisms*[1] $mor(\mathcal{C})$. This may be a set or a proper class. Each *morphism* has a *domain* and a *codomain*. A morphism f with domain X and codomain Y is denoted as

$$f : X \to Y$$

The collection of all morphisms between two objects X and Y in a category $\mathcal{C}$ is denoted[2] $Hom_{\mathcal{C}}(X, Y)$

(3) For each pair of morphisms f, g, if $cod(f) = dom(g)$, that is the codomain of f is the same as the domain of g, then f and g are *composable* and there exists a morphism

$$g \circ f : dom(f) \to cod(g)^3$$

[1]Morphisms are also sometimes called *arrows*. Similarly, the *domain* and *codomain* may be refered to as the *source* and the *target* of the morphism. Some authors will change between these many times in the text. For the sake of consistency, I will stick to the nomenclature as defined above.

[2]$Hom_{\mathcal{C}}(X, Y)$ is pronounced "the hom-set between X and Y". Despite the name, the hom-set need not be a set in general, but may instead be a proper class.

[3]Note that composition works in the opposite order to which it is written. Thus, $g \circ f$ has the same domain as f and the same codomain as g. $g \circ f$ may be pronounced as "g following f"and the author finds this convention useful in remembering the order correctly.

(4) For each object X in the category, there is a designated *identity morphism* $1_X : X \to X$ with the property that for *every* morphism f, g with X as domain and codomain respectively

$$1_X \circ g = g$$

$$f \circ 1_X = f$$

(5) For each pairwise composable triplet of morphisms f, g, h, if f, g are composable and g, h are composable , then

$$f \circ (g \circ h) = (f \circ g) \circ h$$

and this justifies the notation

$$f \circ g \circ h$$

The notions of *objects* and *morphisms* are to be considered *atomic*, and have no intrinsic characteristics other than the ones defined above as far as category theory is concerned. This is a fundamental part of the philosophy of category theory; any inherent structure to the objects and morphisms other than the way they connect with each other is *discarded*.

In general, we may say that it is not the *internal structure* of the objects that is of interest for category theory, hence why objects are to be considered atomic. Rather, it is how the objects and morphisms are *connected* that is of interest. The operation of morphism composition is thus of fundamental import in how the category is built.

Because of this categorical focus on connections rather than intrinsic properties, there arises a *dual* to most concepts in category theory. At its core, the idea is to take whichever concept is currently being studied and then *reverse* the direction of the morphisms. That is, for each morphism, switch the domain and the codomain, but otherwise leave everything untouched. We demonstrate this with the following *dual*[4] to the concept of a category.

Definition 2.1.2 (Opposite Category)**.** Let $\mathcal{C}$ be a category. The *opposite category* $\mathcal{C}^{op}$ of $\mathcal{C}$ is a category where

(1) The collection of objects $ob(\mathcal{C}^{op})$ is the same as $ob(\mathcal{C})$

(2) For each morphism $f : X \to Y$ in $mor(\mathcal{C})$ there is a morphism $f^{op} : X \to Y$ in $mor(\mathcal{C}^{op})$

(3) Each object has 1_Y^{op} as its identity morphism

[4]Many constructions in category theory come in pairs of two. The *dual* of any categorical concept refers in general to a structure that is *identical*, except for the fact that the *domain* and *codomain* of every morphism is reversed.

(4) For each pair of composable morphisms f, g in $\mathcal{C}$, g^{op} and f^{op} are composable in $\mathcal{C}^{op}$ and

$$(g \circ f)^{op} = f^{op} \circ g^{op}$$

Before we move on to some more higher order concepts, there is one more important thing to mention, and that is the *isomorphism*

Any student well versed in math has surely heard of the concept of an isomorphism. As mathematicians, we often like to say that two or more things are the same *up to isomorphism* as a way to signal to the reader that *for the purposes of the current discussion*, they are to be considered the same even if they technically differ. The observant reader may notice parallels between this idea disregarding irrelevant structure, and the category theoretical philosophy of discarding any non-connectional structure. Indeed, category theory does formalize the concept of an isomorphism, as specified below.

Definition 2.1.3 (Isomorphism)**.** A morphism $f : X \to Y$ is an *isomorphism* if there exists another morphism $f^{-1} : Y \to X$ such that

$$f^{-1} \circ f = 1_x, \ f \circ f^{-1} = 1_y$$

Definition 2.1.4 (Functor)**.** Let C and D be categories. A *functor* is a function $F : C \to D$ that meets the following criteria

(1) Each object X in C is mapped to an object Fx in D.

(2) Each morphism $f : X \to Y$ in C is mapped to a morphism $Ff : Fx \to Fy$ in D

(3) For each object c in C, $F(1_c) = 1_{Fc}$

(4) For each pair of composable morphisms f,g in C, $FgFf = F(gf)$

If $F : C \to D^{op}$ we say that F is a *contravariant* functor[5]

Definition 2.1.5 (Natural transformation)**.** Let $F : C \to D$ and $G : C \to D$ be functors. A *natural transformation* $\alpha : F \Rightarrow G$ is a mapping between two functors such that

(1) For each object $c \in C$, a there is a morphism $a_c : Fc \to Gc$ in D such that for every morphism $f : c \to c'$ in C the composite morphisms $a_{c'} \circ Ff$ and $Gf \circ a_c$ are identical.

Additionally, if every α_c is an isomorphism, α is said to be a *natural isomorphism.* We write this $F \cong G$

[5]Note that since every category has an opposite category, every funktor is a contravariant funktor in some sense, since we usually only talk about either a given category and not both the category and its opposite, this is still a useful term.

Remark. The distinction of being *naturally* isomorphic rather than merely being isomorphic but *not* naturally isomorphic is important. The classic example of this is the *linear functionals* that we see more of later in the text. The space of all linear functionals over a vector space A is isomorphic to A but not naturally isomorphic. It is however naturally isomorphic to the underlying field ⅂

Definition 2.1.6 (Equivalence of categories)**.** A category C is *equivalent* to a category D if there exists functors $F : C \to D$, $G : D \to C$ such that

$$FG \cong 1_D \quad GF \cong 1_C$$

If C is instead equivalent to D^{op}, we say that C and D are *anti-equivalent.*

2.2 Algebra

Basic definitions

Definition 2.2.1 (Group)**.** A *group*[6] G consists of a set G along with a binary operation $+$ such that the following *group axioms* hold

(1) For each $f, g \in G$, $f + g \in G$

(2) For each $f, g, h \in G$, $(f + g) + h = f + (g + h)$ and is usually denoted $f + g + h$.

