SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

The Geometry of a Good Cut: Moser, Pizza, and Beyond

av

Olli Pettersson

2025 - No L14

The Geometry of a Good Cut: Moser, Pizza, and Beyond

Olli Pettersson

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Per Alexandersson

2025

The Geometry of a Good Cut: Moser, Pizza, and Beyond

Olli Pettersson

August 6, 2025

Abstract

How many regions can be formed by connecting points on a circle with straight lines? What appears at first to be a simple geometric exercise quickly unfolds into a rich interplay between combinatorics, geometry, and topology. This thesis explores a family of classic problems involving partitions of space—starting with Moser's circle problem, extending through inductive reasoning, binomial identities, Euler's characteristic, and considerations of higher-dimensional analogues. Along the way, we uncover the seductive illusion of exponential patterns, harness the structure of Pascal's triangle, and reflect on how different mathematical perspectives—combinatorial, visual, and topological—can complement one another. A variety of approaches are explored to shed light on the creative and multifaceted nature of mathematics—qualities that often stand in contrast to how the subject is presented in traditional curricula, where time and space for multiple methods are rare. This work aims to celebrate mathematical thinking as an art of variation and discovery.

Sammanfattning

Hur många områden kan bildas genom att dra raka linjer mellan punkter på en cirkel? Det som först verkar vara en enkel geometrisk uppgift visar sig snabbt leda till en rik väv av kombinatorik, geometri och topologi. I detta arbete utforskas en klass av klassiska problem som rör uppdelningar av rummet—med början i Mosers cirkelproblem och vidare genom induktiva resonemang, binomialidentiteter, Eulers karaktäristik och uppdelningar i högre dimensioner. På vägen avslöjas den förföriska illusionen av exponentiella mönster, Pascals triangel utnyttjas, och reflektioner förs kring hur olika matematiska perspektiv—kombinatoriska, visuella och topologiska—kan samverka. Genom att närma sig problemet från flera håll lyfts matematikens kreativa och mångsidiga karaktär fram—något som ofta kontrasterar mot den traditionella undervisningen där tid och utrymme för alternativa metoder sällan ges. Arbetet är ett försök att hylla matematiken som ett variationsrikt utforskande.

Contents

1

Introduction

One of the delights of mathematics lies in how a seemingly simple question can open doors to deep, diverse, and unexpected territories. This thesis is the result of one such journey, beginning with a classic geometric puzzle known as Moser's circle problem — a problem first presented in 1949 in the *Mathematical Miscellany* column of *Mathematics Magazine* [8]. The problem asks: given n points placed on the circumference of a circle, how many regions can be formed by drawing chords between all pairs of points, assuming no three chords intersect at a single interior point? What starts as a visual and intuitive curiosity soon reveals rich combinatorial structure and geometric complexity.

This problem became the point of departure for a broader investigation into how space can be partitioned — first in the plane, then in one and three dimensions, and ultimately in general r-dimensional Euclidean space. Along the way, the thesis explores well-known sequences such as the Lazy Caterer's problem and Cake numbers, examines recursive and combinatorial reasoning, and draws connections to classical tools like Pascal's triangle, binomial identities, and Euler's characteristic.

The work also carries a pedagogical ambition. Much of school mathematics is linear, goal-oriented, and constrained by time — emphasizing mastery of technique over exploration. In contrast, this thesis aims to showcase how mathematical thinking can be creative, multidirectional, and conceptually rich. Multiple methods — from induction and recursive formulae to geometric visualization and algebraic derivation — are used not for redundancy, but to highlight the flexibility and beauty inherent in mathematics. While the thesis does not present new research findings in the traditional sense, it is grounded in careful reading of classical literature, reconstruction of known results, and the synthesis of seemingly unrelated problems into a coherent narrative. The final section briefly discusses how these geometric and combinatorial explorations echo into applied domains such as spatial analysis in geography, tying together mathematical play and practical relevance.

2

Unexpected Patterns

2.1 Moser's Circle

Imagine drawing points on a circle and for every new point, draw a chord joining each point with every other point. The chords drawn this way split the disc into regions bounded by the chords and arcs of the circle in some cases. After two points we have one chord splitting the disc into two regions. Adding a point generates two more chords, and counting the regions we quickly see that the number has increased to four. We can continue with a fourth point, adding three new chords and increasing the number of regions to 8. A fifth point and careful counting gives us 16 regions. It would seem like adding n points, increases the number of regions generated by the chords to 2^{n-1}. The process so far is illustrated in Figure 1.

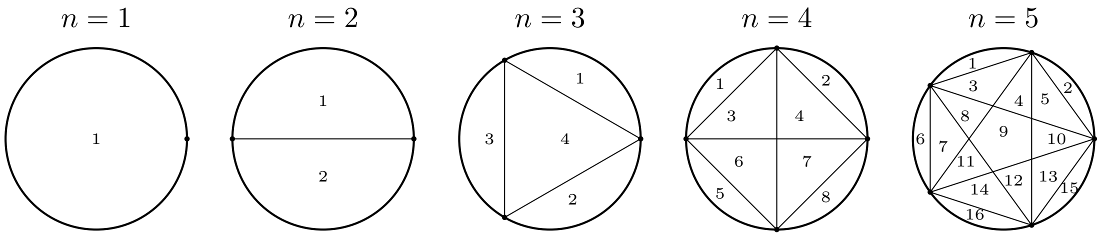

Figure 1: First five cases of Moser's circle

A reasonable guess for the number of regions after $n = 6$ points on the circle would thus be 32. However, after some careful calculations we are only able to find 31. Is there another region hiding somewhere, or could our initial guess for the function giving the number of regions be wrong? What then is the function $f(n)$ that returns the number of regions after n points drawn on the circle?

2.2 From Chords to Regions: Discovering the Pattern

One perfectly adequate way of approaching mathematical problems is to try to break them down into smaller, yet connected pieces. In the case of our circle problem, a natural first step is to look for structure in the numbers — to gather a few examples, organize them, and see if any patterns emerge. To that end, we begin by drawing up a table that records, for a small number of points n, how many chords can be drawn, how many intersections occur, and how many regions are formed. Doing so gives us the following results:

n	$C(n)$	$I(n)$	$R(n)$
1	0	0	1
2	1	0	2
3	3	0	4
4	6	1	8
5	10	5	16
6	15	15	31
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Number of chords $C(n)$, intersections $I(n)$, and regions $R(n)$ formed by connecting n points in general position on a circle.

2.2.1 Looking for Clues

At first glance, the region counts in the rightmost column seem to grow quickly: $1, 2, 4, 8, 16, 31$. There's a whisper of powers of two here — but not quite. Rather than jumping to conclusions, let's take a closer look at the other columns, which may be easier to explain.

