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Topos Theory and Formal Connections to Forcing

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Abstract

We examine the relationship between topos-theoretic and set-theoretic forcing. It is often stated in literature on topos theory that topos-theoretic and set forcing are closely connected and that sheaf toposes generalize set-theoretic forcing models, but the formal details of this relationship are left undiscussed. We aim to bridge this gap and establish direct formal connections between the two approaches. We will define the double negation topos over a poset of forcing conditions, which plays the central role in topos-theoretic forcing. After constructing a cumulative hierarchy in this topos, we show that this hierarchy is isomorphic to a Boolean-valued hierarchy over the same poset. Along the way we observe that the two will produce the same independence results, and that the double negation topos classifies generic filters.

Sammanfattning

Vi undersöker förhållandet mellan toposteoretisk och mängdteoretisk forcing. Även om det ofta påstås i litteraturen att kärvetoposer generaliserar mängdteoretiska forcingmodeller och att de två metoderna för forcing är nära besläktade, lämnas de formella detaljerna i denna koppling odiskuterade. Vi strävar efter att överbrygga denna klyfta genom att etablera direkta formella samband mellan dem. Vi definierar dubbelnegationstoposen över en partiellt ordnad mängd av forcingvillkor och konstruerar en kumulativ hierarki inom den. Därefter visar vi att denna hierarki är isomorf med den boolesk-värderade modellen över samma partiellt ordnade mängd. Längs vägen observerar vi att båda ramverken ger samma oberoenderesultat, och att dubbelnegationstoposen klassificerar generiska filter.

Contents

Introduction

One of the many perspectives on toposes is to regard them as mathematical universes. From this point of view, it is natural to ask how they relate to the familiar set-theoretic universe of **ZF** and to set-theoretic constructions. The method of *forcing* plays a central role in set theory, famously proving independence results such as the independence of the continuum hypothesis by Cohen in [Coh63]. Interestingly, introductions to topos theory are littered with references to forcing alluding to connections – suggestive notation is used with the Kripke-Joyal semantics, a Cohen topos can be defined and independence results are claimed. Yet the precise relationship between these two frameworks remains unspecified, and a set theorist and topos theorists alike may be left questioning how these results translate into their respective foundation. In this thesis, which is in some sense a small exercise in understanding toposes as universes, we aim to bridge some of these gaps, showing formal connections at different points of the two approaches.

To make the comparison, we approach forcing from the view of Boolean-valued models. The focus however will be on the topos-theoretic side. We develop the necessary theory of Grothendieck toposes and their semantics and then construct the double negation topos on a partial order of forcing conditions. We use this topos to get a model of **ZF** and by reframing Boolean-valued models categorically, we draw formal connections between the two approaches. A particular focus will be set on three parts of the forcing technique:

A. the extension of a base model via a poset of forcing conditions,

B. the formulation of the forcing relation itself,

C. and the role and interpretation of a generic filter,

This by no means envelops all of forcing, but only those steps involved in defining the forcing extension that we choose to examine topos-theoretically.

Our aim is to formally connect these components of forcing in set theory and topos theory, emphasizing the parallels rather than reconstructing the full technical machinery behind each approach. For the extension of the base model, the connection is particularly clear: starting from a Grothendieck topos, we construct a model of set theory that can be used to demonstrate independence results over **ZF**, and then show that the resulting extension is isomorphic to the one obtained in the Boolean-valued model approach. As for the generic filter, while its role is central in set-theoretic forcing, its role is not often made explicit in topos-theoretic treatments. We aim to capture the idea of adjoining a generic filter to a base model by showing that the double negation topos is the classifying topos of the theory of generic filters. Finally, we show that the forcing relation aligns in both frameworks: the same elements of the poset force the same formulas, establishing a clear and formal correspondence between the two settings.

The standard textbook [MM94] makes connections to Cohen forcing in its chapter VI on toposes and Logic, but without making formal connections. In [Fou80], Fourman describes how Grothendieck toposes provide models of set theory. This is the approach we use in our specific case of the double negation topos, although we make stronger use of the internal logic of a topos, mostly following the framework of [Joh02].

The outline of the thesis is as follows. The preliminaries introduce and recall essential background from first-order logic, orders and category theory up to the definition of an elementary topos. Section 2 provides an overview of forcing with Boolean-valued models, while Section 3 introduces sites and sheaves, leading to the definition of a Grothendieck topos. These first sections may safely be skipped and simply referred back to by readers familiar with these aspects of set- or topos theory. Section 4 is devoted to the construction and characterization of the double negation topos and the role of generic filters. In Section 5, we develop the internal language of a topos and adapt the Kripke-Joyal semantics to the specific case at hand. Finally, in Section 6, we describe how a Grothendieck topos gives rise to a Boolean-valued model of set theory, and show several equivalences to the Boolean-valued case. We will conclude with some ideas for further connections that were not covered.

I would like to thank my supervisor Peter LeFanu Lumsdaine for his invaluable insight and kind guidance in many meetings that was instrumental in helping me navigate these concepts. Further thanks to Ingo Blechschmidt for interesting conversations on the topic and finally to my friend Anton Christenson for his unwavering support and careful reading of earlier drafts.

1 Preliminaries

This section collects definitions and background needed in the remainder of the text. It begins with a minimal refresher on first-order logic and **ZF**, limited to setting notational conventions and pointing to references. While a basic knowledge of category theory is assumed, we introduce several concepts from category theory less common in introductory courses more carefully, up to the definition of an elementary topos. Finally, we present a somewhat eclectic collection of definitions from order theory that may not be commonly known, focusing on Boolean algebras and locales.

1.1 Logical Preliminaries

Recall the definition of a first-order theory, for instance as defined in [Joh02, D 1.1.1], consisting of sorts, function symbols and relations symbols. We will use Γ to denote a context, that is a finite list of sorts $(x_1 : X_1, ..., x_n : X_n)$ indexed by variables. We write

$$\Gamma \mid \varphi(\overline{x})$$

for a formula in context Γ.

The theory of sets **ZF** on a signature $\Sigma_\in$ has only the sort $\mathcal{V}$ and relation symbol $\in: \mathcal{V}, \mathcal{V}$. Any context will be of the form $\overline{x} : \mathcal{V}^n$, and any term will be a projection. The axioms of set theory can be consulted in [Jec03, I.1] for instance.

1.2 Background on Orders

Boolean algebras in particular will play a recurring role both in topos theory and in forcing. A Boolean algebra is a poset with additional properties. In discussing these orders we will already make use of categorical terminology, as every partially ordered set (poset) can be thought of as a thin category.

Let $\mathbb{P}$ be a poset. When $q \leq p$, we say that q is a *refinement* of p. The following two definitions on posets are perhaps less known but often arise in forcing.

Definition 1.1 A subset $X \subset \mathbb{P}$ is dense below $p \in \mathbb{P}$ if for every $q \leq p$ there exists some $r \in D$ such that $r \leq q$, i.e. every element of $\mathbb{P}$ below p has a refinement in D.

Accordingly, a dense subset of $\mathbb{P}$ is one which contains a refinement for *any* element of the poset.

Definition 1.2 A generic filter on a poset is a subset $G \subseteq \mathbb{P}$ such that

(i) $G \neq \emptyset$,
(ii) G is upward closed,
(iii) any two elements in $\mathbb{P}$ have a common refinement in G,
(iv) and every dense subset of $\mathbb{P}$ has an element in G.

The particulars and consequences of this definition are of course crucial when forcing, but are not further relevant for this thesis, as we will not delve into details of the forcing technique.

We add conditions to the structure of a poset to build up to the definition of a Boolean algebra.

Definition 1.3 A *lattice* is a poset $(L, \leq)$ with binary products (infimums) and coproducts (supremums), called meets and joins.

Definition 1.4 A Heyting algebra $(H, \leq)$ is a bounded lattice with exponentials.

Exponentials for general categories will be discussed in the next section, but for now this amounts to the following. For any $x, y \in L$ there is an element $(x \Rightarrow y) \in L$ such that for all $z \in H$,

$$z \leq (x \Rightarrow y) \;\text{ iff }\; z \wedge x \leq y.$$

In categorical terms one could say that $(x \Rightarrow -) : H \to H$ is right adjoint to $(- \wedge x) : H \to H$.

We can define complements $\neg x$ in a Heyting algebra, but not every element in a Heyting algebra necessarily has one. A complement is an element $\neg x$ such that $x \vee \neg x = \top$ and $x \wedge \neg x = \bot$.

Definition 1.5 A Boolean algebra is a Heyting algebra in which every element has a complement

Of course every Boolean algebra is a Heyting algebra, and there is a forgetful functor

$$U : \text{Bool} \to \text{Heyt}.$$

This functor has a left adjoint

$$\begin{aligned} \neg\neg : \text{Heyt} &\to \text{Bool}, \\ H \mapsto H_{\neg\neg} &= \Big(\{\neg\neg x \mid x \in H\}, \underset{H}{\leq}\Big), \end{aligned}$$

known as the *regularisation of* H or *double negation translation.*

Heyting and Boolean algebras have strong ties to intuitionistic and classical logic respectively, in that for any first-order theory $\mathbb{T}$, one can define the corresponding *Lindenbaum algebra* $L[\mathbb{T}]$, which will be a Heyting- or Boolean algebra, depending on the underlying logic chosen. The connectives then correspond exactly with the operation on the algebra suggested by their notation. See [Bel05, §0] for details.

Example 1.1. Let X be a topological space. Then the poset of its opens $\mathcal{O}$ gives a Heyting algebra, with intersections and unions as its meets and joins, and with the operation

$$U \Rightarrow V := \text{Int}\big(U^C \cup V\big).$$

In fact, due to the small union of opens being open in a topological space, the structure of $\mathcal{O}(X)$ generalizes more accurately as a *frame* or *locale.*

Definition 1.6 A *frame* L is a lattice with all finite meets and all small joins for which the infinite distributive law holds:

$$A \wedge \bigvee_I B_i = \bigvee_I (A \wedge B_i).$$

This is known as a locale in the opposite category. See [MM94, IX] for more on the theory of locales. It is easy to show that every locale is a Heyting algebra.

1.3 Categorical Preliminaries

We are particularly interested in constructions in category theory that generalize properties of set theory, such as subsets, images, preimages, the powerset and characteristic functions.

Throughout this section, we will be using the categories of sets **Set** and the category of presheaves in examples. **Set** will be the inspiration from which we wish to generalize certain concepts, and $\hat{\mathcal{C}}$ being a category nice enough to satisfy these generalized properties. Of course far from all categories support all these constructions, however in this thesis we will be looking almost exclusively at categories "nice enough" to support all concepts introduced. A more detailed analysis of which categorical properties are necessary for exactly which fragments of logic would of course also be possible.

We will present some useful facts around categories that may need to be recalled, and will be useful later on in a specific case, but more appropriately fit in a general categorical context. We will always assume that a category $\mathcal{C}$ is well-powered.

Definition 1.7 Given a small category $\mathcal{C}$, The category of presheaves

$$\hat{\mathcal{C}} := \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

is the category of contravariant functors from $\mathcal{C}$ to **Set**. We will use notation common in algebraic geometry and use letters $U, V, \ldots$ for objects of $\mathcal{C}$ and given a presheaf P on $\mathcal{C}$ and $x \in P(U)$, write

$$x \cdot f := P(f)(x)$$

for any $f : V \to U$, think "the restriction of x along f".

We go through some elementary constructs in the categories $\hat{\mathcal{C}}$ and **Set**.

Recall the definition of (co)limits in a category, and that these correspond to small unions and intersections in **Set**.

Example 1.2. The (co)limit of any presheaf exists, and is given pointwise by the (co)limit in **Set**.

A more explicit account of how a diagram in $\hat{\mathcal{C}}$ gives a diagram in **Set** and a proof of the statement can be found in [Mac78, III.3].

Example 1.3. In **Set**, the pullback of $f : A \to C$ along $g : B \to C$ is given by the fibered product

$$A \times_C B := \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

This makes **Set** a category with all pullbacks.

Example 1.4. In the category $\hat{\mathcal{C}}$, the pullback P of $f : F \to H$ along $g : G \to H$ is determined pointwise by the pullback in **Set**, i.e.

$$P(U) = F(U) \times_{H(U)} G(U).$$

1.3.1 Subobjects

In **Set**, in addition to the notion of objects and morphisms between them, objects can be compared to each other by the subset relation. This is also a common concept in many categories – we may be interested in subgroups, subfields and various subspaces. In set theory a subset is defined pointwise, but often a more general approach is sufficient. Looking at field extensions gives insight into what properties of subobjects we might want in a more general approach. It is possible to define field extensions as an injective ring homomorphism between fields, without clinging to the condition that the subfield be actually contained in the larger one as a set. This does not dramatically change the subject, we can easily brush off the difference as being a simple matter of renaming, and it reminds us to be comfortable in more general cases without having to explicitly move across an isomorphism. The important thing is that the embedding be injective. In Galois theory as well as set theory, subsets and subfields are not primarily interesting on their own, but in relation to each other. The subfields relation gives a way to order the subfields of the given field. The partial order of the powerset also arises by comparing subsets to each other.

Inspired by these ideas to generalize, we first define an order on monics of a category, and then move on to define subobjects and their classification in certain categories.

Definition 1.8 For any category $\mathcal{C}$ and object X, the monics into X form a preorder: For monics $A \overset{f}{\rightarrowtail} X, B \overset{g}{\rightarrowtail} X$,

$$f \leq g \text{ if there exists some } h : A \to B \text{ s.t. } g \circ h = f.$$

There is a canonical way to make an preorder into a partial order by equating any two elements for which $x \leq y$ and $y \leq x$. This is the procedure we follow to define subobjects.

Definition 1.9 Two monics $f : S \rightarrowtail X, g : S' \rightarrowtail X$ are equivalent if $f \leq g$ and $g \leq f$. A subobject is an equivalence class of monics into X. We denote this subobject as $[f]$, or often lazily as $S \rightarrowtail X$ or even $S \subseteq X$, omitting the important datum of the monic between the two objects in favour of imitating the familiar set-theoretic notation.

It is easy to show that two monics are equivalent if and only if there is an isomorphism $h : S \to S'$ such that $g \circ h = f$.

We will write $\mathrm{Sub}_{\mathcal{C}}(X)$ for the partial order of subobjects of X. The notion of a subobject "makes sense" in any category, but for sufficiently nice categories, the subobjects will hold additional interesting structure.

Definition 1.10 For a category $\mathcal{C}$ with all pullbacks, we can define a subobject functor

$$\mathrm{Sub}_{\mathcal{C}}(-) : \mathcal{C} \to \mathbf{Set}$$

which maps objects to their set of subobjects $\mathrm{Sub}_{\mathcal{C}}(X)$ and maps morphisms $X \overset{f}{\to} Y$ to a function $\mathrm{Sub}_{\mathcal{C}}(Y) \to \mathrm{Sub}_{\mathcal{C}}(X)$ that sends any subobject $S \rightarrowtail Y$ to its pullback S' along f.

$$\begin{array}{ccc} S' & \longrightarrow & S \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

It is straightforward to show that this is a well-defined functor.

Example 1.5. In the category of presheaves, monics are pointwise injective natural transformations. A natural way to define a subfunctor of $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ would be as another functor $G : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ such that $G(X) \subseteq F(X)$ for each object $X \in \mathrm{Ob}(\mathcal{C})$ and $G(f)$ is equal to the restriction

$$F(f)\Big|_{F(X)}^{F(Y)}$$

for each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

In the functor category $\hat{\mathcal{C}}$, we have a monic arrow $G \rightarrowtail F$ given simply by the inclusion, so every subfunctor gives a subobject in this category. Conversely, given a subobject $\tau : F \rightarrowtail G$, we can define a functor as $H(X) = \operatorname{im} \tau(X)$. This functor will be both isomorphic to G (and so represent the same subobject) and a subfunctor of F. This justifies using subpresheaf and subobject of presheaf interchangeably.

1.3.2 Images and Preimages

A preimage of a morphism $f : X \to Y$ and subobject $S \subseteq Y$ is defined as the pullback of S along f, called $f^*(S)$. If all pullbacks exist, every morphism $f : X \to Y$ induces a map

$$\begin{aligned} f : \operatorname{Sub}(Y) &\to \operatorname{Sub}(X) \\ S &\mapsto f^*(S). \end{aligned}$$

In certain categories, in particular in categories we will come to know as *toposes*, the preimage functor will have left and right adjoints

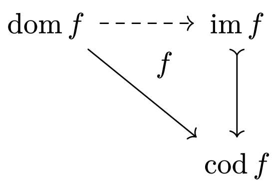
$$\exists_f \dashv f^* \dashv \forall_f.$$

In this case we will be able to see properties similar to images and preimages in **Set**, such as:

Lemma 1.11 *In a category in which the adjoints* $\forall, \exists$ *of the image functor exist for any morphism, if* $\mathrm{e} : X' \twoheadrightarrow X$ *is an epi and* $S, T \subseteq X$*, then*

$$\mathrm{e}^*(A) \subseteq \mathrm{e}^*(B) \text{ implies } A \subseteq B.$$

We say that $\mathcal{C}$ has *images* if there is an operation $\mathbf{im} : \operatorname{Mor}(\mathcal{C}) \to \operatorname{Obj}(\mathcal{C})$ on morphisms such that $\operatorname{im} f \rightarrowtail \operatorname{cod} f$ is the least subobject through which f factors.



1.3.3 The Subobject Classifier

In **Set**, we are able to characterise a subset $S \subseteq X$ by its characteristic function $\varphi_S : X \to \{0, 1\}$, defined as

$$\phi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{o/w.} \end{cases}$$

Crucial to this definition is the role of the truth values $\{0, 1\}$, and our understanding of 1 as “true”. We will generalize the idea of these truth values along with the characterisation of subsets to any category with finite limits with the following definition.

Definition 1.12 In a category with finite limits, a subobject classifier is an object Ω along with a monic $1 \to \Omega$, such that to every monic $S \rightarrowtail X$ in $\mathcal{C}$ there is a unique arrow ϕ_S which with the given monic forms the pullback square

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & & \downarrow \text{true} \\ X & \xrightarrow{\phi_S} & \Omega \end{array}$$

Theorem 1.13 *[MM94, I.3 Proposition 1] A locally small category with finite limits has a subobject classifier if and only if the subobject functor is representable, i.e. there is an object Ω and a natural isomorphism*

$$\mathrm{Sub}_{\mathcal{C}}(-) \cong \mathrm{Hom}_{C}(-, \Omega).$$

We will use the same names for subobjects and their corresponding maps to Ω.

Lemma 1.14 *For any morphism $f : X \to Y$ and subobject $S \subseteq Y$,*

$$\phi_{f^*(S)} = \phi_S \circ f.$$

1.3.4 The Subobject Classifier of Presheaves

For illustrative purposes, we show a false first attempt to define the subobject classifier of presheaves before giving the actual definition.

The subpresheaves of a given presheaf F are determined by how they act on objects, as their action on morphisms is inhereted from F. From this and the pattern of many categorical constructions being pointwise corresponding to those in **Set**, one might think the functor

$$\begin{aligned} W : \mathcal{C}^{\mathrm{op}} &\to \mathbf{Set} \\ U &\mapsto \Omega_{\mathbf{Set}} = \{0, 1\} \\ (f : U \to V) &\mapsto \mathrm{id}_{\{0,1\}} \end{aligned}$$

along with the pointwise injection

$$t : \hat{1} \to W, t(U) = \text{true}$$

may be the subobject classifier of $\hat{\mathcal{C}}$. This is not the case however. While it seems to reasonably classify subpresheaves, by setting

$$\phi_S : X \to W, \phi_S(U) = \phi_{S(U)}$$

pointwise as the classifying morphism for $S(U) \subseteq X(U)$, this classification fails to be a morphism of presheaves, as the following example demonstrates.

Example 1.6 (ϕ_S is not a natural transformation). Consider the category $\bullet \to \bullet$ and the presheaf $F(\bullet \to \bullet) = (2 \leftarrowtail 1)$, as well as its subfunctor $S(\bullet \to \bullet) = (1 \leftarrowtail 0)$. The diagram

$$
\begin{array}{ccc}
F(\bullet) & \xrightarrow{\phi_S(\bullet)} & W(\bullet) \\
\downarrow & & \downarrow \\
F(\bullet) & \xrightarrow{\phi_S(\bullet)} & W(\bullet)
\end{array}
\quad = \quad
\begin{array}{ccc}
1 & \xrightarrow{\phi_0} & \{0,1\} \\
\downarrow & & \downarrow \mathrm{id} \\
2 & \xrightarrow{\phi_1} & \{0,1\}
\end{array}
$$

does not commute.

We can find the actual subobject classifier of $\hat{\mathcal{C}}$ by applying the Yoneda lemma to the characterisation given in Theorem 1.13,

$$\mathrm{Sub}_{\hat{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(-,X)) \cong \mathrm{Hom}_{\hat{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(-,X),\Omega) \cong \Omega(X)$$

suggesting that we write Ω as

$$\Omega(X) = \{S \mid S \text{ is a subfunctor of} \, \mathrm{Hom}_{\mathcal{C}}(-,X)\}$$

We can introduce an easier way to think about these subfunctors by using *sieves*.

Definition 1.15 Let X be an object of a category $\mathcal{C}$. Then a *sieve* on X is a set S of morphisms with codomain X such that

$$f \in S \text{ and } f,h \text{ are composable} \Rightarrow f \circ h \in S.$$

i.e. S is closed under precomposition.

