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Solvability of Dirichlet and Neumann Boudary Value Problems on $C^{1,\alpha}$ Domains

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Abstract

This thesis investigates the solvability of the Dirichlet and Neumann boundary value problems for bounded $C^{1,\alpha}$ domains. Through introducing layer potentials and proving "jump relations" at the boundary, resulting solvability criteria is formulated in terms of operators and thus are investigated using the theory of compact operators and the Fredholm Alternative. The results of the extension of this problem to Lipschitz domains is then introduced and compared to the $C^{1,\alpha}$ case.

Sammanfattning

I denna avhandling undersöks lösbarheten för Dirichlet- och Neumannrandvärdesproblem för begränsade $C^{1,\alpha}$-områden. Genom att introducera lagerpotentialer och studera deras diskontinuiteter vid randen, formuleras resulterande kriterier för lösbarhet i termer av operatorer, som därefter undersöks med hjälp av teorin för kompakta operatorer och Fredholms alternativ. Problemet utvidgas senare till Lipschitzområden och resultatet jämförs med $C^{1,\alpha}$ fallet.

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Contents

1 Introduction

Boundary value problems (BVPs) generally involve finding a function that has certain behavior on the boundary of a domain, and solves a differential equation within the domain. Boundary value problems solving Laplace's equation ($\Delta u = 0$) arise naturally from the study of several natural phenomena, including gravitation, fluid dynamics, electrostatics.

In this thesis, we explore the solvability of Dirichlet and Neumann boundary value problems, focusing on the role that the regularity of the boundary plays. We seek to find conditions that guarantee that solutions exist to the given BVPs. This is done through the introduction of layer potentials and development of "jump relations", which describe their behavior at the boundary, from which we can formulate precise solvability criteria in terms of images of operators. We can then use the theory of Compact Operators and the Fredholm alternative to investigate these operators and discover concrete conditions for solvability.

Finally, the results are compared to that of the case with lower boundary regularity, namely the Dirichlet and Neumann BVPs on Lipschitz domains, and adaptations to the formulation and strategy are introduced to handle the extension to this more challenging case.

The document follows the following structure: Section 2 covers background and notation; Section 3 introduces layer potentials and jump relations; Section 4 presents functional analytic tools; Section 5 applies those tools to the problem; Section 6 proves the main solvability criteria; and Section 7 discusses extensions to lower boundary regularity.

2 Background and Setup to Boundary Value Problems

2.1 $C^{1,\alpha}$ Domains

This thesis involves the study of solutions to boundary value PDEs, where the boundary has a certain amount of "regularity". In order to state the boundary problems and the subsequent theory rigorously, we begin with some preliminary definitions.

Definition 2.1 ($C^{1,\alpha}$ Function). *A function $\gamma : \mathbb{R}^{n+1} \to \mathbb{R}$ is called $C^{1,\alpha}$ continuous if γ is continuously differentiable and $\nabla\gamma$ is Hölder continuous with exponent $0 < \alpha \leq 1$.*

Definition 2.2 ($C^{1,\alpha}$ Domain). *A set $\Omega \subseteq \mathbb{R}^{n+1}$ is a $C^{1,\alpha}$ domain if the boundary $\partial\Omega$ can be locally represented as the graph of a $C^{1,\alpha}$ function, i.e. for every $P^* \in \partial\Omega$ there exists a coordinate system (after potentially rotating or translating the domain) and a $C^{1,\alpha}$ function $\gamma : X \subseteq \mathbb{R}^n \to \mathbb{R}$ such that for some $r > 0$, $\Omega \cap B_r(P^*) = \{(x, y) \in B_r(P^*) : y > \gamma(x)\}$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$.*

As a result, we see that the boundary of $\partial\Omega$ can be expressed in the following way

$$\partial\Omega \cap B_r(P^*) \subseteq \{(x, \varphi(x)) : x \in X\} .$$

Unless otherwise stated, we will assume that Ω is a bounded and connected $C^{1,\alpha}$ domain in $\mathbb{R}^{n+1}$ for $n \geq 2$.

Within this text, we will be integrating over sets Ω and $\partial\Omega$ as manifolds. It is therefore helpful to set the convention of using variables P^*, Q^*, etc. for points on $\partial\Omega$ and P, Q

for points on Ω or Ω^c in order to distinguish them from the coordinates that describe them. Additionally, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$ will be used as the local coordinates that are used to describe those points. For example, $P^* = (x, y)$ follows this convention. Also, $\underline{0} = \{0, \dots, 0\}$ will be used to denote the zero vector.

Proposition 2.3. *Let $P^* \in \partial\Omega$ and let γ be a $C^{1,\alpha}$ function that describes $\partial\Omega$ near P^* so that $P^* = (x_0, \gamma(x_0))$ for $x_0 \in \mathbb{R}^n$. The outward unit normal to $\partial\Omega$ at P^* is given by*

$$n(P^*) = \frac{(\nabla\gamma(P^*), -1)}{\sqrt{1 + |\nabla\gamma(P^*)|^2}} .$$

Proof. Define the function $F(x, y) = \gamma(x) - y$. The points in $\mathbb{R}^{n+1}$ that satisfy $F(x, y) = 0$ forms a level set that happens to be the boundary of $\partial\Omega$. Since the gradient of a function at a point is orthogonal to the level set of the function at that point, we have that

$$\nabla F = \left(\frac{\partial\gamma}{\partial x_1}, \frac{\partial\gamma}{\partial x_2}, \dots, \frac{\partial\gamma}{\partial x_n}, -1 \right) = (\nabla\gamma, -1)$$

is normal to $\partial\Omega$. To verify that this is the outward normal, we note that ∇F points in the direction where F increases fastest. Since F is 0 on $\partial\Omega$, the ∇F points to where F is positive, i.e. where $y < \gamma(x)$, which is away from the domain Ω.

After dividing by the norm, we get that the unit normal is given by

$$n = \frac{(\nabla\gamma, -1)}{\sqrt{1 + |\nabla\gamma|^2}} . \qquad \square$$

Since the normal exists at every point of the boundary, we can define the normal derivative of a function on the boundary.

Definition 2.4 (Normal Derivative)**.** *The normal derivative of $u : \partial\Omega \to \mathbb{R}$ at $P^* \in \partial\Omega$ is*

$$\frac{\partial u}{\partial n}(P^*) = \nabla u \cdot n(P^*)$$

where $n(P^*)$ is the outward normal to $\partial\Omega$ at P^*.

In addition to a normal vector being well defined at every point on the boundary, $C^{1,\alpha}$ domains possess the following useful property.

Proposition 2.5. *For a $C^{1,\alpha}$ domain Ω, there exists a $\mu > 0$ such that if dist$(P, \partial\Omega) < \mu$, then there exists a unique $P^* \in \partial\Omega$ such that P can be written as $P = P^* + n(P^*)$.*

For a proof of this result, see Lemma 14.16 in [1], where a stronger result for C^k domains with $k \geq 2$ is stated. From the proof we see that just the existence of such a μ follows under weaker assumptions, namely when $\partial\Omega$ is C^1. The region surrounding $\partial\Omega$ for which one can express points uniquely in terms of a boundary point and its normal is often referred to as a "tubular neighborhood".

2.2 Statement of Dirichlet and Neumann Boundary Value Problems

We will be investigating four distinct but related boundary value problems (BVPs).

Definition 2.6 (Interior Dirichlet Problem)**.** *For a domain* Ω *and a function* $f \in C(\partial\Omega)$*, the Interior Dirichlet Problem involves finding a solution function* $u : \overline{\Omega} \to \mathbb{R}$ *that satisfies the following conditions*

$$\begin{cases} \Delta u = 0 \text{ on } \Omega \\ u = f \text{ on } \partial\Omega \,. \end{cases}$$

Definition 2.7 (Exterior Dirichlet Problem)**.** *For a domain* Ω *and a function* $f \in C(\partial\Omega)$*, the Exterior Dirichlet Problem involves finding a solution function* $u : \overline{\Omega^c} \to \mathbb{R}$ *that satisfies the following conditions*

$$\begin{cases} \Delta u = 0 \text{ on } \overline{\Omega}^c \\ u = f \text{ on } \partial\Omega \,. \end{cases}$$

Definition 2.8 (Interior Neumann Problem)**.** *For a domain* Ω *and a function* $f \in C(\partial\Omega)$*, the Interior Neumann Problem involves finding a solution function* $u : \overline{\Omega} \to \mathbb{R}$ *that satisfies the following conditions*

$$\begin{cases} \Delta u = 0 \text{ on } \Omega \\ \frac{\partial u}{\partial n_-} = f \text{ on } \partial\Omega \,, \end{cases}$$

where $\frac{\partial u}{\partial n_-}$ *denotes the normal derivative taken with the normal to* $\partial\Omega$ *pointing from* Ω *towards* Ω^c*.*

Definition 2.9 (Exterior Neumann Problem)**.** *For a domain* Ω *and a function* $f \in C(\partial\Omega)$*, the Exterior Neumann Problem involves finding a solution function* $u : \overline{\Omega^c} \to \mathbb{R}$ *that satisfies the following conditions*

$$\begin{cases} \Delta u = 0 \text{ on } \overline{\Omega}^c \\ \frac{\partial u}{\partial n_+} = f \text{ on } \partial\Omega \,, \end{cases}$$

where $\frac{\partial u}{\partial n_+}$ *denotes the normal derivative taken with the normal to* $\partial\Omega$ *pointing from* Ω^c *towards* Ω*.*

In each of the four BVPs, the solution must be harmonic on Ω and satisfy certain properties, expressed as the function f, on the boundary $\partial\Omega$. Frequently, this function f is referred to as the "boundary data" of the BVP. This thesis investigates for which functions $f : C(\partial\Omega) \to \mathbb{R}$ the stated BVPs with boundary data f can be solved.

2.3 Integrating along $C^{1,\alpha}$ domains

This thesis involves extensive integration on our domain Ω as well as its boundary $\partial\Omega$. We will first state useful identities for integrating on domains, as well as how to convert to integrating in local coordinates.

Proposition 2.10 (Green's First Identity for Integrals)**.** *For* $u, v \in C^1(\overline{\Omega})$ *where* Ω *is bounded, we have*

$$\int_{\partial\Omega} u(P^*) \frac{\partial v(P^*)}{\partial n} d\sigma(P^*) = \int_{\Omega} \left(u(P)\Delta v(P) + \nabla u(P) \cdot \nabla v(P)\right) dP \tag{1}$$

where σ *is the surface measure of* $\partial\Omega$*.*

This identity is the result of the use of the divergence theorem on $v\nabla u$. From this, we get the following results specifically for harmonic functions.

Corollary 2.10.1. *If u is harmonic on a bounded domain Ω, we have that*

$$\int_{\Omega} |\nabla u(P)|^2 \, dP = \int_{\partial\Omega} u(P^*) \frac{\partial u(P^*)}{\partial n} d\sigma(P^*) \tag{2}$$

and

$$\int_{\partial\Omega} \frac{\partial u(P^*)}{\partial n} d\sigma(P^*) = 0 \, . \tag{3}$$

Proof. To show (2), we use Green's first identity for integrals with $u = v$. We get that

$$\begin{aligned} \int_{\partial\Omega} u(P^*) \frac{\partial u(P^*)}{\partial n} d\sigma(P^*) &= \int_{\Omega} \left(u(P)\Delta u(P) + \nabla u(P) \cdot \nabla u(P) \right) dP \\ &= \int_{\Omega} |\nabla u(P)|^2 dP \, . \end{aligned}$$

The first term in the integral dissapeared since u is assumed to be harmonic and thus $\Delta u = 0$.

To show (3), we again use Green's first identity, but with $u \equiv 1$ and $v = u$, i.e.

$$\int_{\partial\Omega} 1 \frac{\partial u(P^*)}{\partial n} d\sigma(P^*) = \int_{D} 1\Delta u(P) + \nabla(1) \cdot \nabla u(P) dP = 0 \, .$$

The above integral became 0 since u is harmonic and constant functions have zero gradient. □

From (3), we can immediately conclude that the boundary data f of the interior Neumann problem must satisfy $\int_{\partial\Omega} f(P^*) d\sigma(P^*) = 0$ since if $\frac{\partial u}{\partial n} = f$, then

$$\int_{\partial\Omega} f(P^*) d\sigma(P^*) = \int_{\partial\Omega} \frac{\partial u(P^*)}{\partial n} d\sigma(P^*) = 0 \, .$$

This tempers our expectations for the general solvability of the interior Neumann problem as it tells us already it is not solvable for every $f \in C(\partial\Omega)$.

Sometimes it will become necessary to switch to local coordinates when working on integrals on domains and their boundaries. We will often encounter integrands that have a singularity on the boundary, and the following result is useful for making sense of integrals of such troublesome integrands taken along the boundary as it manages to bound the integral near the singularity.

Proposition 2.11. *Let $F : \partial\Omega \to \mathbb{R}$ be a function of the form*

$$|F(P^*, Q^*)| = \frac{c}{|P^* - Q^*|^\beta}$$

for $c, \beta \geq 0$ such that $\beta \neq n - 1$. Then we have the following bound

$$\int_{\partial\Omega \cap B_\varepsilon(P^*)} |F(P^*, Q^*)| d\sigma(Q^*) \leq C\varepsilon^{n-\beta}$$

for some constant $C > 0$.

Proof. Since $\partial\Omega$ is a $C^{1,\alpha}$ domain, we can find a γ that describes $\partial\Omega$ around P^*. Therefore there exist $x_0, x \in \mathbb{R}^n$ such that $P^* = (x_0, \gamma(x_0))$ and $Q^* = (x, \gamma(x))$.

If $Q^* \in \partial\Omega \cap B_\varepsilon(P^*)$, then $|P^* - Q^*| < \varepsilon$ and

$$|x_0 - x| \leq |(x_0 - x, \gamma(x_0) - \gamma(x))| = |P^* - Q^*| < \varepsilon\,.$$

Therefore $x \in B^n_\varepsilon(x_0) \subseteq \mathbb{R}^n$ (note that $B_\varepsilon(P^*) \subseteq \mathbb{R}^{n+1}$, so $B^n_\varepsilon(x_0)$ is a ball of one dimension lower and therefore distinguished with the superscript n) and thus

$$\partial\Omega \cap B_\varepsilon(P^*) \subseteq \{(x, \gamma(x)) : x \in B^n_\varepsilon(x_0)\}$$

so

$$\int_{\partial\Omega\cap B_\varepsilon(P^*)} |F(P^*, Q^*)| d\sigma(Q^*) \leq \int_{\{(x,\gamma(x)):x\in B^n_\varepsilon(x_0)\}} |F(P^*, Q^*)| d\sigma(Q^*)\,.$$

Since the region we are integrating over can be parametrized as the graph of a function, we can change variables to get

$$\int_{\partial\Omega\cap B_\varepsilon(P^*)} |F(P^*, Q^*)| d\sigma(Q^*) \leq \int_{B^n_\varepsilon(x_0)} |F((x_0, \gamma(x_0)), (x, \gamma(x)))| \sqrt{1 + |\nabla\gamma(x)|^2} dx\,.$$

Further, since γ is a $C^{1,\alpha}$ function, $\nabla\gamma$ is continuous and thus $\sqrt{1 + |\nabla\gamma(x)|^2}$ is bounded on the compact set $B^n_\varepsilon(x_0)$, say by $C' > 0$. All together we now have that

$$\int_{\partial\Omega\cap B_\varepsilon(P^*)} |F(P^*, Q^*)| d\sigma(Q^*) \leq C' \int_{B^n_\varepsilon(x_0)} |F((x_0, \gamma(x_0)), (x, \gamma(x)))| dx\,.$$

Since $|P^* - Q^*| \geq |x_0 - x|$, we have that

$$|F(P^*, Q^*)| = \frac{c}{|P^* - Q^*|^\beta} \leq \frac{c}{|x_0 - x|^\beta}\,.$$

Our integral then becomes

$$\begin{aligned}\int_{\partial\Omega\cap B_\varepsilon(P^*)} |F(P^*, Q^*)| d\sigma(Q^*) &\leq C' \int_{B^n_\varepsilon(x_0)} |F((x_0, \gamma(x_0)), (x, \gamma(x)))| dx \\ &\leq C' \int_{B^n_\varepsilon(x_0)} \frac{c}{|x_0 - x|^\beta} dx\,.\end{aligned}$$

We can express the region we are integrating over in the following way

$$B^n_\varepsilon(x_0) = \{x_0 + rz : z \in \partial B^n_1(0) = S^{n-1}, r \in [0, \varepsilon)\}\,,$$

so we will change variables in order to integrate with respect to z and r separately, i.e.

$$\begin{aligned}\int_{\partial\Omega\cap B_\varepsilon(P^*)} |F(P^*, Q^*)| d\sigma(Q^*) &\leq C' \int_{B^n_\varepsilon(x_0)} \frac{c}{|x_0 - x|^\beta} dx \\ &= C' \int_{S^{n-1}} d\sigma(z) \int_0^\varepsilon \frac{c}{r^\beta} r^{n-1} dr \\ &= C'\sigma(S^{n-1}) \int_0^\varepsilon c r^{n-1-\beta} dr \\ &= \frac{C'\sigma(S^{n-1})}{n - 1 - \beta} \varepsilon^{n-\beta} \\ &= C\varepsilon^{n-\beta}\,,\end{aligned}$$

which gives us the bound we desired. □

3 PDEs Strategies for Finding Solutions

3.1 Newtonian Kernel

In both the Dirichlet and Neumann boundary value problems, we seek solutions that are harmonic. This section investigates possible forms harmonic functions take. From there, we will begin our journey of piecing together a solution to the BVPs.

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function, i.e. $\Delta F = 0$. To see what form F will possibly take, we will first assume that the solution is radial, meaning that there exist a function $R : \mathbb{R} \to \mathbb{R}$ such that F can can be written as $F(x) = R(r)$ where $r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$. We first calculate $\frac{\partial r}{\partial x_i}$ as follows

$$\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\sqrt{x_1^2 + \cdots + x_n^2}\right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_n^2}} = \frac{x_i}{r}\,.$$

Using the chain rule, we calculate $\frac{\partial F}{\partial x_i}$, $\frac{\partial^2 F}{\partial x_i^2}$, and ΔF to be

$$\begin{aligned}\frac{\partial F}{\partial x_i} &= \frac{\partial R}{\partial r}\frac{\partial r}{\partial x_i} = R'(r)\frac{x_i}{r}\\ \frac{\partial^2 F}{\partial x_i^2} &= \frac{R''(r)}{r^2}x_i^2 - \frac{R'(r)x_i^2}{r^3} + \frac{R'(r)}{r}\\ \Delta F &= \sum_{i=1}^{n}\frac{\partial^2 F}{\partial x_i^2} = R''(r) - \frac{R'(r)}{r} + n\frac{R'(r)}{r}\,.\end{aligned}$$

Since we assumed F to be harmonic, we have that

$$\Delta F = R''(r) - \frac{n-1}{r}R'(r) = 0\,.$$

From this, we obtain the second order ODE

$$\frac{R''(r)}{R'(r)} = \frac{1-n}{r}\,,$$

the solution of which is

$$R(r) = \begin{cases} A\ln(r) + B \text{ if } n = 2 \\ \frac{A}{(2-n)r^{n-2}+B} \text{ if } n \geq 3\,. \end{cases}$$

If we take $n \geq 3$, $r = |x-y|$, and choose convenient constants, we arrive at what is called the Newtonian Kernel.

Definition 3.1 (Newtonian Kernel)**.** *For $n \geq 3$, $x, y \in \mathbb{R}^n$, the Newtonian Kernel is given by*

$$N(x,y) = \frac{-1}{(n-2)w_n}\frac{1}{|x-y|^{n-2}} \tag{4}$$

where w_n is the surface area of an n-dimensional unit ball.

From the initial derivation of F, we can see that that $N(x,y) = F(x-y)$ should be harmonic with respect to x when $x \neq y$. In addition, the gradient of the Newtonian kernel will be used frequently in this section, so it has been calculated and now stated for future reference

$$\nabla_y N(x,y) = \frac{-1}{w_n}\frac{x-y}{|x-y|^n}\,. \tag{5}$$

3.2 Layer Potentials

In addition to being harmonic, the solutions to the boundary value problems we are looking at need to satisfy certain conditions at the boundary of the domain Ω. In this section, we introduce the concept of layer potentials, which end up being helpful as they provide a way to use the given information at boundary of the domain to describe points inside the domain while also exploiting the harmonic properties of the Newtonian kernel.

Definition 3.2 (Single Layer Potential)**.** *For $\varphi \in C(\partial\Omega)$, we define the single layer potential of φ as*

$$S[\varphi](P) = \int_{\partial\Omega} N(P, Q^*)\varphi(Q^*)d\sigma(Q^*) \tag{6}$$

where σ is the measure for the boundary $\partial\Omega$.

The Single Layer Potential is an example of what is called an "integral operator" with "kernel" $N(P, Q)$. We will see that this operator is especially relevant to Neumann boundary value problems. Before that, we will confirm that this operator is well defined, continuous, and harmonic.

Proposition 3.3. *For $\varphi \in C(\partial\Omega)$, the Single Layer Potential $S[\varphi]$ is well defined on $\mathbb{R}^{n+1}$.*

Proof. We use u to denote the Single Layer Potential $u = S[\varphi]$.

To show that u is well defined, we must show that $u(P) < \infty$ for every $P \in \Omega$.

For $P \notin \partial\Omega$, the integrand $N(P, Q^*)\varphi(Q^*)$ is continuous for every $Q^* \in \partial\Omega$. Thus, since $\partial\Omega$ is compact, we have that

$$u(P) = \int_{\partial\Omega} N(P, Q^*)\varphi(Q^*)d\sigma(Q^*) < \infty\,.$$

For $P^* \in \partial\Omega$, $N(P^*, Q^*)$ is not continuous on $\partial\Omega$, so we need to do more work to bound the integral. We proceed by splitting up the integral over $\partial\Omega$ into integrals over $\partial\Omega \cap B_\varepsilon(P^*)$ and $\partial\Omega \setminus B_\varepsilon(P^*)$ for $\varepsilon > 0$ as follows

$$\begin{aligned} u(P^*) &= \int_{\partial\Omega} N(P^*, Q^*)\varphi(Q^*)d\sigma(Q^*) \\ &= \int_{\partial\Omega\setminus B_\varepsilon(P^*)} N(P^*, Q^*)\varphi(Q^*)d\sigma(Q^*) + \int_{\partial\Omega\cap B_\varepsilon(P^*)} N(P^*, Q^*)\varphi(Q^*)d\sigma(Q^*) \\ &=: I_1 + I_2\,. \end{aligned}$$

For integral I_1, the integrand $N(P^*, Q^*)\varphi(Q^*)$ is continuous on the compact set $\partial\Omega \setminus B_\varepsilon(P^*)$. It then follows that

$$I_1 = \int_{\partial\Omega\setminus B_\varepsilon(P^*)} N(P^*, Q^*)\varphi(Q^*)d\sigma(Q^*) < \infty.$$

For integral I_2, we have that

$$\begin{aligned}I_2 &= \int_{\partial\Omega\cap B_\varepsilon(P^*)} N(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*)\\ &\leq \int_{\partial\Omega\cap B_\varepsilon(P^*)} |N(P^*,Q^*)||\varphi(Q^*)|d\sigma(Q^*)\\ &\leq \|\varphi\|_\infty \int_{\partial\Omega\cap B_\varepsilon(P^*)} |N(P^*,Q^*)|d\sigma(Q^*)\\ &\leq C' \int_{\partial\Omega\cap B_\varepsilon(P^*)} |N(P^*,Q^*)|d\sigma(Q^*)\\ &= \int_{\partial\Omega\cap B_\varepsilon(P^*)} \frac{C'}{(n-2)w_n}\frac{1}{|P^*-Q^*|^{n-2}}d\sigma(Q^*)\\ &= \int_{\partial\Omega\cap B_\varepsilon(P^*)} \frac{c}{|P^*-Q^*|^{n-2}}d\sigma(Q^*)\end{aligned}$$

where $c = \frac{C'}{(n-2)w_n}$ and we have used that $\|\varphi\|_\infty \leq C'$ for some $C' \in \mathbb{R}$ since φ is continuous on the compact set $\partial\Omega$.