2.2.2 Counting Chords

The second column — the number of chords — is more cooperative. The numbers $0, 1, 3, 6, 10, 15$ might look familiar: they're the so-called triangular numbers, formed by summing consecutive integers. That's no coincidence. Each chord connects a pair of points, and there are $\binom{n}{2}$ such pairs. So:

$$C(n) = \binom{n}{2}.$$

A simple and satisfying formula, arising from the basic structure of the problem.

2.2.3 Counting Intersections

The number of intersections is a little trickier. For $n = 4$, we get our first interior crossing point. Then things accelerate: $1, 5, 15...$ If we consult our mental binomial toolbox again, we might notice that these match:

$$I(n) = \binom{n}{4}.$$

Why choose four? Because in order to create an interior intersection, we need two chords that cross — and each of those chords is defined by a pair of points, which must be disjoint. So we need four distinct points, and every such quadruple yields exactly one intersection, assuming general position. That's an appealing combinatorial insight.

2.2.4 The Temptation of a Pattern

Armed with these two formulas, we might guess at something for the regions. If we try:

$$R(n) = 1 + C(n) + I(n) = 1 + \binom{n}{2} + \binom{n}{4},$$

we find, somewhat miraculously, that it agrees with all the values in our table. But is this just wishful thinking? Can we prove it?

2.3 Euler Characteristic: A Topological Detour

To establish the formula rigorously, we turn to a classic result from the world of planar graphs: Euler characteristic or more specifically Euler's polyhedron formula. This states that for any connected planar graph,

$$V - E + F = 2,$$

where V is the number of vertices, E the number of edges, and F the number of faces — analogous to the regions (R) formed by chords in Moser's circle — including the outer, unbounded region.

In our case, we're only interested in the regions inside the circle, so we subtract the outer face and rewrite the formula as:

$$R(n) = F - 1 = E - V + 1.$$

This will allow us to compute $R(n)$ indirectly, using counts of vertices and edges.

2.3.1 Vertices

The vertices of our graph consist of:

- The n points placed on the circle, and
- The intersection points of the chords inside the circle, of which there are $\binom{n}{4}$.

Thus:

$$V = n + \binom{n}{4}.$$

2.3.2 Edges

Edges are slightly more involved. We count:

- The $\binom{n}{2}$ chords we originally draw,
- The extra segments created by intersections. Each intersection splits two chords into four segments, adding 2 edges per intersection, for a total of $2\binom{n}{4}$,
- The n circular arcs connecting the outer points on the circle.

Adding these together gives:

$$E = \binom{n}{2} + 2\binom{n}{4} + n.$$

2.3.3 A Brief Proof Before Continuing

Before tying everything together in our circle partition problem, we take a brief detour to establish a foundational result from planar graph theory. The Euler characteristic is not only a beautiful result in its own right — revealing a deep relationship between the number of vertices, edges, and faces of a graph — but it also serves as an essential tool for our later derivation of the number of regions formed by intersecting chords.

It is important to note that Euler's formula applies to connected planar graphs — graphs that can be drawn on the plane without edge crossings and where every vertex is reachable from any other through a sequence of edges. To help build some intuition, we first look at two simple examples of connected planar graphs.

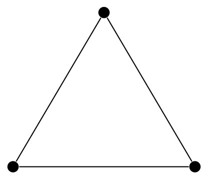

Figure 2: A connected planar graph with $V = 3$, $E = 3$, and $F = 2$ (including the outer face).

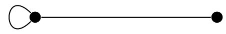

Figure 3: A connected planar graph with a loop: $V = 2$, $E = 2$, $F = 2$.

Theorem 2.1. *Let $G = (V, E)$ be a connected and non-empty planar graph with $|V| = v$ vertices and $|E| = e$ edges. Let r denote the number of regions in the plane formed by embedding G, including the unbounded outer region. Then:*

$$v - e + r = 2.$$

Proof. We proceed by induction on the number of edges e.

Base cases:

If $e = 0$, then the connected graph G consists of a single isolated vertex. There is one region (the entire plane), so:

$$v - e + r = 1 - 0 + 1 = 2.$$

If $e = 1$, there are two possibilities:

- The edge connects two distinct vertices: $v = 2$, $e = 1$, $r = 1$.
- The edge forms a loop on a single vertex: $v = 1$, $e = 1$, $r = 2$.

In both cases, $v - e + r = 2$, so the theorem holds.

Inductive step:

Let $k \in \mathbb{N}$ and assume that the theorem holds for all connected planar graphs with $e \leq k$ edges. Let $G = (V, E)$ be a connected planar graph with v vertices, r regions, and $e = k + 1$ edges.

Choose an edge $\{a, b\} \in E$, and define $H = G - \{a, b\}$ as the graph obtained by removing that edge. We now consider two cases depending on whether H remains connected.

Case 1: H is connected.

In this case, the edge $\{a, b\}$ either connects two existing vertices or is a loop. In either scenario, removing the edge merges two adjacent regions in G into one region in H. Hence, H has:

$$v \text{ vertices}, \quad e = k \text{ edges}, \quad r - 1 \text{ regions}.$$

By the inductive hypothesis on H, we have:

$$v - k + (r - 1) = 2.$$

Solving this gives:

$$v - (k + 1) + r = 2 \quad \implies \quad v - e + r = 2,$$

so the theorem holds for G.

Case 2: H is disconnected.

Then H consists of two connected components, say H_1 and H_2, with:

$$v_1 + v_2 = v, \quad e_1 + e_2 = e - 1 = k, \quad r_1 + r_2 = r + 1.$$

(Note that each disconnected component introduces its own unbounded region, so removing the connecting edge increases the total number of regions by 1.)

By the inductive hypothesis applied to each component, we have:

$$v_1 - e_1 + r_1 = 2, \quad v_2 - e_2 + r_2 = 2.$$

Adding the two:

$$(v_1 + v_2) - (e_1 + e_2) + (r_1 + r_2) = 4.$$

Substituting:

$$v - (e-1) + (r+1) = 4 \quad \Longrightarrow \quad v - e + r = 2.$$

Thus, the theorem holds in both cases, completing the induction. □

2.4 Putting It All Together

With Euler's characteristic now firmly established, we are ready to return to our original geometric setting — n points placed on a circle, with chords drawn between all pairs in general position. This configuration, once interpreted as a planar graph by treating chord intersections as vertices and arcs and segments as edges, falls squarely within the domain where Euler's formula applies. By carefully accounting for the number of vertices and edges in this graph, we can use the formula and substitute our vertices and edges to get

$$\begin{aligned} R(n) &= E - V + 1 \\ &= \left[\binom{n}{2} + 2\binom{n}{4} + n\right] - \left[n + \binom{n}{4}\right] + 1 \\ &= \binom{n}{2} + \binom{n}{4} + 1. \end{aligned}$$

After some convenient cancellations, we arrive at a formula that confirms our earlier guess — and it's no longer just a guess, but a theorem:

$$R(n) = \binom{n}{0} + \binom{n}{2} + \binom{n}{4}.$$

Here, we've written the constant $\binom{n}{0} = 1$ not merely for stylistic elegance, but to emphasize the underlying binomial structure. Each term in this formula corresponds to a particular geometric entity involved in the partitioning process:

- $\binom{n}{0}$: the initial, undivided region — the whole circle before any chords are drawn.
- $\binom{n}{2}$: the number of chords that can be drawn between any pair of the n points.
- $\binom{n}{4}$: the number of interior intersection points formed when chords cross each

other — intersections require exactly four distinct points to define two intersecting chords.