(3) There is a special element called 0 called the *identity* of the group. For each $g \in G$, $0 + g$ =g

(4) For each $g \in G$, there exists an *inverse* $-g$ such that $g+(-g) = -g+g = 0$

A group may have any number of additional properties, but one of particular import of us is that when the group operation is *commutative.*

Definition 2.2.2 (Abelian Group)**.** An *abelian group* is a group G that fulfills the following additional axiom in addition to the ones listed above

(5) For all f, g in G, $f + g = g + f$

Definition 2.2.3 (Ring)**.** A *ring* R is a set R together with two binary operations $+$ and $\times$ such that the following *ring axioms* hold true. It is important to note here that all rings considered will be assumed to be *commutative.*

For $+$

(1) For every $f, g \in G$, $f + g \in G$

(2) For any $f, g, h \in R$, $f + (g + h) = (f + g) + h$, and just as with groups, the parenthesis are usually omitted.

[6] This definition is actually abuse of notation. Formally, this group should be denoted $(G, +)$

(3) For every $f, g \in R$ $f + g = g + f$

(4) There is an element $0 \in R$ such that for all $g \in G$, $0 + g = g$

For $\times$

(5) For every $f, g \in R$, $f \times g \in R$

(6) For every $f, g, h \in G$, $f \times (g \times h) = (f \times g) \times h$

(7) There is an element called 1 that has the property, for all $g \in R$, $1 \times g = g \times 1 = g$

(8) For every $f, g \in R$, $f \times g = g \times f$

and then additionally

(9) For each $f, g, h \in R$, $f \times (g + h) = (f \times g) + (f \times h)$

Definition 2.2.4 (Module)**.** A *module* can be thought of as a generalization of an abelian group. Let R be a commutative ring. An abelian group $(M, +)$ together with a binary operation $\cdot : R \times M \to M$ is a *R-module*[7] if the following holds for every $r, s \in R$ and every $x, y \in M$:

(1) $r \cdot (x + y) = r \cdot x + r \cdot y$

(2) $(r + s) \cdot x = r \cdot x + s \cdot x$[8]

(3) $(r \times s) \cdot x = r \cdot (s \cdot x)$

And if R has a unit

(4) $1 \cdot x = x$

Definition 2.2.5 (Algebra)**.** Let M be an R-module, and let r, s, x, y be as above. We say that M is an *R-algebra* if the following holds in addition to the module properties resume,,

(5) $r \cdot (x \times y) = (r \cdot x) \times y = x \times (r \cdot y)$

For x,y elements in M and a,b elements of R resume,,

(6) [9] $(x + y) \cdot z = x \cdot z + y \cdot z$

Definition 2.2.6 (Ideal)**.** Given a ring R, an *ideal*[10] is a set $I \subseteq R$ such that $(I, +)$ is a subgroup of $(R, +)$ and for every $r \in R$ and $x \in I$, $r \times x \in I$.[11].

[7] In general, right R-modules and left R-modules are not the same, however since we assume that R is commutative, they will coincide

[8] Keep in mind that + on the left side of the equals sign is the operation in R, but + on the right hand side is the operation in M

[9] Since R is commutative, left distributivity follows from right distributivity

[10] usually there is a difference between a *left* and a *right* ideal, but since we are assuming all rings to be commutative, this difference is nil.

[11] We often say that I absorbs multiplication

A *proper ideal* is an ideal that is a proper subset of the ring R.

An ideal I is said to be *maximal* if there is no larger proper ideal containing it, that is for every ideal $J \supseteq I$, either $I = J$ or $R = J$.

If the quotient ring R/I *mby define quotient or nah* has a unit, then the I is said to be *modular*. We denote the set of all ideals that are both maximal, modular and proper by $\mathrm{Max}(R)$

Definition 2.2.7 (subspace)**.** If A is some algebraic construction and $B \subseteq A$ is an algebraic structure of the same kind as A and importantly *with the same algebraic operations as* A[12], then B is a subspace of A. Usually we talk more specifically about *subgroups*, *subalgebras*, etc. but the definitions are analogous.[13]

Definition 2.2.8 (Homomorphism)**.** A *homomorphism* is a mapping preserving algebraic structure. This can mean a few things, depending on exactly which of the above algebraic structures we are considering.

A *group homomorphism* if a function φ between to groups G and H which preserves the *group structure*, that is

$$\varphi(x +_G y) = \varphi(x) +_H \varphi(y) \quad \forall x, y \in G$$

where $+_G$ is the group operation in G and $+_H$ is the group operation in H.

For a *ring homomorphism*, we require that not only that $+$ be preserved, but also that $\cdot$ is.

$$\varphi(x \cdot_G y) = \varphi(x) \cdot_H \varphi(y)$$

Here G and H are of course assumed to be rings.

Similarly, for an *algebra homomorphisms* and later **-homorphisms*, we also require that the structure of the algebra be preserved.

Notation. In the preceding sections, I have tried to be clear with which multiplication belongs to which structure, which ring, which module and so on and so forth. However, this is often clear from context, and for the remainder of this text I will not specify which is which. The reason for this is that it unnecessarily clutters the page with symbols and degrades readability rather than helps it. From now on, both $x \times y$ and $x \cdot y$ will be written simply as xy unless the context is somehow such that it is not obvious which is which.

Definition 2.2.9 (Kernel)**.** Let φ be a homomorphism defined on some algebra A. Then the *kernel* ker of φ is the set of of all points that are mapped to 0, that is $\ker \varphi = \varphi^{-1}(\{0\})$

The codomain $\varphi(A)$ is often denoted $A/\ker \varphi$ and is said to be the *quotient space*.[14]

[12] but restricted to B of course

[13] Of course, this can be generalized with categorical language, but that is not what we are doing here

[14] Kernels and quotients may be defined for most algebraic structures, but I have elected not to do so here. The definitions are analogous and do not really bring any new insight other than the one already gained.

Definition 2.2.10 (*-algebra)**.** Let A be a $\mathbb{C}$-algebra. Let $* : A \to A, x \mapsto x^*$ be function satisfying the following properties

For every $x, y \in A$

(1) $(x + y)^* = x^* + y^*$

(2) $(\lambda x)^* = \overline{\lambda} x^*$ for any $\lambda \in \mathbb{C}$

(3) $(xy)^* = y^* x^*$

(4) $(x^*)^* = x$

Then A is called a *-algebra and $*$ is called *involution.*

Definition 2.2.11 (Linear functional)**.** A *linear functional* is a function eval from an albegra A to the underlying set .

