SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Symmetric Polynomials

av

Sebastian Sundblom

2025 - No L4

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Symmetric Polynomials

Sebastian Sundblom

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Alexander Berglund

2025

Abstract

The theory of symmetric polynomials is one that has applications in several branches of mathematics, and in this paper we will begin to see what a symmetric polynomial means and how they are built by first looking at the monomial, elementary and power sum symmetric polynomials before moving onto proving Newton's identities and the fundamental theorem of symmetric polynomials. We will also look at the transformation matrix from the elementary symmetric polynomials to the power sum symmetric polynomials and vice versa.

Sammanfattning

Teorin om symmetriska polynom har många applikationer för olika grenar av matematiken, och i denna uppsats så utforskar vi vad som menas med ett symmetrisk polynom och hur dessa är uppbyggda genom att först titta på monomial, elementära och power sum symmetriska polynomen innan vi sedan går in på beviset för Newtons identiteter och fundamentalsatsen för symmetriska polynom. I slutet så täcker vi transformationsmatrisen från elementär symmetriska polynomen till power sum polynomen och vice versa.

Contents

1 Introduction

The following paper is a study into the world of symmetric polynomials. These symmetric polynomials is something that I had never heard of before writing this paper, but that I now see in the mathematics that I work with on a regular basis. First, to get a basic understanding of what makes a polynomial symmetric, let us observe the polynomial

$$f(x, y, z) = x^2(y + z) + y^2(x + z) + z^2(x + y).$$

We can see that for any permutation of x, y, z we will receive an identical polynomial, just written in a different order, for example

$$f(y, x, z) = y^2(x + z) + x^2(y + z) + z^2(y + x)$$
$$f(z, x, y) = z^2(x + y) + x^2(z + y) + y^2(z + x).$$

In each permutation we can see that we receive the same polynomial, this is what makes a polynomial symmetric.

Now to explore an example of where we can find these symmetrical polynomials in the mathematics that we are already familiar with, let us observe the polynomial $f(x) = x^3 + ax^2 + bx + c$ for any coefficient $a, b, c \in \mathbb{C}$. As this is a polynomial of degree three we know that

I this has exactly three roots in the complex numbers

II it can be written as a product of its factors.

So let us see what happens if we rewrite the polynomial as a product of factors based on the roots x_1, x_2, x_3, and then expand it.

$$f(x) = x^3 + ax^2 + bx + c \tag{1}$$

$$f(x) = (x - x_1)(x - x_2)(x - x_3)$$

$$f(x) = (x^2 - xx_2 - xx_1 + x_1x_2)(x - x_3)$$

$$f(x) = x^3 - x^2x_3 - x^2x_2 + xx_2x_3 - x^2x_1 + xx_1x_3 + xx_1x_2 - x_1x_2x_3.$$

The given expression might at a first glance seem messy but let us reorganize it into something more comfortable.

$$f(x) = x^3 - x^2x_1 - x^2x_2 - x^2x_3 + xx_1x_2 + xx_1x_3 + xx_2x_3 - x_1x_2x_3$$

$$f(x) = x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3. \quad (2)$$

We can now see in (2) that the polynomial is written in such a way that we can make an easy comparison to the polynomial that we wrote in (1). The coefficients a, b and c can be explained in terms of the polynomial's roots x_1, x_2, x_3.

$$\begin{cases} -a = x_1 + x_2 + x_3 \\ b = x_1x_2 + x_1x_3 + x_2x_3 \\ -c = x_1x_2x_3 \end{cases}.$$

Looking at each of these it is clear that each permutation of the variables results in an identical polynomial, hence these are symmetrical. These three polynomials are also very special in the field of symmetric polynomials as these are the elementary symmetric polynomials and this paper will prove that every symmetric polynomial can be written as a unique combination of these.

The field of symmetric polynomials is vast. It has applications in several different branches of mathematics. In this introduction we made a connection between the roots of a polynomial of degree 3 and the elementary symmetrical polynomials. This connection can be broadened to show this connection for a polynomial of any degree d. The theory of symmetric polynomials is also the entryway to Galois theory, has application in enumerative combinatorics, Lie algebra, algebraic geometry and group theory. [4][2][3][1] The history of this field reaches back hundreds of years and include discoveries made by one of the most brilliant people to have ever walked this earth, Sir Isaac Newton. He made the discovery of what we call Newton's identities around 1666. Though to say that he discovered these is misnomer as Albert Girard had come to the same finding earlier, in 1629, unbeknownst to Newton. For this reason Newton's identities is also known as the Girard-Newton formulae.[2]

We will begin our journey by looking at some of the different symmetric polynomials that we have at our disposal. First we will get a broader overlook of what it means that a polynomial is symmetric and then move on to three specific types of symmetric polynomials. For this we will be working with the book written by Stanley. [4] The first one we will be looking at is called the monomial symmetric polynomials, denoted as m_λ, we will prove that this constitutes a basis for the symmetric polynomial ring. The second is the power sum symmetric polynomials, denoted as p_λ. At last we have the elementary symmetric polynomials, denoted as e_λ. After this we will move onto proving Newton's identities by working through the paper written by Zeilberger. [5] The proof in this paper follows Zeilbergers proof, but is more comprehensive to help the reader understand it and also includes an example to help the reader follow along, as well as expand the proof to show that this relation between the power sum symmetric polynomials and the elementary symmetric polynomials works in both directions. We will then prove the fundamental theorem of symmetric polynomials, following the proof written by Lang. [3] The reader will once again be given an example to more easily follow the proof. With both these proofs completed we can then move onto proving that the elementary symmetric polynomials and power sum symmetric polynomials are bases for the symmetric polynomial ring. We finish the paper with the transformation matrices from the elementary symmetric polynomials to the power sum symmetric polynomials and vice versa.

2 Prerequisite knowledge

To fully understand this paper about symmetric polynomials there are a few concepts that I would like to highlight as important.

2.1 Rings

With symmetric polynomials we will be interested in looking at the ring of all symmetric polynomials and therefore to ensure that the reader can follow the notation used in this paper we will define what these. We will follow the axioms given by Dummit. [1]

Definition 1. *A ring $R[x_1, ..., x_n]$ is a set with the operations addition and multiplication that satisfies the three axioms called ring axioms.*

I *(R,+) is an abelian group.*

- *(a)* $(a + b) + c = a + (b + c)$ *for any* $a, b, c \in R$ *(Associative).*
- *(b)* $a + b = b + a$ *for any* $a, b \in R$ *(Commutative).*
- *(c) R has an element 0 such that* $a + 0 = a$ *for any* $a \in R$ *(Additive identity).*
- *(d) For each* $a \in R$ *there exists an element such that* $a + (-a) = 0$ *(Additive inverse).*

II *R is a monoid.*

- *(a)* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *for any* $a, b, c \in R$ *(Associative).*
- *(b) R has an element 1 such that* $a \cdot 1 = a$ *for any* $a \in R$ *(Multiplicative identity).*

III *Multiplication is distributive with respect to addition.*

- *(a)* $a \cdot (b + c) = a \cdot b + a \cdot c$ *for any* $a, b, c \in R$ *(Left sided distribution).*
- *(b)* $(b + c) \cdot a = b \cdot a + c \cdot a$ *for any* $a, b, c \in R$ *(Right sided distribution).*

We will also define what a subring is.

Definition 2. *A subring is a subset of R which is itself a ring as it is closed under addition and multiplication, and shares the same multiplicative identity as R.*

Let us apply these axioms to polynomials to see if they form a ring. It should be clear that if the polynomials form a ring, then the symmetric polynomials forms a subset of this ring which then has the potential to be its own ring, a subring.