Each of these elements plays a direct role in subdividing the circle, and the total number of regions they produce is precisely what the formula captures.

We may also express this result in closed form, as will be shown later through an inductive proof:

$$R(n) = \frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}.$$

This compact polynomial version, while perhaps less illuminating in terms of geometry, is powerful for computation and analysis. It provides a direct way to calculate the number of regions for any n, without needing to count chords or intersections manually.

What began as a simple table of curious values has now unfolded into a deep connection between geometry, combinatorics, and topology — and that, in many ways, is the essence of mathematics.

2.5 Can We Be Sure?

The notion of *general position* appears frequently throughout this thesis. Intuitively, it describes arrangements that avoid degeneracies — that is, configurations where elements intersect "too nicely," reducing the number of regions formed. Two degenerate cases in two dimensions are illustrated below in Figure 4. Since our main focus lies in lines and regions, we adopt a geometric perspective suited to that context.

Definition 2.2 (General Position)**.**

- In the plane, a collection of lines is in *general position* if:
  - no two lines are parallel, and
  - no three lines intersect at a single point.
- In three-dimensional space, a collection of planes is in *general position* if:
  - no two planes are parallel,
  - no three intersect in a common line,
  - no four intersect at a common point.

Figure 4: Two degenerate configurations in the plane: triple intersection (left), and parallel lines (right).

This condition ensures that each new line (or plane) contributes the maximal number of new regions to the partition — a fundamental assumption in problems like the Moser's circle and beyond.

So far, we have assumed general position for all the chords in the circle as a prerequisite for the solution to hold. How then can we be certain that no three chords intersect in a shared point, or that no two chords run parallel to each other, thus rendering all of our hard work useless and invalid? The task at hand is therefore to ensure that at each step, we can add a new point to the circle while preserving the general position of the configuration. Specifically, when adding a new point A and drawing chords to each of the existing n points O, one of two things can happen for each chord AO:

a) The chord AO passes through a point I where two existing chords intersect.

b) The chord AO intersects all existing chords in novel interior points, creating only new intersections.

We want to ensure that case **b)** holds for all chords from the new point. The following lemma establishes that this is always possible:

Lemma 2.3. *A new point A can always be chosen on the circle such that case* ***b)*** *occurs for all chords AO, where O ranges over the existing points. That is, no chord AO passes through any existing intersection point in the interior of the circle.*

Proof. Suppose we already have n points in general position on the circle. Let O denote any of these existing points, and let I denote any existing interior intersection point formed by two previously drawn chords.

For case **a)** to occur, the chord AO would have to pass through such a point I, meaning the points A, O, and I would be collinear as shown below in Figure 5.

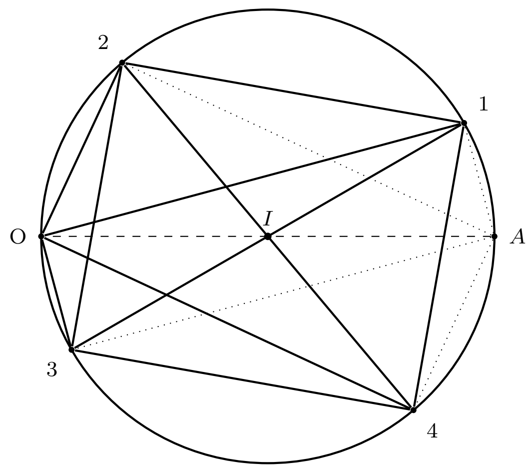

Figure 5: One of the finite points A resulting in an intersection point I of three chords.

Now observe that for each such pair (O, I), there is at most one location on the circle (excluding O) where A can lie on the line $\overline{OI}$. Since there are finitely many existing points O and finitely many intersection points I, the total number of such "bad" locations for A is also finite. But the circle contains infinitely many points. Therefore, there must exist at least one position for A on the circle that does not lie on any such line $\overline{OI}$. Choosing A at such a position guarantees that no chord AO passes through any existing intersection point — that is, case **b)** occurs for every chord. As a consequence, if k chords intersect a given new chord AO, and each does so at a distinct point (since no triple intersections occur), then the chord AO creates $k + 1$ new regions. □

2.6 Inductive Proof

While the formula for $R(n)$ has already been expressed in terms of binomial coefficients, we now present an alternative proof by induction — offering a different perspective on the structure and growth of the number of regions formed when connecting n points in general position on a circle This approach builds on the intuitive idea that each newly added point — when connected to all previous points — introduces a predictable number of new chords, intersections, and regions. Figure 6 illustrates the inductive step from $n = 4$ to $n = 5$, where four new chords (shown as dashed lines) are drawn from the new point to the existing four. Each chord intersects some of the pre-existing chords and creates a number of new regions.

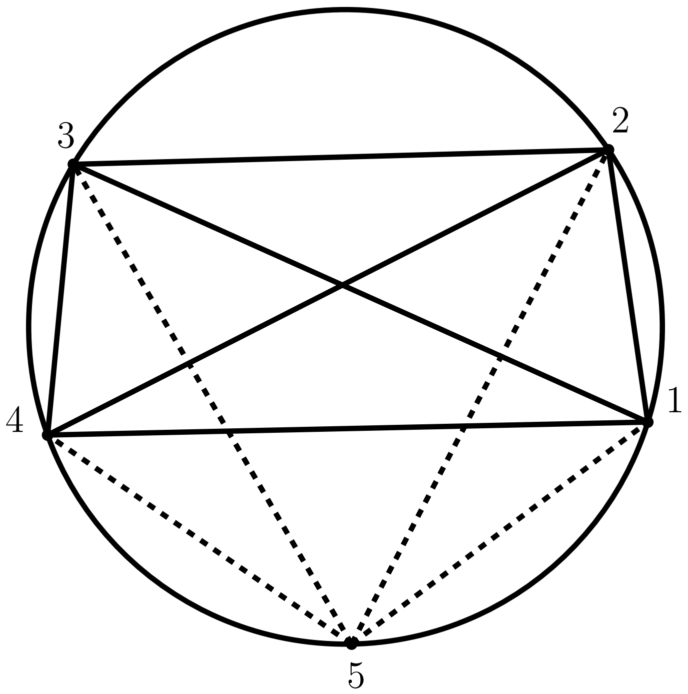

Figure 6: Moser's circle.