Additionally, if the linear functional is an algebra homomorphism, that is to say if

$$\varphi(xy) = \varphi(x)\varphi(y)$$

Then we say that φ is *multiplicative*.

2.3 Topology

Basic definitions

First we begin with some basic definitions, followed by some properties of topological spaces that will be useful later. Most of these definitions are from [Lee11]

Definition 2.3.1 (Topology)**.** A *topology*[15]consists of a set X together with a collection $\mathcal{T}$ of subsets of X called *open sets.* They must fulfill the following axioms;

For $A_i \in \mathcal{T}$

(1) For *any*[16]indexing set I, $\bigcup_{i \in I} A_i \in \mathcal{T}$

(2) $X \in \mathcal{T}$

(3) If I is a finite set, then $\bigcap_{i \in I} A_i \in \mathcal{T}$

Additionally, the complement of an open set A in X, $X \setminus A$, is called a *closed set.*

When one wishes to define a topological space X, straight up defining what the open sets are is the most straightforward way of doing it, but there are many other equivalent ways. Instead of defining the open sets one may, for example, instead choose to define the *neighborhoods* of points. I will not show that this is an equivalent way of defining a topology in this text, but the concept itself is still very useful.

[15]Similarly as with groups, denoting a topological space merely as X is abuse of notation. More proper would be to write (X,$\mathcal{T}$).

[16]It is important to note that $\emptyset$ is *always* a member of the topology because it is the union over $\{A_i\}_{i \in \emptyset}$.

Definition 2.3.2 (Neighborhood)**.** Let X be a topological space, let $q \in X$ and let A be an open set containing a A set N is said to be a *neighborhood* of q if $A \subseteq N$

Remark. Every open set A is a neighborhood of the points contained in A. This follows immediately from the definition. Some authors, notably [Lee11], define neighborhoods as *always* being open sets, however the more common definition is to define neighborhoods as *containing* open sets, while themselves not necessarily being open.

Remark. While open sets can be thought of as a generalization of the *open intervals* from the real numbers, neighborhoods can instead be thought of as a generalization fo the concept of *closeness.* If N_1 and N_2 are both neighborhoods of q and $N_1 \subset N_2$, then the points contained in N_1 may in some sense be thought of as *closer to q* than the points contained in $N_2 \setminus N_1$.

The final way to define a topology I will mention in this text is to define a *basis.*

Definition 2.3.3 (Basis)**.** A *basis* on a set X is a collection of sets $\mathcal{B}$ that fulfill two criteria.

(1) $\mathcal{B}$ forms a cover of X, that is $\bigcup_{B \in \mathcal{B}} B = X$

(2) The intersection of any two basis sets is also a basis set, that is to say $\forall B_1, B_2 \in \mathcal{B}, B_1 \cap B_2 \in \mathcal{B}$

Proposition 2.3.4. *Let $\mathcal{B}$ be a basis on X, and let $\mathcal{T}$ be the collection of all (possible empty) unions of sets from $\mathcal{B}$. Then $\mathcal{T}$ is a topology on X.*

Proof. The proof may be found in [Lee11]. □

Definition 2.3.5 (Continuous function)**.** A continuous function is a function between two topological spaces $f : X \to Y$ such that for every set $A \subseteq Y$ that is open in Y, the preimage of A under f, $f^{-1}(A)$, is open in X.

Example 2.3.6*.* In $\mathbb{R}$ equipped with the default topology, the function $f : x \mapsto x^2$ is continuous.

Remark. In the category of topological spaces Top, the continuous functions are the morphisms and the topological spaces are the objects. From this one might surmise that continuous functions are indeed very important.

Definition 2.3.7 (Product topology)**.** Let A and Y be topological spaces. Let $A \subseteq X$ be a set open in X and $B \subseteq Y$ be a set open in Y. Let $\mathcal{B} \subseteq X \times Y$ be the collection of all sets on the form $A \times B$. Then the topology on $X \times Y$ generated by $\mathcal{B}$ is called the *product topology.*

Example 2.3.8*.* Let $\mathbb{R}$ have the usual topology, and let $\mathbb{R} \times \mathbb{R}$ have the product topology. Then the sets

$$
\begin{aligned}
A &= \{(x, y) : x > 0, y > 0\} \\
B &= \{(x, y) : x^2 + y^2 < 1\} \\
C &= A \cup B
\end{aligned}
$$

are all open in $\mathbb{R} \times \mathbb{R}$, while the sets

$$D = \{(x, y) : x = 0, y \in \mathbb{R}\}$$
$$E = \{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 \leq 1\}$$

are not open in $\mathbb{R} \times \mathbb{R}$.

Properties of topological spaces

When defining a topology, the open sets may be chosen more or less arbitrarily (as long as they fulfill the axioms). Each different choice of opens sets define a different topology over the set X. However, many of these topologies behave sort of "strange".

Example 2.3.9. Consider the set $X = \{1, 2\}$, with $\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}\}$ as the open sets. The reader may verify that this is a proper topological space. It has the "strange" property that the number 2 is "close" to the number 1, as every neighborhood containing 2 also contains 1, yet 1 is *not* "close" to 2 as the set $\{1\}$ is a neighborhood of 1 but not of 2. Examples such as this motivates the definition of the *Hausdorff* property.

Definition 2.3.10 (Hausdorff space)**.** A topological space $X$is considered *Hausdorff* if, for every pair of points q_1, $q_2 \in X$, there exists neighborhoods N_1, N_2, such that $q_1 \in N_1$, $q_2 \in N_2$ and $N_1 \cap N_2 = \emptyset$

We can see now, that X in example 2.3.9 is not Hausdorff, because there is no neighborhood of 2 not also containing 1. The Hausdorff property is one of several so called *separation axioms*, which as the name implies, says something about how well the points in the space are separated from each other. In a similar spirit, I will now state, but not prove, a theorem called *Urysohn's lemma*

Theorem 2.3.11 (Urysohn's lemma)**.** *Let X be a topological space such that for every pair of disjoint closed subsets of X have disjoint open neighborhoods.*[17] *Let $A, B \subseteq X$ be closed. Then, and only then, is there a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$*

Proof. The interested reader may find a proof of the theorem in [Wil04] □

Definition 2.3.12 (Compact set)**.** Let X be a topological space, and $\{A_i\}_{i\in I}$ be a collection of open sets in X such that $\{A_i\}_{i\in I}$ covers X, meaning

$$X \subseteq \bigcup_{i\in I} A_i$$

If in *every* such $\{A\}_{i\in I}$ we can find a *finite* subcollection of sets

$$\{A_j\}_{j\in J} \subseteq \{A\}_{i\in I} \text{ where } J \subseteq I \text{ is finite}$$

[17]This is called being a *normal space* and is another one of the so called separation axioms

that also covers X, then we say that X is *compact*.

