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**Rational Herglotz-Nevanlinna functions of several variables**

av

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# Rational Herglotz-Nevanlinna functions of several variables

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**Abstract**

A Herglotz-Nevanlinna function of several variables is a holomorphic function from the poly-upper half-plane to the closed upper half-plane. A recent development shows that a function of this type admits an integral representation determined by a set of constants and a particular positive Borel measure, known as a Nevanlinna measure. The appearance of the representation has sparked further research on Herglotz-Nevanlinna functions, often with emphasis on the class of Nevanlinna measures.

This thesis is concerned with the class of real-rational functions, which are the rational Herglotz-Nevanlinna functions whose boundary values on $\mathbb{R}^n$ are real. The theory on real-rational functions is rather sparse, and many of the results seem to lack formal proofs. This leads to the first of two aims of this thesis, which is to provide a more complete overview of real-rational functions of several variables than what currently exists in the literature. The results include, but are not limited to, a characterization of the real-rational functions in terms of the structure of the functions themselves, a support theorem for Nevanlinna measures representing real-rational functions, an invariance property of real-rational functions and their representing Nevanlinna measures, and an exact description of the relation between any real-rational function of two variables with an affine denominator (possibly after a biholomorphic change of variables) and its representing Nevanlinna measure.

The second aim of this thesis is to present a result on real-rational functions. The result and its context was communicated to the author by the thesis supervisor and, to the best of our knowledge, has not previously appeared in the literature. Specifically, it is shown that a real-rational function of two variables with a denominator that is a product of two affine factors has a nontrivial decomposition into a sum of two real-rational functions.



**Sammanfattning**

En Herglotz-Nevanlinna funktion av flera variabler är en holomorf funktion från det poly-övre halvplanet till det slutna övre halvplanet. Ett nytt resultat från 2019 visar att en funktion av den här typen har en integralframställning som bestäms av en uppsättning konstanter och ett särskilt positivt Borel-mått, känt som ett Nevanlinna-mått. Introductionen av integral-framställningen har inspirerat fortsatt forskning på Herglotz-Nevanlinna funktioner, ofta med tonvikt på klassen av Nevanlinna-mått.

Detta examensarbete berör klassen av reell-rationella funktioner, vilka är de rationella Herglotz-Nevanlinna funktionerna vars randvärden på $\mathbb{R}^n$ är reella. Teorin kring reell-rationella funktioner är utspridd över flera artiklar och många resultat verkar sakna formella bevis. Detta leder till arbetets första av två syften, nämligen att ge en mer heltäckande översikt av reell-rationella funktioner av flera variabler än vad som finns tillgängligt i litteraturen idag. Resultaten innefattar bland annat en karakterisering av de reell-rationella funktionerna i termer av deras egen struktur, en sats om stödet till de Nevanlinna-mått som representerar reell-rationella funktioner, en invariansegenskap hos reell-rationella funktioner och deras representerande Nevanlinna-mått, samt en exakt beskrivning av sambandet mellan varje reell-rationell funktion av två variabler vars nämnare är affin (eventuellt efter ett biholomorft byte av koordinater) och dess motsvarande Nevanlinna-mått.

Det andra syftet med arbetet är att presentera ett resultat för reell-rationella funktioner. Resultatet och dess kontext har förmedlades till författaren av handledaren och har, såvitt vi vet, inte tidigare publicerats. Mer specifikt visas att en reell-rationell funktion av två variabler vars nämnare är en produkt av två affina faktorer kan skrivas som en summa av två reell-rationella funktioner.

**Acknowledgments**

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# Contents

# 1 Introduction

A Herglotz-Nevanlinna function of several variables is a holomorphic function from the poly-upper half-plane to the closed upper half-plane, see Definition 2.8. A recent development by [LN19] shows that there is a one-to-one correspondence between the class of Herglotz-Nevanlinna functions of $n$ variables and the family of triples $(a, b, \mu)$, where $a \in \mathbb{R}$, $b \in [0, \infty)^n$ and $\mu$ is a positive Borel measure on $\mathbb{R}^n$ satisfying the growth condition (11) and the rather peculiar Nevanlinna condition (13). Such a measure is known as a Nevanlinna measure.

Given a Herglotz-Nevanlinna function $h$ and its corresponding triplet $(a, b, \mu)$, their relation is materialized by the integral representation formula

$$h(z) = a + \sum_{j=1}^{n} b_j z_j + \frac{1}{\pi^n} \int_{\mathbb{R}^n} K_n(z, t)\, d\mu(t).$$

Here, $K_n$ is the kernel function (5). The appearance of this integral representation has inspired further research on Herglotz-Nevanlinna functions of several variables from the perspective of Nevanlinna measures, see e.g [LN20], [Ned20] and [NS24]. In [Ned20] product Nevanlinna measures are studied, whereas the focus of [LN20] and [NS24] leans towards the geometry of the support of Nevanlinna measures.

In the one-dimensional setting, the integral representation has been available for roughly a century. Since its first appearance, the theory of Herglotz-Nevanlinna functions of one variable has had numerous applications within e.g material sciences and electromagnetics; see [LO22] for an overview. A more concrete example is that the driving point impedance in an RLC circuit is essentially a rational Herglotz-Nevanlinna function ([Win08, Theorem 5.2]). For reasons that lie within pure or applied mathematics (usually both), certain classes of Herglotz-Nevanlinna functions have received particular attention. One of these is the class of rational Herglotz-Nevanlinna functions, which has been completely classified in terms of the parameters from the integral representation; given the Lebesgue decomposition of the Nevanlinna measure corresponding to a rational Herglotz-Nevanlinna function of one variable, its singular part is a finite sum of nonnegative multiples of Dirac measures on $\mathbb{R}$, and the Radon-Nikodym derivative of its absolutely continuous part is given by the boundary values on $\mathbb{R}$ of the imaginary part of the function. In the multi-dimensional setting, however, little is still known about rational Herglotz-Nevanlinna functions.

This thesis is concerned with rational Herglotz-Nevanlinna functions of several variables (see Definition 3.1), with a particular focus on those that are real on the distinguished boundary of the poly-upper half-plane. These are the real-rational functions (see Definition 3.4). Perhaps as expected, a rational Herglotz-Nevanlinna function is real-rational if and only if its coefficients are real (Lemma 3.5). Two of the main results of this thesis are tailored for the two variable setting, the first of which is Theorem 3.18 – later referred to as the characterization theorem – as it provides an exact description of the relation between any real-rational function of two variables with an affine denominator and its corresponding Nevanlinna measure. The second result, Theorem 4.2 – later referred to as the decomposition theorem – shows that every real-rational function of two variables with a denominator that is a product of two affine factors decomposes into a sum of real-rational functions.

Both the characterization and decomposition theorems are extended by an automorphism invariance property of Nevanlinna measures of real-rational functions (Proposition 3.14) to a slightly more general setting, where the denominator only need to be as specified after a biholomorphic change of variables. Specifically, under a few assumptions, it is shown that if $\mu$ is the representing measure of a real-rational function $h$ then $h \circ f$ is real-rational with $\det(J_f(t)) f^* \mu$ as its representing measure.

As usually is the case, there are limitations tied to the results. To begin with, the function

$$(z, w) \mapsto -\frac{w}{w^2 + zw - 1}$$

in (50) is an example of a real-rational function with a denominator of total degree 2 that neither Theorem 3.18 or Theorem 4.2 apply to (even after any biholomorphic change of coordinates). This means that a full classification of the subclass of real-rational functions with a denominator of total degree 2 remains open.

Another limitation lies in possible generalizations of the decomposition theorem; the intuition gained from the results presented in this thesis suggests that the real-rational function

$$(z, w) \mapsto -\frac{zw}{(zw-1)(z+w)} \tag{1}$$

can be decomposed into a sum of two real-rational functions, but it turns out that this is not the case.

The structure of the thesis is as follows. Relevant results from recent research is collected in Chapter 2, where some other background material is introduced in an attempt to make these results more assessable. In Chapter 3 the rational Herglotz-Nevanlinna functions are explored; we learn more about the structure of real-rational functions (see e.g Theorem 3.5) and prove the characterization theorem. Chapter 4 is mainly concerned with the decomposition theorem and touches on the topic of extremal Nevanlinna measures. It ends with some concluding remarks and a final example where the function (1) is explored.

# 2 Background

This chapter covers the relevant background and necessary prerequisites for the subsequent chapters of the thesis. Previous exposure to introductory graduate measure theory and complex analysis in one and several variables is assumed.

## 2.1 Measure theory

Most of the following notation and definitions are from either [Coh13] or [Fol13]. To begin with, denote the Borel $\sigma$-algebra by $\Sigma$ and denote the Lebesgue measure on $(\mathbb{R}^n, \Sigma)$ by $\lambda_n$ when $n > 1$, and $\lambda$ otherwise. Denote the Dirac measure on $\mathbb{R}$ with a point mass at $\alpha \in \mathbb{R}$ by $\delta_\alpha$. Let $(X, \Sigma, \mu)$ be a measure space equipped with a positive Borel measure $\mu$.

The *support* of $\mu$ is defined as

$$\operatorname{supp} \mu := \{x \in X : \forall \text{ open } U \ni x,\ \mu(U) > 0\}.$$

The following two lemmas are concerned with the support of $\mu$.

**Lemma 2.1.** *Let $A \subset X$ be open. If $\mu(A) = 0$, then $\operatorname{supp} \mu \subset X \setminus A$.*

*Proof.* Take $x \in A$. Because $A$ is open there is a neighborhood $U$ of $x$ contained in $A$. By assumption $0 \leq \mu(U) \leq \mu(A) = 0$, so $\mu(U) = 0$ and therefore $x \notin \operatorname{supp} \mu$. Hence, $x \in X \setminus A$ whenever $x \in \operatorname{supp} \mu$. □

**Lemma 2.2.** *The support of $\mu$ is closed.*

*Proof.* Consider its complement $\{x \in X : \exists \text{ open } U \ni x,\ \mu(U) = 0\}$. It will suffice to show that this set is open in $X$. To this end, take $x \in X \setminus \operatorname{supp} \mu$ and let $V \subset X$ be a neighborhood of $x$ satisfying $\mu(V) = 0$. Note that $V$ is a neighborhood of *every* point it contains, so then $V \subset X \setminus \operatorname{supp} \mu$. In particular, $x \in \operatorname{int}(X \setminus \operatorname{supp} \mu)$, hence $X \setminus \operatorname{supp} \mu = \operatorname{int}(X \setminus \operatorname{supp} \mu)$ as $x$ was arbitrary. □

Let $(Y, \Sigma)$ be another measure space and suppose that $f : X \to Y$ is measurable. The *push-forward* of $\mu$ by $f$ is the positive Borel measure $f_*\mu := \mu \circ f^{-1}$ on $(Y, \Sigma)$. It satisfies a change of variables formula.

**Theorem 2.3.** *Let $f : X \to Y$ be a measurable function between the measure spaces $(X, \Sigma, \mu)$ and $(Y, \Sigma, f_*\mu)$. Suppose $g : Y \to \overline{\mathbb{R}}$ is measurable. Then $g$ is $f_*\mu$-integrable if and only if $g \circ f$ is $\mu$-integrable, in which case*

$$\int_Y g \, d(f_*\mu) = \int_X g \circ f \, d\mu.$$

Analogously, given a bijection $f : Y \to X$ with a measurable inverse, the *pullback* of $\mu$ by $f$ is the positive Borel measure $f^*\mu := \mu \circ f$ on $(Y, \Sigma)$. It satisfies essentially the same change of variables formula as the pushforward but with $f$ exchanged for $f^{-1}$ in Theorem 2.3, as per the relation $f^*\mu = (f^{-1})_*\mu$.

Recall that when $X = \mathbb{R}^n$, the measure $\mu$ is singular if there is a Borel subset $A \subset \mathbb{R}^n$ such that $\lambda_n(A) = 0$ and $\mu(\mathbb{R}^n \setminus A) = 0$. More generally, given a measurable function $f$ from $\mathbb{R}^n$ to $(X, \Sigma, \mu)$, the pushforward $f_*\mu$ is *singular* if there is a Borel subset $A \subset X$ such that $f_*\lambda_n(A) = 0$ and $\mu(X \setminus A) = 0$. The following example expands on this.

*Example* 2.4. Consider the Cayley transform

$$z \mapsto \frac{z - i}{z + i}, \tag{2}$$

which defines a homeomorphism $\phi : \mathbb{R} \to \mathbb{T} \setminus \{1\}$. Both $\phi$ and its inverse are Borel measurable when the spaces are equipped with their respective Borel $\sigma$-algebras. In particular, the pushforward by $\phi$ of a positive Borel measure $\mu$ on $\mathbb{R}$ is well-defined on the 1-torus $\mathbb{T}$ (upon extending it by zero on $\{1\}$), and $\phi(A)$ is a Borel subset of $\mathbb{T}$ whenever $A \subset \mathbb{R}^n$ is a Borel set. Thus, if $\mu$ is singular, then so is $\phi_*\mu$; by assumption, there is a Borel subset $A \subset \mathbb{R}$ such that $\lambda(A) = 0$ and $\mu(\mathbb{R} \setminus A) = 0$. Then, with $\tilde{A} := \phi(A)$,

$$\phi_*\lambda(\tilde{A}) = \lambda \circ \phi^{-1} \circ \phi(A) = \lambda(B) = 0$$

and

$$\phi_*\mu(\mathbb{T} \setminus \tilde{A}) = \mu \circ \phi^{-1} \circ \phi(\mathbb{R} \setminus A) = \mu(\mathbb{R} \setminus A) = 0,$$

and singularity follows.

The previous argument generalizes naturally to the $n$-dimensional setting. In this case the component-wise Cayley transform

$$(z_1, \ldots, z_n) \mapsto (\phi(z_1), \ldots, \phi(z_n)) =: \Phi(z_1, \ldots, z_n)$$

is considered instead, pushing a positive Borel measure $\mu$ on $\mathbb{R}^n$ to $\Phi_*\mu$ on $\mathbb{T}^n$ (which is extended by zero on $\mathbb{T}^n \setminus (\mathbb{T} \setminus \{1\})^n$). $\Diamond$

Given the measure space $(X, \Sigma)$ from before, further assume that $X$ is a metric space. Let $m \geq 0$ and $\delta > 0$ be arbitrary. For any $A \subset X$, define

$$H_{m,\delta}(A) := \inf\{\sum_{j=1}^{\infty}(\operatorname{diam} B_j)^m : B_j \subset X \text{ open}, \operatorname{diam} B_j < \delta \text{ and } A \subset \bigcup_{j=1}^{\infty} B_j\}.$$

The limit

$$\mathcal{H}^m := \lim_{\delta \to 0} \gamma_m H_{m,\delta}, \quad \gamma_m := \frac{\pi^{m/2}}{2^m \Gamma(\frac{m}{2} + 1)}$$

defines a measure on $(X, \Sigma)$ known as the *$m$-dimensional Hausdorff measure*. The constant $\gamma_m$ is the volume of an $m$-dimensional ball with diameter 1 when $m \in \mathbb{N}$ ([Fol13, Corollary 2.55]) and appears here as to make $\mathcal{H}^m$ the Lebesgue measure on $\mathbb{R}^n$ when $m = n$, see [Fol13, Proposition 11.20]. The utility of the $m$-dimensional Hausdorff measure on $\mathbb{R}^n$ partly lies in that it captures the true size of smooth $m$-dimensional submanifolds of $\mathbb{R}^n$; see Theorem 2.5 below. In particular, the measure $\mathcal{H}^1$ on $\mathbb{R}^n$ with $n > 1$ assigns to curves their lengths, and $\mathcal{H}^2$ on $\mathbb{R}^n$ with $n > 2$ assigns to surfaces their areas, despite the fact that these objects all have Lebesgue measure zero in the ambient space. See Example 2.7 for more, and see [Fol13, Theorem 11.25] for a proof of the theorem.

**Theorem 2.5.** *Let $V \subset \mathbb{R}^m$ be open and let $M$ be a smooth $m$-dimensional submanifold of $\mathbb{R}^n$ parametrized (globally) by $g : V \to M$. If $A \subset V$ is a Borel set, then $f(A) \subset M$ is a Borel set and*

$$\mathcal{H}^m(g(A)) = \int_V \chi_{g(A)}(g(t))\sqrt{\det(J_g^* J_g)}\, d\lambda_m(t),$$

*where $J_g$ is the Jacobian matrix of $g$. Moreover, if $k : M \to \mathbb{R}$ is Borel measurable and either nonnegative or in $L^1(M, \mathcal{H}^m)$, then*

$$\int_M k(t)\, d\mathcal{H}^m(t) = \int_V k(g(t))\sqrt{\det(J_g^* J_g)}\, d\lambda_m(t).$$

*Remark* 2.6. Given a smooth $m$-submanifold $M$ of $\mathbb{R}^n$ and a continuous nonnegative density function $k$, the notation $\mu = k(t)\mathcal{H}^m_M$ is frequently used to denote the positive Borel measure

$$\mu(A) = \int_M \chi_A(t)k(t)\, d\mathcal{H}^m(t) \tag{3}$$

on $\mathbb{R}^n$. Theorem 2.5 give this integral meaning. $\Diamond$

*Example* 2.7. Consider the measure $\mu = \mathcal{H}^1_\ell$, where $\ell$ denotes the 1-dimensional hyperplane $t_1+t_2 = 0$ parametrized by $g(t) = (t, -t)$. Note its volume element

$$\sqrt{J_g^* J_g} = \sqrt{2}.$$

Following Remark 2.6, the $\mu$-measure of any Borel subset $A \subset \mathbb{R}^2$ is computed as

$$\mu(A) = \int_{\mathbb{R}} \sqrt{2}\chi_A(t, -t)\, dt.$$

Clearly $\mu(\mathbb{R}^2 \setminus \ell) = 0$ so $\operatorname{supp}\mu \subset \ell$ by Lemma 2.1. The reverse inclusion is also easy to show and it then follows that the support of $\mu$ is the line $\ell$. This means that $\mu$ only captures the intersection with $\ell$ of any Borel subset, and we expect it to do so by assigning this intersection its length. Take e.g the cube $C := [-1, 1]^2 \subset \mathbb{R}^2$, then

$$\mu(C) = \int_{\mathbb{R}} \sqrt{2}\chi_C(t, -t)\, dt = \sqrt{2} \int_{[-1,1]} dt = 2\sqrt{2},$$

which is precisely the length of the line segment given by $\ell \cap C$. ♢

## 2.2 Herglotz-Nevanlinna functions and the poly-upper half-plane

The poly-upper half-plane is the product of $n$ copies of the upper half-plane $\mathbb{H}$ and we denote it by $\mathbb{H}^n$. An element of $\mathbb{H}^n$, or $\mathbb{C}^n$ in general, is denoted by $z := (z_1, z_2, \dots, z_n)$. In the two-variable setting specifically, the notation $(z, w) \in \mathbb{C}^2$ is also frequently used. Similarly, $x$ and $y$ denote the $n$-dimensional vectors $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$. These are typically used to collect the real and imaginary parts of the entries of an $n$-dimensional complex vector according to $z = x + iy = (x_1 + iy_1, \dots, x_n + iy_n)$.