Suppose we have already drawn all chords between $n-1$ points in general position on a circle. According to the inductive hypothesis, the number of regions formed is $R(n-1)$. Now, when a new n-th point is added, it will be connected to the existing $n-1$ points by new chords. Let us denote the new point as P, and suppose the existing points are labeled $1, 2, \ldots, n-1$ in circular order.

Each new chord $P \to i$ (for $i = 1, \ldots, n-1$) will intersect some of the previously drawn chords. Specifically, a new chord divides the circle into two arcs, and it will intersect chords that connect points from opposite sides of this division.

Lemma 2.4. *Let P be the newly added point on the circle, connected by chords to each of the previous $n-1$ points. For a chord drawn from P to point i (where*

$i = 1, \ldots, n-1$), the number of new intersections created by that chord is given by:

$$(n-i-1)(i-1),$$

where $n-i-1$ is the number of points to the left of point i, and $i-1$ is the number of points to its right. (In Figure 6 we can see that chords from node 5 to 1 and 2 each intersect with zero chords and chords to 3 and 4 each intersect 2 chords, i.e. they have 2 points to one side and 1 point to the other side of each respectively.)

Each of these intersections occurs in a distinct region and subdivides it, adding one new region. In addition, the chord itself introduces a new face, contributing an extra region. Hence, each chord from the new point adds:

$$1 + (n-i-1)(i-1)$$

new regions. This is illustrated below in Figure 7, where the new dashed chord intersects existing chords at points a, b. Each intersection splits an existing region into two, forming new subregions labeled a_1, a_2, b_1, b_2. Furthermore, the chord introduces a new region labeled c in Figure 7.

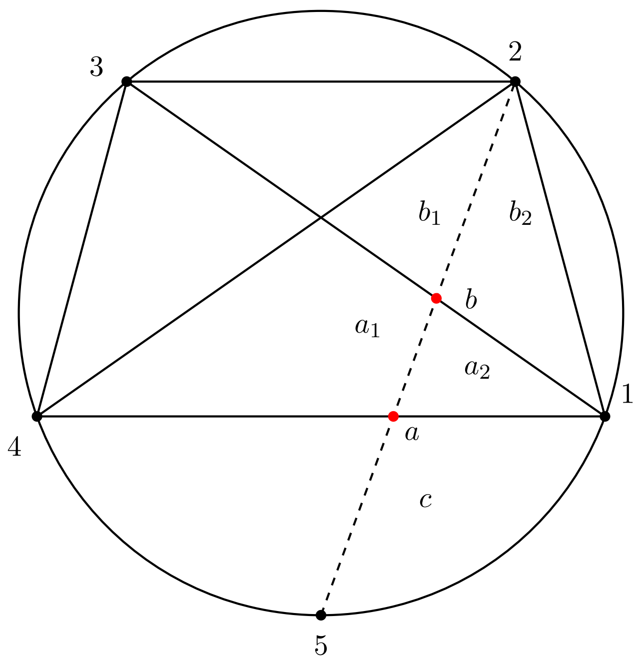

Figure 7: The new chord from point 5 to point 2 intersects two existing chords. Each intersection point (labeled a, b) splits a region into two: e.g., a_1, a_2.

Summing over all $i = 1$ to $n - 1$, the total number of new regions added is:

$$\sum_{i=1}^{n-1} [1 + (n - i - 1)(i - 1)] .$$

This gives the recurrence:

$$R(n) = R(n - 1) + \sum_{i=1}^{n-1} [1 + (n - i - 1)(i - 1)] .$$

We now simplify this sum:

$$\sum_{i=1}^{n-1} [1 + (n - i - 1)(i - 1)] = \sum_{i=1}^{n-1} 1 + \sum_{i=1}^{n-1} (n - i - 1)(i - 1).$$

The first sum yields:

$$\sum_{i=1}^{n-1} 1 = n - 1.$$

For the second sum, expand the product:

$$(n - i - 1)(i - 1) = -i^2 + ni - 2i + 1 - n.$$

Thus, we have:

$$\sum_{i=1}^{n-1} (n - i - 1)(i - 1) = \sum_{i=1}^{n-1} (-i^2 + ni - 2i + 1 - n).$$

Using known formulas:

$$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}, \quad \sum_{i=1}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6}, \quad \sum_{i=1}^{n-1} 1 = n - 1,$$

we simplify and eventually find:

$$R(n) = R(n - 1) + \frac{1}{6}n^3 - n^2 + \frac{17}{6}n - 2.$$

Summing this recurrence from $k = 1$ to n, with initial value $R(0) = 1$, yields the closed-form expression:

$$R(n) = \frac{n}{24}(n^3 - 6n^2 + 23n - 18) + 1.$$

2.7 Why Powers of Two?

The region counts in the circle problem begin with a sequence that strongly resembles powers of two: $R(n) = 1, 2, 4, 8, 16, \ldots$. This resemblance is not coincidental, and to understand it more deeply, we turn to Pascal's triangle.

Each row in Pascal's triangle gives the binomial coefficients for a fixed n. The first six rows, shown both in integer form and binomial form, are displayed below in Figure 8:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

Figure 8: Pascal's Triangle: integer values and binomial notation.

Now, observe the following: for each n, the number of regions $R(n)$ in the circle problem can be expressed as:

$$R(n) = \sum_{k=0}^{4} \binom{n-1}{k}.$$

This means that $R(n)$ equals the sum of the first five entries in the $(n-1)$-th row of Pascal's triangle.

For example:

$$
\begin{aligned}
R(1) &= \binom{0}{0} = 1 \\
R(2) &= \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2 \\
R(3) &= \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4 \\
R(4) &= \sum_{k=0}^{3} \binom{3}{k} = 1 + 3 + 3 + 1 = 8 \\
R(5) &= \sum_{k=0}^{4} \binom{4}{k} = 1 + 4 + 6 + 4 + 1 = 16 \\
R(6) &= \sum_{k=0}^{4} \binom{5}{k} = 1 + 5 + 10 + 10 + 5 = 31.
\end{aligned}
$$

This pattern holds for all $n \geq 1$, and aligns beautifully with the combinatorial formula we have already derived:

$$R(n) = \binom{n}{0} + \binom{n}{2} + \binom{n}{4}.$$

In fact, one can verify that:

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} = \sum_{k=0}^{4} \binom{n-1}{k}.$$

This identity — while not immediately obvious — arises from Pascal's rule:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

and reveals the hidden link between even-indexed binomial sums and truncated rows of the triangle. Thus, the early powers-of-two pattern in $R(n)$ is a projection of deeper combinatorial structure, not mere coincidence.

