SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Galois Groups and Fundamental Groups

av

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2025 - No M1

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Självständigt arbete i matematik 30 högskolepoäng, advancerad nivå

Handledare: Gregory Arone

2025

Abstract

Ever since Galois theory emerged, there has been an open problem called *The inverse Galois problem.* It states whether or not any finite group G can occur as the Galois group of a finite extension over a fixed base field k. This depends on the base field k. For $k = \mathbb{C}(t)$ the problem has a positive answer and a proof will be provided with the help of Riemann surfaces. When $k = \mathbf{Q}$ the problem remains open.

In this project, a promising approach will be presented concerning the *algebraic fundamental group*, which will allows us to turn a finite group G with trivial center and a rigid system of rational conjugacy classes into a Galois group over $\mathbb{Q}$. To obtain the algebraic fundamental group we will present the theory in the following way: In the first chapter the Galois theory for both finite and infinite extensions will be presented and also *Groethendieck's reformulation* of the main Galois theorem will be stated and proved. The latter serves as an abstraction and will allow us to draw analogies between the two central objects of the project, namely the Galois groups and the Fundamental groups. In the second chapter, the focus will be on *covers* and the *fundamental group* of a space and the analogy between them with field extensions and the absolute Galois group. The third chapter is about *Riemann surfaces* for which we will obtain a link between the Galois theory of fields and that of covers. It will be proven that finite etale algebras over the field of meromorphic functions of a fixed Riemann surface corresponds up to isomorphism to finite *branched* covers of Riemann surfaces. The last chapter will be dedicated to obtaining the algebraic fundamental group via the theory of algebraic curves by relating it to the theory presented in the third chapter.

Sammanfattning

Ända sedan Galoisteorin uppstod har det funnits ett öppet problem som kallas *Det inversa Galoisproblemet.* Det handlar om huruvida varje ändlig grupp G kan uppträda som Galoisgruppen för en ändlig utvidgning över ett fixerat basfält k. Detta beror på basfältet k. För $k = \mathbb{C}(t)$ har problemet ett positivt svar, och ett bevis kommer att presenteras med hjälp av Riemannytor. När $k = \mathbf{Q}$ förblir problemet öppet.

I detta projekt kommer en lovande metod att presenteras som berör *den algebraiska fundamentalgruppen*, vilket gör det möjligt att omvandla en ändlig grupp G med trivialt centrum och ett stelt system av rationella konjugatklasser till en Galoisgrupp över $\mathbb{Q}$. För att erhålla den algebraiska fundamentalgruppen kommer vi att presentera teorin på följande sätt: I det första kapitlet kommer Galoisteorin för både ändliga och oändliga utvidgningar att presenteras, och även *Grothendiecks reformulering* av den huvudsakliga Galoissatsen kommer att anges och bevisas. Den senare fungerar som en abstraktion och gör det möjligt för oss att dra analogier mellan de två centrala objekten i projektet, nämligen Galoisgrupperna och fundamentalgrupperna. I det andra kapitlet kommer fokus att ligga på *överlagringar* och *fundamentalgruppen* för ett rum och analogin mellan dessa och fältextensioner samt den absoluta Galoisgruppen. Det tredje kapitlet handlar om *Riemannytor*, där vi kommer att upprätta en koppling mellan Galoisteorin för kroppar och teorin för överlagringar. Det kommer att bevisas att ändliga etaleska algebror över fältet av meromorfa funktioner på en fix Riemannyta motsvarar upp till isomorfism ändliga *förgrenade* överlagringar av Riemannytor. Det sista kapitlet kommer att ägnas åt att erhålla den algebraiska fundamentalgruppen via teorin om algebraiska kurvor genom att relatera den till teorin som presenterades i det tredje kapitlet.

Acknowledgements

I would like to thank my supervisor Gregory Arone for his guidance and support throughout this project. The topic of this thesis combines almost every subject I have encountered in this master‘s program and therefore I want to thank all my previous professors for the knowledge they have helped me obtain. Also, I want to thank my family and friends for their tremendous support throughout this period.

Contents

1 Introduction

The Galois theory originally emerged for studying roots of polynomials over fields. A root a of a polynomial f over a field k defines an extension of finite degree L. For each such finite field extension $k \subseteq L$ we can define the group of automorphisms $Aut(L|k)$ of L that fixes the base field k elementwise. This group sends a root of a polynomial to another root of a polynomial. It is said to be Galois if the field that gets fixed by the action of is exactly the base field k. This gives a correspondence between Galois groups and Galois extensions. It is natural to extend this notion to infinite extensions, but then we fail to have a bijection between groups and fields because the groups end up to be too many. Hopefully, we can remedy this case by setting a topology on the groups and then we get a correspondence. This correspondence allow us then to formulate the main Galois theorem in purely categorical terms which depends on the choice of a separable closure k_s of a field k and the group $Gal(k_s|k)$. In this situation we get a correspondence between finite separable field extensions L of k and sets with left $Gal(k_s|k)$-action . A parallel construction is developed for covering spaces and we get a similar correspondence which depends on the base point x of a space X and the fundamental group $\pi_1(X,x)$ and the correspondence is between covering spaces over X and sets with continuous right $\pi_1(X,x)$ action. This right $\pi(X,x)$-action will be shown that is equivalent to a left $Aut(\hat{X}|X)$ action, where $\hat{X}$ is the universal covering of X. The Riemann surfaces in Chapter 3 are naturally endowed with a field $M(X)$ of meromorphic functions and with covering maps coming from proper holomorphic maps. This structure will allow us to combine both theories to get a correspondence between covers and field extensions of a fixed field. This will enable us to describe the Galois groups from the fundamental groups. Though this is the analogy we want, it restrict us to describe only field extensions over the complex numbers $\mathbb{C}$. In order to extend this correspondence over the rational numbers $\mathbb{Q}$, we introduce algebraic curves and relate them to Riemann surfaces by endowing them with a Riemann structure. We will conclude with an interesting result that finite groups G with trivial center and a rigid system of rational conjugacy classes can arise as a Galois group over $\mathbb{Q}$.

2 Galois Theory for Fields

2.1 Field Theory

In this section we will present the theory of fields needed to develop the results of Galois Theory.

Recall that a field extension L of k is such that k is a subfield of L and we denote it as $L|k$.

Definition 2.1. Let k a field and an extension L of k.

1. An element $a \in L$ is said to be algebraic over k if there exists a non-zero polynomial $f \in k[t]$ such that $f(a) = 0$.

2. An extension L of k is algebraic if every element $a \in L$ is algebraic over k.

3. If the polynomial f is monic and irreducible we call it the minimal polynomial of a over k.

Remark 1. A field extension L over k can be viewed as a vector space over the field k. A finite extension $L|k$ is such that L is a finite dimensional vector space over k. For a finite extension $L|k$ we have that L is also algebraic over k. That comes from the fact that if $a \in L$, then the powers of a $a, a^1, ..., a^n$ can not be linearly independent over k and thus must be the root of a polynomial with coefficients in k. In this situation denote the degree of L over k as $[L : k]$. If we have a tower of finite field extensions $M|L|k$ then we have that $[M : k] = [M : L][L : k]$, because if $\{a_1, ..., a_n\}, \{b_1, ..., b_m\}$ are two basis for the extensions $M|L$ and $L|k$ respectively then $\{a_i b_j\}$ forms a basis for $M|k$.

A monic polynomial is a univariate polynomial (a polynomial in one variable) such that the leading coefficient of the polynomial is 1. We note that k being a field implies that we can assume that a univariate polynomial is monic (otherwise we multiply the leading coefficient with its multiplicative inverse). Also we note that an algebraic extension $L|k$ is a k-algebra that is generated by algebraic elements over k. We write for a k-algebra generated by elements $a_1, ..., a_n$

$$L = k(a_1, ..., a_n)$$

If $L = k(a)$ is generated by one algebraic element over k, then we have that $[L : k] = n$, where n is the degree of the minimal polynomial of a. In fact , when L is a finite extension over a field k of characteristic 0, then we have that $L = k(a_1, .., a_n)$ and by the next proposition we can see that there exists $\gamma \in L$ such that $L = k(\gamma)$.

An embedding of a field k into another field L is a ring homomorphism

$$\sigma : k \to L$$

And for such a homomorphism we have that $\sigma(1) = 1$ and $\sigma(0) = 0$. Also because $xx^{-1} = 1$ then we have that $\sigma(x)\sigma(x)^{-1} = 1$ and therefore if $x \neq 0$, then $xx^{-1} = 1$ so $\sigma(x) \neq 0$. Therefore we get that $\sigma(x) = 0 \Rightarrow x = 0$ so σ

is injective. The image $\sigma(k) \subseteq L$ and is clearly a subfield of L, thus we can identify k with its image. We also have that if p is an irreducible polynomial over $k[t]$ then $\sigma(p)$ is also irreducible in $\sigma(k)[t]$. This follows easily from if $\sigma(p)$ is not irreducible then

$$\sigma(p) = gh \Rightarrow p = \sigma(g)^{-1}\sigma(h)^{-1}$$

and thus p has a factorization in $k[t]$,contradiction. We also have that if $a \in k$ is a root of a polynomial $f \in k[t]$ then we have that $\sigma(a)$ is a root of $\sigma(f) \in \sigma(k)[t]$. Lastly, if $E|L|k$ is a tower of extensions then we say that an embedding $\tau : L \to E$ is an extension of $\sigma : k \to E$ if $\tau(x) = \sigma(x), \forall x \in k$.

Proposition 2.1. ***Primitive Element*** *If L is a finite extension over k and k is of characterisic 0 then we have that $\exists \gamma \in L$ such that $L = k(\gamma)$.*

A proof of the proposition will be given in the more general form of seperable extension.

Definition 2.2. A field k is algebraically closed if it has no proper algebraic extension. An algebraic closure of k is an algebaic extension that is algebraically closed.

The next proposition states that given a field k , there exists an algebraic closure of k and gives some properties of the algebraic closure. Recall that a k-embedding is an injective homomorphism that leaves k-elementwise fixed.

Proposition 2.2. *Let k a field.*

1. *There exists an algebraic closure $\hat{k}$ of k.*
2. *For a finite extension L over k, we have that there exists an embedding $L \to \hat{k}$ that leaves k elementwise fixed.*
3. *For two algebraic closures $\hat{k_1}, \hat{k_2}$ we have that there is a non-unique isomorphism $\hat{k_1} \cong \hat{k_2}$.*
4. *For L algebraic extension of k we have that there exists an k-isomophism of algebraic closures $\hat{L} \cong \hat{k}$ which extends the k-embedding $L \to \hat{k}$.*

Proof. 1. A proof is given in the book [7], Theorem 7.4.

2. Proof in [7] Theorem 2.3 [VII].
3. Proof in [7] Theorem 7.5.

□

Thus henceforth when speaking of an algebraic extension of k, we can think of it as a subfield in a fixed algebraic closure $\hat{k}$.

For a finite extension as given in **Proposition 1.2.2** we have that if k is of characteristic 0 we can prove that the number of embeddings $L \to \hat{k}$ that leaves k elementwise fixed is equal to $[L : k]$, i.e the degree of the extension. But for

positive characteristic, let that be $p \in \mathbf{N}$ we have the following situation:
Let F be a field of characteristic p and $a \in F$. Then for the polynomial $t^p - a \in F[t]$ suppose we have a root $b \in E$ where E is an algebraic extension of F,

$$(t-b)^p = t^p - b^p = t^p - a$$

so $b \in E = F(b)$ is the only root of the polynomial and therefore any embedding $F(b) \to \hat{k}$ that fixes F pointwise must map b to b and thus the only choice is the identity on $F(b)$, but the degree of the algebraic extension is n. Therefore to preserve this important property together with the primitive element property over any characteristic we introduce the notion of separability.

Definition 2.3. A polynomial $f(x) \in k[x]$ is separable if it has no multiple roots in an algebraic closure. An element $a \in L$ inside an algebraic extension $L|k$ is separable over k if its minimal polynomial is separable. The algebraic extension $L|k$ is separable over k if every element in L is separable over k.

Definition 2.4. A splitting field of a polynomial $f \in k[x]$ is the smallest finite extension $S|k$ in which the polynomial f splits into linear factors.

Remark 2. A splitting field of a polynomial f exists, and any two splitting fields of the same polynomial are isomorphic. These are the contents of Theorems 3.1 and 3.2 [VII] in [7]. Thus we can speak of **the** splitting field of a polynomial f.Also, if L is the splitting field of a polynomial $f \in k[t]$ then we can see that it is generated by its roots over k ,i.e $L = k(a_1, .., a_n)$ with $a_1, ..., a_n$ roots of f.

Remark 3. When k is of characteristic 0 we see that separability holds,because of the fact that an irreducible polynomial $f \in k[x]$ has no multiple roots if and only if f' is nonzero. In fact, if f has multiple roots then f and f' have a common root in the splitting field of f, thus a common factor $h = x - a$, from the irreducibility of f we and because $f, f' \in k[t]$ we get $f|f'$, which is absurd as f' has a smaller degree than f, so $f' = 0$. The other direction is trivial over characteristic 0.

Proposition 2.3. *Let $L|k$ be a finite extension of degree n. Then L has at most n distinct k-algebra homomorphism to $\hat{k}$, with equality if and only if the extension is separable.*

Proof. Let L be a finite extension of k, then there exist $a_1, a_2, ..., a_m \in L$ such that $L = k(a_1, ..., a_m)$. If $m = 1$ then we have that $L = k(a_1)$ and a k-algebra homomorphism $\sigma : k(a_1) \to \hat{k}$ is characterized by the image of a_1 in $\hat{k}$ (k is fixed elementwise by such a homomorphism). The image of a_1 has to be a root of the image of the minimal polynomial f of a_1 as was mentioned in the discussion about the embeddings. Because the homomorphism fixes the coefficients of the polynomial then we have that the image $\sigma(a_1)$ is a root of the polynomial f. The number of the roots of f is at most n with equality if and only if the polynomial is separable. Because of the multiplicativity of the degree of the extension discussed above inductively we have that $[L : k] = [L :$

$k(a_1,...,a_{m-1})[k(a_1,..,a_{m-1}):k]$ for $L=k(a_1,..,a_m)$ has at most n k-algebra homomorphisms to $\hat{k}$ with equality if all a_i are separable. To prove the "only if" part assume that we have a non separable element in the extension , i.e $a \in L$, then we proved that the number of distinct k-algebra homomorphisms $k(a) \to \hat{k}$ is strictly less than $[k(a):k]$ and the number of k(a)-algebra homomorphisms $L \to \hat{k}$ is at most $[L:k(a)]$. Then the number of k-algebra homomorphisms $L \to \hat{k}$ is strictly less than $[L:k(a)][k(a):k]=[L:k]=n.$ □

From the proof of the proposition above we immediately get the following

Corollary 2.3.1. *For a tower of finite extensions $L|M|k$ we have that $L|k$ is separable if and only if both $L|M$ and $M|k$ are separable.*

Corollary 2.3.2. *A finite extension $L|k$ is separable if and only if there exist separable elements $a_1,...,a_n \in L$ such that $L=k(a_1,...,a_n)$.*

We will now construct the biggest separable extension of a field k inside of a closure $\hat{k}$ of it. For that we recall that the compositum LM of two field extensions L and M of k which both lie inside a fixed algebraic closure $\hat{k}$ of k, is the smallest subfield of $\hat{k}$ which contains both L and M.

Corollary 2.3.3. *If L and M are two finite separable extensions of k inside $\hat{k}$, then their compositum LM is also a separable extension of k inside $\hat{k}$.*

Proof. As L,M are both finite separable extensions then from corollary 1.3.2, both are of the form $L=k(a_1,..,a_n)$ and $M=k(b_1,...,b_m)$ for separable elements $\{a_i,b_i\}$. Then $LM=k(a_1,..,a_n,b_1,..,b_m)=M(a_1,..,a_n)$ and because a_i are separable over k, i.e their minimal polynomial has no multiple roots in an algebraic closure, then they are separable over M too. Then we have that $LM|M$ is separable and also $M|k$ is separable by assumption . Thus from corollary 1.3.1 we have that $LM|k$ is also separable. □

By the above 3 corollaries we have that the compositum k_S of all finite separable extensions of k in an algebraic closure $\hat{k}$ is a separable extension with the property that for each $a \in k_s$ we have that $k(a)|k$ is a finite separable extension of k inside $\hat{k}$. Also each finite separable subextension of $\hat{k}|k$ is contained inside k_s by definition. The compositum k_s is said to be the **separable closure** of k inside the algebraically closed $\hat{k}$. All subextension of $k_s|k$ (even infinite) will be called separable.
We will now prove the theorem of the primitive element for separable extensions.

Proposition 2.4. *A finite separable extension $L|k$ can be generated by one element.*

Proof. If k is a finite field then so is $L=k(a_1,..,a_m)$ and we also have that the multiplicative subgroup of non-zero elements of L is a cyclic group, thus it is generated by an element $\gamma \in L^*$. Then we have that $k(\gamma)$ contains at least as many elements as $(\gamma) \subseteq L^*$ which are $|L|-1$. Also $k(\gamma)$ contains the zero which is not contained in (γ),as it is a k-vector space,thus $|k(\gamma)|=|L|$ and because

$k(\gamma)$ is a subfield of L then we have that $k(\gamma) = L$ and thus we have a primitive element.
If k is an infinite field then it is enough to prove the statement for $L = k(\alpha, \beta)$ then for the general case induction applies. Let $L = k(\alpha, \beta)$ and $c \in k - \{0\}$ and consider $\gamma = \alpha + c\beta$. We will show that γ is a primitive element for some choice of $c \in k$. We have that $k(\gamma) \subseteq k(\alpha, \beta)$ and if $\beta \in k(\gamma)$ then we are done as $\alpha = \gamma - c\beta \in k(\gamma)$,thus $k(\alpha, \beta) = k(\gamma)$. So we may assume that $\beta \notin k(\gamma)$. As β is algebraic over k then it is also algebraic over $k(\gamma)$ and because $\beta \notin k(\gamma)$ we get that the algebraic extension $k(\gamma, \beta)$ has at least degree 2 over $k(\gamma)$(and by separability distinct roots). Then by **Proposition 1.2** we have that there exists an embedding σ that sends $\sigma(\beta) = \beta'$,where β' is the other root of the minimal polynomial of β over $k(\gamma)$, and that fixes $k(\gamma)$. So we have $\sigma(\gamma) = \gamma$ therefore from $\gamma = \alpha + c\beta$, we get that $\sigma(\alpha) + c\sigma(\beta) = \alpha + c\beta$ which implies

$$c = \frac{\sigma(a) - a}{\beta - \sigma(\beta)} \tag{1}$$

because there are only finitely many such embeddings σ and each one sends α, β to different roots of their minimal polynomial, then we have that there are finitely many c satisfying equation (1). So we have that if $\beta \notin k(\gamma)$, then there exists finitely many c so that this happens, but because the field is infinite we choose any c different from those of relation (1) and we get that $\beta \in k(\gamma)$ which then implies that our extension is generated by one algebraic element. □

2.2 Galois Theory for Finite Extensions

In this section we will present the Galois theory for finite extensions. Let L an extension of k, we denote as $Aut(L|k)$ the group of automorphisms of L that fixes the field k. The elements fixed by the group $Aut(L|k)$ acting on L is an extension of k by definition.

Definition 2.5. An algebraic extension $L|k$ is called Galois extension if the elements of L that get fixed by $Aut(L|k)$ is exactly the field k. In this case we denote $Aut(L|k) = Gal(L|k)$ and call it the Galois group of the extension $L|k$.

Lemma 2.5. *A separable closure k_s of k is always a Galois extension.*

Proof. Let $a \in k_S - k$, we want to prove that a is not fixed by all elements of $Aut(k_s|k)$. So a is a separable element and the extension $k(a)$ is a separable, algebraic extension of k. Any $\sigma \in Aut(L|k)$ sends a root of the minimal polynomial $f \in k[t]$ of a to $\sigma(a)$ which is again a root of f.Assume $a' \neq a$ is another root of the minimal polynomial of a (exists as a is separable). Then from **Proposition 1.2** we have that there exists embeddings $k(a) \to \hat{k}$ and $k(a') \to \hat{k}$ that leave k elementwise fixed and we also have an isomorphism $k(a) \to k(a')$ by sending a to a'. Thus the isomorphism extends to an automorphism of $\hat{k}$. So there exists an element $\hat{\sigma} \in Aut(\hat{k}|k)$ that does not fix a. It remains to show that the restriction of $\hat{\sigma}$ to k_s defines an element of $Aut(k_s|k)$. Indeed, let

$b \in k_s$ a separable element , then $\hat{\sigma}$ maps b to another root of the same minimal polynomial b', which is separable. Therefore only k is fixed by the action of $Aut(k_s|k)$ on k_s. □

The Galois group $Gal(k_s|k)$ is called the **absolute Galois group** of k.

Lemma 2.6. *A Galois extension $L|k$ is a separable extension and the minimal polynomial of every element splits in linear factors in L .*

Proof. Let $a \in L - k$ and consider the polynomial $f = \prod(x - \sigma_i(a))$, where $\sigma_i \in Gal(L|k)$ and $\sigma_i(a) \neq \sigma_j(a)$ for all $i \neq j$. Then, because every $\sigma_i(a)$ is a root of the minimal polynomial g of a, we get that the product is finite. Also, we have that L is the splitting field of f and because every $\sigma_i(a)$ is a root of g then we have that L' is an extension of L , where L' is the splitting field of g. We thus have that $f|g$ inside $L'[t]$ thus we have $g = hf$ for $h \in L'[t]$. We note that $f \in k[t]$. Indeed, the coefficients of f are the elementary symmetric polynomials in its roots and any $\sigma \in Gal(L|k)$ acting on those fixes them, thus by assumption that the extension is Galois we have that $f \in k[t]$. Thus we have that $f|g$ and that $f, g \in k[t]$ and g is irreducible over $k[t]$ as the minimal polynomial of a. Thus the extension is separable by definition of f and the minimal polynomial g splits in linear factors in L. □

Now we state a proposition that characterizes Galois extensions.

Proposition 2.7. *Let k a field , k_s a separable closure and $L \subseteq k_s$ a finite field extension of k. The following are equivalent:*

1. *$L|k$ is a Galois extension.*

2. *$L|k$ is separable extension and the minimal polynomial for every $a \in L$ splits in linear factors in L.*

3. *For every $\sigma \in Aut(k_s|k)$ we have that $\sigma(L) \subseteq L$.*

Proof. The proof (1) ⇒ (2) was given in **Proposition 2.2**. Assume (2) then L contains all the roots of each minimal polynomial g_a for all $a \in L$ and because for every $\sigma \in Aut(k_s, k)$ we have that $\sigma(a)$ is a root of g_a, thus we have $\sigma(L) \subseteq L$. For (3) ⇒ (1),we pick $a \in L - k$ and by **Lemma 2.1** we have that $k_s|k$ is Galois,thus $\exists \sigma \in Gal(k_s|k)$ with $\sigma(a) \neq a$.Now, $\sigma(L) \subseteq L$ by assumption implies that $\sigma|_L \in Aut(L|k)$ and $\sigma(a) \neq a$ implies that only k is fixed by $Aut(L|k)$, thus $L|k$ Galois. □

We now proceed with stating the Galois Theorem for finite extensions. We will denote L^H to be the subfield of L that is fixed under the action of the subgroup $H \subset Gal(L|k)$ on L.

Theorem 2.8. ***Galois Theorem for finite extensions*** *Let $L|k$ be a finite Galois extension with Galois group G. Then the maps*

$$M \to H$$

$$H \to L^H = M$$

yield an inclusion reversing bijection from the subfield $k \subseteq M \subseteq L$ to the subgroups $H \subseteq G$. The extension $L|M$ is always Galois. The extension $M|k$ is Galois if and only if H is a normal subgroup of G and in this case we have $Gal(M,k) \cong G/H$.

Proof. Let $L|k$ be a Galois extension with $Gal(L|k) = G$ and k_s a separable closure of k that contains L as a subfield and $M \subseteq L$. Then any $\sigma \in Aut(k_s|M) = Gal(k_s|M)$ (**Lemma 2.1**) that fixes M, must also fix k , thus $\sigma \in Gal(k_s|k)$ and because $L|k$ is Galois we get from **Proposition 2.3** that $\sigma(L) \subseteq L$ and therefore $L|M$ is a Galois extension. Denoting thus $H = Gal(L|M)$, we have that $L^H = M$ and $H \subseteq G$. Conversely if $H \subseteq G$ then $L|L^H$ is Galois by definition with Galois group $Gal(L|L^H) = H$.

To prove the second statement assume first that $H \subseteq G$ is a normal subgroup, thus we can define the quotient G/H. Any non-zero representative $\hat{\sigma} \in G/H$ comes from an automorphism of L that fixes k but does not fix M elementwise. Therefore each such element $\hat{\sigma}$ acts on $M = L^H$,as any $\sigma \in Aut(L|k)$ acts on M and the action is determined by the class of σ modulo H, therefore $M^{G/H}$ is defined and also the equality $M^{G/H} = L^G$ holds and because G Galois group then $L^G = k$ so $M^{G/H} = k$ thus $M|K$ is Galois extension with Galois group $Gal(M|k) = G/H$.

Conversely, let $M|k$ be a Galois extension . Let $\sigma \in G = Aut(L|K)$ and let $\hat{k}$ an algebraic closure of k that contain L. Then $\sigma : L \to L \subseteq \hat{k}$ extends to a $\sigma' : \hat{k} \to \hat{k}$ by **Proposition 1.2 (4)** that fixes k-elementwise and restricting to k_s by **Lemma 1.5** we have that for any $\sigma \in G$ there exists an extension o $\sigma'_{|k_s} \in Gal(k_s, k)$ and because $M|k$ is Galois then we have from **Proposition 1.7 (3)** that $\sigma'_{|k_s}(M) = \sigma(M) \subseteq M$. Thus the restriction to M defines a natural homomorphism $G \to Aut(M|k) = Gal(M|k)$ $(\sigma \mapsto \sigma_{|M}$), and the kernel of that map are the automorphisms in G that map to the identity on $Gal(M|k)$, i.e that fixes M, but those are $Aut(L|M)$. Thus the kernel is equal to $Aut(L|M)$ and the kernel of a group homomorphism is always a normal subgroup of the domain. Thus $H = Aut(L|M) = Gal(L|M)$ is in fact normal. □

The most common characterization of a Galois extension is the following, which also gives a constuctive way to get a Galois extension.

Lemma 2.9. *Let $L|k$ a finite extension, then $L|k$ is Galois if and only if it is the splitting field of an irreducible separable polynomial.*

Proof. Let $L|k$ finite Galois extension, then by **Proposition 1.7** $L|k$ is separable and the minimal polynomial of each $a \in L$ splits in linear factors. By the primitive theorem for separable extensions we get that $L = k(a)$ and by assumption we have that the minimal polynomial f_a splits in linear factors and this is an irreducible, separable polynomial. Conversely, if L is the splitting field of an irreducible separable polynomial then it is generated by all the distinct roots of the polynomial and every root $a \in L$ must be mapped to another root $\sigma(a)$ by $\sigma \in Gal(k_s, k)$ and therefore by **Proposition 1.7** the extension is Galois. □

Corollary 2.9.1. *For a finite Galois extension $L|K$ we have that $[L : k] = |G|$ where G is the Galois group of the extension.*

Proof. By the previous proposition, we have that L is the splitting field of an irreducible and separable polynomial f and let $deg(f) = n$. Then $L = k(a_1, .., a_n)$ and $[L : k] = n$ and every element $\sigma \in Gal(L|k)$ is a homomorphism of k-algebras to $\hat{k}$ as in **Proposition 1.3** and thus we have $|G| = n$. □

Remark 4. Generally for any finite algebraic extension $L|k$ we have $|Aut(L|k)| \leq [L : k]$ with equality if and only if the extension is Galois. When equality holds, we note that the automorphism group $Gal(L|k)$ acts transitively on the roots of the minimal polynomial. Thus, if the extension is of degree n we have a natural injection $Gal(L|k) \to S_n$. This implies that $|Gal(L|k)| \leq n!$ and more interestingly that any splitting field of a separable irreducible polynomial has at most $[L : k] \leq n!$.

2.3 Infinite Galois Extensions

In this section we will extend the main Theorem of the Galois Theory (**Theorem 1.8**) to infinite Galois extensions. In order to do so, we first have to describe what an infinite Galois extension is. Extending **Definition 1.5** to the infinite case, we see that a Galois extension $L|k$ arises as a k-algebra of infinite degree, i.e. $L = k(a_i|i \in I)$ where a_i are algebraic elements and I any set of indices (even uncountable). The problem with infinite extensions, though we have that there are maps from subgroups of the Galois groups to field extensions of a base field and vice versa, is that it is no longer a bijection, because there are too many subgroups and at least two of them can have the same fixed field. To overcome this fact we will use topology.

A first property that follows from the definition is that any infinite Galois extension arises as a union of finite Galois extensions. Indeed by the following proposition we get that any finite subextension of $K|k$ can be embedded in a Galois extension and thus taking the compositum of those Galois extensions gives us the result.

Proposition 2.10. *Let $K|k$ an infinite Galois extension. Any finite subextension of $K|k$ can be embedded in a finite Galois extension.*

Proof. Any finite subextension of a Galois extension is separable , therefore by the primitive element theorem for separable extension we have that $L = k(a)$ with minimal polynomial $f_a \in k[t]$. Therefore taking the splitting field of the irreducible, separable polynomial f gives us a Galois extension in which L can be embedded. □

This fact has a crucial consequence, that $Gal(K|k)$ can be turned into a profinite group. From the main Galois correspondence we have that if $M|L|k$ is a tower of finite Galois extensions contained in an infinite Galois extension $K|k$, then $Gal(L|k) \cong Gal(M|k)/Gal(M|L)$, where $Gal(M|L)$ is the kernel of the surjective map $\phi_{ML} : Gal(M|k) \to Gal(L|k)$ which was build in the proof of **Theorem 1.8** and if $N|k$ is another Galois extension such that $M \subseteq N$ then

$$\phi_{NL} : Gal(N|k) \to Gal(M|k) \to Gal(L|k)$$

We have that the kernel of the ϕ_{NM} is $Gal(N|M)$ and the kernel of ϕ_{ML} is $Gal(M|L)$ and because any automorphism fixing M also fixes L, thus we have that $\phi_{NM}(Gal(N|M)) \subseteq Gal(M|L)$ and therefore we have that the map indeed factors as above, i.e $\phi_{NL} = \phi_{ML} \circ \phi_{NM}$. We will generalize this construction by the following definition.

Definition 2.6. A (filtered) inverse system of groups (G_a, ϕ_a) consists of:

1. A partially order set $(\Lambda, \leq)$ which is directed in the sense that for all $(\alpha, \beta) \in \Lambda$ there exists $\gamma \in \Lambda$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.
2. For each $\alpha \in \Lambda$ a group G_α.
3. For each $\alpha \leq \beta$ a homomorphism $\phi_{\alpha\beta} : G_\beta \to G_\alpha$ such that we have equalities $\phi_{\alpha\gamma} = \phi_{\alpha\beta} \circ \phi_{\beta\gamma}$ for $\alpha \leq \beta \leq \gamma$.

Definition 2.7.

1. The inverse limit of an inverse system of groups is defined to be a subgroup of the product $\prod_a G_a$ constisting of sequence $(g_a) \in \prod_a G_a$ such that $\phi_{\alpha\beta}(g_\beta) = g_\alpha$ for all $\alpha \leq \beta$.We denote it as $\varprojlim(G_a)$
2. A profinite group is the inverse limit of an inverse system of **finite** groups.

We will now prove that $Gal(K|k)$ can be turned into a profinite group.

Proposition 2.11. *Let $K|k$ be a Galois extension of fields. Then the Galois groups of the finite Galois subextensions of $K|k$ together with $\phi_{NM} : Gal(N|k) \to Gal(M|k)$ can be turned into an inverse system of groups with the inverse limit being isomorphic to $Gal(K|k)$. Thus $Gal(K|k)$ is profinite group.*

Proof. Let the partially ordered set be the finite Galois subextensions of $K|k$ together with the partial order $L \leq M$ if and only if $L \subseteq M$. From **Theorem 1.8** we have that for each such finite Galois sub-extension we have a group $Gal(L|k)$ and the third property of the definition follows from the preceding discussion. Thus indeed we have an inverse system of groups.We denote G_L to be the Galois group corresponding to the finite Galois extension $L|k$.
We now prove the isomorphism of $Gal(K|k)$ with the inverse limit of the system. Let ϕ be

$$\phi : Gal(K|k) \to \prod_L G_L$$

$$\sigma \mapsto (\sigma_{|G_L})$$

The fact that a restriction of an automorphism in $Gal(K|k)$ defines an automorphism on $L|k$ when $L|k$ is Galois follows from **Proposition 1.7**.
We want to prove that ϕ is bijective. First, we find that ϕ is injective. Indeed, if σ is nontrivial, then there exists $a \in K$ such that $\sigma(a) \neq a$ and there exists a

finite Galois extension containing a, which is the splitting field of the minimal polynomial of a, which is also separable as a subextension of $K|k$. Therefore, in order for a sequence (σ_{G_L}) to be trivial, we must have σ to be trivial as an automorphism of $Gal(K|k)$. We also note that $\phi(Gal(K|k)) \subseteq \varprojlim Gal(L|k)$ because we have $\phi_{ML}(\sigma_{|M}) = \sigma_{|L}$ for all $\sigma \in Gal(K|k)$ for finite Galois extensions $M|L|k$. On the other hand, if $(\sigma_L) \in \varprojlim Gal(L|k)$ then we can construct $\sigma \in Gal(K|k)$ by setting $\sigma(a) = \sigma_L(a)$ for a finite Galois extension $L|k$ that contains a, for all a.The fact that $\sigma \in Gal(K|k)$ is a well defined element comes from the fact that σ_L form a compatible system of automorphisms, as $(\sigma_L) \in \varprojlim Gal(L|k)$ and by consturction we have that $\sigma \mapsto (\sigma_L)$. Thus infact we have $Gal(K|k) \cong \varprojlim Gal(L|k)$. □

Corollary 2.11.1. *Projections to the components of the inverse limit yields natural surjections $Gal(K|k) \to Gal(L|k)$ for all finite Galois subextensions $L|k$ of $K|k$.*

Having this proposition, we can demonstrate an example of what means that the Galois subgroups are too many for them to be in a bijection with field extensions.

