SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

The Halperin conjecture

av

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2025 - No M9

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Alexander Berglund

2025

Abstract

The Halperin conjecture is a long-standing open problem in algebraic topology, stating that the rational Serre spectral sequence of orientable fibrations whose fibre is a positively elliptic space degenerates at the E_2 page. In this thesis, we carry out a survey of the conjecture, studying its background, various alternative formulations and the current state of research. Concretely, we first review the basics of fibrations, spectral sequences and rational homotopy theory as a basis for the rest of the work. Then, we move on to state two re-phrasings of the conjecture and show their equivalence with the original one. Our main contribution is to give complete proofs of these equivalences, whose details had been partially omitted in the literature. Especially relevant is the algebraic formulation, which states that the cohomology algebra of a positively elliptic space does not admit non-zero derivations of negative degree. Later, we present a series of specific cases for which the conjecture has been proved and review the techniques used in the latest publications on the topic. We conclude with counterexamples showing that the statement of the conjecture is sharp.

Sammanfattning

Halperins förmodan är ett långvarigt olöst problem inom algebraisk topologi, som postulerar att den rationella Serre-spektralsekvensen för en orienterbar fibration, vars fiber utgörs av ett positivt elliptiskt rum, degenererar på E_2-sidan. I detta arbete ger vi en översikt av förmodan och studerar dess bakgrund, olika alternativa formuleringar och det aktuella forskningsläget. Först repeterar vi de grundläggande begreppen inom fibrationer, spektralsekvenser och rationell homotopiteori, vilka resten av arbetet bygger på. Därefter beskriver vi två alternativa formuleringar av förmodan och bevisar deras ekvivalens med den ursprungliga. Vårt huvudsakliga bidrag är att ge fullständiga bevis för dessa ekvivalenser, vars detaljer delvis utelämnats i litteraturen. Väldigt relevant är den algebraiska versionen, som innebär att kohomologialgebran för ett positivt elliptiskt rum inte medger några icke-triviala derivationer av negativ grad. Vidare presenterar vi en serie specifika fall där förmodan har bevisats och granskar de tekniker som använts i de senaste publikationerna inom ämnet. Vi avslutar med motexempel som visar att villkoren i förmodan är skarpa.

Acknowledgements

This thesis means the final stage of my education as a mathematician over the last seven years. While part of the work has been done on my own, there are (hopefully not uncountably) many people that have helped me reach this point and who deserve some words of gratitude.

First of all, I need to thank Alexander Berglund for being an excellent supervisor in the past few months. His magnificent teaching of the algebraic topology course confirmed my interest in this beautiful field, and later he was truly right when proposing the interesting Halperin conjecture as the topic for my thesis. During our long discussions, he was always pleased to solve my multiple doubts and suggest new ideas and changes to the text, making this work far better than it would have been otherwise.

Igualmente, me es obligado recordar a mis profesores en la Universidad de Cantabria, ya que sin ellos nunca habría llegado a esta segunda etapa de mis estudios. En especial, gracias a Javier, Juanma, Luis y Nuria, tanto por su contribución en el pasado como por su presencia y apoyo cuando he acudido a ellos más adelante. También a Bea y Teresa por dar el banderazo de salida.

Gracias a Alberto por aguantarme día tras día en Albano y ser mi trocito de España en el extranjero. In the same manner, thanks to Andrea, Celia, Fabian, Giacomo, Jette, Maret and Yann for making Stockholm homelier than I could have imagined. Merci à los payasos for all the adventures we had in Switzerland, especially Jonathan and Valeria for being part of the Swedish gang and Ala for her constant support no matter the circumstances. Dziękuję i kocham cię.

A José Ángel por tantas conversaciones y cuadraturas; a Manu y Pablo por las llamadas olímpicas eternas; y a Álex y Joseba por quedar en Santander siempre que nos es posible.

Por último, gracias a mi familia, y en particular a mis padres, por su apoyo incodicional y por ser los principales responsables de que me haya convertido en la persona y el estudiante que soy.

Villaprovedo, agosto de 2025.

Contents

1 Introduction

The Halperin conjecture is a long-standing conjecture in algebraic topology, concretely in rational homotopy theory. It was proposed by S. Halperin in 1976 and then featured as the first of seventeen open problems in the field in [FHT01, §39]. The statement of the conjecture is the following.