3

Other Partitions of Space

Having explored the surprising complexity of partitioning a circle with chords, one might wonder how the act of dividing space generalizes to other dimensions. What happens when we partition a line? A plane? A three-dimensional space? In this section, we investigate the structure and patterns arising from such partitions — revealing a beautiful consistency and surprising parallels across dimensions.

3.1 Lazy Caterer

Imagine you are cutting a pizza — but instead of the usual goal of making evenly sized slices, you opt for a quirkier objective: to create as many individual pieces as possible, completely disregarding their size. How many pieces can n cuts produce? And what conditions must the cuts satisfy to achieve the maximum number of pieces? To simplify things, let us momentarily forget that the pizza has any depth. We'll treat it as a flat disk — a two-dimensional case not unlike the circle problem discussed earlier. An initial cut ($n = 1$) yields two pieces, and a second cut can produce two more — giving a total of four. A conventional approach might then divide two of those slices in half, resulting in six. But if our aim is to maximize the number of pieces, we can position the third cut more strategically, producing seven separate regions as shown below in Figure 9.

Equipped with a ruler and our wits, we can place the next few cuts and calculate that $n = 4$ yields 11 regions, $n = 5$ increases that number to 16, and the sixth cut gives us 22 pieces of pizza to share (quite unevenly) with our hungry friends. By now the drawing gets somewhat messy and hard to count by brute force, no fear, as mathematicians we find a way to organize the results thus far and take a different approach. Let C be the number of cuts, P the number of pieces (regions) obtained, and ΔP the difference between each two consecutive values of P, and finally, let $\Delta^2 P$ be the difference of the difference. In this manner, we create a difference table

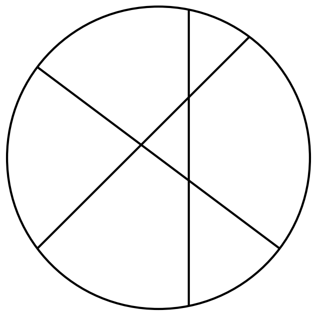

Figure 9: Three straight cuts arranged to generate the maximum number of regions (7).

for the sequence and from there try to discern any emerging patterns.

C	P	ΔP	$\Delta^2 P$
0	1		
1	2	1	
2	4	2	1
3	7	3	1
4	11	4	1
5	16	5	1
6	22	6	1

Table 2: Difference table for the Lazy Caterer's sequence.

Interestingly, the fact that the second differences are constant indicates that the function $P(C)$ — returning the number of pieces after C cuts — is quadratic. We assume a general form:

$$P(C) = aC^2 + bC + c.$$

Using the values $P(0) = 1$, $P(1) = 2$, and $P(2) = 4$, we obtain the system:

$$\begin{aligned} P(0) &= c = 1 \\ P(1) &= a + b + c = 2 \\ P(2) &= 4a + 2b + c = 4, \end{aligned}$$

solving the system, we find $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = 1$, giving us the closed-form formula:

$$P(C) = \frac{C^2 + C + 2}{2}.$$

3.1.1 A Recursive Argument

An alternative derivation — perhaps more intuitive — relies on recursive reasoning. Assume that we have already made $C - 1$ cuts in general position, resulting in $P(C-1)$ regions. The Cth cut intersects all $C - 1$ previous cuts at distinct points (since we are in general position), dividing the new cut into C segments. Each segment passes through a previously undivided region, thus splitting it into two. As a result, C new regions are created:

$$P(C) = P(C-1) + C$$

with the base case:

$$P(0) = 1.$$

Expanding the recurrence:

$$P(C) = 1 + 1 + 2 + \cdots + C = 1 + \sum_{k=1}^{C} k = 1 + \frac{C(C+1)}{2}.$$

So we again arrive at:

$$P(C) = \frac{C^2 + C + 2}{2}.$$

This recursive formulation, although not a rigorous proof of the result, offers a more geometric intuition about how each new line incrementally contributes to the growing number of regions. Together, these derivations highlight the beauty and flexibility of mathematical thinking — how the same truth can emerge from different pathways of reasoning.

When J.Steiner [11] first posed the problem of counting regions in space, he used a similar recursive reasoning but wrote the equation as

$$P(C) = 1 + C + \frac{C(C-1)}{1 \cdot 2}. \tag{1}$$

This style of notation will be helpful in building intuition as we venture on to partitioning space in other dimensions. At the moment — with the binomial coefficients fresh in mind from the previous section — we might notice and ponder on the fact that equation 1 can be written as

$$P(C) = \binom{C}{0} + \binom{C}{1} + \binom{C}{2}.$$

3.2 Other Dimensions

Counting the number of sections after n cuts of a string of spaghetti or licorice (in accordance with our previously established food metaphors) should be quick work, and after a moment's thought, we can conclude that this one-dimensional analogue to the pizza problem reveals that C points on a line divide the line into $C+1$ line segments and rays (in the case of an infinite line).

Moving on to watermelons (as Banks posed the problem in [4]) or birthday cakes, we can follow the recursive reasoning from the two-dimensional analogue above; and with the help of a difference table and by solving a system of equations conclude that C planes in general position divide space into a number of regions given by the function

$$P(C) = 1 + C + \frac{C(C-1)}{1 \cdot 2} + \frac{C(C-1)(C-2)}{1 \cdot 2 \cdot 3},$$

which we again can rewrite as

$$P(C) = \binom{C}{0} + \binom{C}{1} + \binom{C}{2} + \binom{C}{3}.$$

It is worth noting that our use of a difference table implicitly assumes that the function we seek is a polynomial. This is a reasonable and often fruitful assumption when the differences stabilize at a fixed order — as they do here, at the fourth row — but it remains an assumption nonetheless. The true justification for the polynomial form of these functions lies in the combinatorics of how new planes, lines, or points divide existing regions. Still, the alignment between empirical patterns and known binomial expressions provides a strong hint toward a deeper structure — one that generalizes gracefully to higher dimensions.

These observations hint at a general principle: as we move up in dimension, each new term in the binomial expansion contributes to the total number of regions. But can this pattern be generalized further — beyond three dimensions?

Before diving into a formal generalization, it's worth pausing to appreciate a more intuitive route. The mathematician George Pólya made a simple yet powerful observation based on counting regions in lower dimensions.

Pólya [9] tabulated the number of regions generated by dividing one-, two-, and three-dimensional spaces using points, lines, and planes, respectively. He observed a charming recursive pattern: for $C \geq 1$, each entry in Table 3 is the sum of the entry directly above it and the one diagonally above to the right. For instance, the

15 regions formed by four planes in 3D space arise from the sum $8 + 7$, where 8 is the number of regions from three planes in 3D, and 7 is the number from four lines in 2D. This triangle-like growth mirrors Pascal's triangle — and offers an intuitive bridge to the binomial formulation.