To further bound this integral, we use Proposition 2.11 with $F(P^*,Q^*) = \frac{c}{|P^*-Q^*|^{n-2}}$ to get that

$$I_2 \leq \int_{\partial\Omega\cap B_\varepsilon(P^*)} \frac{c}{|P^*-Q^*|^{n-2}}d\sigma(Q^*) \leq C\varepsilon^2 < \infty$$

for some $C > 0$. Therefore $u(P^*) = I_1 + I_2 < \infty$ on $\partial\Omega$ and is thus well defined. □

Proposition 3.4. *For $\varphi \in C(\partial\Omega)$, the Single Layer Potential $S[\varphi]$ is continuous on $\mathbb{R}^{n+1}$.*

Proof. Let $u = S[\varphi]$. If $P \notin \partial\Omega$, then u is continuous at P since $\partial\Omega$ is compact and the integrand $N(P,Q^*)\varphi(Q^*)$ is continuous for $Q^* \in \partial\Omega$.

If $P^* \in \partial\Omega$, the integrand $N(P^*,Q^*)\varphi(Q^*)$ is no longer continuous on $\partial\Omega$, so we proceed with a $\frac{\varepsilon}{3}$ style proof, so let $\varepsilon > 0$. For $\eta > 0$, we have that

$$\begin{aligned}|u(P)-u(P^*)| =& \left|\int_{\partial\Omega}(N(P,Q^*)-N(P^*,Q^*))\varphi(Q^*)d\sigma(Q^*)\right|\\ \leq& \int_{\partial\Omega}|N(P,Q^*)-N(P^*,Q^*)|\|\varphi\|_\infty d\sigma(Q^*)\\ =&\|\varphi\|_\infty \int_{\partial\Omega\cap B_\eta(P_0)}|N(P,Q^*)-N(P^*,Q^*)|d\sigma(Q^*)\\ &+ \|\varphi\|_\infty \int_{\partial\Omega\setminus B_\eta(P_0)}|N(P,Q^*)-N(P^*,Q^*)|d\sigma(Q^*)\\ \leq&\|\varphi\|_\infty \int_{\partial\Omega\cap B_\eta(P_0)}|N(P,Q^*)|d\sigma(Q^*) + \|\varphi\|_\infty \int_{\partial\Omega\cap B_\eta(P_0)}|N(P^*,Q^*)|d\sigma(Q^*)\\ &+ \|\varphi\|_\infty \int_{\partial\Omega\setminus B_\eta(P_0)}|N(P,Q^*)-N(P^*,Q^*)|d\sigma(Q^*)\\ &=: I_1 + I_2 + I_3\,.\end{aligned}$$

Since $\varphi \in C(\partial\Omega)$, and $\partial\Omega$ is compact, we could conclude that φ is bounded on $\partial\Omega$ and thus take out $\|\varphi\|_\infty < \infty$ from the integrand.

Looking first at I_2, we have that

$$\begin{aligned}
I_2 &= \|\varphi\|_\infty \int_{\partial\Omega\cap B_\eta(P_0)} |N(P^*, Q^*)| d\sigma(Q^*) \\
&= \|\varphi\|_\infty \int_{\partial\Omega\cap B_\eta(P_0)} \frac{1}{w_n(n-2)|P^* - Q^*|^{n-2}} d\sigma(Q^*) \\
&= \frac{\|\varphi\|_\infty}{w_n(n-2)} \int_{\partial\Omega\cap B_\eta(P_0)} \frac{1}{|P^* - Q^*|^{n-2}} d\sigma(Q^*) \\
&\leq C_1 \eta^2
\end{aligned}$$

where in the final inequality is the result of applying Proposition 2.11, which involves converting to polar coordinates. We therefore have that for a η small enough, we can guarantee that $I_2 < \frac{\varepsilon}{3}$.

The integral I_1 written explicitly is

$$\begin{aligned}
I_1 &= \|\varphi\|_\infty \int_{\partial\Omega\cap B_\eta(P_0)} |N(P^*, Q^*)| d\sigma(Q^*) \\
&= \|\varphi\|_\infty \int_{\partial\Omega\cap B_\eta(P_0)} \frac{1}{w_n(n-2)|P^* - Q^*|^{n-2}} d\sigma(Q^*).
\end{aligned}$$

We note that for the η found for I_1, if we take $|P - P^*|$ small enough, $|P - Q^*|$ and $P^* - Q^*|$ should be comprable distances, and therefore we should be able bound I_2 in the same way as I_1, with a bound of order η^2. Therefore for this η and $|P - P^*|$ small enough, we should have that $I_2 < \frac{\varepsilon}{3}$.

For I_3, we note that for $Q^* \in \partial\Omega \setminus B_\eta(P_0)$, the integrand $N(P, Q^*)$ is continuous as a function of P. Therefore $|N(P^*, Q^*) - N(P, Q^*)| \to 0$ as $P \to P^*$. Since we are integrating the continuous integrand over the compact set $\partial\Omega \setminus B_\eta(P_0)$, by dominated convergence it follows that

$$I_3 = \|\varphi\|_\infty \int_{\partial\Omega\setminus B_\eta(P_0)} |N(P^*, Q^*) - N(P, Q^*)| d\sigma(Q^*) \to 0 \text{ as } P \to P^* .$$

Therefore for $|P - P^*|$ small enough, we can know that $I_3 < \frac{\varepsilon}{3}$.

All together, we can say that if $|P - P^*|$ is small enough so that the bounds to I_1 and I_3 hold, we have that

$$|u(P) - u(P^*)| \leq I_1 + I_2 + I_3 < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

and therefore u is continuous at $P^* \in \partial\Omega$. □

So far, we have seen that the Single Layer Potential is well defined and continuous on all of $\mathbb{R}$. We will see that the Single Layer Potential ends up central to the solution to the Neumann Boundary problems. With this in mind, since the Neumann Boundary Problem involves the normal derivative of the boundary, we will see what the normal derivative of the Single Layer Potential looks like in the following proposition.

Proposition 3.5. *For $\varphi \in \partial\Omega$, let $u = S[\varphi]$. Then we can take the normal derivative of u as follows*

$$\frac{\partial u(P)}{\partial n} = \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n_P} \varphi(Q^*) d\sigma(Q^*) .$$

Proof. Suppose that the variable P can be written like $P = (x_1, \ldots, x_{n+1})$. Using the Leibnitz rule for integrals, we can take partial derivatives into the integral as follows

$$\frac{\partial u(P)}{\partial x_i} = \frac{\partial}{\partial x_i} \int_{\partial\Omega} N(P, Q^*)\varphi(Q^*) d\sigma(Q^*) = \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial x_i} \varphi(Q^*) d\sigma(Q^*)$$

where the partial derivative in the integral is of $N(P, Q^*)$ as a function of P. We therefore have that

$$\begin{aligned}
\frac{\partial u(P)}{\partial n} &= \nabla u(P) \cdot n(P) \\
&= \left(\int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial x_1} \varphi(Q^*) d\sigma(Q^*), \ldots, \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial x_{n+1}} \varphi(Q^*) d\sigma(Q^*) \right) \cdot n(P) \\
&= \int_{\partial\Omega} \left(\frac{\partial N(P, Q^*)}{\partial x_1}, \ldots, \frac{\partial N(P, Q^*)}{\partial x_{n+1}} \right) \cdot n(P)\varphi(Q^*) d\sigma(Q^*) \\
&= \int_{\partial\Omega} \nabla_P N(P, Q^*) \cdot n(P)\varphi(Q^*) d\sigma(Q^*) \\
&= \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n_P} \varphi(Q^*) d(Q^*)
\end{aligned}$$

as claimed. □

Before proving that the Single Layer Potential is harmonic, we introduce the related Double Layer potential, as we will be able to show that they are harmonic together.

Definition 3.6 (Double Layer Potential)**.** *For $\varphi \in C(\partial\Omega)$, we define the double layer potential of φ on Ω as*

$$D[\varphi](P) = \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n_{Q^*}} \varphi(Q^*) d\sigma(Q^*) \tag{7}$$

where $\frac{\partial N(P,Q^*)}{\partial n_{Q^*}} = \nabla_{Q^*} N(P, Q^*) \cdot n(Q^*)$ with $n(Q^*)$ being the outward unit normal of $\partial\Omega$.

Using the gradient for N we calculated in Equation 5, we have that

$$\frac{\partial N(P, Q^*)}{\partial n_{Q^*}} = \nabla_{Q^*} N(P, Q^*) \cdot n(Q^*) = \frac{-(P - Q^*) \cdot n(Q^*)}{w_n |P - Q^*|^n} .$$

This term is the kernel of the Double Layer Potential when thought of as an integral operator. The Double Layer Potential is written explicitly as

$$D[\varphi](P) = \int_{\partial\Omega} \frac{-(P - Q^*) \cdot n(Q^*)}{w_n |P - Q^*|^n} \varphi(Q^*) d\sigma(Q^*) . \tag{8}$$

When P^* and Q^* both lie on the boundary $\partial\Omega$, kernel is sometimes referred to as $K(P^*, Q^*)$, i.e.

$$K(P^*, Q^*) = \frac{\partial N(P^*, Q^*)}{\partial n_{Q^*}} = \frac{-(P^* - Q^*) \cdot n(Q^*)}{w_n |P^* - Q^*|^n} \tag{9}$$

with

$$D[\varphi](P^*) = \int_{\partial\Omega} K(P^*, Q^*)\varphi(Q^*)d\sigma(Q^*)\,.$$

Proposition 3.7. *For $\varphi \in C(\partial\Omega)$, the Double Layer Potential $D[\varphi]$ is well defined on $\mathbb{R}^{n+1}$.*

If we attempted to show that the Double Layer Potential is well defined in the same way that we showed the Single Layer Potential is in Proposition 3.3, we would encounter difficulties as we end up with an integral of the form $\int_0^\varepsilon \frac{1}{r}dr$, which is unbounded. To get around this, we rely on the fact that Ω is a $C^{1,\alpha}$ domain. This regularity of the boundary will allows us to show that the Double Layer Potential is in fact bounded on $\partial\Omega$. The following lemma expresses the regularity of $\partial\Omega$ in the form of a useful inequality.

Lemma 3.8. *There exists a constant $c > 0$ such that for every $P^*, Q^* \in \partial\Omega$*

$$|(P^* - Q^*) \cdot n(Q^*)| \leq c|P^* - Q^*|^{1+\alpha}\,. \tag{10}$$

Proof. After possibly a translation and rotation of coordinates, we can find a $C^{1,\alpha}$ function $\gamma : \mathbb{R}^n \to \mathbb{R}$ that defines $\partial\Omega$ around P^* and Q^* such that $Q^* = (\underline{0}, 0)$ and $n(Q^*) = (\underline{0}, 1)$. With this γ, there exists an $x_0 \in \mathbb{R}^n$ such that $P^* = (x_0, \gamma(x_0))$ and we have that $\gamma(0) = 0$ and $\nabla\gamma(0) = \underline{0}$. Using this setup, we can then see that

$$|(P^* - Q^*) \cdot n(Q^*)| = |((x_0, \gamma(x_0)) - (\underline{0}, 0)) \cdot (\underline{0}, 1))| = |\gamma(x_0)|\,.$$

As a result of the mean value theorem, there exists a ξ between $\underline{0}$ and x_0 such that $|\gamma(x_0)| = |\gamma(x_0) - \gamma(\underline{0})| = |\nabla\gamma(\xi)||x_0 - \underline{0}|$. Additionally, since γ is a $C^{1,\alpha}$ function on a compact domain, the gradient is Hölder continuous with exponent α, so there exists a $c > 0$ such that $|\nabla\gamma(\xi)| = |\nabla\gamma(\xi) - \nabla\gamma(\underline{0})| \leq c|\xi - \underline{0}|^\alpha \leq c|x_0|^\alpha$. All together we have that

$$|(P^* - Q^*) \cdot n(Q^*)| = |\gamma(x_0)| = |\nabla\gamma(\xi)||x_0 - \underline{0}| \leq c|x_0|^\alpha|x_0| = c|x_0|^{\alpha+1}$$

and thus, as desired,

$$|(P^* - Q^*) \cdot n(Q^*)| \leq c|x_0|^{\alpha+1} \leq c|P^*|^{\alpha+1} = c|P^* - Q^*|^{\alpha+1}\,. \qquad \square$$

With this lemma, we will now be able to show that the Double Layer Potential is in fact well defined.

Proposition 3.7. Let u to denote the Double Layer Potential $u = D[\varphi]$. In order to show that u is well defined, we must show that $u(P) < \infty$ for every $P \in \mathbb{R}^{n+1}$.

This proof begins in the same way as that of the Single Layer Potential is in Proposition 3.3. By the same reasoning from the proof of Proposition 3.3, $u(P) < \infty$ when $P \notin \partial\Omega$.

For $P^* \in \partial\Omega$, we bound and split up the integral as follows

$$\begin{aligned} u(P^*) &= \int_{\partial\Omega} K(P^*, Q^*)\varphi(Q^*)d\sigma(Q^*) \\ &\leq \int_{\partial\Omega} |K(P^*, Q^*)||\varphi(Q^*)|d\sigma(Q^*) \\ &= \int_{\partial\Omega\setminus B_\varepsilon(P^*)} |K(P^*, Q^*)||\varphi(Q^*)|d\sigma(Q^*) + \int_{\partial\Omega\cap B_\varepsilon(P^*)} |K(P^*, Q^*)||\varphi(Q^*)|d\sigma(Q^*) \\ &=: I_1 + I_2\,. \end{aligned}$$

In order to bound I_1, we first see that the integrand $K(P^*, Q^*)\varphi(Q^*)$ is continuous on the compact set $\partial\Omega \setminus B_\varepsilon$ and therefore

$$
\begin{aligned}
I_1 &= \int_{\partial\Omega\setminus B_\varepsilon(P^*)} |K(P^*, Q^*)||\varphi(Q^*)|d\sigma(Q^*) \\
&= \|\varphi\|_\infty \int_{\partial\Omega\setminus B_\varepsilon(P^*)} |K(P^*, Q^*)|d\sigma(Q^*) \\
&< \infty\,.
\end{aligned}
$$

To bound I_2, we proceed as follows

$$
\begin{aligned}
I_2 &= \int_{\partial\Omega\cap B_\varepsilon(P^*)} |K(P^*, Q^*)||\varphi(Q^*)|d\sigma(Q^*) \\
&\leq \int_{\partial\Omega\cap B_\varepsilon(P^*)} |K(P^*, Q^*)|\|\varphi\|_\infty d\sigma(Q^*) \\
&= \int_{\partial\Omega\cap B_\varepsilon(P^*)} \frac{\|\varphi\|_\infty}{w_n}\frac{|(P^* - Q^*)\cdot n(Q^*)|}{|P^* - Q^*|^n} d\sigma(Q^*)\,.
\end{aligned}
$$

At this point, we apply Lemma 3.8 to get that

$$
\begin{aligned}
I_2 &\leq \int_{\partial\Omega\cap B_\varepsilon(P^*)} \frac{\|\varphi\|_\infty}{w_n}\frac{|(P^* - Q^*)\cdot n(Q^*)|}{|P^* - Q^*|^n} d\sigma(Q^*) \\
&\leq \int_{\partial\Omega\cap B_\varepsilon(P^*)} \frac{\|\varphi\|_\infty}{w_n}\frac{c|P^* - Q^*|^{\alpha+1}}{|P^* - Q^*|^n} d\sigma(Q^*) \\
&= \int_{\partial\Omega\cap B_\varepsilon(P^*)} C'\frac{1}{|P^* - Q^*|^{n-\alpha-1}} d\sigma(Q^*)\,.
\end{aligned}
$$

From here, we apply Proposition 2.11 with $F(P^*, Q^*) = C'\frac{1}{|P^*-Q^*|^{n-\alpha-1}}$ to get that

$$
I_2 \leq \int_{\partial\Omega\cap B_\varepsilon(P^*)} C'\frac{1}{|P^* - Q^*|^{n-\alpha-1}} d\sigma(Q^*) \leq C\varepsilon^{\alpha+1} < \infty\,.
$$

Therefore $u(P^*) \leq I_1 + I_2 < \infty$, so $u = D[\varphi]$ is well defined. □

Corollary 3.8.1. *There exists a constant $C > 0$ such that for all $P^* \in \partial\Omega$,*

$$
\int_{\partial\Omega} |K(P^*, Q^*)|d\sigma(Q^*) < C\,.
$$

Proof. Looking at the proof of Proposition 3.7, we see that our integral is exactly

$$
\int_{\partial\Omega} |K(P^*, Q^*)|d\sigma(Q^*) = I_1 + I_2
$$

with $\varphi \equiv 1$. In the proof, we see that the bounds on I_1 and I_2 are independent of the point P^*, and therefore hold for all $P^* \in \partial\Omega$. □

Now is a good time to confirm that the Single and Double Layer Potential indeed preserve the imposed harmonicity of the Newtonian Potential.

Proposition 3.9. *Let $\varphi \in C(\partial\Omega)$. Then $D[\varphi]$ and $S[\varphi]$ are harmonic on $(\partial\Omega)^c$.*

Proof. Let $P \in (\partial\Omega)^c$. Since Ω is open, we can find some ε such that $B_\varepsilon(P) \subseteq B_{2\varepsilon}(P) \subseteq \Omega$. As we are now far enough away from the boundary, where any singularity of N would lie, we see that $N(P, Q^*)$ is smooth (in P) on $B_\varepsilon(P)$. This means that first derivatives and higher with respect to P will be continuous. Using the Leibnitz rule for integrals, we are able to change the order of differentiation and integration in both the case of the Single Layer Potential and that of the Double Layer Potential. We are thus able to take the Laplacian into the integral as follows

$$\begin{aligned}\Delta S[\varphi](P) &= \int_{\partial\Omega} \Delta_P\big(N(P,Q^*)\varphi(Q^*)\big)d\sigma(Q^*)\\ &= \int_{\partial\Omega} \Delta_P N(P,Q^*)\varphi(Q^*)d\sigma(Q^*)\\ &= 0\end{aligned}$$

$$\begin{aligned}\Delta D[\varphi](P) &= \int_{\partial\Omega} \Delta_P\left(\frac{\partial N(P,Q^*)}{\partial v_{Q^*}}\varphi(Q^*)\right)d\sigma(Q^*)\\ &= \int_{\partial\Omega} \frac{\partial(\Delta_P N(P,Q^*))}{\partial v_{Q^*}}\varphi(Q^*)d\sigma(Q^*)\\ &= 0\,.\end{aligned}$$

In each case, the last equality is a result of the Newtonian Kernel being harmonic, so the Laplacian $\Delta_P N(P, Q^*) = 0$. Therefore on $(\partial\Omega)^c$, $S[\varphi]$ and $D[\varphi]$ are harmonic. □

Proposition 3.10. *There exists a constant $C > 0$ such that for every $P \notin \partial\Omega$,*

$$\int_{\partial\Omega}\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right|d\sigma(Q^*) \le C\,.$$

Proof. We begin by selecting a value for $\delta > 0$ such that:

1. We have $\delta < \frac{1}{2c}$, where c is the constant from Lemma 3.8.
2. For all $P \in \mathbb{R}^{n+1}$ such that $dist(P, \partial\Omega) < \frac{1}{2}\delta$ there exists a unique $P^* \in \partial\Omega$ and $t \in (-\frac{\delta}{2}, \frac{\delta}{2})$ such that P can be written as $P = P^* + tn(P^*)$. By Proposition 2.5, since Ω is a $C^{1,\alpha}$ domain, such a $\delta > 0$ exists.

<u>Case 1:</u> $dist(P, \partial\Omega) \ge \frac{1}{2}\delta$

In this case, we have that for $Q^* \in \partial\Omega$,

$$\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| = \left|\frac{1}{w_n}\frac{(P-Q^*)\cdot n(Q^*)}{|P-Q^*|^n}\right| \le \frac{1}{w_n}\frac{1}{|P-Q^*|^{n-1}} \le \frac{1}{w_n}\frac{1}{|\frac{1}{2}\delta|^{n-1}} = C_1\delta^{1-n}\,.$$

The integral over $\partial\Omega$ therefore can be bounded in the following way,

$$\int_{\partial\Omega}\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right|d\sigma(Q^*) \le C_1\delta^{1-n}\int_{\partial\Omega}d\sigma(Q^*) < C_2$$

since Ω is bounded and thus the surface area of $\partial\Omega$ is finite. Note that the bound is independent of P other than the fact that $dist(P, \partial\Omega) \ge \frac{1}{2}\delta$.