Example 1.7 (Sieves on a Poset). Let $\mathbb{P}$ be a poset and $x \in \mathbb{P}$. Then a sieve on x is a subset S such that

$$y \in S \text{ and } z \leq y \Rightarrow z \in S,$$

i.e. a downward closed subset bounded from above by x.

For an object C of $\mathcal{C}$, the set $t(C)$ of all arrows into C is a sieve, called the maximal sieve on C.

We can relate sieves to subfunctors of $\mathrm{Hom}_{\mathcal{C}}(-,X)$. Every subfunctor $Q \subseteq \mathrm{Hom}_{\mathcal{C}}(-,X)$ defines a sieve

$$S := \{f \mid f \in Q(A) \text{ for some } A \in \mathcal{C}\}.$$

and conversely every sieve S a subfunctor Q of $\mathrm{Hom}(-,X)$ by

$$Q(A) := \{f : A \to X \mid f \in S\}.$$

It is easy to check that this is a well-defined 1-1 correspondence.

In categories with pullbacks, we have for any map $g : X \to Y$ and any subobject $S \rightarrowtail \mathrm{Hom}(-,X)$ its preimage $g^*(S)$. One can check that this corresponds to the sieve

$$g^*(S) = \{f \mid g \circ f \in S\}.$$

After this short excursion on sieves, we can rewrite

$$\Omega(X) = \{S \mid S \text{ is a sieve on } X \text{ in } \mathcal{C}\} \tag{1.1}$$

for objects, and on morphisms this functor maps as the preimage

$$S \cdot g = \{h \mid g \circ h \in S\}.$$

Theorem 1.13 guarantees that this is the subobject classifier.

1.3.5 Exponentials

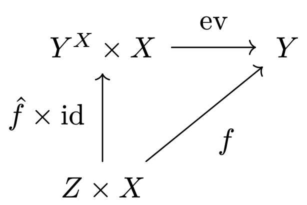
Recall that products and coproducts, along with the initial and terminal elements $0, 1$ of a category satisfy equalities analogous to those of the rules of arithmetic. For instance, $X \times 1 = 1$ and $X \times 0 = 0$ for any object X.

This pattern continues with the introduction of exponentials, motivated by the understanding of an exponential Y^X as the maps from X to Y.

Given objects X, Y of a category, the exponent Y^X will be an object such that $\mathrm{Hom}(Z \times X, Y) \cong \mathrm{Hom}(Z, Y^X)$. Explicitly, we define this as an object together with an evaluation arrow

$$\mathrm{ev} : Y^X \times X \to Y$$

such that for every $f : Z \times X \to Y$ there exists a unique map $\hat{f} : Z \to Y^X$ that makes the following diagram commute:



Example 1.8. For any object X, we have

$$X^0 = 1,$$
$$X^1 = X.$$

1.3.6 The Subobject Lattice of a Presheaf

Properties of the surrounding category will have consequences on the order of subobjects.

Theorem 1.16 *[MM94, I.8 Prop 5] For any presheaf $P \in \hat{\mathcal{C}}$ over a small category $\mathcal{C}$, $\mathrm{Sub}_{\hat{\mathcal{C}}}(P)$ is a Heyting algebra.*

As with the rest of this section, we will not provide a proof of this fact, however we shall give an explicit description of the operations. Most are a pointwise application of the set-theoretic case, but not all.

Let S, T be subpresheaves of P and $C \in \mathcal{C}$. Then

(i) $\top = P$

(ii) $(S \wedge T)(U) = S(U) \cap T(U)$

(iii) $\bot =$ the constant empty presheaf

(iv) $(S \vee T)(U) = S(U) \cup T(U)$

(v) $(S \Rightarrow T)(U) = \{x \in P(U) \mid \text{for all } f : D \to C, \text{if } x \cdot f \in S(U) \text{ then } x \cdot f \in T(U)\}$.

Further, for any $f : X \to Y$, the preimage functor has left adjoint

$$\exists_f(S) = \{y \mid \exists z \in S.(f(z) = y\}$$

and right adjoint

$$\forall_f(S) = \{y \mid \forall z.(f(z) = y \Rightarrow z \in S)\}.$$

See [MM94, I.9] for further details and proofs.

1.3.7 Elementary Toposes

All properties we have met so far are collected in the definition of an (elementary) topos. There are many possible definitions of a topos that take different collections of properties as axioms to derive the rest from.

Definition 1.17 A *topos* is a cartesian closed category with finite limits and a subobject classifier.

It is possible to show properties analogous to those we encountered for sheaves for the subobject lattice of any element of a topos.

Theorem 1.18 *[MM94, I.8 Prop 5] For any elementary topos $\mathcal{E}$ over a small category $\mathcal{C}$ and any object X, $\mathrm{Sub}_{\mathcal{E}}(X)$ is a Heyting algebra.*

Lemma 1.19 *For every arrow $f : X \to Y$ in a topos, the induced map $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ has right and left adjoints*

$$\exists_f \dashv f^{-1} \dashv \forall_f.$$

While the specific definition of these maps is interesting, for our purposes adjointness is the most consequential property. See [MM94, IV] for a proof of this fact and many more constructions for elementary toposes.

2 Forcing with Boolean-valued Models

In this section, we outline how Boolean-valued models can be used to derive independence results via forcing. Cohen originally formulated forcing in terms of set-theoretic models, "extending" a base model by the generic filter of a poset of forcing conditions. However, set-theoretic models are not strictly necessary for establishing independence results. Any structure that serves as a model by satisfying an appropriate form of soundness can be used to compare the consistency of theories and produce independence results – be it a set, categorical, inner or Boolean-valued model.

Since our main focus is topos-theoretic forcing, this section will remain brief and provide only a rough overview, emphasizing aspects that we will later be revisiting in the topos-theoretic setting.

We will soon define more precisely how Boolean-valued models in particular interpret the language of set theory. Generally speaking, we expect a model of a theory $\mathbb{T}$ over Σ to provide us with a predicate $M \vDash$ on formulas over Σ such that $M \vDash \varphi$ holds whenever $\mathbb{T} \vdash \varphi$, and $M \nvDash \bot$.

Theorem 2.1 *Let $\mathbb{T}$ be a first-order theory. Suppose $(M \vDash -)$ is a translation from $\Sigma_{\mathbb{T}}$ to $\Sigma_{\in}$ such that:*

- *If $\mathbb{T} \vdash \varphi$, then $\mathbf{ZF} \vdash (M \vDash \varphi)$.*
- *$\mathbf{ZF} \vdash \neg(M \vDash \bot)$.*

Then $\mathrm{Con}(\mathbf{ZF}) \Rightarrow \mathrm{Con}(\mathbb{T})$.

Any model satisfying the axioms of $\mathbf{ZF} + \varphi$ for some φ over $\Sigma_{\in}$ would therefore be proving that $\mathbf{ZF} \nvdash \neg\varphi$, giving one direction of an independence result. The following corollary accomodates **ZFC** and any other equiconsistent extension.

Corollary 2.2 *Let $\mathbb{T}$ be a theory and $\mathbb{T}'$ an extension of **ZF** such that $\mathrm{Con}(\mathbf{ZF}) \Rightarrow \mathrm{Con}(\mathbb{T}')$. Suppose $(M \vDash -)$ is a translation from $\Sigma_{\mathbb{T}}$ to $\Sigma_{\in}$ such that:*

- *If $\mathbb{T} \vdash \varphi$, then $\mathbb{T}' \vdash (M \vDash \varphi)$.*
- *$\mathbb{T}' \vdash \neg(M \vDash \bot)$.*

Then $\mathrm{Con}(\mathbb{T}') \Rightarrow \mathrm{Con}(\mathbb{T})$.

The inner model L, for instance, gives $\mathrm{Con}(\mathbf{ZFC}) \Rightarrow \mathrm{Con}(\mathbf{ZFC} + \mathbf{CH})$ as well as $\mathrm{Con}(\mathbf{ZFC}) \Rightarrow \mathrm{Con}(\mathbf{ZFC} + V = L)$ through such an argument. We will build a model that satisfies this condition as well, while avoiding some of the technical details in Cohen's original approach with countable transitive set models.[1] Through the Boolean-valued approach, it is possible to get $\mathrm{Con}(\mathbf{ZFC}) \Rightarrow \mathrm{Con}(\mathbf{ZFC} + \neg\mathbf{CH})$ as well as $\mathrm{Con}(\mathbf{ZFC}) \Rightarrow \mathrm{Con}(\mathbf{ZFC} + V \neq L)$, and thereby complete the proof of independence of these two theorems from **ZFC**.

We will not develop a general theory of Boolean-valued models, but instead construct the particular Boolean-valued model $V^{\mathbb{B}}$ over a Boolean algebra $\mathbb{B}$.[2] We will go through the steps required to show that $V^{\mathbb{B}}$ satisfies the properties described above. Then we explain how a poset of forcing conditions can determine such a Boolean algebra, define the forcing relation and sketch the role a generic filter of this poset plays in showing independence results.

[1]To see more precisely how the two methods connect and be translated into one another, see [Bel05, §4].

[2]Heyting-valued models are also possible and follow an analogous setup over a Heyting algebra, see [Bel05, §8].

2.1 The Universe of $\mathbb{B}$-valued Sets

It is common to study inner models that are subclasses of the von Neumann universe V – here, we will consider extensions thereof. In the cumulative hierarchy, the successor step involves adding all subsets of a given set, which can be represented by two-valued characteristic functions. The idea is to generalize this construction to functions taking values in a complete Boolean algebra $\mathbb{B}$.

Definition 2.3 By wellfounded induction on $\alpha \in \mathrm{Ord}$, let $V_\alpha^{\mathbb{B}}$ be the set of functions to $\mathbb{B}$ whose domain is a subset of $V_\beta^{\mathbb{B}}$ for some $\beta < \alpha$, i.e.

$$V_\alpha^{\mathbb{B}} = \left\{x : X \to \mathbb{B} \mid \exists \beta < \alpha . X \subseteq V_\beta^{\mathbb{B}}\right\}$$

and call their union over the class of ordinals

$$V^{\mathbb{B}} = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha^{\mathbb{B}}$$

the *universe* $V^{\mathbb{B}}$ *of* $\mathbb{B}$*-valued sets.*

An equivalent definition setting apart limit and successor ordinals can be found in [Jec03, 14.15]

It is immediate from the definition that

$$\alpha \leq \beta \text{ implies } V_\alpha^{\mathbb{B}} \subseteq V_\beta^{\mathbb{B}}. \tag{2.2}$$

Example 2.1. Both in order to grow accustomed to this hierarchy, and because they are interesting special cases, we will compute the first few levels.

- $V_0^{\mathbb{B}} = \emptyset$.
- $V_1^{\mathbb{B}}$ contains exactly the empty function 0, and is the singleton in **Set**:

$$V_1^{\mathbb{B}} = \{x : X \to \mathbb{B} \mid X \subseteq V_0^{\mathbb{B}}\} \cong 1$$

- $V_2^{\mathbb{B}}$ contains the empty function along with a function for every element in $\mathbb{B}$.

$$\begin{aligned} V_2^{\mathbb{B}} &= \{x : X \to \mathbb{B} \mid X \subseteq V_0^{\mathbb{B}} \vee X \subseteq V_1^{\mathbb{B}}\} = \{x : X \to \mathbb{B} \mid X = \emptyset \vee X = 1\} \\ &= \{x : X \to \mathbb{B} \mid X = \emptyset\} \cup \{x : X \to \mathbb{B} \mid X = 1\} \\ &\cong 1 \cup \mathbb{B}. \end{aligned}$$

At this point it is noticeable that in the case of $\mathbb{B} = 2$, we do *not* get $V_2 \cong V_2^2$, nor can we expect a correspondence for the following levels. There will however be a way to formally relate the two, after we introduce an internal language to reason and make statements about its elements.

Let $\Sigma_{\mathbb{B}}$ be the first-order signature obtained by adding a function symbol for every element of $V^{\mathbb{B}}$ to the signature $\Sigma_\in$ of set theory. Notationally we will make no difference between terms of $\Sigma_{\mathbb{B}}$ and elements of $V^{\mathbb{B}}$ for convenience.

We will now recursively assign to each formula $\varphi(x_1, ..., x_n)$ over $\Sigma_{\mathbb{B}}$ and each $u_1, ..., u_n \in V^{\mathbb{B}}$ a Boolean value $[\![\varphi(\overline{u})]\!] \in \mathbb{B}$ by the following clauses:

(i) $$[\![\top]\!] := \top$$

(ii) $$[\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \wedge [\![\psi]\!]$$

(iii) $$[\![\bot]\!] := \bot$$

(iv) $$[\![\varphi \vee \psi]\!] := [\![\varphi]\!] \vee [\![\psi]\!]$$

(v) $$[\![\, \varphi \Rightarrow \psi \,]\!] := [\![\, \varphi \,]\!] \Rightarrow [\![\, \psi \,]\!]$$

(vi) $$[\![\, \exists x.\varphi(x) \,]\!] := \bigvee_{v \in V^B} [\![\, \varphi(v) \,]\!]$$

(vii) $$[\![\, \forall x.\varphi(x) \,]\!] := \bigwedge_{v \in V^B} [\![\, \varphi(v) \,]\!]$$

It remains of course to assign values to the atomic cases, the element relation and equality relation. These can be related to each other by

$$\begin{aligned} &(u \in v) \Leftrightarrow \exists x \in v.(x = u), \\ &(u = v) \Leftrightarrow \forall x \in v.(x \in u) \wedge \forall x \in u.(x \in v). \end{aligned}$$

With this in mind, define

(viii) $$[\![\, u \in v \,]\!] := \bigvee_{x \in \operatorname{dom}(v)} [v(x) \wedge [\![\, u = x \,]\!]],$$

(ix) $$[\![\, u = v \,]\!] := \bigwedge_{x \in \operatorname{dom}(v)} [v(x) \Rightarrow [\![\, x \in u \,]\!]] \quad \wedge \bigwedge_{x \in \operatorname{dom}(u)} [u(x) \Rightarrow [\![\, x \in v \,]\!]].$$

That this recursive definition is well-defined is not obvious, but can be shown by picking an appropriate well-founded relation, see [Bel05, p.23] for further details.

Example 2.2. Connecting back to the previous example, consider these two Boolean functions in $V_2^{\mathbb{B}}$,

$$0 : \emptyset \to \mathbb{B} \text{ and } b : 1 \to \mathbb{B}, 0 \mapsto \bot.$$

We can show that $V^{\mathbb{B}} \vDash [\![\, 0 = b \,]\!]$ by simply applying the recursive definition:

$$\begin{aligned} [\![\, 0 = b \,]\!] &= \underbrace{\bigwedge_{x \in \emptyset} \dots}_{=\top} \wedge \bigwedge_{x \in 1} (b(x) \Rightarrow [\![\, x \in 0 \,]\!]) \\ &= (\bot \Rightarrow [\![\, x \in 0 \,]\!]) = \top. \end{aligned}$$

This demonstrates that $V^{\mathbb{B}}$ contains a lot of "duplicates" that could get rid of by dividing by the internal equality relation $[\![\, . = . \,]\!]$. As doing so does not bring any particular advantage to Boolean-valued forcing, it is not commonly done, and neither will we - we will be dividing out by this relation later when comparing it to models in a topos.

Remark 2.1. Note that $V^{\mathbb{B}}$ is not a set, and that $[\![\, - \,]\!]$ cannot be formalized within **ZF** as a function. Different forcing methods will have different ways of dealing with this depending on the type of models they wish to create. For our purposes this won't pose further complications.

It is fundamental to check that this assignment of Boolean truth values respects the first-order rules of inference.

Theorem 2.4 *Let φ, ψ be formulas over $\Sigma_{\mathbb{B}}$ and let $\overline{u} \in V^{\mathbb{B}^n}$. If $\varphi(\overline{x}) \vdash \psi(\overline{x})$ in first-order logic, then $[\![\, \varphi(\overline{u}) \,]\!] \leq [\![\, \psi(\overline{u}) \,]\!]$.*

Proof. It is proven carefully by induction in [Bel05, Theorem 1.17] that the assignment $[\![\, - \,]\!]$ respects (a version of) the rules of inference of first-order logic, from which this theorem follows. □

Using this assignment of elements in $\mathbb{B}$ to formulas over $\Sigma_{\mathbb{B}}$, we now consider the validity of formulas over $\Sigma_{\in}$.

Definition 2.5 Let $\varphi(x_1, ..., x_n)$ be a formula over $\Sigma_{\in}$. Then

$$V^{\mathbb{B}} \vDash \varphi(\overline{x}) :\Leftrightarrow \forall u_1, ..., u_n \in V^{\mathbb{B}}.[\![\varphi(u_1, ..., u_n)]\!] = 1.$$

Due to (2.2) we can also write

$$V^{\mathbb{B}} \vDash \varphi(\overline{x}) :\Leftrightarrow \forall \alpha \in \mathrm{Ord}\,.\forall u_1, ..., u_n \in V^{\mathbb{B}}_{\alpha}.[\![\varphi(u_1, ..., u_n)]\!] = 1.$$

In [Bel05, 1.33], it is stated that $V^{\mathbb{B}}$ is a model of **ZFC**, i.e. that all axioms of **ZF** are true in $V^{\mathbb{B}}$. We rephrase this to better match our setup in Theorem 2.1, and refer to [Bel05, 1.33] or [Jec03, II 14.24] for the full proofs of all axioms. From this, Soundness follows:

Corollary 2.6 *For any formula φ over $\Sigma_{\in}$, if* $\mathbf{ZF} \vdash \varphi$, *then* $\mathbf{ZF} \vdash (V^{\mathbb{B}} \vDash \varphi)$.

All this justifies the use of $V^{\mathbb{B}}$ as a model. But we still need more machinery to *force* additional statements before we can apply Theorem 2.1 in a non-trivial way.

2.2 The Poset of Forcing Conditions

We have seen how to construct a model from a Boolean algebra. In Cohen's original formulation of forcing, a poset of forcing conditions, which need not form a Boolean algebra itself, plays a central role. A ground model is extended by adjoining a generic filter over this poset. In the Boolean-valued approach, the poset still plays the same crucial role as it gives rise to a Boolean algebra that underlies the construction. Specifically, the poset first induces a Heyting algebra structure, which can then be transformed into a Boolean algebra via a double negation translation. We will examine some steps of this construction more closely.

Given a poset $\mathbb{P}$ we get the Heyting algebra (and so called *left order topology* on $\mathbb{P}$) of sets from the basis

$$\mathcal{O}_p := \{q \in \mathbb{P} \mid q \leq p\}.$$

This is a basis for a topology on $\mathbb{P}$, i.e. contains every $p \in \mathbb{P}$ and if $x \in \mathcal{O}_p \cap \mathcal{O}_q$, then $\mathcal{O}_x \subseteq \mathcal{O}_p \cap \mathcal{O}_q$. As such it generates a locale and Heyting algebra Dn of down-closed sets.

The Heyting algebra Dn can be turned into a Boolean algebra through the double negation translation. We take a closer look at the negation on Dn in order to characterize this translation.

The interior of an arbitrary $X \subseteq \mathbb{P}$ is

$$\mathrm{Int}(X) = \{x \mid \forall y \leq x.y \in X\},$$

so the negation operation on the Heyting algebra Dn is

$$\neg X := (X \Rightarrow \bot) = \mathrm{Int}\big(X^C\big) = \{x \mid \forall y \leq x.y \notin X\}.$$

The double negation operation on Dn is given by

$$\begin{aligned} \neg\neg X = \mathrm{Int}\big((\neg X)^C\big) &= \{p \mid \forall q \leq p.q \notin (\neg X)\} \\ &= \{p \mid \forall q \leq p.\exists r \leq q.r \in X\} \end{aligned}$$

Note that this corresponds to the order theoretic definition of dense, so

$$\neg\neg X = \{p \in \mathbb{P} \mid X \text{ is dense below } p\}.$$

As discussed in Section 1.2, double negation defines a functor mapping Heyting algebras H to Boolean algebras $H_{\neg\neg}$, pointwise. So the Heyting algebra Dn of down-closed subsets of $\mathbb{P}$ is mapped to the Boolean algebra $\mathrm{Dn}_{\neg\neg}(\mathbb{P})$.

Now we have stepped from posets through locales and Heyting algebras to finally Boolean algebras. At this point it is only natural to express that these steps have been taken in a way that preserve certain properties of $\mathbb{P}$, and that $\mathbb{P}$ relates to $\mathrm{Dn}_{\neg\neg}(\mathbb{P})$ in a reasonable way. Of course, this depends on the use. As we will sketch later, an important factor here is for the generic filter of $\mathbb{P}$ to be preserved. As we will closely examine the corresponding statement on the topos-theoretic side later on, we will only state here the theorem which guarantees this preservation from [Jec03, Cor 14.12].

Theorem 2.7 *For every poset* $(\mathbb{P}, <)$*, there is a unique complete Boolean algebra* $\mathbb{B}$ *and order homomorphism* $j : \mathbb{P} \to \mathbb{B} \setminus \{0\}$ *such that*

(i) $j(\mathbb{P})$ *is dense in* $\mathbb{B}$*,*

(ii) Any $p, q \in \mathbb{P}$ *are compatible if and only if* $j(p) \wedge j(q) \neq \bot$*.*

This Boolean algebra $\mathbb{B}$ is then called the *Boolean completion* of $\mathbb{P}$.