On the topic of the poly-upper half-plane, an automorphism of $\mathbb{H}^n$ is a biholomorphism from $\mathbb{H}^n$ to itself. These are the functions of the form

$$f(z) = P\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \dots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right), \tag{4}$$

where the coefficients in each component function are real and satisfies $a_j d_j - b_j c_j > 0$, and $P$ is a permutation matrix. The set of automorphisms of $\mathbb{H}^n$ form a group under composition, denoted $\operatorname{Aut}\mathbb{H}^n$. It is isomorphic to the group of automorphisms of the unit poly-disc $\mathbb{D}^n := \mathbb{D} \times \cdots \times \mathbb{D}$, denoted $\operatorname{Aut}\mathbb{D}^n$. This latter group and the specifics of its elements is discussed in [Rud69]. The form of any $f \in \operatorname{Aut}\mathbb{H}^n$ as proposed by (4) can be obtained from some $g \in \operatorname{Aut}\mathbb{D}^n$ via the component-wise Cayley transform (introduced in Example 2.4) by writing $f = \Phi^{-1} \circ g \circ \Phi$.

**Definition 2.8.** A holomorphic function $h : \mathbb{H}^n \to \mathbb{C}$ is a *Herglotz-Nevanlinna function* if its imaginary part is nonnegative.

Notice that any positive linear combination $ah_1 + bh_1$, $a, b \geq 0$, of Herglotz-Nevanlinna functions $h_1$ and $h_2$ is itself a Herglotz-Nevanlinna function. Moreover, the composition $h \circ f$ of a Herglotz-Nevanlinna function $h$ and automorphism $f \in \operatorname{Aut}\mathbb{H}^n$ is again a Herglotz-Nevanlinna function. It is also possible to compose different Herglotz-Nevanlinna functions in various ways to obtain new ones, given reasonable assumptions. See below for a demonstration.

*Example* 2.9. Choose $n_1, n_2 \in \mathbb{N}$ and consider two Herglotz-Nevanlinna functions

$$h_1 : \mathbb{H}^{n_1} \to \mathbb{C} \quad \text{and} \quad h_2 : \mathbb{H}^{n_2} \to \mathbb{C}\,.$$

Let $h$ be a Herglotz-Nevanlinna function on $\mathbb{H}^2$. If $\operatorname{Im} h_1 > 0$ and $\operatorname{Im} h_2 > 0$ on their respective domain, meaning that none of $h_1$ and $h_2$ attain a real value, then the composition

$$(z, w) \mapsto h(h_1(z), h_2(w))$$

is a Herglotz-Nevanlinna function on $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$. ♢

In addition to the already mentioned methods of producing new Herglotz-Nevanlinna functions from old, we have the following lemma.

**Lemma 2.10.** *Suppose $h : \mathbb{H}^n \to \mathbb{C}$ does not vanish anywhere. Then $h$ is a Herglotz-Nevanlinna function if and only if $z \mapsto -\frac{1}{h(z)}$ is a Herglotz-Nevanlinna function.*

*Proof.* The function $z \mapsto -\frac{1}{h(z)}$ is well-defined on $\mathbb{H}^n$ by assumption and it is holomorphic if and only if $h$ is holomorphic. Moreover, a simple computation shows that the imaginary part of one function is nonnegative if and only if the imaginary part of the other function is nonnegative:

$$\operatorname{Im} -\frac{1}{h(z)} = \operatorname{Im} -\frac{\overline{h(z)}}{\left|h(z)\right|^2} = \frac{1}{\left|h(z)\right|^2} \operatorname{Im} h(z). \qquad \square$$

## 2.3 The integral representation

The integral representation that associate a set of constants and a particular measure to a Herglotz-Nevanlinna function is presented in Theorem 2.12 below, along with an incredibly useful corollary. Before then, some appropriate theory is introduced, such as the kernel function that appears in the representation and the class of representing measures.

The kernel function $K_n : \mathbb{H}^n \times \mathbb{R}^n \to \mathbb{C}$ in the representation (15) is defined as

$$(z, t) \mapsto \frac{1}{(2i)^{n-1}} \prod_{j=1}^{n} \left( \frac{1}{t_j - z_j} - \frac{1}{t_j + i} \right) - i \prod_{j=1}^{n} \frac{1}{t_j^2 + 1}. \tag{5}$$

It turns out that it is related to the Poisson kernel for the poly-upper half-plane; see (10) below. This is the function $\mathcal{P}_n : \mathbb{H}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$(z, t) \mapsto \prod_{j=1}^{n} \frac{\operatorname{Im} z_j}{|t_j - z_j|^2}, \tag{6}$$

which is essentially a product of $n$ copies of the Poisson kernel from the one-dimensional setting. As a side note, the Poisson kernel $\mathcal{P}_n$ is in some sense invariant under automorphisms of $\mathbb{H}^n$, as proposed by the following.

**Lemma 2.11.** *Let $f \in \operatorname{Aut}(\mathbb{H}^n)$ and denote the Jacobian matrix of $f|_{\mathbb{R}^n}$ by $J_f$. Then*

$$\mathcal{P}_n(z, t) = \det(J_f(t))\, \mathcal{P}_n(f(z), f(t)), \quad \forall t \in \mathbb{R}^n, z \in \mathbb{H}^n .$$

*Proof.* The automorphism $f$ is of the form (4). For now, assume that the permutation matrix $P$ is the identity. Then

$$f(z) = \left( \frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n} \right), \tag{7}$$

where the coefficients in each component function are real and satisfies $a_j d_j - b_j c_j > 0$ for each $j$. It follows that

$$\det J_f(t) = \det \left[ \frac{\partial f_j}{\partial t_k}(t) \right]_{jk} = \prod_{j=1}^{n} \frac{\partial f_j}{\partial t_j}(t). \tag{8}$$

Due to the structure of $\mathcal{P}_n$ it will suffice to show the result in the one-dimensional setting. Thus suppose $n = 1$ and let

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc > 0.$$

The identity

$$\mathcal{P}_1(z, t) = f'(t) \mathcal{P}_1(f(z), f(t)) \tag{9}$$

is obtained by straightforward calculations: with

$$\operatorname{Im} f(z) = \operatorname{Im} \frac{az+b}{cz+d} = \frac{(ad-bc)}{|cz+d|^2} \operatorname{Im} z$$

and

$$f(t) - f(z) = \frac{at+b}{ct+d} - \frac{az+b}{cz+d} = \frac{(ad-bc)(t-z)}{(ct+d)(az+b)},$$

it follows that

$$\begin{aligned} f'(t)\mathcal{P}_1(f(z), f(t)) &= \frac{ad-bc}{(ct+d)^2} \frac{\operatorname{Im} f(z)}{|f(t)-f(z)|^2} \\ &= \frac{ad-bc}{(ct+d)^2} \frac{\frac{(ad-bc)}{|cz+d|^2} \operatorname{Im} z}{\frac{(ad-bc)^2}{(ct+d)^2|cz+d|^2}|t-z|^2} = \frac{\operatorname{Im} z}{|t-z|^2} = \mathcal{P}_1(z,t). \end{aligned}$$

Observe that if $c \neq 0$ then $t \mapsto f'(t)\mathcal{P}_1(f(z), f(t))$ is not defined at $t = -\frac{d}{c}$. However, since the equality (9) holds for all other $t \in \mathbb{R}$, and $\mathcal{P}_1(z,t)$ is defined at every $t \in \mathbb{R}$, the singularity at $t = -\frac{d}{c}$ is removable. By viewing the function $t \mapsto f'(t)\mathcal{P}_1(f(z), f(t))$ as its continuous extension to $t = -\frac{d}{c}$ by $\mathcal{P}_1(z, -\frac{c}{d})$, the identity (9) holds for all $t \in \mathbb{R}$ (and $z \in \mathbb{H}$).

It is easy to see that both $\det J_f$ and the kernel $\mathcal{P}_n$ are invariant under permutation of coordinates, so the conclusion extends to all automorphisms of $\mathbb{H}^n$. □

As demonstrated in [LN19, Proposition 3.3], the two kernels $K_n$ and $\mathcal{P}_n$ are related by

$$\operatorname{Im} K_n = \mathcal{P}_n + R_n, \tag{10}$$

where $R_n$ is a rest term that also depends on both $z \in \mathbb{H}^n$ and $t \in \mathbb{R}^n$. In accordance with [LN19, NS24], we define a *Nevanlinna measure* to be a positive Borel measure $\mu$ on $\mathbb{R}^n$ satisfying the *growth condition*

$$\int_{\mathbb{R}^n} \prod_{j=1}^n \frac{1}{1+t_j^2} \, d\mu(t) < \infty \tag{11}$$

and the *Nevanlinna condition*

$$\int_{\mathbb{R}^n} R_n(z,t) \, d\mu(t) = 0 \quad \forall z \in \mathbb{H}^n \,. \tag{12}$$

Notice that when $\mu$ is a Nevanlinna measure, the Nevanlinna condition yields

$$\int_{\mathbb{R}^n} \operatorname{Im} K_n \, d\mu = \int_{\mathbb{R}^n} \mathcal{P}_n \, d\mu$$

from (10). This identity is a crucial component in the proof of the integral representation and is in fact the reason for the Nevanlinna condition. The one-dimensional setting is simpler as the rest term $R_1$ is zero, which means that all positive Borel measures on $\mathbb{R}$ satisfying the growth condition appear as representing measures of one-variable Herglotz-Nevanlinna functions.

It is shown in [LN19, Theorem 5.1] that the Nevanlinna condition is equivalent to

$$\int_{\mathbb{R}^n} \frac{1}{(t_{j_1} - z_{j_1})^2 (t_{j_2} - \overline{z_{j_2}})^2} \prod_{\substack{j=1 \\ j \neq j_1, j_2}}^n \frac{2 \operatorname{Im} z_j}{|t_j - z_j|^2} \, d\mu(t) = 0 \quad \forall z \in \mathbb{H}^n \tag{13}$$

for every pair of indices $j_1, j_2 \in \mathbb{N}$ with $1 \leq j_1 < j_2 \leq n$, provided that $\mu$ is a positive Borel measure satisfying the growth condition. The same result gives two additional equivalent formulations of the Nevanlinna condition for such measures, one of which is that the function

$$z \mapsto \int_{\mathbb{R}^n} \mathcal{P}_n(z,t) \, d\mu(t) \tag{14}$$

is pluriharmonic on $\mathbb{H}^n$. Among all these formulations, condition (13) appears to be the most common in the literature, see [LN20, Ned20, NS24].

Some remarks on the significance of the growth condition is in order. Firstly, if $\mu$ is a positive Borel measure satisfying the growth condition, then the *Cauchy-type function*

$$z \mapsto \int_{\mathbb{R}^n} K_n(z,t)\, d\mu(t)$$

is well-defined. This is important for the integral representation theorem since the pure-integral part of the representation is such a function (see (15)). Secondly, a positive Borel measure $\mu$ that satisfies the growth condition is finite on compact subsets of $\mathbb{R}^n$. To see this, let $K \subset \mathbb{R}^n$ be compact. Since

$$t \mapsto \prod_{j=1}^{n} \frac{1}{1+t_j^2} > 0$$

is continuous on $\mathbb{R}^n$, it attains a minimum value $M > 0$ on $K$. Thus

$$M\mu(K) = \int_K M\, d\mu(t) \le \int_K \prod_{j=1}^{n} \frac{1}{1+t_j^2}\, d\mu(t) \le \int_{\mathbb{R}^n} \prod_{j=1}^{n} \frac{1}{1+t_j^2}\, d\mu(t) < \infty,$$

which implies $\mu(K) < \infty$. A step further is to note that positive Borel measures on $\mathbb{R}^n$ that are finite on compact subsets are regular (see e.g [Coh13, Theorem 7.2.3]), meaning that Nevanlinna measures are regular.

We are now ready to state the characterization theorem and its corollary from [LN19, Theorem 4.1 and Corollary 4.6].

**Theorem 2.12.** *A function $h : \mathbb{H}^n \to \mathbb{C}$ is a Herglotz-Nevanlinna function if and only if there are constants $a \in \mathbb{R}$ and $b \in [0,\infty)^n$, and a Nevanlinna measure $\mu$, such that*

$$h(z) = a + \sum_{j=1}^{n} b_j z_j + \frac{1}{\pi^n} \int_{\mathbb{R}^n} K_n(z,t)\, d\mu(t). \tag{15}$$

*In this case, the parameters $a$, $b$ and $\mu$ are unique to $h$.*

*Remark* 2.13. Given a Herglotz-Nevanlinna function $h$, its representing parameters are typically presented as a triplet $(a, b, \mu)$, labeled as the *data* of $h$. ◊

**Corollary 2.14.** *Let $h$ be a Herglotz-Nevanlinna function of $n$ variables and let $(a, b, \mu)$ be its data from representation (15). Then the following statements hold.*

*(i) If $h$ attains a real value on $\mathbb{H}^n$, then $h$ is constant.*

*(ii) The constant $a$ satisfies $a = h(i)$.*

*(iii) For any $j \in \{1, 2, \dots, n\}$, the $j$'th entry of $b$ satisfies*

$$b_j = \lim_{z_j \hat{\to} \infty} \frac{h(z)}{z_j}.$$

*(iv) Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$-function satisfying*

$$\big|\varphi(x)\big| \le C \prod_{j=1}^{n} \frac{1}{1+x_j^2} \quad \forall x \in \mathbb{R}^n \tag{16}$$

*for some constant $C \ge 0$. Then*

$$\int_{\mathbb{R}^n} \varphi(x)\, d\mu(x) = \lim_{y \to 0+} \int_{\mathbb{R}^n} \varphi(x)\, \mathrm{Im}\, h(x+iy)\, d\lambda_n(x).$$

*(v) The imaginary part of h satisfies*

$$\operatorname{Im} h(z) = \sum_{j=1}^{n} b_j \operatorname{Im} z_j + \frac{1}{\pi^n} \int_{\mathbb{R}^n} \mathcal{P}_n(z,t)\, d\mu(t).$$

*Remark* 2.15. The notation $z_j \hat{\to} \infty$ that appears in Corollary 2.14(iii) denotes the limit $|z_j| \to \infty$ in any *Stolz domain* $\{z \in \mathbb{H} : \theta \leq \arg z \leq \pi - \theta\}$ where $0 < \theta \leq \frac{\pi}{2}$. It is a nontangential limit. ◊

*Remark* 2.16. The integral identity in Corollary 2.14(iv) is sometimes called the *Stieltjes inversion formula*, which is what it is known as in the one-dimensional setting. Given a Herglotz-Nevanlinna function, the formula provides a way to determine some of the structure of its corresponding Nevanlinna measure from its behavior on the distinguished boundary $\mathbb{R}^n$ – see e.g Theorem 3.9. ◊

The following example of a Herglotz-Nevanlinna function has already been studied in [LN19, Example 3.5], but it is summarized here nonetheless since it is so fundamental in the study of Herglotz-Nevanlinna functions.

*Example* 2.17. By use of the residue theorem it can be shown that

$$\int_{\mathbb{R}} K_n(z,t)\, dt_j = \pi K_{n-1}((z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_n),(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)) \tag{17}$$

for each $j \in \{1,2,\ldots,n\}$. Therefore, when $\mu = \lambda_n$, the Cauchy-type function

$$z \mapsto \frac{1}{\pi^n} \int_{\mathbb{R}^n} K_n(z,t)\, d\mu(t) \tag{18}$$

evaluates to $\pi^n K_0$ by Fubini's theorem and repeated use of the residue theorem. Upon noting that $K_0 = i$, the function (18) is simply the constant function $z \mapsto i$, which is a Herglotz-Nevanlinna function with all constants in its representation (15) set to zero. In particular, if $c \geq 0$, then the measure $c\lambda_n$ is a Nevanlinna measure and corresponds to the Herglotz-Nevanlinna function $z \mapsto ci$. ◊

## 2.4 A closer look at Nevanlinna measures

By [Ned20, Theorem 4.1], a product $\mu_1 \times \mu_2$ on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ of positive Borel measures (not identically zero) on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, is a Nevanlinna measure if and only if one factor is a Nevanlinna measure and the other is the Lebesgue measure in their respective dimensions. From this, the following is obtained.