Case 2: $dist(P, \partial\Omega) < \frac{1}{2}\delta$

Let $P^* \in \partial\Omega$ be the unique P^* such that $P = P^* + tn(P^*)$ for $t \in (-\frac{1}{2}\delta, \frac{1}{2}\delta)$. We will integrate over $\partial\Omega \setminus B_\delta(P^*)$ and $B_\delta(P^*)$ separately, i.e.

$$\begin{aligned}\int_{\partial\Omega}\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| d\sigma(Q^*) &= \int_{\partial\Omega\setminus B_\delta(P^*)}\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| d\sigma(Q^*) + \int_{\partial\Omega\cap B_\delta(P^*)}\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| d\sigma(Q^*) \\ &=: I_1 + I_2\,.\end{aligned}$$

For $Q^* \in \partial\Omega \setminus B_\delta(P^*)$, we have that

$$|P - Q^*| \geq |P^* - Q^*| - |P - P^*| \geq \delta - \frac{1}{2}\delta = \frac{1}{2}\delta\,.$$

With the same workings as in Case 1, we get that

$$\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| \leq C_1\delta^{1-n}$$

and thus

$$I_1 = \int_{\partial\Omega\setminus B_\delta(P^*)}\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| d\sigma(Q^*) \leq C_1\delta^{1-n}\int_{\partial\Omega\setminus B_\delta(P^*)} d\sigma(Q^*) < C_3\,.$$

For $Q^* \in B_\delta(P^*)$, the integrand in I_2 can be bounded as follows

$$\begin{aligned}\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| &= \left|\frac{1}{w_n}\frac{(P-Q^*)\cdot n(Q^*)}{|P-Q^*|^n}\right| \\ &\leq \frac{|(P-P^*)\cdot n(Q^*)| + |(P^*-Q^*)\cdot n(Q^*)|}{w_n|P-Q^*|^n} \\ &\leq \frac{|P-P^*| + c|(P^*-Q^*)|^{1+\alpha}}{w_n|P-Q^*|^n}\,.\end{aligned}$$

When considering at the denominator, we will use that

$$|P-Q^*|^2 = |P-P^*|^2 + |P^*-Q^*|^2 + 2(P-P^*)\cdot(P^*-Q^*)\,. \tag{11}$$

Looking at the final term, we use the fact that $P - P^* = |P - P^*|n(P^*)$, the result from Lemma 3.8, and the first condition that δ must satisfy to get that

$$\begin{aligned}|2(P-P^*)\cdot(P^*-Q^*)| &\leq 2|P-P^*||n(P^*)\cdot(P^*-Q^*)| \\ &\leq 2c|P-P^*||P^*-Q^*|^{1+\alpha} \\ &\leq 2c|P-P^*||P^*-Q^*||P^*-Q^*| \\ &\leq 2c\delta|P-P^*||P^*-Q^*| \\ &< |P-P^*||P^*-Q^*| \\ &\leq \frac{1}{2}|P-P^*|^2 + \frac{1}{2}|P^*-Q^*|^2\,.\end{aligned}$$

The last line is a result of Young's inequality for products. Therefore the final term of (11) becomes

$$2(P-P^*)\cdot(P^*-Q^*) \geq -\left(\frac{1}{2}|P-P^*|^2 + \frac{1}{2}|P^*-Q^*|^2\right)\,.$$

It then follows that

$$|P - Q^*|^2 \geq |P - P^*|^2 + |P^* - Q^*|^2 - \left(\frac{1}{2}|P - P^*|^2 + \frac{1}{2}|P^* - Q^*|^2\right)$$
$$= \frac{1}{2}\left(|P - P^*|^2 + |P^* - Q^*|^2\right) .$$

Putting this into the denominator of the expression above, we have that

$$\frac{|P - P^*| + c|(P^* - Q^*)|^{1+\alpha}}{w_n|P - Q^*|^n} \leq C_4 \frac{|P - P^*| + c|(P^* - Q^*)|^{1+\alpha}}{(|P - P^*|^2 + |P^* - Q^*|^2)^{\frac{n}{2}}}$$
$$\leq \frac{C_4|P - P^*|}{(|P - P^*|^2 + |P^* - Q^*|^2)^{\frac{n}{2}}} + \frac{C_4 c|(P^* - Q^*)|^2}{|P^* - Q^*|^n}$$
$$\leq \frac{C_4|P - P^*|}{(|P - P^*|^2 + |P^* - Q^*|^2)^{\frac{n}{2}}} + \frac{C_4 c}{|P^* - Q^*|^{n-2}} .$$

To show that $\int_{B_\delta(P^*)} \left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| d\sigma(Q^*)$ is finite, we use the bound on the integrand we have found and convert to polar coordinates. To this end, let $r = |P^* - Q^*|$ and let $a = |P - P^*|$. Note that a is constant as we are working with a specific $P \notin \partial\Omega$.

The integral then becomes

$$\int_{B_\delta(P^*)} \left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| d\sigma(Q^*) \leq C_5 \int_0^\delta \left[\frac{a}{(a^2 + r^2)^{\frac{1}{2}}} + \frac{c}{r^{n-2}}\right] r^{n-1} dr$$
$$= C_5 \int_0^\delta \frac{a r^{n-1}}{(a^2 + r^2)^{\frac{1}{2}}} dr + C_5 c \int_0^\delta dr$$
$$< \infty .$$

As we have shown that the integral over $\partial\Omega \setminus B_\delta(P^*)$ and $B_\delta(P^*)$ are bounded with bounds independent of P, and thus the integral is bounded over $\partial\Omega$. □

3.3 Double Layer Potential Behavior at Domain Boundary

So far, we have introduced Single and Double Layer Potentials as we work towards finding solutions to our boundary value problems. These seem like good candidates as they are harmonic on Ω and $\overline{\Omega}^{\mathsf{c}}$, and they take in information about a function defined only on the boundary $\partial\Omega$ and give a function defined on all of $\mathbb{R}^{n+1}$.

In the Dirichlet boundary value problems, we are further looking for a solution that is continuous on $\overline{\Omega}$ (or Ω^{c}). It is disappointing, therefore, that we cannot guarantee that a Double Layer Potential, our strongest candidate so far for a solution, cannot be guaranteed to be continuous on $\overline{\Omega}$ (or Ω^{c}).

Similarly, the Single Layer Potential also cannot be implemented as a solution so simply.

Despair not, however, as we will find a condition for φ that guarantees that $D[\varphi]$ and can be extended continuously to the boundary and what values the extension needs to take on the boundary. We will also be able to find what values the normal derivative of $S[\varphi]$ takes on the boundary of Ω.

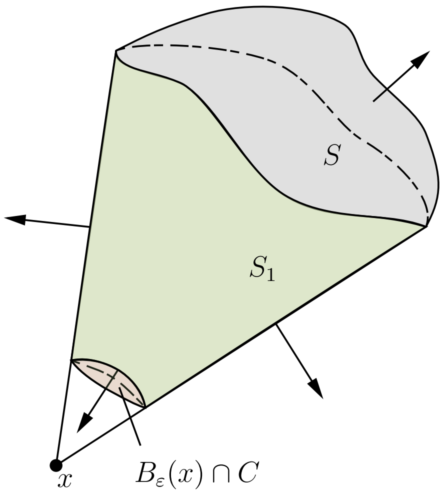
We begin with a somewhat technical result that is crucial for uncovering how the Double Layer Potential behaves on the boundary $\partial\Omega$.

Proposition 3.11. *Let S be a $C^{1,\alpha}$ surface and $x \notin S$ such that each line starting at x and going through $y \in S$ only intersects S once. Let C be the cone containing all such lines between x and S. Then*

$$\int_S \frac{\partial N(x,y)}{\partial n_y} d\sigma(y) = \frac{\omega(C)}{w_n} \tag{12}$$

where $\omega(C) := \int_{C \cap B_1(x)} d\sigma(y)$ is the solid angle of C.

Proof. Let $D \subseteq C$ be the bounded set comprised of the union of all line segments between x and S and let ε small enough that $B_{2\varepsilon}(x) \cap S = \emptyset$. Consider $D' = D \setminus B_\varepsilon(x)$. The boundary of D' is comprised of the surfaces S, $\partial B_\varepsilon(x) \cap C$, and S_1, where S_1 is comprised of the line segments between x and ∂S starting at x minus any points in $B_\varepsilon(x)$.



Using Green's Identity, we have that

$$\int_{\partial D'} 1 \frac{\partial N(x,y)}{\partial n_y} d\sigma(y) = \int_D 1 \Delta N(x,y) + \nabla(1) \cdot \nabla N(x,y) dy = 0\,.$$

Since S_1 is comprised of straight lines, we can see that that $n_y \perp (x-y)$ on S_1. Therefore on S_1,

$$\frac{\partial N(x,y)}{\partial n_y} = \frac{(x-y) \cdot n_y}{w_n |x-y|^n} = 0\,.$$

We therefore have that

$$\int_S \frac{\partial N(x,y)}{\partial n_y} d\sigma(y) + \int_{S_1} 0 \, d\sigma(y) + \int_{\partial B_\varepsilon(x) \cap C} \frac{\partial N(x,y)}{\partial n_y} d\sigma(y) = 0\,.$$

Note that we are integrating along $\partial B_\varepsilon(x) \cap C$ with the normal $v_y = -\varepsilon(y-x)$ pointing outwards towards x. We can therefore calculate the desired integral as follows,

$$\begin{aligned}\int_S \frac{\partial N(x,y)}{\partial n_y} d\sigma(y) &= -\int_{\partial B_\varepsilon(x)\cap C} \frac{\partial N(x,y)}{\partial n_y} d\sigma(y) \\ &= -\int_{\partial B_\varepsilon(x)\cap C} \frac{(x-y)\cdot n_y}{w_n|x-y|^n} d\sigma(y) \\ &= \int_{\partial B_\varepsilon(x)\cap C} \frac{(x-y)\cdot \varepsilon(x-y)}{w_n|x-y|^n} d\sigma(y) \\ &= \frac{1}{w_n\varepsilon^{n-1}} \int_{\partial B_\varepsilon(x)\cap C} 1 \, d\sigma(y) \\ &= \frac{1}{w_n\varepsilon^{n-1}} \varepsilon^{n-1} \int_{\partial B_1(x)\cap C} 1 \, d\sigma(y) \\ &= \frac{\omega(C)}{w_n}. \end{aligned}$$

□

Using this result, we will look at the Double Layer Potential of $\varphi \equiv 1$, i.e. just integrating the kernel, and look at what values $D[1](P)$ takes depending if $P \in \Omega$, $P \in \overline{\Omega}^{\mathsf{c}}$, or $P \in \partial\Omega$.

Proposition 3.12. *For a $C^{1,\alpha}$ domain Ω,*

$$\int_{\partial\Omega} \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} d\sigma(Q^*) = \begin{cases} 1 \text{ if } P \in \Omega \\ 0 \text{ if } P \in \overline{\Omega}^{\mathsf{c}}. \end{cases}$$

Proof. If $P \in \overline{\Omega}^{\mathsf{c}}$, using Green's identity and the fact that $N(P,Q^*)$ is harmonic on Ω when $P \notin \partial\Omega$, we have that

$$\int_{\partial\Omega} 1 \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} d\sigma(Q^*) = \int_\Omega 1\Delta N + \nabla(1)\cdot\nabla N dQ = 0\,.$$

If $P \in \Omega$, find $r > 0$ such that $B_{2r}(P) \subset \Omega$ and consider the domain $\Omega' = \Omega \setminus B_r(P)$. Then, again using Green's identity,

$$\int_{\partial\Omega} \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} d\sigma(Q^*) - \int_{\partial B_r(P)} \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} d\sigma(Q^*) = \int_{\Omega'} 1\Delta N + \nabla(1)\cdot\nabla N dQ^* = 0\,.$$

Therefore, using Proposition 3.11, we have that

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} d\sigma(Q^*) &= \int_{\partial B_r(P)} \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} d\sigma(Q^*) \\ &= \frac{\omega(B_r(P))}{w_n} = \frac{w_n}{w_n} = 1\,. \end{aligned}$$

□

Proposition 3.13. *For a $C^{1,\alpha}$ domain Ω, if $P^* \in \partial\Omega$, then*

$$\int_{\partial\Omega} K(P^*,Q^*) d\sigma(Q^*) = \frac{1}{2}\,.$$

Proof. Consider the region $\Omega' = \Omega \setminus B_\varepsilon(P^*)$ for some $\varepsilon > 0$. The boundary of Ω' is comprised of $S_\varepsilon := (\partial\Omega) \setminus B_\varepsilon(P^*)$ and $\partial B'_\varepsilon := (\partial B_\varepsilon(P^*)) \cap \Omega$ with outward normal pointing towards P^*.

Since $P^* \notin \Omega'$, by applying Proposition 3.12 we have that

$$\int_{S_\varepsilon} \frac{\partial N(P^*, Q^*)}{\partial n_{Q^*}} d\sigma(Q^*) + \int_{B'_\varepsilon} \frac{\partial N(P^*, Q^*)}{\partial n_{Q^*}} d\sigma(Q^*) = 0\,.$$

Therefore

$$\begin{aligned} \int_{\partial\Omega} K(P^*, Q^*) d\sigma(Q^*) &= \lim_{\varepsilon\to 0} \int_{S_\varepsilon} \frac{\partial N(P^*, Q^*)}{\partial n_{Q^*}} d\sigma(Q^*) \\ &= -\lim_{\varepsilon\to 0} \int_{B'_\varepsilon} \frac{\partial N(P^*, Q^*)}{\partial n_{Q^*}} d\sigma(Q^*) \\ &= \lim_{\varepsilon\to 0} \frac{\omega(\partial B'_\varepsilon)}{w_n}\,. \end{aligned}$$

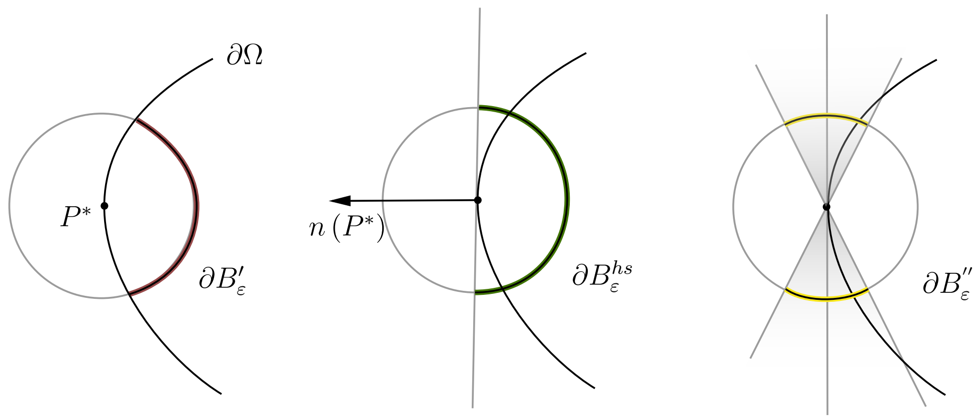
The sign change in the last equality is to take into consideration the fact that the outward normal of B'_ε as part of the boundary of Ω' is pointing towards its center P^*, which is the opposite direction to the normal used to calculate the solid angle.

Let $B^{\text{hs}}_\varepsilon$ denote the hemisphere of radius ε centered at P^* on the same side of the tangent plane at P^* as Ω, i.e. $B^{\text{hs}}_\varepsilon = \{Q \in B_\varepsilon(P^*) : \langle Q, n(P^*)\rangle \le 0\}$. Because Ω is a $C^{1,\alpha}$ domain, $\partial B'_\varepsilon$ and $\partial B^{hs}_\varepsilon$ only differ slightly as sets. Specifically, they can only differ within a "equatorial strip" of the form

$$\partial B''_\varepsilon = \{Q \in \partial B_\varepsilon(P^*) : |\langle Q, n(P^*)\rangle| \le c\varepsilon^{1+\alpha}\}\,,$$

which one can show has surface area of the order $O(\varepsilon^{n-1+\alpha})$.

The following diagram illustrates in two dimensions the sets $\partial B'_\varepsilon$, $\partial B^{hs}_\varepsilon$, and $\partial B''_\varepsilon$.



We have that $\partial B^{\text{hs}}_\varepsilon \setminus \partial B''_\varepsilon \subseteq \partial B'_\varepsilon \subseteq \partial B^{\text{hs}}_\varepsilon \cup \partial B''_\varepsilon$ and by the monotonicity of the solid angle,

$$\omega(\partial B^{\text{hs}}_\varepsilon) - \omega(\partial B''_\varepsilon) \le \omega(\partial B'_\varepsilon) \le \omega(\partial B^{\text{hs}}_\varepsilon) + \omega(\partial B''_\varepsilon)\,.$$

Since, $\partial B^{\text{hs}}_\varepsilon$ and $\partial B''_\varepsilon$ are already subsets of a sphere, their solid angle is simply given by $\omega = \varepsilon^{1-n} \times$ Surface Area. Therefore $\omega(\partial B^{\text{hs}}_\varepsilon) = \varepsilon^{1-n}\frac{w_n}{2}\varepsilon^{n-1} = \frac{w_n}{2}$ and $\omega(\partial B''_\varepsilon) \le$

$\varepsilon^{1-n}C\varepsilon^{n-1+\alpha} = C\varepsilon^{\alpha}$ for some constant C arising from bounding the term of order $O(\varepsilon^{n-1+\alpha})$. Taking the limit $\varepsilon \to 0$, we have that

$$\lim_{\varepsilon\to 0}\left(\frac{\omega(\partial B_\varepsilon^{\text{hs}})}{w_n} - \frac{\omega(\partial B_\varepsilon'')}{w_n}\right) \le \lim_{\varepsilon\to 0}\frac{\omega(\partial B_\varepsilon')}{w_n} \le \lim_{\varepsilon\to 0}\left(\frac{\omega(\partial B_\varepsilon^{\text{hs}})}{w_n} + \frac{\omega(\partial B_\varepsilon'')}{w_n}\right)$$
$$\lim_{\varepsilon\to 0}\left(\frac{1}{2} - \frac{C}{w_n}\varepsilon^\alpha\right) \le \lim_{\varepsilon\to 0}\frac{\omega(\partial B_\varepsilon')}{w_n} \le \lim_{\varepsilon\to 0}\left(\frac{1}{2} + \frac{C}{w_n}\varepsilon^\alpha\right)$$
$$\frac{1}{2} \le \lim_{\varepsilon\to 0}\frac{\omega(\partial B_\varepsilon')}{w_n} \le \frac{1}{2}\,.$$

Therefore

$$\int_{\partial\Omega} K(x,y)d\sigma(y) = \lim_{\varepsilon\to 0}\frac{\omega(\partial B_\varepsilon')}{w_n} = \frac{1}{2}\,. \qquad \square$$

These last propositions solidify that the Double Layer Potential is not generally continuous across the boundary $\partial\Omega$. However, the following lemma gives a scenario to where the Double Layer Potential is continuous at at least a point on the boundary.

Lemma 3.14. *Let $\Phi \in C(\partial\Omega)$ such that $\Phi(P_0^*) = 0$. Then the function*

$$u(P^*) = \int_{\partial\Omega} K(P^*,Q^*)\Phi(Q^*)d\sigma(Q^*)$$

is continuous at P_0^*.

Proof. By Proposition 3.10, there exists a $C > 0$ such that for every $P \in \Omega$,

$$\int_{\partial\Omega}\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| d\sigma(Q^*) < C\,.$$

Additionally, by Corollary 3.8.1, there exists $C' > 0$ such that for every $P^* \in \partial\Omega$

$$\int_{\partial\Omega}|K(P^*,Q^*)|d\sigma(Q^*) < C'\,.$$

Let $\varepsilon > 0$. Since Φ is continuous and 0 at P_0^* , we can find a $\eta > 0$ such that for $P^* \in \partial\Omega$ such that $|P^* - P_0^*| < \eta$, we have that $|\Phi(P^*)| < \frac{\varepsilon}{2(C+C')}$. We therefore have that

$$\begin{aligned}|u(P) - u(P_0^*)| =& \left|\int_{\partial\Omega}\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\Phi(Q^*)d\sigma(Q^*) - \int_{\partial\Omega}K(P_0^*,Q^*)\Phi(Q^*)d\sigma(Q^*)\right|\\ \le& \int_{\partial\Omega}\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}} - K(P^*,Q^*)\right||\Phi(Q^*)|d\sigma(Q^*)\\ \le& \int_{\partial\Omega\cap B_\eta(P_0^*)}\left(\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}\right| + |K(P^*,Q^*)|\right)|\Phi(Q^*)|d\sigma(Q^*)\\ &+ \int_{\partial\Omega\setminus B_\eta(P_0^*)}\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}} - K(P^*,Q^*)\right||\Phi(Q^*)|d\sigma(Q^*)\,.\end{aligned}$$

The first term can be bounded as follows

$$\begin{aligned}&\int_{\partial\Omega\cap B_\eta(P_0^*)}\left(\left|\frac{\partial N(P,Q^*)}{\partial n(Q^*)}\right| + |K(P^*,Q^*)|\right)|\Phi(Q^*)|d\sigma(Q^*)\\ &\quad\le\left(\int_{\partial\Omega\cap B_\eta(P_0^*)}\left|\frac{\partial N(P,Q^*)}{\partial n(Q^*)}\right|d\sigma(Q^*) + \int_{\partial\Omega\cap B_\eta(P_0^*)}|K(P^*,Q^*)|d\sigma(Q^*)\right)\frac{\varepsilon}{2(C+C')}\\ &\quad\le(C+C')\frac{\varepsilon}{2(C+C')} = \frac{\varepsilon}{2}\,.\end{aligned}$$

To bound the second term, first let $M = \int_{\partial\Omega} |\Phi(Q^*)| d\sigma(Q^*) < \infty$ and then find a $\delta > 0$ such that if $|P - P_0^*| \leq \delta$,

$$\sup_{Q^* \in \partial\Omega \setminus B_\eta(P_0^*)} \left| \frac{\partial N(P, Q^*)}{\partial n(Q^*)} - K(P^*, Q^*) \right| \leq \frac{\varepsilon}{2M} .$$

It then follows that

$$\int_{\partial\Omega \setminus B_\eta(P_0^*)} \left| \frac{\partial N(P, Q^*)}{\partial n(Q^*)} - K(P^*, Q^*) \right| |\Phi(Q^*)| d\sigma(Q^*) \leq \frac{\varepsilon}{2M} \int_{\partial\Omega \setminus B_\eta(P_0^*)} |\Phi(Q^*)| d\sigma(Q^*) \leq \frac{\varepsilon}{2} .$$

Therefore if $|P - P_0^*| \leq \delta$, $|u(P) - u(P_0^*)| \leq \epsilon$, i.e. u is continuous at P_0^*. □

Though the Double Layer Potential as defined in (7), often has a discontinuity on $\partial\Omega$ and is thus usually not continuous on $\overline{\Omega}$ (or Ω^c) as the solutions of the Dirichlet BVPs need to be. The following proposition allows us to say that the Double Layer Potential can be extended continuously from Ω to be defined on $\partial\Omega$ (or from $\overline{\Omega}^c$ to $\partial\Omega$) to create form a continuous function on $\overline{\Omega}$ (or Ω^c), and describe what values this extension should take on the boundary. This get us one step closer to considering it as a solution to the Dirichlet BVPs.

This result is often referred to as a "jump relation" for the Double Layer Potential since, as we will see, the continuous extension of $D[\varphi]$ "jumps" up or down by $\frac{1}{2}\varphi$ at the boundary, depending if it is the extension from inside or outside Ω.

Proposition 3.15 (Jump Relations for Double Layer Potential)**.** *Let $\varphi \in C(\partial\Omega)$, then*

$$u_-(P) = \begin{cases} \int_{\partial\Omega} \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} \varphi(Q^*) d\sigma(Q^*) & P \in \Omega \\ \frac{1}{2}\varphi(P) + \int_{\partial\Omega} K(P, Q^*) \varphi(Q^*) d\sigma(Q^*) & P \in \partial\Omega \end{cases} \tag{13}$$

is continuous on $\overline{\Omega}$. Similarly,

$$u_+(P) = \begin{cases} \int_{\partial\Omega} \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} \varphi(Q^*) d\sigma(Q^*) & P \in \overline{\Omega}^c \\ -\frac{1}{2}\varphi(P) + \int_{\partial\Omega} K(P, Q^*) \varphi(Q^*) d\sigma(Q^*) & P \in \partial\Omega \end{cases} \tag{14}$$

is continuous on Ω^c.

Proof. Denote $u(P) := D[\varphi](P) = \int_{\partial\Omega} \frac{\partial N(P,Q^*)}{\partial n(Q^*)} \varphi(Q^*) d\sigma(Q^*)$. Since we know u is continuous on Ω and $\overline{\Omega}^c$, we will focus on showing that it can be continuously extended to the boundary $\partial\Omega$.

Let $P^* \in \partial\Omega$. We will first show that the continuous extention of u to $\overline{\Omega}$ is u_- by showing that

$$\lim_{\substack{P \to P^* \\ P \in \Omega}} u(P) = \frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K(P^*, Q^*) \varphi(Q^*) d\sigma(Q^*) .$$

For $P \in \Omega$, we can re-write $u(P)$ as follows,

$$\begin{aligned} u(P) &= \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n(Q^*)} \varphi(Q^*) d\sigma(Q^*) \\ &= \varphi(P^*) \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n(Q^*)} d\sigma(Q^*) + \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n(Q^*)} [\varphi(Q^*) - \varphi(P^*)] d\sigma(Q^*) \\ &= \varphi(P^*) + \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n(Q^*)} [\varphi(Q^*) - \varphi(P^*)] d\sigma(Q^*) . \end{aligned}$$

The first integral became $\varphi(P^*)$ as a result of Proposition 3.12 , as $P \in \Omega$, so the integral becomes 1.