Example 1 Let $L = Q(\sqrt{2}, \sqrt{3}, \sqrt{5}, ..)$ be an infinite extension over Q by adjoining the square roots of all prime numbers.Then we have the countable sequence of field extensions

$$Q(\sqrt{2}) \subseteq Q(\sqrt{2}, \sqrt{3}) \subseteq ...$$

the irreducible polynomial of $\sqrt{p}$ remain irreducible over any extension not containing it and thus the extension $M(\sqrt{p})|M$ is of degree 2 for any finite subextension of L and is also Galois. That L is Galois follows from the fact that L is the compositum of finite Galois extensions. For each extension $M(\sqrt{p})|M$ we have two choices for an element in the automorphism group, either the identity or the automorphism mapping $\sqrt{p} \mapsto -\sqrt{p}$. Therefore

$$Gal(K|k) \cong \varprojlim Gal(L|k) \cong \prod_p \{-1, 1\} \cong \prod_{n \in N} \{-1, 1\}$$

The number of subgroups of order 2 in $Gal(K|k)$ is thus 2^N which is uncountable and the number of field extensions is countable. Therefore, a bijection can not exist between the two.

We will now define the topology on $Gal(K|k)$ that will "fix" our correspondence. We will get a correspondence between closed subgroups of $Gal(K|k)$ and intermediate fields $K|L|k$. We endow the profinite group $Gal(K|k)$ with a topology as follows: we endow each finite Galois subextension G_a with the discrete topology, the product $\prod_a G_a$ with the product topology and then the subgroup $Gal(K|k) = \varprojlim G_a \subseteq \prod_a G_a$ with the subspace topology. The projections to the components $\prod_a G_a \to G_a$ are continuous, thus also the natural projections $Gal(K|k) \to G_a$ are continuous(subspace topology).

We now state two properties of the above topological construction.

Lemma 2.12. *Let $(G_\alpha, \phi_{\alpha\beta})$ be an inverse system of groups endowed with the discrete topology as above. Then the inverse limit $\varprojlim G_\alpha$ is a closed topological subgroup of the product $\prod_\alpha G_\alpha$.*

Proof. Let $(g_\alpha) \in \prod_\alpha G_\alpha$. We want to prove that if $(g_\alpha) \notin \varprojlim G_\alpha$ then there exists an open set containing (g_α) which has empty intersection with $\varprojlim G_\alpha$. If $(g_\alpha) \notin \varprojlim G_\alpha$, then exists $\phi_{\alpha\beta}$ such that $\phi_{\alpha\beta}(g_\beta) \neq g_\alpha$. We pick the set U consisting of all the points that have their $\alpha - th$ and $\beta - th$ components equal to g_α and g_β respectively. The set U is open as the product of opens in the product topology (the points are open because of the discrete topology). By construction U can not a non-empty intersection with the inverse limit, it contains the point (g_α) and it is open. Thus the complement of $\varprojlim G_\alpha$ in $\prod_\alpha G_\alpha$ is open. □

Recall that a topological group is called totally disconnected if the only connected subsets are the one point subsets.

Corollary 2.12.1. *A profinite group G is compact and totally disconnected. Moreover, the open subgroups are precisely the closed subgroups of finite index.*

Proof. A finite discrete subgroup is compact as the singleton sets form a basis for the topology and they are finitely many. Therefore by Tychonoff's theorem the product $\prod_a G_a$ is compact and thus the inverse limit is also compact as it is a closed subset(Lemma 1.12) of the compact space $\prod_a G_a$. If a subset U of the inverse limit contains two distinct elements $(g_a), (g'_a) \in \varprojlim G_a$ then they differ at one component G_β, i.e $g_\beta \neq g'_\beta$ therefore forming the open subsets containing those points as we did in Lemma 1.12 we get that U can be written as a union of two disjoint open subsets, thus it is disconnected. So the only connected subsets are indeed the one point subsets. For the second statement, let U be an open subset of G. We then have that $G = \cup_{g \in G} gU$, and every set gU is open (being homeomorphic to U) thus the complement of U in G is a union of opens and because of compactness of G we have that there are finitely many of those, thus U is indeed a closed subgroup of finite index. Conversely, for a closed subgroup V of finite index we have $G = \cup_{g \in G} gV$ which are finitely many, so V is the complement of a finite union of closed subgroups gV , so V is open. □

We now state the Galois correspondence for infinite Galois groups.

Theorem 2.13. *Let L be a subetension of the Galois extension $K|k$. Then $Gal(K|L)$ is a closed subgroup of $Gal(K|k)$. Moreover, in this way we get a bijection between subextensions of $K|k$ and closed subgroups of $Gal(K|k)$, where open subgroups correspond to finite extensions of k contained in K. A subextension $L|k$ is Galois over k if and only if $Gal(K|L)$ is normal in $Gal(K|k)$; in this case there is a natural isomorphism $Gal(L|k) \cong Gal(K|k)/Gal(K|L)$.*

Proof. First we assume that $L|k$ is a finite separable extension contained in K. By **Proposition 1.10** we embed it in a finite Galois extension $M|k$ contained in K. Then we have the group $Gal(M|k)$ by **Theorem 1.8**, which also contains $Gal(M|L)$ as a subgroup. Let U_L be the inverse image of $Gal(M|L)$ by the natural projection $Gal(K|k) \to Gal(M|k)$ that was discussed at the beginning of this subsection. The projection is continuous and $Gal(M|k)$ is finite with the discrete topology , therefore its finite subgroup $Gal(M|L)$ is open and so is U_L. We will prove that $U_L = Gal(K|L)$. Clearly, $U_L \subseteq Gal(K|L)$ as any element in U_L fixes L and is contained in $Gal(K|k)$. On the other hand, the image of $Gal(K|L)$ by the projection $Gal(K|k) \to Gal(M|k)$ is contained in $Gal(M|L)$, because of **Proposition 1.7 (3)**.

Let $L|k$ be an arbitrary subextension of $K|k$, then we saw that it can be written as a union of finite subextensions $L_a|k$. From what we have proven each $Gal(K|L_a)$ is an open subgroup of $Gal(K|k)$, hence it is also a closed subgroup of finite index by **Corollary 1.12.1**. The infinite intersection $\cap_a Gal(K|L_a)$ is equal to $Gal(K|L)$ and it is also a closed subgroup of $Gal(K|k)$.

Conversely, let $H \subseteq G$ a closed subgroup of G, then it fixes an extension $L|k$ and therefore is contained in $Gal(K|L)$, thus $H \subseteq Gal(K|L)$. To show equality, let $\sigma \in Gal(K|L)$ and U_M be the kernel of the map $Gal(K|L) \to Gal(M|L)$ where $M|L$ is a finite Galois extension. The kernel U_M is open as the preimage of an open set (the identity on $Gal(M|L)$) and it also contains the identity of $Gal(K|L)$. By the projection $Gal(K|L) \to Gal(M|L)$ we note that $Im(H) \subseteq Gal(M|L)$. In fact we have $Im(H) = Gal(M|L)$, as otherwise according to **Theorem 1.8** we would have that the image fixes an extension strictly larger than L, which would contradict that L is fixed by H. Therefore we have that $Im(h) = Im(\sigma)$ for some $h \in H$, which implies that $\sigma^{-1}h \in U_M$, i.e $h \in \sigma U_M$. Therefore, as U_M was chosen arbitrarily we have that for each U_M there exists $h \in H$ such that $h \in \sigma U_M$. Because σU_M is open and it also contains σ, then σU_M is an open neighborhood of σ and because U_M form an open neighborhood basis of $1 \in Gal(K|L)$, so does σU_M for $\sigma \in Gal(K|L)$. Therefore by the characterization of the topological closure: $a \in \tilde{H}$ if and only if every open neighborhood of a contains a point in H , we get that $\sigma \in \tilde{H}$. By assumption H is closed, thus $\sigma \in H$, so $H = Gal(K|L)$.

By **Corollary 1.12.1** we have that the open subgroups of $Gal(K|k)$ are precisely the closed subgroups of finite index and the closed subgroups of finite index fix an extension $L|k$, which has to be finite .

Finally, if we have that $H \subseteq G$ is a normal subgroup then like in the proof of **Theorem 1.8** we let $M = K^H$ and we get that $M|k$ is Galois with Galois group $Gal(M|k) = G/H$.Conversely, if the extension $M|k$ is Galois, then as in the proof of **Theorem 1.8** we have the natural projection map $Gal(K|k) \to Gal(M|k)$ with kernel $Gal(K|M) = H$ which is a normal subgroup of $Gal(K|k)$ and $M = K^H$. □

2.4 Category Theory

In this subsection we will define basic notions of Category theory,which we will vastly use throughout this project to prove equivalence between Categories. Once we have equivalence of categories, we can "draw" properties from one category to the other and that will lead to intresting results. We first start with the definition of a category.

Definition 2.8. A category consists of *objects* and *morphisms* between pairs of objects; given a pair of objects A, B in the category C the morphisms from A to B form a set, denoted $Hom(A, B)$. These are subject to the following constraints:

1. For any object A, the set $Hom(A, A)$ contains a distinguished element id_A, the identity morphism on A.

2. Given two morphisms $\phi \in Hom(B, C)$ and $\psi \in Hom(A, B)$, there exists a canonical morphism $\phi \circ \psi \in Hom(A, C)$, the composition of ϕ and ψ. The composition satisfies:

   - Given $\phi \in Hom(A, B)$ the relation $\phi \circ id_A = id_B \circ \phi = \phi$ holds.
   - For $\lambda \in Hom(A, B)$, $\psi \in Hom(B, C)$ and $\phi \in Hom(C, D)$ the equality $(\phi \circ \psi) \circ \lambda = \phi \circ (\psi \circ \lambda)$ holds (Associativity).

We continue with more definitions:

A morphism $\phi \in Hom(A, B)$ is an *isomorphism* if there exists $\psi \in Hom(B, A)$ such that $\psi \circ \phi = id_A$ and $\phi \circ \psi = id_B$; we denote the set of isomorphism between A and B by $Isom(A, B)$.

The *opposite category* C^{op} contains the same objects with C but with the arrows reversed, i.e for every pair of objects (A, B) of C, there is an elementwise bijection between the sets $Hom(A, B)$ of C and $Hom(B, A)$ of C^{op} preserving the identity morphism and the composition.

Definition 2.9. A (covariant) functor F between two categories C_1 and C_2 is a rule $A \mapsto F(A)$ on objects and a map $Hom(A, B) \to Hom(F(A), F(B))$ on morphisms which sends the identity to the identity and preserves composition. A *contravariant* functor from C_1 to C_2 is a *covariant* functor from C_1 to C_2^{op}.

Definition 2.10. If F and G are two functors with the same domain C_1 and target C_2, then we define a *morphism* of functors Φ between F and G to be a collection of morphisms $\Phi_A : F(A) \to G(A)$ for each object $A \in C_1$ such that for any morphism $\phi : A \to B$ in C_1 the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "\Phi_A"] \arrow[d, "F(\phi)"'] & G(A) \arrow[d, "G(\phi)"] \\
F(B) \arrow[r, "\Phi_B"] & G(B)
\end{tikzcd}$$

commutes. The morphism Φ is an isomorphism if each Φ_A is an isomorphism for every object A; in this case we write $F \cong G$.

Now we give the most important definition of this section.

Definition 2.11. Two categories C_1 and C_2 are *equivalent* if there exist two functors $F : C_1 \to C_2$ and $G : C_2 \to C_1$ and two isomorphisms of functors $\Phi : F \circ G \to id_{C_2}$ and $\Psi : G \circ F \to id_{C_1}$. Two categories are *isomorphic* if we can find $F \circ G = id_{C_2}$ and $G \circ F = id_{C_1}$. Finally, we say that C_1 and C_2 are $anti-equivalent$ (resp. $anti-isomorphic$) if C_1 is equivalent (resp. isomorphic) to C_2^{op}.

From the definition it follows that in order to prove equivalence of two categories we have to define two functors F and G. We will now give a characterization of category equivalence that requires only the construction of one of the two functors. To do so, we give a definition.

Definition 2.12.

1. A functor $F : C_1 \to C_2$ is $faithful$ if for any two objects A and B of C_1 the map of sets $F_{AB} : Hom(A, B) \to Hom(F(A), F(B))$ induced by F is injective; it is $fully\ faithful$ if the maps F_{AB} are bijective.
2. The functor F is *essentially surjective* if any object of C_2 is isomorphic to an object of the form $F(A)$.

We now have the equivalent characterization of category equivalence using only a functor in one direction.

Lemma 2.14. *Two categories C_1 and C_2 are equivalent if and only if there exists a functor $F : C_1 \to C_2$ which is fully faithful and essentially surjective.*

Proof. Lemma 1.4.9 of [10]. □

Remark 5. The notion of equivalence of categories means that "up to isomorphism the categories have the same objects and morphisms", but that does not mean that there are bijections between objects and morphisms. That comes from the fact that by essentially surjective property there exists an isomorphism $F(A) \cong B$ for $A \in C_1$, $B \in C_2$, but there could be more that one choices such that $F(A') \cong B$. To demonostrate this situation more explicity we will consider the following example:

Consider C_1 to be a category with one object c and one morphism $1_c : c \to c$ and the category C_2 with two objects a, b and four morphisms $1_a, 1_b$ the idenity morphisms and two isomorphisms $i_a : a \to b$ and $i_b : b \to a$. Let F be the functor $F : C_1 \to C_2$ that maps $c \mapsto a$ and $1_c \mapsto 1_a$. Then F is essentially surjective as every object of C_2 is isomorphic to $F(c) = a$. Also F is fully faithful as $1_c \mapsto 1_a$. Therefore we have an equivalence of the two categories, but we do not have a bijection between elements nor morphisms. If we were to drop the isomorphisms from the category C_2 then it would no longer be true that the two categories are equivalent.

We will now close this section with the notion of representability of a functor.

Definition 2.13. A functor F from a category C to the category of **Sets** is *representable* if there is an object $A \in C$ and an isomorphism of functors $F \cong Hom(A,\ )$.

The functor $Hom(A,)$ defined above sends an object $B \in C$ to $Hom(A, B)$ which is a set. The object A is called the *representing object*. If we have two objects B and D of C then every morphism $D \to B$ induces a morphism of sets $Hom(B,) \to Hom(D,)$ via composition, i.e if we have an object $A \in C$ with a morphism $B \to A$, then $D \to B \to A$ defines a morphism from $D \to A$.

Yoneda‘s Lemma empowers us with the fact that the representing object of each functor is unique up to unique isomorphism.

Lemma 2.15. ***Yoneda Lemma*** *If F and G are functors $C \to$ **Sets** represented by objects C and D respectively, then every morphism $\Phi : F \to G$ of functors is induced by a unique morphism $D \to C$ as above.*

Proof. Lemma 1.4.12 of [10]. □

Corollary 2.15.1. *The representing object of a representable functor F is unique up to unique isomorphism.*

Proof. Assume C and D are both representing objects of F then by Yoneda‘s lemma the morphism $Id : F \to F$ is induced by a unique morphism $D \to C$, but also by a unique morphism $C \to D$. The composition of those must be the identity on C or D and therefore there exists a unique isomorphism between the two representing objects.

□

2.5 Groethendieck‘s Reformulation of Galois Theory

In this section we will give a variant of the Galois‘ main theorem.

We first start with a base field k. We fix an algebraic closure $\hat{k}$ and construct the separable closure k_s of k, as we did in the second section. We proved in **Lemma 1.5** that the separable closure is always a Galois extension, thus we have the group $Gal(k_s|k)$. We let L be a finite separable extension of k; not necessarily a subextension of k_s. From **Proposition 1.2 2)** and **Proposition 1.3** we have there exist $[L : k]$ embeddings $L \to \hat{k}$ that leaves k elementwise fixed. The images of those homomorphisms must be contained in the subextension k_s of $\hat{k}$ (separable elements map to separable elements under injective homomorphisms that fix k elementwise). Therefore we can endow the set $Hom_k(L, k_s)$, which is defined to be the set of homomorphisms from L to k_s that fix k elementise, with a left action by $Gal(k_s|k)$ defined to be

$$a : Gal(k_s|k) \times Hom_k(L, k_s) \to Hom_k(L, k_s)$$

$$(\sigma, \phi) \mapsto \sigma \circ \phi$$

The two groups above have also a natural topology defined on them, so our action must respect this topology. We saw that $Gal(k_s, k)$ is a profinite group by **Proposition 1.11** and $Hom_k(L, k_s)$ carries the discrete (finite) topology. We recall the definition:

Definition 2.14. A topological group G acts continuously on a topological space X if the multiplication map $m : G \times X \to X$ is continuous.

We have the following characterization of a topological group G acting on a discrete space X.

Lemma 2.16. *If G is a topological group acting on a discrete topological space X then the action is continuous if and only if the stabilizer* $G_x = \{g \in G | gx = x\}$ *is open for all* $x \in X$.

Proof. Let the action be continuous, i.e the multiplication map is continuous. The restriction of the multiplication map on $G \times \{x\}$ is continuous, $\{x\}$ is an open subset of X in the discrete topology and thus the preimage of $\{x\}$ is open and because it is equal to the stabilizer of x we get that the stabilizer of x is open for all $x \in X$.

Now assume that the stabilizer of x is open for all x. For each $x \in X$ we have that the preimage under the multiplication map is equal to $U_x = \{(g, y) \in G \times X | gy = x\}$. Clearly $G_x \subseteq U_x$. If $y \in U_x - G_x$ then there exists $h \in G$ such that $hy = x$, therefore $y \in h^{-1}G_x \subseteq U_x$. The sets G_x and $h^{-1}G_x$ are homeomorphic and thus open by assumption. Therefore for each $y \in U_x$ there exists an open subset hG_x for some $h \in G$ that contains y and lies inside U_x, thus we conclude that U_x is open. □

We will now state a lemma that gives some properties of the action of $Gal(k_s|k)$ on $Hom_k(L, k_s)$. We still work under the assumption that L is a finite separable extension of k. We recall that a group action is called transitive if for every pair $x, y \in X$ there exists an element $g \in G$ such that $gy = x$.

Lemma 2.17. *The above left action of* $Gal(k_s, k)$ *on* $Hom_k(L, k_s)$ *is continuous and transitive, hence* $Hom_k(L, k_s)$ *as a* $Gal(k_s|k)$*-set is isomorphic to the left coset space of some open subgroup in* $Gal(k_s|k)$*. If L is Galois over k then this coset space is in fact a quotient by an open normal subgroup.*

Proof. By the above proposition we need to show that the stabilizer of each point is open and that will imply that the action is continuous. Let U be the stabilizer of an element $\phi \in Hom(L, k_s)$. Then for every $\sigma \in U$ we have that $\sigma(\phi(L)) = \phi(L)$ and $\phi(L)$ is a finite subextension of the Galois extension $k_s|k$ and therefore applying **Theorem 1.13** we have that U is an open subgroup of $Gal(k_s|k)$, thus the action is continuous. By assumption we have that L is a finite separable extension of k and thus applying the primitive element theorem (**Proposition 1.4**) we have that $L = k(a)$. Let f_a be the minimal polynomial of a. As we saw in the course of the proof of **Proposition 1.3** a k-algebra homomorphism from $L = k(a)$ to $\hat{k}$ (and thus to k_s) is characterized by the image of a, which also has to be a root of f_a. Let $\phi, \psi \in Hom_k(L, k_s)$, then we have a k-isomorphism of fields $k(\phi(a)) \cong k(\psi(a))$ by sending $\phi(a) \mapsto \psi(a)$, which extends to an automorphism σ on k_s. Then for this σ we have $\sigma\phi = \psi$, so the action is transitive. This implies that we can regard a fixed $\phi \in Hom_k(L, k_s)$ and write any $\psi \in Hom_k(L, k_s)$ as $\sigma\phi = \psi$ for an element $\sigma \in Gal(k_s|k)$.

We define the following map by using the previous argument:

$$g_\phi : Hom_k(L, k_s) \to U\backslash Gal(k_s|k)$$

$$\sigma \circ \phi \mapsto \sigma U$$

Which is immediately seen to be an open, surjective and continuous map. Injectivity follows from

$$\sigma U = \sigma' U \Rightarrow \sigma^{-1}\sigma' U = U \Rightarrow \sigma^{-1}\sigma'(\phi) = \phi \Rightarrow \sigma(\phi) = \sigma'(\phi)$$

Thus it is a homeomorphism and so $Hom_k(L, k_s) \cong U\backslash Gal(k_s|k)$, which the left coset space of the open subgroup U of $Gal(k_s|k)$. If U is a normal subgroup, then we obtain $Gal(k_s|k)/U$ and by **Theorem 1.13** this happens if and only if L is Galois over k. □

If we now have another finite separable extension M of k, any k-homomorphism $\mu : L \to M$ induces a map $\phi : Hom_k(M, k_s) \to Hom_k(L, k_s)$ via composition with μ ($a \mapsto a\circ\mu$). Both are $Gal(k_s|k)$-sets and we recall that a $G-equivariant$ map $f : X \to Y$ between two $G-sets$ X and Y is such that $f(gx) = gf(x)$ for all $g \in G$ and all $x \in X$. For ϕ we have that $\phi(\sigma a) = (\sigma a) \circ \mu = \sigma \circ (a \circ \mu) = \sigma\phi(a)$ and thus it is a $Gal(k_s|k)$ equivariant map. So we have obtained a contravariant functor from the category of finite separable extensions to the category of finite sets with continuous transitive left $Gal(k_s|k)$-action. We will now prove that this gives an anti-equivalence between the two categories.

Theorem 2.18. *Let k be a field with fixed separable closure k_s. Then the contravariant functor defined above mapping a finite separable extension $L|k$ to the finite $Gal(k_s|k)$-set $Hom_k(L, k_s)$, gives an anti-equivalence between the category of finite separable extensions of k and the category of finite sets with continuous, transitive left $Gal(k_s|k)$-action. Here Galois extensions give rise to $Gal(k_s|k)$-sets isomorphic to some finite quotient of $Gal(k_s|k)$.*

Proof. We will prove that $Hom_k(-, k_s)$ satisfies the fully faithful and essentially surjective properties given in **Lemma 1.14**. For essentially surjective we have to show that any continuous transitive left $Gal(k_s|k)$-set S is isomorphic to some $Hom_k(L, k_s)$ for some L finite separable extension. We pick a set S as above and a point $s \in S$ and because S has the discrete topology and $Gal(k_s|k)$ acts continuously then by **Lemma 1.16** the stabilizer of s denoted as U_s is an open subgroup of $Gal(k_s|k)$ and thus by **Theorem 1.13** it fixes a finite separable extension $L|k$. We want to show that $Hom_k(L, k_s) \cong S$. We define a map of $Gal(k_s|k)$-sets

$$p : Hom_k(L, k_s) \to S$$

$$\sigma \circ i \mapsto \sigma s$$

where i is the inclusion $L \to k_s$ and $\sigma \in Gal(k_s|k)$ (the action is transitive on both sets, so any element of both sets can be written as $\sigma \circ i$ and σs for some $\sigma \in Gal(k_s|k)$). This map is well defined. Indeed pick $\sigma_1 \circ i = \sigma_2 \circ i$, then

$\sigma_1^{-1}\sigma_2 \in Stab(i) = \{\sigma \in Gal(k_s|k) | \sigma(L) = L\}$, but L was picked as the fixed field under the action of U_s, therefore $Stab(i) = U_s$ and therefore $\sigma_1^{-1}\sigma_2 \in U_s$ implies that $\sigma_1 s = \sigma_2 s$. Injectivity follows from the previous argument in the reverse direction and surjectivity follows from the action on both sets being transitive. Therefore the map is in fact an isomorphism of $Gal(k_s|k)$-sets.

For the fully faithful property we have to show that given two finite separable extensions L and M of k, the set of k-homomorphisms $L \to M$ corresponds bijectively to the set of $Gal(k_s|k)$-maps $Hom_k(M.k_s) \to Hom_k(L, k_s)$. Let $f : Hom_k(M.k_s) \to Hom_k(L, k_s)$ be a $Gal(k_s|k)$-map. Both $Hom_k(M.k_s)$ and $Hom_k(L, k_s)$ are transitive $Gal(k_s|k)$-sets and therefore we can fix $\phi \in Hom_k(M.k_s)$ which will generate the whole set $Hom(M, k_s)$ by writing any element as $\sigma\phi$ for $\sigma \in Gal(k_s|k)$. We saw that f is $Gal(K_s|k)$-equivariant and thus $f(\sigma\phi) = \sigma f(\phi)$, so $f(\phi)$ will be our choice for the generator of $Hom_k(L, k_s)$. If we pick U_ϕ to be the stabilizer of ϕ then we have $\sigma\phi = \phi$ for any $\sigma \in U_\phi$, thus applying the map f we must have

$$f(\sigma\phi) = \sigma f(\phi) = f(\phi)$$

Thus the stabilizer U_ϕ is contained in the stabilizer of $f(\phi)$ which we will denote $U_{f(\phi)}$. Taking the fixed subfields of the action of U_ϕ and $U_{f(\phi)}$ on k_s, then because of the inclusion reversing correspondence, we have that $f(\phi)(L) \subseteq \phi(M)$ as subfields of k_s and both are finite and separable because the stabilizers are open (apply **Theorem 1.13**). Denoting $\psi : \phi(M) \to M$ to be the inverse to ϕ (ϕ is injective and thus an isomorphism on its image) then we have that $\psi \circ f(\phi) : L \to M$ which is an homomorphism that fixes k. So $\psi \circ f(\phi) \in Hom_k(L, M)$ and by composition with ϕ we see that this map induces f, therefore we have checked surjectivity, i.e having a map f we found $\psi \circ f(\phi) \in Hom_k(L, M)$ inducing it. Injectivity follows from the fact that if $g \in Hom_k(L, M)$ is another map inducing f we must have that $\phi \circ g = f(\phi)$ (both ϕ and $f(\phi)$ are generators of their $Gal(k_s|k)$-sets as we mentioned), but that means $g = \psi \circ f(\phi)$. Thus the fully faithful property holds.

For the last statement, let L be a Galois extension, then by **Lemma 1.17** $Hom_k(L, k_s) \cong Gal(k_s|k)/U$ where U is the stabilizer of an element $\phi \in Hom(L, k_s)$, which indeed is a finite quotient of $Gal(k_s|k)$. □

If we wish to extend the above anti-equivalence to $Gal(k_s|k)$-sets with not necessarily transitive action, then we have the natural replacement of finite separable extension by finite etale k-algebras.

Definition 2.15. A finite dimensional k-algebra A is *etale* over k if it is isomorphic to a finite direct product of finite separable extensions of k,i.e $A \cong \prod_{i=1}^{n} L_i$ with L_i finite separable extensions.

We have the following characterisation of finite etale k-algebras.

Proposition 2.19. *Let A be a finite dimensional k-algebra. Then the following are equivalent:*

1. *A is etale.*

2. *$A \otimes_k \hat{k}$ is isomorphic to a finite direct sum of copies of $\hat{k}$ (algebraic closure of k).*

3. *$A \otimes_k \hat{k}$ has no nonzero nilpotent elements.*

Proof. Proof in [10] **Proposition 1.5.5**. □

We now state the version of Groethendieck's reformulation of the Galois main theorem. We note that the $Gal(k_s|k)$ action on k_s, gives an action on $Hom_k(A, k_s)$ as above.

Theorem 2.20. *Let k be a field and k_s a fixed separable closure. Then the functor that maps a finite etale k-algebra A to the finite set $Hom_k(A, k_s)$ gives an anti-equivalence of categories between the category of finite etale k-algebras and the category of finite sets with continuous $Gal(k_s|k)$ action. Here separable field extensions give rise to sets with transitive $Gal(k_s|k)$ action and Galois extensions to $Gal(k_s|k)$ sets isomorphic to finite quotients of $Gal(k_s|k)$.*

Proof. The last part of the theorem is exactly the content of **Theorem 1.18**. Let $A = \prod_{i=1}^{n} L_i$ with L_i finite separable extensions. For essentially surjective property we have to show that any continuous finite $Gal(k_s|k)$-set S is isomorphic to some $Hom_k(A, k_s)$ for a finite etale k-algebra A. Following the proof of **Theorem 1.18** for each point $s \in S$, we consider its stabilizer U_s which is open by assumption of the continuity of the action and therefore applying **Theorem 1.13** we have that it fixes a finite separable extension which we will denote $L_s|k$. We define the same map of $Gal(k_s|k)$-sets

$$p : Hom_k(L_s, k_s) \to S$$

$$\sigma \circ i \mapsto \sigma s$$

For which injectivity and the well defined property hold for the same reason as previously, but surjectivity fails because S is no longer a transitive set. Thus we have injections $p_i : Hom_k(L_i, k_s) \to S$ for each stabilizer U_s of each point $s \in S$. Because of the assumption of S being finite then there are at most finite such injections. Actually, they are equal to the orbits of S under the action of $Gal(k_s|k)$ (as seen immediately from the image of those injections). Thus, each p_i is an isomorphism between $Hom(L_i, k_s)$ and an orbit S_i of S.Therefore

$$S = \cup_i^n S_i \cong \prod_i^n Hom_k(L_i, k_s) = Hom_k(\prod_i^n L_i, k_s)$$

The second isomorphism comes from $(p_1, ..., p_n)$ and the last equality holds because clearly $\prod_i^n Hom_k(L_i, k_s) \subseteq Hom_k(\prod_i^n L_i, k_s)$ as any $(\phi_1, ..., \phi_n)$ defines a homomorphism from $\prod_i^n L_i$ to k_s. For the inverse consider $\phi : \prod_i^n L_i \to k_s$ if $\phi(L_i) \neq 0$ (because ϕ is a non-trivial homomorphism from a field to another) then ϕ is injective and we can not have a homomorphism being non-trivial on the product $L_i \times L_j$, because the later contains zero divisors (consider $(1,0) \times (0,1)$) but k_s does not as it is a field. Thus we have proved the essentially surjective property, as $\prod_i^n L_i$ is a finite etale k-algebra.

For the fully faithful property consider two finite etale k-algebras $A = \prod_i L_i$ and $A' = \prod_j L'_j$. Then a map $f : \prod_i Hom_k(L_i, k_s) \to \prod_j Hom_k(L'_j, k_s)$ identifies with a family of maps

$$f_i : Hom_k(L_i, k_s) \to Hom_k(L'_j, k_s)$$

one for each i, which correspond bijectively to maps $L_i \to L'_j$ by **Theorem 1.19** and thus f is in one-to-one correspondence with a family of maps $L_i \to L'_j$, one for each i, which define uniquely a map $\prod_i L_i \to \prod_j L'_j$. □

This last theorem will be our passage from the Galois theory of fields to the Galois theory of covering spaces, which we will introduce in the next chapter.

3 Galois Theory for Covers

3.1 Covers

In this section we will give some basic definitions and propositions about covers.

Definition 3.1. 1. Let X be a topological space. A *space* over X is a topological space Y together with a continuous map $p : Y \to X$.

2. A morphism between two spaces $p_i : Y_i \to X$ (i=1,2) over X is given by a continuous map $f : Y_1 \to Y_2$ making the diagram

$$\begin{tikzcd} Y_1 \arrow[r, "f"] \arrow[dr, "p_1"'] & Y_2 \arrow[d, "p_2"] \\ & X \end{tikzcd}$$

commute, i.e $p_2 \circ f = p_1$.

3. A *cover* of X is a space Y over X, where the continuous map $p : Y \to X$ is subject to the following condition: each point of x of X has an open neighborhood U for which $p^{-1}(U)$ decomposes as a disjoint union of open non-empty subsets V_i of Y such that the restriction of p on each V_i is a homeomorphism onto U. We say that U in the above situation is evenly covered.

4. A morphism of two covers Y_1, Y_2 of X is a morphism of spaces over X.

Remark 6. From the definition of the cover of X it is an immediate result that $p : Y \to X$ is surjective, as every point $x \in X$ has an open neighborhood which is homeomorphic to an open subset V of Y and thus there exists $y \in V$ such that $p(y) = x$ for all $x \in U$. We call set $p^{-1}(x) = \{y \in Y | p(y) = x\}$ the fibre of x.

A first example of a cover is the following: Let X be a topological space and I a set with the discrete topology. Then the projection $p : X \times I \to X$ is a cover, because for every open subset U of X we have $p^{-1}(U) = U \times I = \cup_i(U \times \{i\})$, which are open (product topology) and disjoint because of the discreteness of I. This cover is called **trivial** cover of the topological space X. The next proposition shows that in fact every cover is locally the trivial cover.