**Theorem 2.18.** *Let $M \subset \mathbb{R}^n$ be a translate by $\alpha \in \mathbb{R}$ of one of the coordinate hyperplanes of $\mathbb{R}^n$. Then $M$ supports a Nevanlinna measure if and only if the measure is a positive constant multiple of*

$$\lambda \times \cdots \times \delta_\alpha \times \cdots \times \lambda, \tag{19}$$

*where the Dirac measure is at the position in the product that corresponds to which coordinate hyperplane that $M$ is a translate of.*

*Proof.* Assume without loss of generality that $M = \{x \in \mathbb{R}^n : x_1 = \alpha\}$. Then the support of the Nevanlinna measure $\mu := c\delta_\alpha \times \lambda_{n-1}$ is

$$\operatorname{supp} \mu = \operatorname{supp} \delta_\alpha \times \operatorname{supp} \lambda_{n-1} = M$$

for every $c > 0$.

Conversely, let $\mu$ be a Nevanlinna measure on $\mathbb{R}^n$ with support $M$. Note that it can be written $\delta_\alpha \times \nu$, where $\nu$ is the pushforward of $\mu$ by the projection $(\alpha, x_2, \ldots, x_n) \mapsto (x_2, \ldots, x_n)$, meaning that $\nu(A) = \mu(\{x \in \mathbb{R}^n : x_1 = \alpha,\ (x_2,\ldots,x_n) \in A\})$ for any Borel subset $A \subset \mathbb{R}^{n-1}$. Furthermore, $\nu$ is a positive Borel measure on $\mathbb{R}^{n-1}$ and $\delta_\alpha$ is a Nevanlinna measure on $\mathbb{R}$. It follows from [Ned20, Theorem 4.1] that $\nu = c\lambda_{n-1}$ for some $c > 0$, which completes the proof. □

Next is a result due to [NS24, Theorem 3.7]. Together with Theorem 2.18 they classify the Nevanlinna measures that are supported on hyperplanes in $\mathbb{R}^n$.

**Theorem 2.19.** *Let $m \in \{1, 2, \ldots, n-1\}$ and suppose $M \subset \mathbb{R}^n$ is an $m$-dimensional hyperplane that is nonparallel to the coordinate hyperplanes of $\mathbb{R}^n$. Then $M$ supports a Nevanlinna measure if and only if $m = n-1$ and there are constants $c \in \mathbb{R}$ and $a := (a_1, \ldots, a_n) \in \mathbb{R}^n$, where each $a_j \geq 0$ and at least two of them are nonzero, such that*

$$M = \{x \in \mathbb{R}^n : a \cdot x = c\}.$$

*In this case, the Nevanlinna measure $\mu$ is given by*

$$\mu = \tilde{k}(x)\mathcal{H}^m_M, \tag{20}$$

*where $\tilde{k} \in \mathbb{R}[x_1, \ldots, x_n]$ is of at most total degree 2, only depends on the variables $x_j$ for which $a_j > 0$, and is nonnegative on $M$.*

*Remark* 2.20. In Theorem 2.19, the total degree of $\tilde{k}$ is the maximum sum of degrees in each variable in any of its terms; see Definition 2.33. For example, the polynomial $x \mapsto x_1^2 x_2 + x_1 x_3$ is of total degree 3 because $\max\{2+1, 1+1\} = 3$. ◊

*Remark* 2.21. The parametrization $g$ of a hyperplane contributes with a constant volume element in the formulas of Theorem 2.5, so with $k := \frac{1}{\pi}\sqrt{\det(J_g^* J_g)}\,\tilde{k} \circ g$ the measure (20) can be written as

$$\mu(A) = \pi \int_{\mathbb{R}^{n-1}} k(t)\chi_A(g(t))\, d\lambda_{n-1}(t)$$

in accordance with Remark 2.6. The factor $\pi$ will make sense later. Note that $k$ is nonnegative and of at most total degree 2, similar to $\tilde{k}$. In the two-variable setting, this means that $k$ can be written $k(t) = b(t+c)^2 + d$ for some $b, d \geq 0$ and $c \in \mathbb{R}$. ◊

In two dimensions, Theorem 2.19 implies that no straight line with positive slope can support a nonzero Nevanlinna measure. A stronger result in this direction is true: due to [LN20, Theorem 3.17], the support of a nonzero Nevanlinna measure on $\mathbb{R}^2$ is not contained in a strip of positive slope; see Figure 1.

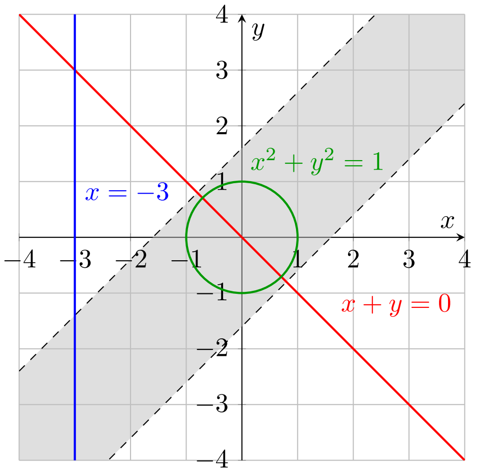


Figure 1: Admissible supporting hyperplanes $x = -3$ (blue) and $x + y = 0$ (red). The 1-sphere (green) does not appear as the support of any Nevanlinna measure.

Furthermore, if the support of a Nevanlinna measure on $\mathbb{R}^2$ do not intersect any cross made from a vertical and horizontal strip, then by [LN20, Theorem 3.25] the support is empty. In particular, if the support of a Nevanlinna measure is bounded or perhaps contained in one of the quadrants of $\mathbb{R}^2$, then it is the zero measure. These results on the geometry of the support of Nevanlinna measures are collected in the following, where they are stated in the $n$-dimensional setting similar to [LN20, Theorem 3.17] and [LN20, Theorem 3.25].

**Theorem 2.22.** *Let $\mu$ be a Nevanlinna measure on $\mathbb{R}^n$. If there exists some $\beta > 0$ and real numbers $\alpha_1 < \alpha_2$ such that*

$$\operatorname{supp}\mu \subset \{x \in \mathbb{R}^n : \alpha_1 < x_{j_1} - \beta x_{j_2} < \alpha_2\}$$

*for some $j_1, j_2 \in \{1, 2, \dots, n\}$, then $\mu \equiv 0$. Similarly, if there exists numbers $\alpha_j < \beta_j$ for $j = 1, 2, \dots, n$, such that*

$$\operatorname{supp}\mu \cap \bigcup_{j=1}^{n} \{x \in \mathbb{R}^n : \alpha_j < x_j < \beta_j\} = \varnothing,$$

*then $\mu \equiv 0$.*

Notice that by Theorem 2.22, no nontrivial Nevanlinna measure is discrete. In fact, no nontrivial Nevanlinna measure has a point mass anywhere on $\mathbb{R}^n$; this is a consequence of [LN20, Corollary 3.8] which states, among other things, that Nevanlinna measures vanish on subspaces of codimension at least 2.

*Example* 2.23. Both $(z, w) \mapsto 2w - \frac{1}{z}$ and $(z, w) \mapsto -\frac{1}{z+w}$ are Herglotz-Nevanlinna functions. Their respective Nevanlinna measures are $\frac{1}{\pi}\delta_{-3} \times \lambda$ and

$$\mu(A) = \pi \int_{\mathbb{R}} \chi_A(t, -t)\, dt,$$

which can be confirmed by either direct calculation of the integral (15) or applying the upcoming Theorem 3.18. The first measure is on the from (19) and the second is on the form (20). Their supporting lines are displayed in Figure 1 as $x = -3$ and $x + y = 0$, respectively. ♢

## 2.5 Briefly on zero sets and function theory

Most of the first half of the following theory is based on [Leb24, Chapter 6] and is concerned with zero sets of holomorphic functions. This is also a good place to mention that we will denote the zero set $f^{-1}(\{0\})$ of a function $f$ by $\mathcal{Z}_f$.

The zero set of a holomorphic function on an open subset $U \subset \mathbb{C}^n$ is an example of a *complex-analytic subvariety* of $U$. In general, a complex-analytic subvariety of an open subset $U \subset \mathbb{C}^n$ is a set $X \subset U$ such that, for each point in $U$, there is a neighborhood $W$ so that $W \cap X$ is the intersection of the zero sets of a family of holomorphic functions on $W$. For convenience we shorten complex-analytic subvariety to subvariety.

A subvariety $X$ can be decomposed into a disjoint union of two sets $X_{reg}$ and $X_{sing}$. The first of these is the set of *regular points* of $X$. These are points near which $X$ is the graph of a holomorphic function. The *dimension* of $X$ at a regular point is the dimension of the domain of the corresponding holomorphic function. A point that is not regular is *singular* and these points make $X_{sing}$. By e.g [Leb24, Lemma 6.5.10], the regular set of a subvariety $X$ is dense in $X$, and so near every singular point there is a regular point. The subvariety $X$ is of *pure dimension* $k$ if the dimension at each regular point is $k$, in which case $X_{reg}$ admits the structure of a complex $k$-submanifold. This submanifold is of codimension $n - k$, where $n$ is the dimension of the ambient space, hence it is natural to define the *codimension* of a subvariety of pure dimension to be this number. Finally, a subvariety of pure dimension and codimension 1 is a *hypervariety*. Due to Theorem 2.24 below, we are really only interested in hypervarieties.

**Theorem 2.24.** *Let $U \subset \mathbb{C}^n$ be a domain and suppose $f : U \to \mathbb{C}$ is holomorphic and nonconstant. Then the zero set of $f$ is a hypervariety.*

*Proof.* If $f$ is nonconstant then $\mathcal{Z}_f$ is not empty nor equals to all of $U$. The rest follows from [Leb24, Theorem 6.5.9]. □

The following, [Leb24, Theorem 6.6.5], is the most important result on hypervarieties for our purposes.

**Theorem 2.25.** *Let $U \subset \mathbb{C}^n$ be open and $X \subset U$ a hypervariety. Then $X_{sing}$ is a subvariety of $U$ of dimension less than or equal to $n-2$.*

The two previous results combine to give the following corollary, which is one of two results that we are really after. The corollary is then used to invoke the open mapping theorem in the proof of Lemma 2.28 below, which is the second result.

**Corollary 2.26.** *Let $q \in \mathbb{C}[z_1, \ldots, z_n]$. Then $\mathbb{C}^n \setminus \mathcal{Z}_q$ is connected.*

*Proof.* The conclusion is immediate if $q$ is constant, so suppose $q$ is nonconstant. Then by Theorem 2.24, the zero set of $q$ is a hypervariety. In particular, $(\mathcal{Z}_q)_{\text{reg}}$ is a $(2n-2)$-dimensional real submanifold of $\mathbb{R}^{2n}$. Because the codimension is $\geq 2$, it is easy to construct a path between any two points in $\mathbb{R}^{2n} \setminus (\mathcal{Z}_q)_{\text{reg}}$, hence the space is connected. By Theorem 2.25, $(\mathcal{Z}_q)_{\text{sing}}$ is even smaller than $(\mathcal{Z}_q)_{\text{reg}}$ in dimension, so the conclusion extends to $\mathbb{R}^{2n} \setminus \mathcal{Z}_q$. □

*Remark* 2.27. Take $p, q \in \mathbb{C}[z_1, \ldots, z_n]$ and set $r(z) := p(z)q(z)$. Then $r \in \mathbb{C}[z_1, \ldots, z_n]$ and $\mathcal{Z}_r = \mathcal{Z}_p \cup \mathcal{Z}_q$ so by Corollary 2.26 it follows that $\mathbb{C}^n \setminus (\mathcal{Z}_p \cup \mathcal{Z}_q)$ is connected. This observation is useful for later. ◊

**Lemma 2.28.** *Let $p, q \in \mathbb{C}[z_1, \ldots, z_n]$ and define a rational function $h$ by $z \mapsto \frac{p(z)}{q(z)}$. If $h$ is unimodular then it is constant.*

*Proof.* We show the contrapositive. To this end, assume that $h$ is nonconstant. Openness of $\mathbb{C}^n \setminus \mathcal{Z}_q$ is immediate upon noting that, by continuity of $q$, $\mathcal{Z}_q$ is closed in $\mathbb{C}^n$. In addition, $\mathbb{C}^n \setminus \mathcal{Z}_q$ is connected as per Corollary 2.26. Since $h$ is holomorphic on $\mathbb{C}^n \setminus \mathcal{Z}_q$, the open mapping theorem yields that $h(\mathbb{C}^n \setminus \mathcal{Z}_q)$ is open in $\mathbb{C}^n$. However, the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ contains no subsets that are open in $\mathbb{C}$ and so $h(\mathbb{C}^n \setminus \mathcal{Z}_q)$ is not contained in $\mathbb{S}^1$. In particular, $h$ is not unimodular. □

Next is a way of rewriting a certain expression using an automorphism of the upper half-plane. It is not particularly interesting by itself, but stating it here will make a later argument easier to follow.

**Lemma 2.29.** *If $f \in \operatorname{Aut}\mathbb{H}$ then*

$$\frac{1}{(t-z)^2} = \frac{f'(t)f'(z)}{(f(t)-f(z))^2} \quad \forall t \in \mathbb{R},\ z \in \mathbb{H}. \tag{21}$$

*Proof.* The function $f$ can be written $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ such that $ad - bc > 0$, so then

$$\begin{aligned}\frac{f'(t)f'(z)}{(f(t)-f(z))^2} &= \frac{\frac{ad-bc}{(ct+d)^2}\frac{ad-bc}{(cz+d)^2}}{\left(\frac{at+b}{ct+d} - \frac{az+b}{cz+d}\right)^2} \\ &= \frac{(ad-bc)^2}{\big((at+b)(cz+d) - (az+b)(ct+d)\big)^2} \\ &= \frac{(ad-bc)^2}{(ad-bc)^2(t-z)^2} = \frac{1}{(t-z)^2}.\end{aligned}$$

Observe that if $c \neq 0$ then $t \mapsto \frac{f'(t)f'(z)}{(f(t)-f(z))^2}$ is not defined at $t = \frac{d}{c}$. However, since (21) holds for all other $t \in \mathbb{R}$, and $t \mapsto \frac{1}{(t-z)^2}$ is defined for all $t \in \mathbb{R}$, the singularity at $t = -\frac{d}{c}$ is removable. By viewing $t \mapsto \frac{f'(t)f'(z)}{(f(t)-f(z))^2}$ as its continuous extension to $t = -\frac{d}{c}$ by $\frac{1}{(-\frac{d}{c}-z)^2}$, the desired conclusion holds for all $t \in R$ (and $z \in \mathbb{H}$). □

Continuing on the theme of holomorphic functions, the following shows that certain complex-valued functions on parts of $\mathbb{R}^n$ can be "complexified" by extending uniquely and holomorphically to somewhere on $\mathbb{C}^n$. A proof can be located in [Leb24, Proposition 3.1.3].

**Lemma 2.30.** *Let $U \subset \mathbb{R}^n$ be open and connected. Suppose $f : U \to \mathbb{C}$ is real-analytic. Then there is an open and connected subset $V \subset \mathbb{C}^n$ such that $U \subset V$ and a unique holomorphic function $\tilde{f} : V \to \mathbb{C}$ such that its restriction to $U$ is $f$.*

Regarding real-analytic functions, the next result states that the zero set of such a function is negligible in a measure-theoretic sense. A proof based on the implicit function theorem is provided by [Mit20].

**Lemma 2.31.** *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a real-analytic function. If $h$ is not identically zero then $\lambda_n(\mathcal{Z}_h) = 0$.*

The above of course applies to real polynomials. We mention it here because complex polynomials with real coefficients - which are real polynomials when restricted to $\mathbb{R}^n$ - end up playing a central role in later chapters. Also, while on the topic of polynomials, two different notions of the degree of a polynomial are appropriately introduced here. First, for any $j \in \{1, 2, \ldots, n\}$, a polynomial $p \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ can be viewed as a polynomial in the single variable $z_j$ with polynomial coefficients in

$$\mathbb{C}[z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n].$$

This motivates the following definition.

**Definition 2.32.** Let $p$ be a polynomial in $\mathbb{C}[z_1, z_2, \ldots, z_n]$ and let $j \in \{1, 2, \ldots, n\}$. The *degree of $p$ in $z_j$*, denoted by $\deg_j p$, is the degree of $p$ as a polynomial in the variable $z_j$.

The above notion of degree ignores an important structure of polynomials of several variables, which are the products of variables $z_j$ for two or more different $j \in \{1, 2, \ldots, n\}$. This is accounted for in the next definition. An example has already been provided in Remark 2.20.

**Definition 2.33.** Let $p$ be a polynomial in $\mathbb{C}[z_1, \ldots, z_n]$. The *total degree of $p$* is the maximum sum of degrees in each variable in any of its terms. That is, the total degree of

$$z \mapsto \sum_{j_n=0}^{d_n} \cdots \sum_{j_1=0}^{d_1} \alpha_{j_1,\ldots,j_n} z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}$$

is the nonnegative integer

$$\max_{\substack{0 \le j_k \le d_k, k=1,\ldots,n \\ \alpha_{j_1,\ldots,j_n} \neq 0}} j_1 + \cdots + j_n.$$

Finally, Partitions of Unity or a smooth version of Urysohn's lemma implies the existence of a smooth bump function. A proof can be found in e.g [Lee03, Proposition 2.25], where it is given in the more general setting of smooth manifolds.

**Lemma 2.34.** *Let $U$ be an open subset of $\mathbb{R}^n$. Suppose $F \subset \mathbb{R}^n$ is closed and contained in $U$. Then there exist a $\mathcal{C}^1$-function $\varphi : \mathbb{R}^n \to [0, 1]$ such that $\varphi \equiv 1$ on $F$ and $\varphi \equiv 0$ on $\mathbb{R}^n \setminus U$.*

Note that for any given compact subset $F \subset \mathbb{R}^n$, the function $x \mapsto \prod_{j=1}^n \frac{1}{1+x_j^2}$ attains a minimum value $\frac{1}{C} > 0$ on $F$ by continuity. It follows that $\varphi$ in Lemma 2.34 satisfies

$$\left|\varphi(x)\right| \le 1 = C\frac{1}{C} \le C \prod_{j=1}^{n} \frac{1}{1 + x_j^2}$$

for all $x \in \mathbb{R}^n$. This means that a bump function from Lemma 2.34 satisfies (16) and therefore qualifies as a test function in Steiltjes inversion formula Corollary 2.14(iv). This is how Lemma 2.34 is applied later.