Conjecture (Halperin conjecture)**.** *Let $F \to E \to B$ be a fibration of simply connected spaces such that F has finite-dimensional rational homology, finite-dimensional rational homotopy and positive Euler characteristic. Then, the Serre spectral sequence with rational coefficients degenerates at the E_2 page.*

A simply-connected space F with these properties is generally called *positively elliptic*.

Although the conjecture remains open, it has been established for various specific cases, which include some of the most common settings. For instance, considering geometric constraints, the claim holds when the fibre is a Kähler manifold [Bla56] or a homogeneous space [ST87]. In turn, the statement is known for certain forms of the cohomology algebra of the fibre, namely when it has at most three generators [Tho81, Lup90], is reduced [PP96], has relations of large degree [CYZ19] or formal dimension at most 20 [KW23].

Interestingly, the conjecture admits several equivalent formulations. First, for fibrations of simply connected spaces (or, more generally, *orientable* fibrations), the Serre spectral sequence degenerates at the E_2 page if and only if the fibration is *totally non-cohomologus to zero* (TNCZ), i.e. the map $H^*(E) \to H^*(F)$ induced in cohomology is surjective. Hence, an alternative statement is that every orientable fibration with a positively elliptic fibre is TNCZ. Two more elaborate formulations were obtained by W. Meier [Mei82], when proving that the following properties are equivalent for a positively elliptic space F:

(1) Every orientable fibration $F \to E \to B$ is TNCZ.

(2) Every derivation of negative degree in $H^*(F;\mathbb{Q})$ is zero.

(3) $\pi_{2k} \operatorname{aut} F \otimes \mathbb{Q} = 0$ for all $k > 0$.

Here $\operatorname{aut} F$ denotes the space of self-homotopy equivalences of F. The statement in terms of property (2) is known as the *algebraic* Halperin conjecture, and has been used to show most of the known cases listed above. Concretely, it states that the cohomology algebra of a positively elliptic space does not admit non-trivial derivations of negative degree. Another relevant re-phrasing was given by G. Lupton [Lup98] in terms of formality relations between the spaces B and E.

The goal of this work is to provide a survey of the Halperin conjecture and serve as a relevant introductory source to the topic. In addition, the text features two original proofs of known results whose details had been omitted in the literature. In this way, the main contribution of the thesis is to give a complete proof of the equivalence between the statements (1), (2) and (3) above. The text is addressed to readers with at least some basic algebraic topology knowledge, who should be able to acquire a clear understanding of the statement of the conjecture, its equivalent formulations and the current state of research. Concretely, we expect the reader to be familiar with the main concepts in homotopy theory, (co)homology and homological algebra. Basic knowledge in commutative algebra and category theory is also assumed.

The main body of the thesis consists of three chapters. In Chapter 2 we review the preliminaries needed to understand the conjecture and prove the main results of subsequent chapters. We begin with the basics of fibrations, including essential concepts such as orientability, homotopy fibres and Stasheff's classification theorem. Next, we discuss properties of algebras, eventually focusing on differential graded algebras and derivations thereof. This allows to introduce spectral sequences, and in particular the Serre spectral sequence associated to a fibration. Lastly, we give a brief introduction to rational homotopy theory and present the minimal model of a simply connected space, a fundamental tool in the theory. Aiming for conciseness, we omit most proofs of the properties reviewed, since they are often covered by standard sources. Nonetheless, we do include some which provide arguments or constructions with relevance later in the work.

In Chapter 3 we prove the equivalence between the properties (1), (2) and (3) above, leading to two reformulations of the Halperin conjecture. To start with, we define positively elliptic spaces and algebras, and prove a classification theorem for orientable fibrations. Then, in Section 3.1, we show (1) $\Leftrightarrow$ (2) when F is a positively elliptic space, yielding the algebraic formulation. Theorem 3.7 provides the more direct implication (2) $\Rightarrow$ (1). In turn, through a series of auxiliary lemmas, we prove the converse direction in Theorem 3.12. This result was stated in [Mei82], but the proof skipped some non-trivial steps that are completed here. We conclude the chapter with Section 3.2, where Proposition 3.18 yields the equivalence (2) $\Leftrightarrow$ (3). Again, we give an alternative proof of this result.

Finally, in Chapter 4 we discuss four cases for which the conjecture is known to hold. We have chosen settings with relevance in other areas of mathematics (Kähler manifolds, homogeneous spaces) or milestones towards a full proof (algebras with at most three generators or formal dimension up to 20), and for which the proofs can be sketched in a couple of pages. In this way, rather than providing complete arguments, we aim to describe the examples and exhibit how the constraints imposed lead to the conclusion in each case. In the final section, we show that counterexamples arise when one assumption at a time is relaxed, demonstrating that the conditions in the statement of the conjecture are sharp.