Proposition 1. *Polynomial functions with complex coefficients form a ring, the polynomial ring.*

Proof. Given three polynomials f, g and h, where $f, g, h \in R$ it should be easy to see that it is an abelian group as it is associative and commutative under addition. R also has an additive identity and additive inverse. It should also be clear that R is a monoid as it is associative under multiplication and has a multiplicative identity. Multiplication is also distributive with respect to addition, and with that we know that polynomials form a ring. □

With that we can move on to our goal, proving that the symmetric polynomials forms a subring.

Proposition 2. *The symmetric polynomials form a subring to the ring of polynomials.*

Proof. Given two symmetric polynomials f and g of degree d, we know right away that these are a subset of the polynomials. It is also clear that if we add two symmetric polynomials, the result is another symmetric polynomial of degree d. It should also be easy to see that if we using multiplication, the result is yet another symmetric polynomial of degree $2d$. This subset also shares the same multiplicative identity as the ring R, which is the last requirement. With that we have proved that the subset of symmetric polynomials forms a subring to the ring of polynomials. We will denote this subring as Λ_R^d, where d is the degree of the symmetric polynomial. □

2.2 Partitions and orders

When it comes to symmetrical polynomials we will naturally run into partitions. In this paper we will be using Stanley's[4] description of partitions and orders to ensure that the reader can follow along the notation that is used.

Definition 3. *A partition λ of d where $d \in \mathbb{N}$, also called an integer partition, is a sequence of integers $(\lambda_1, ..., \lambda_k) \in \mathbb{N}^k$ that satisfies the conditions*

I $\lambda_1 \geq \cdots \geq \lambda_k$,

II $\sum \lambda_i = d$.

We will consider any $\lambda_i = 0$ as irrelevant, then we are left with the infinite sequence $(\lambda_1, ..., \lambda_k, 0, 0, ...)$. We will use the notation $Par(d)$ to describe the set of all partitions λ of nd, with Par(0) being an empty partition with the sequence (0,0,...). We define this as

$$Par := \bigcup_{n \geq 0} Par(d).$$

Worth to note is that there is a shorthand for writing sequences that is used, so 11111 would be equal to the sequence (1,1,1,1,1,0,0,...). This would mean that the results of the $Par(d)$ would be written as

$$Par(1) = \{1\}$$

$$Par(2) = \{2, 11\}$$

$$Par(3) = \{3, 21, 111\}$$

and so on. If $\lambda \in Par(d)$ then this will be denoted as $\lambda \vdash d$, or $|\lambda| = d$. Each sequence will also have a length where the number of elements where $\lambda_i \neq 0$ is the length of λ, denoted as $\ell(\lambda)$.

Definition 4. *A weak composition is a vector of non-negative integers with the sum of each entry equaling d.*

A partition λ can be described as $\lambda = 1^{m_1}2^{m_2}...k^{m_k}$ where m_i describes how many parts of λ is equal to i. So if we have a partition 44333221 this would be written as $1^1 2^2 3^3 4^2$.

In this paper we will be looking at different partial orderings on partitions. The first one is called "dominance order" and is denoted with $\leq$. Given that we have a μ and λ such that $|\mu| = |\lambda|$, then we define $\mu \leq \lambda$ if for all $i \geq 1$ we have that

$$\mu_1 + ... + \mu_i \leq \lambda_1 + ... + \lambda_i.$$

This is defined only on $Par(d)$ for $d \in \mathbb{N}$.

The second and last order that we will define is any linear order that is compatible with the dominance order, in this paper we will use both reverse lexicographic order and lexicographic order. For reverse lexicographic order, denoted $\overset{R}{\leq}$, given $|\mu| = |\lambda|$ we define $\mu \overset{R}{\leq} \lambda$ if $\mu = \lambda$, or for some i

$$\mu_1 = \lambda_1, \quad \cdots, \quad \mu_i = \lambda_i, \quad \mu_{i+1} < \lambda_{i+1}.$$

So what does this mean? This mean that we have a way of sorting the order of our partitions. If we sort $Par(4)$ with reverse lexicographic order we get

$$4 \overset{R}{>} 31 \overset{R}{>} 22 \overset{R}{>} 211 \overset{R}{>} 1111.$$

If we were to look at the same set of partitions in lexicographic order, we would get the exact opposite direction.

3 Symmetric polynomials

We will now start to dive into the world of symmetric polynomials. First we will go through what these symmetric polynomials are and what shapes they come in and discover the connections between them, look at their building blocks in the power sums (p_λ), elementary symmetric polynomials (e_λ) and monomial symmetric polynomials (m_λ). This paper will prove that these are bases for the ring of Λ_R^d by providing proofs for both Newton's identities and the fundamental theorem of symmetric polynomials.

3.1 General about symmetric polynomials

So how do we define our symmetric polynomials? There are several types of symmetric polynomials, we will begin by stating a general definition of a symmetric polynomial following Stanley. [4]

Definition 5. *Let $x = (x_1, x_2, ..., x_n)$ be a set of indeterminates where $n \in \mathbb{N}$. A homogeneous symmetric polynomial of degree d over a commutative ring R is written as the summation*

$$p(x) = \sum_{\alpha} c_\alpha x^\alpha, \tag{3}$$

that fulfills

I α ranges over all the weak compositions $\alpha = (\alpha_1, \alpha_2, ...)$ of d where $d \in \mathbb{N}$,

II $c_\alpha \in R$,

III x^α is the monomial $x_1^{\alpha_1} x_2^{\alpha_2} ...$,

IV $p(x_{w(1)}, x_{w(2)}, ..., x_{w(n)}) = p(x_1, x_2, ..., x_n)$ for every permutation w of the positive integers.

We will denote the set of all homogeneous symmetric polynomials of degree d over the ring R as Λ_R^d.

For many of the proofs in this paper, it would be beneficial to prove that λ_R^d is a vector space. To provide a proof for that we will use what Stanley writes on page 286-287. [4] I have not written this proof myself, but rely on Stanley.

Proposition 3. *Λ_R^d is a vector space.*

Proof. It should be clear that if $f, g \in \Lambda_R^d$, meaning that f and g are symmetric polynomials of degree d and $a, b \in R$ then it naturally follows that $af + bg \in \Lambda_R^d$. This means that if you use scalar multiplication with a symmetric polynomial, the resulting polynomial will be symmetric as well. It also means that if you add two symmetric polynomials of the same degree d, the resulting polynomial will also be symmetric of degree d. With this we can state that Λ_R^d is an R-module and if $R = \mathbb{Q}$ we get that $\Lambda_\mathbb{Q}^d$ is a $\mathbb{Q}$-vector space. Given $f \in \Lambda_\mathbb{Q}^m$ and $g \in \Lambda_{\mathbb{Q}^\ltimes}$, then it is easy to see that $fg \in \Lambda_\mathbb{Q}^{m+n}$. So if we define

$$\Lambda_{\mathbb{Q}} = \Lambda^0_{\mathbb{Q}} \oplus \Lambda^1_{\mathbb{Q}} \oplus \cdots$$

then our $\Lambda_{\mathbb{Q}}$ has structure of a $\mathbb{Q}$-algebra and is therefore a ring with operations compatible with vector space structure. For simplicity we will in the future denote $\Lambda^d_{\mathbb{Q}}$ as Λ^d instead.

□

To put this to the test we will take the example where $d = 3$ for the two functions f and g. If $f(x, y, z) = x^2yz + xy^2z + xyz^2$ then $2f(x, y, z) = 2x^2yz + 2xy^2z + 2xyz^2$ which we can clearly see is still symmetric as every permutation of x, y, z will result in an identical polynomial. Now to handle the case where we add a symmetric polynomial of degree 3 to f. Let g be $g(x, y, z) = x^2y^2 + x^2z^2 + y^2z^2$ then

$$f(x, y, z) + g(x, y, z) = x^2yz + xy^2z + xyz^2 + x^2y^2 + x^2z^2 + y^2z^2$$

which is clearly a homogeneous symmetric polynomial as well. Lastly, if we were to take the multiplication we get

$$f(x, y, z) \cdot g(x, y, z) = 2xyz(x^2y^2 + x^2z^2 + y^2z^2)(x + y + z),$$

which we also can see is a homogeneous symmetric polynomial of degree 6.