Proposition 3.1. *A space Y over X is a cover if and only if each point x of X has an open neighborhood U such that the restriction of the projection $p : Y \to X$ to $p^{-1}(U)$ is homeomorphic (as a space over U) to a trivial cover.*

Proof. For the "if" part, if each point has such a neighborhood U then we have that $p^{-1}(U) \cong U \times I$ for some discrete set I. Then from the previous example we have that $p^{-1}(U)$ is a union of open,disjoint subsets of Y which map homeomorphically onto U and therefore p is a cover. For the other direction we have from the definition of the cover that every $x \in X$ has a neighborhood

U for which $p^{-1}(U)$ decomposes as a disjoint union of open subset V_i mapping homeomorphically onto U. Let $p^{-1}(U) = \cup_{i \in I} V_i$ be such a decomposition, then the map

$$f : \cup_{i \in I} V_i \to U \times I$$

$$v_i \mapsto (p(v_i), i)$$

where I is endowed with the discrete topology is a homeomorphism. Indeed for every $(p(v_i), i)$ there exists a unique $v_i \in V_i$ (the restriction of p on V_i is a homeomorphism) and the continuity of f and its inverse are also immediate from the discreteness of I. □

Remark 7. We note from the previous proof that for any $x \in X$ we have a one to one correspondence between the discrete set I and the set $p^{-1}(x)$ of the fibers of x. Thus the cardinalities satisfy $|I| = |p^{-1}(x)|$. If we set an equivalence relation on X by declaring two points $x \sim x'$ if and only if they have the same cardinality of their fibers, then we see that any open evenly covered neighborhood U of x is in the equivalence class of $[x]$. In particular each such class $[x]$ is open and thus X can be partitioned in disjoint open subsets. If X is connected then it can not be partitioned into two or more disjoint open subsets and thus there is only one class $[x]$. Thus we get the following corollary:

Corollary 3.1.1. *If X is connected then the cardinality of the fibers is the same for every x in X.*

In this chapter we will be mostly interested with covers coming from group actions. If we have a topological group G acting continuously on a topological space X, then we can form the quotient map $q : X \to X \backslash G$ which is a continuous surjective map. Here $G \backslash X$ denotes the set of the equivalence classes of X under the identifications $x \sim y$ if and only if there exists $g \in G$ such that $gx = y$ and the topology on it is defined by declaring a set $U \in G \backslash X$ to be open if and only if $q^{-1}(U)$ is open. To turn the quotient map into a cover we must be sure that the open subsets gU of X which all map homeomorphically to U in $G \backslash X$ are disjoint. That will be given by the next definition.

Definition 3.2. Let G be a group acting from the left on a topological space X. The action of G is *even* (or properly discontinuous) if each point $x \in X$ has an open neighborhood U such that the open sets gU are pairwise disjoint for all $g \in G \backslash \{1\}$.

Therefore we get that:

Lemma 3.2. *If G is a group acting evenly on a topological space X then the quotient map $q : X \to G \backslash X$ is a cover of $G \backslash X$.*

Proof. Let $[x] \in G \backslash X$ and choose an open neighborhood $V = q(U)$ where U is an open neighborhood of $x \in X$ satisfying the property of **Definition 2.2**(V is open from quotient topology). Then $q^{-1}(V) = \cup_{g \in G} gU$ is the union of open (homeomorphic to U), disjoint subsets of X that map homeomorphically on to V by the restriction of the quotient map to each gU. □

Remark 8. For the converse we also need that G acts freely ($gx = x$ only if $g = 1$), but this will follow from **Lemma 2.4** when X is connected by noting that multiplication by g defines an automorphism of the cover $q : X \to G\backslash X$. Then if G acts freely and the quotient map q is a cover then G is acting evenly on X. Indeed, assume that $q : X \to G/X$ is a cover. Then by definition every $x \in G\backslash X$ has an open neighborhood $U \subseteq G\backslash X$, such that $q^{-1}(U) = \cup_i V_i$ where V_i are disjoint opens mapping homeomorphically to U under the restriction of q to each V_i, i.e $q(V_i) \cong U$ for all i. If $x_i \in V_i$ then by definition of q we have that there exists $g \in G$ such that $gx_i = x$. If we assume that $gV_i \cap V_i \neq \emptyset$, then $y \in gV_i \cap V_i$ and thus $q(y) = [y] = [g^{-1}y]$ and both y and $g^{-1}y$ are in V_i. Because the restriction of q on V_i is a homeomorphism then $y = g^{-1}y$ in X. Here is where we need that G acts freely to say that $g = 1$ and so the action is even.

3.2 Galois Covers

In this section we will prove an analogue of **Theorem 1.8** for covers. From now on we will fix a base field X which will be assumed to be locally connected (each point x has a basis of open neighborhoods consisting of connected open subsets). First we need an analogue for the automorphism group of an extension defined in Section 1.2 .

Definition 3.3. Given a cover $p : Y \to X$ its automorphism group denoted $Aut(Y|X)$ are the automorphisms of Y as a space over X, i.e automorphisms $\sigma : Y \to Y$ such that the following diagram

$$\begin{tikzcd}
Y \arrow[r, "\sigma"] \arrow[dr, "p"'] & Y \arrow[d, "p"] \\
 & X
\end{tikzcd}$$

commutes, which means $p\sigma(y) = p(y)$ for all $y \in Y$.

Remark 9. Because of the commutativity of the diagram we note that if $y \in p^{-1}(x)$ for some $x \in X$, then $\sigma(y) \in p^{-1}(x)$, so σ preserves the fibers for a given point x, i.e the restriction to the fiber gives $\sigma : p^{-1}(x) \to p^{-1}(x)$. Therefore we have an action of $Aut(Y|X)$ on the fiber of any given point.

A key technical tool for working with covering spaces is the following proposition called unique lifting property.

Proposition 3.3. *Let $p : Y \to X$ a covering map. Suppose Z is a connected topological space and $f, g : Z \to Y$ are two continuous maps such that $p \circ f = p \circ g$. If there is a point $z \in Z$ with $f(z) = g(z)$, then $f = g$.*

Proof. Suppose $z \in Z$ is as above, that is, $y = f(z) = g(z)$. Let V be a connected open neighborhood of $p(y) = p(f(z))$ satisfying the condition in the definition of the cover (such a V exists, because X is locally connected) and let $U_i \cong V$ be the component of $p^{-1}(V)$ containing y. Due to the continuity of

f and g the preimages of U_i under those maps are open neighborhoods of the point $z \in Z$, therefore, taking the intersection of the two preimages we have an open neighborhood W of z that maps to U_i under both f and g. Because $p \circ f = p \circ g$ and because p is a homeomorphism restricted to U_i we have $f(W) = g(W)$, which shows that if f and g agree on a point, then they agree on a whole neighborhood, implying that the set $A = \{z \in Z | f(z) = g(z)\}$ is open. Also, if we have $z' \in Y$ such that $f(z') \neq g(z')$, then because $p \circ f = p \circ g$ we must have that $f(z')$ and $g(z')$ are in two different open subsets U_i and U_j, whose preimages are again open neighborhoods of z' and thus their intersection W' is an open neighborhood of z' for which $f(W') \neq g(W')$, implying that the set $B = \{z \in Z | f(z) \neq g(z)\}$ is also open (note that B is the complement of A). Therefore the set $A = \{z \in Z | f(z) = g(z)\}$ is non-empty by assumption and also open and closed in Z and because of the connectedness of Z it is the whole set Z, i.e $A = Z$ and so $f(z) = g(z)$ for all $z \in Z$. □

We are now able to prove the following result immediately. We call a cover $p : Y \to X$ connected, if Y is a connected topological space.

Lemma 3.4. *An automorphism ϕ of a connected cover $p : Y \to X$ such that $\phi(y) = y$ for a $y \in Y$ is the identity automorphism on Y.*

Proof. Applying **Proposition 2.3** for $Z = Y$, $f = id$ and $g = \phi$ we have that $\phi = id$ on Y. □

This lemma immediately implies the following important proposition:

Proposition 3.5. *If $p : Y \to X$ is a connected cover, then the group $Aut(Y|X)$ acts evenly on Y.*

Proof. Let y be a point of Y and $x = p(y)$. Because X is assumed to be locally connected then there exists a connected neighborhood V of x such that $p^{-1}(V)$ is a disjoint union of open sets U_i as in the definition of a cover. Let U_i be the one that contains y. We will prove that U_i is a neighborhood of y such that gU_i are pairwise disjoint for all $g \in G \backslash \{1\}$ as in **Definition 2.2**. Indeed, a non-trivial $\phi \in Aut(Y|X)$ maps U_i homeomorphically onto some U_j, because $p \circ \phi(U_i) = p(U_i) = V$ and so $\phi(U_i)$ is one of the sets U_j. Since Y is connected, **Lemma 2.4** applies and shows that for $\phi \neq Id_Y$ we must have $i \neq j$. □

Conversely, by using the fact that by **Lemma 2.2** we have that if G acts evenly on X then the quotient map $q : X \to G \backslash X$ is a cover of $G \backslash X$ we get :

Proposition 3.6. *If G is a group acting evenly on a connected space Y, then the automorphism group of the cover $q : Y \to G \backslash Y$ is precisely G.*

Proof. We note that we can naturally view G as a subgroup of $Aut(Y|(G\backslash Y)$, as any $g \in G$ defines a homeomorphism h_g

$$h_g : Y \to Y$$

$$y \mapsto gy$$

which is also compatible with the projection as y and gy map to the same class in $G\backslash Y$ under q, i.e $q(y) = q(gy) = [y]$ for any $g \in G$. Now let $\phi \in Aut(Y|(G\backslash Y)$ and $y \in Y$. Since the fibers of q are precisely the orbits of G we may find $g \in G$ such that $\phi(y) = gy$ and thus as we view G as a subgroup of $Aut(Y|(G\backslash Y)$ we have $g^{-1}\phi(y) = y$, thus $g^{-1}\phi$ is an automorphism fixing the point y and because Y is connected then **Lemma 2.4** implies that $g^{-1}\phi = id$, i.e $\phi = g$ for some $g \in G$ and so $Aut(Y|(G\backslash Y) \subseteq G$ giving $Aut(Y|(G\backslash Y) = G$. □

Now let $p : Y \to X$ be a connected cover. Then by letting $Aut(Y|X)$ act on Y we can form the quotient space $Aut(Y|X)\backslash Y$ coming from the quotient map $q : Y \to Aut(Y|X)\backslash Y$. If $q(y_1) = q(y_2)$ for $y_1, y_2 \in Y$ then $\phi(y_1) = y_2$ for an element $\phi \in Aut(Y|X)$ and because of the compatibility of ϕ with the covering map p we get $p(\phi(y_1)) = p(y_1) = p(y_2)$, so by passing to the quotient (Theorem 3.73 in [9]) we get that p factors as $p = \hat{p}q$ for a unique continuous map $\hat{p} : Aut(Y|X)\backslash Y \to X$. So

$$Y \xrightarrow{q} Aut(Y|X)\backslash Y \xrightarrow{\hat{p}} X$$

Definition 3.4. A cover $p : Y \to X$ is Galois if Y is connected and the above induced map $\hat{p}$ is a homeomorphism.

Remark 10. Here we note the analogy between a Galois cover and a Galois field extension. In the situation of a Galois extension we had a base field k and we characterized an algebraic extension $L|k$ to be Galois if the fixed field from the action of $Aut(L|k)$ on L was exactly k. Here the analogy is we start with a locally connected topological base space X and then every connected cover $p : Y \to X$ over X is Galois if the action of $Aut(Y|X)$ on Y induces an isomorphism $\hat{p} : Aut(Y|X)\backslash Y \to X$.

The next proposition gives us another analogy between the two categories. We recall that an algebraic extension $L|k$ was Galois if and only if $Aut(L|k)$ acted transitively on the roots of the minimal polynomial (Remark 4). We state now the analogue for Galois covers.

Proposition 3.7. *A connected cover $p : Y \to X$ is Galois if and only if $Aut(Y|X)$ acts transitively on the fibers $p^{-1}(x)$ of p for all $x \in X$.*

Proof. Assume first that $Aut(Y|X)$ acts transitively on the fibers of p, then for any two $y_1, y_2 \in Y$ such that $p(y_1) = p(y_2)$ there exists $\phi \in Aut(Y|X)$ such that $y_1 = \phi(y_2)$, but then $[y_1] = [\phi(y_2)] = [y_2]$ by definition of $Aut(Y|X)\backslash Y$ and thus $\hat{p}$ is injective. It is also surjective and continuous by its definition and also $\hat{p}$ is an open map as both p and q are and so indeed it is a homeomorphism.

For the converse if $\hat{p}$ is a homeomorphism then any $x \in X$ can be uniquely identified with $[y] \in Aut(Y|X)\backslash Y$ by $\hat{p}([y]) = x$, and $[y]$ contains all the elements of the form $\phi(z) = y$ for $\phi \in Aut(Y|X)$ and $z \in Y$. Then indeed for any $x \in X$ we have that $p^{-1}(x) = q^{-1}(\hat{p}^{-1}(x)) = q^{-1}([y]) = \{z \in Y|\phi(z) = y\}$ so $Aut(Y|X)$ acts transitively on the fibers. □

Remark 11. In fact it is enough for $Aut(Y|X)$ to act transitively on one fiber $p^{-1}(x)$, because p being a continuous map and Y being a connected topological space implies that X is also connected by the main theorem of connectedness (Theorem 4.7 , [9]) and so is $Aut(Y|X)\backslash Y$, which becomes a connected cover of X. If $Aut(Y|X)$ acts transitively on one then for this $x \in X$ there exists a unique $[y] \in Aut(Y|X)\backslash Y$ and from **Corollary 2.1.1** it follows that the cardinality of the fibers are the same, i.e $\hat{p}$ becomes a homeomorphism.

We need one more proposition to be able to give the analogue of **Theorem 1.8** for covering space.

Proposition 3.8. *Let X be a locally connected space, $q : Z \to X$ a connected cover and $f : Y \to Z$ a continuous map. If the composite $q \circ f : Y \to X$ is a cover, then so is $f : Y \to Z$.*

Proof. What we need to show is that every point $z \in Z$ has an open neighborhood W such that the preimage $f^{-1}(W)$ decomposes as a disjoint union of opens $V_i \subseteq Y$ which map homeomorphically to W by f. Let $q(z) = x \in X$, because both q and $q \circ f$ are covers then we choose a connected open neighborhood U of x that satisfies the property of the covering space for both maps (exists because of X being locally connected), therefore we get disjoint opens $V_i \subseteq Y$ and $W_j \subseteq Z$ that map homeomorphically to U by $q \circ f$ and q respectively. So we have $q \circ f(V_i) = q(W_j) = U$ for all i and j. Because $V_i \cong U$ and U is connected, then V_i is connected and so is $f(V_i)$ and because it maps homeomorphically to U by q then we have that $f(V_i) \subseteq W_j$ for some j and in fact $f(V_i) = W_j$ because of the fact that $q(f(V_i)) = q(W_j)$. Therefore f is an open map, as every point $y \in Y$ has an open neighborhood that maps homeomorphically to an open neighborhood of Z. In particular, $f(Y)$ is open.

We will now show that f is surjective. From the fact that Z is connected it is enough to check that the complement of the image $f(Y)$ is open. To that extend, let $z \in Z$ such that there does not exist $y \in Y$ that maps to it. Let W_j be the neighborhood that contains z and maps homeomorphically to U, then W_j must be disjoint from the image of $f(Y)$, otherwise we saw that there exists V_i connected open of Y such that $f(V_i) = W_j$ and thus we would have contradiction by the assumption that z is not in the image. So in fact we have that the complement of $f(Y)$ is indeed open and therefore $f(Y)$ is both open and closed subspace of connected Z, so $f(Y) = Z$ and thus f is surjective. It remains to justify that $f^{-1}(W_j)$ is a disjoint union of opens mapping homeomorphically to W_j. By surjectiveness of f we have that there exists at least one $U_i \subseteq Y$. If there are more than one then by the constuction of the sets $V_i \subseteq Y$ they are disjoint, open and map homeomorphically onto W_j which contains z. We thus conclude that f is a covering map. □

We are now ready to prove the Theorem

Theorem 3.9. *Let $p : Y \to X$ be a Galois cover, H a subgroup of $G = Aut(Y|X)$. Then p induces a natural map $\hat{p}_H : H\backslash Y \to X$ which turns $H\backslash Y$ into a cover of X.*

Conversely if $Z \to X$ is a connected cover fitting into a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\
 & X
\end{tikzcd}$$

then $f : Y \to Z$ is a Galois cover and actually $Z \cong H\backslash Y$ for some H subgroup of G. In this way we get a bijection between subgroups of G and intermediate covers Z as above. The cover $q : Z \to X$ is Galois if and only if H is a normal subgroup of G and in this case $Aut(Z|X) \cong G/H$.

Proof. Let H be a subgroup of G, then by passing to the quotient we get that p factors as the composite

$$Y \xrightarrow{p_H} H\backslash Y \xrightarrow{\hat{p}_H} X$$

where the first map is the natural quotient map. Because G acts evenly, then any subgroup of it acts also evenly on Y and therefore applying **Lemma 2.2** we have that the map p_H is a cover of $H\backslash Y$. Now let $x \in X$ and U a connected open neighborhood (X locally connected) of it which is evenly covered by p, we want to show that U is an evenly covered neighborhood under the map $\hat{p}_H$. As U is evenly covered by p then we have disjoint opens subsets $V_i \subseteq Y$ that map homeomorphically to U under p. Because p_H is a covering map then it is a local homeomorphism, open and a quotient map (Proposition 11.1, [9]), in particular each V_i is a connected open subset (homeomorphic to U) and so $p_H(V_i)$ is also a connected open subset of $H\backslash Y$. Suppose $p_H(V_i)$ and $p_H(V_j)$ are two such sets, then if their intersection is not empty then there exists an element $[y]$ for which $p_H(y_1) = p_H(y_2) = [y]$ for two different elements $y_1 \in V_i$ and $y_2 \in V_j$, but then by definition of the quotient map we have that there exists $\phi \in H$ such that $\phi(y_1) = y_2$, but ϕ is an automorphism of the cover $p : Y \to X$ and so we get $\phi(V_i) = V_j$ and so $p_H(V_i) = p_H(\phi(V_i)) = p_H(V_j)$. This shows that any $p_H(V_i)$ and $p_H(V_j)$ are either disjoint or equal. Since p_H is surjective, then $\hat{p}_H^{-1}(U)$ is equal to the disjoint union of the opens $p_H(V_i)$. It remains to show that $p_H(V_i)$ map homeomorphically to U by $\hat{p}_H$. As $p = \hat{p}_H \circ p_H$ and p is injective on each component V_i then so is p_H and because p_H is surjective as the quotient map and also open it follows that p_H is a homeomorphism on each component V_i. Therefore $\hat{p}_H$ is a homeomorphism on each $p_H(V_i)$ and also it maps to U. Thus we showed that $\hat{p}_H$ is a cover.

Conversely, suppose $q : Z \to X$ is a connected cover fitting into the commutative diagram above. We want to show that $f : Y \to Z$ is Galois. From **Proposition 2.8** we have that $f : Y \to Z$ is a connected cover. By **Proposition 2.7** we need to prove that $H = Aut(Y|Z)$ acts transitively on the fibers $f^{-1}(z)$ (by Remark 11 checking for one fiber is enough). By the assumption that $p : Y \to X$ is Galois we have that $Aut(Y|X) = G$ acts transitively on the fibers $p^{-1}(x)$ (**Proposition 2.7**). Because of the commutativity of the diagram we have $p = q \circ f$, so if $y_1, y_2 \in f^{-1}(z)$, then $y_1, y_2 \in p^{-1}(q(z))$, so there exists an element $\phi \in Aut(Y|X) = G$ such that $\phi(y_1) = y_2$, as $p : Y \to X$ is Galois and thus G acts transitively on the fibers. We want to show that $\phi \in H = Aut(Y|Z)$, which is equivalent to saying that the set $S = \{y \in Y : f(\phi(y)) = f(y)\}$ (by definition $Aut(Y|Z)$ is the set of automorphisms compatible with the cover f). We note that $q : Z \to X$ is a connected cover, Y a connected topological space, f and $f \circ \phi$ two continuous maps satisfying $q \circ f = q \circ (f \circ \phi)$ (because $p = p\phi$) and there is a point y_1 such that $f(y_1) = f(\phi(y_1) = f(y_2)$, therefore **Proposition 2.3** applies and gives that $f(y) = f \circ \phi(y)$ for all $y \in Y$ and so the set $S = Y$, implying that $\phi \in H = Aut(Y|Z)$ and therefore $Aut(Y|Z)$ acts transitively on the fibers and we get $f : Y \to Z$ is Galois. It is clear that $Z \cong H\backslash Y$, as the two maps f and the quotient map $Y \to H\backslash Y$ make the same identifications and so by the uniqueness of the quotient (Theorem 3.75, [9]) we get the homeomorphism. The fact that $H = Aut(Y|Z)$ is a subgroup of $G = Aut(Y|X)$ is immediate because any $\phi \in Aut(Y|Z)$ such that $f\phi = f$, implies that $q \circ (f\phi) = q \circ f$ and so $p\phi = p$.

Assume now that H is a normal subgroup of $G = Aut(Y|X)$. Then we form the quotient G/H which acts naturally on $H\backslash Y = Z$ (this is a general statement about group actions and we will prove it in the following remark) and this action preserves the projection $q : H\backslash Y \to X$, because any element $g \in G/H$ preserves the map p. So we obtain a group homomorphism $G/H \to Aut(Z|X)$ which is injective, as any non-trivial element in G/H acts non-trivially on Z. We get $(G/H)\backslash Z \cong G\backslash Y \cong X$, where the first isomorphism comes from noting that the quotient maps $Y \to H\backslash Y \to G/H\backslash(H\backslash Y)$ and $Y \to G\backslash Y$ make the same identifications (both maps identify elements that differ by an element $g \in G$, i.e $gy_1 = y_2$) and the second isomorphism comes from the assumption that p is Galois. By the injection of $G/H \to Aut(Z|X)$, it can be viewed as a subgroup and therefore $Aut(Z|X)\backslash Z \cong X$ so indeed we get a Galois cover $q : Z \to X$ and also $Aut(Z|X) \cong G/H$.

We assume now that $q : Z \to X$ is Galois and we also have $p : Y \to X$ to be Galois. We pick an element $y \in Y$ with image $q \circ f(y) = x$ in X and let $\phi \in Aut(Y|X)$. Then $q \circ f \circ \phi(y) = q \circ f(y) = x$ and therefore both $f \circ \phi(y)$ and $f(y)$ lie in the fiber of x over q, i.e $q^{-1}(x)$. Because q is Galois then there exists $\psi \in Aut(Z|X)$ such that $\psi(f(y)) = f(\phi(y))$. Then each $\phi \in Aut(Y|X)$ induces an automorphism $\psi \in Aut(Z|X)$. We claim that it is unique. Indeed if $\lambda \in Aut(Z|X)$ such that $\lambda(f(y)) = f(\phi(y))$, then $\lambda(f(y)) = \psi(f(y))$ and thus $\psi^{-1}\lambda$ fixes $f(y)$ and by **Lemma 2.4** we have that $\psi^{-1}\lambda = Id$, i.e $\psi = \lambda$. Both $\psi \circ f$ and $f \circ \phi$ are continuous functions from the connected Y (the cover p being Galois) to Z and they agree on y, therefore $\psi \circ f = f \circ \phi$ by **Proposition 2.3**. We thus get a group homomorphism $g : Aut(Y|X) \to Aut(Z|X)$ by mapping $\phi \mapsto \psi$, because $g(\phi_1\phi_2)f(y) = (\psi_1\psi_2)(f(y)) = (f(\phi_1\phi_2)(y)) = \psi_1(f(\phi_2(y)) = \psi_1(\psi_2(f(y))) = g(\phi_1)(g(\phi_2)(f(y)))$ and because both agree on $f(y)$ we thus get $g(\phi_1\phi_2) = g(\phi_1)g(\phi_2)$. It is clearly surjective, because $Z \cong H\backslash Y$ for the subgroup $H = Aut(Y|Z) \subseteq G = Aut(Y|X)$. Now we claim that $Aut(Y|Z) = ker(g)$ which will give us the result that $Aut(Y|Z) = H$ is normal. Let $\phi \in ker(g)$, then $g(\phi) = id_Z$ and therefore $f \circ \phi = f$ and so $\phi \in Aut(Y|Z)$. Conversely, if $\phi \in Aut(Y|Z)$ then $f \circ \phi = f = \psi \circ f$ and because of the uniqueness we stated above we get $\psi = id_Z$, therefore $g(\phi) = \psi = id_Z$ and we get that $\phi \in ker(g)$. We now have the assertion: $ker(g) = Aut(Y|Z)$, so $Aut(Y|Z)$ is a normal subgroup of $Aut(Y|X)$ and because the group homomorphism is surjective we also have $Aut(Y|X)/Aut(Y|Z) \cong Aut(Z|X)$ □

Remark 12. Let G be a group acting on a topological space Y and H a normal subgroup of G. Then the actions of G on Y induces an action of G/H on $H\backslash Y$,i.e

$$G/H \times H\backslash Y \to H\backslash Y$$

$$(gH, [y]) \mapsto gH[y] = [gy]$$

Well defined: let $[x], [y] \in H\backslash Y$, then there exists $h \in H$ such that $hx = y$, multiplying by g we get $ghx = gy$ but H is normal subgroup so $gH = Hg$ and thus exists $h' \in H$ such that $gh = h'g$. It follows that $h'gx = gy$ and therefore $[gx] = [gy]$ as classes of H. Also, if $gH = kH$ then by normality there exists $n \in H$ such that $g = nk$ and so $gx = nkx$ so $[gx] = [kx]$ as classes of H, i.e $gH[x] = [gx] = [kx] = kH[x]$.
Continuity follows from the continuity of $G \times Y \to Y$ by noting that the diagram $q \circ m : G \times Y \to H\backslash Y$ and $m' \circ (p, q) : G \times Y \to H\backslash Y$ commutes, i.e $q \circ m = m' \circ (p, q)$ where $q : Y \to H\backslash Y$, $p : G \to G/H$ are the canonical quotient maps, then because (p, q) is a quotient map as a product of quotients and $q \circ m$ is continuous as composition of continuous the characteristic property of the quotient topology (**Theorem 3.70,[9]**) gives $m' : G/H \times H\backslash Y \to H\backslash Y$ is continuous.
At last $eH[x] = [ex] = [x]$ and $((g_1H)(g_2H))[x] = (g_1g_2)H[x] = [g_1g_2x] = (g_1H)[g_2x] = (g_1H)(g_2H)[x]$.

3.3 Monodromy Action

Our goal is now to prove an analogue of **Theorem 1.20**. Instead of the absolute group $Gal(k_s|k)$ of a given base field k we had there, now we will have the fundamental group on a given base topological space X acting on the fibers of a given point. This is called the monodromy action. We will now recall basic definitions and facts about the theory of the fundamental groups.

Let X be a topological space. A *path* in X is a continuous function $f : I \to X$, where $I = [0, 1]$ is the closed interval. The *endpoints* of f are the points $f(0)$ and $f(1)$ in X; if they coincide then the path is called a *closed path* or a *loop*.

Let $f, g : X \to Y$ be two continuous functions. A *homotopy* from f to g is a continuous function $H : X \times I \to Y$ such that for all $x \in X$

$$H(x, 0) = f(x), H(x, 1) = g(x)$$

A $path - homotopy$ between two paths $f, g : [0, 1] \to X$ in X is a continuous function $H : [0, 1] \times [0, 1] \to X$ such that for all $x \in X$

$$H(s, 0) = f(s), \forall s \in [0, 1]$$

$$H(s,1) = g(s), \forall s \in [0,1]$$

$$H(0,t) = f(0) = g(0), \forall t \in [0,1]$$

$$H(1,t) = f(1) = g(1), \forall t \in [0,1]$$

we see here that a path-homotopy fixes the endpoints $(f(0) = g(0), f(1) = g(1))$. It is an easy fact that path-homotopy is an equivalence relation for two fixed points p, q which are the endpoints of the paths. We are mostly interested when $p = q$, so that we have a closed path. We define $[f]$ to be the path-homotopy class of f, where f is a path in X.

Given two paths $f, g : I \to X$ we say they are composable if $f(1) = g(0)$. For two composable paths we define their product $f \cdot g : I \to X$ to be $f \cdot g(s) = f(2s)$ if $0 \leq s \leq \frac{1}{2}$ and $f \cdot g(s) = g(2s - 1)$ if $\frac{1}{2} \leq s \leq 1$. The condition $f(1) = g(0)$ guarantees that the $f \cdot g$ is continuous. The product of paths is well defined on path classes (Proposition 7.10, [9]).

We now define the *fundamental group* of X based at p to be the set of path classes of loops based at p with the product of paths defined above, i.e

$$\pi_1(X,p) = \{[f] \text{ |f is a loop based at p}\}$$

the fact that it is a group follows from Theorem 7.11 in [9], its identity element is the class of the constant map at the point p denoted $[c_p]$ and the inverse of a class $[f]$ is the class $[f^{-1}]$ where $f^{-1}(s) = f(1-s)$.

Under the assumption that X is path connected, then any two points p, q can be connected via a path g and that gives an isomorphism on the fundamental groups $\Phi_g : \pi_1(X,p) \to \pi_1(X,q)$ by the rule $[f] \mapsto [g^{-1}][f][g]$. Thus we get that the fundamental group in this case does not depend on the base point. If in this case we have that the fundamental group is trivial (i.e contains only the constant path class $[c_p]$) then we say it is *simply connected.*

Our goal now is to give the action of the fundamental group on the fibers, which is called the **monodromy action**. Let $p : Y \to X$ be a cover, pick a point $x \in X$ and let $\pi_1(X,x)$ be the fundamental group on this base point. In order to define such an action we will have to "lift" paths from X to Y. The following proposition enables us to do so.

Proposition 3.10. *Let $p : Y \to X$ a cover, y a point in Y and $x = p(y)$.*

1. *Given a path $f : [0,1] \to X$ in X starting at x(i.e $f(0) = x$), there exists a unique path $\hat{f} : [0,1] \to Y$ such that $\hat{f}(0) = y$ and $p \circ \hat{f} = f$.*
2. *Assume moreover given a second path $g : [0,1] \to X$ homotopic to f. Then the unique lift $\hat{g} : [0,1] \to Y$ with $\hat{g}(0) = y$ and $p \circ \hat{g} = g$ has the same endpoints with $\hat{f}$, i.e we have $\hat{f}(1) = \hat{g}(1)$*

Proof. **Lemma 2.3.2,[10]** □

Let $y \in p^{-1}(x)$ a point in the fiber of the base point x, $[f] \in \pi_1(X,x)$ a path class represented by a path $f : [0,1] \to X$ with $f(0) = f(1) = x$ and

$\hat{f} : [0,1] \to Y$ its unique lift to Y, given by the previous proposition, such that $\hat{f}(0) = y$. We define the **monodromy action** to be the right action of $\pi_1(X, x)$ on y to be $y[f] = \hat{f}(1)$. Because of the second part of the proposition, we note that the action does not depend on the representative f, as any two representatives of a given class have the same endpoints and that $\hat{f}(1) \in p^{-1}(x)$. So in fact we have an action on the fibers $p^{-1}(x)$.

When X is connected and locally simply connected, then we have that there exists a simply connected cover of it called **universal cover** and we will denote it $\hat{X}$. Its existence is guaranteed by the following construction:
Let X be connected and locally simply connected and $x_0 \in X$ a base point. We define the space $\hat{X}_{x_0}$ to be the set that contains all the path classes $[f]$ that starts at x_0 and we define $q : \hat{X} \to X$ to be $q([f]) = f(1)$. This gives a well defined map because homotopic paths have the same endpoints by definition. We set a topology on $\hat{X}$ as follows: for each U simply connected neighborhood of each points $f(1) \in X$, we define $[fU] = \{[f \cdot a] :$ a is a path starting at f(1) in U$\}$. The fact that q then becomes a covering map comes from Theorem 11.43, [9]. The universal cover comes equipped with a universal point $\hat{x}_0 \in \hat{X}_{x_0}$ which is the unique lift of the constant path c_{x_0} based at x_0. We summarize the properties of the universal covering in the following proposition.

Proposition 3.11. *Let X be a connected and locally simply connected space. The following statements hold:*

1. *The universal cover defined above is unique up to isomorphism, i.e any two simply connected covers of X are isomorphic.*

2. *The universal cover $\hat{X}_{x_0}$ is connected.*

3. *There is an isomorphism of groups $Aut(\hat{X}_{x_0}|X) \cong \pi_1(X, x_0)$.*

4. *The cover $\pi : \hat{X}_{x_0} \to X$ is Galois, i.e $Aut(\hat{X}_{x_0}|X)$ acts transitively on the fibers of π.*

Proof. 1) follows from **Proposition 11.41, [9]**.
2) follows from **Theorem 11.43, [9]**.
3) follows from **Corrolary 12.9, [9]**.
4) follows from 3) by noting that the fundamental group $\pi_1(X, x_0)$ acts transitively on the fibers (**Theorem 11.22, [9]**). □

Remark 13. In part 3) the automorphism $Aut(\hat{X}_{x_0}|X) \cong \pi_1(X, x_0)$ is given by the map $\phi_\gamma(e) = e\gamma$ where $\gamma \in \pi_1(X, x_0)$ and $e \in p^{-1}(x)$. So we note that the automorphism group acts from the left, whereas the fundamental group acts from the right on a fiber.