# 3 Real-rational functions

This chapter is devoted to the study of *rational Herglotz-Nevanlinna functions* of several variables (see Definition 3.1), with emphasis on the subclass of *real-rational functions* (see Definition 3.4).

## 3.1 Rational Herglotz-Nevanlinna functions

A proper definition of a rational Herglotz-Nevanlinna function is provided below, followed by two lemmas. The first is a characterization of the class of polynomial Herglotz-Nevanlinna functions and the second gives bounds on the growth of rational Herglotz-Nevanlinna functions.

**Definition 3.1.** A *rational Herglotz-Nevanlinna function* is a fraction of two polynomials in $\mathbb{C}[z_1, \dots, z_n]$, with no nontrivial factor in common, such that its restriction to $\mathbb{H}^n$ is a Herglotz-Nevanlinna function. It is a *polynomial Herglotz-Nevanlinna function* specifically if the denominator is constant.

**Lemma 3.2.** *Let $h$ be a Herglotz-Nevanlinna function of $n$ variables and let $(a, b, \mu)$ be its data from representation (15). Then $h$ is a polynomial if and only if $\mu$ is a nonnegative multiple of $\lambda_n$.*

*Proof.* If $\mu = c\lambda_n$ for some $c \geq 0$ then the pure-integral part of $h$ is constantly equal to $ci$ in accordance with Example 2.17, hence $h(z) = a + \sum_{j=1}^n b_j + i$. This completes one direction. As for the other, suppose $h$ is a complex polynomial. Then

$$h(z) = \sum_{j_n=0}^{d_n} \cdots \sum_{j_1=0}^{d_1} \alpha_{j_1,\dots,j_n} z_1^{j_1} z_2^{j_2} \dots z_n^{j_n},$$

where each $\alpha_{j_1,\dots,j_n} \in \mathbb{C}$ and each $d_k := \deg_k h$. By assumption, $h$ is a Herglotz-Nevanlinna function, so Corollary 2.14(iii) can be applied to conclude three things; firstly, $d_k \leq 1$. Secondly, $\alpha_{j_1,\dots,j_n} = 0$ whenever two or more indices $j_k$ are nonzero. Finally, $\alpha_{0,\dots,1,\dots,0} = b_j$ for every $j \in \{1, 2, \dots, n\}$, where the nonzero index is at position $j$ and $b_j$ is the $j$'th entry of $b$. With this, the above description of $h$ reduces to

$$h(z) = \alpha_{0,\dots,0} + \sum_{j=1}^n b_j z_j.$$

Set $\alpha := \alpha_{0,\dots,0}$. Corollary 2.14(ii) implies that $\operatorname{Re}\alpha = a$, so then $i \operatorname{Im}\alpha$ is the pure-integral part of the integral representation of $h$. Since the pure-integral part is itself a Herglotz-Nevanlinna function, it follows that $\operatorname{Im}\alpha \geq 0$, which - again by Example 2.17 - is exactly the function obtained from the Nevanlinna measure $\operatorname{Im}\alpha\lambda_n$. Thus, $\mu = \operatorname{Im}\alpha\lambda_n$, where $\operatorname{Im}\alpha \geq 0$. □

**Lemma 3.3.** *Let $h = \frac{p}{q}$ be a rational Herglotz-Nevanlinna function of $n$ variables and let $(a, b, \mu)$ be its data from representation (15). Then the following holds for $j = 1, 2, \dots, n$:*

*(i) If $h$ does not vanish anywhere on $\mathbb{H}^n$, then $\left|\deg_j p - \deg_j q\right| \leq 1$,*

*(ii) $b_j = 0$ if and only if $\deg_j q \geq \deg_j p$,*

*(iii) If $h$ does not vanish anywhere on $\mathbb{H}^n$, then $b_j = 0$ if and only if $0 \leq \deg_j q - \deg_j p \leq 1$.*

*Proof.* It is an immediate consequence of Corollary 2.14(iii) that

$$\deg_j p \leq \deg_j q + 1. \tag{22}$$

In particular, if $b_j = 0$ then $\deg_j p \leq \deg_j q$. It is also clear that $b_j = 0$ when $\deg_j p \leq \deg_j q$, so (ii) is done.

By Lemma 2.10, the function $z \mapsto -\frac{q(z)}{p(z)}$ is a Herglotz-Nevanlinna function. Applying Corollary 2.14(iii) to this function yields

$$\deg_j q \leq \deg_j p + 1,$$

which together with (22) yields (i). Item (iii) follows from the first two. □

## 3.2 Real-rational functions

The focus now shifts to the real-rational functions, starting with a formal definition. This is followed by a set of equivalent conditions for when a rational Herglotz-Nevanlinna function is real-rational in terms of the structure of the function itself.

**Definition 3.4.** A rational Herglotz-Nevanlinna function is *real-rational* if

$$\lim_{y \to 0^+} \operatorname{Im} h(x + iy) = 0$$

for $\lambda_n$-a.e $x \in \mathbb{R}^n$.

**Lemma 3.5.** *Suppose $h = \frac{p}{q}$ is a rational Herglotz-Nevanlinna function of $n$ variables. Then the following are equivalent:*

*(a) $h$ is real-rational,*

*(b) The limit of $\operatorname{Im} h(x + iy)$ as $y \to 0^+$ vanishes for whichever $x \in \mathbb{R}^n$ it exists,*

*(c) $\operatorname{Im} p(x)\overline{q(x)} = 0$ for all $x \in \mathbb{R}^n$,*

*(d) The polynomials $p$ and $q$ have real coefficients.*

*Proof.* We show $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$ and $(c) \Leftrightarrow (d)$. Starting with the first of these implications, note that $\lim_{y \to 0^+} \operatorname{Im} h(x + iy) = 0$ for $\lambda_n$-a.e $x \in \mathbb{R}^n$ by assumption. Since

$$\operatorname{Im} h(z) = \frac{\operatorname{Im} p(z)\overline{q(z)}}{q(z)\overline{q(z)}}, \tag{23}$$

it thus follows by continuity of $z \mapsto \operatorname{Im} p(z)\overline{q(z)}$ that $\operatorname{Im} p(x)\overline{q(x)} = 0$ for all $x \in \mathbb{R}^n$. The second implication is immediate from (23). As for the third, consider $q|_{\mathbb{R}^n}$, i.e the restriction of $q$ to $\mathbb{R}^n$, and let $u$ and $v$ be its real and imaginary parts, respectively. Both $u$ and $v$ are real polynomials on $\mathbb{R}^n$ so Lemma 2.31 implies that $\lambda_n(\mathcal{Z}_u) = \lambda_n(\mathcal{Z}_v) = 0$. Because $\mathcal{Z}_{q|_{\mathbb{R}^n}}$ is the intersection of these zero sets, it holds that

$$\lambda_n(\mathcal{Z}_{q|_{\mathbb{R}^n}}) = 0. \tag{24}$$

It reads from (23) that the limit of $\operatorname{Im} h(x + iy)$ as $y \to 0^+$ exists whenever $x \notin \mathcal{Z}_{q|_{\mathbb{R}^n}}$, so then $\lim_{y \to 0^+} \operatorname{Im} h(x + iy) = 0$ for $\lambda_n$-a.e $x \in \mathbb{R}^n$ by (24).

What remains is the equivalence $(c) \Leftrightarrow (d)$. The left implication $(\Leftarrow)$ is immediate. To show the right implication $(\Rightarrow)$, first note that the assumption $\operatorname{Im} p(x)\overline{q(x)} = 0$ implies

$$p(x)\overline{q(x)} = \overline{p(x)\overline{q(x)}} = \overline{p(x)}q(x), \quad \forall x \in \mathbb{R}^n .$$

In fact, $p(x)\overline{q(\overline{x})} = \overline{p(\overline{x})}q(x)$ for $x \in \mathbb{R}^n$. Hence, for $x \in \mathbb{R}^n$ such that $x \notin \mathcal{Z}_p \cup \mathcal{Z}_q$,

$$\frac{p(x)}{\overline{p(\overline{x})}} = \frac{q(x)}{\overline{q(\overline{x})}}. \tag{25}$$

Denote the left and right side of (25) by $\tilde{p}$ and $\tilde{q}$ and note that they naturally extend holomorphically to $\mathbb{C}^n \setminus \mathcal{Z}_{\overline{p}}$ and $\mathbb{C}^n \setminus \mathcal{Z}_{\overline{q}}$, respectively, where $\overline{p}(z), \overline{q}(z) \in \mathbb{C}[z_1, \dots, z_n]$ are the denominators

$$\overline{p}(z) := \overline{p(\overline{z})} \quad \text{and} \quad \overline{q}(z) := \overline{q(\overline{z})}.$$

Set $\Omega := \mathbb{C}^n \setminus (\mathcal{Z}_{\overline{p}} \cup \mathcal{Z}_{\overline{q}})$. At this point, we have two holomorphic functions

$$\tilde{p}(z) = \frac{p(z)}{\overline{p}(z)} \quad \text{and} \quad \tilde{q}(z) = \frac{q(z)}{\overline{q}(z)}$$

that agree on $\Omega \cap \mathbb{R}^n$. It follows from Lemma 2.30 that they agree on all of $\Omega$ by uniqueness of the complexifications $\tilde{p}$ and $\tilde{q}$. Further notice that both fractions are unimodular on their respective domains, so Lemma 2.28 implies that both are constant. Hence, there is some constant $\zeta \in \mathbb{C}$ such that $\tilde{p} \equiv \tilde{q} \equiv \zeta$ on $\Omega$. To put it more concretely,

$$\frac{p(z)}{\overline{p}(z)} = \zeta \quad \text{and} \quad \frac{q(z)}{\overline{q}(z)} = \zeta, \quad \forall z \in \Omega.$$

Multiply each side of both identities with their respective denominator to obtain $p(z) = \zeta\overline{p}(z)$ and $q(z) = \zeta\overline{q}(z)$ on $\Omega$. Each side of both of these identities have holomorphic extensions to $\mathbb{C}^n$, so the identity theorem – which can be applied since $\Omega$ is, in fact, a domain; openness is clear from closedness of $\mathbb{C}^n \setminus \mathcal{Z}_{\overline{p}}$ and $\mathbb{C}^n \setminus \mathcal{Z}_{\overline{q}}$, and connectedness follows from Lemma 2.26 (see the following Remark 2.27) – implies that the identities hold on all of $\mathbb{C}^n$:

$$p(z) = \zeta\overline{p}(z) \quad \text{and} \quad q(z) = \zeta\overline{q}(z), \quad \forall z \in \mathbb{C}^n .$$

Recall, however, that $p$ and $q$ have no nontrivial factor in common by definition, as $\frac{p}{q}$ is assumed to be a rational Herglotz-Nevanlinna function. Hence $\zeta = 1$, which yields

$$p(z) = \overline{p}(z) = \overline{p(\overline{z})} \quad \text{and} \quad q(z) = \overline{q}(z) = \overline{q(\overline{z})}, \quad z \in \mathbb{C}^n .$$

This holds in particular on $\mathbb{R}^n$ where $\overline{z} = z$, so then $p(x) = \overline{p(x)}$ and $q(x) = \overline{q(x)}$ for $x \in \mathbb{R}^n$, meaning that $p$ and $q$ are real on $\mathbb{R}^n$. The desired result follows. □

*Remark* 3.6. This characterization, and especially the equivalence between $(a)$ and $(d)$, is frequently used in the remaining parts of this thesis. It is therefore by convenience that $(a)$ and $(d)$ are used interchangeably in certain contexts without a reference to Lemma 3.5. To be clear, the definition of a real-rational function remains in its current form as suggested by Definition 3.4; it is the most natural definition, partly because its possible generalization to the full class of Herglotz-Nevanlinna functions. ◊

The first application of the above characterization is to show Corollary 3.7 below, which is concerned with restrictions of real-rational functions on certain complex hyperplanes.

**Corollary 3.7.** *Let $\zeta \in \mathbb{R}$ and suppose $h = \frac{p}{q}$ is a real-rational function of $n$ variables such that $q$ has no factor of the form $(z_j - \zeta)$ for some $j \in \{1, 2, \dots, n\}$. Then*

$$h_\zeta(w) := h(w_1, \dots, w_{j-1}, \zeta, w_{j+1}, \dots, w_n), \quad w \in \mathbb{H}^{n-1},$$

*is a real-rational function of $n-1$ variables.*

*Proof.* Assume without loss of generality that $j = 1$, meaning that $z_1$ is the variable fixed at $\zeta$. The function $h_\zeta$ is clearly rational, and its coefficients are real since both $\zeta$ and the coefficients of $h$ are real by assumption (see Lemma 3.5). The desired conclusion follows once it has been proven that $h_\zeta$ is a Herglotz-Nevanlinna function, which amounts to showing that $h_\zeta$ is holomorphic and that $\operatorname{Im} h_\zeta \geq 0$ on $\mathbb{H}^{n-1}$.

Given that $(z_1 - \zeta)$ is not a factor of $q$, and that $q$ have real coefficients, it follows that $q(\zeta, w)$ is nonzero for all $w \in \mathbb{H}^{n-1}$. Thus, $h$ extends continuously to $(\zeta, w)$ for each $w \in \mathbb{H}^{n-1}$. In particular, $h_\zeta$ is well-defined on $\mathbb{H}^{n-1}$. Holomorphy of $h_\zeta$ is now immediate from $h$.

Next note that $\operatorname{Im} h$ is defined and continuous wherever $h$ is (see (23)). Since $h$ extends continuously to $\{\zeta\} \times \mathbb{H}^{n-1}$, and $\operatorname{Im} h \geq 0$ on $\mathbb{H}^n$, it follows that $\operatorname{Im} h(\zeta, w) = \operatorname{Im} h_\zeta$ is well-defined and nonnegative for all $w \in \mathbb{H}^{n-1}$. □

*Remark* 3.8. If $\zeta \in \mathbb{R}$ is replaced by $\zeta \in \mathbb{H}$ in Corollary 3.7, the conclusion is no longer true in general; consider e.g $h(z,w) = z + w$ and let $\zeta = i$, then $h_\zeta(w) = i + w$ which is not a real-rational function. However, $h_\zeta$ is still a rational Herglotz-Nevanlinna function, and is a conclusion that *does* hold in general. Moreover, for this conclusion, the assumption on the factors of $q$ can be removed. ◊

## 3.3 Nevanlinna measures of real-rational functions

It turns out that the zero set of the denominator of a real-rational function is responsible for some of the structure of its corresponding Nevanlinna measure.

**Theorem 3.9** (The support theorem)**.** *Let $h = \frac{p}{q}$ be a real-rational Herglotz-Nevanlinna function of $n$ variables and let $\mu$ be the measure from its representation (15). If $q$ is nonconstant then $\mu$ is singular with its support contained in $\mathcal{Z}_q \cap \mathbb{R}^n$.*

*Proof.* Consider the restriction $q_{\mathbb{R}^n}$ of $q$ to $\mathbb{R}^n$. Since the coefficients are real, it follows from Lemma 2.31 that $\lambda_n(\mathcal{Z}_{q_{\mathbb{R}^n}}) = 0$. The desired result is then obtained by showing that $\mu(\mathbb{R}^n \setminus \mathcal{Z}_{q_{\mathbb{R}^n}}) = 0$; singularity is immediate, and the claim on its support follows from Lemma 2.1. To this end, because $\mathbb{R}^n \setminus \mathcal{Z}_{q_{\mathbb{R}^n}}$ is open, it suffices to show that $\mu(B) = 0$ whenever $B \subset \mathbb{R}^n \setminus \mathcal{Z}_{q_{\mathbb{R}^n}}$ is an open ball. So let $B \subset \mathbb{R}^n \setminus \mathcal{Z}_{q_{\mathbb{R}^n}}$ be an open ball and set $\tilde{\mu} := \mu|_B$, where $\mu|_B(A) := \mu(A \cap B)$ for all Borel subsets $A \subset \mathbb{R}^n$. Choose an arbitrary closed subset $F \subset B$ and set $N := \mathbb{R}^n \setminus B$. By Lemma 2.34 (and the proceeding paragraph), there is a smooth bump function $\varphi : \mathbb{R}^n \to [0,1]$ satisfying the condition (11) such that $\varphi \equiv 1$ on $F$ and $\varphi \equiv 0$ on $N$. Recall that $\lim_{y \to 0^+} \operatorname{Im} h(x + iy) = 0$ by assumption, so for sufficiently small $y \in \mathbb{R}^n$ we have $\left|\operatorname{Im} h(x + iy)\right| \leq 1$ for all $x \in \mathbb{R}^n$. Since $|\varphi| \leq 1$ on $B$ and vanishes outside of $B$ it follows that

$$\left|\varphi(x) \operatorname{Im} h(x + iy)\right| \leq \chi_B(x), \quad \forall x \in \mathbb{R}^n .$$

The indicator function $\chi_B$ is of course $\lambda_n$-integrable over $\mathbb{R}^n$, so Lebesgue's dominated convergence theorem now combines with Corollary 2.14(iv) to give

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi \, d\mu &= \lim_{y \to 0^+} \int_{\mathbb{R}^n} \varphi(x) \operatorname{Im} h(x + iy) \, d\lambda_n(x) \\ &= \int_{\mathbb{R}^n} \varphi(x) \lim_{y \to 0^+} \operatorname{Im} h(x + iy) \, d\lambda_n(x) = 0. \end{aligned}$$

Since $\tilde{\mu} \leq \mu$ and $F \subset \mathbb{R}^n$ we have

$$\tilde{\mu}(F) = \int_F \varphi \, d\tilde{\mu} \leq \int_{\mathbb{R}^n} \varphi \, d\mu,$$

hence $\tilde{\mu}(F) = 0$ by the above. Recall that $F$ was chosen arbitrarily, so all closed subsets of $B$ have $\tilde{\mu}$-measure zero. By regularity of $\tilde{\mu}$ (inherited from $\mu$), it follows that each open $U \subset B$ satisfies

$$\tilde{\mu}(U) = \sup\{\tilde{\mu}(K) : K \subset B \text{ compact and } K \subset U\} = 0.$$

Regularity now further implies that each Borel subset $A \subset B$ satisfies

$$\tilde{\mu}(A) = \inf\{\tilde{\mu}(U) : U \subset B \text{ open and } A \subset U\} = 0.$$

In particular, $\mu(B) = \tilde{\mu}(B) = 0$, which concludes the proof. □

An additional assumption on the structure of the denominator leads to the stronger claim that $\operatorname{supp} \mu = \mathcal{Z}_q \cap \mathbb{R}^n$. Perhaps this is the case for all real-rational functions, but showing it would require a slightly different approach than what is offered in the given proof.