3.2 Monomial symmetric polynomials

We will begin our journey into the different forms of symmetric polynomials by introducing the monomial symmetric polynomials. We will follow the definition given by Stanley. [4]

Definition 6. *Let $\lambda = (\lambda_1, \lambda_2, ...) \vdash d$, we then define the monomial symmetric polynomial $m_\lambda(x) \in \Lambda^d$ by the summation*

$$m_\lambda = \sum_\alpha x^\alpha, \tag{4}$$

where the sum covers all the distinct permutations of $\alpha = (\alpha_1, \alpha_2, ...)$ from our vector λ.

Now what does this mean? For example

$$\begin{aligned} m_0 &= 1, \\ m_1 &= \sum_i x_i, \\ m_2 &= \sum_i x_i^2, \\ m_{111} &= \sum_{i<j<k} x_i x_j x_k. \end{aligned}$$

We can see that the notation in m_λ represents which permutations of the monomials that we are looking for. So in the case of m_{111} we are looking for all the permutations where we have the monomial of three variables where all are of degree one. So to put this into a case. Given that we have a monomial symmetric polynomial of three variables x, y, z then our m_{11} would be all permutations of x, y, z where they all have degree one and the combination of two variables. This would give us

$$m_{11} = xy + xz + yz.$$

To be able to say that the monomial symmetric polynomials make a building block to the symmetric polynomials we have to prove that it constitutes a basis of Λ^d, so that will be our next objective.

Proposition 4. *$\{m_\lambda : \lambda \in Par(d)\}$ is a basis for Λ^d.*

Proof. To prove that such a proposition is true we have to fulfill the normal criteria of what a basis for a vector space is, that it spans the entire vector space and that it is linearly independent. To formalize it we write this as

I For all $f \in \Lambda$ there exists a c_λ such that $f = \sum_{\lambda \vdash d} c_\lambda m_\lambda$. This means that there is a linear combination of monomials that spans Λ.

II $\sum_{\lambda \vdash d} c_\lambda m_\lambda = 0$ must mean that for all $\lambda \vdash d$ we get that $c_\lambda = 0$. This would give us linear independence.

Let f be the homogeneous symmetric polynomial given by (3)

$$f = \sum_{\alpha} c_\alpha x^\alpha.$$

So we set out to prove both of these criteria for f of n variables and of degree d. If $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ then $\sum_{i=1}^{n} \alpha_i = d$. Since this is a symmetric polynomial we get that $c_\alpha = c_{\lambda(\alpha)}$. To see this more clearly we take the example of the symmetric polynomial in two variables, $g(x, y) = c_{20}x^2 + c_{11}xy + c_{02}y^2$, clearly for this to be symmetric $c_{20} = c_{02}$. So with this we get that

$$f = \sum_{\lambda \vdash d} c_\lambda \sum_{\lambda(\alpha) = \lambda} x^\alpha. \tag{5}$$

We can see that the second sum in (5) is something that we recognize from (4) so we can rewrite this as

$$f = \sum_{\lambda \vdash d} c_\lambda m_\lambda$$

and with that we have verified the first criterion, the linear combination and we can move on to proving linear independence.

To prove linear independence we have to prove that if $\sum_{\lambda \vdash d} c_\lambda m_\lambda = 0$, then $c_\lambda = 0$. Let us begin by stating that if there is a $\mu \in Par(d)$ then the monomial

x^μ occurs only in the term m_μ, and its coefficient has to be c_μ. With this we know that $c_\mu = 0$. This means we have found that it is linearly independent and therefore makes a basis for Λ. □

3.3 Power sums

We will introduce another form of symmetric polynomials, the power sum symmetric polynomials. We follow the definition given by Stanley. [4]

Definition 7. *We define the power sum symmetric polynomials p_λ by the summation*

$$p_d = m_d = \sum_i x_i^d, \quad n \geq 1, \tag{6}$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \quad \textit{if} \quad \lambda = (\lambda_1, \lambda_2, ...) \quad \textit{and} \quad \lambda \in Par.$$

We can see that the power sum symmetric polynomial is defined by the summation of the monomial with one variable of degree d. So to give a case to put this into perspective, if we have a power sum of 3 variables x, y, z then we would get

$$\begin{aligned} p_0 &= m_0 = 1 \\ p_1 &= x + y + z \\ p_2 &= x^2 + y^2 + z^2 \end{aligned}$$

and so on. I would invite the reader to consider for a moment the thought that these rather simple polynomials could form a building block for the symmetric polynomials. Let us stay in three variables for simplicity and consider the case $p_1 2$. What we would get is

$$\begin{aligned} p_{12} &= x^3 + x^2y + x^2z + xy^2 + y^3 + y^2z + x^2z + y^2z + z^3, \\ &= x^3 + y^3 + z^3 + x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2. \end{aligned}$$

We can see that the result is $p_{12} = p_3 + m_{21}$ which is an interesting result that we will build upon in the next subsection.

3.4 Elementary symmetric polynomials

We have now been introduced to the monomial and power sum symmetric polynomials. We have also seen that m_λ is a solid building block of the symmetric polynomials as they are a basis for Λ. Now we will introduce the elementary symmetric polynomials. We will follow the definition given by Stanley. [4]

Definition 8. *We define the elementary symmetric polynomials e_λ by the summation*

$$e_d = m_{1^d} = \sum_{i_1 < \cdots < i_d} x_{i_1} \cdots x_{i_d} \tag{7}$$

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \quad \textit{if} \quad \lambda = (\lambda_1, \lambda_2, ...) \quad \textit{and} \quad \lambda \in Par.$$

We can see that the elementary symmetric polynomials is a subset of the monomial symmetric polynomials m_λ for $\lambda = (\lambda_1, \lambda_2, ...)$ where $\lambda_1 \leq 1$. In three variables this gives us

$$e_0 = m_0 = 1$$

$$e_1 = x + y + z$$

$$e_2 = xy + xz + yz$$

$$e_3 = xyz$$

$$e_4 = 0.$$

As our case is the case where $d = 3$, once we reach e_4 we simply have no permutations in λ that fulfills this, making this 0. I would again invite the reader to consider the possibility to use this new set of symmetric polynomials as a building block. If we consider the case e_{12} in three variables we would get

$$\begin{aligned} e_{12} &= x^2y + x^2z + xyz + xy^2 + xyz + y^2z + xyz + xz^2 + yz^2, \\ &= 3xyz + x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 \end{aligned} \tag{8}$$

Let us do the same thing as in the last case and take a moment to observe the result. We then see that $e_{12} = 3e_3 + m_{21}$ which is a very interesting finding. So if we use this result with the result at the end of §3.3 we get

$$\begin{cases} p_{12} = p_3 + m_{21} \\ e_{12} = 3e_3 + m_{21} \end{cases},$$

$$\begin{cases} p_{12} - p_3 = m_{21} \\ e_{12} - 3e_3 = m_{21} \end{cases},$$

$$p_{12} - p_3 = e_{12} - 3e_3,$$

$$p_3 = -e_{12} + 3e_3 - p_{12}.$$

We can see that now we have an equation in terms of only power sums and elementary symmetric polynomials. We will in the next subsection look further into this relationship and develop this interesting result further.