Further, if around every point q it is possible to find a neighborhood $N(q)$ that is compact, then the topological space X is said to be *locally compact*[Lee11][18].

Theorem 2.3.13 (Tychnoff's theorem). *Let $\{X_i\}_{i \in I}$ be a collection of compact topological spaces. Then their cartesian product $\prod X_i$ is compact with respect to the product topology.*

Proof. The interested reader may find the proof in [Kel81] on page 143. □

Definition 2.3.14 (Topological group). A topological space G together with a binary operation $+$ and a unary operation $^{-1}$ is said to be a *topological group* if the following holds

(1) $(G, +)$ is a group, as defined in 2.2.1 (with $^{-1}$ being the inverse operation $^{-1} : x \mapsto x^{-1}$)

(2) $+$ and $^{-1}$ are continuous with respect to the product topology on $G \times G$ and the topology on G respectively

Further, if the topology defined on G is both locally compact and Hausdorff, we say that G is a *locally compact group*[Fol16].

2.4 Measure theory

Definition 2.4.1 (Measure). A *σ-algebra* consists of a set X together with a collection $\mathfrak{A}$ of subsets of X such that the following hold

(1) $X \in \mathfrak{A}$

(2) If $A \in \mathfrak{A}$, then $X \setminus A \in \mathfrak{A}$

(3) If for every $n \in \mathbb{N}$ $A_n \in \mathfrak{A}$, then $\bigcup_{\mathbb{N}} A_n \in \mathfrak{A}$

Let $\mathfrak{A}$ be a σ-algebra, and let $\mu : \mathfrak{A} \to [0, \infty]$. Further let μ be *countably additive*, meaning

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \quad \text{with } A_i \cap A_j = \emptyset \text{ for every } A_i, A_j \in \mathfrak{A}$$

and let $\mu(A_n) < \infty$ for some $n \in \mathbb{N}$. Then μ is said to be a *measure* on $\mathfrak{A}$

The set X together with $\mathfrak{A}$ and μ is called a *measure space*.

[18] There are several different definitions of *local compactness*, the one given here is the most common. The curious reader may look in [Lee11] for several properties of locally compact spaces that other authors sometimes gives as definitions. Unfortunately, the various properties are not in general equivalent to being locally compact as defined here, however in a *Hausdorff space*, they are.

Definition 2.4.2 (Borel measure)**.** Let X be a topological space. Let $\mathfrak{A}$ be the smallest σ-algebra such that for every open set $A \subseteq X$, A is in $\mathfrak{A}$.[19] Then the members of $\mathfrak{A}$ are called the *Borel sets* of X

If X is locally compact and Hausdorff, the measure μ defined on $\mathfrak{A}$ is called a *Borel measure.*

Definition 2.4.3 (Measurable function)**.** Let f be a function from a measure space M to a topological space X. If for every open set $A \subseteq X$, the preimage $f^{-1}(A) \subseteq M$ is a measurable set, then the function f is said to be *measurable.*

3 Gelfand theory

3.1 Definitions

Definition 3.1.1 (Normed algebra)**.** let A be a $\mathbb{C}$-algebra and let $\|\cdot\| : A \to \mathbb{R}$ be a unary mapping with the following properties

For $x, y \in \mathbb{R}$

(1) $\|x + y\| \leq \|x\| + \|y\|$

(2) $\|ax\| = |a|\|x\|$ for all $a \in \mathbb{C}$

(3) If $\|x\| = 0$ then $x = 0$

(4) $\|x\| \geq 0$

Remark. The norm on an algebra induces a so called *metric* on A. Any function f that preserves the metric, that is $\|f(x)\|_2 = \|x\|_1$ is called *isometric* and is an isomorphism of metric spaces.

Definition 3.1.2 (Adjoining unit)**.** Let A be an algebra without unit. Let $A_e = \{(x, \lambda) \in A \times C\}$. Then define

$$(x, \lambda) + (y, \mu) = (x + y, \lambda + \mu) \quad \mu(x, \lambda) = (\mu x, \mu\lambda)$$

and define

$$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$$

Then $e = (0, 1)$ is the identity in A_e

If A is normed, it can be extended to A_e by

$$\|(x, \lambda)\| = \|x\| + |\lambda|$$

Definition 3.1.3 (Vanish at infinity)**.** Let X be a locally compact topological space and let $f : X \to \mathbb{C}$. Let $\varepsilon > 0$. If there exists some compact subset $A \subset X$ such that $|f(x)| < \varepsilon$ for every such ε, then f is said to *vanish at infinity.*

Notation. We denote the set of *all* functions f from X that vanish at infinity with $C_0(X)$

[19] Such $\mathfrak{A}$ is guaranteed to exist [Rud87, Thm. 1.10]

Definition 3.1.4 (Strong separation)**.** Let X be a topological space. Let F be a family of functions from X to $\mathbb{C}$ such that the following holds

$$\forall x \in X \exists f \in F \quad f(x) \neq 0, \quad \forall x, y \in X x \neq y \exists g \in F, g(x) \neq f(x)$$

Then F is said to *strongly separate the points of X*

Definition 3.1.5 (Cauchy sequence)**.** Let A be an algebra, and let $\{x_n\}$ be a sequence of points in A for $n \in \mathbb{N}$. If for every choice of $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $n, m > N$

$$\|x_n - x_m\| < \varepsilon$$

then $\{x_n\}$ is called a *Cauchy sequence*.

If there exists a point $a \in A$ such that for every $\varepsilon > 0$

$$\|x_m - a\| > \varepsilon \quad \text{only for finitely many } x_m \in \{x_n\}$$

then a is said to be a *limit point* of $\{x_n\}$

Definition 3.1.6 (Banach Algebra)**.** Let A be a $\mathbb{C}$-algebra. Let $\|\cdot\| : A \to \mathbb{R}$ be a function satisfying the following properties

For every $x, y \in A$

(1) $\|x\| \geq 0$

(2) If $\|x\| = 0$ then $x = 0$

(3) $\|x + y\| \leq \|x\| + \|y\|$

(4) $\|\lambda x\| = |\lambda| \|x\|$ for any $\lambda \in \mathbb{C}$

(5) $\|e\| = 1$

(6) $\|xy\| \leq \|x\| \|y\|$

Then $\|\cdot\|$ is called a *norm* and A together with $\|\cdot\|$ is called a *normed algebra*.