We will now construct a functor that will be the analogue of the $Hom_k(-, k_s)$ functor defined in **Theorem 1.18** which was seen to send a finite separable extension $L|k$ to the finite $Gal(k_s|k)$-set $Hom_k(L, k_s)$. Given a space X and a base point $x \in X$ define the functor Fib_x from the category of covers of X to the category of sets equipped with a right $\pi_1(X, x)$-action which sends a cover

$p : X \to Y$ to the fiber $p^{-1}(x)$. To see that this indeed defines a factor we note that a homomorphism of covers of X, i.e $q : Y_1 \to Y_2$ by definition respects the fibers over x so we get a map $q : p_1^{-1}(x) \to p_2^{-1}(x)$, where p_1, p_2 are the two covering maps $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$, which is a homomorphism of sets equipped with a right action of $\pi_1(X, x)$ and this right action is respected as the unique lift of a closed path at the point x, starting from a point y_1 in Y_1 gets sent to the unique lift of the same path in Y_2 starting at $q(y_1)$. The next proposition will justify that this functor is indeed the analogue of the $Hom_k(L, k_s)$.

Proposition 3.12. *Given a space X and a point $x \in X$ the functor Fib_x is representable by the universal cover $\hat{X}_x \to X$, i.e $Fib_x(Y) \cong Hom(\hat{X}_x, Y)$ for a cover $Y \to X$.*

Proof. We want to show that given a cover $p : Y \to X$ and a point $x \in X$, a point $y \in p^{-1}(x)$ corresponds in a canonical and functorial manner to a morphism of covers π_y from $\pi : \hat{X}_x \to X$ to the cover $p : Y \to X$. Let $y \in p^{-1}(x)$, then for any path based at x there exists a unique lift of the path to Y starting at y, i.e if f is a path based at x then there exists unique $\hat{f}$ in Y such that $\hat{f}(0) = y$. The space $\hat{X}_x$ contains all the path classes starting at x and so it contains any $[f]$. We define $\pi_y : \hat{X}_x \to Y$ to be the map $[f] \mapsto \hat{f}_y(1)$, where $\hat{f}_y$ denotes the unique lift of f in Y starting at y. This is well defined as path homotopic paths have the same endpoints by **Proposition 2.10 2)**. We note that $\pi = p \circ \pi_y$ and therefore the continuity of π_y follows from the continuity of π and p. Then π_y becomes a covering map via **Proposition 2.8**. So we proved that a point $y \in p^{-1}(x)$ defines a covering homomorphism π_y from the universal cover to any cover Y of X. To construct an inverse to the map $y \mapsto \pi_y$ we send a homomorphism f of coverings to $f(\hat{x})$ where $\hat{x} = [c_x]$ the universal element of $\hat{X}$. Thus we get a point y in the fiber $p^{-1}(x)$. It is an easy verification that

$$y \mapsto \pi_y \mapsto \pi_y(\hat{x}) = \pi_y([c_x]) = \hat{c}_y(1) = y$$

$$f \mapsto f(\hat{x}) \mapsto \pi_{f(\hat{x})} = f$$

Finally we have obtained an isomorphism between the functors $Y \to Fib_x(Y)$ and $Y \to Hom(\hat{X}_x, Y)$, since given a homomorphism of covering spaces $q : Y_1 \to Y_2$ between two cover $q_1 : Y_1 \to X$ and $q_2 : Y_2 \to X$ mapping $y_1 \in Y_1$ to some $y_2 \in Y_2$, the induced map $Hom(\hat{X}_x, Y_1) \to Hom(\hat{X}_x, Y_2)$ maps π_{y_1} to π_{y_2}. Indeed we saw that a morphism is uniquely determined by where it sends the universal element $\hat{x}$ and we have that if $q(y_1) = y_2$ then $q(\pi_{y_1}(\hat{x})) = \pi_{y_2}(\hat{x})$ and therefore $q \circ \pi_{y_1} = \pi_{y_2}$. □

We next recover the monodromy action. Let $\phi : \hat{X}_x \to \hat{X}_x$ be a covering automorphism, then composition from right with ϕ yields a bijection $Hom(\hat{X}_x, Y) \to Hom(\hat{X}_x, Y)$, so we get a right action of $Aut(\hat{X}_x|X)$ on $Hom(\hat{X}_x, Y) \cong Fib_x(Y)$ by the left action of $Aut(\hat{X}_x|X)$ on $\hat{X}_x$. This is exactly the monodromy action on the fibers. To see this, by **Proposition 2.12** each point $y \in p^{-1}(x)$ corresponds to a morphism of covers $\pi_y : \hat{X}_x \to Y$. The map π_y is the map $[f] \mapsto \hat{f}_y(1)$ where $\hat{f}$ is the unique lift of f to Y with $\hat{f}(0) = y$ and $\pi_y([c_x]) = \hat{c}_y(1) = y[c_x]$. Any automorphism acting from the right on π_y gives $\pi_y \circ \phi : \hat{X}_x \to Y$ for which $\phi(\hat{x}) = \hat{x}'$ and so $\pi_y \circ \phi(\hat{x}) = \pi_y(\hat{x}')$. By Remark 12 we have $\phi(\hat{x}) = \hat{x}\gamma$ for $\gamma \in \pi_1(X, x)$ and so $\phi(\hat{x}) = [c_x]\gamma = [c_x\gamma]$ and from this it follows that

$$\pi_y \circ \phi(\hat{x}) = \pi_y[c_x\gamma] = (\widehat{c_x\gamma})_y(1) = y[c_x\gamma] = y[\gamma]$$

Which is indeed the monodromy action on the fiber y. Now we are ready to state the promised theorem.

Theorem 3.13. *Let X be a connected and locally simply connected topological space and $x \in X$ a base point. The functor Fib_x induces an equivalence of categories from the category of covers of X with the category of non-empty right $\pi_1(X, x)$-sets. Here connected covers correspond to $\pi_1(X, x)$-sets with transitive action and Galois covers to cosets of spaces of normal subgroups.*

Proof. We saw that X under these conditions has a simply connected universal cover $\hat{X}_x$ and $Fib_x(-) \cong Hom(\hat{X}_x, -)$. For fully faithfulness we have to show that given two covers $p_1 : Y \to X$ and $p_2 : Z \to X$ any map $f : Fib_x(Y) \to Fib_x(Z)$ of $\pi_1(X, x)$-sets comes from a unique homomorphism of covers $Y \to Z$. We may assume that Y and Z are connected, as otherwise we can split $Y = \cup_i Y_i$ into its disjoint connected components, define covering homomorphisms from each Y_i to a connected component Z_j of Z and then take their disjoint union to form a covering homomorphism from Y to Z. Applying **Proposition 2.12** and **Theorem 2.9** we get that both $\pi_y : \hat{X}_x \to Y$ and $\pi_z : \hat{X}_x \to Z$ (for $y \in Fib_x(Y), z \in Fib_x(Z)$) are Galois covers and that $Y \cong H_1 \backslash \hat{X}_x$ and $Z \cong H_2 \backslash \hat{X}_x$, where $H_1, H_2 \subseteq Aut(\hat{X}_x|X)$ and specifically $H_1 = Aut(\hat{X}_x|Y)$ and $H_2 = Aut(\hat{X}_x|Z)$. Let $y \mapsto f(y) = z$ where y is the image of the universal element $\hat{x}$ under the map π_y, i.e $y = \pi_y(\hat{x})$, and the map f is an arbitrary map on the fibers. Both π_y and π_z are quotient maps and we have for two path loops g, h based at x

$$\pi_y([g]) = \pi_y([h]) \Rightarrow \hat{g}_y(1) = \hat{h}_y(1) \Rightarrow y[g] = y[h] \Rightarrow f(y)[g] = f(y)[h]$$

$$\Rightarrow z[g] = z[h] \Rightarrow \pi_z([g]) = \pi_z([h])$$

so passing to the quotient we get a unique $p : Y \to Z$ such that $p \circ \pi_y = \pi_z$ for $z = f(y)$. This is a covering homomorphism as

$$\pi = p_1 \circ \pi_y = p_2 \circ \pi_z = p_2 \circ p \circ \pi_y \Rightarrow p_1 = p_2 \circ p$$

We note that the above homomorphism of covers is determined uniquely by where the map f sends $y = \pi_y(\hat{x})$. So we start with π_y, construct $\pi_{f(y)} = \pi_z$ and find a unique covering homomorphism p as above. We note also that if f was bijective then by the uniqueness of the quotient we would get $Y \cong Z$.

To prove the essential surjectivity we have to show that each right $\pi_1(X, x)$-set S is isomorphic to the fiber of some cover of X. Let S be a transitive set, then there exists only one orbit. This means that there exists $s \in S$ such that for every other element $s' \in S$ there exists a $[f] \in \pi_1(X, x)$ such that $s[f] = s'$, where $[f]$ is a closed loop. We fix a point $s \in S$ and we denote its stabilizer as $U_s \subseteq \pi_1(X, x)$. We want to show that $Hom(\hat{X}_x, Y) \cong S$ for some Y connected cover of X. Pick $Y = U_s \backslash \hat{X}_x$. We define

$$h : S \to Hom(\hat{X}_x, Y)$$

$$s[f] \mapsto [\hat{x}][f] = q(\hat{x})[f]$$

where $[\hat{x}]$ denotes the class of $\hat{x}$ in $Y = U_s \backslash \hat{X}_x$. Here we note that $Aut(\hat{X}_x|X) \cong \pi_1(X, x)$ acts transitively on $Hom(\hat{X}_x, Y)$ because it is exactly the monodromy action as remarked by the discussion preceeding **Theorem 2.13**, so any such homomorphism can be written as $q \circ \phi$, where $q : \hat{X}_x \to U_s \backslash \hat{X}_x$ the standard quotient map and ϕ an automorphism corresponding to an element $[f] \in \pi_1(X, x)$ under the isomorphism and because any homomorphism is uniquely determined by the image of $\hat{x}$ so infact every homomorphism is of the form defined in the map h $(q(\phi(\hat{x})) = q(\hat{x}[f]) = q(\hat{x})[f])$. Thus the map h is surjective. Injectivity follows because we chose $Y = U_s \backslash \hat{X}_x$, so we get that h is a bijective map between transitive $\pi_1(X, x)$-sets and also Y is connected as the continuous image of $\hat{X}_x$ under q. If S is not transitive then we separate it to its disjoint orbits $S = \cup_i S_i$ and we define isomorphisms

$$h_i : S_i \to Hom(\hat{X}_x, Y_i)$$

as above. Then we get an isomorphism

$$h : \cup_i S_i \to \cup_i Hom(\hat{X}_x, Y_i) = Hom(\hat{X}_x, \cup_i Y_i)$$

where the last equality holds because the image of the connected $\hat{X}_x$ via a continuous homomorphism of covers must lie inside one connected component.Lastly the statement about the Galois covers follows from **Theorem 2.9** in view of $Y = U_s \backslash \hat{X}_x$ defines a Galois cover if and only if U_s is a normal subgroup of $\pi_1(X, x)$.

□

Remark 14. We now compare the above theorem with **Theorem 1.20**. The role of the separable closure k_s is played by the universal cover $\hat{X}_x$. Here the universal cover constructed is depended on the point x which we had to fix, whereas **Theorem 1.20** depended on choosing an algebraic closure $\hat{k}$ in which k_s lies. In either case, choosing another algebraic closure $\hat{k}'$ or another point $y \in X$ yields non-canonical isomorphisms $\hat{k} \cong \hat{k}'_s$ and $\hat{X}_x \cong \hat{X}_y$, the first one is justified by **Proposition 1.2 3)** and the second one is justified because X being connected and locally simply connected implies that X is path-connected and thus we get an isomorphism on the fundamental groups for any two points and an isomorphism on the two universal coverings by a choice of a path connecting x and y. The fundamental group $\pi_1(X, x) \cong Aut(\hat{X}_x|X)$ plays the role of the absolute Galois group $Aut(k_s|k) = Gal(k_s|k)$ and the functor inducing the equivalence is $A \mapsto Hom(A, k_s)$ (a contravariant functor) and here it is $Y \mapsto Fib_x(Y) \cong Hom(\hat{X}_x, Y)$.

The above remark shows us that there is a strong analogy between the two Theorems, but it does not address the finiteness condition that **Theorem 1.20** has. To solve this problem first we will construct the *profinite completion* of the fundamental group $\pi_1(X,x)$, which is denoted as $\widehat{\pi_1(X,x)}$. As the name indicates it arises as a profinite group.

Remark 15. Given a group G, the set of its finite quotients can be turned into an inverse system as follows. Let Λ be the index set formed by the normal subgroups of finite index of G and we set a partial order on the index set by : $U_a \leq U_b$ if and only if $U_b \subseteq U_a$ for $a,b \in \Lambda$. If $U_a \leq U_b$ are two such normal subgroups of G, then we have a natural quotient map $\phi_{ab} : G/U_b \to G/U_a$, which is immediately seen to have the third property of **Definition 1.6**. In this way, by taking the inverse limit of the system we obtain the *profinite completion* of G, denoted $\widehat{G}$. There is also a natural homomorphism $G \to \widehat{G}$ by sending an element $g \in G$ to its class in each quotient.

By the above remark we turn the set of the finite quotients of $\pi_1(X,x)$ to its profinite completion $\widehat{\pi_1(X,x)}$. We call a cover $p : Y \to X$ finite if it has finite fibers for all points. If X is connected, **Corollary 2.1.1** implies that the cardinality of the fibers is the same for all points. We now state a Corollary of **Theorem 2.13** that bears a closer resemblance to **Theorem 1.20**.

Corollary 3.13.1. *Let X a connected and locally simply connected topological space and $x \in X$ a base point. The functor Fib_x induces an equivalence of the category of finite covers of X with the category of finite continuous right $\widehat{\pi_1(X,x)}$-sets. Connected covers correspond to finite $\widehat{\pi_1(X,x)}$-sets with transitive action and Galois covers to coset spaces of open normal subgroups.*

Proof. Everything follows from the previous Theorem, except from the fact that we have a continuous action by the profinite completion of the fundamental group. We assume again that the covers are connected, otherwise we handle it exactly in the same fashion as previously. Let $p : Y \to X$ a finite connected cover and X connected then the fibers have the same cardinality for every point. The action of $\pi_1(X,x)$ on the fiber $p^{-1}(x)$ factors via a finite quotient of $\pi_1(X,x)$, as there are only finitely many points in the fiber, that is, because $\pi_1(X,x)$ acts transitively on connected covers then we set an equivalence relation on $\pi_1(X,x)$ for which $[f] \sim [g]$ if and only if $y[f] = y[g]$ for a fixed point in the fiber $y \in p^{-1}(x)$ or equivalently if $[fg^{-1}]$ is in the stabilizer of y. The stabilizer H_y of y is thus a subgroup of finite index and hence contains a normal subgroup of finite index, the kernel of the natural map $\rho_{H_y} : G \to H_y$, $g \mapsto gU_y$ which we denote N. Then N is open as the preimage of the identity element on H_y which carries the discrete topology and thus is open and we also have that $H_y = \cup_g Ng$ as g runs through all elements of $\widehat{\pi_1(X,x)}$. All Ng sets are homeomorphic to N and thus they are open and that implies H_y is open. So the stabilizer H_y of every point is open (the point was chosen arbitrarily) and applying **Lemma 1.16** yields that the action of $\widehat{\pi_1(X,x)}$ on the discrete fibers is indeed continuous. Conversely, a continuous action of $\widehat{\pi_1(X,x)}$ on a finite set

S factors though a finite quotient as previously, which is also a finite quotient of $\pi_1(X,x)$ ($G \to \widehat{G} \to \widehat{G}/H$) and that gives rise to a cover $p : Y \to X$ by **Theorem 2.13** which has to be finite. □

3.4 Sheaves

In this section we will introduce sheaves which will be used in the following two chapters. We will also give a reformulation of the Galois theory is terms of locally constant sheaves. First we start with some basic definitions.

Definition 3.5. Let X a topological space. A *presheaf* of sets (or any category) $\mathcal{F}$ is a rule which assigns to each open subset U of X, a set (or any other object) $\mathcal{F}(U)$ and to each inclusion of open subsets $U \subseteq V$ of X, a map $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ which has the following two properties:

1. $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity morphism in the category for any $U \subseteq X$.

2. For any tower of opens $U \subseteq V \subseteq W$ we have that $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$.

The elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U and the maps ρ_{VU} are often referred to as restriction morphisms.

Remark 16. Similarly we can define a presheaf of abelian groups, groups, rings or any other category by requiring the rule to send the open subsets of X to abelian groups, groups or rings, that is $\mathcal{F}(U)$ is an object of the category of choice.

Definition 3.6. A morphism of presheaves $\Phi : \mathcal{F} \to \mathcal{G}$ is a collection of maps $\Phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for each inclusion $U \subseteq V$ the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\Phi_V"] \arrow[d, "\rho_{vu}"] & \mathcal{G}(V) \arrow[d, "\rho'_{vu}"] \\
\mathcal{F}(U) \arrow[r, "\Phi_U"] & \mathcal{G}(U)
\end{tikzcd}$$

The most basic example of a presheaf is the one that assigns to each open subset U of a topological space X the set $\mathcal{F}(U)$ of continuous functions $f : U \to \mathbb{R}$. Here the restriction map ρ_{VU} for two open subsets of X with $U \subseteq V$ is just the restiction of the continuous function $f : V \to \mathbb{R}$ to $f|_U : U \to \mathbb{R}$. Motivated by this, for any $s \in \mathcal{F}(V)$ we will denote $\rho_{VU}(s)$ to be $s|_U \in \mathcal{F}(U)$, the restriction of the section to the smaller open subset. The continuous maps have also another important property, that they can be glued together (**Lemma 3.23,** [**9**]). That means that if we have two continuous maps $f_1 : U_1 \to \mathbb{R}$ and $f_2 : U_2 \to \mathbb{R}$ with the property that they agree on the intersection, i.e $f_1(x) = f_2(x), \forall x \in U_1 \cap U_2$ then we can uniquely define the continuous map $f : U_1 \cup U_2 \to \mathbb{R}$ by setting $f(x) = f_i(x)$ if $x \in U_i$. This property leads us to the definition of *sheaves.*

Definition 3.7. A *sheaf* over a topological space X is a presheaf $\mathcal{F}$ that satisfies the two following axioms:

1. Given a non-empty $U \subseteq X$ and an open cover $\{U_i\}_{i \in I}$ of U, if two sections $s, t \in \mathcal{F}(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then we have that $s = t$.
2. For an open covering of U as above, given a family of sections $\{s_i \in \mathcal{F}(U_i) : i \in I\}$ with the property that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, whenever $U_i \cap U_j \neq \emptyset$, then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$. By the previous property such an $s \in \mathcal{F}(U)$ is unique.

The first axiom is referred to as the *identity* axiom and that is because any two sections agreeing on the restrictions to all open subsets are identified. The second axiom is referred to as the *gluing* axiom and it states that any family of sections defined on the open subsets, which agree on the overlaps $U_i \cap U_j$, can be glued together to a unique section over the whole U; just as in the case of continuous functions. We will now demonstrate two important examples of sheaves that will be used in this thesis.

Example 1. Let $\mathbb{C}$ be the complex numbers with the euclidean topology. It is a connected and locally connected space. Let $\{U_i\}_{i \in I}$ be a cover of $\mathbb{C}$ by open connected subsets. We define the *sheaf of holomorphic functions* on a connected open subset D to be the sheaf of rings whose sections over some open subset $U \subseteq D$ are the complex functions holomorphic on U. The restriction maps are given by the restriction of the holomorphic maps to connected open subsets, which are again holomorphic maps. The identity property of sheaves is immediately seen to hold, as any holomorphic map such that $f|_{U_i} = 0$ for every U_i open connected is $f = 0$ on the whole space. The second property comes from gluing holomorphic maps. This example carries on to any complex manifold and we shall see examples in the next chapter about Riemann surfaces.

Example 2. Let X and S be two topological spaces. We define a sheaf $\mathcal{F}_S$ on X by mapping $U \subseteq X$ to the set $\mathcal{F}(U)$ of continuous functions $U \to S$ for all non-empty open $U \subseteq X$. Just as in the case of real valued functions, this defines indeed a sheaf on X. If we now assume that S carries the discrete topology, then we call $\mathcal{F}_S$ the *constant sheaf* on X with value S. The name comes from the fact that over connected subsets $U \subseteq X$ of X, the sections of $\mathcal{F}_S(U)$ are constant, i.e if we have $f : U \to S$ continuous and U is connected, if f is not constant then it maps U to at least two distinct points s_1, s_2 in S, which are open in S and thus their preimages are open subsets of U and disjoint, thus they disconnect U, contradiction. So we get $\mathcal{F}_S(U) = S$.

Given any open subset U of a space X with a sheaf $\mathcal{F}$ defined on it, we can define a sheaf on U by taking the restriction of the sheaf on U, denoted $\mathcal{F}|_U$, by considering only the sections of $\mathcal{F}$ over open subsets of U. We can now give the definition of a locally constant sheaf.

Definition 3.8. A sheaf $\mathcal{F}$ on a topological space X is *locally constant* if every point $x \in X$ has an open neighborhood U of X such that the restriction $\mathcal{F}|_{\mathcal{U}}$ is isomorphic (in the category of sheaves) to a constant sheaf.

We will see that the category of locally constant sheaf is isomorphic to the category of covers. To this end, we will first turn a cover $p : Y \to X$ into a sheaf over X. Henceforth we assume that all spaces are locally connected. The next definition explains why we call elements of $\mathcal{F}(U)$ sections.

Definition 3.9. Let $p : Y \to X$ be a space over X and $U \subseteq X$ an open subset. A *section* of p over U is a continuous map $s : U \to Y$ such that $p \circ s = Id_U$.

We turn the space $p : Y \to X$ into a sheaf $\mathcal{F}_Y$ by mapping each open set $U \subseteq X$ to the set of sections of p over U and we define the restriction maps $\rho_{VU}(s) : \mathcal{F}_Y(V) \to \mathcal{F}_Y(U)$ to be the restriction of the sections $s|_U$. This in fact defines a presheaf on the space X, as the properties of **Definition 2.5** easily follows. The gluing property of **Definition 2.7** follows from the gluing lemma (**Lemma 3.23, [9]**) and the identity property follows from the continuity of the sections, because if there exists a point $x \in X$ such that $s(x) \neq 0$, then there exists an open neighborhood U of x such that $s(y) \neq 0$ for all $y \in U$ contradicting the fact that $s|_{U_i} = 0$. Therefore, $\mathcal{F}_Y$ defines a sheaf on X.

Proposition 3.14. *If $p : Y \to X$ is a cover then $\mathcal{F}_Y$ is a locally constant sheaf. It is constant if and only if the cover is trivial.*

Proof. Let $p : Y \to X$ be a cover, $x \in X$ a point and U a connected open neighborhood (X is locally connected) which is evenly covered, i.e $p^{-1}(U) = \cup_i V_i$ where V_i are connected open and disjoint subsets of Y. Then by **Proposition 2.1** we have that $p^{-1}(U) \cong U \times F$ where F is a discrete set. The image of a section $s : U \to Y$ is connected because of the fact that images of connected subsets under continuous maps are connected. From the definition of a section it follows that $p \circ s(U) = U$ and therefore $s(U)$ maps homeomorphically to U, so it has to be one of the connected components V_i. Therefore sections s are in one to one correspondence with points of the discrete set F, so the restriction of $\mathcal{F}_Y$ to U is isomorphic to the discrete set F, i.e $\mathcal{F}_Y|_U \cong F$ (because U is connected any subset has the same cardinality on the fibers by **Corollary 2.1.1**, i.e isomorphic to the same discrete space F). Thus the restriction is a constant sheaf and so $\mathcal{F}_Y$ is locally constant. The sheaf $\mathcal{F}_Y$ is constant if and only if $\mathcal{F}_Y \cong F$ which happens if and only if $p^{-1}(U) \cong U \times F$ for all U open connected neighborhoods which implies that $Y \cong X \times F$ as covers of X, so $p : Y \to X$ is a trivial cover.

□

Given a morphism $\phi : Y \to Z$ of covers of a space X, there is a natural map $\Phi : \mathcal{F}_Y \to \mathcal{F}_Z$ of the locally constant sheaves defined above, which comes from sending a section $s : U \to Y$ to $\phi \circ s : U \to Z$. This is a continuous map as the composition of two continuous maps. To see that it defines a section of $\mathcal{F}_Z$ over U, we pick a connected open subset $U \subseteq X$ that satisfies the evenly covered property for both covers $p_1 : Y \to X$ and $p_2 : Z \to X$. Then we have $p_1^{-1}(U) \cong U \times F_1$ and $p_2^{-1}(U) \cong U \times F_2$ for two discrete sets F_1, F_2 and we have turned both covers into their respective locally constant sheaves. By the commutativity of the morphism of covers we have $p_2 \circ \phi = p_1$ and for the section

s we have $p_1 \circ s = id_U$. Thus $p_2 \circ (\phi \circ s) = id_U$. We construct an open cover of X by choosing such open neighborhoods of each point of $x \in X$ that satisfies the property for both covers and now it follows that we get a morphism of locally constant sheaves $\Phi : \mathcal{F}_Y \to \mathcal{F}_Z$. Thus $Y \mapsto \mathcal{F}_Y$ defines a functor.

Theorem 3.15. *The above functor defines an equivalence between the category of covers of X and the category of locally constant sheaves on X.*

Though we can prove the theorem by showing the above functor is fully faithful and essentially surjective, it is more informative to construct a functor in the reverse direction. The functor that will be constructed will send a presheaf $\mathcal{F}$ to a space over X and a locally constant sheaf to a cover of X. We will need the notion of *stalks* of a presheaf.

Definition 3.10. Let $\mathcal{F}$ be a presheaf of sets on a topological space X and let $x \in X$ a point. The *stalk* of $\mathcal{F}$ on x, denoted $\mathcal{F}_x$, is defined as the disjoint union of the sets $\mathcal{F}(U)$ for all open neighborhoods U of x, modulo the following equivalence relation: $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are equivalent $(s \sim t)$ if there exists an open neighborhood $W \subseteq U \cap V$ such that $s|_W = t|_W$. That means

$$\mathcal{F}_x = \sqcup_{x \in U} \mathcal{F}(U) / \sim$$

We denote $s_x \in \mathcal{F}_x$ to be the class of a section in $\mathcal{F}(U)$ for an open neighborhood U of x.

Remark 17. We note from the definition that giving a point $s_x \in \mathcal{F}_x$ is the same as giving a pair (U, s) such that $s \in \mathcal{F}(U)$ and that the equivalence relation then becomes $(U, s) \sim (V, t)$ if and only if there exists $W \subseteq U \cap V$ such that $t|_W = s|_W$. From this interpretation we get that for every $U \subseteq X$ open neighborhood of x, we have an induced map

$$\mathcal{F}(U) \to \mathcal{F}_x$$

$$s \mapsto (U, s)$$

and of course this map is compatible with the restriction maps by definition of the stalk, that is $s|_V \mapsto (U, s|_V) = (V, s)$. So for a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ we get an induced morphism on the stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ by $(U, s) \mapsto (U, \phi_U(s))$ for every open neighborhood U of x. Since ϕ is compatible with the restriction maps, we get that this map is well defined, i.e if $(U, s) \sim (U', s')$ then there exists $W \subseteq U \cap U'$ such that $s|_W = s'|_W$ and so

$$\phi_U(s)|_W = \phi_W(s|_W) = \phi_W(s'|_W) = \phi_{U'}(s')|_W$$

where the first and fourth equality hold from the commutativity of the diagram in **Definition 2.6** and thus $(U, \phi_U(S)) \sim (U', \phi_{U'}(s'))$.

We will now construct a functor that sends a presheaf $\mathcal{F}$ to a space $p_{\mathcal{F}} : X_{\mathcal{F}} \to X$ over X. As a set $X_{\mathcal{F}}$ is set to be equal to all disjoint stalks at every point in X, i.e $X_{\mathcal{F}} = \sqcup_{x \in X} \mathcal{F}_X$. We set the natural projection $p_{\mathcal{F}}(\mathcal{F}_x) = \{x\}$ to be the constant map on each x and so $p_{\mathcal{F}}^{-1}(x) = \mathcal{F}_x$. What remains is to define a topology on $X_{\mathcal{F}}$ that should also turn $p_{\mathcal{F}}$ into a continuous map. Let $U \subseteq X$ be an open subset and $s \in \mathcal{F}(U)$ a section. We define a map $i_s : U \to X_{\mathcal{F}}$ by $x \mapsto (U, s)$ and we define a topology on $X_{\mathcal{F}}$ to be the coarsest topology such that each set $i_s(U)$ is open for each section $s \in \mathcal{F}(U)$ and each open $U \subseteq X$. To see that this generates a topology for $X_{\mathcal{F}}$, we first note that $\mathcal{F}_x = \cup_s i_s(U)$ for any open neighborhood U of x. Taking the union over all open subsets of X thus yields that $X_{\mathcal{F}} = \cup_U \cup_s i_s(U)$. Now let $p \in i_s(U) \cap i_t(V)$, then $p \in (U, s) \cap (V, t)$, so there exist open neighborhoods W, Q of p inside U, V such that $p|_W = s|_W$ and $p|_Q = t|_Q$, therefore $W \cap Q \neq \emptyset$ and $p|_{W\cap Q} = s|_{W\cap Q} = t|_{W\cap Q}$ and thus $p \in (W \cap Q, p) = i_p(W \cap Q) \subseteq i_s(U) \cap i_t(V)$. To see that $p_{\mathcal{F}}$ becomes continuous it is enough to note that for each open $U \subseteq X$, $p_{\mathcal{F}}^{-1}(U) = \cup_{x\in U} \mathcal{F}_x = \cup_{x\in U} \cup_s i_s(V)$ which is union of opens.

In case $\mathcal{F}$ is locally constant, then the space $X_{\mathcal{F}}$ becomes a cover of X. Indeed if U is a connected subset of X such that $\mathcal{F}|_U \cong F$ becomes constant, then for any connected subset V of U (a basis of those exists as X is locally connected) we have that $\mathcal{F}|_V \cong F$ for the same discrete set F and thus $\mathcal{F}_x \cong F$ for all $x \in U$ and that implies $p_{\mathcal{F}}^{-1}(U) \cong U \times F$ and from **Proposition 2.1** $X_{\mathcal{F}}$ is a cover.

We saw that a morphism $\phi : \mathcal{F} \to \mathcal{G}$ induces a morphism $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ on the stalks for every $x \in X$ compatible with the projections and thus we get a morphism $\Phi : X_{\mathcal{F}} \to X_{\mathcal{G}}$ compatible with the projections. The next lemma states that this morphism is a morphism of spaces over X.

Lemma 3.16. *Let $\mathcal{F}, \mathcal{G}$ be two presheaves defined on X, then the map $\Phi : X_{\mathcal{F}} \to X_{\mathcal{G}}$ defined above is a morphism of spaces over X.*

Proof. The map Φ is the result of gluing together the maps $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ for every $x \in X$. It is immediate that $p_{\mathcal{F}} = p_{\mathcal{G}}\Phi$ as for all x both map their stalk on x to x. It remains to be proven that Φ is continuous. Pick U open subset of X, a point $x \in X$ and a section $t \in \mathcal{G}(U)$. The basic open set $i_t(U)$ maps x to $t_x \in \mathcal{G}_x$. Each preimage $s_x \in \Phi^{-1}(t_x)$ lies in $\mathcal{F}_x$ by the construction of Φ and comes from a section $s \in \mathcal{F}(V)$ for $V \subseteq U$ neighborhood of x which can be chosen small enough so that $\phi(s) = t|_V$. Then $s_x \in i_s(V)$ which is a basic open subset of $X_{\mathcal{F}}$ by construction and we also have that $\Phi(i_s(V)) \subseteq i_t(V)$. Thus Φ is open as any point s_x in the preimage of $i_t(U)$ contains a basic open neighborhood $i_s(V)$ contained in the preimage of $i_t(V) \subseteq i_t(U)$. □

From the lemma we get that $\mathcal{F} \mapsto X_{\mathcal{F}}$ is a functor from the category of presheaves on X to the category of spaces over X. When $\mathcal{F}$ is locally constant we saw that $X_{\mathcal{F}}$ becomes a cover of X. To get an equivalence of categories we have to assume that we have a locally constant sheaf to utilize **Proposition 2.14**. We are ready to prove **Theorem 2.15**.

Proof. We have to show that $\mathcal{F} \mapsto X_{\mathcal{F}} \mapsto \mathcal{F}_{X_{\mathcal{F}}}$ is isomorphic to the identity morphism on sheaves, i.e $\mathcal{F}_{X_{\mathcal{F}}} \cong \mathcal{F}$ and for covers $Y \mapsto \mathcal{F}_Y \mapsto X_{\mathcal{F}_Y}$ is isomorphic to the identity morphism on the covers, i,e $X_{\mathcal{F}_Y} \cong Y$. For $\mathcal{F}_{X_{\mathcal{F}}} \cong \mathcal{F}$ we consider the map $\mathcal{F} \mapsto F_{X_{\mathcal{F}}}$ which maps a section $s \in \mathcal{F}(U)$ to the local section $i_s : U \to X_{\mathcal{F}}$ for every neighborhood U of x. This is immediately seen to be an isomorphism on the induced maps on the stalks, as $s_x \mapsto s_x$ and therefore we get an isomorphism of sheaves (**Proposition 1.1**,Ch.[II], [4]). To show $X_{\mathcal{F}_Y} \cong Y$ we choose an open connected cover $\{U_i\}_i$ of X such that $\mathcal{F}_Y$ is the constant sheaf on its restriction to each U_i and each U_i satisfies the definition of the covering of $Y \to X$. We saw that in this case $p_{\mathcal{F}_Y}^{-1}(U_i) \cong U_i \times F$, where $F \cong \mathcal{F}_Y|_{U_i}$ and from the proof of **Proposition 2.14** we have $p^{-1}(U_i) \cong U_i \times F$, thus $X_{\mathcal{F}_Y} \cong Y$. □

4 Riemann surfaces

4.1 Basic Concepts

Let X be a Hausdorff space. We recall that a function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at a point $z_0 \in \mathbb{C}$ if it is complex differentiable in an open neighborhood U of z_0 and it is holomorphic on an open set V if it is holomorphic on each point $z \in V$. Holomorphicity is a strong condition and it implies that f is infinitely differentiable in the neighborhood U of z_0 and that f is analytic, that is $f(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n$ for $a_n = \frac{f^{(n)}(z_0)}{n!}$ and every point z in the neighborhood U of z_0 which is given by the definition of holomorphicity (**Theorem 7.16,[5]**).