3.5 Newton's identities

Now that we have explored different types of symmetric polynomials we will be moving our focus over to Newton's identities. This identity forms a connection between the power sums and the elementary symmetric polynomials in such a way that we can rewrite them in terms of each other. We will focus on rewriting the power sums in terms of the elementary symmetric polynomials. Let us first explore this relationship with a case and then move on to put it into terms of a concrete proof. If we begin by exploring how we would find p_2 in terms of the elementary symmetric polynomials, where would we start? Let us constrain ourselves to three variables x, y, z to make it a bit easier to follow. This would mean that our $p_2 = x^2 + y^2 + z^2$ so a natural first step would be to find a way to introduce a power of two with e_k. So let us start it off by taking

$$\begin{aligned} e_{11} &= (x+y+z)(x+y+z) \\ &= x^2 + xy + xz + yx + y^2 + yz + zx + zy + z^2 \\ &= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz. \end{aligned}$$

If we compare this result with our knowledge about elementary symmetric polynomials we see that the result that we got can be rewritten as

$$\begin{aligned} e_{11} &= p_2 + 2e_2 \\ p_2 &= e_{11} - 2e_2. \end{aligned}$$

So with this we have proved that for three variables, p_2 can be written in terms of the elementary symmetric polynomials. Let us take it a step further and take a look at p_3. It would feel natural to begin in a similar way to find $p_3 = x^3+y^3+z^3$ so we start off by doing just that and find

$$\begin{aligned} e_{111} &= (x+y+z)(x+y+z)(x+y+z) \\ &= (x^2 + xy + xz + yx + y^2 + yz + zx + zy + z^2)(x+y+z) \\ &= x^3 + y^3 + z^3 + 6xyz + 3(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2). \end{aligned} \tag{9}$$

We can see that we find the p_3 that we were looking for as well as $3e_3$ and a monomial that we recognize from (8). Using this relation we get that

$$\begin{aligned} e_{12} &= x^2y + x^2z + xyz + xy^2 + xyz + y^2z + xyz + xz^2 + yz^2 \\ e_{12} - 3e_3 &= x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2. \end{aligned}$$

With this we can simplify the monomial in (9) to find the solution to our problem and get that

$$\begin{aligned} e_{111} &= p_3 + 6e_3 + 3(e_{12} - 3e_3) \\ p_3 &= e_{111} - 3e_{12} + 3e_3. \end{aligned}$$

This proves that for at least three variables we can write the power sum symmetrical polynomials in terms of the elementary symmetric polynomials. It should be easy to see that we can expand the number of variables and produce the exact same result. Now that we have an understanding of what Newton's identities are about, we will move on to tackle the general case. This proof follows the work of Zeilberger[5] who has written a short and dense article on it. We will make it easier to understand as we break it down and also apply it to a specific case at the same time.

Proposition 5. *A combinatorial way to write Newton's identities is given by*

$$\sum_{r=0}^{k-1}(-1)^r e_r \cdot p_{k-r} + (-1)^k e_k k = 0 \tag{10}$$

where k is a positive integer. [5]

Proof. We will start by rewriting this in another way as we have defined the power sum symmetric polynomials in §3.3 and the elementary symmetric polynomials in §3.4. So let us handle each of the components to get that

$$e_k = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k},$$
$$e_r = \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r},$$
$$p_{k-r} = \sum_{j=1}^{n} x_j^{k-r},$$

where n and k are positive integers. We can now rewrite the equation as

$$\sum_{r}^{k-1}(-1)^r \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r} \sum_{j=1}^{n} x_j^{k-r} + (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} k = 0$$

where n and k are positive integers and $x_1, ..., x_n$ are commuting indeterminates. Now we have a grasp of what it is that we are working with and can move on to solving the problem. Zeilberger[5] gives us a set to consider. Consider the set $\mathscr{A} = \mathscr{A}(n, k)$ of pairs (A, j^l) for which

- I $A \subseteq \{1, ..., n\}$,
- II $j \in \{1, ..., n\}$,
- III $|A| + l = k$ where $|A|$ is the length of A,
- IV $l \ge 0$ and if $l = 0$ then $j \in A$.

So if $l \ge 0$ we get the relationship that $l = k - |A| \ge 0$ which means that $k \ge |A|$, and if $k = |A|$ then $j \in A$. Now what would this look like if we applied numbers to it? Consider the set $\mathscr{A}(3, 2)$ and we would get that our set A can be $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$. The set $\{1, 2, 3\}$ is left out since $k \ge |A|$ and for this set $|A| = 3$ while our $k = 2$. Our j has to be $1, 2, 3$. So to form these pairs we combine these

$$\begin{array}{ccc} (\emptyset, 1^2) & (\emptyset, 2^2) & (\emptyset, 3^2) \\ (\{1\}, 1^1) & (\{1\}, 2^1) & (\{1\}, 3^1) \\ (\{2\}, 1^1) & (\{2\}, 2^1) & (\{2\}, 3^1) \\ (\{3\}, 1^1) & (\{3\}, 2^1) & (\{3\}, 3^1) \end{array}$$

$$\begin{array}{cc} (\{1,2\},1^0) & (\{1,2\},2^0) \\ (\{1,3\},1^0) & (\{1,3\},3^0) \\ (\{2,3\},2^0) & (\{2,3\},3^0) \end{array}$$

We can note that for the second set of sets where $|A| = 2$ we need j to be a member of A and therefore pairs like $(\{1,2\},3^0)$ have to be excluded. Now if we define the weight of our pair (A, j^l) as $w(A, j^l) = (-1)^{|a|}(\prod_{a\in A} x_a)x_j^l$ given by Zeilberger[5] we can get the weight of each of our pairs to be

$$\begin{array}{ccc} w(\emptyset,1^2) = x_1^2 & w(\emptyset,2^2) = x_2^2 & w(\emptyset,3^2) = x_3^2 \\ w(\{1\},1^1) = -x_1^2 & w(\{1\},2^1) = -x_1x_2 & w(\{1\},3^1) = -x_1x_3 \\ w(\{2\},1^1) = -x_1x_2 & w(\{2\},2^1) = -x_2^2 & w(\{2\},3^1) = -x_2x_3 \\ w(\{3\},1^1) = -x_1x_3 & w(\{3\},2^1) = -x_2x_3 & w(\{3\},3^1) = -x_3^2 \end{array}$$

$$\begin{array}{cc} (\{1,2\},1^0) = x_1x_2 & (\{1,2\},2^0) = x_1x_2 \\ (\{1,3\},1^0) = x_1x_3 & (\{1,3\},3^0) = x_1x_3 \\ (\{2,3\},2^0) = x_2x_3 & (\{2,3\},3^0) = x_2x_3. \end{array}$$

Now if we were to take the sum of these weights we would get the l.h.s. of (10). Let us compare for our case to see what we get

$$\sum_{r=0}^{2-1}(-1)^r e_r \cdot p_{2-r} + (-1)^2 e_2 2 = p_2 - e_1p_1 + 2e_2$$

$$\begin{aligned} \sum_{(A,j^l)\in\mathscr{A}(3,2)} w(A,j^l) = x_1^2+x_2^2+x_3^2-(x_1^2+x_2^2+x_3^2{+}2(x_1x_2+x_1x_3+x_2x_3)) \\ +2(x_1x_2+x_1x_3+x_2x_3) \\ = p_2 - e_1p_1 + 2e_2. \end{aligned} \tag{11}$$

We can see that the both expressions are the same in this specific case. In (11) we can also more easily see that the result is zero, which is what we stated in the proposition. Now let us tackle the general case and prove that these expressions are equal.

$$\sum_{(A,j^l)\in\mathscr{A}(3,2)} w(A,j^l) = \sum_{(A,j^l)\in\mathscr{A}(3,2)} (-1)^{|A|}\cdot\prod_{a\in A} x_a \cdot x_j^l$$