Further, if *every* Cauchy sequence of points in A has a limit point $a \in A$, then A is said to be *complete*.

If A is both normed and complete it is called a *Banach algebra*.

Theorem 3.1.7 (Stone-Weierstrass theorem)**.** *Let X be a locally compact A be a subalgebra of $C_0(X)$ such that A is closed under complex conjugation. Further, assume A strongly separates X. Then for every neighborhood $\varepsilon > 0$ and for every function $f \in C_0(A)$ there is some $a \in A$ such that $\|f - a\|_\infty < \varepsilon$*

Definition 3.1.8 (Spectrum of an element)**.** Let A be and algebra over the complex numbers $\mathbb{C}$ and let e be the identity of A. We say $x \in A$ is *invertible* if there exists and *inverse* $x^{-1} \in A$ such that $xx^{-1} = e$. Then for and $x \in A$, the *spectrum* of x, $\sigma_A(x)$ is defined as follows

$$\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is } \textit{not} \text{ invertible in } A$$

Definition 3.1.9 (Spectral radius)**.** Let A be a normed algebra and let $x \in A$. Then

$$r_A(x) = \inf\{\|x^n\|^{1/n} : n \in \mathbb{N}\}$$

is called the *spectral radius* of x.

Definition 3.1.10. Let X be a locally compact Hausdroff space and let $f \in C_0(X)$ Then the *supremum norm* is the norm defined by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

Definition 3.1.11 (Gelfand topology)**.** Let A be a commutative Banach algebra, and let $\Delta(A)$ denote the set of all nonzero linear functionals on A. Let $\mathcal{T}$ be the smallest topology for which every function

$$\text{eval}_x : \Delta(A) \to \mathbb{C} \quad \varphi \mapsto \varphi(x) \quad \text{for some } x \in A$$

Notation. For the reminder of this text, $\Delta(A)$ will always refer to the set of linear functionals over A an it will be assumed to always have the Gelfand topology as above.

3.2 The Gelfand representation

Theorem 3.2.1 ([Kan09, Lem. 2.1.1])**.** *Let A be an algebra over the real or the complex numbers with identity e, and let φ be a linear functional on A which satisfies*

$$\varphi(e) = 1 \text{ and } \varphi\left(x^2\right) = \varphi(x)^2$$

Then $\varphi(xy) = \varphi(x)\varphi(y)$ for every x,$y \in A$

Proof.

$$\begin{aligned}
\varphi\left(x^2\right) + \varphi(xy + yx) + \varphi\left(y^2\right) &= \varphi\left(x^2 + xy + yx + y^2\right) \\
&= \varphi\left((x+y)^2\right) \\
&= (\varphi(x+y))^2 \\
&= (\varphi(x) + \varphi(y))^2 \\
&= \varphi(x)^2 + 2\varphi(x)\varphi(y) + \varphi(y)^2 \\
&= \varphi\left(x^2\right) + 2\varphi(x)\varphi(y) + \varphi\left(y^2\right)
\end{aligned}$$

Subtracting from both sides, we see that

$$\varphi(xy + yx) = 2\varphi(x)\varphi(y)$$

for any x,$y \in A$. To complete the proof, we need to show that $\varphi(xy) = \varphi(yx)$

Let $a, b \in A$ and note that

$$(ab - ba)^2 + (ab + ba)^2 = 2(a(bab) + (bab)a)$$

implies that

$$\begin{aligned}\varphi(ab-ba)^2+4\varphi(a)^2\varphi(b)^2 &= \varphi\left((ab-ba)^2\right)+\varphi(ab+ba)^2\\ &= \varphi\left((ab-ba)^2+(ab+ba)^2\right)\\ &= 2\varphi(a(bab)+(bab)a)\\ &= 4\varphi(a)\varphi(bab)\end{aligned}$$

Then set $a = x - \varphi(x)e$ so that $\varphi(a) = 0$ and set $b = y$ to obtain

$$\begin{aligned}0 &= \varphi(ay-ya)\\ &= \varphi(ay)-\varphi(ya)\end{aligned}$$

Thus $\varphi(ay) = \varphi(ya)$, so $\varphi(xy) = \varphi(yx)$

□

Definition 3.2.2 (Weight function)**.** Let G be a locally compact group and let ω be a positive function defined on G. Then ω is a *weight function* if it satisfies the following

(1) $\omega(xy) \leq \omega(x)\omega(y) \quad \forall x, y \in G$

(2) ω is Borel measurable

Theorem 3.2.3 ([Kan09, Lem. 1.3.3])**.** *Let C be a compact subset of a G. Then there exists positive real numbers a and b, such that*

$$a \leq \omega(x) \leq b \quad \forall x \in C$$

Proof. The proof may be found in [Kan09]. The reader may find [BCS58] a useful reference for some specific steps in the proof.

□

Theorem 3.2.4 (Gleason-Kahane-Zelasko [Kan09, Thm. 2.1.2])**.** *Let A be a unital Banach algebra. For a linear functional φ on A the following are equivalent:*

(1) φ is nonzero and multiplicative

(2) $\varphi(e) = 1$ and $\varphi(x) \neq 0$ for every $x \in A$ that is invertible.

(3) $\varphi(x) \in \sigma_A(x) \quad \forall x \in A$

Proof. The proof is done in 3 steps.

$1 \to 2$: If φ is nonzero, then $\varphi(e) = 1$ per definition, and if x is invertible, then $1 = \varphi(x)\varphi(x^{-1})$.

$2 \to 3$: Let $\lambda \in A \setminus \sigma_A(x)$. Then $0 \neq \varphi(x - \lambda e) = \varphi(x) - \lambda$

$3 \to 1$: Let $\varphi(x) \in \sigma_A(x)$ $\forall x \in A$. We first show that $\varphi\left(x^2\right) = \varphi(x)^2$. It then follows from 3.2.1 that φ is nonzero and multiplicative.