Let $\Phi(Q^*) = \varphi(Q^*) - \varphi(P^*)$. By Lemma 3.14, since Φ is continuous on $\partial\Omega$ and $\Phi(P^*) = 0$, we have that the second integral is continuous at P^* . We can therefore take the limit $P \to P^*$ and then use Proposition 3.13 to get

$$\begin{aligned}
\lim_{\substack{P \to P^* \\ P \in \Omega}} u(P) &= \varphi(P^*) + \int_{\partial\Omega} K(P^*, Q^*)[\varphi(Q^*) - \varphi(P^*)] d\sigma(Q^*) \\
&= \varphi(P^*) - \varphi(P^*) \int_{\partial\Omega} K(P^*, Q^*) d\sigma(Q^*) + \int_{\partial\Omega} K(P^*, Q^*)\varphi(Q^*) d\sigma(Q^*) \\
&= \varphi(P^*) - \frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K(P^*, Q^*)\varphi(Q^*) d\sigma(Q^*) \\
&= \frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K(P^*, Q^*)\varphi(Q^*) d\sigma(Q^*) \,,
\end{aligned}$$

which is exactly u_- defined on the boundary. Therefore u_- is continuous on all of $\overline{\Omega}$.

The process of showing that u_+ is the continuous extention of u on Ω^{c} is similar. To show that

$$\lim_{\substack{P \to P^* \\ P \in \overline{\Omega}^{\mathrm{c}}}} u(P) = -\frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K(P^*, Q^*)\varphi(Q^*) d\sigma(Q^*) \,,$$

we have that

$$\begin{aligned}
u(P) &= \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n(Q^*)} \varphi(Q^*) d\sigma(Q^*) \\
&= \varphi(P^*) \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n(Q^*)} d\sigma(Q^*) + \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n(Q^*)} [\varphi(Q^*) - \varphi(P^*)] d\sigma(Q^*) \\
&= \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n(Q^*)} [\varphi(Q^*) - \varphi(P^*)] d\sigma(Q^*) \,.
\end{aligned}$$

As a result of Proposition 3.12, the first integral is 0 and thus the first term dissapears. Again, taking limits, with Lemma 3.14 and Proposition 3.13 we get

$$\begin{aligned}
\lim_{\substack{P \to P^* \\ P \in \overline{\Omega}^{\mathrm{c}}}} u(P) &= \int_{\partial\Omega} K(P^*, Q^*)[\varphi(Q^*) - \varphi(P^*)] d\sigma(Q^*) \\
&= -\varphi(P^*) \int_{\partial\Omega} K(P^*, Q^*) d\sigma(Q^*) + \int_{\partial\Omega} K(P^*, Q^*)\varphi(Q^*) d\sigma(Q^*) \\
&= -\frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K(P^*, Q^*)\varphi(Q^*) d\sigma(Q^*) \,.
\end{aligned}$$

□

3.4 Single Layer Potential Behavior at Domain Boundary

We will now introduce a similar result that describes the behavior of the Single Layer Potential at the boundary $\partial\Omega$. We have already showed that the Single Layer Potential is continuous on $\partial\Omega$, so a continuous extension to $\partial\Omega$ is of no use to us. Instead, we will be looking at the normal derivative at the boundary. If we recall from Proposition 3.5, we calculated the normal derivative of the Single Layer Potential and it had the form

$$\frac{\partial u(P)}{\partial n} = \int_{\partial\Omega} \frac{\partial N(P, Q^*)}{\partial n_P} \varphi(Q^*) d\sigma(Q^*) \,.$$

In order to state the following result, we define the kernel term K^* for $P^*, Q^* \in \partial\Omega$ as

$$K^*(P^*, Q^*) = \frac{\partial N(P^*, Q^*)}{\partial n_{P^*}} = \frac{(P^* - Q^*) \cdot n(P^*)}{w_n|P^* - Q^*|^n}.$$

This kernel resembles the kernel K that shows up in the Double Layer Potential, but the key difference is that the normal derivative is taken with respect to the first variable P^* instead of Q^*. This makes a profound difference when used as the kernel of an integral operator.

Now that the kernels K and K^* have been introduced, we will work extensively with their associated integral operators. It will be very useful to show that these integral operators map continuous functions on the boundary to continuous functions on the boundary. The following proposition gives us that result, though it is stated in more generality than is needed for this section, as the more general statement is used later on.

Proposition 3.16. *Let $K(P, Q) : \partial\Omega \times \partial\Omega \to \mathbb{R}$ such that $K(P^*, Q^*) \leq \frac{C}{|P^* - Q^*|^{n-1-\alpha}}$ for $C > 0$ and K is continuous when $P^* \neq Q^*$. Then the integral operator T_K given by*

$$T_K\varphi(P^*) = \int_{\partial\Omega} K(P^*, Q^*)\varphi(Q^*)d\sigma(Q^*)$$

maps bounded functions on $\partial\Omega$ to continuous functions on $\partial\Omega$.

Proof. This proof follows a similar structure to that of the proof of the Single Layer Potential being continuous. Let φ be a bounded function on $\partial\Omega$ and $\varepsilon > 0$. For $P^*, P_0^* \in \partial\Omega$,

$$\begin{aligned}
|T\varphi(P^*) - T\varphi(P_0^*)| &= \left|\int_{\partial\Omega} (K(P^*, Q^*) - K(P_0^*, Q^*))\varphi(Q^*)d\sigma(Q^*)\right| \\
&= \left|\int_{\partial\Omega \cap B_\eta(P_0^*)} (K(P^*, Q^*) - K(P_0^*, Q^*))\varphi(Q^*)d\sigma(Q^*)\right| \\
&\quad + \left|\int_{\partial\Omega \setminus B_\eta(P_0^*)} (K(P^*, Q^*) - K(P_0^*, Q^*))\varphi(Q^*)d\sigma(Q^*)\right| \\
&\leq \|\varphi\|_\infty \int_{\partial\Omega \cap B_\eta(P_0^*)} |K(P^*, Q^*)|d\sigma(Q^*) \\
&\quad + \|\varphi\|_\infty \int_{\partial\Omega \cap B_\eta(P_0^*)} |K(P_0^*, Q^*)|d\sigma(Q^*) \\
&\quad + \|\varphi\|_\infty \int_{\partial\Omega \setminus B_\eta(P_0^*)} |K(P_0^*, Q^*) - K(P^*, Q^*)|d\sigma(Q^*) \\
&=: I_1 + I_2 + I_3
\end{aligned}$$

Looking first at I_2, we have that

$$\begin{aligned}
I_2 &= \|\varphi\|_\infty \int_{\partial\Omega \cap B_\eta(P_0^*)} |K(P_0^*, Q^*)|d\sigma(Q^*) \\
&= \|\varphi\|_\infty \int_{\partial\Omega \cap B_\eta(P_0^*)} \frac{C}{|P_0^* - Q^*|^{n-1-\alpha}} d\sigma(Q^*) \\
&\leq C'\eta^{1+\alpha},
\end{aligned}$$

where the final inequality is the result of applying Proposition 2.11. Therefore for η small enough, we have that $I_2 < \frac{\varepsilon}{3}$.

Now looking at I_1, writing out κ explicitly and using Proposition 3.8, we get that

$$\begin{aligned} I_1 =& \|\varphi\|_\infty \int_{\partial\Omega \cap B_\eta(P_0^*)} |\kappa(P^*, Q^*)| d\sigma(Q^*) \\ =& \|\varphi\|_\infty \int_{\partial\Omega \cap B_\eta(P_0^*)} \frac{C}{|P^* - Q^*|^{n-1-\alpha}} d\sigma(Q^*) \end{aligned}$$

By a similar reasoning to that used in Proposition 3.4, we say that for $|P^* - P_0^*|$ small enough, I_1 can be bounded by something of the order as the bound for I_2, i.e. of order $\eta^{1+\alpha}$, and thus for η and $|P^* - P_0^*|$ small enough, we have that $I_1 < \frac{\varepsilon}{3}$.

Finally, since the integrand is continous on the set $\partial\Omega \setminus B_\eta$ for the η we found working with I_2, by the same argument as in Proposition 3.4, we can say that for $|P^* - P_0^*|$ small enough, $I_3 < \frac{\varepsilon}{3}$. Therefore for a small enough $|P^* - P_0^*|$, we have that

$$|T_K\varphi(P_0^*) - T_K\varphi(P^*)| \leq I_1 + I_2 + I_3 < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

and therefore $T_K\varphi$ is continuous on $\partial\Omega$ for φ bounded. □

Corollary 3.16.1. *For $\varphi \in C(\partial\Omega)$,*

$$T_K\varphi(P^*) = \int_{\partial\Omega} K(P^*, Q^*)\varphi(Q^*) d\sigma(Q^*)$$

and

$$T_{K^*}\varphi(P^*) = \int_{\partial\Omega} K^*(P, Q^*)\varphi(Q^*) d\sigma(Q^*)$$

are continuous on $\partial\Omega$.

Proof. Applying Lemma 3.8, we have that

$$|K(P^*, Q^*)| = \frac{|(P^* - Q^*) \cdot n(Q^*)|}{w_n |P^* - Q^*|^n} \leq \frac{c|P^* - Q^*|^{1+\alpha}}{w_n |P^* - Q^*|^n} = \frac{c}{w_n |P^* - Q^*|^{n-1-\alpha}}$$

and

$$|K^*(P^*, Q^*)| = \frac{|(P^* - Q^*) \cdot n(P^*)|}{w_n |P^* - Q^*|^n} \leq \frac{c|P^* - Q^*|^{1+\alpha}}{w_n |P^* - Q^*|^n} = \frac{c}{w_n |P^* - Q^*|^{n-1-\alpha}}.$$

Therefore both K and K^* are bounded as required to apply Proposition 3.16. Since φ is continuous on the compact set $\partial\Omega$, it is necessarily bounded, so by Proposition 3.16, $T_K\varphi$ and $T_{K^*}\varphi$ are continuous on $\partial\Omega$. □

From the following proposition, we will be able to see what the normal derivative of the Single Layer Potential will be on the boundary for a given φ. From this, we see that the Single Layer Potential is a natural candidate for the solution to the Neumann boundary value problems as it is harmonic and has predictable behavior in its normal derivative at the boundary. We will just have to figure out which φ to use to give us the right values on the boundary.

Proposition 3.17 (Jump Relations for Single Layer Potential)**.** *Let $\varphi \in C(\partial\Omega)$. Then for $u = S[\varphi] : \mathbb{R}^{n+1} \to \mathbb{R}$ we have that*

$$\frac{\partial u}{\partial n_-}(P^*) = -\frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K^*(P^*, Q^*)\varphi(Q^*)d\sigma(Q^*)$$

and

$$\frac{\partial u}{\partial n_+}(P^*) = \frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K^*(P^*, Q^*)\varphi(Q^*)d\sigma(Q^*)\,.$$

Corollary 3.17.1. *Let $\varphi \in \partial\Omega$ and $u = S[\varphi]$. Then we can recover φ from the normal derivatives along $\partial\Omega$ in the following way*

$$\varphi = \frac{\partial u}{\partial n_+} - \frac{\partial u}{\partial n_-}\,.$$

In order to prove this proposition, we first need to formalize some ideas regarding working with "tubular neighborhoods", as introduced in Proposition 2.5. First, we get a result that tells us that as a result of the existence of the tubular neighborhood, proving continuity to the boundary only requires proving continuity from the normal direction. Additionally, the ability to express points P near the boundary in terms of a boundary point P^* and its normal $n(P^*)$ allows us to extend the definition of normal derivatives along the boundary to the idea of normal derivatives at points in a tubular neighborhood of $\partial\Omega$.

Proposition 3.18. *Suppose an η as described in Proposition 2.5 exists for $\partial\Omega$, where Ω is a $C^{1,\alpha}$ domain and suppose $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous when restricted to Ω, $\overline{\Omega}^{\mathsf{c}}$, and $\partial\Omega$ separately. If for every $Q^* \in \partial\Omega$ we have that if $Q_t = Q^* + tn(Q^*)$ then $|f(Q_t) - f(Q^*)| \to 0$ as $t \to 0$, then f is continuous on all of $\mathbb{R}^{n+1}$.*

Proof. Let $U = \{P \in \mathbb{R}^{n+1} : \text{dist}(P, \partial\Omega) < \eta\}$. In order to show that f is continuous on all of $\mathbb{R}^{n+1}$, we must show that the limit of f from Ω or $\overline{\Omega}^{\mathsf{c}}$ to $\partial\Omega$ aligns with how f is defined on $\partial\Omega$. Let $P^* \in \partial\Omega$ and suppose $\{P_n\}_{n\in\mathbb{N}} \subseteq \Omega \cap U$ is a sequence such that $P_n \to P^*$. Since $P_n \in U$, we have that $P_n = P_n^* + t_n n(P_n^*)$, where $P_n^* \in \partial\Omega$. Note that $t_n = \text{dist}(P_n, \partial\Omega)$. In order for P_n to converge to P^*, it must be the case that $P_n^* \to P^*$ and $t_n \to 0$ as $n \to \infty$.

Now let $\varepsilon > 0$. As a result of our assumption that f is continuous along tangents to the boundary, we can conclude that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\text{dist}(Q, \partial\Omega) < \delta$ then $|f(Q) - f(Q^*)| \leq \frac{\varepsilon}{2}$, where Q^* is the point on $\partial\Omega$ that satisfies $Q = Q^* + tn(Q^*)$. If that were not the case, there would exist a $\varepsilon > 0$ such that for all $\delta > 0$ if $\text{dist}(Q, \partial\Omega) < \delta$, then $|f(Q) - f(Q^*)| > \frac{\varepsilon}{2}$ for some $Q \in U$, which would contradict our assumption. Since $t_n = \text{dist}(P_n, \partial\Omega) \to 0$ as $n \to \infty$, there must be a $N_1 > 0$ such that for $n > N_1$, $t_n < \delta$ and thus $|f(P_n) - f(P_n^*)| < \frac{\varepsilon}{2}$.

Additionally, since f is continuous on $\partial\Omega$, there exists a $N_1 > 0$ such that for $n > N_1$, we have that $|f(P_n^*) - f(P^*)| \leq \frac{\varepsilon}{2}$.

All together, we have that

$$|f(P_n) - f(P^*)| \leq |f(P_n) - f(P_n^*)| + |f(P_n^*) - f(P^*)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and therefore f is continuous on $\overline{\Omega}$. Taking a sequence $\{P_n\}_{n\in\mathbb{N}} \subseteq \overline{\Omega}^{\mathsf{c}} \cap U$ and following the same procedure gives us that f is continuous on Ω^{c} and is therefore continuous on all of $\mathbb{R}^{n+1}$. □

As claimed earlier, within a tubular neighborhood of $\partial\Omega$ we can express points P as $P = P^* + tn(P^*)$ for some $P^* \in \partial\Omega$. For this reason, we are able to make sense of an extension of the normal derivative of a function to points outside the boundary, but within the tubular neighborhood. We can therefore define the normal derivative with respect to P as follows

$$\frac{\partial u(P)}{\partial n} := \nabla u(P) \cdot n(P^*)$$

where P^* is the point on the boundary such that $P = P^* + tn(P^*)$. With this notation, we will be able to take limits from either side of $\partial\Omega$ in the pursuit of $\frac{\partial u}{\partial n_-}$ and $\frac{\partial u}{\partial n_+}$.

Lemma 3.19. *Let U denote a tubular neighborhood of $\partial\Omega$ and let $v = D[\varphi]$ and $u = S[\varphi]$ for$\varphi \in C(\partial\Omega)$. The function $F : U \to \mathbb{R}$ is continuous where*

$$F(P) = \begin{cases} v(P) + \frac{\partial u}{\partial n}(P) & P \notin \partial\Omega \\ \int_{\partial\Omega} K(P,Q^*)\varphi(Q^*)d\sigma(Q^*) + \int_{\partial\Omega} K^*(P,Q^*)\varphi(Q^*)d\sigma(Q^*) & P \in \partial\Omega\,. \end{cases}$$

Lemma 3.19. As defined, F is continuous away from $\partial\Omega$, thus to show continuity on all of $\mathbb{R}^{n+1}$, it is enough to show continuity on a neighborhood surrounding $\partial\Omega$.

By Proposition 3.18, if F is continuous when restricted to the boundary $\partial\Omega$, to show continuity on the tubular neighborhood U surrounding $\partial\Omega$, it is enough to show that for $P = P^* + tn(P^*) \in U$, $F(P) - F(P^*) \to 0$ as $t \to 0$.

As it is defined on the boundary, by Corollary 3.16.1, F is continuous when restricted to $\partial\Omega$. Therefore to prove continuity of F, it is enough to prove continuity in the normal direction to $\partial\Omega$. Let $\varepsilon > 0$, $P^* \in \partial\Omega$ and $P = P^* + tn(P^*) \in U$. We have that

$$\begin{aligned} &|F(P) - F(P^*)| \\ &= \left| v(P) + \frac{\partial u}{\partial n}(P) - \int_{\partial\Omega} \frac{\partial N(P^*,Q^*)}{\partial n_{Q^*}}\varphi(Q^*)d\sigma(Q^*) - \int_{\partial\Omega} \frac{\partial N(P^*,Q^*)}{\partial n_{P^*}}\varphi(Q^*)d\sigma(Q^*) \right| \\ &\leq \int_{\partial\Omega} \left| \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} + \frac{\partial N(P,Q^*)}{\partial n_P} - \frac{\partial N(P^*,Q^*)}{\partial n_{Q^*}} - \frac{\partial N(P^*,Q^*)}{\partial n_{P^*}} \right| \varphi(Q^*)d\sigma(Q^*)\,. \end{aligned}$$

To show that this integral goes to zero, we will split the integral into an integral over $\partial\Omega \setminus B_\delta(P^*)$ and $\partial\Omega \cap B_\delta(P^*)$ and show that both integrals can be taken to zero. We will denote the integral over $\partial\Omega \cap B_\delta(P^*)$ as I_A and the integral over $\partial\Omega \setminus B_\delta(P^*)$ as I_B.

Starting with I_A, we have that

$$\begin{aligned} I_A = \int_{\partial\Omega\cap B_\delta(P^*)} &\left| \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} + \frac{\partial N(P,Q^*)}{\partial n_P} - \frac{\partial N(P^*,Q^*)}{\partial n_{Q^*}} - \frac{\partial N(P^*,Q^*)}{\partial n_{P^*}} \right| \varphi(Q^*)d\sigma(Q^*) \\ &\leq \|\varphi\|_\infty \int_{\partial\Omega\cap B_\delta(P^*)} \left| \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} + \frac{\partial N(P,Q^*)}{\partial n_P} \right| d\sigma(Q^*) \\ &\quad + \|\varphi\|_\infty \int_{\partial\Omega\cap B_\delta(P^*)} \left| \frac{\partial N(P^*,Q^*)}{\partial n_{Q^*}} + \frac{\partial N(P^*,Q^*)}{\partial n_{P^*}} \right| d\sigma(Q^*) \\ &=: I_1 + I_2\,. \end{aligned}$$

We will show that we can find an $\eta > 0$ such that both $I_1 < \frac{\varepsilon}{4}$ and $I_2 < \frac{\varepsilon}{4}$. Looking first at I_2 since it is a bit more straightforward, we can see that the integrand written explicitly is

$$\left|\frac{\partial N(P^*,Q^*)}{\partial n_{Q^*}}+\frac{\partial N(P^*,Q^*)}{\partial n_{P^*}}\right| = \left|\frac{-(P^*-Q^*)\cdot n(Q^*)}{w_n|P^*-Q^*|^n}+\frac{(P^*-Q^*)\cdot n(P^*)}{w_n|P^*-Q^*|^n}\right|$$
$$= \frac{|(P^*-Q^*)\cdot(n(P^*)-n(Q^*))|}{w_n|P^*-Q^*|^n}$$
$$\leq \frac{|n(P^*)-n(Q^*)|}{w_n|P^*-Q^*|^{n-1}}\,.$$

Since $\partial\Omega$ is a $C^{1,\alpha}$ domain, we have that $|n(P^*)-n(Q^*)| \leq C|P^*-Q^*|^\alpha$ for some $C>0$. Therefore we have that

$$\left|\frac{\partial N(P^*,Q^*)}{\partial n_{Q^*}}+\frac{\partial N(P^*,Q^*)}{\partial n_P}\right| \leq \frac{|n(P^*)-n(Q^*)|}{w_n|P-Q^*|^{n-1}}$$
$$\leq \frac{C|P^*-Q^*|^\alpha}{w_n|P^*-Q^*|^{n-1}}$$
$$= \frac{C}{w_n}|P^*-Q^*|^{\alpha+1-n}\,.$$

Integrating this, we have

$$I_2 = \|\varphi\|_\infty\int_{\partial\Omega\cap B_\delta(P^*)}\left|\frac{\partial N(P^*,Q^*)}{\partial n_{Q^*}}+\frac{\partial N(P^*,Q^*)}{\partial n_{P^*}}\right|d\sigma(Q^*)$$
$$\leq \frac{C\|\varphi\|_\infty}{w_n}\int_{\partial\Omega\cap B_\delta(P^*)}|P^*-Q^*|^{\alpha+1-n}d\sigma(Q^*)$$
$$\leq C'\delta^{\alpha+1}\to 0.$$

where the last inequality is the result of applying Proposition 2.11. Therefore for a δ small enough, we can guarantee that $I_2<\frac{\varepsilon}{4}$.

Moving on to I_1, let's see what the integrand of the first integral looks like explicitly:

$$\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}+\frac{\partial N(P,Q^*)}{\partial n_P}\right| = \left|\frac{-(P-Q^*)\cdot n(Q^*)}{w_n|P-Q^*|^n}+\frac{(P-Q^*)\cdot n(P^*)}{w_n|P-Q^*|^n}\right|$$
$$= \frac{|(P-Q^*)\cdot(n(P^*)-n(Q^*))|}{w_n|P-Q^*|^n}$$
$$\leq \frac{|n(P^*)-n(Q^*)|}{w_n|P-Q^*|^{n-1}}\,.$$

If Q^* is close enough to P^*, the vector Q^*-P^* should be nearly orthogonal to $n(P^*)$. We can therefore say that for a small enough δ, we have $|P-Q^*|\geq|P^*-Q^*|$.

We then have that the integrand of the first integral is bounded by

$$\left|\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}+\frac{\partial N(P,Q^*)}{\partial n_P}\right| \leq \frac{|n(P^*)-n(Q^*)|}{w_n|P^*-Q^*|^{n-1}}\,.$$

We have now obtained the same bound in the workings for I_2, so we can use that result to say that I_1 can be made less than $\frac{\varepsilon}{4}$ for small enough δ.

Now we proceed seeking to bound I_B, the integral over $\partial\Omega\setminus B_\delta(P^*)$. We note that on the compact set $\partial\Omega\setminus B_\delta(P^*)$, the function $\frac{\partial N(P,Q^*)}{\partial n_{Q^*}}+\frac{\partial N(P,Q^*)}{\partial n_P}$ is continuous in P. Therefore, as $P \to P^*$, the integrand goes to zero, and through dominated convergence

$$I_B = \int_{\partial\Omega \cap B_\delta(P^*)} \left| \frac{\partial N(P,Q^*)}{\partial n_{Q^*}} + \frac{\partial N(P,Q^*)}{\partial n_P} - \frac{\partial N(P^*,Q^*)}{\partial n_{Q^*}} - \frac{\partial N(P^*,Q^*)}{\partial n_{P^*}} \right| \varphi(Q^*) d\sigma(Q^*) \to 0$$

as $P \to P^*$. Therefore for $|P - P^*|$ small enough, we have that $I_B < \frac{\varepsilon}{2}$.