Take a moment to consider what $\prod_{a\in A} x_a$ will produce. We can see that it will create a monomial of $|A|$ variables. So if we were to sum over all $A \subseteq \{1,...,n\}$ we would get the different elementary symmetric polynomials from e_0 to e_n so therefore

$$e_r = \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}\cdots x_{i_r} = \sum_{A\subseteq\{1,\ldots,n\}}\prod_{a\in A} x_a,$$

where $|A| = r$. So we should necessarily be able to rewrite e_k in the same manner, where $|A| = k$. So with this knowledge at hand we can rewrite the original equation as follows

$$\sum_{r}^{k-1}(-1)^r(\sum_{1\leq i_1<\cdots<i_r\leq n} x_{i_1}\cdots x_{i_r})\sum_{j=1}^{n} x_j^{k-r} + (-1)^k(\sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1}\cdots x_{i_k})k =$$

$$= \sum_{r}^{k-1}(-1)^r \sum_{\substack{A\subseteq\{1,\ldots,n\}\\|A|=r}} \prod_{a\in A} x_a \cdot \sum_{j=1}^{n} x_j^{k-r} + (-1)^k \sum_{\substack{A\subseteq\{1,\ldots,n\}\\|A|=k}} \prod_{a\in A} x_a \cdot k$$

$$= \sum_{\substack{A\subseteq\{1,\ldots,n\}\\|A|<k}} (-1)^{|A|} \prod_{a\in A} x_a \cdot \sum_{j=1}^{n} x_j^l$$

$$= \sum_{\substack{A\subseteq\{1,\ldots,n\}\\|A|<k}} \sum_{j=1}^{n}(-1)^{|A|} \prod_{a\in A} x_a \cdot x_j^l$$

$$= \sum_{\substack{A\subseteq\{1,\ldots,n\}\\|A|<k}} \sum_{j=1}^{n} w(A, j^l).$$

So with this we have proved that the sum of the weights of the elements of $\mathscr{A}$ is equal to the l.h.s. of (10). Now we move on to proving that it equals 0. Let us do so by using the mapping $T : \mathscr{A}(n,k) \to \mathscr{A}(n,k)$ [5] such that

$$T(A, j^l) = \begin{cases} (A/j, j^{l+1}), & j \in A \\ (A \cup j, j^{l-1}), & j \notin A \end{cases}$$

If we apply this to $\mathscr{A}(3,2)$ we would get

$$\begin{array}{ccc} T(\emptyset,1^2) = (\{1\},1^1) & T(\emptyset,2^2) = (\{2\},2^1) & T(\emptyset,3^2) = (\{3\},3^1) \\ T(\{1\},1^1) = (\emptyset,1^2) & T(\{1\},2^1) = (\{1,2\},2^0) & T(\{1\},3^1) = (\{1,3\},3^0) \\ T(\{2\},1^1) = (\{1,2\},1^0) & T(\{2\},2^1) = (\emptyset,2^2) & T(\{2\},3^1) = (\{2,3\},3^0) \\ T(\{3\},1^1) = (\{1,3\},1^0) & T(\{3\},2^1) = (\{2,3\},2^0) & T(\{3\},3^1) = (\emptyset,3^2) \end{array}$$

$$\begin{array}{cc} T(\{1,2\},1^0) = (\{2\},1^1) & T(\{1,2\},2^0) = (\{1\},2^1) \\ T(\{1,3\},1^0) = (\{3\},1^1) & T(\{1,3\},3^0) = (\{1\},3^1) \\ T(\{2,3\},2^0) = (\{3\},2^1) & T(\{2,3\},3^0) = (\{2\},3^1). \end{array}$$

Now if we were to apply the weight to each new transformed pairs and we get

$$\begin{array}{ccc} w(\{1\},1^1) = -x_1^2 & w(\{2\},2^1) = -x_2^2 & w(\{3\},3^1) = -x_3^2 \\ w(\emptyset,1^2) = x_1^2 & w(\{1,2\},2^0) = x_1x_2 & w(\{1,3\},3^0) = x_1x_3 \\ w(\{1,2\},1^0) = x_1x_2 & w(\emptyset,2^2) = x_2^2 & w(\{2,3\},3^0) = x_2x_3 \\ w(\{1,3\},1^0) = x_1x_3 & w(\{2,3\},2^0) = x_2x_3 & w(\emptyset,3^2) = x_3^2 \end{array}$$

$$\begin{array}{ll} w(\{2\},1^1) = -x_1x_2 & w(\{1\},2^1) = -x_1x_2 \\ w(\{3\},1^1) = -x_1x_3 & w(\{1\},3^1) = -x_1x_3 \\ w(\{3\},2^1) = -x_2x_3 & w(\{2\},3^1) = -x_2x_3. \end{array}$$

This result is similar to the previous result where we applied the weight of each pair. In fact the only difference is that for each result the sign flipped, meaning that $\sum_{(A,j^l)\in\mathscr{A}(3,2)} T(w(A,j^l)) = \sum_{(A,j^l)\in\mathscr{A}(3,2)} -w(A,j^l)$. Let us move on to prove the general case.

Proposition 6. $w(T(A,j^l)) = -w(A,j^l)$

Proof. We know that $w(A,j^l) = (-1)^{|A|}\prod_{a\in A} x_a x_j^l$ and we have to consider two cases. Case one being $j \in A$ and case two being $j \notin A$, so let us start with case one and we get that

$$\begin{aligned} w(T(A,j^l)) &= w(A/\{j\}, j^{l+1}) \\ &= (-1)^{|A|-1} \prod_{a\in A/\{j\}} x_a x_j^{l+1} \\ &= (-1)^{-1}\cdot(-1)^{|A|} \prod_{a\in A/j} x_a x_j x_j^l \\ &= -(-1)^{|A|} \prod_{a\in A} x_a x_j^l \\ &= -w(A,j^l). \end{aligned}$$

With this we have proved case one and can move onto case two

$$\begin{aligned} w(T(A,j^l)) &= w(A\cup\{j\}, j^{l-1}) \\ &= (-1)^{|A|+1} \prod_{a\in A\cup\{j\}} x_a x_j x_j^{l-1} \\ &= -(-1)^{|A|} \prod_{a\in A\cup} x_a x_j^l \\ &= -w(A,j^l). \end{aligned}$$

This proves that both cases fulfills the proposition and the proof is done. □

From here it should be fairly easy to see that the mapping $T : \mathscr{A}(n,k) \to \mathscr{A}(n,k)$ is an involution. Consider the case that $j \in A$ then in the first transformation j is removed from A and then in the next transformation it is added right back. This holds true for the case where $j \notin A$ as well, where j is first added to A and then removed once again. We know that any involution is also a bijection which is key to solving this problem. We know that

$$\sum_{(A,j^l)\in\mathscr{A}(n,k)} (w(A,j^l) + w(T(A,j^l))) = 0$$
$$\sum_{(A,j^l)\in\mathscr{A}(n,k)} w(A,j^l) + \sum_{(A,j^l)\in\mathscr{A}(n,k)} w(T(A,j^l)) = 0. \tag{12}$$

With the key observation that this is a bijection, we know that

$$\sum_{(A,j^l)\in\mathscr{A}(n,k)} w(A,j^l) = \sum_{(A,j^l)\in\mathscr{A}(n,k)} w(T(A,j^l)).$$

This means that we can rewrite (12) as

$$\sum_{(A,j^l)\in\mathscr{A}(n,k)} w(A,j^l) + \sum_{(A,j^l)\in\mathscr{A}(n,k)} w(A,j^l) = 0$$
$$2 \cdot \sum_{(A,j^l)\in\mathscr{A}(n,k)} w(A,j^l) = 0$$

and with that we have proved that the l.h.s. of (10) is zero and therefore the proposition holds true.