C = number of dividing elements	Number of regions for:		
	space by planes	**plane by lines**	**line by points**
0	1	1	1
1	2	2	2
2	4	4	3
3	8	7	4
4	15	11	5
5	26	16	6
6	42	22	7
7	64	29	8
...	...	...	...
C			$C + 1$

Table 3: Number of regions created by C dividing elements in 1D, 2D, and 3D.

3.3 Generalization and Other Considerations

3.3.1 Partitioning r-Dimensional Space

As Buck [5] showed in 1943, the problem of counting regions created by dividing elements extends far beyond slicing pizzas or drawing chords in a circle. In fact, when C hyperplanes lie in general position in r-dimensional Euclidean space, the number of resulting regions is given by a simple but powerful combinatorial expression:

$$F_r(C) = \sum_{k=0}^{r} \binom{C}{k}.$$

This result unifies a broad class of partitioning problems, including the Lazy Caterer's problem ($r = 2$) and the Cake Numbers ($r = 3$). What begins as a spatial curiosity reveals a deeper structure rooted in binomial identities and topological invariants — a striking example of how seemingly elementary problems echo through the higher reaches of mathematics.

3.3.2 Bounded and Unbounded Regions

Not all regions created by partitions are alike. As Steiner [11] and later Wetzel [12] observed, some regions are bounded — entirely enclosed by dividing elements — while others extend infinitely. For n lines in general position in the plane, the number of *bounded* regions is given by:

$$R'(n) = 1 - n + \binom{n}{2} = \binom{n-1}{2}.$$

This can be understood intuitively. The first three lines form the initial bounded region (a triangle). Each additional line intersects existing segments, creating new bounded regions. In fact, each new line contributes to the count according to the number of finite line segments it crosses. Figure 10 illustrates this growth process. Meanwhile, the number of *unbounded* regions increases by 2 with each line, yielding a total of $2n$ unbounded regions.

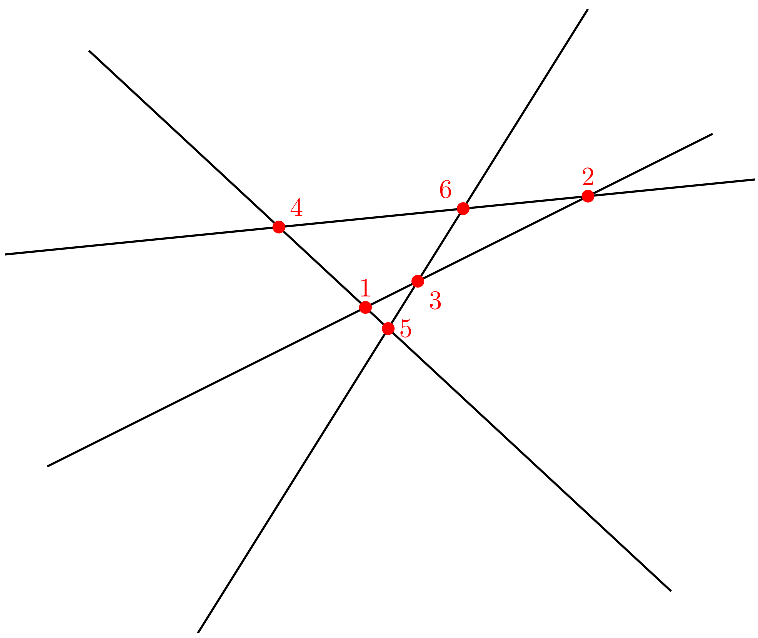

Figure 10: Four lines in general position forming bounded and unbounded regions. Red dots mark intersection points.

3.3.3 Regions Lost to Degeneracies

We have seen that n lines in general position divide the plane into a maximal number of regions, of which $\binom{n-1}{2}$ are bounded. But this bound hinges delicately on the assumption of general position: no two lines are parallel, and no three intersect at a single point.

When lines deviate from this ideal — for instance, by becoming concurrent or parallel — regions are inevitably lost. Wetzel [12] offers an elegant way to quantify these losses by considering the geometric nature of the degeneracies.

Concurrencies. If all n lines intersect at a single point, the configuration fails to produce any bounded regions at all. In fact, the entire count $\binom{n-1}{2}$ is lost. More generally, if k lines intersect at a single point M_i, then $\binom{k-1}{2}$ bounded regions are lost due to that intersection. This reflects the collapse of $\binom{k}{2}$ pairwise intersections into a single point, from which only $k-1$ regions emerge.

Parallels. If a family of x parallel lines in direction d is introduced, one can envision these as the limiting case of lines converging at a distant point M, then displaced to become parallel. In this transformation, two types of regions vanish:

- $\binom{x-1}{2}$ bounded regions are lost at the point M;
- $x-1$ unbounded regions are lost beyond M.

Together, this yields a total loss of $\binom{x}{2}$ regions per parallel family.

General formula. If a configuration contains m multiple intersection points $M_1, M_2, \ldots, M_m$, where k_i lines meet at M_i, and p families of parallel lines of sizes $x_1, x_2, \ldots, x_p$, the total number of regions lost is:

$$\Delta R = \sum_{i=1}^{m} \binom{k_i - 1}{2} + \sum_{j=1}^{p} \binom{x_j}{2}.$$

This formulation offers a geometric and intuitive insight: every violation of general position either collapses regions at a point (concurrency) or hides them at infinity (parallelism).

3.3.4 Brousseau's Sweeper Line

A clever method for counting regions inductively comes from Brousseau [3], using what is often called a *sweeper line* argument. Imagine a vertical line sweeping across the plane from left to right. Each time the line crosses a vertex (an intersection point), it increases the number of regions by one. This technique lends itself not only to elegant proofs of classical formulas — like the Lazy Caterer's formula — but also provides a dynamic visual metaphor: regions are born one by one, as the world unfolds from left to right. Alexanderson and Wetzel [1] extend this reasoning to various formulas in three-space, showing how the sweeping principle adapts to more complex spatial arrangements.

This section ties together the combinatorial beauty, the geometric intuition, and the topological constraints that underlie space partitioning. From Buck's general theorem to the losses due to degeneracy and the dynamism of sweeper lines, each layer adds depth to our understanding of how complexity emerges from structure.

4

Beyond the Circle: Reflections on Spatial Analysis and Binary Words

4.1 Into the Third Dimension: Moser's Problem Extended

If the original Moser problem invites us to slice the circle with chords drawn between n points, then its natural three-dimensional cousin asks: how many regions can be formed by drawing all possible planes determined by triples of n points on a sphere, assuming general position?

This extension has yet to be solved in full generality, but a conjectured sequence exists (OEIS A144841 [7]), beginning

$$1,\ 1,\ 2,\ 11,\ 41,\ 121,\ 316,\ 757,\ 1961,\ \ldots,$$

with an associated (conjectured) formula:

$$R(n) = 280\binom{n}{9} + 10\binom{n}{6} + 6\binom{n}{4} + \binom{n}{3} + 1.$$

While the full derivation remains elusive, we can begin to understand the components. Given that a plane is determined by three non-collinear points in space, the number of planes formed from n points is $\binom{n}{3}$. Each of these planes can intersect with others and form new regions in the ambient three-dimensional space.