It is also shown in [Bel05, Lemma 2.5] that for a *refined* poset[3], this map j will be an order isomorphism into a dense subset of $\mathbb{B}$. The poset used in Cohen Forcing for instance is refined anyway. In this case we could simply identify $\mathbb{P}$ with $j(\mathbb{P})$. In any case will often simply write p for $j(p)$ in $\mathbb{B}$, and the reader is free to add an assumption to the poset or to consider this as a harmless abuse of notation.

Another common assumption is that $\mathbb{P}$ has a maximal element 1, as it will make some notation easier later on.

2.3 The Forcing Relation

Let $\mathbb{P}$ be a poset and $\mathbb{B}$ its Boolean completion. We will define a forcing relation

$$p \Vdash \varphi(\overline{x}) \quad \text{iff} \quad p \leq [\![\, \varphi(\overline{u}) \,]\!]$$

where $p \in \mathbb{P}$ and $\varphi(x_1, ..., x_n)$ is a formula over $\Sigma_\in$ and $a_1, ..., a_n \in V^{\mathbb{B}}$. An immediate consequence of this definition is monotonicity:

Lemma 2.8 *For any formula* φ*,*

$$p \leq q \text{ and } q \Vdash \varphi \text{ implies } p \Vdash \varphi.$$

It also inherits a form of soundness from $[\![\, - \,]\!]$.

Lemma 2.9 *For any formula* φ*,*

$$\varphi \vdash \psi \text{ and } p \Vdash \varphi \text{ implies } p \Vdash \psi.$$

We collect the following recursive clauses for the forcing relation

[3] called *separated* in [Jec03].

Theorem 2.10 *Let φ, ψ be formulas over $\Sigma_\in$ and $p \in \mathbb{P}$. Then*

(i)	$p \Vdash \top$,		
(ii)	$p \Vdash \varphi(\overline{u}) \wedge \psi(\overline{u})$	iff	$p \Vdash \varphi(\overline{u})$ and $p \Vdash \psi(\overline{u})$,
(iii)	$p \Vdash \bot$	iff	$p = 0$,
(iv)	$p \Vdash \varphi(\overline{u}) \vee \psi(\overline{u})$	iff	for any $q \leq p$ there is an $r \leq q$ s.t. $r \Vdash \varphi(\overline{u})$ or $r \Vdash \psi(\overline{u})$,
(v)	$p \Vdash \varphi(\overline{u}) \Rightarrow \psi(\overline{u})$	iff	for any $q \leq p, q \Vdash \varphi(\overline{u})$ implies $q \Vdash \psi(\overline{u})$,
(vi)	$p \Vdash \forall x.\varphi(\overline{u}, x)$	iff	for any $v \in V^{\mathbb{B}}, p \Vdash \varphi(\overline{u}, v)$,
(vii)	$p \Vdash \exists x.\varphi(\overline{u}, x)$	iff	for any $q \leq p$ there is an $r \leq q$ and a $v \in V^{\mathbb{B}}$ s.t. $r \Vdash \varphi(\overline{u}, v)$.

Proof. See [Bel05, Thm 2.5]. □

Lemma 2.11 *Let φ be a formula, $p \in \mathbb{P}$ and suppose $S \subseteq \mathbb{P}$ is dense below p. Then*

$$\forall q \in S.(q \Vdash \varphi) \text{ implies that } p \Vdash \varphi.$$

Proof (idea). This follows from the fact that $p \Vdash \varphi$ if and only if $\{q \mid q \Vdash p\}$ is dense below p. □

Assuming that $1 \in \mathbb{P}$, we can relate the $\vDash$ and $\Vdash$ relation with

$$V^{\mathbb{B}} \vDash \varphi \text{ if and only if } 1 \Vdash \varphi.$$

One can relate the two without assuming this, but the assumption is commonly used in forcing, and greatly simplifies notation.

This forcing relation will play a core part in showing independence results with $V^{\mathbb{B}}$.

2.4 Independence Results in $V^{\mathbb{B}}$

The following is a very brief outline of how independence results can be obtained via forcing in the Boolean-valued universe $V^{\mathbb{B}}$. We roughly describe the approach in [Bel05, §2]. The specific steps vary with the choice of poset $\mathbb{P}$ and the desired goal – forcing is not an automatic procedure. A common procedure however is to embed each "ground-model" set into $V^{\mathbb{B}}$ via its canonical name. For any set x,

$$\check{x} = \{\langle \check{y}, 1\rangle \mid y \in x\} \in V^{\mathbb{B}}$$

is called the canonical $\mathbb{P}$-name of x. The collection of canonical $\mathbb{P}$-names inside $V^{\mathbb{B}}$ is (modulo the Boolean-valued equality) isomorphic to the original model V, and one can show that V and $\mathbb{B}$ satisfy the same first-order sentences about sets in V.

Example 2.3.

(i) $(\emptyset) = \emptyset$ the empty map into B
(ii) $\check{1} = \{\langle\emptyset, 1\rangle\}$,
(iii) $\check{2} = \left\{\left\langle\check{0}, 1\right\rangle, \left\langle\check{1}, 1\right\rangle\right\}$.

Canonical names tie $V^{\mathbb{B}}$ back to the ground model V. For instance, one shows that every ordinal in $V^{\mathbb{B}}$ has the form $\check{\alpha}$ for some ordinal α in V, and forcing preserves the ordinals.

The poset $\mathbb{P}$ and the Boolean algebra $\mathbb{B}$ themselves belong to $V^{\mathbb{B}}$, and one can speak of a generic filter over $\check{\mathbb{P}}$, even though no such filter exists in the ground model.[4] This internally defined

[4]provided the forcing is nontrivial.

generic filter often plays a crucial role in the forcing argument. We illustrate its importance with a concrete example.

Example 2.4. In Cohen forcing, one takes $\mathbb{P}$ to be the set of all finite partial functions $p : \omega \rightharpoonup \{0, 1\}$, ordered by reverse inclusion. A filter $G \subseteq \mathbb{P}$ is generic when $\bigcup G$ is a total function $g : \omega \to \{0, 1\}$. So, vaguely speaking, the generic filter corresponds to a new object (here a function) in the extension that was approximated by elements in the poset. The full forcing proof is of course much more extensive - we meant only to highlight the importance of G.

3 Grothendieck Toposes

In this chapter we will introduce the notion of a site and define sheaves on a site, leading us to Grothendieck toposes. Sites and toposes induced by locales will be an accompanying example. We will then investigate some of the categorical properties of a Grothendieck topos.

3.1 Sites

The reader may already be familiar with the concept of presheaves and sheaves over a topological space. While presheaves over a category $\mathcal{C}$ are simply the contravariant functors to **Set**, defining sheaves on an arbitrary category $\mathcal{C}$ requires a notion of a *covering*, which should generalize the notion of covering on a topological space (or locale).

The language of sieves will be useful in the following definition. Recall that for any map $g : U \to V$ and sieve S on V, we defined the preimage sieve $g^*(S) = \{h \mid g \circ h \in S\}$, and that t_U denotes the maximal sieve on U.

Definition 3.1 (based on [MM94, III.2 Def 1]) A *Grothendieck topology* on a category $\mathcal{C}$ is an assignment of a set $\mathrm{Cov}(U)$ of *covering sieves* to each object U of $\mathcal{C}$, such that the following conditions are satisfied:

(i) $t_U \in \mathrm{Cov}(U)$.

(ii) *Stability axiom*: For any $g : U \to V$, if $S \in \mathrm{Cov}(V)$, then $g^*(S) \in \mathrm{Cov}(U)$,

(iii) *Transitivity axiom*: If $S \in \mathrm{Cov}(V)$ and R is a sieve on V s.t. $h^*(R) \in \mathrm{Cov}(V)$ for all $h : D \to C$ in S, then $R \in \mathrm{Cov}(C)$.

We call the pair $(\mathcal{C}, \mathrm{Cov})$ a *site*, and may say that a sieve *covers* U when $S \in \mathrm{Cov}(U)$.

Example 3.1. Any locale X induces a site (X, Cov) as follows: The category is simply the locale itself, with $u \in \mathcal{O}(X)$ as its objects and $\mathrm{Hom}(u, v) = \{* \mid u \leq v\}$. As Grothendieck topology, we can use

$$\mathrm{Cov}(u) = \Big\{M \mid \bigvee\{v \in \mathcal{O}(X) \mid (* : v \to u) \in M\} = u\Big\}.$$

Notice that to simplify notation, we can identify any morphism in $\mathcal{C}$ with its domain and codomain, and in doing so identify any morphism in a covering of u with its domain. We can then write

$$\mathrm{Cov}(u) = \Big\{M \mid \bigvee M = u\Big\},$$

and get a form more reminiscent of a covering in a topological space. The coverage conditions can then be rephrased and checked:

(i) $L_{\leq u}$ covers u

(ii) If S covers v and $u \leq v$, then $S_{\leq u}$ covers u

(iii) If S covers v and R a sieve on v such that $R_{\leq u}$ covers u for all $u \in S$, then R covers v.

Example 3.2 (The trivial topology). We can make any category $\mathcal{C}$ into a site with the *trivial topology* by setting $\mathrm{Cov}(U) = \{t_U\}$ for all objects U.

3.2 Sheaves on a Site

Definition 3.2 A *sheaf* on a site $\mathcal{C}$ is a functor $\mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ satisfying the following condition.

For any $U \in \mathrm{Ob}(\mathcal{C}), S \in \mathrm{Cov}(U)$, if the family $\{s_f \in F(U') \mid (f : U' \to U) \in S\}$ is compatible, then there exists a unique $s \in F(U)$ such that

$$s \cdot f = s_f$$

for every $f \in S$.

For every $(f : U' \to U) \in M$, let $s_f \in F(U')$. The set $(s_f)_{f \in M}$ is *compatible* if for any $f, g \in S$ and any $a : V \to \mathrm{dom}\, f, b : V \to \mathrm{dom}\, g$,

$$f \circ a = g \circ b \text{ implies } s_f \cdot a = s_g \cdot b.$$

Definition 3.3 The category of sheaves $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ on a site $\mathcal{C}$ is the full subcategory of presheaves $\hat{\mathcal{C}}$ with sheaves as objects.

In particular, a morphism of sheaves is simply a morphism of functors, i.e. a natural transformation.

When the choice of coverage will not change within a discussion, we may omit it from the notation and write $\mathrm{Sh}(\mathcal{C})$.

Example 3.3. The category of sheaves on the site induced by a locale X will be denoted simply as $\mathrm{Sh}(X)$. Reading the sheaf condition on this site will match the definition familiar from topological spaces.

Example 3.4 (Presheaves are sheaves). Given any category $\mathcal{C}$, the category of presheaves $\hat{\mathcal{C}}$ can in fact be thought of as a category of sheaves $\mathrm{Sh}(\mathcal{C})$ by choosing Cov to be the trivial topology. We justify this claim by proving the equivalence

$$\hat{\mathcal{C}} \cong \mathrm{Sh}(\mathcal{C}).$$

Since we have defined $\mathrm{Sh}(\mathcal{C})$ to be the full subcategory of $\hat{\mathcal{C}}$ with sheaves as objects, all we need to show is that every presheaf F in $\hat{\mathcal{C}}$ satisfies the sheaf condition under the trivial topology.

Let $U \in \mathcal{C}$ be arbitrary, $M = \{f \in \mathrm{Hom}_{\mathcal{C}(U',U)} \mid U' \in \mathcal{C}\}$ and let $\{s_f \in F(U')\}$ be a compatible family. Since $\mathrm{id}_U \in M$, this will always give a unique choice for amalgamation

$$s_{\mathrm{id}_U} \cdot \mathrm{id}_U = F(\mathrm{id}_U)(s_{\mathrm{id}_U}) = \mathrm{id}_{F(U)}(s_{\mathrm{id}_U}) = s_{\mathrm{id}_U}$$

and this choice satisfies the equality for any element other element of the family s_f:

$$s_{\mathrm{id}_U} \cdot f = F(f)(s_{\mathrm{id}_U}) = F(\mathrm{id}_{U'})(s_f) = s_f$$

by compatibility of the family and $\mathrm{id}_{U'} \circ f = f \circ \mathrm{id}_U$.

3.3 Categorical Properties of Grothendieck Toposes

In this section we will outline some of the properties of the category of sheaves and come to the conclusion that it is a topos.

The category of sheaves will inherit many of the categorical properties through its relation to the category of presheaves. There is an adjoint functor pair $a \dashv i$. We will not give an explicit construction of a as it will not be of further consequence and can be found in any introduction to sheaves, refer for instance to [MM94, III.5]. The usefulness of the associated sheaf functor for our purposes remains in it being a right adjoint to inclusion.

3.3.1 Limits and Colimits

Recall that the category of presheaves $\hat{\mathcal{C}}$ has all limits and colimits. One can use the adjoint functor theorem (AFT) to argue that $\mathrm{Sh}(\mathcal{C})$ does too, and that limits in these two categories coincide.

Theorem 3.4 *For a site $\mathcal{C}$ and a diagram $J \to \hat{\mathcal{C}}$ of presheaves P_j, if all P_j are sheaves then so is $\lim P_i$.*

In the category $\hat{\mathcal{C}}$, sheaves are closed under limits ([MM94] III.4 Prop 4). So given a diagram (F_i) of sheaves, its limit is preserved by the AFT because i is a right adjoint.

$$\lim^{\hat{\mathcal{C}}} F_i = i\big(\lim^{\mathrm{Sh}(\mathcal{C})} F_i\big) = \lim^{\mathrm{Sh}(C)} F_i$$

This proves not only that $\mathrm{Sh}(\mathcal{C})$ has all limits, but also that limits are defined in the same way as in $\hat{\mathcal{C}}$: point-wise. It follows that the terminal presheaf is a sheaf, and that pullbacks in $\mathrm{Sh}(\mathcal{C})$ are the pullbacks in $\hat{\mathcal{C}}$.

For colimits on the other hand, we can apply the AFT to the left adjoint a and get that for a diagram of sheaves F_i,

$$a\Big(\operatorname{colim}^{\hat{\mathcal{C}}} F_i\Big) = \operatorname{colim}^{\mathrm{Sh}(\mathcal{C})} a(F_i) = \operatorname{colim}^{\mathrm{Sh}(\mathcal{C})} F_i.$$

Theorem 3.5 *Let $\mathcal{C}$ be a category with small colimits and limits, and $\mathcal{D}$ be subcategory with a functor $F : \mathcal{C} \to \mathcal{D}$. Suppose that*

(i) $\mathcal{D}$ is closed under limits in $\mathcal{C}$,

(ii) F is left-adjoint to the inclusion finctor,

$$\mathcal{D} \underset{F}{\overset{i}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathcal{C}$$

(iii) and F restricts to the identity on $\mathcal{D}$.

Then $\mathcal{D}$ has all colimits and limits, which are explicitely given for any diagram X_i in $\mathcal{D}$ by

$$\lim^{\mathcal{D}} X_i = \lim^{\mathcal{C}} X_i,$$
$$\operatorname{colim}^{\mathcal{D}} X_i = F\big(\operatorname{colim}^{\mathcal{C}} X_i\big).$$

In particular this states that the colimit of any small diagram of sheaves exists, and that it is the sheafification of its colimit in the category of presheaves.

It follows that the initial sheaf is the sheafification of the initial presheaf. Since we haven't introduced the associated sheaf functor explicitely, we will give an explicit description of this sheaf.

That the empty presheaf is not a sheaf follows from the following fact about sheaves:

Lemma 3.6 *For any sheaf F over $\mathcal{C}$ and any $U \in \mathcal{C}$, if $\emptyset \in \mathrm{Cov}(U)$, then $F(U) \cong 1$.*

Proof. Given an empty family, the sheaf condition requires that there be a unique $s \in F(U)$ such that $s \cdot f = s_f$ for every $f \in \emptyset$. This condition is vacously true, so $F(U)$ must contain a single unique element, i.e. be the singleton set. □

From this it is easy to see that in the case of a site with an object covered by the empty set, the empty presheaf is not a sheaf. However this is easily fixed.

Lemma 3.7 *The functor*

$$0_{\mathrm{Sh}} : \mathcal{C}^{op} \to \mathbf{Set}$$
$$U \mapsto \begin{cases} 1 \text{ if } \emptyset \in \mathrm{Cov}(U) \\ 0 \text{ o/w} \end{cases}$$

is the initial object in the category $\mathrm{Sh}(\mathcal{C})$

Proof. It is straightforward to check that this is a sheaf. Let F be another sheaf over $\mathcal{C}$. Then by Lemma 3.6, $O_{\mathrm{Sh}(U)} \subseteq F(U)$ for any $U \in \mathcal{C}$. This makes 0_{Sh} a subsheaf[5] of F, with a unique monic $0_{\mathrm{Sh}} \rightarrowtail F$. In fact a morphism $f : 0_{\mathrm{Sh}} \to F$ is componentwise uniquely determined by $f(U) : 1 \to 1$ in case $\emptyset \in \mathrm{Cov}(U)$ and by the unique injection $f(U) : 0 \rightarrowtail F(U)$ otherwise. □

Exponentials do not immediately fit the pattern of limits or colimits, but can still be connected to the exponentials of presheaves. One can prove that if exponentials exist in $\mathrm{Sh}(\mathcal{C})$, they must correspond with the exponentials in $\hat{\mathcal{C}}$. Further, sheaves in $\hat{\mathcal{C}}$ are closed under exponentials. This proves that $\mathrm{Sh}(\mathcal{C})$ has all exponentials and gives their explicit description. All this is done in [MM94, III.6].

3.3.2 The Subobject Classifier of Sheaves

A *subsheaf* of a sheaf F should of course simply be a subobject in $\mathrm{Sh}(\mathcal{C})$. There is however a more hands-on characterisation: Every subsheaf of F can be uniquely represented by a subfunctor $G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ such that for each object $U \in \mathcal{C}$ with $M \in \mathrm{Cov}(U)$ and for each $s \in F(U)$,

$$(F(f)(s) \in G(U') \text{ for every } (f : U' \to U) \in M) \Rightarrow s \in G(U).$$

The subobject classifier of presheaves on $\mathcal{C}$ as given in (3.1) cannot serve as subobject classifier of the corresponding Grothendieck topos, but the set of sieves can be restricted by the following definition.

Definition 3.8 For an object C of category $\mathcal{C}$, a sieve S on C is *closed* if for all $f : D \to C$,

$$f^*(S) \text{ covers } D \Rightarrow f \in S.$$

For instance, the maximal sieve t_X on any X is closed.

The functor

$$\begin{aligned} \Omega : \mathcal{C}^{\mathrm{op}} &\to \mathbf{Set} \\ C &\mapsto \{S \mid S \text{ is a closed sieve on } C\} \\ f &\mapsto f^* \end{aligned} \tag{3.3}$$

is a sheaf, and forms the subobject classifier in the category of sheaves over $\mathcal{C}$ along with the map

[5] Refer to the next subsection

$$\text{true} : 1 \to \Omega,$$
$$\text{true}(C)(*) = t_C$$

For a motivated introduction of this construction, see [MM94, III.7], or perhaps even more helpfully, [MM94, II.8] for a first step over topological spaces.

We conclude this section on the categorical properties of sheaves by summarizing them in terms of an existing concept.

Corollary 3.9 *For any site $\mathcal{C}$, the category* $\mathrm{Sh}(\mathcal{C})$ *is a topos.*

Proof. As argued, $\mathrm{Sh}(\mathcal{C})$ has all limits and exponentials, so is cartesian closed. Furthermore it has a subobject classifier. □

Since Grothendieck toposes are an important class of toposes, we have motivated the following definition.

Definition 3.10 A *Grothendieck topos* is a category equivalent to $\mathrm{Sh}(\mathcal{C})$ for a site $\mathcal{C}$. Given a site $\mathcal{C}$, $\mathrm{Sh}(\mathcal{C})$ is called the *Grothendieck topos over* $\mathcal{C}$.

3.4 The Subobject Lattice of a Sheaf

We already know that for each object X of a Grothendieck topos, the set of subobjects $\mathrm{Sub}(X)$ forms a Heyting algebra. For Grothendieck toposes a slightly stronger statement holds.

Proposition 3.11 *For any Grothendieck topos over a site $\mathcal{C}$ and any sheaf $F \in \mathrm{Sh}(\mathcal{C})$, the set* $\mathrm{Sub}(F)$ *forms a* complete *Heyting algebra.*

One proves this statement by giving constructions for each operation required, showing that they are sheaves and finally checking distributivity. A motivated walkthrough of this can be found in [MM94, III.8 Prop 8]. We list each operation, along with explicit descriptions of f^* along with its left and right adjoints of the preimage functor, which exist by Lemma 1.19.