Let $n \geq 2$ and let

$$P(\lambda) = \varphi((\lambda e - x)^n)$$

Let $\lambda_1, \ldots, \lambda_n$ denote the roots of P. Then

$$0 = P(\lambda_i) = \varphi((\lambda_i e - x)^n) \in \sigma_A((\lambda_i e - x)^n) \quad \text{for each } i$$

But then $\lambda_i \in \sigma_A(x)$ and thus $|\lambda_i| \leq r_A(x)^2$ Further

$$\prod_{i=1}^{n} (\lambda - \lambda_i) = p(\lambda) = \lambda^n - n\varphi(x)\lambda^{n-1} + \binom{n}{2}\varphi\left(x^2\right)\lambda^{n-2} + \cdots + (-1)^n \varphi\left(x^n\right)$$

By comparing the coefficients of each power of λ we see that

$$\sum_{i=1}^{n} \lambda_i = n\varphi(x) \quad \text{and} \quad \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = \binom{n}{2}\varphi(x^2)$$

From this it follows that

$$\left(\sum_{i=1}^{n} \lambda_i\right)^2 = \sum i = 1^n \lambda_i^2 + 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = \sum_{i=1}^{n} \lambda_i^2 + n(n-1)\varphi(x^2)$$

Combining the above we conclude that

$$n^2 \left|\varphi(x)^2 - \varphi\left(x^2\right)\right| = \left|-n\varphi\left(x^2\right) + \sum_{i=1}^{n} \lambda_i^2\right| \leq n\left|\varphi\left(x^2\right)\right| + n r_A(x)^2$$

As this holds for all n we conclude that $\varphi\left(x^2\right) = \varphi(x)^2$

□

Theorem 3.2.5 ([Kan09, Thm. 1.4.6]). *Let X be a locally compact Hausdorff space. Let $E \subseteq X$, and let the following hold for each such E:*

$$I(E) = \{f \in C_0(X) : f(x) = 0 \textit{ for every } x \in E\}$$

Then $g : E \to I(E)$ is a bijection between the collection of nonempty closed subsets of X and the proper closed ideals of $C_0(X)$. $I(E)$ is a modular ideal iff E is compact. $I(E) \in \operatorname{Max}(C_0(X))$ iff E is a singleton set.

The interested reader may find the proof in [Kan09]

Theorem 3.2.6 ([Kan09, Thm. 2.1.8]). *For a commutative Banach algebra A, the mapping*

$$\varphi \to \ker\varphi = \{x \in A : \varphi(x) = 0\}$$

Proof. Let $\varphi \in \Delta(A)$. Then $\ker\varphi$ is an ideal of A and also a losed linear subspace of codimension one in A . Choose $a \in A$ such that $\varphi(a) = 1$. Then for any $x \in A$

$$\varphi(ax - x) = \varphi(a)\varphi(x) - \varphi(x) = 0$$

so $ax - x \in \ker\varphi$. It follows that $\ker\varphi \in \mathrm{Max}(A)$

Let $\varphi_1, \varphi_2 \in \Delta(A)$ such that $\ker\varphi_q = \ker\varphi_2$. Denote this ideal I. Let u be such that u is an identity modulo I . Since the codimension of I in A is one, meaning that I is a proper ideal, any $x \in A$ can be *uniquely* expressed as

$$x = \lambda u + y \quad \text{for some } y \in I, \lambda \in \mathbb{C}$$

□

Theorem 3.2.7 ([Kan09, Thm. 2.2.3]). *Let A be a commutative banach algebra then the following holds*

(1) $\Delta(A)$ *is a locally comapct Hausdorff space*

(2) $\Delta(A_e) = \Delta(A) \cup \{\varphi_\infty\}$ *is the one-point compactification of* $\Delta(A)$

(3) $\Delta(A)$ *is compact if* A *has an identity.*

Proof. 1 Let let A have an identity, and let $\varphi_1, \varphi_2 \in \Delta(A)$ be distinct. Then there is some $x \in A$ such that $0 < \frac{1}{2}|\varphi_1(x) - \varphi_2(x)|$. It follows then that here are disjoint neighbourhoods N_1, N_2 containing φ_1 and φ_2 respectively, so $\Delta(A)$ is Hausdorff. Let N be a neighborhood of $Delta(A)$ and let N_e be a neighborhood of $\Delta(A_e)$. Let $\varphi \in \Delta(A), \varepsilon > 0$ and let $F \subset A$ be finite. Then

$$N_e(\varphi, F, \varepsilon) = \begin{cases} N(\varphi, F, \varepsilon) \cup \{\varphi_\infty\} & \text{if } |\varphi(x)| < \varepsilon \ \forall x \in F \\ N(\varphi, F, \varepsilon) & \text{otherwise} \end{cases}$$

Thus the Gelfand topology on $\Delta(A)$ coincides with the Gelfand topology on $\Delta(A_e)$. Since $\{\varphi_\infty\}$ is closed in $\Delta(A_e)$, it follows that $\Delta(A)$ is open, and thus locally compact .

2 Let $x \in A, \varepsilon > 0$

$$\begin{aligned} N_\varepsilon &= \{\varphi_\infty\} \cup \{\varphi \in \Delta(A) : |\varphi(x)| < \varepsilon\} &(1)\\ &= \Delta(A_e) \setminus \{\psi(\Delta(A_e) : |\phi(x)| \geq \varepsilon\} &(2) \end{aligned}$$

The sets $\{\psi(\Delta(A_e) : |\phi(x)| \geq \varepsilon, x \in A\}$ are closed in $\Delta(A_e)$, so they are compact since $Delta(A)$ is Hausdorff. The complement of a basv neighbourhood N around φ_∞ is the union of finitely many such compact ψ. Thus $Delta(A_e) = \Delta(A) \cup \{\varphi_\infty\}$ is the one point compactification of $\Delta(A)$

3 Let
$$C = \prod_{x\in A} \{z \in \mathbb{C} : |z| \leq \|x\|\}$$

and let C have the Product topology. Then by Tychnoff's theorem, C is compact. Note that $|\varphi(x)| \leq \|x\|$ for every $\varphi \in \Delta(A), x \in A$. Define

$$\phi : \Delta(A) \to \mathbb{C} \quad \varphi \mapsto \varphi(x) \quad \text{for } x \in A$$

Then ϕ is injective and per definition a homeomorphism.

To show that $\Delta(A)$ is compact, it suffices to show that $\varphi(Delta(A))$ is closed (recall that this is true).