We have also found a δ such that $I_1 < \frac{\varepsilon}{4}$ and $I_2 < \frac{\varepsilon}{4}$, so all together we have that for $|P - P^*|$ small enough

$$|F(P) - F(P^*)| \le I_A + I_B \le I_1 + I_2 + I_B < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore F is continuous on the tubular neighborhood surrounding $\partial\Omega$. Since away from $\partial\Omega$ we know F is continuous, we can conclude that F is continuous on all of $\mathbb{R}^{n+1}$. □

After establishing all this, we can now proceed with the proof of Proposition 3.17.

Proposition 3.17. We will use the continuity of the function F from Lemma 3.19 in order to find $\frac{\partial u}{\partial n_-}$ and $\frac{\partial u}{\partial n_+}$.

Using Proposition 3.15, we know what the limit of the Double Layer Potential v to the boundary should look like, depending on if we take the limit from inside or outside Ω. We can use this information, along with what values we know F takes on $\partial\Omega$ to solve for $\frac{\partial u}{\partial n_-}$ and $\frac{\partial u}{\partial n_+}$.

If we take the limit from within Ω to $\partial\Omega$, we see that the following equation must hold on $\partial\Omega$:

$$\begin{aligned}
\int_{\partial\Omega} K(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*) &+ \int_{\partial\Omega} K^*(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*) \\
&= F(P^*) \\
&= \lim_{\Omega \ni P \to P^*} F(P) \\
&= \lim_{\Omega \ni P \to P^*} \left[v(P) + \frac{\partial u}{\partial n}(P) \right] \\
&= \frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*) + \frac{\partial u}{\partial n_-}(P^*)\,.
\end{aligned}$$

Solving for $\frac{\partial u}{\partial n_-}(P^*)$, we get that

$$\frac{\partial u}{\partial n_-}(P^*) = -\frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K^*(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*)\,.$$

Similarly, taking the limit from $\overline{\Omega}^{\mathsf{c}}$ to $\partial\Omega$ instead, we get that

$$\begin{aligned}
\int_{\partial\Omega} K(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*) &+ \int_{\partial\Omega} K^*(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*) \\
&= -\frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*) + \frac{\partial u}{\partial n_+}(P^*)\,.
\end{aligned}$$

meaning

$$\frac{\partial u}{\partial n_+}(P^*) = \frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} K^*(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*)\,,$$

as desired. □

3.5 Dirichlet and Neumann BVP Solvability Criteria

If there is a function $\varphi \in C(\partial\Omega)$ such that $f(P^*) = \frac{1}{2}\varphi + \int_{\partial\Omega} \frac{\partial N(P^*,Q^*)}{\partial n_{Q^*}}\varphi(Q^*)d\sigma(Q^*)$, then u_- from Proposition 3.15 is a solution to the interior Dirichlet boundary value problem with boundary data f. Similarly, if there exists a function $\varphi \in C(\partial\Omega)$ such that $f(P^*) = -\frac{1}{2}\varphi(P^*) + \int_{\partial\Omega} \frac{\partial N(P^*,Q^*)}{\partial n_{P^*}}\varphi(Q^*)d\sigma(Q^*)$, then u from Proposition 3.17 solves the interior Neumann boundary value problem. Similar such conditions exist for the exterior analogues of these boundary value problems.

To make things more concise as we move towards arguing in terms of images of operators, we will define the ubiquitous yet unwieldy integrals we used frequently as the operators T and T^*, where for $\varphi \in C(\partial\Omega)$,

$$
\begin{aligned}
T\varphi(P^*) &= \int_{\partial\Omega} \frac{\partial N(P^*,Q^*)}{\partial n_{Q^*}}\varphi(Q^*)d\sigma(Q^*) && (15)\\
&= \int_{\partial\Omega} K(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*) \\
T^*\varphi(P^*) &= \int_{\partial\Omega} \frac{\partial N(P^*,Q^*)}{\partial n_{P^*}}\varphi(Q^*)d\sigma(Q^*) && (16)\\
&= \int_{\partial\Omega} K^*(P^*,Q^*)\varphi(Q^*)d\sigma(Q^*)\,.
\end{aligned}
$$

With this notation, the efforts of the previous sections can be condensed into the following proposition.

Proposition 3.20. *Let Ω be a $C^{1,\alpha}$ domain and consider $f \in C(\partial\Omega)$ as BVP boundary data*

$$
\begin{aligned}
f \in \textit{range}\left(\frac{1}{2}I + T\right) &\implies \textit{there exists a solution to the interior Dirichlet BVP}\\
f \in \textit{range}\left(-\frac{1}{2}I + T\right) &\implies \textit{there exists a solution to the exterior Dirichlet BVP}\\
f \in \textit{range}\left(-\frac{1}{2}I + T^*\right) &\implies \textit{there exists a solution to the interior Neumann BVP}\\
f \in \textit{range}\left(\frac{1}{2}I + T^*\right) &\implies \textit{there exists a solution to the exterior Neumann BVP}
\end{aligned}
$$

From the statement of the above proposition, we see that if the operators $\pm\frac{1}{2}+T,\ \pm\frac{1}{2}+T^* : C(\partial\Omega) \to C(\partial\Omega)$ are surjective, we can ensure that solutions exist to all the boundary value problems we have introduced. In the following section, we will develop tools from functional analysis to see whether this is the case, and if not, for which f solutions to the boundary problems exist.

4 Functional Analysis

We will begin with a review of some basic yet important ideas from functional analysis.

4.1 Foundational Definitions and Results for the Study of Operators

In the study of operators, the space on which the operator is defined is often a very important characteristic. Certain results for operators hold when we know they are defined on a specific type of space. In our case, we will be using that $L^2(\partial\Omega)$ is a Hilbert space.

Let us begin with the following basic definitions.

Definition 4.1 (Banach Space)**.** *A Banach Space is a normed linear space that is complete.*

Definition 4.2 (Hilbert Space)**.** *A Hilbert Space is a Banach Space with an inner product that satisfies* $||x|| = \langle x, x\rangle^{1/2}$.

Definition 4.3. *The space $L^p(X)$ consists of all Lebesgue measurable functions on X such that* $||f||_{L^p(X)} := \left(\int_X |f(x)|^p dx\right)^{1/p} < \infty$.

Proposition 4.4 (Riesz-Fischer Theorem)**.** *The space $L^p(X)$ with the norm $||f||_{L^p(X)} := \left(\int_X |f(x)|^p dx\right)^{1/p}$ is a Banach space.*

For a proof of this result, see Theorem 4.17 in [2]. In this case, we can define the inner product

$$\langle f, g\rangle_{L^2(X)} = \int_X f(x)g(x)dx\,.$$

Since $\langle f, f\rangle_{L^2(X)} = \int_X |f(x)|^2 dx = ||f||^2_{L^2(X)}$, we can see that $L^2(X)$ is a Hilbert space.

Definition 4.5 (Adjoint Operator)**.** *Let $A : H \to H$ be an operator between a Hilbert space H and itself. If there is an operator A^* such that for every $x, y \in H$*

$$\langle Ax, y\rangle = \langle x, A^*y\rangle\,,$$

then A^* is called the adjoint of A.

Definition 4.6 (Bounded Operator)**.** *An operator $A : X \to Y$ between two Banach spaces is bounded if there exists a constant $C \geq 0$ such that for all $x \in X$*

$$||Ax||_Y \leq C||x||_X\,.$$

Definition 4.7 (Operator Norm)**.** *The norm of a bounded operator is*

$$||A|| = \sup_{x\neq 0} \frac{||Ax||_Y}{||x||_X}\,.$$

The following lemma is a useful result relating to the invertibility of operators of the form $I - A$, which resemble the operators we are looking at. This will be a useful result later on.

Lemma 4.8. *Let X be a Banach space and $A : X \to X$ an operator such that $\|A\| < 1$. Then $I - A$ is invertible with*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k\,.$$

Proof. For $x \in X$ and $N \in \mathbb{N}$, consider

$$(I - A)\sum_{k=0}^{N} A^k x = (I - A)(I + A + \cdots + A^N)x = (I - A^{N+1})x = x - A^{N+1}x$$

We will show that $\{\sum_{k=0}^{N} A^k x\}_{N\in\mathbb{N}}$ is a Cauchy sequence. Let $\lambda = \|A\| < 1$ and $N < M \in \mathbb{N}$. Then

$$\begin{aligned}\left|\sum_{k=0}^{M} A^k x - \sum_{k=0}^{N} A^k x\right| &= \left|\sum_{k=N+1}^{M} A^k x\right| \\ &\leq \sum_{k=N+1}^{M} |A^k x| \\ &\leq \sum_{k=N+1}^{M} \lambda^k |x| \\ &= |x|\frac{\lambda^{N+1} - \lambda^{M-1}}{1-\lambda} \\ &\leq |x|\frac{\lambda^{N+1}}{1-\lambda} \to 0 \text{ as } N, M \to \infty.\end{aligned}$$

Therefore $\{\sum_{k=0}^{N} A^k x\}_{N\in\mathbb{N}}$ is a Cauchy sequence and since X is a Banach space, the sequence converges in X and necessarily $A^k x \to 0$ as $k \to \infty$. Thus

$$\lim_{N\to\infty}(I - A)\sum_{k=0}^{N} A^k x = x - \lim_{N\to\infty} A^{N+1}x = x.$$

Therefore

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$ □

4.2 Compact Operators

The theory of compact operators is central to the analysis of the operators T and T^*. Using the results from this section, we will show that the operators are in fact compact, and we will be able to use strong results, like the Fredholm Alternative, which hold for compact operators.

Definition 4.9 (Compact Operator). *An operator $A : X \to Y$ from one Banach space to another is a compact operator if for every bounded sequence $\{x_n\} \subseteq X$, the sequence $\{Ax_n\} \subseteq Y$ has a convergent subsequence.*

Though the definition of a compact operator is quite straightforward, it can sometimes be a challenge to prove that an operator is compact directly from the defintion. Therefore, we will show the following results in order to give us some alternative ways to show that an operator is compact.

Proposition 4.10. *Let $A : X \to Y$ be a bounded operator between Banach Spaces X and Y. Furthermore, suppose $A(X) \subseteq Y$ is finite dimensional. Then A is a compact operator.*

Proof. Let $\{x_n\}$ be a bounded sequence in X. Since A is a bounded operator, the sequence $\{Ax_n\}$ in Y is also bounded. As $\{Ax_n\}$ lies in the finite dimensional image of A, by the Bolzano-Weierstrass theorem there is a convergent subsequence. Thus A is compact. □

Proposition 4.11. *Let X and Y be Banach spaces and suppose that $A : X \to Y$ is the limit of a sequence of compact operators $A_n : X \to Y$, i.e. $||A_n - A|| \to 0$. Then T itself is a compact operator.*

Proof. To show that A is compact, we take a sequence $\{x_i\}$ that is bounded, meaning there exists an $M > 0$ such that $\|x_i\| < M$ for all i. We will show that the sequence $\{Ax_i\}$ has a convergent subsequence.

Since $A_n \to A$, for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|A_n - A\| \leq \frac{\varepsilon}{3M}$ for $n \geq N$. Since A_N is assumed to be compact, $\{A_N x_i\}$ has a convergent subsequence $\{A_N x_{i_j}\}$, meaning the subsequence is Cauchy and there exists $J \in \mathbb{N}$ such that $|A_N x_{i_j} - A_N x_{i_k}| \leq \frac{\varepsilon}{3}$ for $j, k > J$. All together we have that

$$\begin{aligned}|Ax_{i_j} - Ax_{i_k}| &\leq |Ax_{i_j} - A_N x_{i_j}| + |A_N x_{i_j} - A_N x_{i_k}| + |A_N x_{i_k} - Ax_{i_k}| \\ &\leq \|A_N - A\||x_{i_j}| + \frac{\varepsilon}{3} + \|A_N - A\||x_{i_k}| \\ &\leq \frac{\varepsilon}{3M}M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M}M = \varepsilon\,.\end{aligned}$$

Thus the subsequence $\{Ax_{i_j}\}$ is a Cauchy sequence, and since Y is a Banach space and therefore complete, it converges. Therefore A is a compact operator. □

Proposition 4.12. *If $A : H \to H$ is a compact operator on a Hilbert space H, so is $A^* : H \to H$.*

Proof. We will first show that A^* is bounded. We will use the fact that since A is compact, it is neccesarily bounded. For $x \in H$,

$$\|A^*x\|^2 = \langle A^*x, A^*x\rangle = \langle x, AA^*x\rangle \leq \|x\|\|AA^*x\| \leq \|x\|\|A\|\|A^*x\|\,.$$

If we divide both sides by $\|x\|\|A^*x\|$, we get that

$$\frac{\|A^*x\|}{\|x\|} \leq \|A\|\,.$$

It then follows that $\|A^*\| \leq \|A\|$, meaning A^* is bounded.

Now, to show that A^* is compact, take a sequence $\{x_n\} \in H$ such that $\|x_n\| \leq M$ for some $M \in \mathbb{R}$. We will show that the sequence $\{A^*x_n\}$ has a convergent subsequence.

Since A^* is a bounded operator, we know that the sequence $\{A^*x_n\}$ is bounded. Thus, since A is compact, the sequence $\{AA^*x_n\}$ has a convergent subsequence. It is therefore a Cauchy sequence, meaning that for every $\varepsilon > 0$ there exists some $N \in \mathbb{R}$ such that for $l, m \geq N$,

$$\|AA^*x_{n_l} - AA^*x_{n_m}\| < \frac{\varepsilon^2}{2M}\,.$$

Now considering the subsequence $\{A^* x_{n_m}\} \subseteq \{A^* x_n\}$, we see that

$$\begin{aligned}
\|A^* x_{n_l} - A^* x_{n_m}\|^2 &= \|A^*(x_{n_l} - x_{n_m})\|^2 \\
&= \langle A^*(x_{n_l} - x_{n_m}), A^*(x_{n_l} - x_{n_m})\rangle \\
&= \langle x_{n_l} - x_{n_m}, AA^*(x_{n_l} - x_{n_m})\rangle \\
&\leq \|x_{n_l} - x_{n_m}\| \|AA^*(x_{n_l} - x_{n_m})\| \\
&\leq 2M \frac{\varepsilon^2}{2M} = \varepsilon^2 .
\end{aligned}$$

All together, we have that $\|A^* x_{n_l} - A^* x_{n_m}\| \leq \varepsilon$ for $l, m \geq N$, meaning $\{A^* x_{n_m}\}$ is a Cauchy sequence and thus convergent in H. Therefore A^* is compact. □

4.3 Fredholm Alternative

We have introduced the notion of compact operators in order to use the following useful result, the Fredholm Alternative. This result relevant to our situation because if we are able to show that the operator T is compact, we will gain valuable information about operators of the form $\lambda I + T$ and $\lambda I + T^*$. Recalling the statement Proposition 3.20 summarizing the solvability of our boundary value problems in terms of images of operators, we see that investigating the operators $\lambda I + T$ and $\lambda I + T^*$ with $\lambda = \pm\frac{1}{2}$ is central to our problem.

Proposition 4.13 (Fredholm Alternative). *Let $A : H \to H$ be a compact operator from a Hilbert space H to itself. For each $0 \neq \lambda \in \mathbb{R}$, exactly one of the following alternatives hold:*

1. *$\lambda I + A$ is invertible.*
2. *$\dim(\ker(\lambda I + A)) = \dim(\ker(\lambda I + A^*))$ are non-zero and finite. Plus,*

$$\textit{range}\,(\lambda I + A) = \ker(\lambda I + A^*)^{\perp} \tag{17}$$

$$\textit{range}\,(\lambda I + A^*) = \ker(\lambda I + A)^{\perp} . \tag{18}$$

For the proof, see Theorem 4.3 in [3]. We will now work to show that our operators T and T^* are compact.

4.4 Hilbert-Schmidt Operators

In order to show that T is compact, we must introduce the notion of Hilbert-Schmidt kernels and operators.

Definition 4.14 (Hilbert-Schmidt Kernel). *The function $K : X \times X \to \mathbb{R}$ is a Hilbert-Schmidt Kernel if*

$$\|K\|_{L^2(X\times X)} := \left(\int_X \int_X |K(x,y)|^2 dx dy\right)^{\frac{1}{2}} < \infty.$$

Definition 4.15 (Hilbert-Schmidt Operator). *Let K be a Hilbert-Schmidt Kernel, then the associated Hilbert-Schmidt Operator $A_K : L^2(X) \to L^2(X)$ is given by*

$$A_K f := \int_X K(x,y) f(y) dy.$$

It is not immediate to see that A_K maps into $L^2(X)$, so we will show that that is the case in the following proposition.

Proposition 4.16. *The operator $A_K : L^2(X) \to L^2(X)$ is well defined and bounded with $\|A_K\| \leq \|K\|_{L^2(X\times X)}$.*

Proof. For $f \in L^2(X)$ have that

$$\begin{aligned}
|A_k(f)|^2 &= \left| \int_X K(x,y) f(y) dy \right|^2 \\
&= |\langle K(x,y), f(y) \rangle_{L^2(X)}|^2 \\
&\leq \|K(x,y)\|^2_{L^2(X)} \|f(y)\|^2_{L^2(X)} \\
&= \|f(y)\|^2_{L^2(X)} \int_X |K(x,y)|^2 dy \,,
\end{aligned}$$

where we used Cauchy-Schwarz to get the above inequality. Therefore

$$\begin{aligned}
\|A_k(f)\|^2_{L^2(X)} &= \int_X |A_k(f)|^2 dx \\
&\leq \|f(y)\|^2_{L^2(X)} \int_X \int_X |K(x,y)|^2 dy dx \\
&= \|f(y)\|^2_{L^2(X)} \|K\|^2_{L^2(X\times X)} < \infty
\end{aligned}$$

and thus $A_K f \in L^2(X)$.

It then follows that

$$\|A_K\| = \sup_{f\in L^2(X)} \frac{\|A_K f\|_{L^2(X)}}{\|f\|_{L^2(X)}} \leq \sup_{f\in L^2(X)} \frac{\|A\|_{L^2(X\times X)} \|f\|_{L^2(X)}}{\|f\|_{L^2(X)}} = \|A\|_{L^2(X\times X)} < \infty \,.$$

□

Proposition 4.17 (Hilbert-Schmidt Operators are Compact)**.** *Let X be a Hilbert space with a countable orthonormal basis. If $K : X \times X \to \mathbb{R}$ is a Hilbert-Schmidt Kernel with associated Hilbert-Schmidt Operator A_K, then A_K is compact.*

In order to prove that Hilbert-Schmidt operators are compact, we will use the following two standard results from analysis.

Lemma 4.18. *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(X)$ and let $\{\psi_m\}_{m\in\mathbb{N}}$ be an orthonormal basis of $L^2(Y)$. Then $\{\varphi_n \oplus \psi_m\}_{m,n\in\mathbb{N}}$ where $\varphi_n \oplus \psi_m = \varphi_n(x)\psi_m(y)$ forms an orthonormal basis of $L^2(X \times Y)$.*

Proof. To show that the elements in $\{\varphi_n \oplus \psi_m\}_{m,n\in\mathbb{N}}$ are orthonomal we check that

$$\begin{aligned}
\langle \varphi_n \oplus \psi_m, \varphi_i \oplus \psi_j \rangle &= \int_X \int_Y \varphi_n(x)\psi_m(y)\varphi_i(x)\psi_j(y) dy dx \\
&= \int_X \varphi_n(x)\varphi_i(x) dx \int_Y \psi_m(y)\psi_j(y) dy \\
&= \begin{cases} 1 & \text{if } n = i \text{ and } m = j \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

which is equivalent to the elements being orthonormal.

To show that $\{\varphi_n \oplus \psi_m\}$ is a basis of $L^2(X \times Y)$, we need to show that span$(\{\varphi_n \oplus \psi_m\}) = L^2(X \times Y)$, which is equivalent to the condition span$(\{\varphi_n \oplus \psi_m\})^\perp = \{0\}$. Thus it is enough to show that for every $f \in L^2(X \times Y)$ such that $\langle f, \varphi_n \oplus \psi_m \rangle = 0$ for all $m, n \in \mathbb{N}$, $f = 0$ almost everywhere with respect to the measure μ on $X \times Y$. If $\langle f, \varphi_n \oplus \psi_m \rangle = 0$ for all $m, n \in \mathbb{N}$, we have that

$$\begin{aligned} 0 &= \int_{X \times Y} f(x,y)\varphi_n(x)\psi_m(y)d\mu \\ &= \int_X \left[\int_Y f(x,y)\psi_m(y)dy \right] \varphi_n(x)dx \\ &= \left\langle \int_Y f(x,y)\psi_m(y)dy, \varphi_n \right\rangle . \end{aligned}$$

Since $\{\varphi_n\}$ forms a basis of X, the fact that $\left\langle \int_Y f(x,y)\psi_m(y)dy, \varphi_n \right\rangle = 0$ for every n implies that $\int_Y f(x,y)\psi_m(y)dy = 0$ a.e. in X. Take $E_n = \left\{ x \in X | \int_Y f(x,y)\psi_m(y)dy \neq 0 \right\}$. This is a measure 0 set in X, and thus the countable union $E = \bigcup_{n \in \mathbb{N}} E_n$ is also of measure 0. We have that for every $x \notin E$, $\int_Y f(x,y)\psi_m(y)dy = \langle f(x,\cdot), \psi_m \rangle = 0$, meaning, since $\{\psi_m\}$ forms a basis of Y, $f(x,y) = 0$ a.e. on Y as a function of y.

To show that $f = 0$ a.e on $X \times Y$, consider

$$\begin{aligned} \int_{X \times Y} |f(x,y)|^2 d\mu &= \int_X \left[\int_Y |f(x,y)|^2 dy \right] dx \\ &= \int_{X \setminus E} \left[\int_Y |f(x,y)|^2 dy \right] dx \\ &= \int_{X \setminus E} \int_Y 0 \, dydx \\ &= 0. \end{aligned}$$

Thus $f(x,y) = 0$ a.e. on $X \times Y$ and therefore $\{\varphi_n \oplus \psi_m\}_{m,n \in \mathbb{N}}$ forms a basis of $L^2(X \times Y)$. □

Proposition 4.19 (Parseval's Identity)**.** *Let H be a Hilbert space with orthonormal basis $\{\varphi_n\}_{n \in \mathbb{N}}$. For every $x \in H$*

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, \varphi_n \rangle|^2 .$$

For a proof of Parseval's Identity, see Theorem 4.18 in [2].

We can now proceed with the proof that Hilbert-Schmidt operators are compact.

Proposition 4.17. To show that A_K is compact, we will show that it is the limit of operators of finite rank and employ Propositions 4.10 and 4.11 to conclude that A_K is compact. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be the orthonormal basis of $L^2(X)$. By Lemma 4.18, we know that $\{\varphi_n \oplus \varphi_m\}$ forms a basis of $L^2(X \times X)$. We can therefore write K as

$$\begin{aligned} K(x,y) &= \lim_{N \to \infty} \sum_{n,m \leq N} \langle K, \varphi_n \oplus \varphi_m \rangle \, \varphi_n \oplus \varphi_m \\ &= \lim_{N \to \infty} K_N(x,y) \end{aligned}$$

where $K_N(x,y) := \sum_{n,m\leq N} \langle K, \varphi_n \oplus \varphi_m\rangle \, \varphi_n \oplus \varphi_m$.