Let S, T be subobjects of a sheaf F and $f : F \to G$ be a morphism of sheaves. Then

(i) $$\top = F,$$

(ii) $$\left(\bigwedge_I S_i\right)(U) = \bigcap_I S_i(U),$$

(iii) $$\bot = 0_{\mathrm{Sh}(\mathcal{C})},$$

(iv) $$x \in \left(\bigvee_I S_i\right)(U) \text{ iff } \{f : V \to U \mid x \cdot f \in S_i(D) \text{ for some } i\} \in \mathrm{Cov}(C),$$

(v) $$x \in (S \Rightarrow T)(U) \text{ iff } \forall (f : V \to U).[x \cdot f \in S(V) \Rightarrow x \cdot f \in T(V)],$$

(vi) $$x \in f^*(S)(U) \text{ iff } f(S)(x) \in S(U),$$

(vii) $$y \in \exists_g(S)(U) \text{ iff } \{f : V \to U \mid \exists x \in S(V).f(x) = y \cdot f\} \in \mathrm{Cov}(C),$$

(viii) $$y \in \forall_g(S)(U) \text{ iff for all } f : V \to U, g^*(V)(y \cdot f) \subseteq S(V).$$

This concludes our discussion of Grothendieck toposes. In the following section, we will encounter our first non-trivial example of such a topos – one that also introduces our formal connection to forcing.

4 The Double Negation Topology on a Poset

Given a poset $\mathbb{P}$, we now define a particular Grothendieck topos that will play a central role in topos-theoretic forcing. We will see how this topos can be used to derive forcing results in a later chapter. For now, we observe several similarities to the construction of the Boolean-valued model $V^{\mathbb{B}}$ from a poset $\mathbb{P}$ in Chapter 2.

The most manageable construction – and the one we will be using in all subsequent discussion – is to take sheaves over a dense topology on $\mathbb{P}$, yielding the topos $\mathrm{Sh}(\mathbb{P}, \neg\neg)$. Alternatively, defining the same topos via a locale induced by $\mathbb{P}$ mirrors the connection between $\mathbb{P}$ and the Boolean algebra underlying $V^{\mathbb{B}}$. This route also justifies calling $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ the "double negation topos". Finally we will prove that $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ is the classifying topos of the theory of generic filters over $\mathbb{P}$, hinting that generic filters play a similar role in this topos as they do in forcing extensions.

Aside from shedding light on its role in forcing, considering different descriptions of this particular topos exemplifies the fact that toposes in general can be approached in many ways, and alludes to some of these approaches – toposes as categories, as spaces and as theories.

For the rest of this chapter, let $\mathbb{P}$ be a poset. We may use category theoretic constructions and definitions on it by considering it a small, thin category.

4.1 The Dense Topology on $\mathbb{P}$

The dense topology can be defined on arbitrary categories[6], but here we will present the dense topology on a poset, which is also the motivating origin of this topology.

Recall that sieves on posets are exactly the down-closed subsets, and that a down-closed subset D is dense below $p \in \mathbb{P}$ if for every $q \leq p$ there exists some $d \in D$ such that $d \leq q$.

Definition 4.1 The dense sieves on $\mathbb{P}$ form the *dense topology on* $\mathbb{P}$ through the coverage

$$\mathrm{Cov}(p) = \{D \subseteq \mathbb{P} \mid D \text{ is a sieve on } p \text{ and dense below } p\}.$$

It is routine to check that this is in fact a Grothendieck topology.

We will call the Grothendieck topos of sheaves over this site $\mathrm{Sh}(\mathbb{P}, \neg\neg)$. A justification for this name will follow.

For future use we take a closer look at the subobject classifier, which following Equation (4.3), for any $p \in \mathbb{P}$ is given by

$$\Omega(p) = \{S \mid S \text{ is closed sieve on } p\}.$$

A sieve on p is just a down-closed set below p, and is closed if for all $q \leq p$,

$$S_{\leq q} \text{ is dense below } q \Rightarrow q \in S.$$

But this is exactly the definition of $\neg\neg$closed sieve. So

$$\Omega(p) = \{S \mid S \text{ is } \neg\neg\text{closed sieve on } p\}.$$

[6]see [MM94, III.2 e] for this more general definition.

For $p \leq q$, the map $\Omega(p \leq q) : \Omega(q) \to \Omega(p)$ is the restriction of the down-closed set to p, which is equivalent to meeting the set with p. We will therefore write this map as $(-) \wedge p$. The map $\mathrm{true} : 1 \to \Omega$ is given by $\mathrm{true}(p)(0) = t_p = p$.

We get the following pointwise characterisation of subobjects: For a subsheaf $S \subseteq X \in \mathrm{Sh}(\mathbb{P}), p \in \mathbb{P}$ and $x \in X(p)$,

$$x \in S(p) \Leftrightarrow \Phi_S(p)(x) = p.$$

4.2 $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ as a Localic Topos

It is possible to arrive to the same topos by first defining a locale over $\mathbb{P}$. The site induced by this locale will not be the same as the dense topology, but the category of sheaves over this locale and the dense topology respectively will be equivalent.

We proceed exactly as in Section 2.2, defining the Heyting Algebra $\mathrm{Dn}(\mathbb{P})$ of down-closed sets, and then applying the double negation translation to obtain the Boolean Algebra $\mathrm{Dn}_{\neg\neg}(\mathbb{P})$. Every Boolean Algebra is of course also a locale. The precise nature of the embedding of $\mathbb{P}$ into $\mathrm{Dn}_{\neg\neg}(\mathbb{P})$ is relevant in order to be able to force the wanted object, but is of no further significance in the construction of the forcing extension.

We will denote the topos of sheaves over (the induced site over) this locale simply as $\mathrm{Sh}(\mathrm{Dn}_{\neg\neg}(\mathbb{P}))$.

Theorem 4.2 *For any poset $\mathbb{P}$ there is a category equivalence*

$$\mathrm{Sh}(\mathbb{P}, \neg\neg) \simeq \mathrm{Sh}(\mathrm{Dn}_{\neg\neg}(\mathbb{P})).$$

Proof. The idea is to first make a connection on the level of sites.

$$D : \mathbb{P} \to \mathrm{Dn}_{\neg\neg}(\mathbb{P})$$
$$p \mapsto \neg\neg\mathcal{O}_p.$$

Now we define a functor

$$\Phi : \mathrm{Sh}(\mathrm{Dn}_{\neg\neg}(\mathbb{P})) \to \mathrm{Sh}(\mathbb{P}, \neg\neg)$$

by setting for any sheaf $F : \mathrm{Dn}_{\neg\neg}(\mathbb{P})^{\mathrm{op}} \to \mathbf{Set}$,

$$\Phi(F) := F \circ D.$$

and for any morphism of sheaves, $\tau : F \to G$, and for any $p \in \mathbb{P}$,

$$\Phi(\tau)(p) = \tau(D(p)).$$

Let $F \in \mathrm{Sh}(\mathrm{Dn}_{\neg\neg}(\mathbb{P}))$, $p \in \mathbb{P}$ and $x \in \Phi(F)(p)$. Then

$$\Phi(\mathrm{id}_F)(p)(x) = (F \circ D)(\mathrm{id}_F)(x) = \mathrm{id}_F(D(p))(x) = x,$$

so $\Phi(\mathrm{id}_F) = \mathrm{id}_{\Phi(F)}$. For composable morphisms of sheaves τ, σ and any $p \in \mathbb{P}$,

$$(\Phi(\tau)(p) \circ \Phi(\sigma)(p)) = \tau(D(p)) \circ \sigma(D(p)) = (\tau \circ \sigma)(D(p)) = \Phi(\tau \circ \sigma)(p).$$

We have proven that Φ is a functor, without so far making use of the definition of D, but that is about to change.

It would now be possible to start proving that Φ is an equivalence of categories by hand, but we choose to take this opportunity to connect to a general theorem in topos theory, known as the *comparison lemma.*

Lemma 4.3 (Comparison Lemma) *[MM94, Appendix 3] Let $(\mathcal{C}, J)$ and $(\mathcal{A}, J')$ be sites such that:*

(i) All representable presheaves on $\mathcal{C}$ are sheaves.
(ii) There is a full and faithful functor $U : \mathcal{A} \to \mathcal{C}$.
(iii) Every object of $\mathcal{C}$ has a cover of objects from $\mathcal{A}$.
(iv) For any object $X \in \mathrm{Obj}(\mathcal{A})$ and sieve S on X,

$$S \in J'(X) \ \text{ iff } \ U(S) \in J(U(X)).$$

In this case the restriction functor $\hat{\mathcal{C}} \to \hat{\mathcal{A}}$ induces an equivalence of categories

$$\mathrm{Sh}(\mathcal{C}, J) \simeq \mathrm{Sh}(\mathcal{A}, J').$$

Applying this lemma to our particular case tells us that, in order to show that Φ is an equivalence of categories, it suffices to show that

(i) for any $\neg\neg$closed X, $\mathrm{Hom}_{\mathrm{Dn}_{\neg\neg}(\mathbb{P})}(-, D)$ is a sheaf,
(ii) the functor $D : \mathbb{P} \to \mathrm{Dn}_{\neg\neg}(\mathbb{P})$ is full and faithful,
(iii) every $\neg\neg$closed X has a cover of objects from $D(\mathbb{P})$,
(iv) for every $p \in \mathbb{P}$ and sieve S on p,

$$S \text{ is dense below } p \ \text{ iff } \ \{\mathcal{O}_q \mid q \in S\} \in \mathrm{Cov}(\mathcal{O}_p).$$

Property (i) will require the poset to be assumed as refined, just as we saw in the analogous setting Section 2.2. The interested reader may refer to [MM94, Appendix 4 (c)]. □

Once a formal connection between $\mathrm{Sh}(\mathbb{P})$ and the Boolean-valued universe has been made, one could continue exploring how this equivalence relates to the relation between a poset and its associated Boolean algebra, and the role that plays in showing that Boolean-valued and set-valued models over the same poset produce the same independence results.

4.3 $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ as a Classifying Topos

We claim that $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ is the classifying topos of the theory of generic filters. On this basis, one could attempt to relate the role of generic filters in the topos-theoretic approach to forcing to their role in other versions of forcing.

While not a central topic, we give an idea of the implications viewing $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ as a classifying topos has. A classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos, denoted $\mathbf{Set}[\mathbb{T}]$, which has the property that any model of $\mathbb{T}$ in a Grothendieck topos $\mathcal{E}$ corresponds to a geometric morphism $\mathcal{E} \to \mathbf{Set}[\mathbb{T}]$. One such morphism is associated with what is known as the universal model of $\mathbb{T}$, and – as reflected in the notation – one can think of $\mathbf{Set}[\mathbb{T}]$ as the category of sets with this universal model of $\mathbb{T}$ "adjoined" to it. This is particularly suggestive in connection with forcing, where the generic filter plays, more or less directly, the role of the forced object, as we discussed in Section 2.4. All of this could be made precise once the connection between $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ and the Boolean-valued model $V^{\mathbb{B}}$ is established.

The construction depends on the type of theory to be classified. Typically, one defines a syntactic category associated with the theory, which is then equipped with a coverage. The category of sheaves over this syntactic site yields the classifying topos. In [Joh02, D3], a comprehensive background on classifying toposes is given. We will focus on how, given a geometric theory, one can construct its classifying topos via a "syntactic site".

After setting up the theory of generic filters we construct the classifying topos and finally show that it is equivalent to $\mathrm{Sh}(\mathbb{P}, \neg\neg)$.

4.3.1 The Theory of Generic Filters on $\mathbb{P}$

Given any structure, there will be different ways to formalize a first-order theory defining it. A priori we may not be sure how to formalize generic filters as a first-order theory, what signature and what sorts to choose, and which fragment of first-order or infinitary logic to use. When describing subsets of a given structure, such as generic filters of a given poset $\mathbb{P}$, *propositional geometric theories* are a typical choice for axiomatisation. This matches our goal – we have seen that $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ is localic, and it is a known result that localic toposes exactly classify propositional geometric theories (see [Joh02, D 3.1.14]).

A propositional signature is defined as a first-order signature which has no sorts (and hence no function symbols). Geometric formulas over this theory are those recursively built up from equality and atomic relation symbols and closed under truth, binary conjunction, falsity, infinitary disjunction and existential quantification. In the propositional case, existential quantification cannot be applied, leaving the connectives

$$\top, \wedge, \bot, \bigvee .$$

The rules of inference attached to these connectives are collected in [Joh02, D 1.3]. The axioms of a propositional geometric theory are sequents of the form $\varphi \vdash \psi$ where φ, ψ are geometric formulas.

Definition 4.4 The theory $\mathbb{T}$ of generic filters on $\mathbb{P}$ has as signature a collection of nullary relations G_p for every $p \in \mathbb{P}$ (which we can think of as saying $p \in G$), as well as the axioms

(i) $\top \vdash G_1$,

(ii) $G_p \vdash G_q$ for every pair $p \leq q$,

(iii) $G_p \wedge G_q \vdash \bigvee_{r \leq p, r \leq q} G_r$ for every pair p, q,

(iv) $\top \vdash \bigvee_{p \in D} G_p$ for every dense subset $D \subseteq \mathbb{P}$.

The following lemma will be useful later, but also provides a useful exercise in working with geometric theories.

Lemma 4.5 *For any $p \in \mathbb{P}$ and X be dense below p,*

$$G_p \vdash \bigvee_{q \in X} G_q.$$

Proof. Let $p \in \mathbb{P}$ and X dense below p. Set

$$D := \mathbb{P} \setminus \underbrace{\{q \in \mathbb{P} \mid \exists x \in X. x \leq q\}}_{=: X^+} \cup X.$$

An arbitrary $q \in \mathbb{P}$ is either in D itself, or it is in X^+. In the first case it has itself as refinement, and in the second case it has a refinement in $X \subset D$ by definition of X^+. We can conclude that D is dense. By axiom (iv) with D and conjunction, we have

$$\begin{aligned} G_p &\vdash \left(\bigvee_{d \in D} G_d \right) \wedge G_p \\ &\vdash \bigvee_{d \in D} (G_d \wedge G_p) \end{aligned}$$

after applying the distributive law. Now we can split the disjunction into two parts

$$G_p \vdash \left(\bigvee_{d \in \mathbb{P} \setminus X^+} G_d \wedge G_p \right) \vee \left(\bigvee_{x \in X} G_x \wedge G_p \right).$$

We will show that both parts imply $\bigvee_{x \in X} G_x$, and thus $G_p \vdash \bigvee_{x \in X} G_x$ as we wanted.

On the right side we can simply apply conjunction and disjunction elimination rules to get

$$\bigvee_{x \in X} (G_x \wedge G_p) \vdash \bigvee_{x \in X} G_x.$$

For the left side, our plan is to conclude $\bot$ and then use ex falso quod libet. Let $d \in \mathbb{P} \setminus X^+$ be arbitrary. By axiom (ii),

$$G_p \wedge G_d \vdash \bigvee_{r \leq p, r \leq d} G_r.$$

We will now argue that $\{r \in \mathbb{P} \mid r \leq d \wedge r \leq p\} = \emptyset$. Suppose $r \leq p$ and $r \leq d$. Because X is dense below p, there exists a $r' \in X$ such that $r' \leq r$. But then $r' < d$, contradicting $d \notin X^+$.

In conclusion we have

$$G_p \wedge G_d \vdash \bigvee \emptyset \vdash \bot,$$

and thus by ex falso quod libet,

$$\bigvee_{d \in \mathbb{P} \setminus X^+} G_d \wedge G_p \vdash \bot \vdash \bigvee_{x \in X} G_x.$$

□

A propositional geometric theory $\mathbb{T}$ has as syntactic site a locale, called its classifying locale $L_{\mathbb{T}}$.[7] The classifying topos will be $\mathrm{Sh}(L_{\mathbb{T}})$ (See [Joh02] 3.1.14.). This classifying locale follows a construction we encountered in passing in Section 1.2 for intuitionistic and classical logic.

Theorem 4.6 *Given a propositional geometric theory* $\mathbb{T}$*, the preorder*

$$\{\varphi \mid \varphi \text{ is formula over } \mathbb{T}\} \text{ with } \varphi \leq \psi :\Leftrightarrow \varphi \vdash \psi$$

induces a frame called the Lindenbaum Algebra $L_{\mathbb{T}}$.

Proof. Every preorder induces a partial order when dividing out the equivalence $\varphi \sim \psi :\Leftrightarrow \varphi \leq \psi \wedge \psi \leq \varphi$. It is straightforward to show that

[7]More than this: Every locale classifies a propositional geometric theory, so the two structures are closely connected, as one could already guess from the similarity of their definitions and common notation.

$$\{\varphi \mid \varphi \text{ is formula over } \mathbb{T}\}/ \dashv\vdash$$

is a frame. □

Let $L_{\mathbb{T}}$ be the locale given by the Lindenbaum algebra of the theory of generic filters $\mathbb{T}_G$. We have:

$$\mathbf{Set}[\mathbb{T}_G] = \mathrm{Sh}(L_{\mathbb{T}}),$$

and our goal is to prove the equivalence

$$\mathrm{Sh}(L_{\mathbb{T}}) \cong \mathrm{Sh}(\mathbb{P}, \neg\neg).$$

Since we have already proven $\mathrm{Sh}(\mathrm{Dn}_{\neg\neg}(\mathbb{P})) \cong \mathrm{Sh}(\mathbb{P}, \neg\neg)$, and $\mathrm{Dn}_{\neg\neg}(\mathbb{P})$ is a locale, we can prove the equivalence on the level of locales instead of sheaves.

Theorem 4.7 *The Lindenbaum Algebra $L_{\mathbb{T}}$ of generic filters on $\mathbb{T}$ is isomorphic to $\mathrm{Dn}_{\neg\neg}(\mathbb{P})$.*

Corollary 4.8 *The double negation topos $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ is the classifying topos of the theory of generic filters on $\mathbb{P}$.*

The proof of this theorem will rely on the following lemma:

Lemma 4.9 *For any $p \in \mathbb{P}$,*

$$\left\{q \in \mathbb{P} \mid G_q \vdash G_p\right\} = \neg\neg\mathcal{O}_p.$$

Proof (of Lemma 4.9). It suffices to show that $\left\{q \in \mathbb{P} \mid G_q \vdash G_p\right\}$ is $\neg\neg$closed. Supposing $\left\{q \in \mathbb{P} \mid G_q \vdash G_p\right\}$ is dense below d, we want to show that $G_d \vdash G_p$. By Lemma 4.5,

$$G_d \vdash \bigvee_{\{q\in\mathbb{P} \mid G_q \vdash G_p\}} G_q$$

and this in turn by the disjunction rule entails G_p. □

Proof (of Theorem 4.7).

We give morphisms of locales in both directions that will compose to the identity.

In one direction take the map

$$\begin{aligned} f : \mathrm{Dn}_{\neg\neg}(\mathbb{P}) &\to L(\mathbb{T}_G) \\ X &\mapsto \bigvee_{p\in X} [G_p] \end{aligned}$$

We check that this is a frame homomorphism. It naturally preserves small joins by definition of the map. To see that it preserves finite meets reduces to showing that in $\mathbb{T}_G$ for $X, Y \in \mathrm{Dn}_{\neg\neg}(\mathbb{P})$,

$$\bigvee_{X\cap Y} G_r \dashv\vdash \bigvee_X G_x \wedge \bigvee_Y G_y.$$

The implication from left to right is obvious by logical rules of inference. From right to left we get, using distributivity and the common refinement axiom,

$$\begin{aligned}\bigvee_X G_x \wedge \bigvee_Y G_y &\vdash \bigvee_Y G_y \wedge \bigvee_X G_x \\ &\vdash \bigvee_X \bigvee_Y G_y \wedge G_x \\ &\vdash \bigvee_X \bigvee_Y \bigvee_{r \leq x, r \leq y} G_r.\end{aligned}$$

Because X, Y are downward closed, $\{r \in \mathbb{P} \mid \exists x \in X, y \in Y. r \leq x \wedge r \leq y\} = X \cap Y$ and we are done.

For the other direction, consider the following map of preorders:

$$\begin{aligned} g : \mathbb{T} &\to \mathrm{Dn}_{\neg\neg}(\mathbb{P}) \\ G_p &\mapsto \neg\neg\mathcal{O}_p. \end{aligned}$$

Lemma 4.9 tells us that

$$g(G_p) = \{q \in \mathbb{P} \mid G_q \vdash G_p\}.$$

In this form it is particularly clear that this is a map of preorders, i.e. $G_p \vdash G_q$ implies $g(G_p) \leq g(G_q)$.

We saw that $L_{\mathbb{T}}$ is induced by the preorder $(\mathbb{T}, \vdash)$ by definition. Accordingly a map of preorders induces a map from this poset, as long as equivalent formulas are mapped to the same set. Checking this amounts to checking that the axioms are preserved.

(i) $g(G_1) = \neg\neg\mathbb{P} = \mathbb{P} \supseteq \mathbb{P} = g(\top)$.

(ii) $\{x \in \mathbb{P} \mid G_x \vdash G_p\} \subseteq \{x \in \mathbb{P} \mid G_x \vdash G_q\}$ is obvious for every pair $p \leq q$.