Definition 4.1. A *complex atlas* (or *smooth atlas*) on X is an open covering $\mathcal{U} = \{U_i : i \in I\}$ of X together with maps $f_i : U_i \to \mathbb{C}$ mapping U_i homeomorphically onto an open subset of $\mathbb{C}$ and such that $f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j) \subseteq \mathbb{C}$ is holomorphic for every pair $(i, j) \in I \times I$. The maps f_i are called the *complex charts* and the maps $f_j \circ f_i^{-1}$ are called the *transition maps.*

We now set an equivalence relation on the complex atlases of a space X by declaring $(\mathcal{U}, f_i) \sim (\mathcal{U}', f'_j)$ if $(\mathcal{U} \cup \mathcal{U}', f_i \cup f'_j)$ is a complex atlas on X. Here we have that $\mathcal{U} \cup \mathcal{U}'$ is a cover, $(f_i \cup f'_j)$ gives a family of complex charts and so the condition left to be checked is that $f'_j \circ f_i$ should be a holomorphic function from $f_i(U_i \cap U'_j) \to f'_j(U_i \cap U'_j)$ for all pairs $(i, j) \in I \times J$ and $U_i \in \mathcal{U}$, $U'_j \in \mathcal{U}'$. It is a simple verification that this defines an equivalence relation.

Definition 4.2. A *Riemann surface* is a Hausdorff space together with an equivalence class of complex atlases. This equivalence relation is called *complex structure* of the Riemann surface.

Example 3. A trivial example of a Riemann surface is any open subspace U of $\mathbb{C}$. We have the trivial open covering by the space U is self and the complex chart is the inclusion map $U \to \mathbb{C}$ which is an embedding and of course $f(z) = z$ is a holomorphic map.

The simplest non-trivial example is the projective line $\mathbf{P^1}(\mathbb{C})$ which comes from the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ with the following topology: the open sets are the opens sets of $\mathbb{C}$ together with the sets of the form $V \cup \{\infty\}$ where V is the complement of a compact set K in $\mathbb{C}$. We have $U_0 = \mathbb{C}$ and $U_1 = \mathbb{C}^* - \{0\}$ to be an open cover of $\mathbb{C}^*$ and we define complex charts $f_0(z) = z$ and $f_1(z) = \frac{1}{z}$. Those are both homeomorphisms $f_0(U_0) = \mathbb{C}$ and $f_1(U_1) = \mathbb{C}$ and we also have $f_0 \circ f_1^{-1} : f_1(U_0 \cap U_1) = f_1(\mathbb{C} \backslash \{0\}) \to f_0(U_0 \cap U_1) = f_0(\mathbb{C} \backslash \{0\})$ which is given by $f_0 \circ f_1^{-1}(z) = \frac{1}{z}$ and the same goes for $f_1 \circ f_0^{-1}(z) = \frac{1}{z}$ both being holomorphic functions. This space is connected because U_1, U_2 are connected with a common point and compact.

We now turn our focus on holomorphic maps between Riemann surfaces.

Definition 4.3. Let Y and X be Riemann surfaces. A *holomorphic* (or *analytic*) map $\phi : Y \to X$ is a continuous map such that for each pair $U_i \subseteq X$ and $V_j \subseteq Y$ of open subsets satisfying $\phi(V_j) \subseteq U_i$ and complex charts $f_i : U_i \to \mathbb{C}$

and $g_j : V_j \to \mathbb{C}$ the functions $f_i \circ \phi \circ g_j^{-1} : g_j(V_j) \to \mathbb{C}$ are holomorphic for all pair $(i, j) \in I \times J$.

We note that the above definition is independent of the complex charts f_i and g_j, because if we were to have other equivalent complex structures on X and/or Y, then the functions $f'_k \circ \phi \circ g'^{-1}_s$ would still be holomorphic.

We define a *holomorphic function* on an open subset $U \subseteq X$ to be a holomorphic map $\phi : U \to \mathbb{C}$ between Riemann surfaces, where $\mathbb{C}$ has the natural complex structure (Example 3). A complex chart is a holomorphic function. Indeed for a complex chart $f : U \to \mathbb{C}$ we have that the maps $f_i \circ f \circ g_j^{-1}$ are the transition maps which are holomorphic by definition (g_j are the charts of X and f_i are the identity charts of $\mathbb{C}$).

4.2 Finite Branched Covers

In this section we will study holomorphic maps $\phi : Y \to X$ between Riemann surfaces from a topological viewpoint. From now on we shall assume that the maps under consideration are non-constant on all connected components, i.e they do not map a whole component to a point. The next proposition states that any holomorphic map of Riemann surfaces is locally of the form $z \to z^k$ for $k \in \mathbb{Z}$

Proposition 4.1. *Let $\phi : Y \to X$ be a holomorphic map between Riemann surfaces (non-constant) and a point $y \in Y$ with image $\phi(y) = x \in X$. There exists open neighborhoods $V_y \subseteq Y$ and $U_x \subseteq X$ of y and x respectively such that $\phi(V_y) \subseteq U_x$ and complex charts $g_y : V_y \to \mathbb{C}$ and $f_x : U_x \to \mathbb{C}$ satisfying $f_x(x) = g_y(y) = 0$ such that the diagram*

$$\begin{tikzcd}
V_y \arrow[r, "\phi"] \arrow[d, "g_y"] & U_x \arrow[d, "f_x"] \\
\mathbb{C} \arrow[r, "z \to z^{e_y}"] & \mathbb{C}
\end{tikzcd}$$

commutes for an appropriate integer e_y, which is independent of the choice of complex charts.

Proof. **Theorem 2.1, [3]** □

Definition 4.4. The integer e_y above is called the *ramification index* or *branching order* of ϕ at y. The points y with $e_y > 1$ are called *branch points.* We denote the set of branch points of ϕ by S_ϕ.

An immediate corollary to **Proposition 3.1** is the following. We denote $p_n(z) = z^n$

Corollary 4.1.1. *A holomorphic map between Riemann surfaces is open.*

Proof. Because of the commutativity of the diagram in **Proposition 3.1** and because f_x and g_y are homeomorphisms, then we have $\phi = f_x^{-1} p_n g_y$ and all three maps are open, so every point $y \in Y$ has an open neighborhood V_y that maps to an open neighborhood of $\phi(y)$ and therefore ϕ maps open sets to open sets. □

Corollary 4.1.2. *The fibers of ϕ and the set S_ϕ are discrete closed subsets.*

Proof. Let $y \in \phi^{-1}(x)$ then $\phi(y) = x$ and the proposition implies that there exists an open neighborhood V_y of y such that the only element in the fiber of x in V_y is y ($z^n \mapsto 0$ if and only if $z = 0$). Therefore, the fiber of a given point is a discrete subset. They are also closed because ϕ is a continuous map so the preimage of a closed is closed. For the set S_ϕ let $y \in Y$ with $e_y > 1$, then the derivative of ϕ is defined locally around y and it is non-constant and holomorphic thus its set of zeroes is discrete and closed in Y. □

We now consider *proper* holomorphic maps. We recall that a map is *proper* if the preimage of each compact set is a compact set. The Riemann surfaces have the property that they are locally compact, as locally they are homeomorphic to open subsets of $\mathbb{C}$. For locally compact spaces we have that a proper map is closed (**Theorem 4.95,[9]**). Combining the fact that a proper holomorphic map is open (**Corollary 3.1.1**) with the fact that it is closed, gives us that the map is surjective under the assumption that X is a connected space. This is an important fact as we want to view holomorphic maps between Riemann surfaces as covering maps.

Remark 18. A finite cover $p : Y \to X$ is a proper map. Indeed, given a compact set $K \subseteq X$ we consider $\mathcal{U}$ to be an open cover of $p^{-1}(K)$, then because p is an open map we have that $p(U_i)$ form an open cover of the compact space K, thus it can be refined to a finite cover. The preimage of each $p(U_i)$ has to be finite because our cover is finite, thus we have refined $\mathcal{U}$ to a finite cover of $p^{-1}(K)$.

Proposition 4.2. *Let $\phi : Y \to X$ a proper holomorphic map of Riemann surfaces and X a connected space. Then ϕ is surjective with finite fibers and the restriction of ϕ to $Y \backslash \phi^{-1}(\phi(S_\phi))$ is a finite topological cover of $X \backslash \phi(S_\phi)$.*

Proof. We saw that ϕ is surjective because it is an open and closed map mapping to the connected X, i.e $\phi(Y)$ both open and closed subset of X thus the whole set. Finiteness of fibers follows from the fact that the preimage of the compact set $\{x\}$ is compact under the assumption that ϕ is proper and it is also discrete and closed by **Corollary 3.12** and thus finite. For the last statement, because ϕ is a closed map then $\phi(S_\phi)$ is closed, thus $X \backslash \phi(S_\phi)$ is open and so every point has an open neighborhood disjoint from $\phi(S_\phi)$. Any such x is the image of a $y \in Y$ with ramification index $e_y = 1$ and so locally it is a homeomorphism from a neighborhood V_y to a neighborhood U_x. From the fact that $\{x\}$ is compact, then there exist finitely many y_i in its preimage, thus also finitely many open neighborhoods V_i. Taking the intersection $\cap \phi(V_i) = \cap U_i$ we get that $\cap U_i$ is an open neighborhood of x that satisfies the definition of a cover. □

Definition 4.5. A *finite branched cover* is a proper surjective map that restrict to a finite cover outside a discrete closed subset S.

By the previous proposition a proper holomorphic map turns into a finite branched cover. In fact, the next theorem states states that there is an equivalence of categories between the category of Riemann surfaces Y equipped with a proper holomorphic map $Y \to X$ where X is a connected Riemann surfaces and S is a discrete closed subset of X such that all the branch points of Y lie in S with the category of finite topological covers of $X \backslash S$. We denote the category of Riemann surfaces Y with a proper holomorphic map $Y \to X$ whose branch points lie above S by $Hol_{X,S}$. A morphism f in this category is a holomorphic map compatible with the projections onto X, i.e the following diagram

$$\begin{tikzcd}
Y_1 \arrow[r, "f"] \arrow[dr, "\phi_2"'] & Y_2 \arrow[d, "\phi_1"] \\
& X
\end{tikzcd}$$

commutes.

Theorem 4.3. *In the above situation mapping a Riemann surface $\phi : Y \to X$ over X to the topological cover $Y \backslash \phi^{-1}(S) \to X \backslash S$ obtained by the restriction of ϕ on $Y \backslash \phi^{-1}(S)$, induces an equivalence of the category $Hol_{X,S}$ with the category of finite topological covers of $X \backslash S$.*

We first prove that starting with a Riemann space X and a topological cover, we can endow Y with a complex structure turning it into a Riemann surface. For all the following propositions and lemma‘s we assume that the covers and the Riemann surfaces are connected, if they are not we can split them into their connected components and do the same construction for each connected component.

Lemma 4.4. *Let X be a Riemann surface and $p : Y \to X$ a connected cover of X as a topological space. Then Y can be endowed with a unique complex structure for which p becomes a holomorphic mapping.*

Proof. The map p is surjective and for every point $x \in X$ we have a evenly covered neighborhood V of x and also a neighborhood W that satisfies the complex chart definition. Taking the intersection of the two, we can assume that each $x \in X$ has an open neighborhood satisfying both properties. Therefore, for each $y \in Y$ there exists an open neighborhood, the component of the preimage of the set $V \cap W$ under p, that maps homeomorphically onto $V \cap W$ and $V \cap W$ is homeomorphic through the complex chart $f : W \cap V \to \mathbb{C}$ to an open subset of $\mathbb{C}$. Therefore, we endow Y with a complex chart for each point by composing $f \circ p$. In this way we obtain a complex atlas on Y, because of surjectiveness of p and the holomorphicity of the transition maps follow from the holomorphicity of the transition maps of X and the connectedness of Y. The fact that p becomes a holomorphic map follows easily because $f_i p(p^{-1} f_j^{-1}) = f_i f_j^{-1}$

are holomorphic, as they are transition maps. The uniqueness of the complex structure follows from the fact that for any complex structure on Y we must have that p is an analytic isomorphism when restricted to preimages of evenly covered neighborhoods. □

Now we will prove that the functor of **Theorem 3.3** is essentially surjective.

Proposition 4.5. *Assume given a connected Riemann surface X, a discrete closed subset S of points of X and a finite connected cover $\phi' : Y' \to X'$, where $X = X \backslash S$. There exists a Riemann surface Y containing Y' as an open subset and a proper holomorphic man $\phi : Y \to X$ of Riemann surfaces such that $\phi|_{Y'} = \phi'$ and $Y' = Y \backslash \phi^{-1}(S)$.*

Proof. We fix a point $s \in S$. Because X is a Riemann surface then we have that s is inside an open neighborhood mapping homeomorphically to an open neighborhood of $\mathbb{C}$ by a complex chart. As $\mathbb{C}$ is locally connected space then we may take a connected neighborhood of the image of s in $\mathbb{C}$, we pick an open ball $B_\epsilon(im(s))$ of radiues ϵ and centered at $im(s)$. Then the complex chart maps homeomorphically a connected open neighborhood U_s of s to the open connected subset $B_\epsilon(im(s))$ of $\mathbb{C}$. By performing affine linear transformation in $\mathbb{C}$ ($z \mapsto \frac{1}{\epsilon}(z - im(s))$ we may assume that U_s is mapped homeomorphically onto the open unit disc $D \subseteq \mathbb{C}$. The restriction of ϕ' on $\phi'^{-1}(U_s - \{s\})$ is a finite cover and thus $\phi'^{-1}(U_s - \{s\})$ decomposes as a finite disjoint union of connected open components V_s^i which are all covers of U_s and map homeomorphically to it. We have $U_s - \{s\} \cong D - \{0\}$ by the complex chart and thus each V_s^i is a connected cover of $D - \{0\}$. The fundamental group of $D - \{0\}$ is $\pi_1(D - \{0\}) \cong \mathbb{Z}$ and therefore by **Theorem 2.9** V_s^i is isomorphic to a cover $D - \{0\} \to D - \{0\}$ by $z \mapsto z^n$ for some $n \in \mathbb{N}$ (the universal cover here being $L = \{z \in \mathbb{C} | Re(Z) < 0\}$ by the exponential map). Now we want to extend Y' and ϕ' so that they have the properties stated in the Theorem. We choose non-existing point in Y', one for each i and s, we denote them y_s^i and we take $Y = Y' \cup_{i,s} \{y_s^i\}$ and we define an extension of ϕ' by setting $\phi(y_s^i) = s$. We also extend the holomorphic isomorphisms $V_s^i \cong D - \{0\}$ to $\rho_s^i : V_s^i \cup \{y_s^i\} \cong D$ by mapping y_s^i to zero for each i and s and we define the topology on Y to be the one that turns these isomorphisms to homeomorphism. We set a complex structure on Y by keeping the same unique complex structure inherited by the cover ϕ' on Y' (**Lemma 3.4**) and setting for each point of the new points y_s^i the complex charts to be the maps ρ_s^i. In this way we form a complex atlas on Y. Then the extension ϕ of ϕ' is a holomorphic map, since away from y_s^i it is ϕ' which is holomorphic by **Lemma 3.4** and in the neighborhood of y_s^i it looks like $z \mapsto z^n$ which is a holomorphic map. Finally, by Remark 18 ϕ' is proper and so the extension ϕ is also proper as the compact sets of $X = X' \cup S$ differ from those of X' by finitely many points in S (because of compactness). □

Now only the fully faithful property is left to be proven for **Theorem 3.3**.

Proof. To prove fully faithfulness we have to show that for two Riemann surfaces Y and Z equipped with proper holomorphic maps $\phi_Y : Y \to X$ and $\phi_Z : Z \to X$

with no branch points above S and a morphism of covers $\rho' : Y' \to Z'$ over $X' = X\backslash S$, with $Y' = Y\backslash\phi_Y^{-1}(S)$ and $Z' = Z\backslash\phi_Z^{-1}(S)$, there exists a unique holomorphic map $\rho : Y \to Z$ extending ρ'. Again we assume that Y and Z are connected Riemann surfaces, as otherwise we can do the same for the connected components and then take the disjoint union. From **Proposition 2.8**, because $\phi'_Z \circ \rho' = \phi'_Y$ from the assumption that ρ' is a covering homomorphism, then $\rho' : Y' \to Z'$ becomes a connected cover ($Y' = Y\backslash S$ remains connected for S closed and discrete subset as it can be seen that it is path-connected due to its local complex structure). Applying **Lemma 3.4** then Y' can be endowed with a unique complex structure that turns ρ' to a holomorphic map. This complex structure must be compatible with that of Y because $\phi_Y|_{Y'} = \phi'_Y = \phi'_Z \circ \rho' = \phi_Z \circ \rho'$ is holomorphic with respect to both complex structures. It follows from the definition of a cover that for each point $y \in \phi_Y^{-1}(S)$ the map ρ' must send a sufficiently small neighborhood of y, which has been seen in **Proposition 3.5** to be holomorphically isomorphic to $D - \{0\}$, homeomorphically to an open neighborhood of a point $z \in \phi_Z^{-1}(S)$ which is also holomorphically isomorphic to $D - \{0\}$. We then extend ρ' to ρ by setting $\rho(y) = z$ for each such pair and keeping ρ' everywhere else. We end up with a unique holomorphic map ρ extending ρ' by similar arguments as in **Proposition 3.5**. □

An immediate result we get from the above Theorem is that the automorphism group of $\phi : Y \to X$ as an object of $Hol_{X,S}$ is the same as the automorphism group of the cover $Y' \to X'$. Therefore we call Y a *finite Galois branched cover* of X if Y' is Galois cover of X'.

Proposition 4.6. *Let $\phi : Y \to X$ be a proper holomorphic map of connected Riemann surfaces that is topologically a Galois branched cover. Then the following hold:*

1. *The group $Aut(Y|X)$ acts transitively on the fibers of ϕ.*

2. *If $y \in Y$ is a branch point with ramification index e, then so are all the points in the fibre $\phi^{-1}(\phi(y))$.*

Proof. From the fact that $\phi' : Y' \to X'$ is Galois then we have that $Aut(Y'|X')$ acts transitively on the fibers of ϕ' (**Proposition 2.7**) and because of the continuity of the automorphisms we have that $Aut(Y|X)$ acts transitively on the fibers of ϕ. The second statement comes from the fact $Aut(Y|X)$ acts transtively on the fibers and that an element in $Aut(Y|X)$ is a holomorphic homeomorphism, so locally they have the same ramification index e. □

4.3 Relation with Field theory

In the previous subsection **Theorem 3.3** gave us an equivalence of categories between Riemann surfaces with proper holomorphic maps and finite topological covers of a space $X\backslash S$. This gives us a relation between the theories developed in Chapter 2 and so far in Chapter 3. In this subsection, we will develop an equivalence of categories between Riemann surfaces (under conditions) and field extensions of a fixed field. We begin with the notion of meromorphic functions which will play the role of field extension.

Definition 4.6. Let X be a Riemann surface. A *meromorphic* function on X is a holomorphic function on $X \backslash S$, where S is a discrete closed subset of X such that moreover for all complex charts $\phi : U \to \mathbb{C}$ of X the complex function $f \circ \phi^{-1} : \phi(U) \to \mathbb{C}$ is meromorphic.

The meromorphic functions on a Riemann surface X define a ring with respect to the usual addition and multiplication of functions, we will denote it $M(X)$. On connected Riemann surfaces $M(X)$ turns into a field.

Lemma 4.7. *If X is a connected Riemann surface then the ring $M(X)$ is a field.*

Proof. **Lemma 3.3.2, [10]** □

It is easily seen that constant maps are meromorphic functions and therefore $M(X)$ contains a copy of $\mathbb{C}$ inside of it. In the previous subsection, we developed the theory by using non-constant holomorphic map, so we want to continue with this approach. On compact Riemann surfaces every holomorphic function is constant (**Corollary 2.8, [3]**), so it is not clear that meromorphic functions on compact Riemann surfaces can be non-constant. Conveniently, the next theorem enables us to assume so, as it gives that there exist non-constant meromorphic functions on compact Riemann surfaces.

Theorem 4.8. *Let X be a compact Riemann surface, $x_1, ..., x_n \in X$ a finite set of points in X. Then for any $c_1, ..., c_n \in \mathbb{C}$ set of complex numbers, there exists a meromorphic function $f \in M(X)$ that is holomorphic on x_i for every i such that $f(x_i) = c_i$ for all $1 \leq i \leq n$.*

Proof. **Corollary 14.13, [3]** □

We consider now a non-constant holomorphic map $\phi : Y \to X$ between Riemann surfaces. For each $f \in M(X)$ we have that f is holomorphic away from a discrete set S in X, i.e f is holomorphic on $X \backslash S$. We have that the restriction of ϕ to any open subset is holomorphic and therefore we get that $f \circ \phi$ is holomorphic away from $Y \backslash \phi^{-1}(S)$ so it is a meromorphic function on Y as $\phi^{-1}(S)$ is a discrete closed subset of Y. We thus get an induced ring homomorphism $\phi^* : M(X) \to M(Y)$ by sending $f \mapsto f \circ \phi$. Under the assumption that Y is compact and X is compact and connected, we have that ϕ is a proper surjective map with finite fibers. From the compactness of Y, if Y is not connected then it must be a finite union of its connected components Y_i. In this case $M(Y) = M(\sqcup_i Y_i) \cong \prod_i M(Y_i)$. We will now prove that for X, Y connected compact Riemann surfaces the field extension $M(Y)|M(X)$ is finite and therefore even for non-connected Riemann surface Y we will get that $M(Y)|M(X)$ will be finite by the above discussion. The field extension $M(Y)|M(X)$ has meaning when both X, Y are connected, because ϕ^* is a non-trivial field homomorphism (ϕ non-constant) and as such it must be injective and in the general case when Y is not connected we will realise $M(Y)$ as a finite etale algebra over $M(X)$.

Lemma 4.9. *Let $\phi : Y \to X$ be a proper holomorphic map of connected Riemann surfaces which has degree d as a branched cover. Then each meromorphic function $f \in M(Y)$ satisfies a (not necessary irreducible) polynomial equation of degree d over $M(X)$.*

Proof. Let S denote the branch points of ϕ. For each $x \notin \phi(S)$ there exists an open neighborhood U of x such that $\phi^{-1}(U)$ decomposes as a disjoint union of $V_1, ..., V_d$ which are homeomorphic to U, because of **Proposition 3.2**. Let $s_i : U \to V_i$ the inverse of $\phi : V_i \to U$, which is a biholomorphic map, and we set $f_i = f \circ s_i$ which is a meromorphic function defined on U (composition of holomorphic function followed by meromorphic function is meromorphic). We consider

$$F(T) = \prod(T - f_i) = T^d + a_{d-1}T^{d-1} + ... + a_0$$

where a_i are the elementary symmetric polynomials of the f_i, which are additions and multiplications between f_i and thus meromorphic on U. For another $x_1 \notin \phi(S)$, the same construction gives $s'_i : U' \to V'_i$ sections and $f'_i = f \circ s'_i$ meromorphic functions and a polynomial

$$F'(T) = \prod(T - f'_i)$$

In the intersection $U \cap U'$ we have that the sections agree and thus $f_i = f'_i$ and that implies that the polynomials $F'(T) = F(T)$ agree. Therefore we can extend the construction to a_i being meromorphic functions on the whole $X\backslash\phi(S)$. Our goal is to extend them to the whole space X.

To that end, we pick $x \in \phi(S)$ and a coordinate chart $f_x : U_x \to \mathbb{C}$, with U_x an open neighborhood of x, such that $f_x(x) = 0$. By composing with ϕ, we get that $f_x \circ \phi$ is a holomorphic function on a neighborhood of each $y \in \phi^{-1}(x)$, with $(f_x \circ \phi)(y) = 0$. Since f is a meromorphic function on every Y, then its poles have some finite degree. By picking a sufficiently large k and because $(f_x \circ \phi)(y) = 0$ for each $y \in \phi^{-1}(x)$, we have that $(f_x \circ \phi)^k f$ is holomorphic on every $y \in \phi^{-1}(x)$ and in particular, bounded in a punctured neighborhood of each y. We then get that $((f_x \circ \phi)^k f) \circ s_i) = ((f_x \circ \phi)^k \circ s_i)(f \circ s_i)) = ((f_x \circ \phi \circ s_i)^k (f \circ s_i)) = f_x^k f_i$ is a holomorphic map and thus also bounded in a punctured neighborhood U_x of x, i.e $f_x^k f_i$ is bounded on $U_x\backslash\{x\}$. That implies that $f_x^{kd} a_i$ is bounded on $U_x\backslash\{x\}$ and by Riemann‘s singularity theorem we can extend $f_x^{kd} a_i$ to a holomorphic function on all of U_x. Then we have that each a_i extends to a meromorphic function on the whole space X and thus $F(T) \in M(X)[T]$. Now it remains to show that f satisfies $\phi^*(F(T))$, which is the identification of $F(T)$ in $M(Y)$. We have that

$$\phi^* \circ F(f) = f^d + (a_{d-1} \circ \phi)f^{d-1} + ... + a_0 \circ \phi$$

We note that for every s_i we have $(\phi^* \circ F(f)) \circ s_i = (\phi^* \circ F \circ s_i)(f \circ s_i) = F(f_i) = 0$ and so we have that the function $(\phi^* \circ F(f)) \circ s_i$ is identically zero on U and because s_i is biholomorphism we get that $\phi^* \circ F(f)$ is identically zero on V_i for each i. Thus $f \in M(Y)$ satisfies the polynomial equation F (more accurately the image of it in $M(Y)$). □

We will now prove that we can find an irreducible polynomial so that $M(Y)|M(X)$ has exactly degree d.

Proposition 4.10. *Let $\phi : Y \to X$ a non-constant holomorphic map of compact connected Riemann surfaces, which has degree d as a branched cover. Then the induced field extension $M(Y)|M(X)$ is finite of degree d.*

Proof. We will show that in the case when X and Y are compact Riemann surfaces, we can find $f \in M(Y)$ satisfying an irreducible polynomial equation of degree d over $M(X)$. The key theorem is **Theorem 3.8**. Let $x \in X \backslash \phi(S)$ and let $y_1, .., y_d$ be the preimages of x under ϕ (ϕ is branched cover of degree d). By **Theorem 3.8** we find $f \in M(Y)$ with $f(y_i) = c_i$ such that $c_i \neq c_j$ for all $i \neq j$ and such that f is holomorphic on y_i for all i. By **Lemma 3.9** f satisfies a polynomial equation of degree d, thus it must satisfy an irreducible polynomial equation of degree $n \leq d$ over $M(X)$. Let $a_i \in M(X)$ be the coefficients of the irreducible polynomial equation. If all a_i are holomorphic on x, then $a_i(x) \in \mathbb{C}$ and so the polynomial $a_n(x)t^n + ... + a_0(x) \in \mathbb{C}[t]$ has d distinct roots, namely the $f(y_i)$, because the polynomial $F(t)$ is created by the product of $(t - f(y_i))$ and thus we get $n = d$. If one of the a_i have a pole in x, then we choose a point x' in a neighborhood of x, which again is not the image of a branch point, because of the discreteness of the branch points. Moreover, f is holomorphic at its preimages and takes distinct values at all of their preimages. So we can always find x' such that all $a_i(x')$ are holomorphic at x', because of the discreteness of the poles. Finally, we have that $M(Y) \cong M(X)(f)$, because if there existed another $g \in M(Y)$ such that $M(X)(f, g) \cong M(Y)$ then by the primitive element theorem we would have that there exists $h \in M(Y)$ such that $M(X)(f, g) = M(X)(h)$, implying $M(X)(f) \subseteq M(X)(h)$ and from the fact that h would satisfy an irreducible polynomial of degree at most d over $M(X)$ then we would have that $M(X)(f) = M(X)(h)$. Therefore f generates $M(Y)$ over $M(X)$ and the degree of the extension is d. □

By what we have mentioned in this subsection so far we have that the rule $Y \mapsto M(Y)$ gives a contravariant functor from the category of compact Riemann surfaces mapping holomorphically onto a fixed connected,compact Riemann surface X to the category of finite etale algebras over $M(X)$, the field of meromorphic functions of X. In fact, the next theorem states that this is an anti-equivalence of categories.

Theorem 4.11. *The above functor is an anti-equivalence of categories. In this anti-equivalence, finite Galois branched covers of X correspond to finite Galois extensions of $M(X)$ of the same degree.*

Proof. **Theorem 3.3.7,[10]** □

From the fact that $M(X)$ contains a copy of $\mathbb{C}$ inside it, we know that $M(X)$ is of characteristic 0 and from **Remark 3** we see that separability holds for any extension of $M(X)$. Thus the algebraic closure $\widehat{M(X)}$ is a separable extension and therefore the group $Gal(\widehat{M(X)}|M(X))$ is defined and it is also equal to Galois group of the separable closure of $M(X)$. Therefore **Theorem 1.20** applies and states that the category of finite etale algebras over $M(X)$ is anti-equivalent with the category of finite sets with continuous left $Gal(\widehat{M(X)}|M(X))$ action. Combining this with the above theorem yields:

Corollary 4.11.1. *Let X be a connected compact Riemann surface. The category of compact Riemann surfaces mapping holomorphically to X is equivalent to the category of finite sets with continuous left $Gal(\widehat{M(X)}|M(X))$ action.*

For the case when the base space $X = \mathbf{P^1}(\mathbb{C})$ the above Theorem yields an interesting result. First we have that $M(\mathbf{P^1}(\mathbb{C})) \cong \mathbb{C}(t)$, because every meromorphic function on $\mathbf{P^1}(\mathbb{C})$ is rational (**Corollary 2.9, [3]**) and any rational function defines a unique meromorphic function on $\mathbf{P^1}(\mathbb{C})$. We now have the following proposition:

Proposition 4.12. *Let Y be a connected compact Riemann surface. There exists a non-constant holomorphic map $Y \to \mathbf{P^1}(\mathbb{C})$. Consequently, $M(Y)$ is a field extension of $C(t)$.*

Proof. From **Theorem 3.8** we have that there exists a non-constant $f \in M(Y)$. We define $\phi_f : Y \to \mathbf{P^1}(\mathbb{C})$ to be

$$\phi_f(y) = f(y), \text{if y is not a pole of f.}$$

$$\phi_f(y) = \infty, \text{if y is a pole of f.}$$

for each $y \in Y$ we choose a complex chart $g : U \to \mathbb{C}$ such that f is holomorphic at $U - \{y\}$. From Example 3, we have that the two complex charts on $P^1(\mathbb{C})$ are $f_0(z) = z$ and $f_1(z) = \frac{1}{z}$ where f_0 is defined on $U_0 = \mathbb{C}$ and f_1 on $U_1 = \mathbb{C}^* - \{0\}$. If f is holomorphic at y, then $f_0 \circ \phi_f \circ g^{-1}$ is holomorphic on $g(U)$. If not, then it has a pole on y and then we have that $f_1 \circ \phi_f \circ g^{-1} = \frac{1}{f(z)}$ maps $g(U - \{y\})$ to a bounded open subset of $\mathbb{C}$ and therefore from Riemann's removable singularity Theorem we have that $f_1 \circ \phi_f \circ g^{-1}$ extends to a holomorphic function on $g(U)$. Thus ϕ_f is holomorphic and non-constant. Combining with **Proposition 3.10** we get that $M(Y)|M(P^1(\mathbb{C}))$ is a finite extension and from the preceding discussion we get that $M(Y)|\mathbb{C}(t)$ is a finite extension with degree same as the degree of the degree of the holomorphic map as a branched cover. □

Combining the above proposition with **Theorem 3.11** we get the following Corrolary:

Corollary 4.12.1. *The contravariant functor $Y \mapsto M(Y)$, $\phi \mapsto \phi^*$ induces an anti-equivalence between the category of connected compact Riemann surfaces with non-constant holomorphic maps and that of fields finitely generated over $\mathbb{C}$ of transcedence degree 1.*

4.4 Absolute Galois Group of $\mathbb{C}$(t)

We saw in **Corollary 2.13.1** that for a connected and locally simply connected topological space X and a base point $x \in X$, the functor Fib_x induced an equivalence between the categories of finite covers of X with the category of finite sets with right continuous $\widehat{\pi_1(X,x)}$ action. A connected Riemann space is such a space X and we saw that a holomorphic map restricts to a finite topological cover over $X' = X \backslash S$, where S is a closed and discrete space. When we restrict to compact connected Riemann surfaces, S becomes finite. So X' is a cofinite open subset of X and it also bears the properties of connectedness and locally simply connectedness from the space X. In **Corollary 3.11.1** we saw that the category of compact Riemann surfaces mapping holomorphically to X, which is equivalent to the category of finite topological covers over X' by **Theorem 3.3**, is equivalent to the category of finite sets with continuous left $Gal(\widehat{M(X)}|M(X))$ action. So we expect to have an isomorphism between $\widehat{\pi_1(X',x')}$ with a base point $x' \in X'$, and a quotient of $Gal(\widehat{M(X)}|M(X))$. The following Theorem confirms this intuition.