We can show that K_N is a Hilbert-Schmidt Kernel like K since, using Parseval's identity, we have that

$$
\begin{aligned}
|K_N(x,y)|^2 &= \left| \sum_{n,m\leq N} \langle K, \varphi_n \oplus \varphi_m\rangle \, \varphi_n \oplus \varphi_m \right|^2 \\
&= \sum_{n,m\leq N} |\langle K, \varphi_n \oplus \varphi_m\rangle|^2 \\
&\leq \sum_{n,m=1}^{\infty} |\langle K, \varphi_n \oplus \varphi_m\rangle|^2 \\
&= |K(x,y)|^2
\end{aligned}
$$

and thus

$$
\begin{aligned}
\|K_N\|_{L^2(X\times X)} &= \left(\int_X \int_X |K_N(x,y)|^2 dxdy \right)^{\frac{1}{2}} \\
&\leq \left(\int_X \int_X |K(x,y)|^2 dxdy \right)^{\frac{1}{2}} = \|K\|_{L^2(X\times X)} < \infty .
\end{aligned}
$$

Therefore A_{K_N} is a Hilbert-Schmidt operator. To show that A_{K_N} has finite rank, we see that for $f \in L^2(X)$

$$
\begin{aligned}
A_{K_N} f &= \int_X K_N(x,y) f(y) dy \\
&= \int_X \sum_{n,m\leq N} \langle K, \varphi_n \oplus \varphi_m\rangle \, \varphi_n(x)\varphi_m(y) f(y) dy \\
&= \sum_{n\leq N} \underbrace{\sum_{m\leq N} \langle K, \varphi_n \oplus \varphi_m\rangle \int_X \varphi_m(y) f(y) dy}_{a_n = \text{constant}} \, \varphi_n(x) \\
&= \sum_{n\leq N} a_n \varphi_n(x) \, .
\end{aligned}
$$

The image of A_{K_N} has thus been shown to be the span of $\{\varphi_1, \ldots, \varphi_N\}$, meaning the image is finite dimensional. Finally, using the fact that Hilbert-Schmidt Operators are bounded by the $L^2(X\times X)$ norm of the Hilbert-Schmidt Kernel, we have that $A_{K_N} \to A_K$ as $N \to \infty$ since

$$
\|A_K - A_{K_N}\| = \|A_{K-K_N}\| \leq \|K - K_N\| \to 0 \, .
$$

Therefore, as the limit of operators of finite rank, A_K is compact. □

The following result ensures that we are able to use the theory of Hilbert-Schmidt operators when looking at our operators T and T^* by making sure that the assumptions in Proposition 4.17 hold in the context of our problem.

Corollary 4.19.1. *Hilbert-Schmidt operators on $L^2(\partial\Omega)$ are compact.*

Proof. In order to apply Proposition 4.17, we need that $L^2(\partial\Omega)$ has a countable orthonormal basis. Since $\Omega \subseteq \mathbb{R}^{n+1}$ is a bounded $C^{1,\alpha}$ domain, it is separable and by Theorem 4.3 in [4], $L^2(\partial\Omega)$ is separable. By Theorem 5.11 also in [4], if a space is separable, it has a countable orthonormal basis, therefore $L^2(\partial\Omega)$ has a countable orthonormal basis and we can say that Hilbert-Schmidt operators on $L^2(\partial\Omega)$ are compact. □

Lemma 4.20. *Let $K : X \times X \to \mathbb{R}$ be a map such that for some $C \geq 0$ both*

$$\sup_{x\in X}\int_X |K(x,y)|dy \leq C \text{ and } \sup_{y\in X}\int_X |K(x,y)|dx \leq C.$$

Then, for the associated operator $A_K(f) = \int_X K(x,y)f(y)dy$ where $f \in L^2(X)$,

$$\|A_K\| \leq C.$$

Proof. Using Hölder's inequality and the bounds from the statement, we get that

$$\begin{aligned}
|A_K f(x)| &= \left|\int_X K(x,y)f(y)dy\right| \\
&\leq \int_X |K(x,y)||f(y)|dy \\
&= \int_X |K(x,y)|^{1/2}\left(|K(x,y)|^{1/2}|f(y)|\right)dy \\
&\leq \left(\int_X |K(x,y)|dy\right)^{1/2}\left(\int_X |K(x,y)||f(y)|^2 dy\right)^{1/2} \\
&\leq C^{1/2}\left(\int_X |K(x,y)||f(y)|^2 dy\right)^{1/2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|A_K f\|^2_{L^2(X)} &= \int_X |A_K f(x)|^2 dx \\
&\leq C\int_X\int_X |K(x,y)||f(y)|^2 dydx \\
&= C\int_X\left[\int_X |K(x,y)|dx\right]|f(y)|^2 dy \\
&\leq C^2\int_X |f(y)|^2 dy \\
&= C^2\|f\|^2_{L^2(X)}.
\end{aligned}$$

We therefore get the desired result that

$$\|A_K\| = \sup_{f\in L^2(X)}\frac{\|A_K f\|_{L^2(X)}}{\|f\|_{L^2(X)}} \leq \sup_{f\in L^2(X)}\frac{C\|f\|_{L^2(X)}}{\|f\|_{L^2(X)}} = C\,.$$ □

5 Properties of the Operators T and T^*

Using the theory from the previous section, we will apply the results to gain insight into the properties of the operators T and T^*. This is key, as the solvability criteria from Proposition 3.20 is stated in terms of these operators.

Because many of the results from the previous section relied on the operators being defined on Banach and Hilbert spaces, we will be considering T and T^* as operators on $L^2(\partial\Omega)$. Since Ω is bounded, we have that $C(\partial\Omega) \subseteq L^2(\partial\Omega)$ and therefore we will still be able to gain insight on how T and T^* act on $C(\partial\Omega)$.

Proposition 5.1. *The operators T and T^* as defined in equations (15) and (16) are in fact adjoint operators on $L^2(\partial\Omega)$.*

Proof. Let $\varphi, \psi \in L^2(\partial\Omega)$. Then, using Fubini's Theorem, we can change the order of integration to get that

$$\begin{aligned}\langle T\varphi, \psi\rangle &= \int_{\partial\Omega} \left[\int_{\partial\Omega} \frac{\partial N(x,y)}{\partial v_y} \varphi(y) d\sigma(y)\right] \psi(x) d\sigma(x) \\ &= \int_{\partial\Omega} \varphi(y) \left[\int_{\partial\Omega} \frac{\partial N(x,y)}{\partial v_y} \psi(x) d\sigma(x)\right] d\sigma(y)\,.\end{aligned}$$

Let's look at the term in the bracket. If we switch the variables $x \leftrightarrow y$, and notice that $N(x,y) = N(y,x)$, it becomes clear to see that the term in the bracket becomes

$$\int_{\partial\Omega} \frac{\partial N(y,x)}{\partial v_x} \psi(y) d\sigma(y) = \int_{\partial\Omega} \frac{\partial N(x,y)}{\partial v_x} \psi(y) d\sigma(y) = T^*\psi(x)\,.$$

Switching back the variables $y \leftrightarrow x$ and returning it to the equation above, we get that

$$\langle T\varphi, \psi\rangle = \int_{\partial\Omega} \varphi(y) T^*[\psi](y) d\sigma(y) = \langle \varphi, T^*\psi\rangle\,.$$

Thus T^* is the adjoint of T. □

5.1 T and T^* Compact

Proposition 5.2. *The operators T and T^* as defined in (15) and (16) respectively are compact operators.*

Proof. To show that T is compact, we will find a sequence of Hilbert-Schmidt operators T_{K_ε} that converge to T as $\varepsilon \to 0$. Consider the kernel

$$K_\varepsilon(P^*, Q^*) = \psi_\varepsilon(P^*, Q^*) K(P^*, Q^*)$$

where ψ_ε is a smooth function that satisfies $0 \leq \psi_\varepsilon(P^*, Q^*) \leq 1$, $\psi_\varepsilon(P^*, Q^*) = 0$ for $|P^* - Q^*| \leq \varepsilon$, and $\psi_\varepsilon(P^*, Q^*) = 1$ for $|x - y| \geq 2\varepsilon$. We will show that K_ε is a Hilbert-Schmidt Kernel.

We have that

$$\begin{aligned}\sup_{(P^*,Q^*)\in\partial\Omega\times\partial\Omega}|K_\varepsilon(P^*,Q^*)| &= \sup_{(P^*,Q^*)\in\partial\Omega}|\psi_\varepsilon(P^*,Q^*)K(P^*,Q^*)| \\ &\le \sup_{\substack{(P^*,Q^*)\in\partial\Omega\times\partial\Omega\\ |P^*-Q^*|\ge\varepsilon}}|K(P^*,Q^*)| \\ &= \sup_{\substack{(P^*,Q^*)\in\partial\Omega\times\partial\Omega\\ |P^*-Q^*|\ge\varepsilon}}\frac{|(P^*-Q^*)\cdot n(Q^*)|}{w_n|P^*-Q^*|^n} \\ &\le \sup_{\substack{(P^*,Q^*)\in\partial\Omega\times\partial\Omega\\ |P^*-Q^*|\ge\varepsilon}}\frac{1}{w_n|P^*-Q^*|^{n-1}} \\ &\le \frac{1}{w_n\varepsilon^{n-1}}\,.\end{aligned}$$

Therefore K_ε is bounded on $\partial\Omega\times\partial\Omega$ and since $\partial\Omega\times\partial\Omega$ itself is a bounded set, we have that

$$\|K_\varepsilon\|_{L^2_{\partial\Omega\times\partial\Omega}} = \int_{\partial\Omega}\int_{\partial\Omega}|K(P^*,Q^*)|^2 d\sigma(Q^*)d\sigma(P^*) < \infty\,.$$

Therefore K_ε is a Hilbert-Schmidt Kernel, and by Proposition 4.17, T_{K_ε} is a compact operator.

Now we will work to find the bounds needed in order to apply the result in Lemma 4.20. To bound the supremum over $P^*\in\partial\Omega$, we employ a similar argument to that in Corollary 3.7, using the constant c that arises from Lemma 3.8. We then have that

$$\begin{aligned}&\sup_{P^*\in\partial\Omega}\int_{\partial\Omega}|K(P^*,Q^*)-K_\varepsilon(P^*,Q^*)|\,d\sigma(Q^*) \\ &\quad= \sup_{P^*\in\partial\Omega}\int_{\partial\Omega}(1-\psi_\varepsilon(P^*,Q^*))\,|K(P^*,Q^*)|\,d\sigma(Q^*) \\ &\quad\le \sup_{P^*\in\partial\Omega}\int_{\partial\Omega\cap B_{2\varepsilon}(P^*)}|K(P^*,Q^*)|\,d\sigma(Q^*) \\ &\quad\le \sup_{P^*\in\partial\Omega}\frac{c}{w_n}\int_{\partial\Omega\cap B_{2\varepsilon}(P^*)}\frac{1}{|P^*-Q^*|^{n-1-\alpha}}d\sigma(Q^*) \\ &\quad\le \sup_{P^*\in\partial\Omega} C\varepsilon^{\alpha+1} \\ &\quad= C\varepsilon^{\alpha+1}\,.\end{aligned}$$

Noticing the symmetry present in $|K(P^*,Q^*)-K_\varepsilon(P^*,Q^*)|$, we note that the same bound should hold for the supremum over $Q^*\in\partial\Omega$, i.e.

$$\sup_{Q^*\in\partial\Omega}\int_{\partial\Omega}|K(P^*,Q^*)-K_\varepsilon(P^*,Q^*)|\,d\sigma(Q^*) < C\varepsilon^{\alpha+1}\,.$$

Therefore, Lemma 4.20 holds, and we can say that

$$\|T_{K-K_\varepsilon}\| = \|T-T_{K_\varepsilon}\| \le C''\varepsilon^{\alpha+1}$$

and thus $\|T-T_{K_\varepsilon}\|\to 0$ as $\varepsilon\to 0$. As the limit of compact operators T_{K_ε}, by Proposition 4.11, T must be compact as well. By Proposition 4.12, since T is compact, so is the adjoint operator T^*. □

After all that work, we now have that the operator T is compact. We can now invoke the Fredholm Alternative and hopefully gain some valuable insight into the operators $\pm\frac{1}{2}I+T$ and $\pm\frac{1}{2}I+T^*$.

5.2 Application of Fredholm Alternative

Now that we know that T and T^* are compact, we can go about investingating the operators of the form $\pm\frac{1}{2}I+T$ and $\pm\frac{1}{2}I+T$. Of particular importance is the image of these operators, for as stated in Proposition 3.20, the solvability criteria is stated in terms of f being in the image of these operators. With the Fredholm Alternative, we see that there is a connection between images and kernels of these operators, so we will proceed by first investigating the kernels of these operators.

Proposition 5.3. *If Ω is a bounded and connected $C^{1,\alpha}$ domain, then $\ker(-\frac{1}{2}I+T) \subseteq L^2(\partial\Omega)$ is comprised of the constant functions on $\partial\Omega$.*

Before proving this proposition, we state the Maximum Principle for harmonic functions. Additionally, since the Maximum Principle only applies to harmonic functions restricted to compact domains, we will prove a lemma that shows that we can use it for the Single Layer Potential on the unbounded set Ω^{c}.

Proposition 5.4 (Maximum Principle for Harmonic Functions)**.** *Let $\overline{X}$ be compact. If f is harmonic on X and continuous on $\overline{X}$, then the maximum and minimum values of f on $\overline{X}$ are achieved on the boundary ∂X.*

For the proof, see Proposition 2.13 and Corollary 2.14 in [5].

Corollary 5.4.1. *For $\varphi \in L^2(\partial\Omega)$, let $u = S[\varphi]$. If $u|_{\partial\Omega} = 0$ then $u \equiv 0$ on Ω^{c}.*

Proof. Since Ω is bounded, there exists an $R_0 > 0$ such that $\Omega \subseteq B_{R_0}(\underline{0})$. Now for $R > R_0$, consider the set $\Omega_R = \Omega^{\mathsf{c}} \cap B_R(\underline{0})$. By the maximum principle for harmonic functions, the maximum of u on Ω_R must be attained on its boundary. The boundary of Ω_R is comprised of $\partial\Omega$ and $\partial B_R(\underline{0})$. By assumption, u is 0 on $\partial\Omega$, so it follows that

$$\sup_{P\in\Omega_R(\underline{0})} |u(P)| = \sup_{\partial B_R(\underline{0})} |u(P)|.$$

When $P \in B_R(\underline{0})$ and $Q^* \in \partial\Omega$ we have that $|P-Q^*| > \text{dist}(P,\partial\Omega) > |R-R_0|$. Using this, we can bound the values that $u(P)$ takes in the integral as follows

$$\begin{aligned}
\sup_{P\in\partial B_R(\underline{0})} |u(P)| &\leq \sup_{P\in\partial B_R(\underline{0})} \int_{\partial\Omega} |N(P,Q^*)| \|\varphi\|_\infty d\sigma(Q^*) \\
&= \sup_{P\in\partial B_R(\underline{0})} \int_{\partial\Omega} \frac{1}{(n-2)w_n} \frac{\|\varphi\|_\infty}{|P-Q^*|^{n-2}} d\sigma(Q^*) \\
&= \sup_{P\in\partial B_R(\underline{0})} \frac{1}{|R-R_0|^{n-2}} \int_{\partial\Omega} \frac{\|\varphi\|_\infty}{(n-2)w_n} d\sigma(Q^*) \\
&= \frac{C_1}{|R-R_0|^{n-2}} \to 0 \text{ as } R \to \infty.
\end{aligned}$$

It then follows that

$$\sup_{P\in\Omega^{\mathsf{c}}} |u(P)| = \lim_{R\to 0} \sup_{P\in\Omega_R(\underline{0})} |u(P)| = \lim_{R\to 0} \sup_{\partial B_R(\underline{0})} |u(P)| = 0$$

and thus u should be 0 on the whole of Ω^{c}. □

We can now move forward with the proof that the kernel of $-\frac{1}{2}I+T$ is entirely comprised of the constant functions.

Proposition 5.3. We will first show that the constant functions are in the kernel of $-\frac{1}{2}I+T$. Consider the constant function $\varphi \equiv c \in \mathbb{R}$ on $\partial\Omega$. We have that

$$\begin{aligned}\left(-\frac{1}{2}I+T\right)\varphi(P^*) &= -c\frac{1}{2}+c\int_{\partial\Omega}K(P^*,Q^*)d\sigma(Q^*)\\ &= -c\frac{1}{2}+c\frac{1}{2}=0\,,\end{aligned}$$

where we used Proposition 3.13, which states that $\int_{\partial\Omega}K(P^*,Q^*)d\sigma(Q^*)=\frac{1}{2}$ when $P^*\in\partial\Omega$.

We will now show that the constant fuctions are the only functions on the kernel by showing that $\dim\left(\ker\left(-\frac{1}{2}+T\right)\right)\leq 1$.

Since $\ker\left(-\frac{1}{2}+T\right)\neq\{0\}$ as we know it contains the constant functions, we know that $-\frac{1}{2}+T$ is not invertible. Therefore the second alternative of the Fredholm Alternative holds, meaning

$$\dim\left(\ker\left(-\frac{1}{2}+T\right)\right)=\dim\left(\ker\left(-\frac{1}{2}+T^*\right)\right)\,.$$

It is therefore enough for our purposes to show that $\dim\left(\ker\left(-\frac{1}{2}+T^*\right)\right)\leq 1$.

Let $\varphi\in\ker\left(-\frac{1}{2}+T^*\right)$. From Proposition 3.17, we see that $u:\Omega\to\mathbb{R}$ given by $u(P)=S[\varphi](P)=\int_{\partial\Omega}N(P,Q^*)\varphi(Q^*)d\sigma(Q^*)$ satisfies

$$\frac{\partial u}{\partial n_-}(P^*)=\left(-\frac{1}{2}I+T^*\right)\varphi(P^*)=0$$

for $P^*\in\partial\Omega$. Then, using Green's first identity, we have that

$$\int_\Omega|\nabla u(P)|^2dP=\int_{\partial\Omega}u(P^*)\frac{\partial u(P^*)}{\partial n_-}d\sigma(P^*)=0.$$

This can only be the case if $\nabla u=0$ on Ω, i.e. u must be constant on Ω. Say $u\equiv k$ on Ω. Since we know $u=S[\varphi]$ is continuous, we can further say that $u\equiv k$ on $\overline{\Omega}$. Consider the map $F:\ker\left(-\frac{1}{2}+T^*\right)\to\mathbb{R}$ given by $F:\varphi\mapsto k$, where k is the constant value $u=S[\varphi]$ takes on $\overline{\Omega}$. By the workings above, F is well defined and linear. Therefore, if we show that F is injective, we can conclude that $\dim\left(\ker\left(-\frac{1}{2}+T^*\right)\right)\leq\dim(\mathbb{R})=1$.

The condition for injectivity we will check is $F^{-1}(0)=\{0\}$. Let $\varphi\in F^{-1}(0)$, i.e. $u=S[\varphi]\equiv 0$ on $\overline{\Omega}$.

By Corollary 5.4.1, we have that $u\equiv 0$ on Ω^c and therefore $u\equiv 0$ on all of $\mathbb{R}^n$.

Using Corollary 3.17.1, we can recover φ from u, so we have that

$$\varphi=\frac{\partial u}{\partial n_-}-\frac{\partial u}{\partial n_+}=0\,.$$

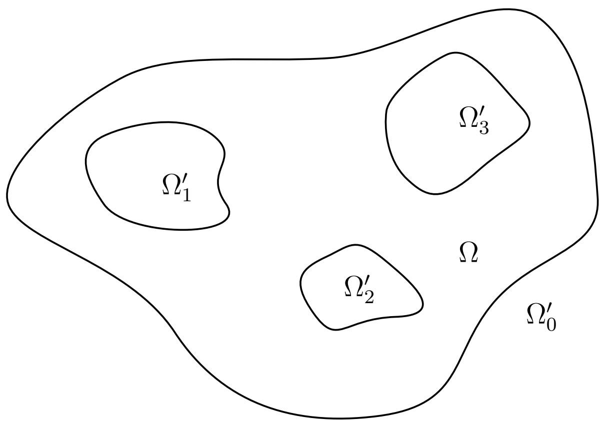
Therefore $F^{-1}(0)=\{0\}$. With this, we have shown that the constant functions on $\partial\Omega$ make up the entirety of $\ker\left(-\frac{1}{2}I+T\right)$. □

Corollary 5.4.2. *For each $k \in \mathbb{R}$, there exists a $\psi \in \ker(-\frac{1}{2}I + T^*)$ such that $v = S[\psi]$ satisifies $v|_{\partial\Omega} = k$.*

Proof. From the proof of Proposition 5.3 above, we had the linear map $F : \ker(-\frac{1}{2} + T^*) \to \mathbb{R}$, which we showed was injective. Additionally, since we were able to conclude that $\dim\left(\ker\left(-\frac{1}{2}I + T\right)\right) = \dim\left(\ker\left(-\frac{1}{2}I + T^*\right)\right) = 1$, we have that F must be surjective onto $\mathbb{R}$. Therefore F is invertible and thus for each $k \in \mathbb{R}$, there is a ψ such that $v = S[\psi] = k$ on Ω. Since the Single Layer Potential is continuous, v must also be k on the boundary $\partial\Omega$. Therefore $v|_{\partial\Omega} = k$. □

Proposition 5.5. *Let Ω be a bounded and connected $C^{1,\alpha}$ domain where Ω^{c} has m bounded components in addition to the one unbounded component, i.e. $\Omega^{\mathsf{c}} = \Omega'_0 \cup \Omega'_1 \cup \cdots \cup \Omega'_m$, where each Ω'_i is connected and Ω'_0 is unbounded. Then the kernel of $\frac{1}{2}I + T$ in $L^2(\partial\Omega)$ is spanned by functions $a_i : \partial\Omega \to \mathbb{R}$, $a \in \{1, \ldots, m\}$, of the form*

$$a_i(P^*) = \begin{cases} 1 \text{ if } P^* \in \partial\Omega'_i \\ 0 \text{ otherwise.} \end{cases}$$



The proof of this proposition follows a similar structure to that of Proposition 5.3, which found the kernel of $-\frac{1}{2}I + T$. In that proof, we used Green's first identity on Ω. In this proof, we will need to do the same thing, except we would need to use Green's first identity on the unbounded region Ω^{c}. In general, Green's first identity does not hold on unbounded regions, so before we proceed with the proof of Proposition 5.5, we will prove the following lemma that allows us to use Green's first identity specifically with the Single Layer Potential $u = S[\varphi]$ on the complement of a bounded domain.

Lemma 5.6. *For $\varphi \in C(\partial\Omega)$ $u = S[\varphi]$, we have that*

$$\int_{\Omega^{\mathsf{c}}} |\nabla u(P)|^2 dP = \int_{\partial\Omega} u(P^*) \frac{\partial u(P^*)}{\partial n_+} d\sigma(P)$$

Proof. Since Ω is bounded, there exists an $R_0 > 0$ such that $\Omega \subseteq B_{R_0}(\underline{0})$. Now consider the region $\Omega^{\mathsf{c}} \cap B_R(\underline{0})$ for $R > R_0$. Using Green's first identity on this bounded region, we have that

$$\int_{\Omega^{\mathsf{c}} \cap B_R(\underline{0})} |\nabla u(P)|^2 dP = \int_{\partial B_R(\underline{0})} u(P) \frac{\partial u(P)}{\partial n_-} d\sigma(P) + \int_{\partial\Omega} u(P^*) \frac{\partial u(P^*)}{\partial n_+} d\sigma(P^*) \quad (19)$$

We will now show that the first term on the right side above goes to 0 as $R \to \infty$.