(iii)
$$\begin{aligned}\{x \in \mathbb{P} \mid G_x \vdash G_p\} \cap \{x \in \mathbb{P} \mid G_x \vdash G_q\} &= \{x \in \mathbb{P} \mid G_x \vdash G_p \wedge G_q\} \\ &\subseteq \left\{x \in \mathbb{P} \mid G_x \vdash \bigvee_{r \leq p, r \leq q} G_r\right\}.\end{aligned}$$

(iv) Let $q \in \mathbb{P}$. D is dense, so there exists some $p \leq q$ in D. Then since $G_p \vdash G_p$ and

$$p \in \bigcup_{p \in D} \{x \in \mathbb{P} \mid G_x \vdash G_p\},$$

we have shown that

$$\bigcup_{p \in D} \{x \mid G_x \vdash G_p\}$$

is dense below q. So

$$\neg\neg \bigcup_{p \in D} \{x \in \mathbb{P} \mid G_x \vdash G_p\} = \mathbb{P}.$$

Finally we show that f and g are bijections. Let For $X \in \mathrm{Dn}_{\neg\neg}(\mathbb{P})$ arbitrary,

$$(g \circ f)(X) = g\left(\bigvee_{p \in X} [G_p]\right) = \bigcup_{p \in X} \neg\neg\mathcal{O}_p.$$

This is equal to X: Let $x \in \bigcup_{p \in X} \neg\neg\{q \leq p\}$, so $\{q \leq p\}$ is dense below x. Then since $\{q \leq p\} \subseteq X$, X is also dense below x. So $x \in X$. For the reverse inclusion, let $x \in X$. Then $\mathcal{O}_x$ is dense below x, so

$$x \in \{q \leq x\} \subseteq \bigcup_{p \in X} \neg\neg\mathcal{O}_p.$$

With this we have shown that $g \circ f$ is the identity, and check the same for $f \circ g$. Showing this on the level of posets suffices, as the isomorphism will extend to frames. Let $p \in \mathbb{P}$.

$$(f \circ g)(G_p) = f\big(\{q \in \mathbb{P} \mid G_q \vdash G_p\}\big) = \bigvee_{\{p \in \mathbb{P} \mid G_q \vdash G_p\}} [G_q] = [G_p]$$

It is easy to see that $\bigvee_{\{p \in \mathbb{P} \mid G_q \vdash G_p\}} G_q \dashv\vdash G_p$, and so $f \circ g$ is the identity as well. □

We have encountered the double negation topos in various forms and caught a glimpse of how it relates to the construction of forcing extensions over set- or Boolean-valued models, as well as the role of generic filters play. From this point on in the thesis, only the form $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ will be relevant, and we will often abbreviate it as $\mathrm{Sh}(\mathbb{P})$.

5 First-Order Logic in a Topos

In this section, we return to some background on toposes. While in the previous section we explored how geometric theories interact with classifying toposes, we now shift focus to first-order logical theories and explain how they can be interpreted within a topos. After introducing the notion of a model in a topos, we will define our central concept: the forcing relation in a topos, both from a semantic perspective and via a recursive definition over formulas. Throughout, we gradually narrow our scope from general elementary toposes, to Grothendieck toposes, and finally to the specific case of the double negation topos $\mathrm{Sh}(\mathbb{P})$.

We conclude by introducing the internal language of a topos, which will provide some powerful language and results for the proofs to come.

5.1 Categorical Semantics

The reader may already be familiar with set models, and we introduced Boolean-valued models in Section 3. We now generalize these ideas to models in categories – specifically toposes. This begins with interpreting a signature, followed by a recursive interpretation of terms and formulas. The categorical properties of toposes discussed in Sections 1 and 3 will be implicitly relied upon throughout although we will not explicitly reference them at every step.

This section is not intended as a comprehensive introduction to categorical semantics, it should merely provide the necessary background for readers already acquainted with first-order logic. Our presentation broadly follows [Joh02, D 1.1 - 1.3], and any further details, proofs and background may be found there.

Definition 5.1 A Σ-structure M in $\mathcal{E}$ assigns

(i) to every sort X in Σ an object $[\![X]\!]_M$ of $\mathcal{E}$,

(ii) to every context $\Gamma = (x_1 : X_1, ..., x_n : X_n)$, the object $[\![\Gamma]\!]_M = [\![X_1]\!]_M \times ... \times [\![X_n]\!]_M$,

(iii) to every function symbol $f : \Gamma \to X$ in Σ a morphism $[\![f]\!]_M : [\![\Gamma]\!]_M \to [\![X]\!]_M$

(iv) to every relation symbol $R : \Gamma$ a subobject $[\![\Gamma \mid R]\!]_M \subseteq [\![\Gamma]\!]_M$.

We will often omit the subscript M when dealing with a single model. Given a Σ-structure, we can of course also define the interpretation of terms and formulas over Σ. For the rest of this subsection, assume M is a Σ-structure in a topos $\mathcal{E}$.

A term in context $t : \Gamma \to X$ over Σ is interpreted in M as a morphism $[\![t]\!] : [\![\Gamma]\!] \to [\![X]\!]$ by the following clauses:

(i) If $t : \Gamma = (x_1 : X_1, ..., x_n : X_n) \to X$ is a variable, and hence of the form x_i for some index $i \leq n$, then

$$[\![t]\!] = \pi_i : [\![\Gamma]\!] \to [\![X_i]\!].$$

(ii) If $t = f(t_1, ..., t_m)$ for some function symbol $f : (Y_1, ..., Y_m) \to Y$ and terms $t_i : \Gamma \to Y_i$, then

$$[\![t]\!] = [\![f]\!] \circ ([\![t_1]\!], ..., [\![t_m]\!]).$$

We will often use $\bar{t} := (t_1, ..., t_n)$ as a shorthand for a tuple of terms, and set $[\![\bar{t}]\!] := ([\![t_1]\!], ..., [\![t_n]\!])$. Substitution for terms is properly interpreted by composition in the category.

A formula $\Gamma \mid \varphi(\overline{x})$ over Σ is interpreted in M by a subobject

$$[\![\Gamma \mid \varphi(\overline{x})]\!] \subseteq [\![\Gamma]\!],$$

sometimes shortened to $[\![\Gamma \mid \varphi]\!]$ or $[\![\varphi]\!]$ when the context is clear, and defined by the following clauses:

(i) If $\varphi(\overline{x}) = R(t_1, ..., t_m)$, where $R : \Gamma$ is a relation symbol and $\overline{t} : \Gamma' \to \Gamma$ are terms, then $[\![\Gamma' \mid \varphi(\overline{x})]\!]$ is the pullback

$$\begin{array}{ccc} [\![\Gamma' \mid \varphi(\overline{x})]\!] & \longrightarrow & [\![R]\!] \\ \downarrow & & \downarrow \\ [\![\Gamma']\!] & \xrightarrow{([\![\overline{t}]\!])} & [\![\Gamma]\!]. \end{array}$$

Note that this corresponds to setting $[\![\Gamma' \mid R(\overline{t})]\!] = [\![\overline{t}]\!]^*([\![\Gamma \mid R(\overline{x})]\!])$

(ii) If $\varphi(\overline{x}) = (t_1 = t_2)$, where $t_i : \Gamma \to X$ are terms, then $[\![\Gamma \mid \varphi(\overline{x})]\!]$ is the pullback of the diagonal $\Delta : [\![X]\!] \rightarrowtail [\![X]\!] \times [\![X]\!]$ along $([\![t_1]\!], [\![t_2]\!])$:

$$\begin{array}{ccc} [\![\Gamma \mid \varphi(\overline{x})]\!] & \longrightarrow & [\![X]\!] \\ \downarrow & & \downarrow \Delta \\ [\![\Gamma]\!] & \xrightarrow{([\![t_1]\!], [\![t_2]\!])} & [\![X, X]\!]. \end{array}$$

This matches the previous clause, in that equality is a special family of relation symbols for which the interpretation is required to be the diagonal.

Given formulas $\Gamma \mid \psi_1, \psi_2$ with interpretations $[\![\Gamma \mid \psi_1]\!]$ and $[\![\Gamma \mid \psi_2]\!]$, the interpretation of φ when built up from logical connectives is given by applying the operations defined on $\mathrm{Sub}([\![\Gamma]\!])$ of the same name:

(iii) $[\![\Gamma \mid \top]\!] = [\![\Gamma]\!] = \top \in \mathrm{Sub}([\![\Gamma]\!])$,

(iv) $[\![\Gamma \mid \psi_1 \wedge \psi_2]\!] = [\![\Gamma \mid \psi_1]\!] \wedge [\![\Gamma \mid \psi_2]\!]$,

(v) $[\![\Gamma \mid \bot]\!] = [\![ ]\!] (= 0) = \bot \in \mathrm{Sub}([\![\Gamma]\!])$,

(vi) $[\![\Gamma \mid \psi_1 \vee \psi_2]\!] = [\![\Gamma \mid \psi_1]\!] \vee [\![\Gamma \mid \psi_2]\!]$,

(vii) $[\![\Gamma \mid \psi_1 \Rightarrow \psi_2]\!] = [\![\Gamma \mid \psi_1]\!] \Rightarrow [\![\Gamma \mid \psi_2]\!]$.

Given a formula $\Gamma, y : Y \mid \psi$ with interpretation $[\![\Gamma, y : Y \mid \psi]\!]$, the interpretation of φ when built up from quantifiers is given by applying the functors $\forall_\pi, \exists_\pi : \mathrm{Sub}([\![\Gamma]\!] \times [\![y : Y]\!]) \to \mathrm{Sub}([\![\Gamma]\!])$, where π is the projection $[\![\Gamma]\!] \times [\![y : Y]\!] \to [\![\Gamma]\!]$:

(viii) $[\![\Gamma \mid \exists y : Y . \psi(\overline{x}, y)]\!] = \exists_\pi([\![\Gamma, y : Y \mid \psi(\overline{x}, y)]\!])$,

(ix) $[\![\Gamma \mid \forall y : Y . \psi(\overline{x}, y)]\!] = \forall_\pi([\![\Gamma, y : Y \mid \psi(\overline{x}, y)]\!])$.

Equivalently we could of course say that a formula is interpreted by the classifying morphism $[\![\Gamma]\!] \to \Omega$ of the subobject.[8]

Lemma 5.2 (Substitution Lemma [Joh02, D 1.2.7]) *Let Σ be a signature interpretable in a topos $\mathcal{E}$. Let $\overline{t} : \Gamma' \to \Gamma$ be a tuple of terms, and $\Gamma \mid \varphi$ be a formula over Σ. Then*

$$[\![\overline{t}]\!]^*([\![\Gamma \mid \varphi(\overline{x})]\!]) = [\![\Gamma' \mid \varphi(\overline{t})]\!].$$

[8] As done in [MM94, VI.5], where an alternate notation $\{x \in \Gamma \mid \varphi(x)\}$ is used, drawing a parallel to set models.

Example 5.1 (Weakening Property). Recall that for any morphism, and so in particular for a projection $\pi : [\![\, \Gamma \,]\!] \times [\![\, y : Y \,]\!] \to [\![\, \Gamma \,]\!]$,

$$\exists_\pi \dashv \pi^* \dashv \forall_\pi .$$

So for any formulas $\Gamma \mid \varphi$ and $\Gamma, y : Y \mid \psi$, Lemma 5.2 and adjointness give

$$\frac{[\![\, \Gamma \mid \varphi(\overline{x}) \,]\!] \leq [\![\, \Gamma \mid \forall y : Y . \psi(\overline{x}, y) \,]\!]}{[\![\, \Gamma, Y \mid \varphi(\overline{x}) \,]\!] \leq [\![\, \Gamma, Y \mid \psi(\overline{x}, y) \,]\!],} \tag{5.2}$$

and

$$\frac{[\![\, \Gamma \mid \exists y : Y . \psi(\overline{x}, y) \,]\!] \leq [\![\, \Gamma \mid \varphi(\overline{x}) \,]\!]}{[\![\, \Gamma, Y \mid \psi(\overline{x}, y) \,]\!] \leq [\![\, \Gamma, Y \mid \varphi(\overline{x}) \,]\!].} \tag{5.3}$$

The horizontal line denotes equivalence and is often used with adjoints.

Definition 5.3 We say that M *believes* $\Gamma \mid \varphi(\overline{x})$, written $M \underset{\Gamma}{\vDash} \varphi$, if $[\![\, \Gamma \mid \varphi \,]\!] = \top$.[9] If M believes all axioms of a theory $\mathbb{T}$, it is called a model of that theory.

The use of this terminology is justified by the following well-known theorem, which will be left without proof.

Theorem 5.4 (Soundness Theorem [Joh02, D 1.3.2]) *Let $\mathbb{T}$ be a first-order theory over a signature Σ and M be a model of $\mathbb{T}$ in a topos $\mathcal{E}$. If a formula $\Gamma \mid \varphi(\overline{x})$ is provable in intuitionistic first-order logic over $\mathbb{T}$, then $M \vDash \varphi$.*

[9]It would be more general to speak of sequents at this stage, but because we intend to use full first-order logic, we can assume sequents of the form $\top \vdash \varphi$ and skip the sequent notation.

Lemma 5.5 *For a Grothendieck topos* $\mathrm{Sh}(\mathcal{C})$ *and any* $U \in \mathcal{C}$*,*

(i) If $\varphi(\overline{x}) = R(t_1, ..., t_m)$*, where* $R : (Y_1, ..., Y_m)$ *is a relation symbol and* $t_i : \Gamma \to Y_i$ *are terms, then*

$$[\![\Gamma \mid R(t_1(\overline{x}), ..., t_m(\overline{x})]\!](U) = \left\{ \left(\overline{b}, \overline{a}\right) \in [\![R]\!](U) \times [\![\Gamma]\!](U) \mid [\![t_i]\!](U)(\overline{a}) = b_i \text{ for all } 0 \leq i \leq m \right\}.$$

(ii) If $\varphi(\overline{x}) = (t_1 = t_2)$*, where* $t_i : \Gamma \to X$ *are terms, then*

$$[\![\Gamma \mid t_1(\overline{x}) = t_2(\overline{x})]\!](U) = \{(a, \overline{a}) \in X(U) \times \Gamma(U) \mid [\![t_1]\!](U)(\overline{a}) = a = [\![t_2]\!](U)(\overline{a})\}.$$

Given formulas $\Gamma \mid \psi_1, \psi_2$ *with interpretations* $[\![\Gamma \mid \psi_1]\!]$ *and* $[\![\Gamma \mid \psi_2]\!]$*,*

(iii) $[\![\Gamma \mid \top]\!] = [\![\Gamma]\!]$,

(iv) $[\![\psi_1 \wedge \psi_2]\!](U) = [\![\psi_1]\!](U) \cap [\![\psi_2]\!](U)$,

(v) $[\![\Gamma \mid \bot]\!] = 0_{\mathrm{Sh}(\mathcal{C})}$,

(vi) $a \in [\![\Gamma \mid \psi_1 \vee \psi_2]\!](U)$ iff $\{f : V \to U \mid a \cdot f \in [\![\psi_1]\!](V) \text{ or } a \cdot f \in [\![\psi_2]\!](V)\}$ covers U,

(vii) $a \in [\![\Gamma \mid \psi_1 \Rightarrow \psi_2]\!](U)$ iff for all $(f : V \to U)$, if $a \cdot f \in [\![\psi_1]\!](V)$ then $\alpha \cdot f \in [\![\psi_2]\!](V)$.

Given a formula $\Gamma, y : Y \mid \psi$ *and the projection* $\pi : [\![\Gamma, Y]\!] \to [\![Y]\!]$*,:*

(viii) $a \in [\![\Gamma \mid \exists y : Y . \psi(\overline{x}, y)]\!](U)$ iff $\{f : V \to U \mid \exists b \in [\![\Gamma, y : Y \mid \psi]\!](V) . \pi(b) = y \cdot f\}$ covers U,

(ix) $a \in [\![\Gamma \mid \forall y : Y . \psi(\overline{x}, y)]\!](U)$ iff for all $(f : V \to U)$, $\pi^*(V)(a \cdot f) \subseteq [\![\Gamma, y : Y \mid \psi(\overline{x}, y)]\!](V)$.

Proof. This is a direct application of Section 3.4 to the interpretation of formulas defined in the previous subsection. □

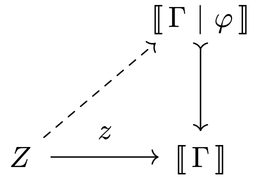
5.2 Kripke-Joyal Semantics

We now take a different perspective on this setup, one that gives a finer relation for the validity of formulas in a Σ-structure than the relation $\vDash$. This relation is often presented under the name *Kripke–Joyal semantics*, typically in the context of the *internal language* of a topos, a concept we will introduce shortly. These concepts are discussed in a wide selection of papers and introductions to topos theory and logic, we refer the reader to [MM94, IV], [Osi75], [Mak77] to just name a few.

Previously, we defined a formula φ to be believed by a model M when it corresponds to the top element in the subobject lattice $\mathrm{Sub}([\mid \Gamma \mid])$. Now, rather than working purely within the subobject lattice, we refer explicitly to objects and morphisms in $\mathcal{E}$.

Lemma 5.6 *For any model M in $\mathcal{E}$ and any formula $\Gamma \mid \varphi(\overline{x})$ we have*

$$M \underset{\Gamma}{\vDash} \varphi(\overline{x}) \quad \text{iff} \quad \text{every } z : Z \to [\![\, \Gamma \,]\!] \text{ factors through } [\![\, \Gamma \mid \varphi \,]\!] :$$



Note that due to $[\![\, \varphi \,]\!] \rightarrowtail [\![\, \Gamma \,]\!]$ being monic, the factorisation is unique.

Proof. This follows directly from the definition of the order relation of an object. □

This at first glance somewhat arbitrary observation is what motivates our definition of a forcing relation, that we can think of using "generalized elements" $z : Z \to X$ to "test" satisfaction, and to express truth more locally.

Definition 5.7 In a topos $\mathcal{E}$ and a model M of $\mathbb{T}$ in $\mathcal{E}$, we define a *forcing relation* between a generalized element $z : Z \to [\![\, \Gamma \,]\!]$ and a formula $\Gamma \mid \varphi(\overline{x})$ over $\mathbb{T}$ as follows:

$$Z \Vdash \varphi(\overline{x}) \text{ for } z :\Leftrightarrow z \text{ factors through } [\![\, \varphi \,]\!].$$

Read: Z *forces* $\varphi(\overline{x})$ *for* z. A handy abuse of notation is to write $Z \Vdash \varphi(z)$.

Another concise way to put this definition, as done in [MM94, VI.6] for instance, is to set

$$Z \Vdash \varphi(z) :\Leftrightarrow \operatorname{im} z \leq [\![\, X \mid \varphi(x) \,]\!].$$

Clearly this is just another way to write the more explicit definition we have chosen.

Remark 5.1. For a closed formula φ without free variables,

$$M \vDash \varphi \Leftrightarrow 1 \Vdash \varphi.$$

A first basic property of the forcing relation is monotonicity.

Theorem 5.8 *[MM94, VI.6] If $U \Vdash \varphi(x)$ for $z : Z \to X$, then for any morphism $f : Z \to Z'$ in $\mathcal{E}$, $Z' \Vdash \varphi(x)$* for *$z \circ f$.*

It is also possible to define the forcing relation inductively over formulas and then show that it correponds to our definition above. But first we notice that in the case of Grothendieck toposes, we can restrict the generalized elements to those that have a representable sheaf as a domain, i.e are of the form $ayU \to [\![\, \Gamma \,]\!]$, where a is the sheafification functor and y the Yoneda embedding. This restriction is justified by the following lemma, which reframes Lemma 5.6 in terms of representable sheaves.

Lemma 5.9 *For a model M of $\mathbb{T}$ in $\hat{\mathcal{C}}$ and any formula φ over $\mathbb{T}$,*

$$M \vDash \varphi \;\text{ iff }\; yU \Vdash \varphi \text{ for every } U \in \hat{\mathcal{C}} \text{ and } z : yU \to [\![\, \Gamma \,]\!].$$

Proof. The forward direction is trivial. Now assume that for any generalized element of the form $z : yU \to [\![\, \Gamma \,]\!]$, z factors through $[\![\, \varphi \,]\!]$.

Let $U \in \mathcal{C}$. As a subobject, $[\![\, \varphi \,]\!](U) \subseteq [\![\, \Gamma \,]\!]$ is clear – now we will show that $[\![\, \Gamma \,]\!](U)$ injects into $[\![\, \varphi \,]\!](U)$ and can then conclude that the two presheaves are isomorphic. By the Yoneda lemma,

$$[\![\,\Gamma\,]\!](U) \cong \mathrm{Hom}_{\hat{\mathcal{C}}}(yU, [\![\,\Gamma\,]\!]),$$

and further by the inclusion-sheafification functor adjoint pair,

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(yU, [\![\,\Gamma\,]\!] \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(ayU, [\![\,\Gamma\,]\!]).$$

Analogously, $[\![\,\Gamma\,]\!](U) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(ayU, [\![\,\Gamma\,]\!])$, and the assumed (unique) factoring of any element of $\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(ayU, [\![\,\Gamma\,]\!])$ through $[\![\,\varphi\,]\!]$ defines an injection

$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(ayU, [\![\,\Gamma\,]\!]) \hookrightarrow \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(ayU, [\![\,\varphi\,]\!]).$$

With $[\![\,\varphi\,]\!] \cong [\![\,\Gamma\,]\!] \cong 1_{\mathrm{Sub}([\![\,\Gamma\,]\!])}$ we have proven $M \vDash \varphi$. □

For any morphism $z : ayU \to [\![\,\Gamma\,]\!]$, we can now shorten $ayU \Vdash \varphi(z)$ to

$$U \Vdash \varphi(z).$$

The Yoneda lemma also suggests the equivalent formulation

$$U \Vdash \varphi(z) \ \text{ iff } \ z \in [\![\,\varphi\,]\!](U),$$

where z refers both to the morphism in $\mathrm{Hom}(ayU, [\![\,\Gamma\,]\!])$ and the corresponding element in $[\![\,\varphi\,]\!](U)$.