In fact, we might consider a simpler candidate formula based on recursive reasoning. Starting with the initial region (before any plane is added), we add $\binom{n}{3}$ planes, each of which creates a new region. Furthermore, each pairwise intersection of these planes can yield additional regions, depending on the structure of their overlap. If we assume — quite optimistically — that each pair of planes intersects in such a way that a new region is added, then the total number of regions could be approximated as:

$$R(n) = 1 + \binom{n}{3} + \binom{\binom{n}{3}}{2}.$$

This expression captures the idea that planes are determined by triples of points and that their intersections can themselves produce further complexity. Of course, this naive approximation likely overestimates the number of regions, especially as n grows, but it offers an intriguing lower-order model to compare against the more intricate OEIS conjecture.

Whether or not this or the OEIS formula is correct in general, both highlight that extending the Moser circle problem to higher dimensions involves exponential growth in both geometric complexity and combinatorial explosion — a rich terrain for further mathematical exploration.

4.2 Spatial Patterns and Proximity

Mathematical problems like Moser's circle challenge us to think creatively about how space can be divided, connected, and interpreted. Though rooted in pure mathematics, these problems resonate with spatial reasoning in other disciplines — especially geography, where understanding spatial patterns is essential.

As Robert Banks [4] cleverly notes; one such example is Dacey's Nearest Neighbor Analysis (NNA), a method developed to assess the spatial distribution of real-world points — such as cities, settlements, or phenomena like earthquakes. NNA compares the observed average distance between nearest neighbors to what would be expected under a random distribution, thereby helping geographers determine whether a pattern is clustered, random, or dispersed.

In contrast to the idealized general position assumption used in problems like Moser's circle — where no three chords intersect at a single point — NNA grounds us in the irregular, often clustered distributions found in reality. While the combinatorial structure of region-counting problems seeks maximal fragmentation under strict rules, nearest neighbor analysis explores emergent structure from actual spatial arrangements.

This contrast opens up a valuable pedagogical reflection: mathematics and geography offer complementary lenses for thinking about space. One idealizes and abstracts; the other observes and interprets. And yet, both ask deep questions about relationships between points, the spaces they define, and the structures that arise.

Even algorithms like Dijkstra's [6] — which operate on graphs to find shortest paths — fit into this broader conversation. Where this thesis explores how edges divide space and create complexity, algorithms like Dijkstra's traverse those edges in search of efficiency. Both approaches remind us that space is not static, but something shaped by the relationships we choose to study.

In teaching, these parallels can be powerful. They suggest that topics like chord intersections, space partitioning, and region-counting are not only intellectually satisfying, but also provide bridges to applied disciplines — encouraging students to see mathematics as both a tool for precision and a medium for spatial imagination.

This bridge between mathematical abstraction and spatial reasoning becomes even more vivid when we explore how the regions formed by lines or planes in general position can be encoded — not geometrically, but algebraically. One surprising and elegant encoding arises through *binary words*, where each spatial region corresponds uniquely to a binary string that reflects its positional relationship to a given set of dividing elements. In what follows, we examine this bijective correspondence and reveal how it links region-counting formulas to familiar objects from discrete mathematics and computer science.

4.3 Binary Words and Geometric Partitions

We have seen that placing n lines in general position in the plane yields a maximum number of regions given by:

$$R(n) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} = 1 + n + \frac{n(n-1)}{2}.$$

This is the familiar sequence $1, 2, 4, 7, 11, \ldots$, but what might come as a surprise is that it also counts certain types of binary strings (P. Alexandersson, personal communication, July 28, 2025).

Let us explore this connection by constructing a visual encoding.

4.3.1 From Regions to Binary Words

Imagine drawing n lines in general position across a square or the first quadrant of a coordinate system. Each line stretches from the y-axis to the x-axis, and we label the lines from top to bottom as lines 1 through n. These lines divide the plane into regions — and each region lies either above or below each line.

We now assign to each region a binary string of length n where:

- a digit 1 in position i means that the region lies *below* line i,
- a digit 0 means that the region lies *above* line i.

For example, in a diagram with four lines, the region below lines 1 and 2 but above lines 3 and 4 would be labeled 1100. An example is shown below in Figure 11, showing the case for $n = 4$.

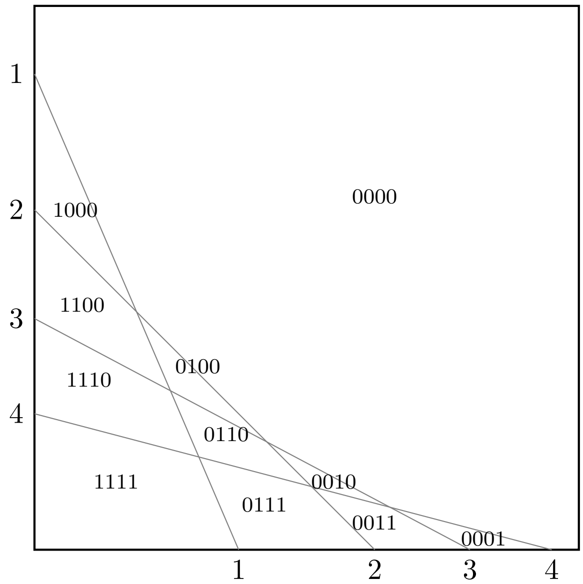

Figure 11: Each region corresponds to a binary word indicating which lines it lies below.

This gives a natural bijection between the regions and binary strings of length n with at most one contiguous block of ones — that is, binary words of the form:

$$00\ldots011\ldots100\ldots0.$$

Proof. We provide an intuitive proof by counting how many binary strings of this form exist for a given length n.

1. There is exactly one word with no ones at all: $00\ldots0$. This contributes $\binom{n}{0} = 1$.

2. There are exactly n words with a single one in any of the n positions: $\binom{n}{1} = n$.

3. For words with a single contiguous block of $k \geq 2$ ones, the number of ways to place such a block in a binary word of length n is $n - k + 1$. Summing over all such k, we get:

$$\sum_{k=2}^{n}(n-k+1) = \binom{n}{2}.$$

Adding these cases together, the total number of binary strings with at most one block of consecutive ones is:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} = R(n),$$

which confirms the bijection between these binary words and the regions formed by n lines in general position. □

4.3.2 The 3D Case: Cake Numbers and Patterned Words

In our exploration of the connections between spatial partitioning and binary words, we now turn our attention to the three-dimensional analogue. Just as lines divide the plane into regions, planes divide three-dimensional space into a growing number of convex polyhedral regions. When these planes are placed in general position—meaning no three planes meet along a common line, and no four intersect at a single point—the number of resulting regions is given by the so-called *cake numbers*:

$$R(n) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}.$$

Each new plane increases the number of regions by slicing through the existing ones, in a way analogous to how a new line slices the plane. But in three dimensions, the resulting structure becomes vastly more intricate. Still, we may ask: can the same kind of binary encoding we used in the planar case offer insight into the structure of these 3D regions?