In the proof of Corollary 5.4.1, we showed that, with the same setup, we have that

$$\sup_{P \in B_R(\underline{0})} |u(P)| \leq \frac{C_1}{|R - R_0|^{n-2}}.$$

Similarly, we can bound $\left|\frac{\partial u(P)}{\partial n_-}\right|$. Using Proposition 3.5, which gives us the normal derivative of the Single Layer Potential, we have that

$$\begin{aligned}
\left|\frac{\partial u(P)}{\partial n_-}\right| &= \left|\int_{\partial\Omega} \frac{\partial N(P, Q^*)}{dn_P} \varphi(Q^*) d\sigma(Q^*)\right| \\
&\leq \|\varphi\|_\infty \int_{\partial\Omega} \left|\frac{\partial N(P, Q^*)}{dn_P}\right| d\sigma(Q^*) \\
&\leq \|\varphi\|_\infty \int_{\partial\Omega} \frac{1}{w_n |P - Q^*|^{n-1}} d\sigma(Q^*) \\
&\leq \|\varphi\|_\infty \frac{1}{w_n |R - R_0|^{n-1}} \int_{\partial\Omega} d\sigma(Q^*) \\
&= \frac{C_2}{|R - R_0|^{n-1}}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\left|\int_{\partial B_R(\underline{0})} u(P) \frac{\partial u(P^*)}{\partial n_-} d\sigma(P)\right| &\leq \int_{\partial B_R(\underline{0})} |u(P)| \left|\frac{\partial u(P)}{\partial n_-}\right| d\sigma(P^*) \\
&\leq \int_{\partial B_R(\underline{0})} \frac{C_1}{|R - R_0|^{n-2}} \frac{C_2}{|R - R_0|^{n-1}} d\sigma(P^*) \\
&= \frac{C_1}{|R - R_0|^{n-2}} \frac{C_2}{|R - R_0|^{n-1}} \sigma(B_R(\underline{0})) \\
&= \frac{C R^n}{|R - R_0|^{2n-3}} \to 0 \text{ as } R \to \infty
\end{aligned}$$

where we used that $\sigma(B_R(\underline{0}))$, the surface area of the $n+1$ dimensional sphere, is of order $O(R^n)$. Taking the limit $R \to \infty$ in (19) gives

$$\int_{\Omega^c} |\nabla u(P)|^2 dP = \int_{\partial\Omega} u(P^*) \frac{\partial u(P^*)}{\partial n_+} d\sigma(P^*). \qquad \square$$

We can now proceed with the proof of Proposition 5.5, which characterizes the kernel of the operator $\frac{1}{2} + T$ in terms of an explicit basis.

Proposition 5.5. We will first show that $a_i \in \ker\left(\frac{1}{2}I + T\right)$. We have that

$$\begin{aligned}
\left(\frac{1}{2}I + T\right) a_i(P^*) &= \frac{1}{2} a_i(P^*) + \int_{\partial\Omega} K(P^*, Q^*) a_i(Q^*) d\sigma(Q^*) \\
&= \frac{1}{2} a_i(P^*) + \int_{\partial\Omega_i'} K(P^*, Q^*) d\sigma(Q^*).
\end{aligned}$$

Since we are only looking at P^* in $\partial\Omega$, if $P^* \notin \partial\Omega_i'$, necessarily $P^* \in (\Omega_i')^c$. Therefore, from Proposition 3.12, for $P^* \notin \partial\Omega_i'$,

$$\int_{\partial\Omega_i'} K(P^*, Q^*) d\sigma(Q^*) = 0.$$

On the other hand, if $P^* \in \partial\Omega_i'$, from Proposition 3.13 we have that

$$\int_{\partial\Omega_i'} K(P^*, Q^*) d\sigma(Q^*) = -\frac{1}{2} \text{ for } P^* \in \partial\Omega_i'.$$

The negative sign occurs because $\partial\Omega_i'$ is the boundary one of the bounded components of Ω^c "inside" Ω, its orientation as a component of $\partial\Omega$ opposite to if we were considering it the boundary the domain Ω_i' on its own. Together, we have that for $P^* \in \partial\Omega$,

$$\begin{aligned} \int_{\partial\Omega_i'} K(P^*, Q^*) d\sigma(Q^*) &= \begin{cases} -\frac{1}{2} \text{ if } P^* \in \partial\Omega_i' \\ 0 \text{ otherwise} \end{cases} \\ &= -\frac{1}{2} a_i(P^*)\,. \end{aligned}$$

Therefore

$$\left(\frac{1}{2}I + T\right) a_i(P^*) = \frac{1}{2} a_i(P^*) - \frac{1}{2} a_i(P^*) = 0$$

and thus $a_i \in \ker\left(\frac{1}{2}I + T\right)$. It then follows that $\dim\left(\ker\left(\frac{1}{2}I + T\right)\right) \geq m$. To show that $\{a_i\}_{i=1}^m$ spans $\ker\left(\frac{1}{2}I + T\right)$, it is enough to show that $\dim\left(\ker\left(\frac{1}{2}I + T\right)\right) \leq m$.

As in the proof of Proposition 5.3, from the fact that $\ker\left(\frac{1}{2}I + T\right) \neq \{0\}$, we see that $\frac{1}{2}I + T$ cannot be invertible, and thus the Fredholm Alternative guarantees that

$$\dim\left(\ker\left(\frac{1}{2}I + T\right)\right) = \dim\left(\ker\left(\frac{1}{2}I + T^*\right)\right)\,.$$

It is therefore enough to proceed by showing $\dim\left(\ker\left(\frac{1}{2}I + T^*\right)\right) \leq m$.

Let $\varphi \in \ker\left(\frac{1}{2}I + T^*\right)$. From Proposition 3.17, we have that $u = S[\varphi] : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfies

$$\frac{\partial u}{\partial n_+} = \left(\frac{1}{2}I + T^*\right) \varphi = 0\,. \tag{20}$$

Then, using Green's first identity on this unbounded region, we have from Lemma 5.6 that

$$\int_{\Omega^\mathrm{c}} |\nabla u(P)|^2 dP = \int_{\partial B_R(\underline{0})} u(P) \frac{\partial u(P)}{\partial n_-} d\sigma(P) = 0\,.$$

This can only be the case if $\nabla u(P) = 0$ on all of Ω^c. Therefore u must be constant on each component of Ω^c. Additionally, since $u(P) = S[\varphi](P)$ approaches 0 as $|P| \to \infty$, as shown in the proof of Corollary 5.4.1, it must be that u must be constantly 0 on the unbounded component Ω_0'.

Now consider the map $F : \ker\left(\frac{1}{2}I + T^*\right) \to \mathbb{R}^m$ where $\varphi \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_m)$, where α_i is the constant value that $u = S[\varphi]$ takes on the component Ω_i'. If we show that F is injective, we can conclude that $\dim\ker\left(\frac{1}{2}I + T^*\right) \leq \dim(\mathbb{R}^m) = m$, as desired.

We will therefore work to show that $F^{-1}(\underline{0}) = \{0\}$ to show that F is injective. Let $\varphi \in F^{-1}(\underline{0})$. This means that $u = S[\varphi] \equiv 0$ on each component Ω_i', including the unbounded Ω_0', i.e. $u \equiv 0$ on all of Ω^c. Since $u = S[\varphi]$ is continuous, we can further conclude that $u \equiv 0$ on $\partial\Omega$ as well. Additionally, since $u = S[\varphi]$ is harmonic, by the Maximum Principle the maximum and minimum value that u takes on Ω must be attained on its boundary $\partial\Omega$. Thus since $u \equiv 0$ on the boundary, we can conclude that $u \equiv 0$ on all of Ω and thus $u \equiv 0$ on $\mathbb{R}^{n+1}$.

Using Corollary 3.17.1, we can recover φ from u as follows

$$\varphi = \frac{\partial u}{\partial n_-} - \frac{\partial u}{\partial n_+} = 0\,.$$

Therefore $F^{-1}(\underline{0}) = \{0\}$. It then follows that $\dim\ker\left(\frac{1}{2}I + T\right) = m$, and thus $\{a_i\}_{i=1}^m$ span all of $\ker\left(\frac{1}{2}I + T\right)$. □

Corollary 5.6.1. *For every function on $\partial\Omega$ of the form $\sum_{i=1}^m \alpha_i a_i$ for $\alpha_i \in \mathbb{R}$, there exists a $\psi \in \ker(\frac{1}{2} + T^*)$ such that $v = S[\psi]$ satisifies $v|_{\partial\Omega} = \sum_{i=1}^m \alpha_i a_i$.*

Proof. In the proof of Proposition 5.5, we used the linear map $F : \ker(\frac{1}{2}I + T^*) \to \mathbb{R}^m$ and showed that it is injective. In addition, as we know that $\dim\left(\ker\left(\frac{1}{2}I + T\right)\right) = \dim\left(\ker\left(\frac{1}{2}I + T^*\right)\right) = m$, we can conclude that F is surjective onto $\mathbb{R}^m$. Therefore F is invertible and thus for each $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, there exists a $\psi \in \ker(\frac{1}{2} + T^*)$ such that $v = S[\psi]$ has $v|_{\Omega'_j} = \alpha_j$. Since the Single Layer Potential is continuous everywhere, the constant value of v on Ω'_j extends to the boundary and thus $v|_{\partial\Omega'_j} = \alpha_j$. In the proof we also showed that if $\psi \in \ker(\frac{1}{2}I + T^*)$, then $v|_{\Omega'_0} = 0$ and again, since $v = S[\psi]$ is continuous, $v|_{\partial\Omega'_0} = 0$. All together, we have that $v|_{\partial\Omega} = \sum_{i=1}^m \alpha_i a_i$. □

As used in the proofs of Propositions 5.3 and 5.5, the Fredholm alternative connects the dimension of $\ker\left(-\frac{1}{2}I + T\right)$ with that of $\ker\left(-\frac{1}{2}I + T^*\right)$, and the dimension of $\ker\left(\frac{1}{2}I + T\right)$ with that of $\ker\left(\frac{1}{2}I + T^*\right)$. The following corollary summarizes this fact.

Corollary 5.6.2. *If Ω is a bounded and connected $C^{1,\alpha}$ domain where Ω^c has m bounded components in addition to the one unbounded component, then*

$$\dim\left(\ker\left(-\frac{1}{2}I + T\right)\right) = \dim\left(\ker\left(-\frac{1}{2}I + T^*\right)\right) = 1$$

and

$$\dim\left(\ker\left(\frac{1}{2}I + T\right)\right) = \dim\left(\ker\left(\frac{1}{2}I + T^*\right)\right) = m.$$

5.3 Partitioning $L^2(\partial\Omega)$ into Images and Kernels of Operators

Since the operators we are looking at, $\pm\frac{1}{2}I + T$ and $\pm\frac{1}{2}I + T^*$ map between $L^2(\partial\Omega)$ and itself, we are able to think of the images and kernels of these operators as subspaces of the same space $L^2(\partial\Omega)$. Additionally, the Fredholm Alternative gives us the ability to find relationships between these subspaces and their orthogonal complements.

Though we are currently thinking of these operators as operators on $L^2(\partial\Omega)$, we must not lose sight of the fact that most of the work we have done so far regarding T and T^* has relied on the assumptions of the form $\varphi \in C(\partial\Omega) \subseteq L^2(\partial\Omega)$. For example, Proposition 3.20, which condensed the Dirichlet and Neumann BVP problem data in terms of these operators, assumed that the boundary data was continuous.

The following proposition will allow us to extend the continuity of a function in the image of these operators to the preimage of that function. This is an important step that will allow us to use the conclusions from Proposition 3.20 regarding the solvability of the BVPs more broadly.

Proposition 5.7. *Let* $\varphi \in L^2(\partial\Omega)$ *such that* $\pm\frac{1}{2}\varphi + T\varphi \in C(\partial\Omega)$ *or* $\pm\frac{1}{2}\varphi + T^*\varphi \in C(\partial\Omega)$. *Then* $\varphi \in C(\partial\Omega)$.

Proof. We will prove the case where $\frac{1}{2}\varphi + T\varphi \in C(\partial\Omega)$. The proofs of the other cases follow a very similar procedure.

Let $\psi : \Omega \times \Omega \to \mathbb{R}$ be a smooth function such that if $|P^* - Q^*| \leq \delta$ then $\psi_0(P^*, Q^*) = 0$ and if $|P^* - Q^*| \geq 2\delta$ then $\psi_0(P^*, Q^*) = 1$. Then, for K as defined in (9), let $K_1 = \psi K$ and $K_0 = (1 - \psi)K$. We note that K_1 is continuous and bounded on $\partial\Omega \times \partial\Omega$ since the diagonal region that includes the singularities of K is brought continuously to zero in K_1.

We will first show that $T_{K_1}\varphi$ is continuous when $\varphi \in L^2(\partial\Omega)$. For $P_0^*, P^* \in \partial\Omega$, we use Cauchy-Schwarz to get that

$$\begin{aligned} |T_{K_1}\varphi(P_0^*) - T_{K_1}\varphi(P^*)| &= \left| \int_{\partial\Omega} \left(K_1(P_0^*, Q^*) - K_1(P^*, Q^*)\right) \varphi(Q^*) d\sigma(Q^*) \right| \\ &\leq \|\varphi\|_2 \sqrt{\int_{\partial\Omega} |K_1(P_0^*, Q^*) - K_1(P^*, Q^*)|^2 \, d\sigma(Q^*)} \,. \end{aligned}$$

Since K_1 is continuous on $\partial\Omega \times \partial\Omega$, we can take the limit $P^* \to P_0^*$ to inside the integral and conclude that $|T_{K_1}\varphi(P_0^*) - T_{K_1}\varphi(P^*)| \to 0$ as $P^* \to P_0^*$ and thus $T_{K_1}\varphi$ is continuous. Let $F = \left(\frac{1}{2}\varphi + T\varphi\right) - T_{K_1}\varphi$. This function F is continuous as it is the sum of the continuous function $T_{K_1}\varphi$ and $\frac{1}{2}\varphi + T\varphi$, which is continuous by assumption. We have that

$$F = \left(\frac{1}{2}\varphi + T\varphi\right) - T_{K_1}\varphi = \frac{1}{2}\varphi + (1 - \psi)T\varphi = \frac{1}{2}\varphi + T_{K_0}\varphi$$

As a result of Proposition 3.16, T_0 maps bounded functions to continuous and therefore bounded functions on $\partial\Omega$. We can therefore think of T_0 as an operator on $L^\infty(\partial\Omega)$ and thus we can consider the norm $\|T_0\|_{L^\infty(\partial\Omega)}$. We have that for $\varphi \in L^\infty(\partial\Omega)$,

$$\begin{aligned} |T_0\varphi(P^*)| &= \left| \int_{\partial\Omega} K_0(P^*, Q^*)\varphi(Q^*) d\sigma(Q^*) \right| \\ &\leq \|\varphi\|_\infty \int_{\partial\Omega} |K_0(P^*, Q^*)| d\sigma(Q^*) \\ &\leq \|\varphi\|_\infty \int_{\partial\Omega \cap B_{2\delta}(P^*)} |K(P^*, Q^*)| d\sigma(Q^*) \\ &\leq \|\varphi\|_\infty C\delta^{\alpha+1} \end{aligned}$$

where getting the final bound is done in the same way as in the proof of Proposition 3.7. Note that the constant C is independent of φ. We can thus see that $T_0\varphi$ is bounded since $\|T_0\varphi\|_\infty \leq |\varphi\|_\infty C\delta^{\alpha+1}$.

We therefore have that

$$\|T_0\|_{L^\infty(\partial\Omega)} = \sup_{\varphi \in L^\infty(\partial\Omega)} \frac{\|T_0\varphi\|_\infty}{\|\varphi\|_\infty} \leq \sup_{\varphi \in L^\infty(\partial\Omega)} \frac{\|\varphi\|_\infty C\delta^{\alpha+1}}{\|\varphi\|_\infty} = C\delta^{\alpha+1} \,,$$

so if we take a δ small enough, we can guarantee that $\|T_0\|_{L^\infty(\partial\Omega)} < \frac{1}{2}$.

Using Lemma 4.8 with the operator $-2T_0$, since $\|-2T_0\|_{L^\infty(\partial\Omega)} < 1$, we have that $I+2T_0$ is invertible with inverse

$$(I+2T_0)^{-1} = \sum_{k=0}^{\infty}(-2T_0)^k$$

and therefore

$$\varphi = \left(\frac{1}{2}I + T_{K_0}\right)^{-1} F = \frac{1}{2}\left(I + 2T_{K_0}\right)^{-1} F = \frac{1}{2}\sum_{k=0}^{\infty}(-2T_0)^k F = -\sum_{k=0}^{\infty}(-2)^{k-1}T_0^k F\,.$$

Since F is continuous and therefore bounded, we can use Proposition 3.16 to conclude that T_0F is continuous, and by induction so is T_0^kF. Additionally, since the series converges with respect to the $L_\infty(\partial\Omega)$ norm, the convergence is uniform and thus, as the uniformly convergent sum of continuous functions, φ is continuous.

The proofs for when $-\frac{1}{2}\varphi + T\varphi \in C(\partial\Omega)$ and $\pm\frac{1}{2}\varphi + T^*\varphi \in C(\partial\Omega)$ work the same way. □

Proposition 5.8. *For the operators T and T^* as defined in equations (15) and (16), we have that*

$$L^2(\partial\Omega) = \ker\left(\frac{1}{2}I + T\right)^{\perp} \oplus \ker\left(\frac{1}{2}I + T^*\right)$$

and

$$L^2(\partial\Omega) = \ker\left(-\frac{1}{2}I + T\right)^{\perp} \oplus \ker\left(-\frac{1}{2}I + T^*\right)\,.$$

Proof. From Corollary 5.6.2, we know that $\dim\left(\ker\left(\frac{1}{2}I + T^*\right)\right) = m$ and we can conclude that $\operatorname{codim}\left(\ker\left(\frac{1}{2}I + T\right)^{\perp}\right) = m$. Therefore, to show that their direct sum makes up $L^2(\partial\Omega)$, it is enough to show that

$$\ker\left(\frac{1}{2}I + T\right)^{\perp} \cap \ker\left(\frac{1}{2}I + T^*\right) = \{0\}.$$

Let $\varphi \in \ker\left(\frac{1}{2}I + T\right)^{\perp} \cap \ker\left(\frac{1}{2}I + T^*\right)$. By the Fredholm Alternative, we know $\varphi \in \ker\left(\frac{1}{2}I + T\right)^{\perp} = \operatorname{range}\left(\frac{1}{2}I + T^*\right)$. This guarantees that there exists a $\psi \in L^2(\partial\Omega)$ such that $\left(\frac{1}{2}I + T^*\right)\psi = \varphi$. By Proposition 5.7, both φ and ψ are continuous since their images in these operators are continuous. Let $u = S[\varphi]$ and $v = S[\psi]$.

Because $\varphi \in \ker\left(\frac{1}{2}I + T^*\right)$, Proposition 3.17 gives us that $\frac{\partial u}{\partial n_+} = 0$ and $\frac{\partial v}{\partial n_+} = \varphi$. We further have that

$$\frac{\partial v}{\partial n_+} = \varphi - 0 = \varphi - \left(\frac{1}{2}\varphi + T^*\varphi\right) = \frac{1}{2}\varphi - T^*\varphi = -\frac{\partial u}{\partial n_-}$$

where the final equality again comes from Proposition 3.17. Using that $\frac{\partial u}{\partial n_+} = 0$ and $\frac{\partial v}{\partial n_+} = -\frac{\partial u}{\partial n_-}$ in Green's Identities gives us

$$\begin{aligned}\int_{\Omega^c}(u\Delta v - v\Delta u)dP &= \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n_+} - v\frac{\partial u}{\partial n_+}\right)d\sigma(P^*) \\ &= \int_{\partial\Omega}\left(-u\frac{\partial u}{\partial n_-} - 0\right)d\sigma(P^*) \\ &= -\int_{\Omega}|\nabla u|^2 dP\,.\end{aligned}$$

Since u and v are harmonic everywhere, $\Delta u = \Delta v = 0$ and thus

$$0 = \int_{\Omega^c}(u\Delta v - v\Delta u)dP = -\int_{\Omega}|\nabla u|^2 dP\,.$$

This is only possible if $\nabla u = 0$ on Ω, i.e. u is constant on Ω. Therefore

$$\varphi = -\frac{\partial u}{\partial n_-} = 0\,.$$

Therefore $\ker\left(\frac{1}{2}I + T\right)^{\perp} \cap \ker\left(\frac{1}{2}I + T^*\right) = \{0\}$ and thus $L^2(\partial\Omega) = \ker\left(\frac{1}{2}I + T\right)^{\perp} \oplus \ker\left(\frac{1}{2}I + T^*\right)$. The proof of the second part of the proposition follows the same strategy. □

Corollary 5.8.1. *For the operators T and T^* as defined in equations (15) and (16), we have that*

$$L^2(\partial\Omega) = \ker\left(\frac{1}{2}I + T\right) \oplus \ker\left(\frac{1}{2}I + T^*\right)^{\perp}$$

and

$$L^2(\partial\Omega) = \ker\left(-\frac{1}{2}I + T\right) \oplus \ker\left(-\frac{1}{2}I + T^*\right)^{\perp}\,.$$

Proof. For the same reason as in Proposition 5.8, to show that $L^2(\partial\Omega) = \ker\left(\frac{1}{2}I + T\right) \oplus \ker\left(\frac{1}{2}I + T^*\right)^{\perp}$, it is enough to show that $\ker\left(\frac{1}{2}I + T\right) \cap \ker\left(\frac{1}{2}I + T^*\right)^{\perp} = \{0\}$.

Suppose that $\varphi \in \ker\left(\frac{1}{2}I + T\right) \cap \ker\left(\frac{1}{2}I + T^*\right)^{\perp}$. By the first result of Proposition 5.8, we can write φ as $\varphi = \varphi_1 + \varphi_2$ for $\varphi_1 \in \ker\left(\frac{1}{2}I + T\right)^{\perp}$ and $\varphi_2 \in \left(\frac{1}{2}I + T^*\right)$. Since $\varphi \in \ker\left(\frac{1}{2}I + T\right)$ and is thus perpendicular to φ_1, we have that $\langle\varphi, \varphi_1\rangle = 0$. Similarly, since $\varphi \in \ker\left(\frac{1}{2}I + T^*\right)^{\perp}$, we have that $\langle\varphi, \varphi_2\rangle = 0$. Therefore

$$\langle\varphi, \varphi\rangle = \langle\varphi, \varphi_1 + \varphi_2\rangle = \langle\varphi, \varphi_1\rangle + \langle\varphi, \varphi_2\rangle = 0$$

and thus $\varphi = 0$.

By a similar argument, we can show that $L^2(\partial\Omega) = \ker\left(-\frac{1}{2}I + T\right) \oplus \ker\left(-\frac{1}{2}I + T^*\right)^{\perp}$. □

Because of the Fredholm Alternative, we have established relationships between the images and kernels of the operators we are looking at that involve the orthogonal complement. Using them with the previous results, namely the Fredholm alternative, Proposition 5.8, and Corollary 5.8.1, we are able to arrive at the following proposition, which gives us four very convenient ways to "partition" the space $L^2(\partial\Omega)$ into a direct sum of images and kernels of the operators. This is indeed helpful, since if you recall the statement of Proposition 3.20, the solvability criteria given for the four BVPs was stated in terms of the boundary data f being in the range of the associated operators.