Conventions and notation

We will use the standard notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ for the basic number sets. Here, $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_{>n}$ denotes the set of natural numbers larger than n. The closed interval $[0, 1] \subseteq \mathbb{R}$ is denoted by I.

Throughout the work, rings are assumed to be commutative with unit and denoted by R, whereas fields are assumed to have characteristic zero and denoted by $\Bbbk$. Note that the Halperin conjecture is stated in terms of rational properties of the fibre, so we will eventually work over the field $\mathbb{Q}$. Thus, we consider objects (modules, algebras) over an arbitrary ring R as long as the greater generality does not involve more complexity, and we specialize to a field $\Bbbk$ or even $\mathbb{Q}$ when this allows to establish simpler or additional structures on the objects.

When taking tensor products, we indicate explicitly the base ring only if there is room for ambiguity. Otherwise, tensor products of modules are taken over the underlying ring, and tensor products of abelian groups are taken over the integers.

By space we mean a topological space with the homotopy type of a finite CW complex. Note that, as a consequence, all spaces have finite-dimensional (co)homology. Maps between spaces are assumed to be continuous. We write $f \simeq g$ when two maps f and g are homotopic. Likewise, given two spaces X and Y, we write $X \sim Y$ when they are weakly homotopy equivalent, $X \simeq Y$ when they are homotopy equivalent, and $X \cong Y$ when they are homeomorphic. More generally, the same notation is used for weak equivalences, homotopy equivalences and isomorphisms in any category with some notion of homotopy, e.g. that of cdg algebras. Finally, $[X, Y]$ denotes the set of homotopy classes of maps (or in general morphisms) $X \to Y$.

Given an object X in any category, we write $\mathrm{Aut}(X)$ for the group of isomorphisms $X \to X$. The identity morphism on X is denoted by $\mathbb{1}_X$ or simply $\mathbb{1}$. In turn, if X is a topological space, we will use $\mathrm{aut}(X)$ for the space of self-homotopy equivalences $X \to X$, which is not a group in general.

Finally, when citing references, we use the abbreviations *Apx.* (Appendix), *App.* (Application), *Cor.* (Corollary), *Ex.* (Example), *Prop.* (Proposition) and *Thm.* (Theorem). We use the symbol § to indicate the highest subdivision of a work, usually *Chapter* or *Section* depending on the type of publication.

Now we are ready start.

2 Preliminaries

2.1 Fibrations

Fibrations are the central object in the statement of the Halperin conjecture. Let us start by giving a formal definition of this notion.

Definition 2.1. A map $\pi : E \to B$ between topological spaces is said to satisfy the **homotopy lifting property** for a space X if the following holds: for any map $s : X \to E$ and homotopy $H : X \times I \to B$ such that $\pi s(x) = H(x, 0)$ for $x \in X$, the homotopy H can be lifted to a homotopy $\tilde{H} : X \times I \to E$, that is $\pi\tilde{H} = H$ and $\tilde{H}(\mathbb{1} \times 0) = s$,

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "\mathbb{1}\times 0"'] & E \arrow[d, "\pi"] \\ X \times I \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B. \end{tikzcd}$$

A **Hurewicz fibration** is a map $\pi : E \to B$ satisfying the homotopy lifting property for all spaces. In turn, a **Serre fibration** is a map $\pi : E \to B$ satisfying the homotopy lifting property for all CW complexes, or equivalently for all disks, as discussed in [Hat$_1$, p. 376].

Henceforth we will refer to Hurewicz fibrations just as **fibrations**. We call B the **base** (**space**) and E the **total space** of the fibration π.

One basic example of a fibration is the projection $Y \times Z \to Y$. With X as above, the lift of a homotopy $H : X \times I \to Y$ is given by $\tilde{H}(x, t) = (H(x, t), s_2(x))$, where s_2 is the second component of the map $s = (s_1, s_2) : X \to Y \times Z$. It is a known result that covering maps satisfy the homotopy lifting property for any space, providing a second example.