5.3 The Forcing Relation in the Double Negation Topos

We finally apply the forcing relation to the Grothendieck topos $\mathrm{Sh}(\mathbb{P})$. We will assume M to be a model of a theory $\mathbb{T}$ in this topos. We will adapt [MM94, VI.7 Thm 1] to this special case, presenting a recursive characterisation of the forcing relation for M in $\mathrm{Sh}(\mathbb{P})$.

Theorem 5.10 *Let $\Gamma \mid \varphi_1, \varphi_2$ and $\Gamma, y \mid \psi$ be formulas over $\mathbb{T}$, $p \in \mathbb{P}$ and let $z \in [\![\,\Gamma\,]\!](p)$. Then*

(i) $p \Vdash \top$

(ii) $p \Vdash \varphi(z) \wedge \psi(z)$ iff $p \Vdash \varphi(z) \wedge p \Vdash \psi(z)$,

(iii) $p \Vdash \bot$ iff $\emptyset \in \mathrm{Cov}(p)$

(iv) $p \Vdash \varphi(z) \vee \psi(z)$ iff for any $q \leq p$ there is an $r \leq q$ s.t. $r \Vdash \varphi(z \cdot r)$ or $r \Vdash \psi(z \cdot r)$,

(v) $p \Vdash \varphi(z) \Rightarrow \psi(z)$ iff for any $q \leq p, q \Vdash \varphi(z \cdot q)$ implies $q \Vdash \psi(z \cdot q)$.

(vi) $p \Vdash \exists y : Y.\varphi(z, y)$ iff for any $q \leq p$ there are $r \leq q$ and $w \in [\![\,Y\,]\!](r)$ such that $r \Vdash (z \cdot r, w)$,

(vii) $p \Vdash \forall y : Y.\varphi(z, y)$ iff for any $q \leq p$ and any $w \in [\![\,Y\,]\!](q), q \Vdash \varphi(z \cdot q, w)$,

Proof. For any formula $\Gamma \mid \varphi$ and $z \in [\![\,\Gamma\,]\!](p)$, we can rewrite $p \Vdash \varphi(z)$ as $z \in \{x \mid \varphi(x)\}(p)$. We will rely heavily on the operations presented in Section 3.4 as well as Lemma 5.5.

(i) $p \Vdash \top$ iff $ay(p)$ factors into $[\![\,\top\,]\!] = 1$. This will always be the case for the terminal element.

(ii) We have the equalities

$$\{x \mid \varphi(x)\}(p) \cap \{x \mid \psi(x)\}(p) = (\{x \mid \varphi(x)\} \wedge \{x \mid \psi(x)\})(p) = \{x \mid (\varphi \wedge \psi)(x)\}(p).$$

Thus,

$$
\begin{aligned}
p \Vdash \varphi(z) \wedge \psi(z) \ &\text{iff} \ \ z \in \{x \mid (\varphi \wedge \psi)(x)\}(p) \\
&\text{iff} \ \ z \in \{x \mid \varphi(x)\}(p) \cap \{x \mid \psi(x)\}(p) \\
&\text{iff} \ \ p \Vdash \varphi(z) \text{ and } p \Vdash \psi(z).
\end{aligned}
$$

(iii) $p \Vdash \bot$ iff $ayp \leq [\![\bot]\!] = \mathcal{O}$, that is if $ayp = \mathcal{O}_{\mathrm{Sh}}$. If we can show that $\mathrm{Hom}(-, p)$ is empty, then we are done. And this is clearly the case whenever p is a least element, i.e. $\emptyset \in \mathrm{Cov}(p)$.

(iv) We pull out disjunction to apply the join on subsheaves.

$$
\begin{aligned}
p \Vdash (\varphi \vee \psi)(z) \ &\text{iff} \ \ \alpha \in \{z \mid (\varphi \vee \psi)(z)\}(p) \\
&\text{iff} \ \ z \in (\{x \mid \varphi(x)\} \vee \{x \mid \psi(x)\})(p) \\
&\text{iff} \ \ \{r \leq p \mid z \cdot r \in [\![x.\varphi]\!](q) \vee z \cdot r \in [\![x.\psi]\!](q)\} \in \mathrm{Cov}(p) \\
&\text{iff} \ \ \{r \leq p \mid r \Vdash \varphi(z \cdot r)\} \cup \{r \leq p \mid r \Vdash \psi(z \cdot r)\} \in \mathrm{Cov}(p)
\end{aligned}
$$

A set is a covering of p if it is bounded by p, down-closed and dense below p. Our set is clearly bounded by p and down-closed by monotonicity of $\Vdash$. So we can rewrite the last line to get the equivalence stated in the theorem.

(v) Again, this follows directly from applying the $\Rightarrow$-operation on subsheaves, following an argument analogous to that made in (iv).

(vi) Once more this follows a similar pattern, but we elaborate a final time, as the quantifiers might feel a little different at first glance.

$$
\begin{aligned}
p \Vdash \exists y : Y.\varphi(z, y) \ &\text{iff} \ \ z \in [\![\overline{x} : \Gamma \mid \exists y : Y.\varphi(\overline{x}, y)]\!](p) \\
&\text{iff} \ \ z \in \exists_{\pi([\![\overline{x}, y:\Gamma, Y \mid \varphi(\overline{x}, y)]\!])} \\
&\text{iff} \ \ \{r \in \mathbb{P} \mid \exists \overline{u}, w \in [\![\overline{x}, y : \Gamma, Y \mid \varphi(\overline{x}, y)]\!].\overline{u} = z \cdot r\} \\
&\text{iff} \ \ \text{for any } q \leq p \text{ there are } r \leq q \text{ and } w \in [\![Y]\!](r) \text{ such that } r \Vdash (z \cdot r, w).
\end{aligned}
$$

(vii) Analogous to previous arguments using the subsheaf operations.

□

5.4 The Internal Language of a Topos

The categorical structure of a topos induces a first-order signature, called its "internal language". This can then in turn be interpreted in the topos, giving us a way to internally reason over statements concerning $\mathcal{E}$.

Definition 5.11 (signature of the internal language) We define the internal signature $\Sigma_{\mathcal{E}}$ of a topos $\mathcal{E}$ by choosing

- the collection of objects in $\mathcal{E}$ to be the sorts of Σ,
- Σ to have a function symbol $f : (X_1, ..., X_n) \to Y$ for every tuple of objects $(X_1, ..., X_n, Y)$ and every morphism $X_1 \times ... \times X_n \to Y$,
- Σ to have a relation symbol $R : X_1, ..., X_n$ for every tuple of objects $(X_1, ..., X_n)$ and every subobject $R \subseteq X_1 \times ... \times X_n$,.

We can make the internal language into a theory $T_{\mathcal{E}}$: Since the internal language has a canoncial interpretation in $\mathcal{E}$, we can add sequent $\varphi \vdash \psi$ as an axiom of the internal theory whenever $[\![\varphi]\!] \leq [\![\psi]\!]$.

This internal language can be interpreted by the topos in the obvious way in that every sort, function symbol and relation is interpreted by the exact object, morphism and subobject it is named after. The $\Sigma_{\mathcal{E}}$-structure is the topos $\mathcal{E}$ itself, which is also trivially a model of the internal theory $\mathcal{E}$. Using the tools developed in the previous subsection, we now have a way to express whether $\mathcal{E}$ believes certain internal statements about itself.

Here are some simple examples, in which we will carefully step between the internal language as a signature and its interpretation in $\mathcal{E}$. In the future we will become more comfortable with thinking of morphisms and function symbols of the internal language interchangably.

Example 5.2. Let $X \xrightarrow{h} Y \xrightarrow{g} Z$ be morphisms in $\mathcal{E}$ and $f = g \circ h$. Then

$$\mathcal{E} \underset{X}{\vDash} f(x) = g(h(x)).$$

Here f, g, h are function symbols interpreted by the morphisms in $\mathcal{E}$. From clause (ii) of the interpretation of terms, we see that $g(h(x))$ is interpreted by the composition $f \circ h$. So now we have two terms interpreted by the same morphism $X \xrightarrow{f} Z$. The equalizer will therefore clearly be the top element X itself.

Example 5.3. Consider a morphism $f : Z \times X \to Y$ and the induced morphism $\hat{f} : Z \to Y^X$. We can relate the two with the internal statement

$$\mathcal{E} \underset{x:X,z:Z}{\vDash} f(z,x) = \mathrm{ev}\Big(\hat{f}(z), x\Big),$$

a direct application of the previous example. We can rewrite this as

$$f(z,x) = y \Leftrightarrow \mathrm{ev}\Big(\hat{f}(z), x\Big) = y,$$

coming closer to how we think of the evaluation function.

The following is an example application of the internal language, but will also become useful later on. It holds for elementary toposes, but we take a shortcut in the proof by assuming it is a Grothendieck topos.

Lemma 5.12 *In a (Grothendieck) topos $\mathcal{E}$, a morphism $f : X \to Y$ is monic if and only if*

$$\mathcal{E} \vDash \forall x, x' : X.(fx = fx' \Rightarrow x = x').$$

Proof. We work with the definitions of logical connectives in the subobject lattice and adjointness.

$$
\begin{aligned}
& [\![- \mid \top]\!] \leq [\![\forall x, x' : X.(fx = fx' \Rightarrow x = x')]\!]) \\
\text{iff}\quad & [\![X \times X \mid \top]\!] \leq [\![x, x' : X \mid (fx = fx' \Rightarrow x = x')]\!] && \text{by Equation (5.2)} \\
& = [\![x, x' : X \mid fx = fx']\!] \Rightarrow [\![x, x' : X \mid x = x']\!] \\
\text{iff}\quad & \underbrace{[\![x, x' : X \mid \top \wedge fx = fx']\!]}_{=[\![x, x' : X \mid fx = fx']\!]} \leq [\![x, x' : X \mid x = x']\!]
\end{aligned}
$$

Note that all these steps were equivalences. This final inequality in turn can be equivalently rephrased using the characterisation of subsheaves: $[\![x, x' : X \mid fx = fx']\!]$ is a subsheaf of $[\![x, x' : X \mid x = x']\!]$ iff for every $U \in \mathcal{C}$,

$$[\![x, x' : X \mid fx = fx']\!](U) \subseteq [\![x, x' : X \mid x = x']\!](U),$$

which then by expanding definitions using Lemma 5.5 is equivalent to

$$\{x, x' \in X(U) \mid f(U)(x) = f(U)(x')\} \subseteq \{x, x' \in X(U) \mid x = x'\}$$

for all $U \in \mathcal{C}$. This holds iff $f(U)$ is injective for every $U \in \mathcal{C}$, that is iff f is monic □

This last lemma will also be useful later, and expresses that two "functions" (elements of Y^X) are equal if they are equal in all points.

Lemma 5.13 *For objects X, Y in a topos,*

$$\mathcal{E} \underset{X,Y}{\models} \forall z : X.(\mathrm{ev}(x, z) = \mathrm{ev}(y, z)) \Rightarrow (x = y)$$

Proof (sketch). This can be approached in a similar way to the internal proofs seen so far, while making use of the universal property of ev. □

6 Interpreting Set Theory in a Grothendieck Topos

So far, we have seen that first-order logic can be interpreted within any topos, sharing many of the familiar properties of Set. In [MM94, IV.1], toposes are examined as universes generalizing Set: one finds analogues of the axiom of choice, a natural numbers object, and power-objects mirroring the corresponding set-theoretic constructions. In [MM94, IV.2], it is shown that $\mathrm{Sh}(\mathbb{P})$ satisfies a categorical form of the failure of the continuum hypothesis. These independence results can be seen through intuitionistic higher-order logic, but relating them formally to first-order set theory requires some more work.

Although a topos may satisfy statements that resemble axioms of **ZF** – such as possessing a natural numbers object – the topos itself does not constitute a model of **ZF** – or formally satisfy the first-order axiom of infinity – simply because the topos is not a structure over the language of set theory. However, **ZF** can be interpreted in any Grothendieck topos by defining a cumulative hierarchy of objects and element relations that generalizes the von Neumann hierarchy. This approach appears in [Fou80] as well as [Hay81]. We define this hierarchy following Fourman's approach, but make stronger use of the internal language of the topos to define an interpretation of **ZF** in V.

We will then see that the Boolean-valued model $V^{\mathbb{B}}$ can be seen as a sheaf in $\mathrm{Sh}(\mathbb{P})$ and show that this model and the cumulative hierarchy V are naturally isomorphic. Finally we compare the interpretation of formulas and the forcing relation in the internal hierarchy to their Boolean-valued semantics.

An analysis of how this first-order model interacts with the corresponding external categorical properties of the topos through intuitionistic higher-order logic would be interesting, but here we will focus on the more modest goal of extracting a first-order model inside $\mathrm{Sh}(\mathbb{P})$, as outlined above.

6.1 The Cumulative Hierarchy in a Grothendieck Topos

Given a Grothendieck topos $\mathcal{E}$, our goal is to define a cumulative hierarchy of objects in $\mathcal{E}$, imitating the von Neumann hierarchy in **Set**. Its construction will be based on [Fou80, Section 2], though the proof has been significantly elaborated upon.

Of course, to interpret **ZF** an $\mathcal{E}$, our universe must be equipped with an element relation. Every object V_α in the hierarchy should be equipped with an *extensional* relation $\in_\alpha$. A binary relation $R \subseteq X \times X$ is said to be extensional if

$$\mathcal{E} \vDash \forall x, x' : X.[\forall z : X.(zRx \Leftrightarrow zRx') \Rightarrow (x = x')],$$

or equivalently, rewriting the subobject R as $\varphi_R : X \times X \to \Omega$, if

$$\mathcal{E} \vDash \forall x, x' : X.[\forall z : X.(\varphi_R(x, z) = \varphi_R(x', z) \Rightarrow (x = x')]. \tag{6.4}$$

It will be useful to have both an internal and external way to characterize extensionality. The following characterisation is true for elementary toposes, but we will take a shortcut and prove the statement for our setup of Grothendieck toposes.

Lemma 6.1 *In a (Grothendieck) topos, a binary relation $\varphi_R : X \times X \to \Omega$ is extensional if and only if the map $\widehat{\varphi_R} : X \to \Omega^X$ is monic.*

Proof. By Lemma 5.12, the map $\widehat{\varphi_R}$ is monic if and only if

$$\forall x, x' : X.(\widehat{\varphi_R}(x) = \widehat{\varphi_R}(x') \Rightarrow x = x') \tag{6.5}$$

holds internally. If we can show internally that for any $x, x' : X$,

$$[\forall z : X.(\varphi_R(x, z) = \varphi_R(x', z))] \Leftrightarrow \widehat{\varphi_R}(x) = \widehat{\varphi_R}(x')$$

we will have shown the equivalence of (6.4) and (6.5) and will be done.

Rewriting φ_R as a composition, we get

$$[\forall z : X.(\mathrm{ev}(\widehat{\varphi_R}(x), z) = \mathrm{ev}(\widehat{\varphi_R}(x'), z)] \Leftrightarrow \widehat{\varphi_R}(x) = \widehat{\varphi_R}(x').$$

The backward direction of this equivalence is trivial function application, and the forward direction is an exact application of Lemma 5.13. □

We will now both recursively define the cumulative hierarchy as a functor, and inductively prove some of its fundamental properties in the following theorem.

Theorem 6.2 *There is a continuous functor*

$$\begin{aligned} V : \mathrm{Ord} &\to \mathcal{E} \\ \alpha &\mapsto V_\alpha \\ (\beta \leq \alpha) &\mapsto i^\alpha_\beta \end{aligned}$$

with extensional maps $\in_\alpha : V_\alpha \times V_\alpha \to \Omega$ *for every* $\alpha \in \mathrm{Ord}$, *such that*

$$\begin{array}{ccc} V_\beta \times V_\beta & & \\ {\scriptstyle i^\alpha_\beta \times i^\alpha_\beta}\downarrow & \searrow^{\in_\beta} & \\ V_\alpha \times V_\alpha & \xrightarrow[\in_\alpha]{} & \Omega \end{array}$$

commutes for every $\beta \leq \alpha$.

Proof. We will construct this functor and the maps $\in_\alpha$ by transfinite recursion. Alongside the construction, we will prove by transfinite induction that for any ordinal α,

(i) for any ordinals $\gamma \leq \beta \leq \alpha$, $i^\alpha_\beta \circ i^\beta_\gamma = i^\alpha_\gamma$,
(ii) $i^\alpha_\alpha = \mathrm{id}_\alpha$,
(iii) $\in_\alpha$ is extensional,
(iv) for any $\beta \leq \alpha$, $\in_\alpha \circ \left(i^\alpha_\beta \times i^\alpha_\beta\right) = \in_\beta$.

Functoriality will follow from (i) + (ii). Continuity will require no further proof as it will arise from the construction.

successor case Set $V_{\alpha+1} := P(V_\alpha) = \Omega^{V_\alpha}$ and $i^\alpha_\alpha = \mathrm{id}_{V_\alpha}$. By the universal property of the exponential, the map $\in_\alpha : V_\alpha \times V_\alpha \to \Omega$ uniquely determines a map $i : V_\alpha \to V_{\alpha+1}$ such that

$$\begin{array}{ccc} V_\alpha \times V_\alpha & & \\ {\scriptstyle \mathrm{id} \times i}\downarrow & \searrow^{\in_\alpha} & \\ V_\alpha \times \Omega^{V_\alpha} & \xrightarrow[\mathrm{ev}]{} & \Omega \end{array}$$

commutes. This map i must be monic by Lemma 6.1 and the induction hypothesis (iii). We set $i^{\alpha+1}_\alpha := i$, and for any $\beta < \alpha + 1$, we can now define

$$i_\beta^{\alpha+1} := i_\alpha^{\alpha+1} \circ i_\beta^\alpha.$$

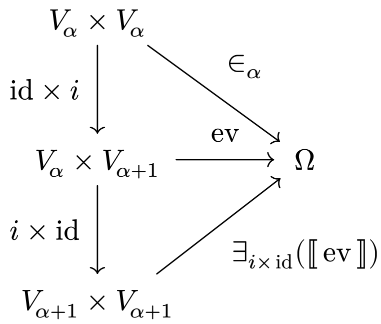
For a chain $\gamma \leq \beta \leq \alpha + 1$, and setting aside the trivial identity case (so $\beta \leq \alpha$),

$$i_\gamma^{\alpha+1} = i_\alpha^{\alpha+1} \circ i_\gamma^\alpha = i_\alpha^{\alpha+1} \circ i_\beta^\alpha \circ i_\gamma^\beta = i_\beta^{\alpha+1} \circ i_\gamma^\beta.$$

We define a new relation $\in_{\alpha+1} \colon V_{\alpha+1} \times V_{\alpha+1} \to \Omega$ as the subobject

$$[\![\, x, y : V_{\alpha+1} \mid \exists x' \in V_\alpha.(x = ix' \wedge \mathrm{ev}(x', y)) \,]\!] \subseteq V_{\alpha+1} \times V_{\alpha+1},$$

which is definitionally equal to $\exists_{i \times \mathrm{id}}([\![\, (x, y).\, \mathrm{ev}(x, y) \,]\!])$. With this we can complete the diagram as follows, and know that it commutes:



Finally we will prove internally that $\in_{\alpha+1}$ is extensional. Let $x, y : V_{\alpha+1}$ be arbitrary and suppose

$$\forall z : V_{\alpha+1}.(z \in_{\alpha+1} x \Leftrightarrow z \in_{\alpha+1} y).$$

Replacing $\in_{\alpha+1}$ with its definition gives

$$\forall z : V_{\alpha+1}.[(\exists z' : V_\alpha.z = i(z') \wedge \mathrm{ev}(x, z')) \Leftrightarrow (\exists z' : V_\alpha.z = i(z') \wedge \mathrm{ev}(y, z'))].$$

Since i is monic, and the corresponding internal function injective (see Lemma 5.12), the existence of a z' such that $z = iz'$ is unique, and we can rewrite the formula as

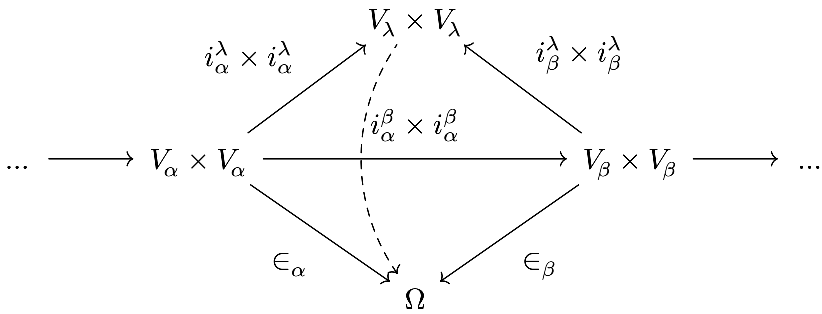
$$\forall z : V_{\alpha+1}.[\exists z' : V_\alpha.z = i(z') \wedge (\mathrm{ev}(x, z') \Leftrightarrow \mathrm{ev}(y, z'))].$$

Applying this statement only to elements of $V_{\alpha+1}$ of the form $i(z')$, we get

$$\forall z' : V_\alpha.(\mathrm{ev}(x, z') \Leftrightarrow \mathrm{ev}(y, z')),$$

so by Lemma 5.13, $x = y$. This concludes the proof that $\in_{\alpha+1}$ is extensional, and with that all we needed to prove in the successor case.

limit case For a limit ordinal λ set $V_\lambda := \mathrm{colim}_{\alpha<\lambda} V_\alpha$ and i_α^λ to be the arising maps from each $\alpha < \lambda$ to the colimit. It is here that we make real use of requiring $\mathcal{E}$ to be a Grothendieck topos, which guarantees the existence of this colimit. This definition naturally satisfies functoriality and continuity, it only remains to define an extensional relation $\in_\lambda \colon V_\lambda \times V_\lambda \to \Omega$. We illustrate how $\in_\lambda$ arises in the following diagram.