Each plane divides space into two half-spaces, and any given region lies entirely on one side of each plane. We can record this information using a binary string of length n, where:

- A `1` in position i indicates that the region lies *above* plane i,
- A `0` in position i indicates that it lies *below* plane i.

In this way, each region can be encoded as a binary word of length n. However, not all binary words correspond to regions that actually appear. The requirement that the planes be in general position imposes constraints on how regions can be situated in space. It turns out that the set of binary words corresponding to valid regions in this setting is exactly the set of words that match the regular expression

`1*0*1*0*.`

In other words, each valid word consists of at most two runs of 1s and two runs of 0s, in alternating order. Examples include:

`0000,    1111,    110000,    001111,    11001100.`

This set corresponds precisely to the number of regions formed by planes in general position, and is listed in the OEIS as sequence A000125 [10].

The bijection arises naturally: each region is identified by its orientation with respect to the planes, and the structure of the spatial division ensures that only certain combinations of above/below orientations can appear. Just as in the planar case, this restriction limits the number of "switches" between being above and below. The regular expression `1*0*1*0*` captures exactly this.

This perspective not only offers a compact encoding of complex geometric structures, but also underscores the power of binary representations in combinatorics. As in the 2D case, we find that spatial reasoning and symbolic patterns—geometry and language—are more deeply connected than they first appear. This perspective not only offers a compact encoding of complex geometric structures, but also underscores the power of binary representations in combinatorics. As in the 2D case, we find that spatial reasoning and symbolic patterns—geometry and language—are more deeply connected than they first appear.

For those seeking a rigorous foundation for this correspondence, Alexandersson and Nabawanda provide a formal proof of the bijection in their 2021 paper [2], showing that the number of binary words of length n matching the regular expression `1*0*1*0*` coincides exactly with the number of regions created by n planes in general position in three-dimensional space.

5

Conclusion

From slicing pizzas and drawing chords to dissecting higher-dimensional space, this thesis has explored how simple geometric constructions lead to rich combinatorial structures. By tracing the progression from the Lazy Caterer's Problem to Moser's Circle Problem and beyond, we have encountered elegant formulas, surprising connections to binary words, and deeper topological principles such as Euler's characteristic.

Beyond their intrinsic mathematical beauty, these problems offer powerful pedagogical opportunities. They illustrate how abstract ideas emerge from concrete reasoning, how structure arises from simplicity, and how mathematics can bridge seemingly unrelated representations — diagrams, graphs, algebraic expressions, and even binary code.

Importantly, the exploration has not been confined to idealized cases. We have reflected on what happens when general position assumptions break down, and how the loss of structure reveals hidden regularities. Tools like difference tables, recursive reasoning, and bijections to binary words all serve not just to count, but to understand the logic behind the count.

In the classroom, such problems invite students into a world where mathematical thinking is creative, visual, and deeply connected. As educators, we can use them not just to teach counting, but to foster spatial reasoning, abstraction, and an appreciation for the elegance of mathematics as a language for structure — whether in two, three, or even higher dimensions.

Acknowledgements

I would like to extend my sincere thanks to my supervisor, Per Alexandersson, for his insightful comments, generous support, and contagious enthusiasm throughout this work. His guidance has sharpened both the structure and substance of this thesis.

I am also grateful to my fellow students for valuable feedback, proof-reading, and conversations that helped clarify many ideas along the way.

Finally, I wish to acknowledge the long mathematical tradition that this thesis builds upon. From Pascal and Euler to Moser and beyond, it is a privilege to explore — and attempt to extend — the ideas of those giants on whose shoulders we now stand.

References

[1] Gerald L. Alexanderson and John E. Wetzel. "Simple Partitions of Space". In: *Mathematics Magazine* 51.4 (1978), pp. 220–225. ISSN: 0025570X, 19300980. URL: http://www.jstor.org/stable/2689466 (visited on 07/26/2025).

[2] Per Alexandersson and Olivia Nabawanda. *Peaks are preserved under run-sorting*. 2021. arXiv: 2104.04220 [math.CO]. URL: https://arxiv.org/abs/2104.04220.

[3] Brother U. Alfred. "A Mathematician's Progress". In: *The Mathematics Teacher* 59.8 (1966), pp. 722–727. DOI: 10.5951/MT.59.8.0722. URL: https://pubs.nctm.org/view/journals/mt/59/8/article-p722.xml.

[4] Robert B. Banks. "Slicing Things Like Pizzas and Watermelons". In: *Slicing Pizzas, Racing Turtles, and Further Adventures in Applied Mathematics*. Princeton University Press, 1999, pp. 23–33. ISBN: 9780691154992. URL: http://www.jstor.org/stable/j.ctt7sbwv.7 (visited on 07/11/2025).

[5] R. C. Buck. "Partition of Space". In: *The American Mathematical Monthly* 50.9 (1943), pp. 541–544. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2303424 (visited on 07/11/2025).

[6] Ralph P. Grimaldi. *Discrete and combinatorial mathematics :* 5th ed. Boston : Pearson Addison Wesley, c2004.

[7] Jorik and Noud Thijssen. *Sequence A144841 in the On-Line Encyclopedia of Integer Sequences (n.d.)* https://oeis.org/A000125. Accessed on July 23 2025.

[8] Leo Moser and W. Bruce Ross. "Mathematical Miscellany". In: *Mathematics Magazine* 23.2 (1949), pp. 109–114. ISSN: 0025570X, 19300980. URL: http://www.jstor.org/stable/3219224 (visited on 07/11/2025).

[9] G. Polya. "INDUCTION IN SOLID GEOMETRY". In: *Mathematics and Plausible Reasoning, Volume 1: Induction and Analogy in Mathematics*. Princeton University Press, 1954, pp. 35–58. ISBN: 9780691080055. URL: http://www.jstor.org/stable/j.ctv14164db.7 (visited on 07/11/2025).

[10] N. J. A. Sloane. *Sequence A000125 in the On-Line Encyclopedia of Integer Sequences (n.d.)* https://oeis.org/A000125. Accessed on July 1 2025.

[11] J. Steiner. “Einige Gesetze über die Theilung der Ebene und des Raumes.” ger. In: *Journal für die reine und angewandte Mathematik* 1 (1826), pp. 349–364. URL: http://eudml.org/doc/183037.

[12] John E. Wetzel. “On the Division of the Plane by Lines”. In: *The American Mathematical Monthly* 85.8 (1978), pp. 647–656. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2320333 (visited on 07/11/2025).