A more interesting one is given by **fibre bundles**. Recall that a fibre bundle is a surjection $\pi : E \to B$ such that each $b \in B$ has a neighbourhood U for which there is a homeomorphism $h : \pi^{-1}(U) \xrightarrow{\cong} U \times F$ with $\pi|_U = p_U h$, where $p_U : U \times F \to U$ is the projection. Here, the space F is called the fibre. Note that $\pi^{-1}(b) \cong F$ for each $b \in B$, and E can then be described as a "bundle of fibres" along B. We may denote the bundle as a sequence $F \hookrightarrow E \to B$. In general, we have that fibre bundles are Serre fibrations. Under further assumptions, namely if the base B is a paracompact space, then fibre bundles are actually Hurewicz fibrations (see [Hat$_1$, p. 379]). Notable examples of fibre bundles are the Hopf fibrations

$$S^0 \hookrightarrow S^1 \to S^1, \quad S^1 \hookrightarrow S^3 \to S^2, \quad S^3 \hookrightarrow S^7 \to S^4 \text{ and } S^7 \hookrightarrow S^{15} \to S^8.$$

For the details of the construction, see [GWZ86].

In a fibre bundle, we can speak of *the* fibre because all fibres are homeomorphic. This need not be the case for a fibration π. Instead, we define pointwise the **fibre** of $b \in B$ as $F_b = \pi^{-1}(b)$. Nonetheless, we do have a weaker statement relating the fibres over the same path component at the level of homotopy (see [Hat$_1$, Prop. 4.61]). Although it is a standard result, we give an outline of the proof since it provides a construction for the action of the fundamental group of the base on the cohomology of the fibre.

Proposition 2.2. *For a fibration $\pi : E \to B$, the fibres F_b over each path component of B are homotopy equivalent.*

Proof (outline). For every path $\gamma : I \to B$ we can construct a homotopy $H : F_{\gamma(0)} \times I \to B$ by setting $H(x,t) = \gamma(t)$ for all $x \in F_{\gamma(0)}$. Together with the inclusion $F_{\gamma(0)} \hookrightarrow E$, we have a diagram like the one in Definition 2.1, with $F_{\gamma(0)}$ in the role of X. Thus, the lifting property provides a homotopy $\tilde{H} : F_{\gamma(0)} \times I \to E$. By commutativity, $\tilde{H}(-,1)$ is a map $L_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$. The association $\gamma \mapsto L_\gamma$ has the following properties:

(1) If $\gamma \simeq \gamma'$ relative to ∂I, then $L_\gamma \simeq L_{\gamma'}$.

(2) For a concatenation $\gamma\gamma'$, we have $L_{\gamma\gamma'} \simeq L_{\gamma'} L_\gamma$.

One can show these by repeatedly using the homotopy lifting property. The details can be consulted in the given reference.

Now, it follows that L_γ is a homotopy equivalence. Indeed, taking γ' as the inverse path of γ, we have $L_{\gamma'} L_\gamma \simeq L_{\gamma'\gamma} \simeq L_* = \mathbb{1}_{F_{\gamma(0)}}$, using (2) and (1), respectively. Here $*$ denotes the constant path at $\gamma(0)$. Arguing similarly in the opposite order, we conclude that $L_{\gamma'}$ and L_γ are homotopy inverses. $\square$

With this, if B is path-connected we may define the fibre F of a fibration as a representative of the homotopy type of F_b. Thus, picking a base point $b_0 \in B$, we can set $F = F_{b_0}$ and write the fibration as $F \hookrightarrow E \to B$. *Henceforth we assume that B and F are path-connected spaces.* The lifting property implies that then E is path-connected as well. It may be equally reasonable to consider fibrations with disconnected fibres, allowing for common examples like non-trivial covering maps. However, the Halperin conjecture being stated for simply connected spaces, we do not need that generality and will later state results requiring the assumption.

In the proof of the proposition, we associate a homotopy equivalence L_γ to each path γ in B. If γ is a loop on the base point b_0, we get a homotopy equivalence $L_\gamma : F \to F$, which induces an isomorphism in cohomology. This yields an **action** of $\pi_1(B)$ on $H^*(F; R)$, where R is any ring. The first use of this action is the following definition.

Definition 2.3. A fibration $F \to E \to B$ is **orientable** (over the ring R) if the action of $\pi_1(B)$ on $H^*(F; R)$ is trivial.

Notice that every fibration of simply connected spaces is orientable. Hence, in the context of the Halperin conjecture, we will eventually focus on this class of fibrations. In particular, orientable fibrations allow for a simpler expression of the Serre spectral sequence, as we discuss in Section 2.3.