Theorem 4.13. *Let X be a connected compact Riemann surface and let X' be the complement of a finite set of points in X. Let $K_{X'}$ be the composite in a fixed algebraic closure $\widehat{M(X)}$ of all finite subextensions which arise from holomorphic maps of connected compact Riemann surfaces $Y \to X$ that restrict to a finite cover over X'. Then $K_{X'}$ is a Galois extension of $M(X)$ and its Galois group is isomorphic to $\widehat{\pi_1(X',x')}$ with a base point $x' \in X'$.*

We need the following Lemma to prove the above Theorem.

Lemma 4.14. *Every finite subextension of $K_{X'}|M(X)$ comes from a connected compact Riemann surface that restricts to a cover over X'.*

Proof. First we show that given two subextensions $L_i|M(X)$ for $i = 1, 2$ coming from connected compact Riemann surfaces $Y_i \to X$ that restricts to covers $p_i : Y_i' \to X'$, their composite L_1L_2 comes from a connected compact Riemann surface Y_{12}. For this we introduce the fibre product of covers $Y_1' \times Y_2' \to X'$, which is the subspace of $Y_1' \times Y_2'$ that consists of points (y, y') that satisfies $p_1(y) = p_2(y')$. It is equipped with natural projections $\pi_{Y_1'} : Y_1' \times_{X'} Y_2' \to Y_1'$ and $\pi_{Y_2'} : Y_1' \times_{X'} Y_2' \to Y_2'$ such that the following diagram

$$\begin{tikzcd}
Y_1' \times_{X'} Y_2' \arrow[r, "\pi_{Y_1'}"] \arrow[d, "\pi_{Y_2'}"] & Y_1' \arrow[d, "p_1"] \\
Y_2' \arrow[r, "p_2"] & X'
\end{tikzcd}$$

commutes. It can be proven that the fiber product of covers of a space is itself a cover of the space, thus $Y_1' \times_{X'} Y_2'$ becomes a cover of X'. From the equivalence of **Theorem 3.3**, we get a proper surjective holomorphic map and a Riemann surface Y_{12} with $\phi : Y_{12} \to X$ and because X is compact then Y_{12} is compact by properness of ϕ. The projections are continuous map and the spaces Y_i' are connected, thus the projections become covering maps from **Proposition 2.8** (note X' is locally connected being locally homeomorphic to an open subset of $\mathbb{C}$ which can be chosen to be connected as $\mathbb{C}$ is locally connected). Applying now **Theorem 3.3** we get a commutative diagram of compact Riemann surfaces

$$\begin{tikzcd}
Y_{12} \arrow[r] \arrow[d] & Y_1 \arrow[d] \\
Y_2 \arrow[r] & X
\end{tikzcd}$$

and applying **Theorem 3.11** we get a commutative diagram

$$\begin{tikzcd}
M(X) \arrow[r, "i_1"] \arrow[d, "i_2"] & M(Y_1) \arrow[d] \\
M(Y_2) \arrow[r] & M(Y_{12})
\end{tikzcd}$$

Because of the universal property of the fiber product, which characterizes it as a space $S \to X'$ over X' with a pair of morphisms $(\phi : S \to Y_1', \psi S \to Y_2')$ such that $p_1\phi = p_2\psi$ and because of the equivalence of categories, this universal property carries on to $M(Y_{12})$ and characterizes it by pairs of morphisms in the category of $M(X)$- algebras $(\gamma_1 : M(Y_1) \to M(Y_{12}), \gamma_2 : M(Y_2) \to M(Y_{12}))$ such that $\gamma_1 i_1 = \gamma_2 i_2$. But that is exactly the tensor product $M(Y_1) \otimes_{M(X)} M(Y_2)$, and thus $M(Y_{12}) \cong M(Y_1) \otimes_{M(X)} M(Y_2)$. We have from **Theorem 3.11** that $M(Y_1) \otimes_{M(X)} M(Y_2)$ is a finite etale algebra over $M(X)$. Let a decomposition $M(Y_1) \otimes_{M(X)} M(Y_2) = C_1 \times ... \times C_n$ as a product of finite separable extensions of $M(X)$ and let $g : L_1 \otimes_{M(X)} L_2 \to L_1L_2$ be the surjective map $a \otimes b \mapsto a \cdot b$. Then $ker(g)$ is a maximal ideal, because the compositum is a field and maximal ideals of a finite product of fields are of the form $C_1 \times ... \times \{0\} \times ..C_n$, thus $M(Y_1) \otimes_{M(X)} M(Y_2)/kerg = C_i \cong L_1L_2$ for some i. Because Y_{12} is compact, we decompose it into its disjoint connected components $Y_{12} = K_1 \sqcup K_2 \sqcup ... \sqcup K_M$ and we get $M(Y_{12}) = \prod_i M(K_i)$, which are finite separable extensions of $M(X)$ and so we may assume $M(K_j) = C_i = L_1L_2$ for some j. Connected components are closed subsets and closed subsets of compact spaces are compact, thus we have that K_j is a compact and connected Riemann surface and $M(K_j) = L_1L_2$.

From what we have proved we get that $K_{X'}$ can be written as a union of finite subextensions $L_1 \subseteq L_1L_2 \subseteq ...$ of $M(X)$ coming from connected compact Riemann surfaces that restrict to a cover over X'. To conclude we have to show that if L is a finite subextension $K_{X'}|L|M(X)$ which comes from a connected compact Riemann surface that restricts to a finite cover over X', then any subextension $L|K|M(X)$ has also this property. We have that $L = M(Y)$ for a connected compact Riemann surface and $K = M(Z)$ for a compact Riemann surface $Z \to X$ (**Theorem 3.11**). We have holomorphic maps $Y \to X$ and $Z \to X$ to the connected and compact space X and also we get that $Y \to X$ factors through $Z \to X$ because of the inclusions coming from $M(X) \to M(Z) \to M(Y)$. Those maps are proper holomorphic maps and thus surjective, so we get that the map $Y \to Z$ is surjective and so Z is connected, as the continuous image of the connected Y. We want to show that $Z \to X$ restricts to a finite cover over X'. Indeed we get that $Y' \to X'$ factors via $Y' \to Z' \to X'$ so if $Y \to X$ contains its branch points over S, where $S = X \backslash X'$, then so does $Z \to X$ and therefore by **Theorem 3.3** $Z \to X$ restricts to a cover over X' and it has to be finite, because of the finiteness of $Y' \to X'$. □

We will now prove **Theorem 3.13**.

Proof. First we will prove that $K_{X'}$ is a Galois extension of $M(X)$. First we note that for any finite extension $L|M(X)$ coming from a connected Riemann surface that restricts to a cover over X' there exists a finite Galois extension $M|L|M(X)$ and this corresponds to a finite Galois branched cover of X of a compact Riemann surface (**Theorem 3.11**), thus is contained in $K_{X'}$. Let $L = M(Y)$ be an arbitrary extension of $M(X)$ that comes from a connected Riemann surface that restricts to a cover over X' and let $\sigma \in Gal(\widehat{M(X)}|M(X))$, then $\sigma(L) \cong L$ as field extensions of $M(X)$ inside the algebraic closure $\widehat{M(X)}$ and therefore $\sigma(L)$ corresponds to an isomorphic Riemann surface to Y by **Theorem 3.11**, thus it corresponds to a connected compact Riemann surface that restricts to a cover over X', so all the Galois conjugates of L are inside $K_{X'}$ which implies that $K_{X'}$ is a Galois extension of $M(X)$.

We will now show that $Gal(K_{X'}|M(X)) \cong \pi_1\widehat{(X', x')}$. First we note that by **Lemma 3.14** and **Proposition 1.11** we have that $Gal(K_{X'}|M(X))$ can be turned into the inverse limit taken over all Galois subextensions, which come from Galois branched covers by **Theorem 3.11**. Therefore we have that

$$Gal(K_{X'}|M(X)) \cong \varprojlim_{\text{L Galois}} Gal(L|M(X))$$

From the fact that $Gal(L|M(X)) = Aut(L|M(X))$ then we have from the equivalence of the categories that

$$Gal(L|M(X)) \cong Gal(Y|X)$$

Where the spaces Y are all finite Galois branched covers restricting to finite covers over X'. Thus we get from **Theorem 3.3** that

$$Gal(Y|X) \cong Gal(Y'|X')$$

Combing we get

$$Gal(K_{X'}|M(X)) \cong \varprojlim_{\text{Y' Galois}} Gal(Y'|X')$$

From **Corollary 2.13.1** we get that finite Galois covers correspond to finite coset spaces of open normal subgroups, thus

$$Gal(K_{X'}|M(X)) \cong \varprojlim_{\text{N normal open}} \pi_1(X', x')/N$$

The last inverse limit is the profinite completion of $\pi_1(X', x')$ by **Remark 15**, thus

$$Gal(K_{X'}|M(X)) \cong \pi_1\widehat{(X', x')}$$

□

We will now give an application to the "inverse Galois problem" over the field $\mathbb{C}(t)$. First we recall that the complex projective line (or *Riemann Sphere*) $P^1(\mathbb{C})$ is homeomorphic to the sphere S^2 as a topological space and $S^2 - \{p\}$ is homeomorphic to $\mathbb{R}^2$ by stereographic projection (**Example 3.21,[9]**). From the fact that homeomorphic spaces have isomorphic fundamental groups (**Corollary 7.26, [9]**) we have that

$$\pi_1(P^1(\mathbb{C}) - \{p_1, ..., p_n\}) \cong \pi_1(\mathbb{R}^2 - \{r_1, ..., r_{n-1})\}$$

And the latter fundamental group can be seen to be a free group on $n-1$ generators by covering $\mathbb{R}^2 - \{r_1, ..., r_{n-1})$ with $n-1$ open subsets $U_i - \{r_i\}$, where each U_i is a connected open subset containing only one of the points r_i. Those opens are path-connected and their intersections are simply connected spaces, thus applying inductively Seifert Van-Kampen theorem for simply connected intersection (**Corollary 10.4**,[9]) we get that the fundamental group $\pi_1(\mathbb{R}^2 - \{r_1, ..., r_{n-1})$ is isomorphic to the amalgamated free product of the groups $\pi_1(U_i - \{r_i\}) \cong Z$. Thus we get that

$$\pi_1(P^1(\mathbb{C}) - \{p_1, ..., p_n\}) \cong \mathbb{Z} \star ... \star \mathbb{Z} \cong F_{n-1}$$

We have also seen that $\mathbb{C}(t) = M(P^1(\mathbb{C}))$.

Theorem 4.15. *Every finite group G arises as the Galois group of some Galois extension $L|\mathbb{C}(t)$.*

Proof. Let G be a finite group with cardinality $|G| = n$. Then we get a surjection of the free group F_n on n elements to G, i.e $p : F_n \to G$. By the preceding discussion we get that there is a surjection

$$s : \pi_1(P^1(\mathbb{C}) - \{p_1, ..., p_{n+1}\}) \to G$$

and let $ker(s) = H$ which is a normal subgroup of $\pi_1(P^1(\mathbb{C}) - \{p_1, ..., p_n\})$. We thus get

$$G \cong \pi_1(P^1(\mathbb{C}) - \{p_1, ..., p_{n+1}\})/H$$

We set $X' = P^1(\mathbb{C}) - \{p_1, ..., p_n\}$ and apply **Theorem 3.13** so we get

$$Gal(K_{X'}|M(X)) = Gal(K_{X'}|\mathbb{C}(t)) \cong \pi_1(P^1(\mathbb{C}) \widehat{- \{p_1}, ..., p_{n+1}\})$$

We denote by S the last profinite completion group for simplicity. Then we get a surjection by the definition of the profinite group

$$S \to \pi_1(P^1(\mathbb{C}) - \{p_1, ..., p_{n+1}\})/H \cong G$$

Let H' be its kernel, so we get that $S/H' \cong G$ and that implies

$$Gal(K_{X'}|M(X))/H' \cong G$$

If we set L to be the field fixed by the action of H' on $K_{X'}$, then $Gal(K_{X'}|L) = H'$ and from **Theorem 1.13** we get that

$$Gal(L|M(X)) \cong Gal(K_{X'}|M(X))/Gal(K_{X'}|L) = Gal(K_{X'}|M(X))/H' \cong G$$

□

So far we have from **Theorem 1.13** that $Gal(\widehat{M(X)}|M(X))/Gal(\widehat{M(X)}|K_{X'}) \cong Gal(K_{X'}|M(X))$ for $X' = X\backslash S$ where S is a finite set of points, which justifies our initial intuition that $\pi_1\widehat{(X', x')}$ is a quotient of $Gal(\widehat{M(X)}|M(X))$. So we have described the quotients of the absolute Galois group, but not the group it self. We note that given a finite set of points T in X such that $S \subseteq T$, we get $X\backslash T \subseteq X\backslash S$ we denote those as X_T and X_S respectively, i.e $X_T \subseteq X_S$. For two such open cofinite subsets of X we have that if $X_T \subseteq X_S$ then $K_{X_S} \subseteq K_{X_T}$, as any connected compact Riemann surface restricting to a finite cover over the larger space must restrict to a finite cover over the smaller space. The fields K_{X_S} are all Galois extensions of $M(X)$ and they contain any finite Galois field subextension of $\widehat{M(X)}$ for a sufficiently large S. To see this, by **Proposition 3.5** any holomorphic map $Y \to X$ of connected compact Riemann surfaces restricts to a cover over a suitable $X' = X\backslash S$ (where S is the set of branch points of the holomorphic map) and then **Theorem 3.11** gives us that any finite Galois subextension of $\widehat{M(X)}$ is contained in $K_{X'}$. So we can turn the absolute Galois group over $M(X)$ to the inverse limit

$$Gal(\widehat{M(X)}|M(X)) \cong \varprojlim_{\text{S}} Gal(K_{X_S}|M(X)) \cong \varprojlim_{\text{X'}} \pi_1\widehat{(X', x)}$$

With this in mind, we will describe the absolute Galois group of $\mathbb{C}(t)$. We first give a definition and a proposition.

Definition 4.7. Let X be a set and let $F(X)$ be the free group with basis X. The *free profinite group* $\widehat{F}(X)$ with basis X is defined to be inverse limit formed by the natural system of quotients $F(X)/U$, where $U \subseteq F(X)$ is a normal subgroup of finite index containing all but finitely many elements of X.

Proposition 4.16. *Let X be a set and $\mathcal{S}$ the system of finite sets $S \subseteq X$ partially ordered by inclusion. Let (G_S, λ_{ST}) be an inverse system of profinite groups indexed by S satisfying:*

1. *The λ_{ST} are all surjective for all $S \subseteq T$.*
2. *Each G_T has a system $\{g_t : t \in T\}$ of elements such that the map $\widehat{F}(T) \to G_T$ induced by the inclusion $T \to G_T$ is an isomorphism and moreover for every $S \subseteq T$ we have $\lambda_{ST}(g_t) = 1$ for all $t \notin S$.*

Then $\varprojlim G_S \cong \widehat{F}(X)$.

Proof. **Proposition 3.4.9 ,[10]** □

We now describe the absolute Galois group of $\mathbb{C}(t)$.

Theorem 4.17. *There is an isomorphism of profinite groups*

$$Gal(\widehat{\mathbb{C}(t)}|\mathbb{C}(t)) \cong \widehat{F}(\mathbb{C})$$

of the absolute Galois group of $\mathbb{C}(t)$ *with the free profinite group on the set* C *of complex numbers.*

Proof. Let S be a finite subset of $\mathbb{C}$ of cardinality $|S| = m$. We let $X = P^1(\mathbb{C})$ and define $X_S = P^1(\mathbb{C})\backslash(S \cup \{\infty\})$. From **Theorem 3.13** we get that

$$Gal(K_{X_S}|\mathbb{C}(t)) \cong \pi_1\widehat{(X_S, x)} \cong \widehat{F}_m$$

If we have T finite subset of $\mathbb{C}$ of cardinality $|T| = n$ such that $S \subseteq T$ then by the preceding discussion we have that $K_{X_T} \subseteq K_{X_S}$, both being extensions of $M(X) = \mathbb{C}(t)$ and therefore by Galois correspondence we get a surjection

$$\lambda_{ST} : Gal(K_{X_T}|M(X)) \to Gal(K_{X_S}|M(X))$$

The groups $Gal(K_{X_T}|\mathbb{C}(t))$ together with the maps λ_{ST} form an inverse system indexed by the finite sets S of $\mathbb{C}$ partially ordered by inclusion. From the inclusion $K_{X_T} \subseteq K_{X_S}$ we get an induced map on the fundamental groups μ_{ST} : $\pi_1(X_T, x) \to \pi_1(X_S, x)$ which gives an induced map $\pi_1\widehat{(X_T, x)} \to \pi_1\widehat{(X_S, x)}$ on the profinite completions which sends an element γ_x to 1 for $x \in T\backslash S$, i.e $\mu_{ST}(\gamma_x) = 1$ for $x \in T\backslash S$. We note that the maps λ_{ST} correspond to the maps μ_{ST} because of the isomorphisms $Gal(K_{X_T}|\mathbb{C}(T)) \cong \pi_1\widehat{(X_T, x)}$ and therefore $\lambda_{ST}(g_x) = 1$ for $x \in T\backslash S$ and because of the isomorphisms $Gal(K_{X_S}|\mathbb{C}(t)) \cong \widehat{F}_m$ we get that the properties of **Proposition 3.16** are satisfied and so

$$\varprojlim_{\text{S}} Gal(K_{X_S}|M(X)) \cong \widehat{F}(\mathbb{C})$$

from the discussion prior to **Definition 3.7** we have that

$$Gal(\widehat{M(X)}|M(X)) \cong \varprojlim_{\text{S}} Gal(K_{X_S}|M(X))$$

combining we get

$$Gal(\widehat{\mathbb{C}(T)}|\mathbb{C}(t)) \cong \widehat{F}(\mathbb{C})$$

□

5 Fundamental Groups of Algebraic Curves

5.1 Background in Commutative Algebra

We begin with the basic definition of this chapter.

Definition 5.1. Given an extension $A \subseteq B$ of rings, an element $b \in B$ is said to be *integral* over A if it is a root of a monic polynomial $x^n + a_{n-1}x^{n-1} + ... + a_0 \in A[x]$. The *integral closure* of A in B constists of all elements in B that are integral over A,i.e $\tilde{A} = \{b \in B|\text{b is integral over A}\}$. If $\tilde{A} = B$ then we say that the extension $A \subseteq B$ is *integral.* Finally, A is *integrally closed* in B if $\tilde{A} = A$. When A is an integral domain we can form the fraction field of A, denoted $K(A)$ and it is an extension of A. If A is integrally closed in $K(A)$ then we say that A is *integrally closed.*

Example 4. A unique factorization domain A is integrally closed. Indeed, let $\frac{a}{b}$ be an element of the fraction field with a,b coprime (can be assumed because of UFD) such that $\frac{a}{b}$ is an integral element over A. Then there exists monic polynomial with coeffiecients in A such that

$$\frac{a^n}{b^n} + c_{n-1}\frac{a^{n-1}}{b^{n-1}} + ... + c_0 = 0$$

multiplying by b^n we get

$$a^n + c_{n-1}ba^{n-1} + ... + c_0b^n = 0 \Rightarrow a^n = -b(c_{n-1}a^{n-1} + ... + c_0b^{n-1})$$

so b divides a^n, but because they were chosen to be coprime be get that b is a unit, thus $\frac{a}{b} = a \in A$.

We now state the basic properties of integral extensions.

Proposition 5.1. *Let $A \subseteq B$ be an extension of rings.*

1. *An element $b \in B$ is integral over A if and only if the subring $A[b]$ of B is finitely generated as an $A-module$.*

2. *The integral closure $\tilde{A}$ of A in B is a subring of B and moreover it is integrally closed in B.*

3. *Given a tower of ring extensions $A \subseteq B \subseteq C$ with $A \subseteq B$ and $B \subseteq C$ being integral extensions, then $A \subseteq C$ is an integral extension.*

4. *If B is integral over A and $P \subseteq A$ a prime ideal of A, then there exists a prime ideal $Q \subseteq B$ in B such that $Q \cap A = P$. Here P is a maximal ideal in A if and only if Q is a maximal ideal in B.*

Proof. 1) INT 2 ,[8], Ch. VII, §1, 2) Proposition 1.4 ,[8], Ch. VII, §1.
3)Proposition 1.3 ,[8], Ch. VII, §1, 4) Proposition 1.10-1.11 ,[8], Ch. VII, §1. □

For property 4 we say that a the prime ideal Q *lies above* P. Assume now that $A \subseteq B$ is an integral extension ($\tilde{A} = B$ in B) of two integrally closed integral domains, i.e $A \subseteq K(A)$ with $\tilde{A} = K(A)$ in $K(A)$ and $B \subseteq K(B)$ with $\tilde{B} = K(B)$ in $K(B)$. We have an induced field extension $K(A) \subseteq K(B)$ of the fraction fields and let this extension be Galois with Galois group $G = Gal(K(B)|K(A))$. Then B is invariant under the action of G on $K(B)$, as every element of it is a root of a polynomial over A and we have seen that a root gets sent to a root by an element $\sigma \in G$. Given a maximal ideal $P \subseteq A$ we denote by S_P the set of maximal ideals Q of B that lie over P, i.e they satisfy $Q \cap A = P$. Then for each $Q \in S_P$ and $\sigma \in G$ we have that $\sigma(Q) \in S_P$, because σ defines an automorphism of B, thus sends maximal ideals in B to maximal ideals.

Let D_Q be the stabilizer of Q in G. We denote $\kappa(Q)$ and $\kappa(P)$ to be the residue fields B/Q and A/P. For an element $a \in A$, we can naturally view it as an element in B because of the inclusion $A \subseteq B$. We denote $\hat{a} \in \kappa(Q)$ to be its class in the residue field. For each $\sigma \in D_Q \subseteq G$ we saw that $\sigma : B \to B$ defines an automorphism, thus $\hat{a} \mapsto \widehat{\sigma(a)}$ defines an automorphism $\widehat{\sigma} : B/Q \to B/Q$ for which $\hat{\sigma}(\hat{a}) = \widehat{\sigma(a)}$, when $\sigma \in D_Q$ (i.e $\sigma(Q) = Q$). Moreover, we get a map $\sigma \mapsto \hat{\sigma}$ which defines a group homomorphism $D_Q \to Aut(\kappa(Q)|\kappa(P))$, because $\hat{\sigma_1}\hat{\sigma_2}(\hat{a}) = \widehat{\sigma_1\sigma_2(a)} = \widehat{\sigma_1\sigma_2}(\hat{a})$. We define the *inertia subgroup* of Q, denote I_Q, to be the kernel of this homomorphism (normal subgroup by definition). The following are true about the inertia subgroup.

Proposition 5.2. *1. The group G acts transitively on the set S_P; in particular, S_P is finite.*

2. The subgroups D_Q are conjugate for all $Q \in S_P$. The same holds for the subgroups I_Q.

3. If the extension $\kappa(Q)|\kappa(P)$ is separable, then it is a Galois extension and the homomorphism $D_Q/I_Q \to Aut(\kappa(Q)|\kappa(P))$ defined above is an isomorphism.

Proof. 1) *Proposition 2.1, [8], Ch. VII, §1.*
2) Let $\sigma \in D_Q$, then $\sigma(Q) = Q$. Let D_R be another group, then from 1) we have that there exists $\gamma \in G$ such that $\gamma(R) = Q = \sigma(Q)$, therefore $\sigma^{-1}\gamma(R) = Q$ and composing by γ^{-1} yields $\gamma^{-1}\sigma\gamma(R) = R$, thus $\gamma^{-1}\sigma\gamma \in D_R$ so $\gamma^{-1}D_Q\gamma = D_R$. The result about inertia subgroups follow now from the fact I_Q being the kernel of $\gamma D_R \gamma^{-1}$ and so is $\gamma I_R \gamma^{-1}$, thus they are conjugate.
3)*Proposition 2.5, [8], Ch. VII, §1.* The proof shows that the map $D_Q \to Aut(\kappa(Q)|\kappa(P))$ is surjective, thus the result. □

Now we move on to Dedekind rings. A *Dedekind* ring A is an integral domain (no zero divisors), Noetherian (all ideals are finitely generated), integrally closed ring and such that all non-zero prime ideals in A are maximal. Examples include $k[t]$ for k field, the integers Z and the integral closure of Z in a field K where K is a finite extension of $\mathbb{Q}$. We recall that the localization of a ring A at an ideal I is the subring $S^{-1}A$ of the fraction field of A, where $S = A \backslash I$ is a multiplicatively closed subset. The following hold for Dedekind rings.

Proposition 5.3. *Let A be a Dedekind ring, then:*

1. *Every non-zero ideal $I \subseteq A$ decomposes uniquely as a product $I = P_1^{e_1} \cdots P_n^{e_n}$ where P_i are prime ideals.*
2. *For each prime ideal $P \subseteq A$, the localization A_P is a principal ideal domain.*

Proof. 1) Corollary 9.4, [1].
2)Theorem 9.3, [1]. □

Lemma 5.4. *The integral closure of a Dedekind ring A in a finite separable extension $L|K(A)$ of its fraction field is a Dedekind ring.*

Proof. We want to show that $\tilde{A}$ in L is a Dedekind ring. The integral closure $\tilde{A}$ in L is a subring of L and also integrally closed by **Proposition 4.1 2)**. From the same Proposition 4) we get that $\tilde{A}$ has the property that every non-zero prime ideal is maximal (because A has it and $\tilde{A}$ integral over it) and that it is an integral domain (the 0 ideal is prime). The Noetherian property follows from **Facts 4.1.4 a), [10]**. □

The following Proposition relates the degree of the extension $L|K(A)$ with the degrees of the induced extensions on the residue fields.

Proposition 5.5. *Let A be a Dedekind ring with fraction field $K(A)$ and let B be the integral closure of A in a finite separable extension $L|K(A)$. For a non-zero prime ideal A we consider a decomposition $PB = Q_1^{e_1} \cdots Q_n^{e_n}$ (B is Dedekind).Then*

$$\sum_{i=1}^{r} e_i[\kappa(Q_i) : \kappa(P)] = [L : K(A)]$$

Proof. **Proposition 4.1.6 , [10]** □

Corollary 5.5.1. *Let $A \subseteq B$ an integral extension of Dedekind rings such that the induced extension of the fraction fields $K(A) \subseteq K(B)$ is a finite Galois extension with Galois group G and let P be a maximal ideal of A. Assume that the extensions $\kappa(Q_i)|\kappa(P)$ are separable for all $Q_i \in S_P$. Then the integers e_i in the formula of **Proposition 4.5** are all same for all i and they are equal to the order $|I_{Q_i}|$ of the intertia subgroups at Q_i.*

Proof. Let K_1 be the field fixed by the action of D_{Q_1} on $K(B)$, so $K_1|K(A)$ is an extension, A_1 the integral closure of A in K_1 and $P_1 = Q_1 \cap A_1$. We have constructed P_1 so that the only maximal ideal lying above it is Q_1 and thus from **Proposition 4.5** we get $e_1[\kappa(Q_1) : \kappa(P)] = [K_1 : K(A)] = |D_{Q_1}|$ (where the last equality holds from finite Galois theory) . Also from **Proposition 4.2 3)** we have that $|D_{Q_1}| = |I_Q|[\kappa(Q_i) : \kappa(P_1)]$, since the extension being

Galois implies that $[\kappa(Q_1):\kappa(P)] = Gal(\kappa(Q_i)|\kappa(P))$ by **Corollary 1.9.1**. So we have that $e_1 = |I_{Q_1}|$ and because all inertia subgroups are conjugate from **Proposition 4.2 2)**, so $|I_{Q_1}| = |I_{Q_i}|$ for all i, then the integers e_i are indeed the same. □

Recall that a *local* ring A, is a ring that has only one maximal ideal. A local Dedekind ring is called *Discrete Valuation Ring.* Equivalent definitions are that A is a local principle ideal domain that is not a field or A is a Noetherian local domain with non-zero principal maximal ideal (**Fact 4.1.8,[10]**). Discrete valuation rings will be important in the theory we will develop in this chapter. The next propostion gives some properties of such rings.

Proposition 5.6. *Let A be a discrete valuation ring and t a generator of its maximal ideal.*

1. *Every non-zero element $a \in A$ can be written as $a = ut^n$ for some unit $u \in A$. Here n does not depend on the choice of the generator t.*
2. *If x is an element of the fraction field $K(A)$ of A, then either x or x^{-1} is contained in A.*
3. *if $A \subseteq B$ and B is a discrete valuation ring with the same fraction field, i.e $K(A) = K(B)$, then $A = B$.*

Proof. **Proposition 4.1.9, [10]** □

5.2 Curves over Algebraically Closed Fields

In this Section we will introduce the main object of study in this Chapter, that of affine curves, over algebraically closed fields. In the case of complex numbers $\mathbb{C}$ we will obtain a relation with Riemann surfaces. In the next chapter we will develop a theory over arbitrary fields. We begin with defining affine varieties. Throughout this section we will assume that k is algebraically closed field.

We define the affine space over an algebraically field k to be

$$\mathbb{A}^n(k) = \{(a_1, ..., a_n) : a_i \in k\}$$

Given an ideal $I \subseteq k[x_1, ..., x_n]$ we define the *affine closed* set defined by I to be

$$V(I) = \{P = (a_1, ..., a_n) \in \mathbb{A}^n(k) : f(P) = 0, \forall f \in I\}$$

From Hilbert‘s Basis Theorem (**Theorem 4.1,[8],Ch IV, §4**) we have that any finitely generated polynomial ring is Noetherian, thus every ideal is finitely generated, so $I = (f_1, ..., f_k)$ and so

$$V(I) = \{P = (a_1, ..., a_n) \in \mathbb{A}^n(k) : f_i(P) = 0, \text{i=1,...,k}\}$$

The sets $V(I)$ form a topology, when they are defined to be the closed sets, called the *Zariski* topology. For an element $f \in k[x_1, ..., x_n]$ we define the set

$D(f) = \{P \in \mathbb{A}^n(k) : f(P) \neq 0\}$, which is the complement of $V(f)$ and we call it *distinguished* open subset. The sets $D(f)$ form a basis for the Zariski topology when f runs through all polynomials. Recall that for an ideal $I \subseteq R$, we define $\sqrt{I} = \{f \in R | f^n \in I, \text{for some } n \in N^*\}$, called the *radical* ideal. We define $\mathcal{I}(X) = \{f \in k[x_1, .., x_n] : f(x) = 0, \forall x \in X\}$. Then for an ideal I we have that $\mathcal{I}(V(I)) = \sqrt{I}$ by Hilbert's Nullstellensatz strong form (**Proposition 3.7, [6],Ch I**).

Definition 5.2. 1. We call a set $X = V(I)$ an *affine variety* if $I = \sqrt{I}$.

2. For an affine variety $X = V(I)$ we define its *coordinate ring* to be the quotient $\mathcal{O}(X) = k[x_1, ..., x_n]/I$. The elements of $\mathcal{O}(X)$ are called *regular functions* on X and the images of x_i in $\mathcal{O}(X)$, denoted $\hat{x_i}$, are called *coordinate functions.*

We may evaluate a function $f \in \mathcal{O}(X)$ at a point $p = (a_1, ..., a_n) \in X$ by setting $f(p) = \hat{f}(a_1, ..., a_n)$ with a preimage $\hat{f}$ of f in $k[x_1, .., x_n]$. The value does not depend on the choice of $\hat{f}$. To see this we use that $I = \mathcal{I}(X)$ and thus for two representatives we have $\hat{f}_1 - \hat{f}_2 \in \mathcal{I}(X)$, i.e $(\hat{f}_1 - \hat{f}_2)(p) = 0$ for all $p \in X$ implying that $\hat{f}_1(p) = \hat{f}_2(p)$.

By definition the coordinate ring $\mathcal{O}(X)$ has no nilpotent elements and such a ring is called **reduced**. It becomes an integral domain if and only if I is a prime ideal. In that case we get the following definition.

Definition 5.3. An affine variety $X = V(I)$ is *integral* if I is a prime ideal.

Remark 19. The mappings $X \mapsto \mathcal{I}(X)$ and $I \mapsto V(I)$ give mutually inverse one-to-one inclusion reversing correspondenced between the following objects.
$\{$Maximal ideals of $k[x_1, ..., x_n]\} \leftrightarrow \{$Points in $\mathbb{A}^n(k)\}$
$\{$Prime ideals of $k[x_1, ..., x_n]\} \leftrightarrow \{$Integral affine varieties in $\mathbb{A}^n(k)\}$
$\{$Radical ideals of $k[x_1, ..., x_n]\} \leftrightarrow \{$Affine varieties in $\mathbb{A}^n(k)\}$
$\{$Maximal ideals of $k[x_1, ..., x_n]/I\} \leftrightarrow \{$Points in $V(I)\}$

We have created the objects of our category, i.e the affine varieties and now we want to construct the morphisms of the category. We recall that a polynomial map $f : \mathbb{A}^n(k) \to \mathbb{A}^m(k)$, is a map $f = (f_1, ..., f_m)$ such that each f_i is a polynomial in $k[x_1, ..., x_n]$.

Definition 5.4. Let $Y = V(J) \subseteq \mathbb{A}^m(k)$ and $X = V(I) \subseteq \mathbb{A}^n(k)$. A morphism $\phi : Y \to X$ is the restriction of a polynomial map to $\mathcal{O}(Y)$, i.e $\phi = (f_1, ..., f_m)$ with $f_i \in \mathcal{O}(Y)$, such that for all $p \in Y$, $\phi(p) = (f_1(p), ..., f_m(p)) \in X$..