□

Now with our proof of Newton's identities done we know that there exists a relationship between the power sum symmetric polynomials and the elementary symmetric polynomials in the form of a formula that gives us an expression of only p_λ and e_λ. We will move on by providing proof that this formula lets us write p_λ in terms of e_λ and vice versa.

Proposition 7. *Newton's identities work in both directions and can be used to write the power sum symmetric polynomials, p_λ, in terms of elementary symmetric polynomials, e_λ, and vice versa.*

Proof. Most of the work to prove this is already done as we have confirmed the formula stated in (10), we will use this to prove this relationship. We will only solve it for writing p_λ in terms of e_λ as the proof for the reverse relationship is analogous. We start by calculating $k = 1$ and get

$$p_1 - e_1 = 0$$
$$p_1 = e_1.$$

We move on to $k = 2$ and get

$$
\begin{aligned}
p_2 - e_1 p_1 + 2e_2 =& 0 \\
p_2 =& -2e_2 + e_1 p_1 \\
p_2 =& -2e_2 + e_1^2
\end{aligned}
$$

By using induction on k we can prove the relationship. Having proved both $k = 1$ and $k = 2$ and confirmed that it holds true we move on by assuming that it holds true for any $k = d$ and get that

$$\sum_{r=0}^{d-1} (-1)^r e_r \cdot p_{d-r} + (-1)^d e_d d = 0.$$

We make a note of two things

I $\sum_{r=0}^{d-1} (-1)^r e_r \cdot p_{d-r}$ will always provide us with a power sum symmetric polynomial of degree d since the first iteration the elementary symmetric polynomial will equal 0,

II $\sum_{r=0}^{d-1} (-1)^r e_r \cdot p_{d-r}$ any further iteration of this sum gives us a power sum of a lower degree than d, which we can rewrite in terms of the elementary symmetric polynomials,

III $(-1)^d e_d d$ will always provide us with an elementary symmetric polynomial of degree d.

With this we have finished our proof and now know that each power sum symmetric polynomial can be written in terms of only elementary symmetric polynomials, and we also consider the reverse relationship proved. □

3.6 Fundamental theorem of symmetric polynomials

We have arrived at the main theorem of this paper, the fundamental theorem of symmetric polynomials, which states that any symmetric polynomial can be written in terms of the elementary symmetric polynomials. In §3.5 we proved that any elementary symmetric polynomial can be written in terms of the power sum symmetric polynomials, and therefore we also know that any symmetric polynomial can be written in terms of the power sum symmetric polynomials once we have proved the fundamental theorem of symmetric polynomials.

We will be working with the proof written by Lang in the book Undergraduate Algebra, page 159-160. [3] Unlike the previous example where we worked the proof and our case side by side we will this time work through the proof and then complement it with a specific case after the proof is done.

The monomial of n variables, $X_1^{k_1} \cdots X_n^{k_n}$, has its weight defined as $k_1 + ... + nk_n$ and the weight of the polynomial $g(X_1, ..., X_n)$ is then defined as the maximum of the weights of monomials in g.

Theorem 1. *Let $f(x) \in R[x_1, ..., x_n]$ be a symmetric polynomial of degree d. Then there exists a polynomial $g(X_1, ...X_n)$ of a weight $\leq d$ such that*

$$f(x) = g(e_1, ..., e_n). \tag{13}$$

Proof. We begin by using induction on n. It is easy to see that the solution is trivial for $n = 1$ since $e_1 = x_1$. We assume that our theorem is proved for $n-1$ variables and move on. If we form a polynomial $p(x)$ of n variables such that

$$\begin{aligned} p(x) =&(x - x_1)(x - x_2) \cdots (x - x_n), \\ p(x) =&x^n - (x_1 + x_2 + ... + x_n)x^{n-1} + ... + (-1)^n x_1 x_2 \cdots x_n, \end{aligned}$$

we get a polynomial where the coefficients of x are all written as our symmetric polynomials e_1 to e_n. If we substitute $x_n = 0$ in $p(x)$ we get

$$p(x) = x^n - (x_1 + x_2 + ... + x_{n-1})x^{n-1} + \cdots + (-1)^{n-1} \cdot x_1 x_2 \cdots x_{n-1} x.$$

We can see that each of the coefficients in x^{n-1} to x is exactly the elementary symmetric polynomials from e_1 to e_{n-1} written in terms of $x_1, x_2, ..., x_{n-1}$. So we do not have the elementary symmetric polynomials of n variables, but $n-1$ variables, and will write these as $(e_1)_0, ..., (e_{n-1})_0$. Let us rewrite it as

$$p(x) = x^n - (e_1)_0 x^{n-1} + \cdots + (-1)^{n-1}(e_{n-1})_0.$$

If we use induction over the degree d, then the case $d = 0$ is trivial. We assume that our $d > 0$ and assume that this is proved for all polynomials of degree $< d$. With this we know that there has to be a polynomial $g_1(X_1, ...X_{n-1})$ with a weight $\leq d$ such that

$$f(x_1, ..., x_{n-1}, 0) = g_1((e_1)_0, ..., (e_{n-1})_0) \tag{14}$$

We can see that $g_1(e_1, ..., e_{n-1})$ has degree $\leq d$ in our variables $(x_1, x_2, ..., x_n)$, so if we define a new symmetric polynomial of $x_1, ..., x_n$ as

$$f_1(x_1, ..., x_n) := f(x_1, ..., x_n) - g_1(e_1, ..., e_{n-1}). \tag{15}$$

We know that the resulting polynomial f_1 has a degree $\leq d$ and is symmetric as well. If we apply the result that we got in (14) we get

$$\begin{aligned} f_1(x_1, ...x_{n-1}, 0) &= f(x_1, ...x_{n-1}, 0) - g_1((e_1)_0, ..., (e_{n-1})_0) \\ &= 0. \end{aligned}$$

We now know that our g_1 removes all the terms in $f(x_1, ..., x_n)$ that do not include our x_n, so this gives us that our f_1 must contain the factor x_n, and since it is symmetric it will also therefor contain e_n as a factor. This must mean that there is another symmetric polynomial $f_2(x_1, ..., x_n)$, such that

$$f_1 = e_n f_2(x_1, ..., x_n). \tag{16}$$

It is clear that our f_2 has to be symmetric as well and its degree be $\leq d-n < d$. By induction, there must exist a polynomial g_2 in n variables such that

$$f_2(x_1, ..., x_n) = g_2(e_1, ..., e_n). \tag{17}$$

Let us use the results given to us in (15), (16) and (17) and we get that

$$\begin{aligned}
f_1(x_1, ..., x_n) &= f(x_1, ..., x_n) - g_1(e_1, ..., e_{n-1}) \\
f(x_1, ..., x_n) &= g_1(e_1, ..., e_{n-1}) + f_1(x_1, ..., x_n) \\
&= g_1(e_1, ..., e_{n-1}) + e_n f_2(x_1, ..., x_n) \\
&= g_1(e_1, ..., e_{n-1}) + e_n g_2(e_1, ..., e_n).
\end{aligned}$$

With this we have proved that for each symmetric polynomial $f(x_1, ..., x_n) \in R$ there exists a $g(e_1, ...e_n)$ meaning that each symmetric polynomial can be written in terms of elementary symmetric polynomials and through our proof at the end of §3.5 that they also can be written in terms of the power sum symmetric polynomials. □