**Corollary 3.10.** *Let $h = \frac{p}{q}$ be a real-rational Herglotz-Nevanlinna function of $n$ variables and let $\mu$ be the measure from its representation* (15)*. If $q$ is nonconstant and affine, then $\mu$ is singular and* $\operatorname{supp}\mu = \mathcal{Z}_q \cap \mathbb{R}^n$*.*

*Proof.* Theorem 3.9 confirms that $\mu$ is singular and that $\operatorname{supp}\mu \subset \mathcal{Z}_q \cap \mathbb{R}^n$. The equality $\operatorname{supp}\mu = \mathcal{Z}_q \cap \mathbb{R}^n$ is now obtained by showing that $\operatorname{supp}\mu$ is dense in $\mathcal{Z}_q \cap \mathbb{R}^n$ as $\operatorname{supp}\mu$ is closed in $\mathbb{R}^n$ (and therefore in the subspace topology) by Lemma 2.2.

Suppose for contradiction that $\operatorname{supp}\mu$ is not dense in $\mathcal{Z}_q \cap \mathbb{R}^n$. Then there is a nonempty open subset of $\mathcal{Z}_q \cap \mathbb{R}^n$, call it $U$, such that $U \cap \operatorname{supp}\mu = \varnothing$. Moreover, $U$ contains a nonempty open cube that we project onto the coordinate axes of $\mathbb{R}^n$ to obtain an $n$-dimensional open rectangle $C := (\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n)$ satisfying $C \cap (\mathcal{Z}_q \cap \mathbb{R}^n) \subset U$. Since $\operatorname{supp}\mu \subset \mathcal{Z}_q \cap \mathbb{R}^n$, it follows that

$$C \cap \operatorname{supp}\mu = (\mathcal{Z}_q \cap \mathbb{R}^n) \cap C \cap \operatorname{supp}\mu \subset U \cap \operatorname{supp}\mu = \varnothing. \tag{26}$$

Since the coefficients of the denominator $q$ are real, its zero set is a $(n-1)$-dimensional hyperplane. By construction of $C$, this hyperplane makes a main diagonal of $C$ so that the $n$-dimensional cross of stripes

$$\tilde{C} := \bigcup_{j=1}^{n} \{x \in \mathbb{R}^n : \alpha_j < x_j < \beta_j\}$$

satisfies $\tilde{C} \cap \mathcal{Z}_q = C \cap \mathcal{Z}_q$. But then (26) yields $\tilde{C} \cap \operatorname{supp}\mu = \varnothing$, which by Theorem 2.22 implies that $\mu \equiv 0$. In accordance with Theorem 2.12, this means that $h$ is a polynomial. In particular, $q$ is constant, and we arrive at a contradiction. □

The support theorem has many significant implications, one of which is the following.

**Corollary 3.11.** *Let $h$ be a Herglotz-Nevanlinna function and let $\mu$ be the measure from its representation* (15)*. If $h$ is real-rational with an affine denominator $q$, then $\mu$ is either the zero measure or on one of the forms* (19) *or* (20)*. Specifically,*

*(i)* $\mu \equiv 0$ *if $q$ is constant,*

*(ii)* $\mu$ *is on the form* (19) *if $q$ only depends on one variable,*

*(iii)* $\mu$ *is on the form* (20) *if $q$ depends on two or more variables.*

*Proof.* Since $q$ is affine, it is of at most total degree 1. If $q$ is of total degree 0 then $h$ is a polynomial, which by Lemma 3.2 implies that $\mu$ is a nonnegative constant multiple of the Lebesgue measure. The coefficients of $q$ are real by assumption, so a quick comparison with the measure in Example 2.17 then shows that $\mu \equiv 0$ as to avoid a nonzero imaginary part of the constant term of $h$.

Now suppose $q$ is of total degree 1. Due to its real coefficients, the set $\mathcal{Z}_q \cap \mathbb{R}^n$ is a real $(n-1)$-dimensional hyperplane which by Theorem 3.9 supports $\mu$. The structure of $\mu$ is therefore given by either Theorem 2.18 or Theorem 2.19, depending on the specifics of the hyperplane; the former if $q$ depends on one variable, and the latter if $q$ depends on at least two. □

*Remark* 3.12. As evident from Corollary 3.11, Theorem 3.9 is useful for determining some of the structure of the Nevanlinna measures associated with real-rational functions. Beyond this, it can be combined with Theorem 2.22 to give additional information about the denominators of such functions directly. For example, there is no real-rational function $h$ of $n$ variables with a denominator of the form $q(z) := z_1^2 + \cdots + z_n^2 - 1$; supposing the contrary, the support of the corresponding Nevanlinna measure is contained in the $(n-1)$-sphere $\mathcal{Z}_q \cap \mathbb{R}^n = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$, which is bounded. By Theorem 2.22 the measure is the zero measure (see also Figure 1), so then by Theorem 2.12 the function $h$ is a polynomial (i.e $q$ is constant) – a contradiction. ◊

*Remark* 3.13. Notice that when $q \in C[z_1, \dots, z_n]$ is affine then $\mathcal{Z}_q = \mathcal{Z}_{q^m}$ for all $m \in \mathbb{N}$, so the conclusion of Corollary 3.11 technically extends to all real-rational functions with a denominator of the form $q^m$ for some $m \in \mathbb{N}$ and affine $q \in C[z_1, \dots, z_n]$. However, this observation does not contribute any additional insights; as it turns out, there are no real-rational functions with a denominator of the form $q^m$ for any integer $m \geq 2$ and affine $q \in C[z_1, \dots, z_n]$. This follows from the upcoming Theorem 3.18, from which it is clear that none of the measures listed in the corollary correspond to a real-rational function with a denominator of total degree 2 or more, and the support theorem. ◊

This subchapter concludes with an invariance property of real-rational functions and their Nevanlinna measures. It asserts that the composition of a real-rational function with an automorphism of $\mathbb{H}^n$ is itself real-rational. It also expresses the Nevanlinna measure of such a composition in terms of the automorphism and the measure of the original function.

Given a polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$, the notation $d_j^p$ is used to denote the degree in $z_j$ of the term in $p$ of lowest degree in $z_j$. For example, if $p(z_1, z_2, z_3) = z_1^2 z_2(1 + z_3)$ then $d_1^p = 2$, $d_2^p = 1$ and $d_3^p = 0$.

**Proposition 3.14** (The automorphism invariance property)**.** *Let $h = \frac{p}{q}$ be a real-rational function of $n$ variables and let $f \in \operatorname{Aut} \mathbb{H}^n$. Then $h \circ f$ is real-rational. Further let $(0, 0, \mu)$ be the data of $h$ from representation (15) and write*

$$f(z) = P\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \dots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right)$$

*for some permutation matrix $P$ in accordance with (4). If*

$$d_j^p \geq d_j^q \text{ for each } j \in \{1, 2, \dots, n\} \text{ that satisfies } a_j = 0, \tag{27}$$

*then the data of $h \circ f$ is $(a, b, \nu)$ with $a = \operatorname{Re} h(f(i))$, $b = 0 \in \mathbb{R}^n$ and $\nu = \frac{1}{\det J_f(t)} f^* \mu$, where $J_f$ denotes the holomorphic Jacobian matrix of $f$.*

*Proof.* It is clear that $h \circ f$ is a Herglotz-Nevanlinna function. In fact, since each component function of $f$ is rational with real coefficients (recall (4)), the composition is real-rational.

By Corollary 2.14(i), the parameter $a$ from the data $(a, b, \nu)$ of $h \circ f$ is given by $h(f(i))$. Next to address is the claim $b = (b_1, \dots, b_n) = 0$, which is a little more technical. First define

$$N(z) := \prod_{j=1}^{n} (c_j z_j + d_j)^{\deg_{j_0} q} p(f(z)) \quad \text{and} \quad D(z) := \prod_{j=1}^{n} (c_j z_j + d_j)^{\deg_{j_0} q} q(f(z)),$$

where $j_0 \in \{1, 2, \dots, n\}$ is the index $j$ for which $\deg_j q$ is maximized. These two functions are complex polynomials by construction, as the denominators of $p \circ f$ and $q \circ f$ are cleared by multiplication with a factor of sufficiently high degree in each variable. We show this formally for clarity and to make it easy to read the degree of the polynomials, which is important for later. For now, focus on $D$. Assume temporarily that the permutation matrix $P$ is the identity and write $q(z) = \sum_{l=0}^{\deg_1 q} k_l(z_2, \dots, z_n) z_1^l$ with polynomial coefficients $k_l \in \mathbb{C}[z_2, \dots, z_n]$. Then

$$\begin{aligned} D(z) &= \prod_{j=1}^{n} (c_j z_j + d_j)^{\deg_{j_0} q} \left[ \sum_{l=0}^{\deg_1 q} k_l(f_2(z), \dots, f_n(z)) \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}\right)^l \right] \\ &= \prod_{j=2}^{n} (c_j z_j + d_j)^{\deg_{j_0} q} \left[ \sum_{l=0}^{\deg_1 q} k_l(f_2(z)), \dots, f_n(z))(a_1 z_1 + b_1)^l (c_1 z_1 + d_1)^{\deg_{j_0} q - l} \right], \end{aligned}$$

where $\deg_{j_0} q - l \geq 0$, $l = 0, 1, \dots, \deg_1 q$. Since each component function $f_j$ only really depends on the variable $z_j$, $D$ is a polynomial in the variable $z_1$. Iterating this procedure $n$ times yields that

$D$ is separately polynomial, which also means that $D \in \mathbb{C}[z_1, \ldots, z_n]$ (see e.g [KL22] for how this follows). This holds true even in the case of a nontrivial permutation matrix $P$; the component functions are permuted in the above and the necessary factor from the product is extracted and put into the sum to obtain the polynomial structure. An analogous argument yields $N \in \mathbb{C}[z_1, \ldots, z_n]$ after invoking Lemma 3.3(ii) on $h$ to obtain $\deg_j q \geq \deg_j p$ for each $j \in \{1, 2, \ldots n\}$.

The polynomials $N$ and $D$ clearly satisfies $h \circ f = \frac{N}{D}$. We can assume that they have no factor in common (if they do, simply cancel the common factors and continue with the remaining rational function), so then $\frac{N}{D}$ is the canonical form of the real-rational function $h \circ f$. Applying Lemma 3.3(ii) yields for any $j \in \{1, 2, \ldots, n\}$ that $b_j = 0$ whenever $\deg_j D \geq \deg_j N$. To reach the desired conclusion $b = 0$, this inequality is the target. As indicated earlier, $P$ effectively only permutes the variables that $N$ and $D$ depend on. Since the process of showing $\deg_j D \geq \deg_j N$ is the same for each index $j$, we can assume that $P$ is trivial and settle for the case $j = 1$.

The degree of $D$ in $z_1$ is obtained from the above, where $D$ is written as a polynomial in $z_1$. To see this, denote $z \mapsto k_l(f_2(z), \ldots, f_n(z))$ by $\tilde{k_l}$ and consider the case $a_1 = 0$. Then $c_1 \neq 0$ and

$$\deg_1 q = \max_{\substack{l=1,2,\ldots,\deg_1 q \\ \tilde{k_l} \not\equiv 0}} \{\deg_{j_0} q - l\} = \deg_{j_0} q - l_{\min},$$

where $l_{\min}$ is taken as the smallest index $l$ for which $\tilde{k_l} \not\equiv 0$. It is, however, desirable to specify $\deg_1 q$ in terms of the original function $h$. Seeing that $z \mapsto (f_2(z), \ldots, f_n(z))$ is surjective onto $\mathbb{H}^{n-1}$, $\tilde{k_l} \equiv 0$ if and only if $k_l \equiv 0$. This means that $l_{\min}$ also is the smallest index $l$ for which $k_l \not\equiv 0$. Since the functions $k_l$ are the polynomial coefficients of $q$ when $q$ is written as a polynomial in $z_1$, we arrive at $l_{\min} = d_1^q$ (recall $d_1^q$ from the preceding paragraph of the proposition statement).

If $c_1 = 0$ instead, then $a_1 \neq 0$ and

$$\deg_1 q = \max_{\substack{l=1,2,\ldots,\deg_1 q \\ \tilde{k_l} \not\equiv 0}} \{l\} = l_{\max},$$

where $l_{\max}$ is taken as the largest index $l$ for which $\tilde{k_l} \not\equiv 0$. Similar to the previous case, $l_{\max}$ is also the largest index for which $k_l \not\equiv 0$, hence $l_{\max} = \deg_1 q$.

Finally, if both coefficients $a_1$ and $c_1$ of $f$ are nonzero, then

$$\deg_1 q = \max_{\substack{l=1,2,\ldots,\deg_1 q \\ \tilde{k_l} \not\equiv 0}} \{l + \deg_{j_0} q - l\} = \deg_{j_0} q.$$

In conclusion,

$$\deg_1 D = \begin{cases} \deg_{j_0} q - d_1^q, & a_1 = 0, \\ \deg_1 q, & c_1 = 0, \\ \deg_{j_0} q, & \text{otherwise.} \end{cases}$$

An analogous argument yields

$$\deg_1 N = \begin{cases} \deg_{j_0} q - d_1^p, & a_1 = 0, \\ \deg_1 q, & c_1 = 0, \\ \deg_{j_0} q, & \text{otherwise.} \end{cases}$$

These combine to give

$$\deg_1 D - \deg_1 N = \begin{cases} d_1^p - d_1^q, & a_1 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\deg_1 D \geq \deg_1 N$ if $d_1^p \geq d_1^q$ whenever $a_1 = 0$, concluding the proof of the claim $b = 0$ under the given assumption.

With the parameters $a$ and $b$ determined, the representing measure $\nu$ is next. Given $b = 0$ it follows from Corollary 2.14(v) that

$$\operatorname{Im} h(z) = \frac{1}{\pi^n} \int_{\mathbb{R}^n} \mathcal{P}_n(z,t)\, d\mu \quad \text{and} \quad \operatorname{Im} h \circ f(z) = \frac{1}{\pi^n} \int_{\mathbb{R}^n} \mathcal{P}_n(z,t)\, d\nu. \tag{28}$$

Since $f(z) \in \mathbb{H}$ on $\mathbb{H}^n$, the representation on the left in (28) implies that

$$\operatorname{Im} h \circ f(z) = \frac{1}{\pi^n} \int_{\mathbb{R}^n} \mathcal{P}_n(f(z),t)\, d\mu. \tag{29}$$

The integral on the right in (29) can be rewritten as

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{P}_n(f(z),t)\, d\mu &= \int_{\mathbb{R}^n} \det(J_{f^{-1}}(t))\, \mathcal{P}_n(z, f^{-1}(t))\, d\mu \\ &= \int_{\mathbb{R}^n} \frac{1}{\det J_f(f^{-1}(t))}\, \mathcal{P}_n(z, f^{-1}(t))\, d\mu \\ &= \int_{\mathbb{R}^n} \frac{1}{\det J_f(t)}\, \mathcal{P}_n(z,t)\, d(f^*\mu), \end{aligned}$$

where the first equality follows from the automorphism invariance of the Poisson kernel (Lemma 2.11) and the third is due to Theorem 2.3. With (29) we arrive at

$$\operatorname{Im} h \circ f(z) = \frac{1}{\pi^n} \int_{\mathbb{R}^n} \frac{1}{\det J_f(t)}\, \mathcal{P}_n(z,t)\, d(f^*\mu). \tag{30}$$

A comparison to the representation on the right in (28) yields

$$\int_{\mathbb{R}^n} \mathcal{P}_n(z,t)\, d\nu = \int_{\mathbb{R}^n} \frac{1}{\det J_f(t)}\, \mathcal{P}_n(z,t)\, d(f^*\mu).$$

Once it is shown that $\frac{1}{\det J_f(t)} f^*\mu$ is a Nevanlinna measure, it follows by uniqueness of the Nevanlinna measure $\nu$ that

$$\nu = \frac{1}{\det J_f(t)} f^*\mu,$$

completing the proof.