Let $\lambda = (\lambda_x)_{x\in A} \in C$ be a point in the smallest closed set containing $\phi(\Delta(A))$. Let $x, y \in A, \alpha, \beta \in \mathbb{C}$. For any $\varepsilon > 0$, if $\varphi(a) - \lambda_a \leq \varepsilon$ whenever $a = \alpha x + \beta y$, then

$$\begin{aligned}|\alpha\lambda_x + \beta\lambda_y - \lambda_{\alpha x+\beta y}| \leq &|\alpha||\lambda_x - \varphi(x)|+ \\ &|\beta||\lambda_y - \varphi(y)| + |\varphi(\alpha + \beta y) - \lambda_{\alpha x+\beta y}| \\ \leq &\varepsilon(|\alpha| + |\beta| + 1)\end{aligned}$$

and

$$\begin{aligned}|\lambda_{xy} - \lambda_x\lambda_y| &\leq |\lambda_{xy} - \varphi(xy)| + |\varphi(y)||\varphi(x) - \lambda_x| + |\lambda_x||\varphi(y) - \lambda_y| \\ &\leq \varepsilon(1 + \|y\| + \|x\|)\end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, it follows that the function $\psi : A \to \mathbb{C}, x \mapsto \lambda_x$ is a homomorphism, and that $\psi \in \Delta(A)$

□

Remark. The other direction of 3 is not relevant for this text, but it does follow from Shilov's idempotent theorem which may be found in

Theorem 3.2.8 ([Kan09, Them. 2.2.5])**.** *Let A be a commutative Banach algebra. Then for each $x \in A$*

$$\sigma_A(x)\backslash\{0\} \subseteq \widehat{x}(\Delta(A)) = \{\varphi(x) : \varphi \in \Delta(A)\} \subseteq \sigma_A(x)$$

Proof. Let A be a Banach algebra with identity e. Then by 3.2.4 it follows that for every $\varphi \in \Delta(A)$, $\varphi(x) \in \sigma_A(x)$. For the other direction, let $\lambda \in \sigma_A(x)$. Then the ideal
$$I = (\lambda e - x)A$$
is proper, so it must be the kernel of some homomorphism $\varphi \in \Delta(A)$. But then $\varphi(\lambda e - x) = 0$, so $\lambda \in \widehat{x}(\Delta(A))$ □

Theorem 3.2.9 ([Kan09, Thm. 2.2.7])**.** *Let A be a commutative Banach algebra and let Γ be the Gelfand representation of A . Then the following holds*

(1) Γ *maps* A *into* $C_0(\Delta(A))$ *and is norm decreasing.*

(2) $\Gamma(A)$ *strongly separates the points of* $\Delta(A)$*.*

(3) Γ *is isometric iff* $\|x\|^2 = \|x^2\|$ *for every* $x \in A$

Proof. 1 By 3.2.7 $\Delta(A_e)$ is the one point compactification of A of $\Delta(A)$, and since $\widehat{x}(\varphi_\infty) = 0$ it follows that $\widehat{x} \in C_0(\Delta(A))$. By 3.2.8

$$\|\widehat{x}\|_\infty = r_A(x) \leq \|x\|$$

3 Since

$$\Gamma(A)(\varphi) \neq \{0\} \quad \forall \varphi \in \Delta(A)$$

and since if $\varphi_1 \neq \varphi_2$ it follows that $\varphi_1(x9 \neq \varphi_2(x)$ for some $x \in A$, it is obvious that $\Gamma(A)$ strongly separates the points in$\Delta(A)$

3 Let $\|y\|^2 = \|y^2\|$ for every $y \in A$. Then by induction,

$$\|y\|^{2^n} = \|y^{2^n}\|$$

for every $n \in \mathbb{N}$. It follows that

$$\|\widehat{x}\|_\infty = r_A(x) = \lim_{n\to\infty} \|x^{2^n}\|^{1/2^n} = \|x\|$$

For the other direction, assume Γ is isometric. But then

$$\|x^2\| = \|\widehat{x}^2\|_\infty = \|\widehat{x}\|_\infty^2 = \|x\|^2$$

□

Theorem 3.2.10 ([Kan09, Lem. 2.4.4])**.** *Let* A *be a commutative C*-algebra. Then the Gelfand homomorphism is a *-homomorphism*

Proof. Let A be unital[20]. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let $\varphi(x) = \alpha + i\beta, \varphi(x^*) = \gamma + i\delta$

Assume that $\beta + \delta \neq 0$. Since $\beta + \gamma$ isn't zero, it must be invertible. Let $y = (\beta+\delta)^{-1}(x + x^* - (\alpha+\gamma)e) \in A$. But then $y = y^*$, so

$$\begin{aligned} \varphi(y) &= (\beta+\delta)^{-1}(\alpha + i\beta + \gamma + i\delta - (\alpha+\gamma)) \\ &= (\beta+\delta)^{-1}(i\beta + i\delta) \\ &= i(\beta+\delta)^{-1}(\beta+\delta) && = i \end{aligned}$$

Thus, for all $r \in R$

$$\varphi(y + rie) = \varphi(y) + ri = i + ri = (1+r)i$$

[20]This is allowed according to Kaniuth

so $|1+r| \leq \|y+rie\|$. But since $y = y^*$ and A is a C*-alegbra, it follows that

$$\begin{aligned}(r+1)^2 &\leq \|y+rie\|^2 \\ &= \|(y+rie)(y+rie)^*\| \\ &= \|y+rie)(y-rie)\| \\ &= \|y^2+r^2e\| \\ &\leq \|y^2\|+r^2\end{aligned}$$

However, since y does not depend on r, there will be a sufficiently large r such that $2r+1 > \|y^2\|$ and the inequality fails. This is a contradiction, so $\beta+\delta = 0$, meaning $\beta = -\delta$.

It follows that

$$\varphi((ix)^*) = \varphi(-ix^*) = -i\varphi(x^*) = -i(\gamma+i\delta) = -\beta - i\gamma$$

However, from the definition of φ above, we know that $\varphi(ix) = i(\alpha+i\beta) = -\beta+i\alpha$. From this we conclude that

$$-\beta - i\gamma = -\beta - i\alpha$$

so $\gamma = \alpha$. But since $\varphi(x)$ and $\varphi(x^*)$ have the same real part but opposite imaginary part, we conclude that $\varphi(x^*) = \overline{\varphi(x)}$ for arbitrary $\varphi \in \Delta(A)$ and $x \in A$ □

Theorem 3.2.11 ([Kan09, Thm. 2.4.5])**.** *For a commutative C*-algebra A, the Gelfand homomorphism* Γ *is an isometric *-isomorphism from A onto* $C_0(\Delta(A))$.