For a morphism $\phi : Y \to X$ of affine varieties, there is an induced k-algebra homomorphism on the coordinate rings $\phi^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ given by $\phi^*(f) = f \circ \phi$. If $f \in \mathcal{O}(X)$ vanishes at a point $p \in X$, then $\phi^*(f)$ vanishes at the points $\phi^{-1}(p) \subseteq Y$ so the map is well defined sending $f \in I$ to $\phi^*(f) \in J$. A point of $\phi^{-1}(p)$ in Y corresponds to a maximal ideal Q and the point $p \in X$ corresponds to a maximal ideal P by Remark 19 and therefore $(\phi^*)^{-1}(Q) = P$. A morphism of affine varieties is continuous with respect to the Zariski topology, because the equality $\phi^{-1}(D(f)) = D(\phi^*(f))$ holds.

Proposition 5.7. *The functor $Y \mapsto \mathcal{O}(Y)$, $\phi \mapsto \phi^*$ is an antiequivalence between the category of affine varieties over the algebraically closed field k and that of finitely generated reduced k-algebras.*

Proof. **Proposition 4.2.10,[10]** □

Let X be an integral affine variety, we saw that $\mathcal{O}(X)$ is an integral domain. We define the fraction field $K(X)$ of X to be the fraction field of the integral domain $\mathcal{O}(X)$. By definition, any element in $K(X)$ is represented by a quotient of polynomials, $\frac{f}{g}$ such that $g \notin I$, i.e it is non-zero at the points of X. Two fraction are identified, i.e $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ if $f_1g_2 - f_2g_1 \in I$.

Next, lets consider a point $p = (a_1, ..., a_n)$ in X, by Remark 15 we saw that it corresponds to a maximal ideal P of $k[x_1, ..., x_n]$. In fact, $P =< x_1 - a_1, ..., x_n - a_n >$ and it is maximal because $k[x_1, ..., x_n]/P \cong k$ is a field. For an open subset $U \subseteq X$ we define the ring of *regular* functions on U to be

$$\mathcal{O}_X(U) = \cap_{p \in U} \mathcal{O}(X)_P$$

where $\mathcal{O}(X)_P$ is the localization of the coordinate ring of X on the maximal ideal P corresponding to the point $p \in X$. For $U = X$ we have that $\mathcal{O}_X(X) = \cap_{p \in X} \mathcal{O}(X)_P = \mathcal{O}(X)$ (**Lemma 4.2.11,[10]**), so our two definitions agree on X.

For two integral affine varieties X and Y and open subsets $U \subseteq X$ and $V \subseteq Y$ we define a morphism $\phi : V \to U$ to be an m-tuple $\phi = (f_1, .., f_m) \in \mathcal{O}(Y)^m$ such that $\phi(p) \in U$ for all points $p \in V$.

We will now restrict our category further to integral affine curves instead of integral affine varieties. We first give a definition.

Definition 5.5. The dimension of an integral affine k-variety X is the trancedence degree of its function field $K(X)$ over k.

Definition 5.6. An *integral affine curve* is an integral affine variety of dimension 1.

For an integral affine curve X we have that any non-zero prime ideal in $\mathcal{O}(X)$ is maximal(**Corollary 4.1.11,[10]**). It is also an integral domain and Noetherian by construction. We are missing the integrally closed property for X to be a Dedekind ring. This will be given by the following definiton.

Definition 5.7. A point p of an integral affine variety X is normal if the local ring $\mathcal{O}_{X,p}$ is integrally closed. We say that X is normal if and only if all points are normal.

In fact X is normal if and only if $\mathcal{O}(X)$ is integrally closed. This comes from the fact that $K(\mathcal{O}(X)) = K(\mathcal{O}(X)_P)$, so if $\mathcal{O}(X)$ is integrally closed, then so is every localization and if every localization is integrally closed then from $\mathcal{O}(X) = \cap_{p \in X} \mathcal{O}(X)_P$ we have that $\mathcal{O}(X)$ is integrally closed.

Therefore a normal integral affine curve is Dedekind. The localizations then become local Dedekind rings and so $\mathcal{O}_{X,P}$ are discrete valuation rings. We will now prove that over dimension 1, the normal property is equivalent to smoothness property. We first give a motivating example.

Example 5. Let $X = V(f) \in \mathbf{A}^2$ be an integral affine variety. We write x and y to be the coordinate functions on X and we assume that $p \in X$ is a point such that one of the partial derivatives $\partial_x f(p)$ or $\partial_y f(p)$ is non-zero; such a point is called a *smooth* point. We will prove that $\mathcal{O}_{X,P}$ is a discrete valuation ring, i.e p is a normal point.

To see this let $p = (a_1, a_2)$ with non-zero $\partial_y f(p)$ partial derivative (if $\partial_x f(p) \neq 0$ we compose $(x, y) \mapsto (y, x)$ with f). After performing affine translation $(x, y) \to (x - a_1, y - a_2)$ we may assume that $p = (0, 0)$ and the partial $\partial_y f(p)$ is still non-zero. The point $p = (0, 0)$ corresponds to the maximal ideal $M_p = (x, y)$ in $k[x, y]/(f)$. Localizing at M_p gives that the maximal ideal M of the local ring $\mathcal{O}_{X,P}$ is the functions that vanish at p and thus $M = (x, y)$. After regrouping terms we write $f = \phi(x)x + \psi(x, y)y$ (note f vanishes on $(0, 0)$ thus no constant terms). Applying ∂_y we get $\partial_y f = \partial_y \psi y + \psi$ and evaluating at the point $(0, 0)$ gives $\partial_y f(p) = \psi(0, 0)$ and thus ψ has a constant term non-zero and equal to the partial derivative of f evaluated at p. Thus in $\mathcal{O}_{X,p}$ we get $0 = x\phi + y\psi$, i.e $y = x\frac{\phi}{\psi}$ (ψ non-zero in $\mathcal{O}_{X,p}$ since $\partial_y f(p) \neq 0$) and therefore $y = gx$, where $g = \frac{\phi}{\psi}$. Therefore $M = (x)$. We have that $\mathcal{O}_{X,p}$ is a Noetherian local domain with non-zero principal maximal ideal, therefore it is a local Dedekind ring (**Fact 4.1.8,[10]**), which implies that $\mathcal{O}_{X,p}$ is integrally closed an thus normal.

In characteristic 0 every normal integral affine curve is locally isomorphic to an integral affine plane curve.

Proposition 5.8. *Assume k is of characteristic 0 and let X be an integral affine curve. Every normal point P of X has a Zariski open neighborhood isomorphic to an open neighborhood of a smooth point on an affine plane curve.*

Proof. **Proposition 4.2.18, [10]** □

In the case of $k = \mathbb{C}$ we can equip a normal affine curve with a complex structure of a Riemann surface. First we start with equipping the subset $V(f) \subseteq \mathbb{C}$ with a complex structure.

We can endow $V(f)$ with a complex structure as follows: Let (x_0, y_0) a point such that one of the partial derivatives is non-zero, let it be $\partial_y f(x_0, y_0) \neq 0$ then from **Theorem 1,[2]** we can find discs D_1, D_2 centered at x_0 and y_0 respectively and a holomorphic map $\phi : D_1 \to D_2$ such that $X \cap (D_1 \times D_2) = \{(z, \phi(z)) \in \mathbb{C} : z \in D_1\}$. We pick $f_i : X \cap (D_1 \times D_2) \to D_1$ to be the restriction of the projection $D_1 \times D_2 \to D_1$. Symmetrically, if (x_1, y_1) is a point where $\partial_x f$ does not vanish, i.e $\partial_x f(x_1, y_1) \neq 0$ then we find open discs C_1, C_2 and a holomorphic map $\psi : C_2 \to C_1$ such that $X \cap (C_1 \times C_2) = \{(\psi(w), w) \in \mathbb{C} : w \in C_2\}$ and $g_i : X \cap (C_1 \times C_2) \to C_2$. The maps f_i and g_j are analytic isomorphisms, thus complex charts. Over points with $\partial_y f \neq 0$ we can check that $f_i f_j^{-1} = Id$ on the intersection on the open discs. Indeed, $z \mapsto (z, \phi(z)) \mapsto z$ by the composition $f_i f_j^{-1}$. The same holds for two points with $\partial_x f \neq 0$. If one point has non-zero $\partial_y f$ and the other has non-zero $\partial_x f$ then we get that $g_i f_j^{-1}$ is $z \mapsto (z, \phi(z)) \mapsto$

$\phi(z)$, thus it is holomorphic, since ϕ is and thus we have a complex structure on $V(f)$. We note here that the complex charts defined on $V(f)$ are the projection maps $x : (x, y) \to x$ when $\partial_y f \neq 0$ and $y : (x, y) \mapsto y$ when $\partial_x f \neq 0$.

Now let X be an integral affine normal curve over $\mathbb{C}$ and P a point (normal by assumption). Then we pick a generator t of the maximal ideal of $\mathcal{O}_{X,P}$ and **Proposition 4.8** gives as an open neighborhood U of P and a function $u \in \mathcal{O}_X(U)$ such that the map $(t, u) \mapsto (x, y)$ yields an isomorphism ρ of U to an open Zariski subset of $V(f)$ satisfying $\partial_y f(\rho(P)) \neq 0$. We choose a neighborhood V of $\rho(P)$, small enough to be contained in $\rho(U)$, so that the restriction to the first coordinate $x : (x, y) \mapsto x$ defines a complex chart on $V(f)$ when it is equipped with the subspace topology inherited from $\mathbb{C}$. We now define a topology on $\rho^{-1}(V)$ by declaring its open subsets to be those that come from open subsets $W \subseteq V$ in the complex topology, so $Q \subseteq \rho^{-1}(V)$ is open if $\rho(Q) \subseteq V$ is open in the complex topology and we declare $x \circ \rho$ to be a complex chart in the neighborhood of $\rho^{-1}(V)$. It can be proven that this construction for all $P \in X$ yields a well defined topology on X and a complex atlas.

5.3 Affine Curves over General Base field

We will now define an *integral affine curve* over a general base field k. To motivate the definition, let $A = \mathcal{O}(X)$ the coordinate ring of an affine variety $X = V(I)$. Then by **Remark 19** the points in $X = V(I)$ correspond bijectively to maximal ideals of A, therefore there is no loss of information if we replace X by the set $\{m|\text{m maximal ideal of A}\}$. A point $x \in X$ lies in a $V(a)$ if and only if $a \subseteq m_x$, where m_x is the maximal ideal corresponding to x. Indeed, $x \in V(a)$ if and only if $f(x) = 0, \forall f \in a$ if and only if $a \subseteq m_x$. Therefore under the identification of X with the set of maximal ideals of A, the Zariski closed subsets become $V(a) = \{m|a \subseteq m, \text{m maximal}\}$. Thus the coordinate ring $A = \mathcal{O}(X)$ determines both the set X and the Zariski topology on it. Moreover, when X is an integral affine variety, the function field $K(X)$, the local rings $\mathcal{O}_{X,p}$ and the ring of regural functions on an open subset $U \subseteq X$, which is defined $\mathcal{O}_X(U) = \cap_{p \in U} \mathcal{O}_{X,p}$ are all constructed from the coordinate ring $A = \mathcal{O}(X)$. Also for each pair of opens $U \subseteq V$, we have an inclusion $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ that is immediately seen to satisfy the presheaf axioms, so we get a presheaf of rings on the space X. In fact, it defines a sheaf of rings on X. To see this the set of regular functions on an open subset U is the set $\mathcal{O}_X(U) = \{\frac{f}{g} \in K(X)|g(p) \neq 0, \forall p \in U\}$ and now it is immediate that the gluing property and the identity property of the sheaf definition holds, just as in the case of continuous functions.

Definition 5.8. A *ringed space* $(X, \mathcal{O}_X)$ is a pair such that X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

We now construct the an *integral affine curve* over a general base field k. We start by picking an integral domain A over a field k, such that A is finitely generated over k and of transcendence degree 1. Then every non-zero prime ideal is maximal (Corollary 4.1.11,[10]). We define the set $Spec(A) = \{p \subseteq$

$A|$p is prime$\}$, which in our setting is the set of non-zero maximal ideals together with the zero ideal (0) which is prime as we are in an integral domain. We equip $Spec(A)$ with the topology such that the closed subsets are of the form $V(a) = \{p \in Spec(A) | a \subseteq p\}$ and again we get a basis on the topology by distinguished open subsets $D(f) = \{p \in Spec(A) | f \notin p\}$. It can be proven in the above setting that a point $p \in Spec(A)$ is closed if and only if p is a maximal ideal. Thus all points in $Spec(A)$ are closed, except the zero ideal (0), so every open subset of $Spec(A)$ must contain the zero ideal. Also that implies that all open subsets in $Spec(A)$ are those that their complement is a finite set of closed points.

Given a point $p \in Spec(A) = X$ we define the local ring $\mathcal{O}_{X,p}$ to be the localization A_p. For $p = (0)$ we get that $\mathcal{O}_{X,(0)} = A_0 = K(A)$ the fraction field of A. Finally for an open subset $U \subseteq X$ we define

$$\mathcal{O}_X(U) = \cap_{p \in U} \mathcal{O}_{X,p} = \cap_{p \in U} A_p$$

We note here that because every open contains (0), that implies that every $\mathbf{O}_X(U)$ is contained in the fraction field $K(A)$. This again defines a sheaf of rings on X.

Definition 5.9. An *integral affine curve* over an arbitrary k is a ringed space $(Spec(A), \mathcal{O}_X)$, where A is a finitely generated k-algebra and of trancedence degree 1 and the sheaf of rings $\mathcal{O}_X$ is defined as above.

Now we have to describe a morphism of ringed spaces.

Definition 5.10. A morphism $(Y, \mathcal{G}) \to (X, \mathcal{F})$ of ringed spaces is a pair (ϕ, ϕ^*), where ϕ is a continuous map of the underlying topological spaces $\phi : Y \to X$ and $\phi^* : \mathcal{F} \to \phi_* \mathcal{G}$ is a morphism of sheaves on X. Here $\phi_* \mathcal{G}$ denotes the sheaf on X defined by $\phi_* \mathcal{G}(U) = \mathcal{G}(\phi^{-1}(U))$.

For integral affine affine curves we have a continuous map $\phi : Spec(B) \to Spec(A)$ and the map ϕ^* defined above as a rule that for each $f \in \mathcal{O}_X(U)$ defined over an open subset U, we get a map $\phi^*_U(f)$ in $\mathcal{O}_Y(\phi^{-1}(U))$. We should think of $\phi^*_U(f)$ as the composition $f \circ \phi$.

We want to establish now an anti-equivalence between the category of integral domains finitely generated over a field with transcendence degree 1 and the category of integral affine curves. First we note that given an integral domain A as above we can construct $Spec(A)$ which by definition is an integral affine curve. Also, given an integral affine curve, we set $A = \mathcal{O}_X(X) = \cap_{p \in Spec(A)} A_p$ and we saw that all $p \in Spec(A)$ except (0) are maximal ideals then $A \subseteq A_{(0)} = K(A)$ and the intersection of A_p for p maximal ideal is given by elements $\frac{f}{g}$ such that g is not in any maximal ideal, but that implies that g is an invertible element in A and therefore $\frac{f}{g} \in A$, so the equality $A = \cap_{p \in Spec(A)} A_p$ indeed holds for X integral affine curve. So we have established a correspondence between the objects of the two categories. For two integral affine curves $X = Spec(A)$, $Y = Spec(B)$ we have that a morphism (ϕ, ϕ^*) with $\phi : Spec(B) \to Spec(A)$ gives a ring homomorphism $\phi^*_X : \mathcal{O}_X(X) \to \mathcal{O}_Y(\phi^{-1}(X)) = \mathcal{O}_Y(Y)$ which is

$\phi_X^* : A \to B$. The converse is a bit more involved and will be given by the next proposition.

Proposition 5.9. *Given a homomorphism $\rho : A \to B$ with A,B as integral domains, finitely generated above a field k of transcendence degree 1, there is a unique morphism $Spec(\rho) : Spec(B) \to Spec(A)$ such that $Spec(\rho)_X^* : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ is equal to ρ.*

Proof. For each prime ideal $P \subseteq B$, the subring $\rho^{-1}(P) \subseteq A$ is a prime ideal, since $A/\rho^{-1}(P) \to B/P$ is injective and B/P is a integral domain, thus the subring $A/\rho^{-1}(P)$ is a integral domain and so $\rho^{-1}(P)$ is prime. Thus we get a map $Spec(\rho) : Spec(B) \to Spec(A)$ by $P \mapsto \rho^{-1}(P)$, which is continuous with respect to the Zariski topology because preimages of closed sets are closed. There are two cases, either ρ is injective or not.

If ρ is injective then A can be considered as a subring of B and we have that $\rho^{-1}(P) = P \cap A$ is a maximal ideal by **Proposition 4.1 4)** and $\rho^{-1}((0)) = (0)$ because of injectivity.

If ρ is not injective then we have that the kernel of the map,i.e $\rho^{-1}(0)$ is a not equal to (0) and therefore it has to be a non-zero prime ideal in A, but that implies that $ker(\rho)$ is a maximal ideal M in A and so the preimage $\rho^{-1}(P) = M$ for all P prime ideals in B. That gives a constant morphism $Spec(B) \to \{M\}$.

In the first case we have $A \subseteq B$ and also $K(A) \subseteq K(B)$ and $A_{(P\cap A)} = A_{\rho^{-1}(P)} \subseteq B_P$ for each maximal $P \subseteq B$, so taking intersections over open $U \subseteq Spec(A)$ we get $\cap_{Q\in U} A_Q \subseteq \cap_{P \in Spec(\rho)^{-1}(U)} B_P$ which by definition are inclusion $\mathcal{O}_X(U) \to \mathcal{O}_Y(Spec(\rho)^{-1}(U))$ and taken over the whole space X we get the map $\rho : A \to B$. In the second case we define $\mathcal{O}_X(U) \to \mathcal{O}_Y(Spec(\rho)^{-1}(U))$ to be

$$\mathcal{O}_X(U) \to A_M \to A_M/MA_M \cong A/M \to B$$

if $M \in U$ or $\mathcal{O}_X(U) \to 0$ otherwise. This sheaf is called a skyscrapper sheaf. The above is seen to be a morphism of sheaves, as it is compatible with the projection maps (If $U \subseteq V$ and $M \in U$, then $M \in V$). □

We note here that being finitely generated over the same field k had no impact on any of the arguments given, thus the statements hold for any fields. From the above discussion we get that

Proposition 5.10. *The rules $A \mapsto Spec(A)$, $\rho \mapsto Spec(\rho)$ and $X \mapsto \mathcal{O}(X)$, $\phi \mapsto \phi_X^*$ yield mutually inverse contravariant functors between the category of integral domains finitely generated and of transcendence degree 1 over a field and the category of integral affine curves.*

Here we actually have a stronger result that the anti-equivalence of categories, because we have that there is a bijection between objects and morphisms of the objects. Such categories are called *anti-isomorphic*.

We say that an integral affine curve is *normal* if its local rings are integrally closed. Just as in the previous subsection, this is equivallent to $\mathcal{O}_X(X)$ being integrally closed.

We will now prove an analogue to **Theorem 3.11** for integral affine curves. First we have to restrict our focus on morphisms of integral affine curves that resembles the properness of the holomorphic map in **Theorem 3.11**. We say that a morphism $\phi : Y \to X$ of integral affines curves is *finite* if $\mathcal{O}_Y(Y)$ becomes a finitely generated $\mathcal{O}_X(X)$-module via the map $\phi^*_X : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$. In that case, ϕ has finite fibers. To see this, let assume $\phi : Spec(B) \to Spec(A)$ and $\phi^* : A \to B$. We have that for any maximal ideal $V(m) = m$ by definition and also $\phi^{-1}(V(a)) = V(\phi^*(a)B)$, thus $\phi^{-1}(m) = V(\phi^*(m)B) \cong Spec(B/\phi^*(m)B)$. From the fact that B becomes a finitely generated A-module via ϕ^*, we thus have that $B/\phi^*(m)B$ becomes a finitely generated A/m module and because A/m if a finite algebraic extension of k, thus $B/\phi^*(m)B$ becomes a finite dimensional k-algebra (dim=0 + Noetherian is equivalent to Artinian) and as such has only finitely many maximal ideals and thus the fibers of a given maximal ideal m are finitely many. This property is shared by proper holomorphic maps of Riemann surfaces. We note that the zero ideal in A can not pullback to a maximal ideal q in B, because for the same reason B/qB has only finitely many maximal ideals and those correspond to maximal ideal $V(qB) = V(\phi^*((0))B) = \phi^{-1}(V((0))) = \phi^{-1}(Spec(A)) = Spec(B)$, thus $Spec(B)$ finite and thus $Spec(A)$ finite and so (0) open in $Spec(A)$ (as a complement of finite union of closed points) and the preimage of (0) is a maximal ideal in B, i.e closed, contradiction. Thus ϕ^* is injective and so we are in the first case of **Proposition 4.9** and that implies that $A \subsetneq B$ and by **Proposition 4.1 4)** we have that there exists for every prime ideal $P \subseteq A$ a prime $Q \subseteq B$ lying over it, so we get that ϕ is **surjective**. Another property shared by proper holomorphic map.

Now assume we have a finite morphism of integral affine curves $Y \to X$, we saw that the induced $\mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ is injective and therefore we get an inclusion of function fields $K(X) \subseteq K(Y)$. We are ready now to state the analogue of **Theorem 3.11**.

Theorem 5.11. *Let X be an integral normal affine curve. The rule $Y \mapsto K(Y)$, $\phi \mapsto \phi^*$ induces an anti-equivalence between the category of normal affine curves eqquiped with a finite morphism $\phi : Y \to X$ and that of finite field extension of the function field $K(X)$.*

Proof. **Theorem 4.3.10, [10]** □

5.4 Proper Normal Curves

We will now obtain algebraically the case of Compact Riemann Surfaces. The starting point is the study of the local rings $\mathcal{O}_{X,p}$ of an integral normal affine curve X over a field k. We have seen that they are discrete valuation rings with the same fraction field $K(X)$ and they all contain the the base field k (see discussion after **Definition 4.7**, the same arguments extend to normal integral affine curves over a general base field of Section 4.3). In fact, those properties characterize the local rings of an integral normal affine curve by the next lemma.

Lemma 5.12. *The local rings of an normal integral affine curve X are exactly the discrete valuation rings R with fraction field $K(X)$ that contain $\mathcal{O}(X)$.*

Proof. Let R be such a ring and let M its unique maximal ideal (R is local Dedekind). Because of the inclusion $\mathcal{O}(X) \subseteq R$ we have that $P = M \cap \mathcal{O}(X)$ is a prime ideal of $\mathcal{O}(X)$ and it has to be non-zero, for otherwise the restriction of the natural projection $R \to R/M$ to $\mathcal{O}(X)$ would be injective (P is the kernel of the map) and therefore $\mathcal{O}(X) \subseteq R/M$. We have that R/M is a finite field extension of the base field k and thus the inclusion $\mathcal{O}(X) \subseteq R/M$ would induce an inclusion $K(X) \subseteq R/M$, but $K(X)$ is of transcendence degree 1 over k while R/M finite field extension of k, thus contradiction. Any non-zero prime ideal in $\mathcal{O}(X)$ is maximal and therefore P is a maximal ideal. Then we have that $\mathcal{O}_{X,P} \subseteq R_M$ but because R is local any element not in its maximal ideal M is already invertible in R thus $R_M = R$ and so $\mathcal{O}_{X,P} \subseteq R$ both being discrete valuation rings with the same fraction field $K(X)$, then **Proposition 4.6 3)** applies and gives $R = \mathcal{O}_{X,P}$. □

We will now analyze an easy example that will motivate our construction of proper normal curves.

Example 6. Let $X = \mathbf{A}^1_k = Spec(k[x])$ be the affine line over k, with k algebraically closed. We have that $k(x)$ is the fraction field of $\mathcal{O}(\mathbf{A}^1_k) = k[x]$, but we also have that $k(x)$ is the fraction field of $k[x^{-1}]$ which is another copy of $\mathcal{O}(\mathbf{A}^1_k)$ with coordinate function $[x^{-1}]$. Let R be a discrete valuation ring with $k \subseteq R$ with fraction field $k(x)$, then by **Proposition 4.6 2)** we have that either $x \in R$ or $x^{-1} \in R$ and hence by the previous Lemma R is the local ring of one of the two copies of X. We know that the maximal ideals of $k[x^{-1}]$ are of the form $(x^{-1} - a)$ and localizing the ring by those we invert any function that has not a zero on a. When $a \neq 0$ we get that $x \in k[x^{-1}]_{(x-a)}$, but when $a = 0$ we have that $x \notin k[x^{-1}]_{(x^{-1})}$. So there is only one discrete valuation ring R that does not contain x in the $k[x^{-1}]$. This corresponds to the point at infinity. This whole construction is parallel to the construction of the Riemann surface $\mathbf{P}^{-1}(\mathbb{C})$, where we had to copies of $\mathcal{C}$, one around 0 and the other around ∞ and we identified the two complex charts by the isomorphism $z \mapsto z^{-1}$ on the intersection. Thus we may regard the discrete valuation rings R with fraction fields $k(x)$ as the local rings of the projective line over $\mathbf{C}$. The same construction holds for algebraically non-closed field k.

We now generalize the construction above. Let X be a normal integral affine curve $X = Spec(A)$ over a field k, pick $f \in \mathcal{O}(X) = A$ such that A is a finitely generated module over $k[f]$ (it is possible because A is an integral domain finitely generated over k of transcendence degree 1 and thus the result holds from Noether‘s Normalization). Let X^- be the normal integral affine curve corresponding to the integral closure of $k[f^{-1}]$ in $K(X)$ (the correspondence is given by **Proposition 4.10** and normality holds because we take the integral closure). Then **Lemma 4.12** gives us that the discrete valuation rings with fraction field $K(X)$ are either local rings of X or X^-. Moreover, there are only finitely many R that are not local rings of X, namely the localizations of $\mathcal{O}(X^-)$ at the finitely many maximal ideals lying above $(f^{-1}) \subseteq k[f^{-1}]$. To see that they are finitely many, first we have that $k[f^{-1}] \subseteq \mathcal{O}(X^-)$ and the latter ring was constructed to be integrally closed over $k[f^{-1}]$ and thus by **Proposition 4.1 4)** every maximal ideal in $k[f^{-1}]$ has a maximal ideal in $\mathcal{O}(X^-)$ lying above it. The maximal ideals in $k[f^{-1}]$ that contain (f^{-1}) are in one to one correspondence with maximal ideals in $k[f^{-1}]/(f^{-1})$ which is a finite algebraic extension of k (dim=0 over k and Noetherian if and only if Artinian , thus only finitely many maximal ideals), therefore the maximal ideals in $k[f^{-1}]$ that contain (f^{-1}) are finitely many. Generally for an inclusion $A \to B$ of rings we have an inclusion $A/m \to B/m_b$ where m_b lies above m ([8],Ch. VII,§1) and thus indeed we get that there are finitely many maximal ideals of $\mathcal{O}(X^-)$ that lie above the maximal ideals containing (f^{-1}). For the same reason, there are only finitely many R that are not local rings of X^-.

We now give a construction that is independent of the choice of f above, which will lead us to the definition of an integral **proper** normal curve.

Let k be a field and $K|k$ a finitely generated field extension of trancedence degree 1. We define the set X^K to be the set of discrete valuation rings with fraction field K containing k. We define the topology on X^k to be such that the proper closed subsets are the finite subsets and we define a sheaf of rings on X^K by the rule $\mathcal{O}^K(U) = \cap_{R \in U} R$ for an open set $U \subseteq X^K$. We call the ringed space (X^K, O^K) constructed an **integral proper normal curve** over k with function field K.

Remark 20. From how we have defined the topology on X^K, we see that because there are only finitely many R that are not local rings of X, we get that the set of local rings of X is an open subset of X^K and for the same reason the set of local rings of X^- is an open subset of X^K. From the fact that any discrete valuation ring is either a local ring of X or X^-, we get that the union of the two defines an open covering of X^K.

A morphism of proper normal curves $Y^L \to X^K$ is defined as a morphism of ringed spaces. Note here that we must have that they are both defined the same base field k. Given an integral proper normal curve X^K we say that an open subset $U^K \subseteq X^K$ is affine if $O^K(U^K)$ is a finitely generated k-algebra. The ringed space $(U^K, O^K|_{U^K})$ is the same as the integral affine curve corresponding to $O^K(U^K)$ via **Proposition 4.10**. Indeed, $O^K(U^K) = \cap_{R \in U^K} R$ by definition and it is also finitely generated k-algebra and of transcendence degree 1, thus $A = O^K(U^K)$ with $K(A) = K$ and so we get the ringed space $(Spec(A), \mathcal{O}_{Spec(A)})$, where $\mathcal{O}_{Spec(A)}(U) = \cap_{p \in U} A_p$ which is an integral affine curve. It is not very hard to verify that $Spec(A) \cong U^K$ ($p \mapsto A_p$) as topological spaces and $O^K|_{U^K} \cong \mathcal{O}_{Spec(A)}$ as sheaves (this is an abuse of notation, we should really check that the pushforward of the sheaf gives an isomorphism, which is equivalent to showing that the induced maps on the stalks are isomorphisms). Conversely, we have seen that the set of local rings of an integral normal affine curve with function field K is a non-empty open subset of X^K (Remark 20). Thus we have established an equivalence of categories.

Proposition 5.13. *The category of integral affine normal curves is equivalent to that of affine open subsets of integral normal proper curves.*

In particular, every integral affine normal curve X can be embedded as an affine open subset in an integral proper normal curve X^K and every morphism $Y \to X$ of integral affine normal curves extends uniquely to a morphism $Y^L \to X^K$ of proper normal curves.

Remark 21. The last statement holds because for every morphism of integral affine normal curves $Y \to X$, we can form a morphism $Y^- \to X^-$ of integral normal curves and then we can glue them along their non-empty open intersection in Y^L to form a morphism $Y^L \to X^K$.

Given a surjective morphism $Y^L \to X^K$ of integral proper normal curves, then L becomes a finite extension of K as both are finitely generated and of transcendence degree 1 over k. Indeed, it is enough to check on affine open subsets which are equivalent to integral normal affine curves by **Proposition 4.13**. So let $Y \to X$ be a surjective morphism of integral affine normal curves then we get an injection on the corresponding rings $\mathcal{O}(X) \subseteq \mathcal{O}(Y)$, because of the anti-isomorphism of categories given by **Proposition 4.10** and that induces an injection of the function fields,i.e $K \subseteq L$. That leads us to.

Proposition 5.14. *The above functor induces an anti-equivalence between the category of integral proper normal curves equipped with a surjective morphism on X^K and the finite field extensions of K.*

Proof. **Proposition 4.4.6, [10]** □

A morphism $\phi : Y^L \to X^K$ of proper normal curves is said to be *finite* if for all affine open subsets $U^K \subseteq X^K$ the preimage $\phi^{-1}(U^K) \subseteq Y^L$ is affine and moreover $\phi_*\mathcal{O}(U^K)$ becomes a finitely generated $\mathcal{O}^K(U^K)$ module. The restriction of ϕ to each open affine $\phi^{-1}(U^K)$ is identified with a morphism of integral proper normal curves (**Proposition 4.13**), which is finite from the condition given above and that implies that ϕ is surjective from the discussion preceding **Theorem 4.11**. We have also that the converse holds, i.e every surjective morphism of proper normal curves is always finite (**Lemma 4.4.7, [10]**). So we get

Corollary 5.14.1. *A morphism $Y^L \to X^K$ of proper normal curves is surjective if and only if it is finite.*

Let X be an integral proper normal curve with function field K and an element $f \in K$ trancedental over k, then we have an inclusion $k(f) \subseteq K$. From Example 6 we know we can realise $\mathbf{P}^1(k)$ as a proper normal curve with function field $k(f)$. Then we get from **Proposition 4.14** a surjective morphism of proper normal curves $X^K \to \mathbf{P}^1(k)$. We note the similarity with **Proposition 3.12**. Just as in the case of **Corolarry 3.12.1** we get the following corrolary here.

Corollary 5.14.2. *Mapping an integral normal proper curve to its function field induces an anti-equivalence between the categories of integral proper normal curves with finite surjective morphisms and that of finitely generated field extensions of k of transcendence degree 1.*

5.5 Finite Branched Covers of Normal Curves

In this section we will get an analogue of the finite branched covers of Riemann surfaces. We will first develop the theory for integral affine curves and then extend it to proper normal curves. We begin with some definitions. Recall that we showed that a finite morphism of integral affine curves $Spec(B) \to Spec(A)$ is surjective and that implies that there is an injection on the corresponding rings $A \subseteq B$ from the anti-isomorphism of categories given in **Proposition 4.10** and so we get an inclusion $K(A) \subseteq K(B)$ of fraction fields.

Definition 5.11. 1. A finite morphism of integral affine curves is *separable* if the induced field extension of fraction field $K(Y)|K(X)$ is separable.

2. We say that a finite separable ϕ is *etale* over a closed point $P \in X$, if B/PB is a finite etale algebra over the residue field $\kappa(P) = A/P$. It is etale over an open subset $U \subseteq X$ if it is etale over all $P \in U$.