(6.6)

A crucial fact here is that here, the product commutes with the colimit, i.e. that

$$\operatorname{colim}(V_\alpha \times V_\alpha)_{\alpha<\lambda} = \operatorname{colim} V_\alpha \times \operatorname{colim} V_\alpha = V_\lambda \times V_\lambda,$$

While this is not general true in general, here we can make use that the ordinals up to λ, being linearly ordered, are *filtered*, and that finite limits (in this case a product) commute with filtered colimits in a Grothendieck topos[10]. By the IH, the maps $(\in_\alpha\colon V_\alpha \times V_\alpha \to \Omega)_{\alpha<\lambda}$ form a cocone of the same diagram, so by the UP of $V_\lambda \times V_\lambda$, we get a unique map $\in_\lambda\colon V_\lambda \times V_\lambda \to \Omega$ making all diagrams commute.

Finally we must prove this map is extensional, i.e. that

$$[\![\, x, x' : V_\lambda \mid \top \,]\!] = [\![\, x, x' : V_\lambda \mid (\forall z. z \in_\lambda x \Leftrightarrow z \in_\lambda x') \Rightarrow x = x' \,]\!]. \tag{6.7}$$

The key intuition here should be that $\mathcal{E}$ believes that for every $x : V_\lambda$, $x = i_\alpha^{\lambda(y)}$ for some α and some $y : V_\alpha$, at which point we will be able to apply the induction hypothesis. It is not straightforward to connect the external intuition on ordinals to this internal statement. The following map will be helpful:

$$\begin{aligned} I : \operatorname{Sub}(V_\lambda \times V_\lambda) &\to \prod_{\alpha<\lambda} \operatorname{Sub}(V_\alpha \times V_\alpha), \\ S &\mapsto \left(i_\alpha^\lambda \times i_\alpha^\lambda\right)^*(S)_{\alpha<\lambda}. \end{aligned}$$

First notice that I is injective: Suppose that for every $\alpha < \lambda$,

$$\left(i_\alpha^\lambda \times i_\alpha^\lambda\right)^*(S) = \left(i_\alpha^\lambda \times i_\alpha^\lambda\right)^*(T)$$

in $\operatorname{Sub}(V_\alpha \times V_\alpha)$. By Lemma 1.14, this is equivalent to supposing

$$\varphi_S \circ \left(i_\alpha^\lambda \times i_\alpha^\lambda\right) = \varphi_T \circ \left(i_\alpha^\lambda \times i_\alpha^\lambda\right)$$

By its universal property, the maps $\left(i_\alpha^\lambda\right)_{\alpha<\lambda}$ into the colimit V_λ are jointly epi, (i.e. epimorphisms for each $\alpha < \lambda$), so we get $S = T$.

All this was to argue that to get (6.7), it is enough to show that

$$I([\![\, x, x' : V_\lambda \mid \top \,]\!]) = I([\![\, x, x' : V_\lambda \mid (\forall z. z \in_\lambda x \Leftrightarrow z \in_\lambda x') \Rightarrow x = x' \,]\!]),$$

or rather that for all $\alpha < \lambda$,

$$\begin{aligned} [\![\, x, x' : V_\alpha \mid \top \,]\!] &= \left(i_\alpha^\lambda \times i_\alpha^\lambda\right)^*([\![\, x, x' : (\forall z. z \in_\lambda x \Leftrightarrow z \in_\lambda x') \Rightarrow x = x' \,]\!]) \\ &= [\![\, x, x' : V_\alpha \mid \left(\forall z : V_\lambda . z \in_\lambda i_\alpha^\lambda(x) \Leftrightarrow z \in_\lambda i_\alpha^\lambda(x')\right) \Rightarrow i_\alpha^\lambda(x) = i_\alpha^\lambda(x') \,]\!]. \end{aligned} \tag{6.8}$$

[10] We won't disrupt this proof with an explanation of this fact, consult [Mac78, IX.2 Thm 1] for the details.

The last equality holds by Lemma 5.2. Using the extensionality of $\in_\alpha$ for each $\alpha < \lambda$, we now get

$$
\begin{aligned}
&[\![x, x' : V_\alpha \mid \forall z : V_\lambda.\big(z \in_\lambda i^\lambda_\alpha(x) \Leftrightarrow z \in_\lambda i^\lambda_\alpha(x')\big)]\!] \\
&\qquad \le [\![x, x' : V_\alpha \mid \forall z : V_\alpha.\big(i^\lambda_\alpha(z) \in_\lambda i^\lambda_\alpha(x) \Leftrightarrow i^\lambda_\alpha(z) \in_\lambda i^\lambda_\alpha(x')\big)]\!] \\
&\qquad \le [\![x, x' : V_\alpha \mid \forall z : V_\alpha.z \in_\alpha x \Leftrightarrow z \in_\alpha x']\!] && \text{by IH (iii)}, \\
&\qquad \le [\![x, x' : V_\alpha \mid x = x']\!] && \text{by IH (iv)}, \\
&\qquad \le [\![x, x' : V_\alpha \mid i^\lambda_\alpha(x) = i^\lambda_\alpha(x')]\!].
\end{aligned}
$$

Using the adjointness of $\wedge$ and $\Rightarrow$, this gives Equation (6.8), concluding the proof.

□

6.2 The Interpretation of ZF in V

It is immediately apparent that we will not be able to frame the hierarchy V as a model or even a $\Sigma_\in$-structure in $\mathcal{E}$ in the sense introduced in Section 5.1, as V is not a small limit. There can be many approaches to expressing V as a model anywayin the setting of producing independence results in particular, such as assuming the existence of a set model to work in from the beginning, or an inaccessible cardinal, or using language able to deal with large limits in a topos. We will work with a formal approach that imitates the definition we would get if the colimit $\mathrm{colim}_{\alpha \in \mathrm{Ord}} V_\alpha$ existed. This won't give us a single structure serving as a model, but it will give us a translation satisfying the conditions of Theorem 2.1.

In fact we do have a $\Sigma_\in$-structure in $(V_\alpha, \in_\alpha)$ for every $\alpha \in \mathrm{Ord}$ - only that we do not expect these to satisfy all axioms of **ZF**. We *would* expect this of an object such as V, only that it does not exist.We will use the relation $V_\alpha \vDash -$, as defined in Definition 5.3, to define $V \vDash -$ formally as follows:

Definition 6.3 Let $\Gamma \mid \varphi(\overline{x})$ be a formula over **ZF**. We say that

$$V \vDash \varphi(\overline{x}) \quad \text{iff} \quad V_\alpha \vDash \varphi(\overline{x}) \text{ for every } \alpha \in \mathrm{Ord}\,.$$

Read V *believes* φ.

Note that none of this depended on the precise definition of V, and that the same method could be used for any functor $V' : \mathrm{Ord} \to \mathcal{E}$, as long as it provided a $\Sigma_\in$ structure in every step. If this would then be a model of **ZF** is another matter – in this case, all axioms can be shown.

We have done this for the axiom of extensionality already. We won't show the other axioms. They can be found in [Fou80,], albeit using a separate definition of interpretation that should be equivalent. In any case we will get them for free when connecting the relation to the Boolean-valued model $V^{\mathbb{B}}$.

We omit the proof that $V^{\mathbb{B}}$ under this definition satisfies soundness.

6.3 The Forcing Relation in $V^{\mathrm{Sh}(\mathbb{P})}$

Similarly to how we defined the *believes* relation on V through that relation on all V_α, in the case of $\mathcal{E} = \mathrm{Sh}(\mathbb{P})$ we now define a forcing relation "on $V^{\mathrm{Sh}(\mathbb{P})}$".[11]

[11] Of course this could be set up for Grothendieck toposes in general in the same way.

We define a *forcing relation* $\Vdash$ on $V^{\mathrm{Sh}(\mathbb{P})}$ between a formula $\Gamma \mid \varphi(\overline{x})$ over $\Sigma_{\in}$, an element $p \in \mathbb{P}$ and $\overline{u} \in [\![\Gamma]\!]_{V_\alpha^{\mathrm{Sh}(\mathbb{P})}})$ by the same recursive clauses as in Theorem 5.10, but adapt the quantifiers to range over the entire hierarchy. We also add cases for the atomic cases - they are exactly the same as $p \Vdash -$ in $V_\alpha^{\mathrm{Sh}(\mathbb{P})}$, but can be simplified now that we have the specific relation $\in$, and that we know that there are no function symbols over $\mathbf{ZF}_{\in}$, so the only terms will be variables.

Let $\Gamma \mid \varphi_1, \varphi_2$ and $\Gamma, y \mid \psi$ be formulas over $\mathbb{T}$, $p \in \mathbb{P}$ and let $z \in [\![\Gamma]\!](p)$. Then

(i) $p \Vdash \top$

(ii) $p \Vdash \varphi(z) \wedge \psi(z)$ iff $p \Vdash \varphi(z) \wedge p \Vdash \psi(z)$

(iii) $p \Vdash \bot$ iff $p = 0$

(iv) $p \Vdash \varphi(z) \vee \psi(z)$ iff for any $q \leq p$ there is an $r \leq q$ s.t. $r \Vdash \varphi(z)$ or $r \Vdash \psi(z)$,

(v) $p \Vdash \varphi(z) \Rightarrow \psi(z)$ iff for any $q \leq p, q \Vdash \varphi(z)$ implies $q \Vdash \psi(z)$.

(vi) $p \Vdash \exists y. \varphi(z, y)$ iff for any $q \leq p$ there are $r \leq q, \beta \in \mathrm{Ord}$ and $w \in V_{\beta(r)}^{\mathrm{Sh}(\mathbb{P})}$ such that $r \Vdash (z, w)$,

(vii) $p \Vdash \forall y. \varphi(z, y)$ iff for any $q \leq p$, any $\beta \in \mathrm{Ord}$ and any $w \in V_{\beta(q)}^{\mathrm{Sh}(\mathbb{P})}, q \Vdash \varphi(z, w)$,

(viii) $p \Vdash z_i = z_j$ iff $z_i = z_j$,

(ix) $p \Vdash z_i \in z_j$ iff $(\in_\alpha)(p)\big(z_i, z_j\big) = p$.

We now attempt to relate $V^{\mathbb{B}}$ and $V^{\mathrm{Sh}(\mathbb{P})}$, as well as their respective forcing relations and interpretation of formulas. We will do this level-wise for each ordinal. The first step for this is to make $V_\alpha^{\mathbb{B}}$ into sheaves. We will then set up a hierarchy in $\mathrm{Sh}(P)$ similarly to how we did for $V^{\mathrm{Sh}(\mathbb{P})}$ and finally set these hierarchies in relation. In a conclusion we will discuss how to extend this connection to the models $V^{\mathbb{B}}$ and $V^{\mathrm{Sh}(\mathbb{P})}$ themselves and how the forcing relations relate to each other.

6.4 The Sheaf $V^{\mathbb{B}}$

Think first of the special case $\mathbb{P} = \{*\}$. In this trivial setup, $\mathrm{Sh}(\mathbb{P}) = \mathbf{Set}$ and $V^{\mathrm{Sh}(\mathbb{P})}$ would be the usual von Neumann universe V. For Boolean-valued models, we'd be looking at V^2. As we saw in Section 2, these two do not directly correspond to each other, as V^2 has "more" elements - however we saw that for $V_2^{\mathbb{B}}$, these functions would overlap when factored by equality. We will continue with this approach by making two elements x, y equal whenever $p \Vdash x = y$ for a fixed p.

Definition 6.4 For any ordinal α, the quotient $V_\alpha^{\mathbb{B}}/(- \Vdash . = .)$ is a functor $\mathbb{P} \to \mathbf{Set}$. It maps $p \mapsto V_\alpha^{\mathbb{B}}/(p \Vdash . = .)$. A morphism $p \leq q$ is mapped to the inclusion

$$V_\alpha^{\mathbb{B}}/(q \Vdash . = .) \to V^{\mathbb{B}}/(p \Vdash . = .),$$
$$[u]_q \mapsto [u]_p.$$

This definition works because if $p \leq q$ and $q \Vdash \varphi$, then $p \Vdash \varphi$.

For better readability we will shorten the notation of this functor to $V_\alpha^{\mathbb{B}}/(-)$.

Lemma 6.5 *For any ordinal α, the functor $V_\alpha^{\mathbb{B}}/(-)$ is a sheaf.*

Proof. We will prove this by following the definition, simplifying whenever the thin structure of $\mathbb{P}$ allows. Let $p \in \mathbb{P}$, S be a sieve (down-closed) of $\mathbb{P}$ and suppose that S covers p, i.e. is dense below p. Further, let

$$\left\{[x_s]_s\right\}_{s \in S}$$

be a compatible family of Boolean functions $x_s : X_s \to \mathbb{B}$, where $X_s \subseteq V_\beta$ for some $\beta < \alpha$. In our case, compatibility means that for any $s, t \in S$ and any $r \leq s$ and $r \leq t$,

$$[x_s]_r = [x_t]_r.$$

In particular, $[x_s]_{s \wedge t} = [x_t]_{s \wedge t}$, or equivalently,

$$s \wedge t \Vdash x_s = x_t. \tag{6.9}$$

We want to show that there exists a unique gluing $[x]_p \in V^{\mathbb{B}}_{\alpha(p)}$ such that for every $s \in S$,

$$[x]_s = [x_s]_s.$$

We first prove uniqueness.Supposing we have two maps $[x]_p$ and $[x']_p$ that satisfy the gluing property. Then of course $[x']_s = [x]_s$ for any $s \in S$. By Lemma 2.11, $s \Vdash x' = x$ for all elements of the covering sieve S implies that $p \Vdash x = x'$, i.e. $[x]_p = [x']_p$.

As putative amalgamation of the compatible family $\left\{[x_s]_s\right\}_{s \in S}$ consider $[x]_p$ with

$$x : \bigcup_{s \in S} X_s \to \mathbb{B}, \tag{6.10}$$

$$y \mapsto \bigvee_{s \in S} (\llbracket y \in x_s \rrbracket \wedge s). \tag{6.11}$$

In a setting very similar to ours but without using the term *sheaf*, Bell calls the amalgamation x in (6.10) a *mixture*, and the proof of the *Mixing Lemma* [Bel05, 1.25] exactly gives the gluing property that we require. We spell out the argument here for completeness.

Let $s \in S$. We first expand and then break down the desired gluing property

$$[x]_s = [x_s]_s$$

into the conjunction of two conditions ($\subseteq$) and ($\supseteq$) using the compatibility assumption as well as common inequalities in $V^{\mathbb{B}}$.

$$\begin{aligned}
& s \Vdash x = x_s, \\
\Leftrightarrow\ & s \leq \llbracket x = x_s \rrbracket, \\
\Leftrightarrow\ & s \leq \left(\bigwedge_{y \in \operatorname{dom} x} x(y) \Rightarrow \llbracket y \in x_s \rrbracket \right) \quad \wedge \quad \left(\bigwedge_{y \in X_s} x_{s(y)} \Rightarrow \llbracket y \in x \rrbracket \right), \\
\Leftrightarrow\ & \forall y \in \operatorname{dom} x . s \leq (x(y) \Rightarrow \llbracket y \in x_s \rrbracket) \text{ and } \forall y \in X_s . s \leq \left(x_{s(y)} \Rightarrow \llbracket y \in x \rrbracket \right), \\
\Leftrightarrow\ & \underbrace{\forall y \in \operatorname{dom} x . (s \wedge x(y)) \leq \llbracket y \in x_s \rrbracket}_{(\subseteq)} \text{ and } \underbrace{\forall y \in X_s . \left(s \wedge x_{s(y)} \right) \leq \llbracket y \in x \rrbracket}_{(\supseteq)}.
\end{aligned}$$

To get the gluing property we simply need to check these two conditions.
($\subseteq$) Let $y \in \operatorname{dom} x$ and $t \in S$. Then

$$
\begin{aligned}
s \wedge x(y) &= \bigvee_{t\in S} (s \wedge t \wedge [\![\, y \in x_t \,]\!]),\\
&\leq \bigvee_{t\in S} ([\![\, x_s = x_t \,]\!] \wedge [\![\, y \in x_t \,]\!]) \quad \text{by (6.9)},\\
&\leq [\![\, y \in x_s \,]\!].
\end{aligned}
$$

($\supseteq$) Let $y \in \operatorname{dom} x$ and $t \in S$. Then

$$
\begin{aligned}
s \wedge x_{s(y)} &\leq s \wedge [\![\, y \in x_s \,]\!],\\
&\leq x(y),\\
&\leq [\![\, y \in x \,]\!].
\end{aligned}
$$

□

Corollary 6.6 *The Boolean-valued hierarchy defines a functor*

$$
\begin{aligned}
V^{\mathbb{B}}_{\bullet}/(-) : \mathrm{Ord} &\to \mathrm{Sh}(\mathbb{P}),\\
\alpha &\mapsto V^{\mathbb{B}}_{\alpha}/(-),\\
(\beta \leq \alpha) &\mapsto \iota^{\alpha}_{\beta}.
\end{aligned}
$$

Proof. The assignment of objects is reasonable by Theorem 6.2. Let $\beta \leq \alpha$. As we know (6.2) that $V^{\mathbb{B}}_{\beta} \subseteq V^{\mathbb{B}}_{\alpha}$, we can take ι^{α}_{β} to be

$$
\begin{aligned}
\iota^{\alpha}_{\beta}(p) : V^{\mathbb{B}}_{\beta}/p &\to V^{\mathbb{B}}_{\alpha}/p,\\
[x]_p &\mapsto [x]_p.
\end{aligned}
$$

This clearly defines a functor. □

We will relate the sheaves in this hierarchy in a way analogous to the definition of $V^{\mathrm{Sh}(\mathbb{P})}$. We can also define an element relation $\in'$ on `VBs` that corresponds to the element relation in $V^{\mathbb{B}}$, but will also match up with the maps $\in_\alpha$ in $V^{\mathrm{Sh}(\mathbb{P})}$.

Definition 6.7 For any $\alpha \in \mathrm{Ord}$,

$$
\in' : V^{\mathbb{B}}_{\alpha}/(-) \times V^{\mathbb{B}}_{\alpha}/(-) \to \Omega
$$

is a map of sheaves defined by

$$
\in'\ (p)\big([x]_p, [y]_p\big) \mapsto [\![\, x \in y \,]\!]_{\mathbb{B}} \wedge p.
$$

Let's unpack what is happening in this map. We have

$$
V^{\mathbb{B}}_{\alpha}/p \times V^{\mathbb{B}}_{\alpha}/p \xrightarrow{[\![\, x \in y \,]\!]_{\mathbb{B}}} \mathbb{B} = \Omega(1) \xrightarrow{(-)\wedge p} \Omega(p).
$$

We must check that this maps sends different representatives to the same sieve. When $p \Vdash x = x' \wedge y = y'$, we can use the soundness in Lemma 2.9 to get

$$
\begin{aligned}
[\![x \in y]\!]_{\mathbb{B}} \wedge p &= [\![x \in y]\!]_{\mathbb{B}} \wedge p \wedge p, \\
&\leq [\![x \in y]\!]_{\mathbb{B}} \wedge [\![x = x']\!]_{\mathbb{B}} \wedge [\![y = y']\!]_{\mathbb{B}} \wedge p \quad \text{by assumption,} \\
&= [\![x \in y \wedge x = x' \wedge y = y']\!]_{\mathbb{B}} \wedge p \\
&\leq [\![x' \in y']\!]_{\mathbb{B}} \wedge p \quad \text{by soundness.}
\end{aligned}
$$

The reverse inequality is of course exactly analogous, resulting in $[\![x \in y]\!]_{\mathbb{B}} \wedge p = [\![x' \in y']\!]_{\mathbb{B}} \wedge p$ as we expected.

Showing that putting together these maps for every $p \in \mathbb{P}$ makes ev a morphism of sheaves is a straightforward argument with commutative diagrams.

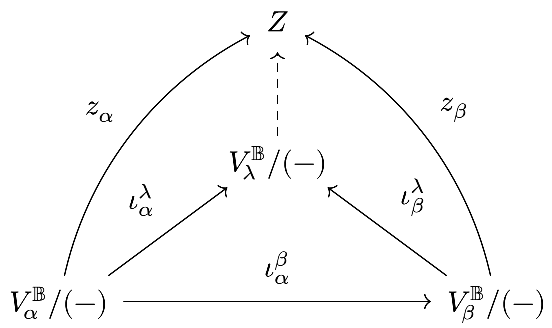
We will make use of this map when setting up $V_{\bullet}^{\mathbb{B}}/(-)$ as a cumulative hierarchy.