The pullback itself is a positive Borel measure on $\mathbb{R}^n$, so then $\frac{1}{\det J_f(t)} f^*\mu$ is one too since the density function $\det J_f(t)$ is positive; this is seen in (8), given that each partial derivative is positive. It remains to check that the measure satisfies the growth condition (11) and Nevanlinna condition (13). With a slight abuse of notation, the inverse function $f^{-1} \in \operatorname{Aut} \mathbb{H}^n$ can be written

$$f(z) = P\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right)$$

for some permutation matrix $P$, where the coefficients $a_j, b_j, c_j$ and $d_j$ are not to be confused with those used for $f$ previously in this proof. As usual, we can assume without loss of generality that

$P$ is the identity. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{1}{\det J_f(t)} \prod_{j=1}^{n} \frac{1}{1+t_j^2}\, d(f^*\mu)(t) &= \int_{\mathbb{R}^n} \frac{1}{\det J_f(f^{-1}(t))} \prod_{j=1}^{n} \frac{1}{1+(f_j^{-1}(t))^2}\, d\mu \\
&= \int_{\mathbb{R}^n} \det J_{f^{-1}}(t) \prod_{j=1}^{n} \frac{1}{1+(f_j^{-1}(t))^2}\, d\mu \\
&= \int_{\mathbb{R}^n} \prod_{j=1}^{n} \frac{a_j d_j - b_j c_j}{(c_j t_j + d_j)^2} \prod_{j=1}^{n} \frac{1}{1+(\frac{a_j t_j + b_j}{c_j t_j + d_j})^2}\, d\mu \\
&= \int_{\mathbb{R}^n} \prod_{j=1}^{n} \frac{a_j d_j - b_j c_j}{(a_j t_j + b_j)^2 + (c_j t_j + d_j)^2}\, d\mu \\
&\leq C \int_{\mathbb{R}^n} \prod_{j=1}^{n} \frac{1}{1+t_j^2}\, d\mu < \infty,
\end{aligned}$$

where the final inequality follows from the fact that $\mu$ satisfies the growth condition. It is a fairly straightforward (but tedious) task to check that the second-to-last inequality holds when $C$ is chosen as $\prod_{j=1}^{n} C_j$, with each

$$C_j := \frac{(a_j^2 + b_j^2 + c_j^2 + d_j^2) - \sqrt{(a_j^2 + b_j^2 - c_j^2 - d_j^2)^2 + 4(a_j b_j + c_j d_j)^2}}{2(a_j d_j - b_j c_j)}.$$

In any case, the measure $\frac{1}{\det J_f(t)} f^*\mu$ satisfies the growth condition. As for the Nevanlinna condition, let $1 \leq j_1 < j_2 \leq n$ be arbitrary and take any $z \in \mathbb{H}^n$. The goal is to show that

$$\int_{\mathbb{R}^n} \frac{1}{\det J_f(t)} \frac{1}{(t_{j_1} - z_{j_1})^2 (t_{j_2} - \overline{z_{j_2}})^2} \prod_{\substack{j=1 \\ j \neq j_1, j_2}}^{n} \frac{2 \operatorname{Im} z_j}{|t_j - z_j|^2}\, d(f^*\mu)(t) = 0. \tag{31}$$

Rewrite the integral in (31) in the usual fashion for the pullback as

$$\int_{\mathbb{R}^n} \frac{1}{\det J_f(f^{-1}(t))} \frac{1}{(f_{j_1}^{-1}(t) - z_{j_1})^2 (f_{j_2}^{-1}(t) - \overline{z_{j_2}})^2} \prod_{\substack{j=1 \\ j \neq j_1, j_2}}^{n} \frac{2 \operatorname{Im} z_j}{\left|f_j^{-1}(t) - z_j\right|^2}\, d\mu(t), \tag{32}$$

where

$$\frac{1}{\det J_f(f^{-1}(t))} = \det J_{f^{-1}}(t) = \prod_{j=1}^{n} \frac{\partial (f^{-1})_j}{\partial z_j}(t) \tag{33}$$

as in the calculations for the growth condition. These derivatives will soon be seen to cancel out. In the following, observe that each component function $f_j$ is an automorphism of $\mathbb{H}$ of the single variable $z_j$. Lemma 2.29 yields

$$\begin{aligned}
\frac{1}{(f_{j_1}^{-1}(t) - z_{j_1})^2} &= \frac{1}{(f_{j_1}^{-1}(t_{j_1}) - z_{j_1})^2} \\
&= \frac{\frac{\partial f_{j_1}}{\partial z_{j_1}}(f^{-1}(t_{j_1}))\frac{\partial f_{j_1}}{\partial z_{j_1}}(z)}{(t_{j_1} - f_{j_1}(z_{j_1}))^2} = \frac{\frac{\partial f_{j_1}}{\partial z_{j_1}}(z)}{(t_{j_1} - f_{j_1}(z_{j_1}))^2} \frac{1}{\frac{\partial (f^{-1})_{j_1}}{\partial z_{j_1}}(t_{j_1})}.
\end{aligned}$$

In this way we obtain

$$\frac{1}{(f_{j_1}^{-1}(t) - z_{j_1})^2 (f_{j_2}^{-1}(t) - \overline{z_{j_2}})^2} = \frac{\frac{\partial f_{j_1}}{\partial z_{j_1}}(z)\frac{\partial f_{j_2}}{\partial z_{j_2}}(z)}{(t_{j_1} - f_{j_1}(z_{j_1}))^2 (t_{j_2} - \overline{f_{j_2}(z_{j_2})})^2} \frac{1}{\frac{\partial (f^{-1})_{j_1}}{\partial z_{j_1}}(t)\frac{\partial (f^{-1})_{j_2}}{\partial z_{j_2}}(t)}. \tag{34}$$

Moreover, since

$$\frac{2\operatorname{Im} z_j}{\left|f_j^{-1}(t)-z_j\right|^2} = 2\mathcal{P}_1(z_j, f_j^{-1}(t)),$$

Lemma 2.11 yields

$$\prod_{\substack{j=1 \\ j\neq j_1,j_2}}^{n} \frac{2\operatorname{Im} z_j}{\left|f_j^{-1}(t)-z_j\right|^2} = \prod_{\substack{j=1 \\ j\neq j_1,j_2}}^{n} \frac{1}{\frac{\partial (f^{-1})_j}{\partial z_j}(t)} \frac{2\operatorname{Im} f_j(z_j)}{\left|t_j - f_j(z_j)\right|^2}. \tag{35}$$

Notice that the partial derivatives in (33) cancel with those in (34) and (35), so with $w = f(z) \in \mathbb{H}^n$ the integral (32) is equal to

$$\int_{\mathbb{R}^n} \frac{\frac{\partial f_{j_1}}{\partial z_{j_1}}(z)\frac{\partial f_{j_2}}{\partial z_{j_2}}(z)}{t_{j_1}-f(z_{j_1}))^2(t_{j_2}-\overline{f(z_{j_2})})^2} \prod_{\substack{j=1 \\ j\neq j_1,j_2}}^{n} \frac{2\operatorname{Im} f_j(z_j)}{\left|t_j - f_j(z_j)\right|^2}\, d\mu(t) =$$

$$\frac{\partial f_{j_1}}{\partial z_{j_1}}(z)\frac{\partial f_{j_2}}{\partial z_{j_2}}(z)\int_{\mathbb{R}^n} \frac{1}{(t_{j_1}-w_{j_1})^2(t_{j_2}-\overline{w_{j_2}})^2} \prod_{\substack{j=1 \\ j\neq j_1,j_2}}^{n} \frac{2\operatorname{Im} w_j}{\left|t_j - w_j\right|^2}\, d\mu = 0.$$

The final equality follows from the fact that $\mu$ satisfies the Nevanlinna condition, completing the proof. □

*Remark* 3.15. Note that Proposition 3.14 also holds for the larger class of rational Herglotz-Nevanlinna functions; simply change "real-rational" for "rational Herglotz-Nevanlinna function" in the proposition statement and its proof. ♢

*Example* 3.16. Consider the one-variable Herglotz-Nevanlinna function $h(z) = -\frac{1}{z} =: \frac{p(z)}{q(z)}$ and note that its data from representation (15) is $(0, 0, \mu)$ (the specifics of the measure are superfluous here; see Theorem 3.18 below for more). Let $f(z) = \frac{az+b}{cz+d}$ be the automorphism of $\mathbb{H}$ with $a = d = 0$, $b = -1$ and $c = 1$. Then $h \circ f$ is the identity. In particular, it is a real-rational function with the representing data $(0, 1, 0)$. Further note that $d^p = 0$ and $d^q = 1$, meaning that $d^p < d^q$. Hence, every assumption in Proposition 3.14 is satisfied with the exception of (27). Since the data of $h \circ f$ is inconsistent with the conclusion in Proposition 3.14, this shows that the assumption (27) cannot be omitted from the proposition statement.

If $a \neq 0$ instead then assumption (27) is automatically satisfied, meaning that we expect $b = 0$. Indeed, by Corollary 2.14(iii), we obtain

$$b = \lim_{z \hat{\to} \infty} \frac{h(f(z))}{z} = \lim_{z \hat{\to} \infty} -\frac{z}{z(az-1)} = 0. \quad ♢$$

*Example* 3.17. On the theme of the previous example, consider the two-variable function $h(z_1, z_2) = \frac{p(z_1,z_2)}{q(z_1,z_2)} = -\frac{1}{z_1+z_2}$ and note that its representing data is $(0, 0, \mu)$ (the specifics of the measure are superfluous here; see Theorem 3.18 below for more). Take any

$$f(z) = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}\right) \in \mathbb{H}^2.$$

Then $h \circ f$ is a real-rational function given by

$$h(f(z)) = -\frac{1}{\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1} + \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}} = -\frac{(c_1 z_1 + d_1)(c_2 z_2 + d_2)}{(a_1 z_1 + b_1)(c_2 z_2 + d_2) + (a_2 z_2 + b_2)(c_1 z_1 + d_1)}.$$

Denote the data of $h \circ f$ by $(a, b, \mu)$. Since $d_1^p = d_2^p = d_1^q = d_2^q = 0$, assumption (27) is satisfied regardless of the coefficients $a_1$ and $a_2$, so we expect $b = 0$ by Proposition 3.14. Indeed, by Corollary 2.14(iii), we obtain

$$b_j = \lim_{z_j \to \infty} \frac{h(f(z))}{z_j} = \lim_{z_j \to \infty} -\frac{(c_1 z_1 + d_1)(c_2 z_2 + d_2)}{z_j((a_1 z_1 + b_1)(c_2 z_2 + d_2) + (a_2 z_2 + b_2)(c_1 z_1 + d_1))} = 0, \quad j = 1, 2,$$

regardless of the values of $a_1$ and $a_2$. ♢

## 3.4 The characterization theorem

In the following, a complete characterization is given in the two-dimensional setting for the class of real-rational functions with an affine denominator by the corresponding class of Nevanlinna measures. An exact description of the relation between function and measure is also provided. Note that half of the characterization has already been addressed by the support theorem via Corollary 3.11. The other half requires residue calculus and involves the automorphism invariance property of real-rational functions.

**Theorem 3.18** (The characterization theorem)**.** *Let $h$ be a Herglotz-Nevanlinna function of two variables $(z, w)$ and let $\mu$ be the measure from its representation (15). Then $h$ is real-rational with an affine denominator $q$ if and only if $\mu$ is the zero measure or of the form (19) or (20). Specifically,*

*(i) $q$ is constant if and only if $\mu \equiv 0$,*

*(ii) $q$ depends on one variable if and only if $\mu$ is of the form (19), in which case the pure-integral part of $h$ is (up to permutation of coordinates) a positive constant multiple of $(z, w) \mapsto -\frac{1}{w-\alpha} + C$, $C := \frac{\alpha}{1+\alpha^2}$,*

*(iii) $q$ depends on two variables if and only if $\mu$ is of the form (20), in which case the pure-integral part of $h$ is given by*

$$(z, w) \mapsto \frac{b(z - \beta + c)(\alpha w - c) - d}{z + \alpha w - \beta} + D, \quad D = \frac{b((c - \beta)(\alpha^2 + \beta c) - c) + \beta d}{\beta^2 + (\alpha + 1)^2},$$

*when $\mu$ is written as*

$$\mu(A) = \frac{\pi}{\alpha} \int_{\mathbb{R}} k(t) \chi_A(t + \beta, -\frac{1}{\alpha} t)\, dt, \quad k(t) = b(t + c - \beta)^2 + d,$$

*with $\alpha > 0$, $b, d \geq 0$ and $\beta, c \in \mathbb{R}$.*

*Proof.* First assuming that $h$ is real-rational with an affine denominator, the desired result is obtained from Corollary 3.11. As for the reverse direction, we consider each case (i)-(iii) separately.

It follows from Theorem 2.12 that $h$ is a polynomial when $\mu \equiv 0$, meaning that $q$ is constant. Suppose next that $\mu$ is of the form (19). Up to permutation of factors, the measure is a positive multiple of $\lambda \times \delta_\alpha$ for some $\alpha \in \mathbb{R}$. Compare with (17) to see that

$$\frac{1}{\pi^2} \int_{\mathbb{R}^2} K_2((z, w), (t_1, t_2)\, d\mu(t) = \frac{1}{\pi} \int_{\mathbb{R}} K_1(w, t_2)\, d\delta_\alpha(t_2) = K_1(w, \alpha) = -\frac{1}{w - \alpha} + \frac{\alpha}{1 + \alpha^2}.$$

Hence, the pure-integral part of representation (15) of $h$ is (up to permutation of coordinates) a positive constant multiple of $(z, w) \mapsto -\frac{1}{w-\alpha} + \frac{\alpha}{1+\alpha^2}$.

Finally suppose that $\mu$ is of the form (20). The supporting hyperplane of $\mu$ is a line in $\mathbb{R}^2$ that – due to Theorem 2.22 – has a negative slope. For now, assume that the line is given by

$\ell_1 := \{t_1 + t_2 = 0\}$ and parametrize it by $t \mapsto (t, -t)$. In accordance with Remark 2.21, the $\mu$-measure of a Borel subset $A \subset \mathbb{R}^2$ is then

$$\mu(A) = \pi \int_{\mathbb{R}} k(t)\chi_A(t, -t)\, dt, \quad k(t) = b(t + c)^2 + d, \tag{36}$$

where $b, d \geq 0$ and $c \in \mathbb{R}$.

With (36) we rewrite the pure-integral part of $h$ as

$$\frac{1}{\pi^2} \int_{\mathbb{R}^2} K_2((z, w), (t_1, t_2))\, d\mu(t_1, t_2) = \frac{1}{\pi} \int_{\mathbb{R}} (b(t + c)^2 + d) K_2((z, w), (t, -t))\, dt, \tag{37}$$

where

$$K_2((z, w), (t, -t)) = \frac{-i(i + z)(i + w)}{2(t - z)(t + i)(t + w)(t - i)} + \frac{-i}{(t^2 + 1)^2}. \tag{38}$$

Denote the first and second term of the kernel (as functions of $t$) by $A(t)$ and $B(t)$, respectively. Note that $A$ and $B$ extend holomorphically to neighborhoods of $\overline{H} \setminus \{z, i\}$ and $\overline{H} \setminus \{i\}$. In the case that $z \neq i$, these two points are simple poles of $A$; otherwise they collapse into a double pole. Meanwhile, $B$ has a double pole at $\zeta = i$.

Consider the integral

$$\int_{\Gamma_R} k(\zeta)(A(\zeta) + B(\zeta))\, d\zeta, \tag{39}$$

where $\Gamma_R$ is the semi-circle in the closed upper half-plane $\overline{H}$ centered at the origin and with radius $R$. Write $\Gamma_R = I_R \cup C_R$, where $I_R$ is the segment of $\Gamma_R$ on the real-axis and $C_R$ is the arc of the semi-circle. Then

$$\int_{\Gamma_R} k(\zeta)(A(\zeta) + B(\zeta))\, d\zeta = \int_{I_R} k(\zeta)(A(\zeta) + B(\zeta))\, d\zeta + \int_{C_R} k(\zeta)(A(\zeta) + B(\zeta))\, d\zeta.$$

The second of these integrals tends to zero as $R$ grows arbitrarily large; on $|\zeta| = R \geq 1$ the density $k$ satisfies

$$\left|k(\zeta)\right| \leq K|\zeta|^2 = KR^2, \quad K := b(1 + c)^2 + d.$$

Similarly, when $\frac{1}{2}R \geq \max_{j=1,\ldots,6}\left|D_j\right|$ for some appropriate constants $C_j$ and $D_k$, we find that

$$\begin{aligned}
\left|A(\zeta) + B(\zeta)\right| &\leq \frac{C_1}{\prod_{j=1}^4 \left|\zeta - D_j\right|} + \frac{C_2}{\left|\zeta - D_5\right|^2 \left|\zeta - D_6\right|^2} \\
&\leq \frac{C_1}{\prod_{j=1}^4 (|\zeta| - \left|D_j\right|)} + \frac{C_2}{(|\zeta| - |D_5|)^2 (|\zeta| - |D_6|)^2} \\
&\leq \frac{C_1}{\prod_{j=1}^4 \frac{1}{2}R} + \frac{C_2}{(\frac{1}{2}R)^4} = \frac{C}{R^4}, \quad C := 16(C_1 + C_2).
\end{aligned}$$

It follows that

$$\begin{aligned}
\left|\int_{C_R} k(\zeta)(A(\zeta) + B(\zeta))\, d\zeta\right| &\leq \sup_{|\zeta|=R} \left|k(\zeta)(A(\zeta) + B(\zeta))\right| \cdot \ell(C_R) \\
&\leq KR^2 \frac{C}{R^4} \pi R = \frac{\pi KC}{R} \to 0 \quad \text{as} \quad R \to \infty.
\end{aligned}$$

What is left of (39) when $R$ grows arbitrarily large is – a multiple by $\pi$ of – the integral of interest, i.e (37), which is now determined using residue calculus. To this end, assume that $z \neq i$ (the final result still applies when $z = i$ by continuity). By the residue theorem, the integral evaluates to

$$2\pi i \big( \operatorname{Res}_{\zeta=z} k(\zeta)A(\zeta) + \operatorname{Res}_{\zeta=i} k(\zeta)A(\zeta) + \operatorname{Res}_{\zeta=i} k(\zeta)B(\zeta) \big),$$

where

$$\operatorname{Res}_{\zeta=z} k(\zeta)A(\zeta) = k(z)\frac{-i(i+w)}{2(z+w)(z-i)}$$

and

$$\operatorname{Res}_{\zeta=i} k(\zeta)A(\zeta) = k(i)\frac{i+z}{4(z-i)}.$$

Due to the double pole of $B$ at $\zeta = i$ the computation of $\operatorname{Res}_{\zeta=i} k(\zeta)B(\zeta)$ is divided into cases. The following two will suffice:

$\underline{b = 0,\ d = 1}$:

$$\operatorname{Res}_{\zeta=i} k(\zeta)B(\zeta) = \frac{d}{d\zeta}\bigg|_{\zeta=i} \frac{-i}{(\zeta+i)^2} = -\frac{1}{4}.$$

$\underline{b = 1,\ d = 0}$:

$$\operatorname{Res}_{\zeta=i} k(\zeta)B(\zeta) = \frac{d}{d\zeta}\bigg|_{\zeta=i} \frac{-i(\zeta+c)^2}{(\zeta+i)^2} = -\frac{1+c^2}{4}.$$

The sum of the residues is

$$2\pi i\left(\frac{i(i+w)}{2(z+w)(i-z)} + \frac{i+z}{4(i-z)} - \frac{1}{4}\right) = -\frac{\pi}{z+w}$$

in the case where $b = 0$ and $d = 1$, and

$$2\pi i\left((z+c)^2\frac{i(i+w)}{2(z+w)(i-z)} + (i+c)^2\frac{i+z}{4(i-z)} - \frac{1+c^2}{4}\right) = \frac{\pi(z+c)(w-c)}{z+w}$$

in the other case. By linearity of the integral (37) it follows from here that the pure-integral part of $h$ is given by

$$(z,w) \mapsto \frac{b(z+c)(w-c)-d}{z+w}. \tag{40}$$

This concludes the proof in the case that $\operatorname{supp}\mu = \ell_1 = \{t_1 + t_2 = 0\}$.