Proof. Let $y \in A$ such that $y = y*$. Then

$$\|y\|^2 = \|yjy\| = \|y^2\|$$

and by induction

$$\|y\|^{2^n} = \left\|y^{2^n}\right\| \quad n \in \mathbb{N}$$

but then

$$r_A(y) = \lim_{n\to\infty} \left\|y^{2^n}\right\|^{1/2^n} = \|y\|$$

Let $x \in A$, but then

$$r_A(x^*x = \|x*x\| = \|x\|^2$$

as per above. But from 3.2.10 we know that $\widehat{x^*} = \overline{\widehat{x}}$ and together with 3.2.8 we conclude that

$$\|\widehat{x}\|_\infty^2 = \|\overline{\widehat{x}}\widehat{x}\|_\infty = \|(x^*x)^2\|_\infty = r_A(x^*x) = \|x\|^2$$

This means that $x \mapsto \widehat{x}$ is isometric and that $\widehat{A}$ every Cauchy sequence with respect to $\|\cdot\|_\infty$ has a limit point in $\widehat{A}$, so it is closed inf $C_0(\Delta(A))$. Then by 3.2.9 $\widehat{A}$ is a *-subalgebra of $C_0\Delta(A))$ which strongly separates the points of $\Delta(A)$ By 3.1.7 it follows that Neighbourhoods of every point of $C_0(\Delta(A))$ contain points of $\widehat{A}$. But then $\widehat{A} = C_0(\Delta(A))$. Since we already knew that the Gelfand transform was a homorphism, this shows that it is an isometric *-homomorphism ☐

Theorem 3.2.12 ([Kan09, Lem. 2.2.12]). *Let A and B be commutative Banach algebras. If there exists an algebra isomorphism $\phi : A \to B$, then $\Delta(A)$ and $\Delta(B)$ are homomorphic.*

Proof. Let $\phi^* : \Delta(B) \to \Delta(A)$ be the dual of ϕ, that is to say

$$\phi^*(\varphi)(a) = \varphi(\phi(a)) \quad \text{for } a \in A, \varphi \in \Delta(B)$$

then ϕ^* is a bijection and continuous as per the definition of Δ. Similarly, ϕ^{*-1} is also continuous. ☐

Theorem 3.2.13 ([Kan09, Cor. 2.4.6]). *For any two C*-algebras A, B, the following are equivalent:*

(1) $\Delta(A)$ and $\Delta(B)$ are homeomorphic.

*(2) There exists an isometric *-isomorphism between A and B.*

(3) There esists an algebra isomorphism between A and B.

Proof. The proof is shown in three steps:

1→2: Let $\phi : \Delta(A) \to \Delta(B)$ be a homeomorphism. But then $f \mapsto f \circ \phi$ is an isometric isomorphism $C_0(\Delta(A)) \to C_0(\Delta(B))$ and $\overline{f} \mapsto \overline{f \circ \phi}$, so $C_0(\Delta(A))$ is isometrically isomorphic to $C_0(\Delta(B))$. By 3.2.11 we know that A is isometrically isomorphic to $C_0(\Delta(A))$, similarly for B. Since isomorphisms are transitive due to the transitivity of morphism composition in general, A is isometrically isomophic to B.

2→3: Is trivially true as all isometric *-isomorphisms are algebra isomorphisms

3→1: This a special case of 3.2.12

☐

Theorem 3.2.14 (Anti-equivalence of categories). *Let* C^*-Alg *denote the category of all C*-algebras with *-algebra homomorphisms as the morphisms and let* Haus *denote the category of all locally compact Hausdorff spaces with continuous functions where the preimage $f^{-1}(K)$ of a compact subset K is compact as morphisms. There exists natural isomorphisms $S : X \to \Delta(C_0(X))$ with $x \mapsto \mathrm{eval}_x$ and $D : A \to C_0\Delta(A))$*[21] *with $a \mapsto \widehat{a}$, such that C_0 and Δ are contravariant functors and*

$$C^*\text{-Alg} \cong \text{Haus}^{op}$$

Proof. This is a consequence of 3.2.13 ☐

[21] It is the existance of D which Gelfand and Neumark originally proved in [GN94]

3.2.1 Dictionary

By this point the reader has hopefully gained some insight into how intimately connected Hausdorff spaces and commutative C*-algebras are. One may in fact construct a dictionary of sorts of which algebraic terms correspond to which topological ones. While I have not in this text defined or even mentioned some of the following terms, I still wish to include them so that they may perhaps serve as inspiration for future reading withing the field.

The following is from [Kha13]

Topological spaces	**Algebraic spaces**
compact	unital
1-point compactification	unitization
Stone-Chech compactification	multiplier algebra
closed subspace	closed ideal
surjection[22]	injection[22]
injection[22]	surjection[22]
homeomorphism	automorphism
Borel measure	positive functional
probability measure	state
disjoin union	direct sum
cartesian product	minimal tensor product

[22] Notice how the dual properites of *surjectivity* and *injectivity* swap with each other when transformed by the contravariant functors

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These are some corrections to The Gelfand Duality bacherlor's thesis by Jesper Flodmark.

Please note that there are a number of spelling mistakes I have deliberately chosen to ignore for the sake of this correction document. This is because I judge that the correct meaning is apparent from context and wish to keep this list of corrections free from unnecessary clutter to the extent it is possible.

Page 8

Definition 2.1.2

(2) The correct definition of the morphisms of the opposite category should be $f^{op} : Y \to X$

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Definition 2.3.2

The corrected definition is as follows

Let X be a topological space, let $q \in X$ and let A be an open set containing q. A set N is said to be a *neighborhood* of q if $A \subseteq N$

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Definition 3.1.1

This definition is incorrect and may be disregarded completely. 3.1.6 contains the correct definition of a normed algebra as considered in this text.

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Theorem 3.2.6

The correct statement of the theorem is as follows

For a commutative Banach algebra A, the mapping

$$\varphi \to \ker \varphi = \{x \in A : \varphi(x) = 0\}$$

is a bijection between $\Delta(A)$ and the set of all maximal modular ideals over A

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Proof of theorem 3.2.6

This proof is incomplete. For a full version of the proof, see [Kan09a, Thm. 2.1.8], page 49.

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In the remark right before Theorem 3.2.8 there is a missing reference of where to find Shilov's Idempotent theorem. The theorem may be found in [Kan09b]

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Theorem 3.2.12

The correct statement of the theorem is as follows:

Let A and B be commutative Banach algebras. If there exists an algebra isomorphism $\phi : A \to B$, then $\Delta(A)$ and $\Delta(B)$ are homeomorphic.

This is the updated reference list

References

[Kan09a] Eberhard Kaniuth. *A course in commutative Banach algebras*. English. Vol. 246. Grad. Texts Math. New York, NY: Springer, 2009. ISBN: 978-0-387-72475-1. DOI: 10.1007/978-0-387-72476-8.

[Kan09b] Eberhard Kaniuth. *A course in commutative Banach algebras*. English. Vol. 246. Grad. Texts Math. New York, NY: Springer, 2009. ISBN: 978-0-387-72475-1. DOI: 10.1007/978-0-387-72476-8.