Following Proposition 2.2, we can define maps between fibrations. Consider fibrations $\pi_0 : E_0 \to B_0$ and $\pi_1 : E_1 \to B_1$, and a pair of maps $f : B_0 \to B_1$ and $g : E_0 \to E_1$ such that $f\pi_0 = \pi_1 g$. We pick base points $b \in B_0$ and $b' := f(b) \in B_1$, and define fibres $F_0 = \pi_0^{-1}(b)$ and $F_1 = \pi_1^{-1}(b')$. Then g induces a map $F_0 \to F_1$, since for any $e \in F_0$, we have $\pi_1(g(e)) = f(\pi_0(e)) = f(b) = b'$, thus $g(e) \in \pi_1^{-1}(b') = F_1$. The following commutative diagram represents the resulting map of fibrations.

$$\begin{tikzcd}
F_0 \arrow[r, hook] \arrow[d, dashed] & E_0 \arrow[r, "\pi_0"] \arrow[d, "g"] & B_0 \arrow[d, "f"] \\
F_1 \arrow[r, hook] & E_1 \arrow[r, "\pi_1"] & B_1
\end{tikzcd}$$

The next result describes how a map reaching the base space of a fibration π induces a new fibration, called the *pullback* of π.

Proposition 2.4. *Given a fibration $\pi : E \to B$ and a map $f : B' \to B$, consider the following pullback diagram in the category of topological spaces*

$$\begin{tikzcd}
E' \arrow[r, dashed, "g"] \arrow[d, dashed, "\pi'"'] & E \arrow[d, "\pi"] \\
B' \arrow[r, "f"] & B
\end{tikzcd}$$

Then π' is a fibration, and the map $h : F' \to F$ induced between the fibres is a homeomorphism.

Proof. Consider a space X and maps s and H as in the diagram

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "\mathbb{1}\times 0"'] & E' \arrow[r, "g"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\
X \times I \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}'"] \arrow[urr, dashed, "\tilde{H}"'] & B' \arrow[r, "f"] & B
\end{tikzcd}$$

extended with the pullback defining π'. Since π is a fibration, we get a homotopy $\tilde{H} : X \times I \to E$. With this, we have maps $\tilde{H}$ and H from $X \times I$ to E and B', respectively, such that $\pi\tilde{H} = fH$. The universal property of pullbacks gives a map $\tilde{H}' : X \times I \to E'$ such that $\pi'\tilde{H}' = H$. Furthermore, notice that there are maps gs and $H(\mathbb{1} \times 0)$ from X to E and B' with $\pi gs = fH(\mathbb{1} \times 0)$, and both s and $\tilde{H}'(\mathbb{1} \times 0)$ are maps $X \to E'$ making commutative the corresponding triangles. By uniqueness in the universal property, we deduce that $\tilde{H}'(\mathbb{1} \times 0) = s$.

For the second claim, note that E' has the form $\{(e, x) \in E \times B' \mid \pi(e) = f(x)\}$, and π' maps $(e, x) \mapsto x$. Then, picking base points $b' \in B'$ and $b \in B$,

$$F' = (\pi')^{-1}(b') = \{(e, x) \in E \times \{b'\} \mid \pi(e) = f(b')\} = F \times \{b'\} \cong F,$$

since $f(b') = b$. $\square$

Next, we present one of the most relevant properties of fibrations: the arising of a long exact sequence of homotopy groups. This result holds not only for Hurewicz fibrations, but also for Serre fibrations [Hat_1, Thm. 4.41]. In whole generality, it is stated as follows.

Theorem 2.5. *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a Serre fibration. Let $b \in B$ and $e \in F = \pi^{-1}(b)$. Then, there is an exact sequence of homotopy groups*

$$\begin{aligned} \dots \to \pi_n(F, e) \xrightarrow{i_*} \pi_n(E, e) \xrightarrow{\pi_*} \pi_n(B, b) \to \pi_{n-1}(F, e) \to \dots \\ \dots \to \pi_0(F, e) \to \pi_0(E, e) \to 0. \end{aligned}$$

Observe that this is not an exact sequence of abelian groups as in homology, since π_0 and π_1 need not be such. The inclusion of π_1 is justified because exactness is defined for groups in general. In turn, one can define exact sequences of pointed sets to address the case of π_0. However, this has no relevance in our setting of path-connected spaces, as the sets π_0 are singletons and the sequence stops at the fundamental groups. Moreover, under this assumption the homotopy groups are independent of the base points chosen, so we will omit them from now on.