In general, as previously mentioned, the supporting line of $\mu$ when $\mu$ is of the form (20) can be written $t_1 + \alpha t_2 = \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Define a function $\tilde{h}$ according to (40). By this point we know that its corresponding Nevanlinna measure, denoted $\tilde{\mu}$, is given by

$$\tilde{\mu}(A) = \pi\int_{\mathbb{R}} k(t)\chi_A(t,-t)\,dt, \quad k(t) = b(t+c)^2 + d,$$

for any Borel subset $A \subset \mathbb{R}^2$, where $b, d \geq 0$ and $c \in \mathbb{R}$. Consider the linear transformation $f : (z,w) \mapsto (z-\beta, \alpha w)$ of $\mathbb{H}^2$ with $\det J_f = \alpha$. Its inverse $f^{-1}$ takes the line $\ell_1$ to the line $\ell_2 := \{t_1 + \alpha t_2 = \beta\}$, the latter of which is parametrized by $t \mapsto f^{-1}(t,-t) = (t+\beta, -\frac{1}{\alpha}t)$. Then, for any Borel subset $A \subset \mathbb{R}^2$,

$$\begin{aligned} f^*\tilde{\mu}(A) &= \tilde{\mu}(f(A)) \\ &= \pi\int_{\mathbb{R}} k(t)\chi_{f(A)}(t,-t)\,dt \\ &= \pi\int_{\mathbb{R}} k(t)\chi_A(f^{-1}(t,-t))\,dt \\ &= \pi\int_{\mathbb{R}} k(t)\chi_A(t+\beta, -\frac{1}{\alpha}t)\,dt, \end{aligned}$$

hence for an appropriate choice of admissible $b, c$ and $d$ the measure $\mu$ can be written $\mu = \frac{1}{\det J_f} f^*\tilde{\mu} = \frac{1}{\alpha} f^*\tilde{\mu}$. It is easy to check that the representing data of $\tilde{h}$ is $(0, 0, \tilde{\mu})$, so Proposition 3.14 applies to give the pure-integral part of $h$ as $z \mapsto \tilde{h}(f(z)) - \operatorname{Re} \tilde{h}(f(i))$, where

$$\operatorname{Re} \tilde{h}(f(i)) = \frac{b((c-\beta)(\alpha^2 + \beta c) - c) + \beta d}{\beta^2 + (\alpha+1)^2}$$

and

$$\tilde{h}(f(z)) = \frac{b(z - \beta + c)(\alpha w - c) - d}{z + \alpha w - \beta}. \qquad \square$$

The automorphism invariance property of real-rational functions as proposed by Proposition 3.14 extends the characterization theorem to a larger subclass of real-rational functions than first considered in Theorem 3.18.

*Example* 3.19. Consider the function

$$(z, w) \mapsto -\frac{z}{zw - 1} = -\frac{1}{w - \frac{1}{z}} \tag{41}$$

and note that it transforms from

$$(z, w) \mapsto -\frac{1}{z + w} \tag{42}$$

via the automorphism $f \in \mathbb{H}^2$ given by $(z, w) \mapsto (-\frac{1}{z}, w)$. The function (42) is a real-rational function with corresponding Nevanlinna measure

$$\mu(A) = \pi \int_{\mathbb{R}} \chi_A(t, -t)\, dt$$

by Theorem 3.18. By Proposition 3.14 the function (41) is therefore real-rational with corresponding Nevanlinna measure $\nu := \frac{1}{\det J_f(t)} f^*\mu$, where $\det J_f(t) = \frac{1}{t_1^2}$ and

$$\begin{aligned} f^*\mu(A) &= \mu(f(A)) \\ &= \pi \int_{\mathbb{R}} \chi_{f(A)}(t, -t)\, dt \\ &= \pi \int_{\mathbb{R}} \chi_A(f(t, -t))\, dt. \end{aligned}$$

To see that assumption (27) of Proposition 3.14 is indeed satisfied, compare with Example 3.17. In any case, it follows that

$$\nu(A) = \pi \int_{\mathbb{R}} \frac{1}{t^2} \chi_A(f(t, -t))\, dt$$

for any Borel subset $A \subset \mathbb{R}^2$. Notice that the support of the measure is precisely the zero set of the denominator of the function, which is a smooth submanifold of $\mathbb{R}^2$, just as in the case when the denominator is affine; see Corollary 3.10. The submanifold in this case is $t_1 t_2 = 1$ with parametrization $t \mapsto f(t, -t) = (-\frac{1}{t}, -t)$. $\lozenge$

The challenging part of generalizing the characterization theorem to arbitrary dimensions is to show that $h$ is real-rational with an affine denominator when $\mu$ is of the form (20). In general, with a Nevanlinna measure of the form in (20), the integral on $\mathbb{R}^n$ in representation (15) reduces to an integral on $\mathbb{R}^{n-1}$. In the two-variable setting this yields an integral on $\mathbb{R}$, in which case the residue theorem applies nicely, as seen in the proof of Theorem 3.18. In higher dimensions, it gets trickier. Nevertheless, it is expected that the characterization theorem does generalize. This is particularly the case if the affine structure of the denominator is ignored and we settle for the claim that $h$ is real-rational when $\mu$ is of the form (20). A suggestion on how to approach this with a proof is to show that $h$ is a rational Herglotz-Nevanlinna function separately from the $\lambda_n$-a.e vanishing of its imaginary part on the distinguished boundary $\mathbb{R}^n$. The latter follows from Lemma 3.20 below, upon first noting that the desired conclusion is equivalent to (43) by Corollary 2.14(v), and then recalling that a measure of the form (20) is singular. The former remains to be explored in future work.

**Lemma 3.20.** *Let $\mu$ be a singular positive measure on $\mathbb{R}^n$. Then*

$$\lim_{y\to 0^+}\int_{\mathbb{R}^n}\mathcal{P}_n(x+iy,t)\,d\mu(t)=0 \quad \textit{for} \quad \lambda_n\text{-a.e } x\in\mathbb{R}^n\,. \tag{43}$$

*Proof.* Denote the Poisson kernel on the unit polydisc by $\tilde{\mathcal{P}}_n$ and recall the component-wise Cayley transform $\Phi$ from Example 2.4. Since $\mathcal{P}_n=\tilde{\mathcal{P}}_n\circ\Phi$, it follows from Theorem 2.3 that

$$\int_{\mathbb{R}^n}\mathcal{P}_n(z,t)\,d\mu(t)=\int_{\mathbb{R}^n}\tilde{\mathcal{P}}_n\circ\Phi\,d\mu=\int_{\mathbb{T}^n}\tilde{\mathcal{P}}_n(w,t)\,d(\Phi_*\mu)(t).$$

The pushforward is singular because $\mu$ is so by assumption. An old result on the polydisc states that if $\nu$ is a singular measure on $\mathbb{T}^n$, then the radial limit

$$\lim_{r\to 1^-}\int_{\mathbb{T}^n}\tilde{\mathcal{P}}_n(w,t)\,d\nu(t)$$

vanishes for $\Phi_*\lambda$-a.e $w\in\mathbb{T}^n$. A proof can be located in [Rud69, 2.3.1], and the desired result then follows.. □

As mentioned in the discussion preceding Lemma 3.20, the condition (43) is equivalent to the vanishing of the imaginary part of the Herglotz-Nevanlinna function that the measure represents. Following this, any Nevanlinna measure that is singular represents a Herglotz-Nevanlinna function with this boundary property by Lemma 3.20. It would be interesting to see if a singular Nevanlinna measure always represents a rational Herglotz-Nevanlinna function, as it does in the two-variable setting in the special case where its support is a hyperplane (Theorem 3.18). If so, then any Nevanlinna measure that is singular represents a real-rational function. Since the converse holds by the support theorem, this would finalize a characterization of the full class of real-rational functions in terms of the corresponding class of Nevanlinna measures. A detailed description of the singular Nevanlinna measures would be desirable in this context – a result that does not seem to currently exist.

# 4 On the decomposition of real-rational functions

The aim of this chapter is to give a decomposition theorem (Theorem 4.2) for certain real-rational functions of two variables. The result and its general context (including Chapter 4.3) were communicated to the author by the thesis supervisor and, to the best of our knowledge, has not previously appeared in the literature.

## 4.1 Extremal Nevanlinna measures

To set the stage, recall that any positive linear combination $ah + bf$, $a, b \geq 0$, of two Herglotz-Nevanlinna functions $h$ and $f$ is itself a Herglotz-Nevanlinna function. The same holds for Nevanlinna measures, meaning that the set of Nevanlinna measures on $\mathbb{R}^n$ forms a convex cone $\mathcal{C}$ in the real vector space of signed Borel measures on $\mathbb{R}^n$. An *extremal ray* of a convex cone is a ray in the ambient vector space that is contained in a face of the cone. In other words, given a Nevanlinna measure $\mu$, the ray $\{\alpha\mu : \alpha \geq 0\}$ is extremal if every Nevanlinna measure $\nu_1$, for which there is another Nevanlinna measure $\nu_2$ such that their sum is on the ray (meaning $\nu_1 + \nu_2 = \alpha\mu$ for some $\alpha > 0$), is also on the ray (meaning that $\nu_1$ is a positive multiple of $\mu$). A measure on an extremal ray of $\mathcal{C}$ may be referred to as an *extremal measure*.

A extremal Nevanlinna measure that represents a real-rational function has a strong connection to the existence of a particular type of decomposition of the function. Specifically, let a *real-rational decomposition* of a real-rational function $h$ with representing data $(0, 0, \mu)$ refer to any decomposition of $h$ into a sum of two nonconstant real-rational functions $h_1$ and $h_2$ such that none of $h_1$ and $h_2$ are a positive constant multiple of $h$. Then $h$ has a real-rational decomposition if and only if $\mu$ is not extremal. This dual nature is exemplified in the two-dimensional setting in the following, where the classification of extremal measures of the form (20) is explored.

*Example* 4.1. Consider a Nevanlinna measure $\mu$ on $\mathbb{R}^2$ of the form (20). Write its density function as $k(t) = b(t + c)^2 + d$ for some $b, d \geq 0$ and $c \in \mathbb{R}$ as per Remark 2.21. In accordance with [NS24, Corollary 3.9], the measure is extremal if and only if either $b = 0$ or $d = 0$. This makes sense; if $b = 0$ or $d = 0$, then the only way to write the density as a sum of two nonnegative polynomials is to make each term in the sum a nonnegative constant multiple of $k$ itself. It follows that the terms in any decomposition of $\mu$ lies on the ray spanned by $\mu$, so $\mu$ is extremal.

For the converse, consider the contrapositive and take $b, d > 0$. Then $\mu = \mu_1 + \mu_2$ where $\mu_1$ and $\mu_2$ are Nevanlinna measures on the form (20) with densities $b(t + c)^2$ and $d$, respectively. Clearly, $\mu$ is not extremal.

By Theorem 3.18, the real-rational function

$$h(z, w) = \frac{b(z - \beta + c)(\alpha w - c) - d}{z + \alpha w - \beta} + D, \quad \alpha > 0,\, \beta \in \mathbb{R},$$

is represented by the data $(0, 0, \mu)$, where $\mu$ is of the form (20) with density function $k(t) = b(t + c)^2 + d$ for some $b, d \geq 0$ and $c \in \mathbb{R}$. In accordance with the dual theory of $\mu$, $h$ has a real-rational decomposition if and only if $b, d > 0$, in which case one possibility is

$$\frac{b(z - \beta + c)(\alpha w - c) - d}{z + \alpha w - \beta} = \left(\frac{b(z - \beta + c)(\alpha w - c)}{z + \alpha w - \beta}\right) + \left(\frac{-d}{z + \alpha w - \beta} + D\right).$$

The above can be extended by the automorphism invariance property (Proposition 3.14); let $h$ be a real-rational function and take $f \in \operatorname{Aut} \mathbb{H}^n$, then $h$ has a real-rational decomposition if and only if $h \circ f$ has one. In particular, if $h$ depends on two variables such that $h \circ f$ has an affine denominator, then the real-rational decomposition of $h$ is obtained from that of $h \circ f$ given above. ♢

In the one-dimensional setting, the problem of the existence of a real-rational decomposition for any given real-rational function was resolved long ago; as touched upon in the introduction, the singular part of a Nevanlinna measure on $\mathbb{R}$ is a finite sum of nonnegative multiples of Dirac measures on $\mathbb{R}$. Each term in this sum is itself a Nevanlinna measure that represents a real-rational function of the form $z \mapsto -\frac{c}{z-\alpha}$ for some $c \geq 0$ and $\alpha \in \mathbb{R}$. Since the Nevanlinna measure of a real-rational function is singular (apply Theorem 3.9 to the case $n = 1$), it follows by the dual nature of real-rational decompositions that every real-rational function with representing data of the form $(0, 0, \mu)$ can be written as a finite sum of terms of the form $z \mapsto -\frac{c}{z-\alpha}$ (possibly including a constant term). In particular, a real-rational function $h$ with such data has no real-rational decomposition if and only if $h(z) = -\frac{c}{z-\alpha} + \frac{\alpha}{1+\alpha^2}$ for some $c > 0$ and $\alpha \in \mathbb{R}$ (i.e $\mu = c\delta_\alpha$). In the multi-variable setting, the situation is not yet completely understood. The decomposition theorem below is important in this regard.

## 4.2 The decomposition theorem

The following shows that most real-rational functions with a denominator that reduces to a product of nonconstant affine factors – perhaps all, but this remains to be investigated; see the paragraph proceeding Lemma 4.6 – has a real-rational decomposition. Perhaps this result is expected; after all, these factors appear as the affine denominator of a number of real-rational functions (see Theorem 3.18). However, when at least one of the factors is not affine, this is not always the case, as evident from Chapter 4.3 where the function (51) is given as a counterexample. Nevertheless, the guaranteed existence of a real-rational decomposition for the functions in Theorem 4.2 is still rather remarkable when considering the more general setting of rational functions with denominators of similar structure, where only a few have a nontrivial decomposition into rational terms.

**Theorem 4.2** (The decomposition theorem)**.** *Let $h = \frac{p}{q_1 q_2}$ be a real-rational function of two variables $(z, w)$ such that*

$$q_1(z, w) = \alpha_1 z + \alpha_2 w + \alpha_3 \quad \textit{and} \quad q_2(z, w) = \beta_1 z + \beta_2 w + \beta_3 \tag{44}$$

*are both nonconstant. Suppose that the representing data of $h$ is $(0, 0, \mu)$ and assume that $p$ is of at most total degree 2. Then $h$ has a real-rational decomposition according to*

$$h(z, w) = \frac{A}{q_1(z, w)} + \frac{B}{q_2(z, w)}, \tag{45}$$

*where $A$ and $B$ are two negative constants.*

*Proof.* The proof consists of two main parts. The first is dedicated to provide a much more detailed description of $q$. In particular, $q$ is essentially on one of two forms: $q(z, w) = zw$ or $q(z, w) = (z + w)(z + \alpha w)$ for some $\alpha > 0$. The second part follows this by giving a real-rational decomposition of $q$ in each of these cases via Lemma 4.5 and 4.6, respectively.

First note that, as usual, the coefficients of $q_1$ and $q_2$ are real by Lemma 3.5. By Lemma 3.3(ii), $\deg_j q \geq \deg_j p$ for $j = 1, 2$, so $q$ depends on both variables $z$ and $w$. This means that for $j = 1, 2$, $\alpha_j$ and $\beta_j$ are not both zero. Moreover, $q_1$ is nonconstant, so $\alpha_1$ and $\alpha_2$ are not both zero. Similar with $\beta_1$ and $\beta_2$. Two things follow: first, at most two of the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ are zero. Second, up to permutation of coordinates, two of these coefficients are zero only in the event that $\alpha_2 = \beta_1 = 0$. In all other cases, at most one coefficient is zero. This yields, up to permutation of coordinates, two paths to continue on:

$\underline{\alpha_2 = \beta_1 = 0}$: The denominator reduces to $q(z, w) = (\alpha_1 z + \alpha_3)(\beta_2 w + \beta_3)$ with $\alpha_1, \beta_2 \neq 0$ and $\alpha_3, \beta_3 \in \mathbb{R}$. After the linear change of coordinates $(z, w) \mapsto (\frac{z-\alpha_3}{\alpha_1}, \frac{w-\beta_3}{\beta_2})$, $q$ can be written $q(z, w) = zw$. This function has a real-rational decomposition according to Lemma 4.5 below. Transforming back to the original function $h$ yields the desired real-rational decomposition (45).