Lemma 6.8 *For a limit ordinal α, $V_{\lambda}^{\mathbb{B}}/(-)$ is the colimit of diagram $V_{<\lambda}^{\mathbb{B}}/(-)$.*

Proof. The maps $\iota_{\alpha}^{\lambda} : V_{\alpha}^{\mathbb{B}}/(-) \to V_{\lambda}^{\mathbb{B}}/(-)$ for every $\alpha < \lambda$ form a cocone of the diagram $V_{<\lambda}^{\mathbb{B}}/(-)$, all triangles in the cocone can easily be verified to commute.

Supposing we had some further cocone Z with maps $z_{\alpha} : V_{\alpha}^{\mathbb{B}}/(-) \to Z$, we can define a map $V_{\lambda}^{\mathbb{B}}/p \to Z(p)$ componentwise as follows: For any $[x]_p \in V_{\lambda}^{\mathbb{B}}/p$, there exists some $\alpha < \lambda$ such that $x \in V_{\alpha}^{\mathbb{B}}$. Then map $[x]_p \mapsto z_{\alpha}(p)[x]_p$. This mapping required a few choices for both representative x and ordinal α, but regardless of choice, the result will be the same - suppose $p \Vdash x = y$, and $y \in V_{\beta}^{\mathbb{B}}$, wlog $\alpha \leq \beta$. Then

$$
z_{\alpha}(p)\big([x]_p\big) = z_{\alpha}(p)\big([y]_p\big) = z_{\beta}(p)\big(\iota_{\alpha}^{\beta}\big([y]_p\big)\big) = z_{\beta}(p)\big([y]_p\big).
$$



By a similar argument, all triangles between the cocones commute. Checking that this map is componentwise a morphism of sheaves i.e. a natural transformation, is a straightforward exercise in commutative diagrams, as is checking naturality of the isomorphism for every $\alpha \leq \lambda$ (see (6.13)). □

Lemma 6.9 *For any ordinal α,*

$$
V_{\alpha+1}^{\mathbb{B}}/(-) = \Omega^{V_{\alpha}^{\mathbb{B}}/(-)}
$$

Proof.

The first step is to define an evaluation map

$$
\text{ev} : V_{\alpha}^{\mathbb{B}}/(-) \times V_{\alpha+1}^{\mathbb{B}}/(-) \to \Omega,
$$

by setting, for any $p \in \mathbb{P}$,

$$\mathrm{ev}(p)\big([x]_p, [y]_p\big) = \in' \big(\iota([x]_p), [y]_p\big)$$

We prove that this suggested evaluation map does in fact satisfy the universal property of the exponential, a diagrammatic reminder of which can be seen below.

$$\begin{array}{ccc} V_\alpha^{\mathbb{B}}/(-) \times Z & & \\ {\scriptstyle \mathrm{id}\times\hat{f}}\ \downarrow & \searrow^{f} & \\ V_\alpha^{\mathbb{B}}/(-) \times V_{\alpha+1}^{\mathbb{B}}/(-) & \xrightarrow[\mathrm{ev}]{} & \Omega \end{array}$$

Let $f : V_\alpha^{\mathbb{B}}/(-) \times Z \to \Omega$ and $p \in \mathbb{P}$, so

$$f(p) : V_\alpha^{\mathbb{B}}/p \times Z(p) \to \Omega(p).$$

The function $f(p)(-, z) : V_\alpha^{\mathbb{B}}/(-) \to \Omega(p)$ can be made into a Boolean function by including $\Omega(p) \hookrightarrow \Omega(1)$

$$\begin{aligned} \hat{f} : Z(p) &\to V_{\alpha+1}^{\mathbb{B}}/p \\ z &\mapsto [f(p)(-, z)]_p \end{aligned}$$

After seeing that this gives a natural transformation, we can check whether the diagram above commutes, that is, whether for any $(x, z) \in V_\alpha^{\mathbb{B}}(p) \times Z(p)$,

$$[\![\, x \in f(p)(-, z) \,]\!]_{\mathbb{B}} \wedge p = f(p)\big([x]_p, z\big)$$

The $\geq$-direction follows immediately from properties of Boolean-valued interpretation, as collected in [Bel05, Thm 1.17 (ii)]. For the $\leq$-direction, first expand by the definition of [|- $\in$ -|] to get

$$[\![\, x \in f(p)(-, z) \,]\!] \wedge p = \bigvee_{y \in V_\alpha(p)} [f(p)(y, z) \wedge [\![\, y = x \,]\!]_{\mathbb{B}} \wedge p].$$

Notice that it's enough to show that for all $q \in \mathbb{P}$ and $y \in V_{\alpha(p)}$,

$$q \leq f(p)\big([y]_p, z\big) \wedge [\![\, x = y \,]\!]_{\mathbb{B}} \wedge p \;\text{ implies }\; q \leq f(p)(x, z).$$

Suppose $q \leq f(p)\big([y]_p, z\big) \wedge [\![\, x = y \,]\!]_{\mathbb{B}} \wedge p$. Then $q \leq [\![\, y = x \,]\!]_{\mathbb{B}}$, so $[y]_q = [x]_q$. Applying the naturality of f twice, we get

$$\begin{aligned} q &\leq f(p)\big([y]_p, z\big) \wedge q = f(q)\big([y]_q, z\big) = f(q)\big([x]_q, z\big), \\ &\leq f(q)\big([x]_q, z\big) \wedge q = f(p)\big([x]_p, z\big) \end{aligned}$$

and with that we have proven the universal property of the exponential.

□

Just as we defined a forcing relation for $\big(V^{\mathrm{Sh}(\mathbb{P})}, \in\big)$, we can define one for $\big(V^{\mathbb{B}}, \in'\big)$. We will write it as $\Vdash'$ to differentiate them and avoid bulky indices. We prove that this forcing relation lines up with the forcing relation on the Boolean-valued model $V^{\mathbb{B}}$, which we will now denote $\underset{\mathbb{B}}{\Vdash}$.

Key fact to remember before we show this is the inequality for any formula $\Gamma \mid \varphi$ over **ZF** and any $u, v \in {V^{\mathbb{B}}}^n$,

$$p \underset{B}{\Vdash} u = v \text{ and } p \underset{\mathbb{B}}{\Vdash} \varphi(u), \text{then } p \underset{\mathbb{B}}{\Vdash} \varphi(v).$$

This is a direct application of Lemma 2.9.

Theorem 6.10 *For any formula $V^n \mid \varphi$ over* **ZF**, *$p \in \mathbb{P}$ and any $\overline{u} \in V^{\mathbb{B}^n}$,*

$$p \Vdash' \varphi\big([\overline{u}]_p\big) \quad \text{iff} \quad p \underset{\mathbb{B}}{\Vdash} \varphi(\overline{u}).$$

Proof. We prove this by induction over the formula φ. If we compare the definition of $\Vdash'$ with the characterisation of $\Vdash$ in Theorem 2.10, we see that they are entirely identical aside from their base cases. If we can prove the equivalence for the atomic cases, we are done.

For equality, the terms will necessarily give a formula of the form $\overline{x} : V^n \mid x_i = x_j$. Let $p \in \mathbb{P}$ and $\overline{u} \in \left(V^{\mathbb{B}}\right)^n$. Then

$$\begin{aligned} p \Vdash' [u_i]_p = \left[u_j\right]_q \;\; &\text{iff} \;\; [u_i]_p = \left[u_j\right]_p \\ &\text{iff} \;\; p \underset{\mathbb{B}}{\Vdash} u_i = u_j. \end{aligned}$$

Similarly for a formula $\overline{x} : V^n \mid x_i \in x_j$, for $p \in \mathbb{P}$ and $\overline{u} \in \left(V^{\mathbb{B}}\right)^n$,

$$\begin{aligned} p \Vdash' [u_i]_p \in \left[u_j\right]_p \;\; &\text{iff} \;\; (\in')(p)\Big([u_i]_p, \left[u_j\right]_p\Big) = p \\ &\text{iff} \;\; [\![u_i \in u_j]\!] \wedge p = p \\ &\text{iff} \;\; [\![u_i \in u_j]\!] \leq p \\ &\text{iff} \;\; p \underset{\mathbb{B}}{\Vdash} u_i \in u_j. \end{aligned}$$

□

6.5 Relating $V^{\mathbb{B}}$ and $V^{\mathrm{Sh}(\mathbb{P})}$

Theorem 6.11 *There is a natural isomorphism between the functors $V^{\mathrm{Sh}(\mathbb{P})}, V_{\bullet}^{\mathbb{B}}/(-) : \mathrm{Ord} \to \mathrm{Sh}(\mathbb{P})$,*

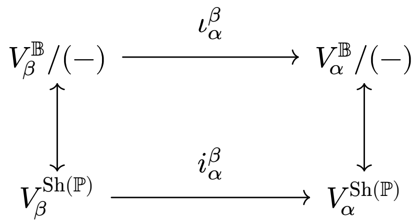
$$\theta : V^{\mathrm{Sh}(\mathbb{P})} \xrightarrow{\cong} V_{\bullet}^{\mathbb{B}}/(-).$$

Furthermore, for every $\alpha \in \mathrm{Ord}$,

$$\begin{array}{ccc} V_\alpha^{\mathrm{Sh}(\mathbb{P})} \times V_\alpha^{\mathrm{Sh}(\mathbb{P})} & & \\ \downarrow \theta_\alpha & \searrow^{\in_\alpha} & \\ V_\alpha^{\mathbb{B}}/(-) \times V_\alpha^{\mathbb{B}}/(-) & \xrightarrow[\in']{} & \Omega \end{array} \tag{6.12}$$

commutes.

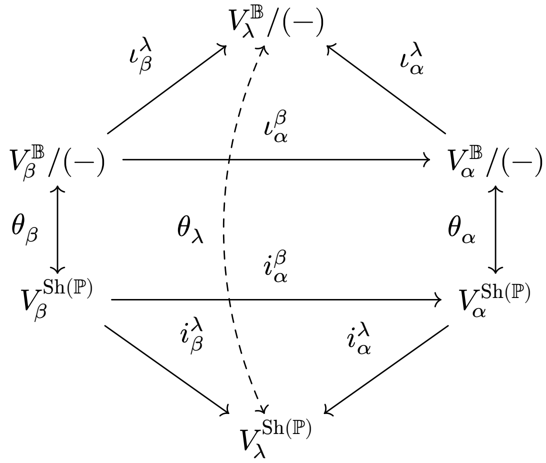
Proof. We will prove this by transfinite induction over Ord. In every step β of the induction we will show the object-wise isomorphism, as well as naturality up to that object, i.e. that the following diagram commutes for every $\alpha \leq \beta$.



(6.13)

limit step If we can show that $V^{\mathbb{B}}_{\lambda}/(-)$ is the colimit of the diagram $V^{\mathrm{Sh}(\mathbb{P})}_{<\lambda}$, we will have proven $V^{\mathbb{B}}_{\lambda}/(-) \cong V^{\mathrm{Sh}(\mathbb{P})}_{\lambda}$. By the induction hypothesis, the diagrams $V^{\mathrm{Sh}(\mathbb{P})}_{<\lambda}$ and $V^{\mathbb{B}}_{<\lambda}/(-)$ are isomorphic, and by Lemma 6.8, $V^{\mathbb{B}}_{\lambda}/(-)$ is the colimit of this diagram.

So any $\alpha \leq \beta$, the following diagram commutes:



From this it is easy to read that two colimits are colimits of each others diagrams, and thus isomorphic. The commutativity of (6.13) follows as it is a subdiagram of the above.

The commutativity of (6.12) follows from the universal property of $V_{\lambda} \times V_{\lambda}$. We defined $\in_{\lambda}$ as the *unique* map such that all triangles in diagram (6.6), would commute. Clearly by passing through isomorphism $\theta_{\lambda} \times \theta_{\lambda}$,

$$\in' \circ (\theta_{\lambda} \times \theta_{\lambda})$$

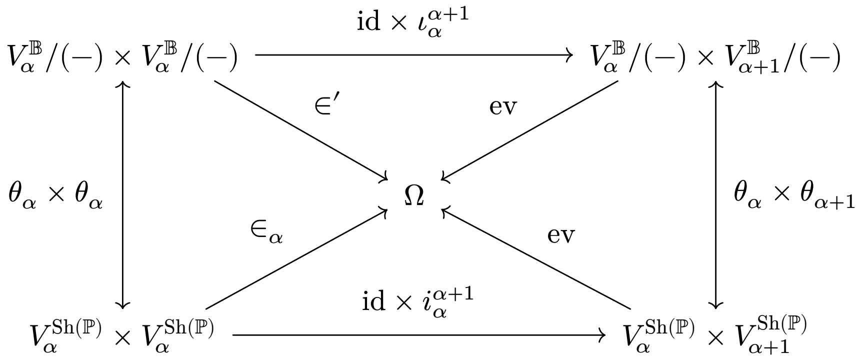
makes the same triangles commute, and so

$$\in' \circ (\theta_{\lambda} \times \theta_{\lambda}) = \in_{\lambda} .$$

successor step By the induction hypothesis, $V^{\mathrm{Sh}(\mathbb{P})}_{\alpha} \cong V^{\mathbb{B}}_{\alpha}/(-)$, and the exponential functor preserves isomorphisms, so $\Omega^{V^{\mathrm{Sh}(\mathbb{P})}_{\alpha}} \cong \Omega^{V^{\mathbb{B}}_{\alpha}/(-)}$. Then by Lemma 6.9,

$$V^{\mathrm{Sh}(\mathbb{P})}_{\alpha+1} \cong V^{\mathbb{B}}_{\alpha+1}/(-).$$

By property of this exponential, and by the induction hypothesis (6.12), all triangles in the following diagram commute:



It is easy to see that the outer square commutes when composed with ev, i.e.

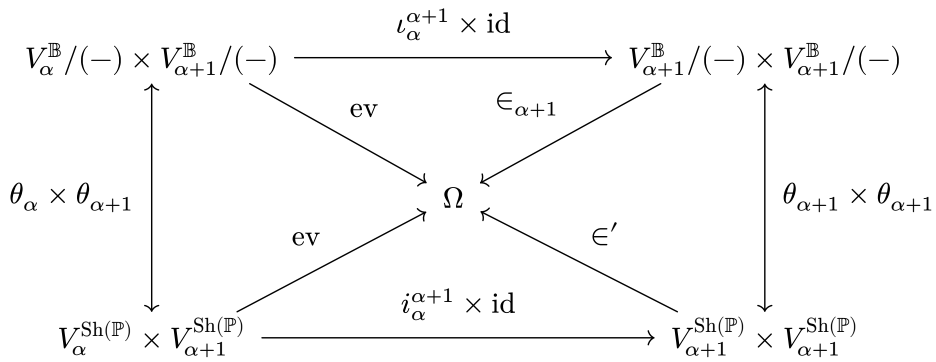
$$\mathrm{ev} \circ (\theta_\alpha, i_\alpha^{\alpha+1} \circ \theta_\alpha) = \mathrm{ev} \circ (\theta_\alpha, \theta_{\alpha+1} \circ \iota_\alpha^{\alpha+1}))$$

Lemma 5.13 then applies exactly and gives

$$i_\alpha^{\alpha+1} \circ \theta_\alpha = \theta_{\alpha+1} \circ \iota_\alpha^{\alpha+1},$$

that is the commutativity of (6.13).

By extending this same diagram to the right, we get:



This time the outer square is known to commute, as are all but the rightmost triangle, which corresponds to (6.12). Simply following this diagram and using that we know several of these morphisms to be monics, we can conclude that this last triangle also commutes.

□

Finally we prove that this natural isomorphism preserves the forcing relations.

Theorem 6.12 *Let* $\Gamma \mid \varphi(\overline{x})$ *with be a formula over* **ZF**, $p \in \mathbb{P}$, α *be an ordinal and* $\overline{u} \in \left(V_\alpha^{\mathrm{Sh}(\mathbb{P})}(p)\right)^n$. *Then*

$$p \Vdash \varphi(\overline{u}) \;\text{ iff }\; p \Vdash' \varphi(\theta_\alpha^n(p)(\overline{u}))$$

Proof. We prove this by induction over the formula φ. Due to the analogous definition of $\Vdash$ and $\Vdash'$, it is clearly sufficient to show the equivalence for the atomic cases.

For equality, the terms will necessarily give a formula of the form $\overline{x} : V^n \mid x_i = x_j$. Let $p \in \mathbb{P}$ and $\overline{u} \in \left(V_\alpha^{\mathrm{Sh}(\mathbb{P})}(p)\right)^n$. Then

$$
\begin{aligned}
p \Vdash x_i = x_j \text{ for } \overline{u} \quad &\text{iff} \quad \pi_i(\overline{u}) = \pi_j(\overline{u}) \\
&\text{iff} \quad \theta_\alpha(p)(\pi_i(\overline{u})) = \theta_\alpha(p)\big(\pi_j(\overline{u})\big) \\
&\text{iff} \quad \pi_i(\theta_\alpha^n(\overline{u})) = \pi_j(\theta_\alpha^n(\overline{u})) \\
&\text{iff} \quad p \Vdash' x_i = x_j \text{ for } \theta^n(\overline{u}).
\end{aligned}
$$

Similarly, for a formula $\overline{x} : V^n \mid x_i \in x_j$, for $p \in \mathbb{P}$ and $\overline{u} \in \left(V^{\mathbb{B}}\right)^n$,

$$
\begin{aligned}
p \Vdash x_i \in x_j \text{ for } \overline{u} \quad &\text{iff} \quad (\in_\alpha)(p)\big(u_i, u_j\big) = p \\
&\text{iff} \quad (\in' \circ\, \theta_\alpha^2)(p)\big(u_i, u_j\big) = p \qquad\qquad \text{(by Theorem 6.11)} \\
&\text{iff} \quad \in' (p)\big(\theta_\alpha(p)(u_i), \theta_\alpha(p)\big(u_j\big)\big) = p \\
&\text{iff} \quad p \Vdash' x_i \in x_j \text{ for } \theta_\alpha^n(\overline{u})
\end{aligned}
$$

□

Since a model believes a statement whenever it is forced by the top element, we get the following immediate corollary.

Corollary 6.13 *For any closed formula φ over* **ZF**, $V^{\mathrm{Sh}(\mathbb{P})} \vDash \varphi$ *if and only if* $V^{\mathbb{B}} \vDash \varphi$.

7 Conclusion

We briefly summarize what we have managed to show witihn this thesis and mention the various further directions that could be taken concerrning topos theory in its formal connections to forcing.

Starting with a poset $\mathbb{P}$, we have constructed the Grothendieck topos $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ and connected it to forcing in various ways:

A. We have seen that a cumulative hierarchy can be defined in any Grothendieck topos, and that in $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ this hierarchy models **ZF** and there is a natural isomorphism

$$V^{\mathrm{Sh}(\mathbb{P})} \cong V^{\mathbb{B}}_{\bullet}/(-)$$

between this hierarchy and the Boolean-valued model $V^{\mathbb{B}}$ when understood as a sheaf.

B. We established that the Kripke-Joyal semantics in $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ reproduces the usual Boolean forcing relation: the recursive clauses defining truth in $V^{\mathrm{Sh}(\mathbb{P})}$ agree with those for $V^{\mathbb{B}}$, so that both models force exactly the same formulas at each condition $p \in \mathbb{P}$.

C. In Section 4 we saw that $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ is the classifying topos of the theory of generic filters

$$\mathrm{Sh}(\mathbb{P}, \neg\neg) \cong \mathbf{Set}[\mathbb{T}_G]$$

and saw some similarities to the relation between $\mathbb{P}$ and the Boolean algebra $\mathbb{B}$ in

$$\mathrm{Sh}(\mathbb{P}, \neg\neg) \cong \mathrm{Sh}(\mathrm{Dn}_{\neg\neg}(\mathbb{P})).$$

Beyond making these connections, we have also studied $\mathrm{Sh}(\mathbb{P}, \neg\neg)$ as a special case of a Grothendieck topos, for which a cumulative hierarchy and a forcing relation can more generally be defined. This has allowed us to understand Grothendieck toposes as a generalization of set-theoretic forcing models.

This work has only scratched the surface of the formal connections between forcing in topos theory and set theory, and opens several paths to further exploration.

The most immediate is perhaps to develop a more comprehensive understanding of the role generic filters. Although Section 4 establishes the generic filter of $\mathbb{P}$ as a universal model for $\mathbb{T}_G$, for a deeper understanding, we would track how the filter G itself arises internally when one carries out the forcing argument, and how it relates to the independence result. It would be interesting to see how the language of classifying toposes could be helpful in forcing, as this is not a view explored in [MM94] when investigating the Cohen Topos for instance.

Other elements in forcing, while not strictly necessary when only proving independence results, have paralels in topos theory as well. The passage from Boolean-valued models to classical set-theoretic models for instance can be compared to making a topos two-valued by the filter quotient construction. One may also see that different set-theoretic approaches to managing size issues and using countable transitive models could be imitated in the topos theoretic setting. We didn't include these steps as they aren't necessary to produce independence results, but it may still be useful and direct connection in other use-cases.

One could also analyse how classical forcing arguments show up in the categorical setting, examining how combinatorial properties, the countable chain condition or the preservation of ordinals manifest in $\mathrm{Sh}(\mathbb{P}, \neg\neg)$. This is quite thoroughly explored in [MM94, IV]. But a study of the connections from the point of view of intuitionistic higher order logic, based on theory presented in [LSS94] for instance, could provide the bigger picture.

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