Proposition 5.9. *For the operators T and T^* as defined in equations (15) and (16), we have that*

$$\begin{aligned} L^2(\partial\Omega) &= \textit{range}\left(\frac{1}{2}I + T\right) \oplus \ker\left(\frac{1}{2}I + T\right) \\ &= \textit{range}\left(-\frac{1}{2}I + T\right) \oplus \ker\left(-\frac{1}{2}I + T\right) \\ &= \textit{range}\left(-\frac{1}{2}I + T^*\right) \oplus \ker\left(-\frac{1}{2}I + T^*\right) \\ &= \textit{range}\left(\frac{1}{2}I + T^*\right) \oplus \ker\left(\frac{1}{2}I + T^*\right) . \end{aligned}$$

Proof. From the Fredholm Alternative, we have that $ker\left(\frac{1}{2}I + T^*\right)^{\perp} = \text{range}\left(\frac{1}{2}I + T\right)$, $ker\left(-\frac{1}{2}I + T^*\right)^{\perp} = \text{range}\left(-\frac{1}{2}I + T\right)$, $ker\left(-\frac{1}{2}I + T\right)^{\perp} = \text{range}\left(-\frac{1}{2}I + T^*\right)$, and $ker\left(\frac{1}{2}I + T\right)^{\perp} = \text{range}\left(\frac{1}{2}I + T^*\right)$. Substituting these into the results from Proposition 5.8 and Corollary 5.8.1, we get the desired results. □

6 Main Result for $L^{1,\alpha}$ Domains

We have now come to the point where we now have all the results we need to examine the solvability of the interior and exterior Dirichlet and Neumann BVPs. The first of the central results we proved that lead to these conclusions is Proposition 3.20, which condensed the work regarding the layer potentials and jump relations for the solvability criteria of the four BVPs in terms of the boundary data f being in the image of the four operators. The second key result is Proposition 5.9, that gave us that $L^2(\partial\Omega)$ could be written as the direct sum of the images and kernels of these operators. From these two results, we will be able to answer concretely the question about the solvability of the interior and exterior Dirichlet and Neumann BVPs.

6.1 Solvability of the Interior Dirichlet Problem

Proposition 6.1 (Solvability of the Interior Dirichlet Problem)**.** *Let Ω be a bounded, connected $C^{1,\alpha}$ domain. For every $f \in C(\partial\Omega)$, there exists a continuous solution $u : \overline{\Omega} \to \mathbb{R}$ to the Dirichlet boundary problem*

$$\begin{cases} \Delta u = 0 & \textit{on } \Omega \\ u = f & \textit{on } \partial\Omega. \end{cases}$$

Proof. From Propostition 5.9, we see that $L^2(\partial\Omega) = \ker(\frac{1}{2}I + T) \oplus \text{range}(\frac{1}{2}I + T)$, and from Proposition 5.5, we know that the kernel is spanned by functions of the form

$$a_i(x) = \begin{cases} 1 \text{ for } x \in \partial\Omega_i' \\ 0 \text{ otherwise.} \end{cases}$$

Since $f \in C(\partial\Omega) \subseteq L^2(\partial\Omega)$, f can be decomposed into $f = \sum_{i=1}^m \alpha_i a_i + \overline{f}$, where $\alpha_i \in \mathbb{R}$, and $\overline{f} \in \text{range}(\frac{1}{2} + T)$. This means there exists a $\varphi \in L^2(\partial\Omega)$ such that $\overline{f} = (\frac{1}{2}I + T)\varphi$. By Proposition 5.7, since $\overline{f} = f - \sum_{i=1}^m \alpha_i a_i$ is continuous, $\varphi \in C(\partial\Omega)$.

Therefore, using Proposition 3.15, we know that $\overline{u} = D[\varphi]$ with its continuous extension to $\partial\Omega$ is the solution to the interior Dirichlet boundary problem

$$\begin{cases} \Delta u & = 0 \text{ on } \Omega \\ u & = \overline{f} \text{ on } \partial\Omega. \end{cases}$$

Additionally, by Corollary 5.6.1, we have a $\psi \in \ker(\frac{1}{2}I + T^*)$ such that $v = S[\psi]$ satisfies $v|_{\partial\Omega} = \sum_{i=1}^m \alpha_i a_i$. Since $\frac{1}{2}\psi + T^*\psi = 0 \in C(\partial\Omega)$, we can conclude that $\psi \in C(\partial\Omega)$ from Proposition 5.7.

It then follows that $u = v + \overline{u}$ is a solution to the original Dirichlet boundary problem as

$$\Delta u = \Delta\left(v + \overline{u}\right) = \Delta\left(S[\psi] + D[\varphi]\right) = 0$$

since the Single and Double Layer Potentials are harmonic for $\psi, \varphi \in C(\partial\Omega)$, and

$$u|_{\partial\Omega} = \left(v + \overline{u}\right)|_{\partial\Omega} = \sum_{i=1}^m \alpha_i a_i + \overline{f} = f\,. \qquad \square$$

6.2 Solvability of the Exterior Dirichlet Problem

Proposition 6.2 (Solvability of the Exterior Dirichlet Problem)**.** *Let Ω be a bounded, connected $C^{1,\alpha}$ domain. For every $f \in C(\partial\Omega)$, there exists a continuous solution $u : \Omega^c \to \mathbb{R}$ to the Dirichlet boundary problem*

$$\begin{cases} \Delta u = 0 & \textit{on } \overline{\Omega}^c \\ u = f & \textit{on } \partial\Omega. \end{cases}$$

Proof. This proof is very similar to the interior case. We have that

$$L^2(\partial\Omega) = \ker(-\frac{1}{2}I + T) \oplus \text{range}(-\frac{1}{2}I + T)$$

from Proposition 5.9 and that $\ker(-\frac{1}{2}I + T) \subseteq L^2(\partial\Omega)$ is comprised of the constant functions. Therefore f can be decomposed into $f = c + \overline{f}$ where c we know is in $\ker(-\frac{1}{2}I + T)$ and $\overline{f} \in \text{range}(-\frac{1}{2}I + T)$. Since $\overline{f} \in \text{range}(-\frac{1}{2}I + T)$ we know $-\frac{1}{2}\varphi + T\varphi = \overline{f}$ for some $\varphi \in L^2(\partial\Omega)$. Since $-\frac{1}{2}\varphi + T\varphi = \overline{f} = f - c \in C(\partial\Omega)$, from Proposition 5.7 we know that $\varphi \in C(\partial\Omega)$.

By Proposition 3.20, we know that $\overline{u} = D[\varphi]$ defined on $\overline{\Omega}^c$ with its continuous extension to $\partial\Omega$ solves the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{on } \overline{\Omega}^c \\ u = \overline{f} & \text{on } \partial\Omega. \end{cases}$$

Additionally, by Corollary 5.6.1, there exists a $\psi \in \ker(-\frac{1}{2}I+T^*)$ such that for $v = S[\psi]$ on the boundary we have $v|_{\partial\Omega} = c$. Since $-\frac{1}{2}\psi + T^*\psi = 0 \in C(\partial\Omega)$, by Proposition 5.7, ψ is continuous on $\partial\Omega$.

We can then say that $u = v + \overline{u}$ is a solution to the original boundary problem as

$$\Delta u = \Delta\left(v + \overline{u}\right) = \Delta\left(S[\psi] + D[\varphi]\right) = 0$$

and

$$u|_{\partial\Omega} = \left(v + \overline{u}\right)|_{\partial\Omega} = c + \overline{f} = f\,. \qquad \square$$

6.3 Solvability of the Interior Neumann Problem

Proposition 6.3 (Solvability of the Interior Neumann Problem)**.** *Let Ω be a bounded, connected $C^{1,\alpha}$ domain. For $f \in C(\partial\Omega)$, there exists a continuous solution $u : \overline{\Omega} \to \mathbb{R}$ to the interior Neumann boundary problem*

$$\begin{cases} \Delta u = 0 \text{ on } \Omega \\ \frac{\partial u}{\partial n_-} = f \text{ on } \partial\Omega\,, \end{cases}$$

if and only if

$$\int_{\partial\Omega} f(P^*)d\sigma(P^*) = 0.$$

Proof. We will first show that $\int_{\partial\Omega} f(P^*)d\sigma(P^*) = 0$ implies the existence of a solution. We have that

$$\int_{\partial\Omega} f(P^*)d\sigma(P^*) = \langle f, 1\rangle = 0$$

Since $\ker(-\frac{1}{2}I+T)$ is spanned by $\{1\}$, and $\langle f, 1\rangle = 0$ we have that $f \in \ker(-\frac{1}{2}+T)^{\perp} =$ range$(-\frac{1}{2}I + T^*)$. Therefore, by Proposition 3.20, there exists a solution to the given boundary problem.

We will now show that the condition $\int_{\partial\Omega} f(P^*)d\sigma(P^*) = 0$ is necessary. If u is a solution to the given boundary problem, then $\frac{\partial u}{\partial n_-} = f$. By Corollary 2.10.1, we have that

$$\int_{\partial\Omega} \frac{\partial u(P^*)}{\partial n_-} d\sigma(P^*) = \int_{\partial\Omega} f(P^*)d\sigma(P^*) = 0\,. \qquad \square$$

6.4 Solvability of the Exterior Neumann Problem

Proposition 6.4 (Solvability of the Exterior Neumann Problem)**.** *Let Ω be a bounded, connected $C^{1,\alpha}$ domain. For $f \in C(\partial\Omega)$, there exists a continuous solution $u : \Omega^{\mathrm{c}} \to \mathbb{R}$ to the exterior Neumann boundary problem*

$$\begin{cases} \Delta u = 0 \text{ on } \Omega^{\mathrm{c}} \\ \frac{\partial u}{\partial n_+} = f \text{ on } \partial\Omega\,, \end{cases}$$

if for $j = 1, \ldots, m$

$$\int_{\partial\Omega'_j} f(P^*)d\sigma(P^*) = 0.$$

Proof. We have that

$$\int_{\partial\Omega_j'} f(P^*)d\sigma(P^*) = \int_{\partial\Omega} f(P^*)a_j(P^*)d\sigma(P^*) = \langle f, a_j\rangle = 0\,,$$

i.e. f is orthogonal to every a_j. From Proposition 5.5, we know that $\ker(\frac{1}{2}I + T)$ is spanned by $\{a_j\}_{j=1}^m$, and therefore $f \in \ker(\frac{1}{2}I + T)^\perp = \text{range}(\frac{1}{2}I + T^*)$. By Proposition 3.20, there exists a solution to the Exterior Neumann BVP. □

Like the Interior case, the condition that f must satisfy $\int_{\partial\Omega_j'} f(P^*)d\sigma(P^*) = 0$ is a necessary condition for very much the same reason as in the interior case. The difference in this case, and the reason Proposition 6.4 is stated as a single direction implication, is that the proof requires the use of Green's first identity on the unbounded region Ω^c. In this work, we showed Lemma 5.6, which was Green's first identity holding for Ω^c, but in that case we knew that the function we would be using it on was the Single Layer Potential $u = S[\varphi]$, so we were able to use the known rate of decay at infinity of u in our proof. It can be shown that general harmonic functions have a predictable decay rate at infinity and that Lemma 5.6 actually holds in much more generality, but the machinery behind it lies beyond the focus of this thesis.

7 Necessity of $C^{1,\alpha}$ Domain Through the Introduction of the Lipschitz Case

Throughout the process of developing these results, the fact that the domain is assumed to be $C^{1,\alpha}$ was crucial. If the boundary of our domain Ω could no longer be thought of as $C^{1,\alpha}$, but as only Lipschitz, the methods from the previous sections would quickly break down.

Let us introduce the setup of the Dirichlet and Neumann for such domains, and in the process we will encounter certain concessions we must make in order to make sense of the problem in this new context of lower boundary regularity. We will begin by recalling the definition of a Lipschitz continuity and then defining what is meant by a Lipschitz domain.

Definition 7.1 (Lipschitz Function)**.** *A function $\gamma : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz if there exists a $C > 0$ such that for all $x, y \in \mathbb{R}^n$*

$$|\gamma(x) - \gamma(y)| \leq C|x - y|.$$

Definition 7.2 (Lipschitz Domain)**.** *A set $\Omega \subseteq \mathbb{R}^{n+1}$ is a Lipschitz domain if the boundary $\partial\Omega$ can be locally represented as the graph of a Lipschitz function, i.e. for every $P^* \in \partial\Omega$ there exists a coordinate system (after potenially rotating or translating the domain) and a Lipschitz function $\gamma : X \subseteq \mathbb{R}^n \to \mathbb{R}$ such that for some $r > 0$, $\Omega \cap B_r(P^*) = \{(x, y) \in B_r(P^*) : y > \gamma(x)\}$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$.*

In order to simplify the statement and subsequent explanation, we will only be considering the case where the whole boundary can be expressed as the graph of a single Lipschitz function. This type of domain is called a special Lipschitz domain. Note that we are no longer requiring that our domain be bounded, so it gives us no problems to express the boundary in this way.

Definition 7.3 (Special Lipschitz Domain). *A domain Ω is a Special Lipschitz Domain if $\Omega = \{(x, y) : y > \gamma(x)\}$ and $\partial\Omega = \{(x, y) : y = \gamma(x)\}$ for a Lipschitz function γ.*

If we recall from the previous sections, the $C^{1,\alpha}$ regularity of Ω played a crucial role and was relied on heavily. In this section, we will highlight the key uses of this property of the boundary, and then introduce how this can be dealt with in the Lipschitz case.

7.1 Existence of Normal to $\partial\Omega$

In the $C^{1,\alpha}$ case, we were able to assume that for every point on the boundary $P^* \in \partial\Omega$, we could find a vector normal to $\partial\Omega$ at P^*, and even calculate it in terms of local coordinates. The Neumann BVP is stated in terms of normal derivatives, and the kernel of the Double Layer Potential and the associated adjoint operator T^* when defined on the boundary both involve normal derivatives. The first immediate complication that arises when considering the problem on a Lipschitz domain is that a Lipschitz boundary could have corners or cusps, so the normal to $\partial\Omega$ exists only almost everywhere (with respect to the surface measure σ of $\partial\Omega$), meaning there could be a measure zero subset of $\partial\Omega$ from which we cannot define a normal vector. As a result, we are often only able to guarantee the existence of limits almost everywhere. We will see that in the Dirichlet and Neumann result for Lipschitz domains, we require that the solution u is a.e. "equal" to the boundary data f.

7.2 Existence of Limits

In the $C^{1,\alpha}$ case, limits of functions to the boundary of $\partial\Omega$ existed with regard to our usual definition of the limit, i.e. the limit of f at P exists if the limit to P from every direction agree. We got a hint that the regularity of the boundary facilitated the existence of limits on the boundary from Propostition 3.18, which relied on the fact that for P close enough to $\partial\Omega$, we can find a $P^* \in \partial\Omega$ such that $P = P^* + tn(P^*)$, a property of $C^{1,\alpha}$ domains.

In the Lipschitz case, the usual conditions for the existence of limits are too strong, so we introduce a weaker notion of a limit, the Non-Tangential Limit. We will see that this is a weaker limit to the usual understanding of a limit because it only requires the limit to exist when approaching "non-tangentially", i.e. not too close to the boundary. These allowed regions of approach are formalized in the following definition.

Definition 7.4 (Non-Tangential Cone). *A cone Γ with vertex $P^* \in \partial\Omega$ is non-tangential if there exists a cone Γ' and a $\delta > 0$ such that*

$$\emptyset \neq (\overline{\Gamma} \cap B_\delta(P^*)) \setminus \{P^*\} \subseteq (\Gamma' \cap B_\delta(P^*)) \subseteq \Omega.$$

Definition 7.5 (Non-Tangential Limit). *Let $f : \Omega \to \mathbb{R}$. If for every non-tangential cone Γ, $f(Q) \to L$ for as $Q \to P$ where $Q \in \Gamma$, then L is the non-tangential limit of f at P.*

By diregarding the limits approaching very near along the boundary, the non-tangential limit is weaker, but within the context of Lipschitz domains is necessesary to make sense of otherwise non-existent limits.

We can create an explicit family of non-tangential cones in the following way, where for $\beta > 1$

$$\Gamma_\beta(P^*) = \{P \in \Omega : |P - P^*| < \beta \mathrm{dist}(P, \partial\Omega)\}$$

is a non-tangential cone. We can see that points that are closer to the boundary $\partial\Omega$ than to the vertex P^* are excluded. The higher the β, the wider the cone will open, meaning it includes points closer to the boundary.

From here we can define a type of maximal function for points on the boundary.

Definition 7.6 (Non-Tangential Maximal Function)**.** *Let $f : \Omega \to \mathbb{R}$, where Ω is a Lipschitz domain. The non-tangential maximal function of f at $P^* \in \partial\Omega$ is given by*

$$M_\beta f(P^*) = \sup\{f(Q) : |P^* - P| < \beta \mathit{dist}(P, \partial\Omega),\ P \in \Omega\}.$$

7.3 Severity of Singularities

Another key use of the regularity of the boundary in the $C^{1,\alpha}$ case was in tempering the severity of singularities, most often those of the kernel of the Double Layer Potential integral operator. Using Lemma 3.8, we were able to bound the kernel K as follows

$$|K(P, Q^*)| \leq \frac{C}{|P - Q^*|^{n-1-\alpha}},$$

which we were then able to integrate with ease thanks to the $\alpha > 0$.

In the Lipschitz case, the best we can do is

$$|K(P, Q^*)| \leq \frac{C}{|P - Q^*|^{n-1}},$$

which proves troublesome when we try to integrate it as the kernel of the Double Layer Potential. However, through the introduction and study of so-called Principal Value Operators with Calderón-Zygmund type kernels, as done in [6], one can show that finite non-tangential limit of the Double Layer Potential at the boundary $\partial\Omega$ exists almost everywhere.

7.4 Statement of Results for Special Lipschitz Domains

With these concessions, and an introduction to the concepts needed to address them, we are able to state the analogous results to the Dirichlet and Neumann boundary value problems for the case of the special Lipschitz domain. Note that they are not split up into interior and exterior cases since special Lipschitz domains are necessarily unbounded and the notion of interior versus exterior carry less weight as a result.

Proposition 7.7 (Dirichlet Boundary Problem for Lipschitz Domain)**.** *For a special Lipschitz domain Ω and $f \in L^2(\partial\Omega)$, there exists a solution u to the boundary value problem*

$$\begin{cases} \Delta u = 0 & \textit{on } \Omega \\ u = f & \textit{on } \partial\Omega. \end{cases}$$

where the boundary values are taken a.e. in a non-tangential sense and when restricted to the boundary, $M_\beta u \in L^2(\partial\Omega)$ with $\|M_\beta u\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}$ where the constant C only depends on β and the characteristics of the domain Ω.

Proposition 7.8 (Neumann Boundary Problem for Lipschitz Domain)**.** *For a special Lipschitz domain Ω and $f \in L^2(\partial\Omega)$, there exists a solution u to the boundary value problem*

$$\begin{cases} \Delta u = 0 & \textit{on } \Omega \\ \frac{\partial u}{\partial n} = f & \textit{on } \partial\Omega. \end{cases}$$

where the boundary values are understood as $n(P^*) \cdot \nabla u(P) \to f(P^*)$ as $P \to P^*$ non tangentially a.e. on $\partial\Omega$ and $M_\beta(\nabla u) \in L^2(\partial\Omega)$ with $\|M_\beta(\nabla u)\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}$ where the constant C only depends on β and the characteristics of the domain Ω.

7.5 Comments on the Methods used for Proving the Lipschitz Case

The intention of this section is not to prove the results above, for that is outside the scope of this thesis, but to highlight the role that the boundary regularity of Ω plays in the strategies used to work towards these results. See [6] for a more complete treatment of this case. The methods used in the Lipschitz case are quite a bit more involved than in the $C^{1,\alpha}$ case, but builds on the theory and strategies from it. The Double Layer Potential, though not so easily thought of as an integral operator defined on $\partial\Omega$, is still a central concept, and remarkably familiar jump relations, that hold almost everywhere, are developed and used to describe behavior at the boundary.

References

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Corrections:

Page 2:

- Proposition 2.3: The expression for the normal vector should be written in terms of $\gamma(x_0)$ instead of $\gamma(P^*)$.

$$n(P^*) = \frac{(\nabla\gamma(x_0), -1)}{\sqrt{1+|\nabla\gamma(x_0)|^2}}\,.$$

Page 6:

- Section 3.1: Working this section in R^n resulted in confusion and errors when applying the results to $\mathbb{R}^{n+1}$. The section has been re-written to apply to $\mathbb{R}^{n+1}$ directly.

Newtonian Kernel

Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be a harmonic function, i.e. $\Delta F = 0$. To see what form F will possibly take, we will first assume that the solution is radial, meaning that there exist a function $R : \mathbb{R} \to \mathbb{R}$ such that F can can be written as $F(P) = R(r)$ where $r = |P| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$. We first calculate $\frac{\partial r}{\partial x_i}$ as follows

$$\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\sqrt{x_1^2+\cdots+x_{n+1}^2}\right) = \frac{2x_i}{2\sqrt{x_1^2+\cdots+x_{n+1}^2}} = \frac{x_i}{r}\,.$$

Using the chain rule, we calculate $\frac{\partial F}{\partial x_i}$, $\frac{\partial^2 F}{\partial x_i^2}$, and ΔF to be

$$\begin{aligned}
\frac{\partial F}{\partial x_i} &= \frac{\partial R}{\partial r}\frac{\partial r}{\partial x_i} = R'(r)\frac{x_i}{r}\\
\frac{\partial^2 F}{\partial x_i^2} &= \frac{R''(r)}{r^2}x_i^2 - \frac{R'(r)x_i^2}{r^3} + \frac{R'(r)}{r}\\
\Delta F &= \sum_{i=1}^{n+1}\frac{\partial^2 F}{\partial x_i^2} = R''(r) - \frac{R'(r)}{r} + (n+1)\frac{R'(r)}{r}\,.
\end{aligned}$$

Since we assumed F to be harmonic, we have that

$$\Delta F = R''(r) + \frac{n}{r}R'(r) = 0\,.$$

From this, we obtain the second order ODE

$$\frac{R''(r)}{R'(r)} = \frac{n}{r}\,,$$

the solution of which is

$$R(r) = \begin{cases} A\ln(r) + B \text{ if } n = 1\\ \frac{A}{(2-n)r^{n-2}} + B \text{ if } n \geq 2\,. \end{cases}$$

If we take $n \geq 2$, $r = |P - Q|$, and choose convenient constants, we arrive at what is called the Newtonian Kernel.

Definition 0.1 (Newtonian Kernel). *For $n \geq 2$, $P, Q \in \mathbb{R}^{n+1}$, the Newtonian Kernel is given by*

$$N(x, y) = \frac{-1}{(n-2)w_n} \frac{1}{|P - Q|^{n-2}}$$

where w_n is the surface area of an n-dimensional unit ball.

From the initial derivation of F, we can see that that $N(P, Q) = F(P - Q)$ should be harmonic with respect to P and Q when $P \neq Q$. In addition, the gradient of the Newtonian kernel will be used frequently in this section, so it has been calculated and now stated for future reference

$$\nabla_Q N(P, Q) = \frac{-1}{w_n} \frac{P - Q}{|P - Q|^n} .$$

Page 8:

- Proof of Proposition 3.4: The ball $B_\eta(P_0)$ that shows up in regions of integration should be $B_\eta(P^*)$.