$\underline{\alpha_1, \alpha_2, \beta_1 \neq 0 \text{ and } \beta_2 \geq 0}$: After the linear change of coordinates $(z, w) \mapsto (\frac{z-\alpha_3}{\alpha_1}, \frac{w}{\alpha_2})$, $q$ can be written $q(z, w) = (z + w)(\beta_1 \frac{z-\alpha_3}{\alpha_1} + \beta_2 \frac{w}{\alpha_2} + \beta_3)$. Extracting $\frac{\beta_1}{\alpha_1}$ from the second factor and "hiding" it in the numerator thus yields $q(z, w) = (z + w)(z + \alpha w + \beta)$, where $\alpha := \frac{\alpha_1 \beta_2}{\beta_1 \alpha_2}$ and $\beta := \frac{\alpha_1 \beta_3}{\beta_1} - \alpha_3$. Note that $\alpha$ is nonnegative because the sign of $\alpha_1$ and $\alpha_2$, as well as the sign of $\beta_1$ and $\beta_2$, are the same due to holomorphy of $h$ on $\mathbb{H}^2$. If $\beta \neq 0$ and $\alpha \neq 1$ then after another linear change of coordinates $(z, w) \mapsto (z - t_0, w + t_0)$ with $t_0 := \frac{\beta}{1-\alpha}$, $q$ can be written $q(z, w) = (z + w)(z + \alpha w)$. If $\beta = 0$ instead then $q(z, w) = (z + w)(z + \alpha w)$ with $\alpha \neq 1$ in accordance with Remark 3.13. In both of these cases, a real-rational decomposition is obtained from Lemma 4.6. Note that Lemma 4.6 assumes that the total degree of the numerator of the function in consideration is at most 2, but this is not a problem; the numerator is a linear transformation of $p$ which by assumption is of at most total degree 2. In any case, transforming back to the original function $h$ yields the desired real-rational decomposition (45). The remaining case where $\beta \neq 0$ and $\alpha = 1$ has actually already been handled; with $q(z, w) = (z + w)(z + \beta)$ we can write $q(z, w) = (z + w)z$ after the linear change of coordinates $(z, w) \mapsto (z - \beta, w + \beta)$. But this is only a permutation of coordinates away from $q(z, w) = (z + w)(z + \alpha w)$ with $\alpha = 0$, which is a structure of $q$ that was considered in a previous case. □

*Remark* 4.3. The assumption on the total degree of $p$ in Theorem 4.2 is only really necessary in the second case of the proof, where $\alpha_1, \alpha_2, \beta_1 \neq 0$ and $\beta_2 \geq 0$. Moreover, it is not as restrictive as it first may appear; given the representing data $(0, 0, \mu)$ of $h$, Lemma 3.3(ii) implies that the degree of $p$ in each variable is at most 2 (or at most 1 in $w$ if $\alpha = 0$), as this is the case for $q$. As such, the total degree of $p$ is in general at most 4 (3 if $\alpha = 0$).

*Remark* 4.4. Since the denominator of each term in the decomposition of $h$ in Theorem 4.2 is affine, the Nevanlinna measure of $h$ is obtained from the characterization theorem (Theorem 3.18) as a sum of two Nevanlinna measures. Given that the affine factors in the denominator are different (see Remark 3.13), these Nevanlinna measures are not constant multiples of one another. In particular, the Nevanlinna measure corresponding to $h$ is not extremal. ◊

**Lemma 4.5.** *Let $h = \frac{p}{q}$ be a real-rational function of two variables with $q(z, w) = zw$ and let $(0, 0, \mu)$ be its data from representation (15). Then $p(z, w) = dz + ew$ with $d, e < 0$. In particular, $h$ decomposes into a sum of real-rational functions according to*

$$h(z, w) = \frac{dz + ew}{zw} = \frac{d}{w} + \frac{e}{z}.$$

*Proof.* Since the degree of the denominator is 1 in each variable it follows from Lemma 3.3(ii) that the numerator is of at most total degree 2. In other words,

$$p(z, w) = az^2 + bw^2 + czw + dz + ew + f,$$

where each coefficient is real. The idea now is to show that $a = b = c = f = 0$ as well as $d, e < 0$ – the desired decomposition then follows.

Note that

$$\operatorname{Re} h(i) = \operatorname{Re} \frac{-a - b - c + (d + e)i + f}{-1} = a + b + c - f.$$

Since $\operatorname{Re} h(i) = 0$ by assumption (see Corollary 2.14(ii)), we conclude that $a + b + c = f$. From here we consider particular restrictions of $h$ to obtain more conditions on the coefficients of $p$. First note that $z \mapsto h(z, -\frac{1}{z})$ is a real-rational function of one variable, where

$$h(z, -\frac{1}{z}) = \frac{az^2 + b\frac{1}{z^2} - cz\frac{1}{z} + dz - e\frac{1}{z} + f}{-z\frac{1}{z}} = -az^2 - dz + (c - f) + \frac{e}{z} - \frac{b}{z^2}.$$

Lemma 3.3(iii) implies that $a = b = 0$, and $d, e \leq 0$ is obtained by nonnegativity of the imaginary part of the function on $\mathbb{H}$. Next, for any $t \neq 0$, the function $z \mapsto h(z, t)$ is by Corollary 3.7 real-rational of one variable. With $a = b = 0$ we can write $h(z, t)$ as

$$h(z, t) = \frac{czt + dz + et + f}{tz} = (c + \frac{d}{t}) + \frac{e + \frac{f}{t}}{z}.$$

Nonnegativity of the imaginary part of the function on $\mathbb{H}$ shows that $e + \frac{f}{t} \leq 0$. Since $t \neq 0$ was arbitrary (and $e \leq 0$), it follows that $f = 0$. Combining this with the previously obtained identities $a = b = 0$ and $a + b + c = f$ yields $a = b = c = f = 0$, hence $p(z, w) = dz + ew$ with $d, e \leq 0$. Finally, note that the inequalities $d, e \leq 0$ are strict; if $e = 0$ then $\frac{p(z)}{q(z)} = \frac{d}{w}$, which is a contradiction regardless of the value of $d$. The case is similar if $d = 0$. ☐

**Lemma 4.6.** *Let $h = \frac{p}{q}$ be a real-rational function of two variables with $q(z, w) = (z + w)(z + \alpha w)$ for some $0 \leq \alpha \neq 1$ and let $(0, 0, \mu)$ be its data from representation (15). Assume that the total degree of $p$ is at most 2. Then $p(z, w) = dz + ew$ with*

$$\begin{cases} d < e < d\alpha \leq 0, & \text{if} \quad 0 \leq \alpha < 1, \\ d\alpha < e < d < 0, & \text{if} \quad \alpha > 1, \end{cases} \tag{46}$$

*where the inequality $d\alpha \leq 0$ is strict unless $\alpha = 0$. In particular, $h$ decomposes into a sum of real-rational functions according to*

$$h(z, w) = \frac{dz + ew}{(z + w)(z + \alpha w)} = \frac{A}{z + w} + \frac{B}{z + \alpha w}, \quad \begin{cases} A := \frac{d-e}{1-\alpha} < 0, \\ B := \frac{e-\alpha d}{1-\alpha} < 0. \end{cases} \tag{47}$$

*Proof.* The approach in this proof resembles that of the previous Lemma 4.5. The numerator $p$ is of at most total degree 2, so it can be written

$$p(z, w) = az^2 + bw^2 + czw + dz + ew + f,$$

where each coefficient is real. The idea now is to show that $a = b = c = f = 0$ as well as the conditions (46) – the desired decomposition then follows upon noting that these conditions on $d$ and $e$ imply that $\frac{d-e}{1-\alpha}$ and $\frac{e-\alpha d}{1-\alpha}$ are negative in both cases of $0 < \alpha < 1$ and $\alpha > 1$. Also, the conditions (46) are at first only derived as $\leq$-inequalities. Once $p(z, w) = dz + ew$ has been obtained, the strict inequalities are implied; if not, we arrive at contradictions similar to those in the end of the proof of Lemma 4.5.

A simple computation shows that

$$\operatorname{Re} h(i) = \operatorname{Re} \frac{-a - b - c + (d + e)i + f}{2i(i + \alpha i)} = \frac{a + b + c - f}{2(1 + \alpha)}.$$

Since $\operatorname{Re} h(i) = 0$ by assumption (see Corollary 2.14(ii)), it follows that $a + b + c = f$. From here we consider particular restrictions of $h$ to obtain more conditions on the coefficients of $p$. To begin with, the map $z \mapsto h(z, 0)$ is by Corollary 3.7 a real-rational function of one variable. Since

$$h(z, 0) = \frac{az^2 + dz + f}{z^2} = a + \frac{d}{z} + \frac{f}{z^2},$$

Lemma 3.3(iii) implies that $f = 0$. Together with $a + b + c = f$ we obtain

$$a + b + c = 0. \tag{48}$$

By nonnegativity of the imaginary part of the function on $\mathbb{H}$, the inequality $d \leq 0$ is also obtained. Similarly, if $\alpha \neq 0$, the function $w \mapsto h(0, w)$ is by Corollary 3.7 a real-rational functions of one variable that can be used to obtain the inequality $e \leq 0$ from

$$h(0, w) = \frac{bw^2 + ew + f}{\alpha w^2} = \frac{1}{\alpha}\big(b + \frac{e}{w} + \frac{f}{w^2}\big).$$

If $\alpha = 0$ then $h$ is not defined at any point $(z, w) = (0, w)$. However, the inequality $e \leq 0$ is soon confirmed in this case as well.

Next take any $t > 0$ and consider the one-variable real-rational function given by $z \mapsto h(z, -\frac{t}{z})$. With $f = 0$ we have

$$h(z, -\frac{t}{z}) = \frac{az^2 + b\frac{t^2}{z^2} - ct + dz - e\frac{t}{z}}{(z - \frac{t}{z})(z - \alpha\frac{t}{z})} = \frac{az^4 + dz^3 - ctz^2 - etz + bt^2}{(z^2 - t)(z^2 - \alpha t)}. \tag{49}$$

The remaining conditions on the coefficients of $p$ are derived from here in two different cases.

$\underline{\alpha \neq 0}$: In this case, polynomial division and partial fraction decomposition of (49) yields

$$h(z, -\frac{t}{z}) = a + \frac{A}{z - \sqrt{t}} + \frac{B}{z + \sqrt{t}} + \frac{C}{z - \sqrt{\alpha t}} + \frac{D}{z + \sqrt{\alpha t}},$$

where

$$\begin{aligned} A &= \frac{(a + b - c)\sqrt{t} + (d - e)}{2(1 - \alpha)}, \\ B &= \frac{-(a + b - c)\sqrt{t} + (d - e)}{2(1 - \alpha)}, \\ C &= \frac{-(a\alpha^2 - c\alpha + b)\sqrt{t} - \sqrt{\alpha}(d\alpha - e)}{2\sqrt{\alpha}(1 - \alpha)}, \\ D &= \frac{(a\alpha^2 - c\alpha + b)\sqrt{t} - \sqrt{\alpha}(d\alpha - e)}{2\sqrt{\alpha}(1 - \alpha)}. \end{aligned}$$

Similar to before, nonnegativity of the imaginary part of the function on $\mathbb{H}$ implies that each of these constants are nonnegative. Since this is the case for all choices of $t > 0$ it follows that

$$\begin{cases} a + b - c = 0, \\ a\alpha^2 - c\alpha + b = 0, \end{cases} \quad \text{and} \quad \begin{cases} d \leq e \leq d\alpha, & \text{if} \quad 0 < \alpha < 1 \\ d\alpha \leq e \leq d, & \text{if} \quad \alpha > 1. \end{cases}$$

The first set of these conditions imply together with (48) that $a = b = c = 0$. Since also $f = 0$ the numerator is reduced to $p(z, w) = dz + ew$. The second set of these conditions combine with the previously established $d, e \leq 0$ to give (46).

$\underline{\alpha = 0}$: In this case, polynomial division and partial fraction decomposition of (49) yields

$$h(z, -\frac{t}{z}) = a + \frac{A}{z - \sqrt{t}} + \frac{B}{z + \sqrt{t}} + \frac{e}{z} - \frac{bt}{z^2},$$

where

$$\begin{aligned} A &= \frac{1}{2}((a + b - c)\sqrt{t} + (d - e)), \\ B &= \frac{1}{2}(-(a + b - c)\sqrt{t} + (d - e)). \end{aligned}$$

Note that $b = 0$ by Lemma 3.3(iii). Furthermore, nonnegativity of the imaginary part of the function on $\mathbb{H}$ implies that $e, A, B \leq 0$ for all choices of $t > 0$. Thus, since also $d \leq 0$, we obtain

$$\begin{cases} a + b - c = 0, \\ b = 0, \\ d \leq e \leq 0. \end{cases}$$

Combining the first two of these conditions with (48) yields $a = b = c = 0$, hence $p(z, w) = dz + ew$ due to $f = 0$. The last of the three conditions is precisely (46) when $\alpha = 0$. □

There are a few generalizations of Theorem 4.2 to consider. Perhaps the most natural is to verify if it remains valid when the assumption on the degree of the numerator is removed. Another obvious generalization is to higher dimensions. In addition, it would be interesting to investigate if the decomposition theorem extends to the case where the denominator is a product of three affine factors, and eventually arbitrarily many.

## 4.3 An example and concluding remarks

The next two paragraphs summarize the main results presented in this thesis. This summary is followed by one final example that reveals certain limitations of these results and, in doing so, also emphasizes the intricate nature of real-rational functions of several variables.

Consider a real-rational function of two variables. If the denominator is affine (possibly after a biholomorphic change of variables) then the corresponding Nevanlinna measure is obtained from the characterization theorem (Theorem 3.18), potentially via the automorphism invariance property (Proposition 3.14). As an elementary example, the representing measure of the real-rational function

$$(z,w) \mapsto -\frac{1}{z+w}$$

is determined as

$$\mu(A) = \pi \int_{\mathbb{R}} \chi_A(t,-t)\,dt$$

directly from the characterization theorem. A more involved example was seen in Example 3.19, where the representing measure of the real-rational function

$$(z,w) \mapsto -\frac{z}{zw-1}$$

was determined as

$$\mu(A) = \pi \int_{\mathbb{R}} \frac{1}{t^2}\chi_A(f(t,-t))\,dt$$

from the characterization theorem via the automorphism invariance property.

If, instead, the denominator is a product of two nonconstant affine factors, and the numerator is of at most degree 2, then it has a real-rational decomposition by the decomposition theorem (Theorem 4.2). Its representing measure is therefore not extremal, which was noted in Remark 4.4.

The intuition gained so far points in the direction that the representing measure of a real-rational function with a numerator of total degree 2, and the specific denominator $(z,w) \mapsto (zw-1)(z+w)$, is not extremal. In the following, however, we construct a counterexample (51). This occurrence is unique to the multi-dimensional setting, as every real-rational function of one variable that we expect to have a real-rational decomposition indeed has one.

To begin with, consider the function

$$h(z,w) = \frac{p(z,w)}{q(z,w)} := -\frac{w}{w^2+zw-1}. \tag{50}$$

Its imaginary part is given by $\operatorname{Im} h = \frac{1}{|q(z,w)|^2}\operatorname{Im} p(z,w)\overline{q(z,w)}$, where

$$\begin{aligned}\operatorname{Im} p(z,w)\overline{q(z,w)} &= -\operatorname{Im} w(\overline{w}^2 + \overline{zw} - 1)\\ &= -|w|^2 \operatorname{Im}\overline{w} - |w|^2\operatorname{Im}\overline{z} + \operatorname{Im} w\\ &= |w|^2\operatorname{Im} w + |w|^2 \operatorname{Im} z + \operatorname{Im} w \geq 0\end{aligned}$$

for every $(z,w) \in \mathbb{H}^2$. Thus, $h$ is a Herglotz-Nevanlinna function. Rewrite $h$ as

$$h(z,w) = -\frac{1}{w - \frac{1}{w} + z}$$

and note that $z - \frac{1}{z} \in \mathbb{H}^n$ whenever $z \in \mathbb{H}^n$. It follows that the function

$$\tilde{h}(z,w) := h(z - \frac{1}{z}, w) = -\frac{1}{w - \frac{1}{w} + z - \frac{1}{z}} = -\frac{zw}{(zw-1)(z+w)} \tag{51}$$

is a Herglotz-Nevanlinna function. In particular, $\tilde{h}$ is real-rational with representing data $(0, 0, \mu)$. To see that there is no real-rational decomposition of $\tilde{h}$, suppose the contrary in pursuit of a contradiction. Then there are nonzero polynomials $p_1, p_2 \in \mathbb{C}[z, w]$ such that

$$h(z, w) = \frac{p_1(z, w)}{zw - 1} + \frac{p_2(z, w)}{z + w}.$$

It follows that

$$(z + w)p_1(z, w) + (zw - 1)p_2(z, w) = -zw, \quad (z, w) \in \mathbb{H}^2\,. \tag{52}$$

This identity extends to all of $\mathbb{C}^2$ and is satisfied on the complex line $z + w = 0$ in particular. On this line, (52) reduces to

$$(-z^2 - 1)p_2(z, -z) = z^2.$$

This, however, means that $z \mapsto p_2(z, -z)$ is not a polynomial, which contradicts $p_2 \in \mathbb{C}[z, w]$. In conclusion, the function (51) has no real-rational decomposition. This also means that its representing Nevanlinna measure $\mu$ – whichever it may be – is extremal.

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