Under the assumption that X and Y are also normal, then we saw in Section 3 that the induced rings are Dedekind rings and from **Proposition 4.3 1)**, we have that PB has a decomposition $PB = P_1^{e_1} \cdots P_n^{e_n}$ where P_i are non zero prime ideals in B and thus maximal. Also, B is a finitely generated A-module (the morphism is finite) and therefore B/PB becomes a finitely generated A/P-module and from the fact that A/P is a field, then B/PB becomes a finitely generated $\kappa(P)$-algebra. We have then from the Chinese Remainder Theorem (*Proposition 1.7,Ch. II,[6]*)

$$B/PB \cong B/P_1^{e_1} \oplus ... \oplus B/P_n^{e_n}$$

From etaleness each summand $B/P_i^{e_i}$ should be a finite separable extension of $\kappa(P)$,i.e a field and if $e_i > 1$ then $B/P_i^{e_i}$ has nilpotents and thus fails to be a field. Therefore that means that

$$B/PB \cong B/P_1 \oplus ... \oplus B/P_n$$

and the residue fields $\kappa(P_i)$ are all finite separable extension of $\kappa(P)$, where $\kappa(P_i) = B/P_i$. Also the P_i are the fiber of the map $\phi : Spec(B) \to Spec(A)$ on P, i.e $\phi^{-1}(P) = \{P_i : i \in \{1, ..n\}\}$. To see this, first consider the commutative diagram where the two vertical arrows are surjective and the horizontal arrows are injective

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A/p \arrow[r] & B/PB
\end{tikzcd}$$

taking the spectra gives a commutative diagram

$$\begin{tikzcd}
\{P_i\} \arrow[r] \arrow[d] & \{P\} \arrow[d] \\
Spec(B) \arrow[r, "\phi"] & Spec(A)
\end{tikzcd}$$

Because the maximal ideals containing PB are exactly the P_i and the maximal ideal in A containing P is P. Now it follows $\phi^{-1}(P) = \{P_i\}$. We give a characterization of a etale morphism in the next Lemma.

Lemma 5.15. *The morphism ϕ is etale above a point $P \in X$ if and only if a generator of the maximal ideal of A_P generates the maximal ideal of B_{P_i} for all $i \in I$ and the field extensions $\kappa(P_i)|\kappa(P)$ are separable.*

Proof. We saw that if ϕ is etale then the extension $\kappa(P_i)|\kappa(P)$ are separable. Also, we have inclusions induced by ϕ at the localized rings $A_P \subseteq B_{P_i}$ (which are discrete valuation rings), which implies that the preimage of the maximal ideal in B_{P_i} is the maximal ideal in A_P, so the generator of A_P must get mapped to the generator of B_{P_i}. For the converse we note that separability forces the $e_i = 1$ in the decomposition of B/PB and the first condition turns each summand to a finite algebra over A/P of trancedence degree 0,thus a finite extension and so their direct sum is a finite etale algebra over A/P. □

We will now give an example that resembles the local nature of a holomorphic map as given in **Proposition 3.1**.

Example 7. Let $\mathbb{C}[x^n] \to \mathbb{C}[x]$ be the inclusion map for $n > 0$, then this induces a surjective map on the spectra $\rho_n : Spec(\mathbb{C}[x]) \to Spec(\mathbb{C}[x^n])$ by **Proposition 4.10**. The prime ideals of $\mathbb{C}[x^n]$ are of the form $(x^n - a)$ where $a \in \mathbb{C}$ ($\mathbb{C}[x^n] \cong \mathbb{C}[t]$) and they pullback to prime ideals of the form $(x - a_i)$. The map ρ_n is readily seen to be finite and separable (they have the same fraction field). To check if it etale we consider the ideal $\mathbb{C}[x]/(x^n - a)\mathbb{C}[x]$. When $a \neq 0$ then as we are in an algebraically closed field we factor it $x^n - a = (x - a_1) \cdots (x - a_n)$. Then we get that

$$(x^n - a)\mathbb{C}[x] = (x - a_1) \cdots (x - a_n)$$

all of which are maximal ideals and so this is a decomposition and from the Chinese Remainder Theorem we get

$$\mathbb{C}[x]/(x^n - a)\mathbb{C}[x] \cong \mathbb{C}[x]/(x - a_1) \oplus ... \oplus \mathbb{C}[x]/(x - a_n) \cong \mathbb{C}^n$$

which is an etale algebra over $\mathbb{C} \cong \mathbb{C}[x]/(x-)$. If $a = 0$ then the decomposition is just $(x^n)\mathbb{C}[x] = (x)^n$ and so $\mathbb{C}[x]/(x^n)\mathbb{C}[x] \cong \mathbb{C}[x]/(x)^n$ which contains nilpotents and thus is not even a field. This shows that the map ρ_n is etale at every point, except the one corresponding to 0. This is an analogue of the local branching behavior of the morphism $z \mapsto z^n$ of Riemann Surfaces.

We can extend the above example as follows. Let k be an algebraically closed field and $f \in k[x]$, then we get an inclusion $k[f] \subseteq k[x]$ (f is a polynomial so it can be generated by x) corresponding to a surjection ρ_f on the spectra and for the same reason as above we get that the maximal ideals in the spectrum of $k[f]$ are of the form $(f - a)$ and they pull back to $(x - a_i)$. We have that f is a polynomial in the algebraically closed field and so the number of it roots is equal to its degree n. The problem in the previous situation was that x^n had multiple roots over 0. The same reasoning shows that $(f-a)\mathbb{C}[x]$ decomposes in $(x-a_1)\cdots(x-a_n)$, where n is the degree of f as a polynomial, if $g = f-a$ has no multiple roots (the a_i are the roots of g). If it does, then it admits a nilpotent function as previously and can not be etale over the point corresponding to a. So we get that ρ_f is etale over a point P if and only if g has no multiple roots if and only if $g'(a_i) \neq 0$ and the a_i are the points that correspond to the maximal ideals Q_i which are the preimage of the point P that corresponds to a. Since $g' = f'$, we get ρ_f is etale if and only if $f'(Q) \neq 0$.

Now we want to relate the theory developed so far with the theory developed for finite branched covers of Riemann surfaces. Firstly, let $\phi : Y \to X$ be a finite morphism of normal integral affine curves over $\mathbb{C}$. We saw in Section 2 that we can endow the spaces Y and X with a complex structure of Riemann surfaces, when the spaces are normal. With these complex structures ϕ can be viewed as a holomorphic map as given in **Definition 3.6**(**Remark 4.2.1,[10]**). Let $P \in X$ a closed point and consider the decomposition $PB = P_1^{e_1} \cdots P_n^{e_n}$ given above. The next proposition states that the indexes e_i in the decomposition are the ramification indexes when ϕ is considered as a holomorphic map.

Proposition 5.16. *The integer e_i in the decomposition of PB is the same as the ramification index of ϕ at P_i when considered as a holomorphic map. In particular, ϕ as an algebraic map is etale above P if and only if as a holomorphic map it restricts to a cover over a complex neighborhood of P.*

Proof. **Proposition 4.5.6,[10]** □

For the second statement, we recall from Chapter 3 that a holomorphic map restricts to a cover over a complex neighborhood of P if and only if P is not the image of a branch point (**Theorem 3.3**). The spaces X and Y with the "complex" topology constructed in Section 2 are Hausdorff spaces as any two points can be separated by disjoint opens. By the discussion above we have that $\phi^{-1}(P)$ is finite, i.e ϕ has finite fibers for each point. Therefore we have that ϕ becomes a finite cover when considered as a holomorphic map away from its branch points. That implies from **Theorem 3.3** that ϕ is proper when considered as a holomorphic map. The next proposition resembles the property that holomorphic maps have to restrict to a cover outside a discrete closed subset.

Proposition 5.17. *Let $\phi : Y \to X$ be a finite separable morphism of integral affine curves. Then there is a non-empty open subset $U \subseteq X$ such that ϕ is etale over U.*

Proof. **Proposition 4.5.9,[10]** □

Just like in the case of Riemann surfaces, we call a morphism $\phi : Y \to X$, as given in the above proposition, a *finite branched cover*. Moreover, if the induced finite separable extension on the function fields $K(Y)|K(X)$ by ϕ is Galois, then the above is called *Galois branched cover*. When the curves are normal, we have that $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are integrally closed and also we have an inclusion $\mathcal{O}(X) \subseteq$

$\mathcal{O}(Y)$ and from the finiteness condition on the morphism we have that $\mathcal{O}(Y)$ is an integral extension of $\mathcal{O}(X)$, because it is a finitely generated $\mathcal{O}(X)$-module. Then from **Proposition 4.2** and the discussion preceding it we get that $G = Gal(K(Y)|K(X))$ acts transitively on the set S_P which was defined to be the maximal ideals lying over P, which are exactly the fibers $\phi^{-1}(P)$. Therefore G acts transitively on the fibers $\phi^{-1}(P)$ for all $P \in X$. From the characterization of etaleness in **Lemma 4.15** we have that the extensions $\kappa(P_i)|\kappa(P)$ are separable, when ϕ is etale above P. Combining this with **Corollary 4.5.1**, we have that $e_i = |I_{P_i}|$ are all equal for all i and because for points P over which ϕ is etale we saw that $e_i = 1$, then the groups $|I_{P_i}| = 1$ are trivial because they are also normal subgroups of D_{P_i} and the identity automorphism of G fixes P_i. The other direction is also true when k is perfect as any finite extension is separable, so $\kappa(P_i)|\kappa(P)$ is separable and the inertia groups being trivial means $PB = P_1 \cdots P_n$, thus the generator of the maximal ideal of A_p generates each maximal ideal of B_{P_i}, so we get:

Proposition 5.18. *Let $\phi : Y \to X$ a finite Galois branched cover of normal integral affine curves defined over a perfect field k. Then ϕ is etale over a point P of X if and only if the inertia subgroups I_{P_i} are trivial for all P_i of Y lying above P.*

The above notions can naturally be extended to integral **proper** normal curves where we get the the next important Theorem. In what follows $X(\mathbb{C})$ denotes the proper normal curve X constructed in Chapter 4, together with a complex structure by endowing each affine integral open curve of X with the complex structure defined in Chapter 2.

Theorem 5.19. *Let X be an integral proper normal curve over $\mathbb{C}$ with function field K. Then the first two categories are equivalent and the third one is anti-equivalent to the first two*

1. *Integral proper normal curves equipped with a finite morphism onto X.*
2. *Compact connected Riemann surfaces equipped with a proper holomorphic map onto $X(\mathbb{C})$.*
3. *Finite extension of K.*

Moreover, a finite morphism $Y \to X$ is etale above a point $P \in X$ if and only if the induced holomorphic map $Y(\mathbb{C}) \to X(\mathbb{C})$ restricts to a cover in a neighborhood of P.

Proof. Preceding discussion from **Proposition 4.5.13,[10]** □

5.6 Algebraic Fundamental Group

The last Theorem in the previous Section yields a strong connection between Riemann Surfaces and integral proper normal curves. Thus, it is reasonable to expect to have an analogue Theorem to **Theorem 3.13**. That is the following:

Theorem 5.20. *Let k be a perfect field, X an integral proper normal curve over k with function field K and $U \subseteq X$ a non-empty open subset. We choose K_s to be a separable closure of the function field K. The composite K_U of all finite subextensions $L|K$ of K_s such that the corresponding finite morphism of integral proper normal curves is etale above all $P \in U$ is a Galois extension of K and each finite subextension of $K_U|K$ comes from a curve etale over U.*

Proof. **Proposition 4.6.1,[10]** □

Definition 5.12. In the situation above we define the **algebraic fundamental group** of U, denoted $\pi_1(U)$, to be $Gal(K_U|K)$.

It is evident from how we defined $\pi_1(U)$ that it is a profinite group. It depends on the choice of the separable closure K_s and since we are over a perfect field, this corresponds to a choice of the algebraic closure of K, just as in the case of Section 1.5. We now state the main result of this section, but before that we have to give some definitions. We define a *proper* (not necessarily integral) normal curve X to be the disjoint union of integral proper normal curves, i.e $X = \sqcup X_i$. The morphisms of proper normal curves are to be the disjoint union of morphisms on each component. The ring of rational functions of it is defined to be the direct sum of the function fields of its components,so $K(\mathcal{O}(X)) = \bigoplus_i K(\mathcal{O}(X_i))$. A finite morphism $\phi : Y \to X$ of a proper normal curve Y equips each component Y_i with an inclusion of fields $K(X) \subseteq K(Y_i)$, we say that such finite morphism is separable if $\bigoplus_i K(\mathcal{O}(X_i))$ is an etale algebra over $K(X)$.

Theorem 5.21. *Let X be an integral proper normal curve over a perfect field k and let $U \subseteq X$ be a non-empty open subset. The category of proper normal curves Y equipped with a finite separable morphism $\phi : Y \to X$ etale over U is equivalent to the category of non-empty finite left $\pi_1(U)$-sets.*

Proof. Let Y be a proper normal curve, i.e $Y = \sqcup_i Y_i$ where Y_i are integral proper normal curves, then we get finite separable morphisms $\phi_i : Y_i \to X$ of integral proper normal curves etale over U. Then by **Proposition 4.14** we get for each component a field extension of $K(X)$, i.e $K_i|K(X)$ and because the corresponding finite morphisms of integral proper normal curves are etale over U, we get that each $K_i \subseteq K_U$ are finite separable extensions of $K(X)$ living inside K_U. Thus we get the finite etale algebra $\bigoplus_i K_i = A$ over $K(X)$ and a set $Hom_k(A, K_U)$, which by **Theorem 1.20** gives an anti-equivalence with the category of finite sets with continuous left $Gal(K_U|K)$ action. So we have an anti-equivalence between the category of proper normal curves Y equipped with a finite separable morphism $\phi : Y \to X$ etale over U with the category of finite etale alebras A over $K(X)$ and the latter is anti-equivalent with the category of finite sets with left $Gal(K_U|K(X))$ action. Therefore we get the equivalence stated in the theorem. □

We note that we may even say more than what stated in the Theorem. That is, if we have a finite separable morphism of integral proper normal curves $\phi : Y \to X$ etale over U, then this corresponds to sets with transitive $Gal(K_U|K)$ action and if the $\phi : Y \to X$ is a finite branched cover etale over U (i.e $K(Y)|K(X)$ Galois) then this corresponds to a finite quotient of $Gal(K_U|K)$. These facts follow from the above proof and **Theorem 1.20**.

Let now U be an integral normal affine curve over a perfect field k. Then we saw in **Proposition 4.13** that we can embed it in an integral proper normal curve X as an affine open subset. Then the fundamental group $\pi_1(U) = Gal(K_U|K)$ is defined by **Theorem 4.20** and it does not depend on the embedding of U in X. We define a proper normal affine curve (again not necessarily integral) to be a disjoint union of integral normal affine curves and the morphisms extend naturally as we did for proper normal curves above. Just as in the proof of **Theorem 4.21** we get:

Corollary 5.21.1. *The category of normal affine curves V equipped with a finite etale morphism $\phi : V \to U$ is equivalent to the category of finite continuous left $\pi_1(U)$ sets.*

Proof. From **Proposition 4.13** we can extend each morphism $V_i \to U$ of integral affine normal curves to a morphism $\phi_i : V_i^K \to U^K$ of integral proper normal curves. Taking the disjoint union of the integral proper normal curves we get a finite separable morphism etale over U and thus the result follows from **Theorem 4.21**. □

We now want to describe the algebraic fundamental group $\pi_1(U)$. Over the complex numbers $\mathbb{C}$ we have the very strong property coming from **Theorem 4.19**. We will use that to get a presentation of the algebraic fundamental group in the following theorem.

Theorem 5.22. *Let X be an integral proper normal curve over $\mathbb{C}$ and let $U \subseteq X$ a non-empty open subset. Then the algebraic fundamental group $\pi_1(U)$ is isomorphic to the profinite completion of the topological group of the Riemann surface associated with U. Hence as a profinite group it has a presentation*

$$< a_1, b_1, ..., a_g, b_g, \gamma_1, ..., \gamma_g | [a_1, b_1] \cdots [a_g, b_g]\gamma_1 \cdots \gamma_g = 1 >$$

where n is the number of points of X lying outside U and g is the genus of the compact Riemann surface $X(\mathbb{C})$ associated with X. The $[a_1, b_1]$ are the commutators defined by the relation $a_1 b_1 = b_1 a_1$.

Proof. From **Theorem 4.20** we have that $K_U|K$ arises as the composite of all finite subextension $L|K$ of K_s coming from finite morphisms of proper normal curves etale over U. Those correspond to all finite subextension $L|K$ of K_s coming from holomorphic maps of the associated connected compact Riemann surfaces $Y \to X$ that restrict to a cover over U from **Theorem 4.19**. But that is exactly the definition of the K'_X composite defined in **Theorem 3.13**, so we have an isomorphism of the Galois extensions $K_U|K$ and $K'_X|M(X)$, therefore $Gal(K_U|K) \cong Gal(K'_X|M(X))$. The latter group, as proved in **Theorem 3.13**, is isomorphic to $\pi_1\widehat{(X', x')}$. Here X' is a compact Riemann surface such that

$X' = X(\mathbb{C})\backslash S$, where $S = X(\mathbb{C})\backslash U$ (since branch point correspond to points outside U). The statement about the presentation now follows from **Remark 3.6.4,[10]**. □

So far we have described the algebraic fundamental group only for integral proper normal curves over $\mathbb{C}$. We now want to extend to any algebraically closed field k of characteristic 0. This will be possible by *base change*.

Let X be an integral affine curve over a field k, recall that this means that $\mathcal{O}(X)$ is an integral domain, finitely generated over k of transcendence degree 1. Let $L|k$ be a finite extension of k, such that $\mathcal{O}(X) \otimes_k L$ is an integral domain. Then we have that $\mathcal{O}(X) \otimes_k L$ is a finitely generated $L-algebra$, integral domain and of trancedence degree 1 and therefore from the map $A \to A \otimes_k L$ which sends $a \mapsto a \otimes 1$ we get and induced map on the spectra $Spec(\mathcal{O}(X) \otimes_k L) \to Spec(\mathcal{O}(X))$ in view of **Proposition 4.10**, which is an integral affine curve. We denote $X_L = Spec(\mathcal{O}(X) \otimes L)$. When $A \otimes_k \hat{k}$ is an integral domain, for $\hat{k}$ an algebraic closure of k, then for every finite subextension $L|k$ of $\hat{k}$ we have that $A \otimes_k L$ is integral domain and therefore for a fixed L as above we get a functor $X \mapsto X_L$. In this case we say that $X = Spec(A)$ is a **geometrically integral**. Moreover, when $L|k$ is a finite separable extension of k then the morphism $X_L \to X$ is finite and etale over X. Finite because $A \otimes_k L$ becomes a finitely generated A-module, since $L|k$ is finite and etale because $A \otimes_k L/P(A \otimes_k L) \cong A/P \otimes_k L \cong \kappa(P) \otimes_k L$ which is a separable algebra over $\kappa(P)$ since L is separable.

Now we want to extend this construction to integral proper normal curves over a field k (algebraically closed). Let X^K be an integral proper normal curve with function field K and $L|k$ a finite extension. Then K is a finite extension of the field $k(t)$ as we saw in Section 4.4 and $K \otimes_k L$ becomes a finitely generated $L(t)$ algebra. Under the assumption that k is algebraically closed then $K \otimes_k L$ becomes a field and in fact a direct product of fields L_i. Each L_i is then finitely generated and of transcendence degree 1 over L and thus corresponds to an integral proper normal curve (Section 4.4 definition of proper normal curve) X^{L_i} over L with function field L_i. We have inclusions $L_i|K$ for each i and therefore by **Proposition 4.14** we get surjective morphisms $\phi_i : X^{L_i} \to X^K$. We define the **base change** X_L to be the disjoint union of the X^{L_i} together with the disjoint union of the morphisms. Thus we get a natural surjective morphism $X^L \to X^K$ of proper normal curves.

When $U \subseteq X^K$ is an open subset we define U_L to be the inverse image of U in X^L and when U is affine this results to $U_L = Spec(\mathcal{O}(U) \otimes_k L)$, from the fact $\mathcal{O}(U) \otimes_k L$ integral domain if and only if $K \otimes_k L$ is a field.

Theorem 5.23. *Let $k \subseteq L$ be an extension of algebraically closed fields of characteristic 0, X an integral proper normal curve over k and $U \subseteq X$ an open subset. The base change functor $Y \mapsto Y_L$ induces an equivalence between the finite covers of X etale over U and those of X_L etale over U_L. Consequently, we have an isomorphism of algebraic fundamental groups $\pi_1(U_L) \cong \pi_1(U)$.*

Proof. See **Theorem 4.6.10,[10]** □

Corollary 5.23.1. *Let k be an algebraically closed field of characteristic 0, X an integral proper normal curve over k and $U \subseteq X$ an open subset. The $\pi_1(U)$ has a presentation as in* ***Theorem 4.22****.*

Proof. □

When k is **not** algebraically closed then we have the result that the absolute Galois group of the base field, i.e $Gal(\hat{k}|k)$ where $\hat{k}$ is the algebraic closure of k and k is a perfect field, arises as a quotient of the algebraic fundamental group, since we have a surjection $\pi_1(U) \to Gal(\hat{k}, k)$ given in the following Theorem.

Theorem 5.24. *Let X be a geometrically integral proper normal curve over a perfect field k and $U \subseteq X$ an open subset. Then there is an exact sequence of profinite groups.*

$$1 \to \pi_1(U_{\hat{k}}) \to \pi_1(U) \to Gal(\hat{k}|k) \to 1$$

Proof. **Proposition 4.7.1,[10]** □

5.7 Application To the Inverse Galois Problem

This section will be dedicated to showing that the theory we have developed in this Project has applications to the inverse Galois problem. First we state the problem:

Problem 1: Let G be a finite group. Construct a finite Galois extension $K|\mathbb{Q}$ such that $G \cong Gal(K|\mathbb{Q})$.

This is called *inverse Galois problem* over Q. All the methods we have developed so far in Chapters 3 and 4 were considering extensions of the fraction fields $k(T)$ for fields k, where T was transcendental over k. Thus, we will reformulate the problem to what is called *regular Inverse Galois* problem over $\mathbb{Q}$.

Problem 2: Let G be a finite group. Construct a regular Galois extension $K|\mathcal{Q}(T)$ such that $Gal(K|\mathcal{Q}(T)) \cong G$.

The regularity condition means that there is no subextension of K of the form $L(T)$ such that $K|L(T)|\mathbb{Q}(T)$ and L is a non-trivial extension of $\mathbb{Q}$. A positive answer to **Problem 2** gives a positive answer to **Problem 1**, because of the following theorem:

Theorem 5.25. *Consider a finite regular Galois extension $K|\mathbb{Q}(T)$ with Galois group G. Let $x^m + a_{m-1}x^{m-1} + ... + a_0$ be a minimal polynomial of the Galois extension with $a_i \in \mathbb{Q}(T)$. There exist infinitely many $t \in \mathbb{Q}$ such that none of the a_i has a denominator vanishing at t and $x^m + a_{m-1}(t)x^{m-1} + ... + a_0(t) \in \mathbb{Q}(x)$ defines a minimal polynomial that gives a Galois extension of Q with Galois group G.*

The only close positive solution we have gotten so far to **Problem 2** was **Theorem 3.15** which stated that every finite group G arises as the Galois group of some extension $L|\mathbb{C}(t)$. We would like to manipulate this case to give a Galois group of some extension over $\mathbb{Q}(T)$. In order to do so, we will utilize the theory of algebraic curves and more specifically the *base change* introduced in the previous Section. We start of by recalling some basic facts we have seen so far.

We start off with **Theorem 3.15** were we got a surjection

$$\pi_1^{top}(P^1(\mathbb{C}) - \{p_1, ..., p_{n+1}\}) \to G$$

Where G was a finite group of order $|G| = n$ and the fundamental group was isomorphic to the free group on n generators. That gave us an isomorphism

$$G \cong \widehat{\pi_1^{top}(P^1(\mathbb{C}) - \{p_1, ..., p_{n+1}\})}/H' \tag{2}$$

by utilizing that $Gal(K_{X'}|M(X))$ was isomorphic to the above profinite completion of the fundamental group in view of **Theorem 3.13**. For this profinite completion we got in **Theorem 4.22** that $\pi_1(U) \cong \widehat{\pi_1(P^1(\mathbb{C}) - \{p_1, ..., p_{n+1}\})}$ when X was an integral proper normal curve over $\mathbb{C}$. Since those facts hold for arbitrary set of points $\{p_i\}$, we choose point p_i such that they are closed point of P^1_{Q}, so that their complement is open, and with $\kappa(P_i) \cong \mathbb{Q}$. From the fact that $\hat{Q} \subseteq \mathbb{C}$ is an extension of algebraically closed field and $P^1_{\mathbb{Q}}$ is an integral proper normal curve over $\mathbb{Q}$ and $P^1_{\hat{\mathbb{Q}}} - \{p_1, ..., p_{n+1}\}$ is an open subset, then applying **Theorem 4.23** we get $P^1_{\hat{\mathbb{Q}}} - \{p_1, ..., p_{n+1}\} \cong P^1_{\mathbb{C}} - \{p_1, ..., p_{n+1}\}$ and from **Theorem 4.22** we get that

$$P^1_{\mathbb{C}} - \{p_1, ..., p_{n+1}\} \cong \widehat{\pi_1^{top}(P^1(\mathbb{C}) - \{p_1, ..., p_{n+1}\})}$$

So combining we get

$$P^1_{\hat{\mathbb{Q}}} - \{p_1, ..., p_{n+1}\} \cong \widehat{\pi_1^{top}(P^1(\mathbb{C}) - \{p_1, ..., p_{n+1}\})} \tag{3}$$

For simplicity we denote $\Pi(n) = P^1_{\mathbb{Q}} - \{p_1, ..., p_{n+1}\}$, $\pi(n) = \widehat{\pi_1^{top}(P^1(\mathbb{C}) - \{p_1, ..., p_{n+1}\})}$ and $\pi_1^{top}(n) = \pi_1^{top}(P^1(\mathbb{C}) - \{p_1, ..., p_{n+1}\})$. By **Theorem 4.24** and the isomorphism above, we get the exact sequence of profinite groups

$$1 \to \pi(n) \to \Pi(n) \to Gal(\hat{\mathbb{Q}}|\mathbb{Q}) \to 1$$

By the exactness of the above sequence we can view $\pi(n)$ as a subgroup of $\Pi(n)$ and moreover it is normal and closed in $\Pi(n)$ (equal to the kernel of $\Pi(n) \to Gal(\hat{\mathbb{Q}}|\mathbb{Q})$). Our goal will be to extend the surjection $\phi : \pi(n) \to G$ (coming from (2)), to a continuous homomorphism $\hat{\phi} : \Pi(n) \to G$ which will automatically be surjective (since $\pi(n)$ injects to $\Pi(n)$ and $\pi(n)$ surjects on G). Therefore we will get that $\Pi(n)/ker(\hat{\phi}) \cong G$. Since by the construction of $\Pi(n)$ in **Theorem 4.20** we have that it is a Galois group contained in $Gal(\mathbb{Q}\hat{(t)}|\mathbb{Q}(t))$, then it is a quotient of the absolute Galois group in view of **Theorem 1.13** and therefore G will arise as the Galois group of a finite Galois extension $K|\mathbb{Q}(t)$.

The extension will be regular since $\Pi(n)/\pi(n) \cong Gal(\hat{\mathbb{Q}}|\mathbb{Q})$ and $ker(\phi) \subseteq ker(\hat{\phi})$ where both are surjective on G. The construction of $\hat{\phi}$ will depend on a group theoretic construction.

Assume given a profinite group Γ $(= \Pi(n)))$ and a closed normal subgroup $N \subseteq \Gamma$ $(=\pi(n))$ and a finite group G. The set $Hom(N, G)$ of continuous homomorphisms $N \to G$ is equipped with two natural actions. One is a left action by G given by $(g, \phi) \mapsto g\phi(n)g^{-1}$ for all $n \in N$ and the other is a right action by Γ on $Hom(N, G)$ given by $(\phi, \sigma) \mapsto \phi(\sigma n \sigma^{-1})$ for $\sigma \in \Gamma$(since N is a normal subgroup of Γ this action is well defined). The two actions are also compatible, i.e $(g, \phi) \circ \sigma = g \circ (\phi, \sigma)$.

Lemma 5.26. *In the above situation, let $S \subseteq Hom(N, G)$ be a subset stable by both actions of G and Γ and such that moreover G acts freely ($(g, s) = s \Rightarrow g = 1$, this means that the stabilizers are trivial) and transitively on S. Then every $\phi \in S$ extends to a continuous homomorphism $\hat{\phi} : \Gamma \to G$.*

Proof. **Lemma 4.8.3,[10]** □

We want to apply the lemma to the groups $\Gamma = \Pi(n)$ and $N = \pi(n)$ so that we can extend the surjection ϕ. This amounts to specifying a set $S \subseteq Hom(\pi(n), G)$ with the properties in **Lemma 4.26**. We had seen in the proof of **Theorem 3.15** that $\pi(n) = \hat{F_n}$ and so the surjection $\phi : \pi(n) \to G$ is determined by the images $\phi(\gamma_i)$ where γ_i the generators of the free group F_n on n generators, we shall call such a tuple $(\phi(\gamma_1), ..., \phi(\gamma_n)) \in G^n$ a generating n-tuple. If $\phi \in S$ and S is stable by the action of $\Gamma = \Pi(n)$, then $(\phi, \sigma) \in S$ for all $\sigma \in \Pi(n)$ and in particular for all $\sigma \in \pi(n) \subseteq \Pi(n)$. For $\sigma \in \pi(n)$ we have $(\phi, \sigma) = \phi(\sigma\gamma_i\sigma^{-1})$ and $(\phi(\sigma), \phi) = \phi(\sigma)\phi(\gamma_i)\phi(\sigma)^{-1}$ and because ϕ is a group homomorphism we get $(\phi, \sigma) = (\phi(\sigma), \phi)$. Conversely, for each $g \in G$ we get that the map $(g, \phi) \mapsto g\phi(\gamma_i)g^{-1}$ defines a surjective homomorphism $\pi(n) \to G$. Therefore it is natural to fix $C_1, ..., C_n$ conjugacy classes in G, with $\phi(\gamma_i) \in C_i$ for each i such that $(\phi(\gamma_1), ..., \phi(\gamma_n))$ is a generating tuple of G. So we consider the set

$$S = \{\phi \in Hom(\pi(n), G) : \phi(\gamma_i) \in C_i, (\phi(\gamma_1), ..., \phi(\gamma_n)) \in G^n \text{ a generating n-tuple}\}$$

Since each $\phi(\gamma_i) \in C_i$ and C_i is a conjugacy class, therefore S is stable by the action of G. Also, if $\sigma \in \pi(n)$ then for an element $\phi \in S$ we have $(\phi, \sigma) = \phi(\sigma\gamma_i\sigma^{-1}) = \phi(\sigma)\phi(\gamma_i)\phi(\sigma^{-1}) \in C_i$ for each i and thus S is also stable by the action of $\pi(n)$.

What remains now is to force the set S to have a free and transitive action by G and a stable action by $\Pi(n)$. The next two definitions will give us those properties.

Definition 5.13. Let G be a finite group. An n-tuple $C_1, .., C_n$ of conjugacy classes in G is called *rigid* if there exists a generating n-tuple $(g_1, ..., g_n) \in G^n$ such that $g_i \in G_i$ and moreover G acts transitively on the set of all such generating n-tuples.

This forces the action of G to be transitive on S. If we also assume that the center of G, $Z(G) = \{z \in G | zgz^{-1} = g, \forall g \in G\}$, is trivial then the action is **free**. Indeed, let $\phi \in S$, $g \in G$ then $(g, \phi) = g\phi(\gamma_i)g^{-1} = \phi(\gamma_i)$ if and only if $g \in Z(G)$ but since $Z(G)$ trivial then $g = 1$ and so the action is free. The next definition give us that S is stable by $\Pi(n)$.

Definition 5.14. A conjugacy class C in a finite group is called *rational* if $g \in C$ implies $g^m \in C$ for all $m \in Z$ prime to the order of G.

The next lemma shows that the rationality implies that S is stable by the action of G.

Lemma 5.27. *Assume that $C_1, ..., C_n$ are rational conjugacy classes in a finite group G and $\phi : \pi(n) \to G$ is a continuous homomorphism with $\phi(\gamma_i) \in C_i$ for all i. Then the same holds for $(\phi, \sigma) = \phi(\sigma\gamma_i\sigma^{-1})$ for all $\sigma \in \Pi(n)$. If moreover ϕ is surjective, so is (ϕ, σ).*

Proof. **Lemma 4.8.6,[10]** □

To sum up the construction , we wanted to extend the the surjection $\phi : \pi(n) \to G$ to a continuous surjection $\hat{\phi} : \Pi_n \to G$. By the properties of the two groups, i.e that $\pi(n)$ is a closed normal subgroup of $\Pi(n)$ we got two compatible actions. Under the conditions on **Lemma 4.26** we showed that it is possible to get such an extension of ϕ. To force the set S to have the properties in the Lemma, we showed that it is enough to force conditions on G. Those were that G has a *rigid* system of *rational* conjugacy classes $C_1, ..., C_n$ and that G has a trivial center. We include all those facts in the Theorem below.

Theorem 5.28. *Let G be a finite group with trivial center such that it has a rigid system of rational conjugacy classes $C_1, ..., C_n$. Then G arises as a finite quotient of $\pi_1(P^1_{\mathbb{Q}} - \{p_1, ..., p_n\})$ where p_i are $\mathbb{Q}$-rational points. In particular, it is the Galois group of a regular Galois extension over $\mathbb{Q}(t)$.*

The problem then shifts to the group theoretic problem of finding a group G with trivial center and with a rigid system of rational conjugacy classes as above. It is not to be assumed an easier problem, but there are groups known from the classification of finite simple groups which has been shown to have a rigid system of rational conjugacy classes and a trivial center. For example Thompson has verified that the *Monster* group and the *baby monster* group have a rigid system of three rational conjugacy groups classes of order $(2, 3, 29)$ and $(2, 3, 71)$ in **[11]**.

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