Now, with the general case proven, we can move onto a specific case and apply the steps formulated in the proof. To make it more manageable, we will limit ourselves to three variables, x_1, x_2, x_3. Let our symmetric polynomial be $f(x_1, x_2, x_3) = x_1^2(x_2 + x_3) + x_2^2(x_1 + x_3) + x_3^2(x_1 + x_2)$. The first step is to substitute $x_3 = 0$ to get

$$\begin{aligned}
f(x_1, x_2, 0) &= x_1^2 x_2 + x_2^2 x_1 \\
&= x_1 x_2 (x_1 + x_2) \\
&= (e_{12})_0 \\
&= g_1((e_1)_0, (e_2)_0)
\end{aligned}$$

where g_1 is just a polynomial of elementary symmetric polynomials of two variables. We then use the definition in (15) and we know that

$$\begin{aligned}
f_1(x_1, x_2, x_3) &= f(x_1, x_2, x_3) - g_1(e_1, e_2) \\
&= x_1^2(x_2 + x_3) + x_2^2(x_1 + x_3) + x_3^2(x_1 + x_2) - e_{12} \\
&= x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 - e_{12}.
\end{aligned} \tag{18}$$

The monomial in (18) has been known to us since before from (8) and we can apply it to our case to rewrite it as

$$\begin{aligned} x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 - e_{12} &= e_{12} - 3e_3 - e_{12} \\ &= -3e_3. \end{aligned}$$

With this we can see that our solution for g_2 is trivial and has to be the -3 infront of the $e_n = e_3$. This gives us the solution for our symmetric polynomial $f(x_1, x_2, x_3)$. We know that our solution can be written as

$$\begin{aligned} f(x_1, x_2, x_3) &= g_1(e_1, e_2) + e_3g_2(e_1, e_2, e_3) \\ x_1^2(x_2 + x_3) + x_2^2(x_1 + x_3) + x_3^2(x_1 + x_2) &= e_{12} - 3e_3. \end{aligned}$$

To move on from here and apply the knowledge that we have won from §3.5 and §3.6 to prove that both elementary symmetric polynomials and power sum symmetric polynomials constitute bases for our Λ we have some work to do.

Proposition 8. *Any homogeneous symmetric polynomial f of degree d can be written in the terms of elementary symmetric polynomials and can therefore be written as*

$$f(x_1, ..., x_n) = g(e_1, ..., e_n) = \sum_{\lambda \vdash d} c_\lambda e_\lambda \tag{19}$$

where c_λ is a complex coefficient.

Proof. We already know what our g looks like as this is just a function of elementary symmetric polynomials. This means that we can write it as

$$g(e_1, ..., e_n) = \sum c_{\alpha_1,...,\alpha_n} e_1^{\alpha_1} \cdots e_n^{\alpha_n}.$$

We know a few things for certain, that

I $n\alpha_n + (n-1)\alpha_{n-1} + \cdots + \alpha_1 = d$

II $\lambda = (n_n^\alpha, ..., 1_1^\alpha) \vdash d$.

With this we know that we can write it as $\sum_{\lambda \vdash d} c_\lambda e_\lambda$ and our proof is done. □

Proposition 9. $\{e_\lambda : \lambda \in Par(d)\}$ *is a basis for* Λ^d.

Proof. We now know from the fundamental theorem of symmetric polynomials that we can write any symmetric polynomial $f(x_1, ..., x_n)$ in terms of the elementary symmetric polynomials. We also proved in proposition 8 that any symmetric polynomial could be written as stated in (19). So given that we know that

$$f(x_1, ..., x_n) = \sum_{\lambda \vdash d} c_\lambda e_\lambda. \tag{20}$$

It also proves that we have a set $E : \{e_\lambda : \lambda \in par(d)\}$ that spans Λ^d where E is a set of vectors in Λ^d. We know that $dim(\Lambda^d) =$ the number of partitions of d and therefore E has the size of $dim(\Lambda^d)$ and such E must be a basis. □

Proposition 10. *$\{p_\lambda : \lambda \in Par(d)\}$ is a basis for Λ^d.*

Proof. As we have previously stated we know that any symmetric polynomial $f(x_1, ..., x_n)$ can be written in terms of the elementary symmetric polynomials. We proved in proposition 8 that these can be written as $\sum_{\lambda \vdash d} c_\lambda e_\lambda$. We also know from proposition 7 that any elementary symmetric polynomial, e_λ can be written as a power sum symmetric polynomial, p_λ. If we combine these two bits of information we get a powerful combination and get that any symmetric polynomial $f(x_1, ..., x_n)$ can be written as

$$f(x_1, ..., x_n) = \sum_{\lambda \vdash d} c_\lambda p_\lambda. \tag{21}$$

It also proves that we have a set $P : \{p_\lambda : \lambda \in par(d)\}$ that spans Λ^d where P is a set of vectors in Λ^d. We know that $dim(\Lambda^d) =$ the number of partitions of d and therefore P has the size of $dim(\Lambda^d)$ and such P must be a basis. □

4 Transformation matrices

In the field of symmetric polynomials there are transformation matrices that lets us quickly transform a basis of all different partitions $Par(d)$ to another basis of the different partitions $Par(d)$, meaning we can write any basis of degree d as another basis of degree d. These transformation matrices have interesting combinatorial implications, this paper will not provide a proof for these but will lay the groundwork of proving that the matrices exists. We will look at the matrices that takes us from the elementary symmetric polynomial basis to the power sum symmetric basis, and vice versa.

4.1 Elementary symmetric polynomials to power sum symmetric polynomials

Now having proved Newton's identities and proposition 7 we have discovered that we can describe any given power sum symmetric polynomial, p_λ, in terms of elementary symmetric polynomials. In proposition 8 we stated that any symmetric polynomial can be written as

$$f(x_1, ..., x_2) = \sum_{\lambda \vdash d} c_\lambda e_\lambda.$$

Now, if we state that our f is the power sum symmetric polynomial we can write this as

$$p_\lambda = \sum_{\mu \vdash d} c_{\lambda\mu} e_\mu \tag{22}$$

where our $c_{\lambda\mu}$ are complex coefficients. It should be clear that for p_λ and e_μ to have the same degree, we know that $|\lambda| = |\mu| = d$, otherwise the different symmetric polynomials would have different degrees and not be equal. We know from proposition 3 that Λ^d is a vector space, V, and therefore we know that there exists a transformation matrix A that takes us from one basis to another such that $x = Ay$ where x and y are column vectors of our different bases of our vector space V, and A is a $n \times n$ matrix. These column vectors x and y holds all different partitions of the degree d, $Par(d)$, of both p_λ and e_μ. Normal convention is that we sort these column vectors in opposite order, so for x we will use lexicographic order and for y we will use reverse lexicographic order. This gives us

$$x = \begin{bmatrix} p_{\lambda_n} \\ p_{\lambda_{n-1}} \\ \vdots \\ p_{\lambda_1} \end{bmatrix},$$

$$y = \begin{bmatrix} e_{\mu_1} \\ e_{\mu_2} \\ \vdots \\ e_{\mu_n} \end{bmatrix}.$$

Now we have to look at our matrix A. We know that it is a $n \times n$ matrix where n is the number of different partitions of $Par(d)$. We know that A holds all of the coefficients, $c_{\lambda\mu}$. Due to the orders we have chosen for x and y this would mean that

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix} = \begin{bmatrix} c_{\lambda_n,\mu_1} & \cdots & c_{\lambda_n,\mu_n} \\ \vdots & \ddots & \\ c_{\lambda_1,\mu_n} & & c_{\lambda_1,\mu_n}. \end{bmatrix}$$

With this we now have all the parts needed to finish our work with the transformation matrix. We know that

$$\begin{bmatrix} p_{\lambda_n} \\ p_{\lambda_{n-1}} \\ \vdots \\ p_{\lambda_1} \end{bmatrix} = \begin{bmatrix} c_{\lambda_n,\mu_1} & \cdots & c_{\lambda_n,\mu_n} \\ \vdots & \ddots & \\ c_{\lambda_1,\mu_n} & & c_{\lambda_1,\mu_n} \end{bmatrix} \begin{bmatrix} e_{\mu_1} \\ e_{\mu_2} \\ \vdots \\ e_{\mu_n} \end{bmatrix}.$$

To put this into a specific case, let us look at the transformation matrix for $V = \Lambda^3$, meaning the vector space of the ring of symmetric polynomials of degree three. We know that $Par(3)$ has three different partitions, $(3), (21), (111)$ so therefore we get

$$\begin{bmatrix} p_{111} \\ p_{21} \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 1 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} e_3 \\ e_{21} \\ e_{111} \end{bmatrix},$$

$$\begin{bmatrix} p_{111} \\ p_{21} \\ p_3 \end{bmatrix} = \begin{bmatrix} e_{111} \\ -2e_{21} + e_{111} \\ 3e_3 - 3e_{21} + e_{111} \end{bmatrix}.$$

4.1.1 Table for the transformation matrix: Elementary symmetric polynomials to power sum symmetric polynomials

We can find the results in this table by using the formula in (10).

$$p_1 = e_1$$

$$\begin{aligned} p_{11} &= e_{11} \\ p_2 &= -2e_2 + e_{11} \end{aligned}$$

$$\begin{aligned} p_{111} &= e_{111} \\ p_{21} &= -2e_{21} + e_{111} \\ p_3 &= 3e_3 - 3e_{21} + e_{111} \end{aligned}$$

$$\begin{aligned} p_{1111} &= e_{1111} \\ p_{211} &= -2e_{211} + e_{1111} \\ p_{22} &= 4e_{22} - 4e_{211} + e_{1111} \\ p_{31} &= 3e_{31} - 3e_{211} + e_{1111} \\ p_4 &= -4e_4 + 4e_{31} + 2e_{22} - 4e_{211} + e_{1111} \end{aligned}$$

4.2 Power sum symmetric polynomials to elementary symmetric polynomials

Most of the work that we need to do is already done in §4.1, and we will closely follow the same structure. If we use our equation from (22) and the knowledge from proposition 7 we know that

$$e_\lambda = \sum_{\mu \vdash d} c_{\lambda\mu} p_\mu \tag{23}$$

where our $c_{\lambda\mu}$ are complex coefficients. Just like before we know that $|\lambda| = |\mu| = d$. There exists a transformation matrix A that takes us from one basis to another such that $x = Ay$ where x and y are column vectors of our different bases of our vector space V, and A is a $n \times n$ matrix, just like previously. The difference this time is that we will change our column vectors x and y to reflect the same ordering that we had in the previous transformation matrix, where the basis that we change to is in lexicographic order and the basis we change from is in reverse lexicographic order. This gives us that

$$x = \begin{bmatrix} e_{\lambda_n} \\ e_{\lambda_{n-1}} \\ \vdots \\ e_{\lambda_1} \end{bmatrix},$$

$$y = \begin{bmatrix} p_{\mu_1} \\ p_{\mu_2} \\ \vdots \\ p_{\mu_n} \end{bmatrix}.$$

Our matrix A stays the same, but the coefficients will potentially take on different values from the one before. This gives us

$$\begin{bmatrix} e_{\lambda_n} \\ e_{\lambda_{n-1}} \\ \vdots \\ e_{\lambda_1} \end{bmatrix} = \begin{bmatrix} c_{\lambda_n,\mu_1} & \cdots & c_{\lambda_n,\mu_n} \\ \vdots & \ddots & \\ c_{\lambda_1,\mu_n} & & c_{\lambda_1,\mu_n} \end{bmatrix} \begin{bmatrix} p_{\mu_1} \\ p_{\mu_2} \\ \vdots \\ p_{\mu_n} \end{bmatrix},$$

and our work is done.

If we apply this to a specific case, say $V = \Lambda^3$, we know that partitions of $Par(3)$ is $(3), (21), (111)$. That would mean that our solution would be

$$\begin{bmatrix} e_{111} \\ e_{21} \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/2 & 1/2 \\ 1/3 & -1/2 & 1/6 \end{bmatrix} \begin{bmatrix} p_3 \\ p_{21} \\ p_{111} \end{bmatrix},$$

$$\begin{bmatrix} e_{111} \\ e_{21} \\ e_3 \end{bmatrix} = \begin{bmatrix} p_{111} \\ \frac{-p_{21}+p_{111}}{2} \\ \frac{2p_3-3p_{21}+p_{111}}{6} \end{bmatrix}.$$

4.2.1 Table for the transformation matrix: Power sum symmetric polynomials to elementary symmetric polynomials

We can find the results in this table by using the formula in (10).

$$e_1 = p_1$$

$$\begin{aligned} e_{11} &= p_{11} \\ e_2 &= \frac{-p_2 + p11}{2} \end{aligned}$$

$$\begin{aligned} e_{111} &= p_{111} \\ e_{21} &= \frac{-p_{21} + p_{111}}{2} \\ e_3 &= \frac{2p_3 - 3p_{21} + p_{111}}{6} \end{aligned}$$

$$\begin{aligned} e_{1111} &= p_{1111} \\ e_{211} &= \frac{-p_{211} + p_{1111}}{2} \\ e_{22} &= \frac{p_{22} - 2p_{211} + p_{1111}}{4} \\ e_{31} &= \frac{2p_{31} - 3p_{211} + p_{1111}}{6} \\ e_4 &= \frac{-6p_4 + 8p_{31} + 3p_{22} - 6p_{211} + p_{1111}}{24} \end{aligned}$$

5 Conclusion

We have now taken some time to really dig into different ways to express the symmetric polynomials. Not only have we seen how to write a symmetric polynomial but we have proved through propositions 4, 9 and 10 that m_λ, e_λ and p_λ make up bases for our ring of symmetric polynomials Λ^d. The proof of Newton's identities also gives us a way to rewrite any power sum symmetric polynomial in terms of elementary symmetric polynomials. We can also rewrite any symmetric polynomial in terms of the elementary symmetric polynomials through the proof of the fundamental theorem, though it might take some time for more difficult symmetric polynomials, due to the induction steps.

This is just the beginning of the journey into the symmetric polynomials. From here there are a multitude of areas which once could explore. One does not have to stay solely in the realm of symmetric polynomials, but can move on to different fields which are touched by this one. However, if one were interested in staying in the symmetric polynomials, there are still other bases that were not covered in this paper, such as the Schur polynomials and complete homogeneous symmetric polynomials. There are also interesting combinatorial implications of the transformation matrices that could be interesting to go further into.

6 Acknowledgments

I would like to thank my supervisor Alexander Berglund for the help he has provided me during this paper. Not only by the suggestion of a very interesting field, but also support in learning about said field and acting as someone with whom I could bounce ideas off of. I would also like to thank my wife, mother and father for their support which allowed me to focus more on this paper. Without either of these I could not have done it.

References

[1] D.S. Dummit and R.M. Foote. *Abstract Algebra*. Wiley, 2003. ISBN: 9780471433347. URL: https://books.google.se/books?id=KJDBQgAACAAJ.

[2] H.M. Edwards. *Galois Theory*. Graduate texts in mathematics. Springer, 1984. ISBN: 9780387909806. URL: https://books.google.se/books?id=6IjBQgAACAAJ.

[3] S. Lang. *Undergraduate Algebra*. Undergraduate Texts in Mathematics. Springer New York, 2005. ISBN: 9780387220253. URL: https://books.google.se/books?id=PdBirmNTwu0C.

[4] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.

[5] Doron Zeilberger. "A combinatorial proof of Newton's identities". In: *Discrete Mathematics* 49.3 (1984), p. 319. ISSN: 0012-365X. DOI: https://doi.org/10.1016/0012-365X(84)90171-7. URL: https://www.sciencedirect.com/science/article/pii/0012365X84901717.