Often, it is precisely this property of fibrations that one is interested in. In fact, one can associate a fibration to any arbitrary map $f : A \to B$ preserving the homotopy types, allowing to obtain a long exact sequence of homotopy groups containing the induced maps $\pi_n(A) \to \pi_n(B)$. Let us see the details of the construction.

We define E_f as the space of pairs (a, γ) where $a \in A$ and $\gamma : I \to B$ is a path such that $\gamma(0) = f(a)$. This space can be topologized as a subspace of $A \times B^I$, where B^I has the compact-open topology. Then, the map $\pi : E_f \to B$ given by $\pi(a, \gamma) = \gamma(1)$ is a fibration (see [Hat_1, Prop. 4.64]). If we regard A as the subspace of E consisting of pairs $(a, *)$, where $*$ represents the constant path at $f(a)$, then A is a deformation retract of E_f. The relevant retraction is obtained by restricting each path γ to the initial segment $\gamma_{[0,1-t]}$ for $t \in I$. In other words, we have a homotopy equivalence $g : A \to E_f$. Now, pick a base point $b \in B$ and consider the fibre $F = \pi^{-1}(b)$. The homotopy inverse of g allows to obtain a map $F \to A$, as in the diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "g"', "\simeq"] & B \\ F \arrow[ur, dashed] \arrow[r, hook] & E_f \arrow[ur, "\pi"'] & \end{tikzcd}$$

where the right triangle commutes and the left triangle commutes up to homotopy. The map f, or the composite $F \to A \to B$, is said to represent the fibration $\pi : E_f \to B$, and F is called the **homotopy fibre** of f. If f is already a fibration, the construction provides another fibration with base B whose fibre is homotopy equivalent to the fibre of f. Note that then the pair

$(\mathbb{1}_B, g)$ defines a map of fibrations. The relation between the two fibrations is generalized in the following definition.

Definition 2.6. Let B be a space and consider fibrations $\pi_0 : E_0 \to B$ and $\pi_1 : E_1 \to B$.

1. A map $f : E_0 \to E_1$ is **fibre-preserving** with respect to B if $\pi_1 f = \pi_0$.
2. A map $f : E_0 \to E_1$ is a **fibre homotopy equivalence** if it is fibre-preserving and a homotopy equivalence.

In the following, we use these notions to introduce the concept of *universal* fibration with a given fibre F, and give an explicit construction thereof in terms of F. We follow the ideas in the seminal papers of J. Stasheff [Sta63] and J. P. May [May75] and state two key results related to them.

First, one can check that the notion of fibre homotopy equivalence defines an equivalence relation. With that, given two spaces B and F, we consider the set $LF(B)$ of fibre homotopy equivalence classes of fibrations $E \to B$ with fibres of the homotopy type of F. This defines a contravariant functor $LF(-)$ from the category of CW complexes and homotopy classes of maps to the category of sets and functions. In addition, Stasheff constructed a space B_F that serves as a universal base for fibrations whose fibre is homotopy equivalent to F, in the sense of Theorem 2.7 below. The space B_F is itself the base of a **universal fibration** $E_F \xrightarrow{u} B_F$. Given a space X, recall that $[X, B_F]$ denotes the set of homotopy classes of maps $X \to B_F$. Like $LF(-)$, the functor $[-, B_F]$ maps CW complexes and homotopy classes of maps into the category of sets and functions. The main result relates both functors.

Theorem 2.7 (Classification theorem, [Sta63])**.** *The functors $[-, B_F]$ and $LF(-)$ are naturally equivalent.*

In other words, for a space B, every homotopy class of maps $B \to B_F$ corresponds precisely to one fibre homotopy equivalence class of fibrations with base B and fibre F, up to homotopy type. Similarly as in Proposition 2.4, the fibration associated to $f : B \to B_F$ can be obtained as a homotopy pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"'] & E_F \arrow[d, "u"] \\ B \arrow[r, "f"] & B_F \end{tikzcd}$$

Notice that the universal fibration is unique up to fibre homotopy type.

Later, an explicit construction of a universal fibration with fibre X was given in terms of the group of self-homotopy equivalences of X. First, May [May75] introduced the (two-sided) geometric bar construction for group-like topological monoids. A topological monoid G is a monoidal object in the category of topological spaces, and it is called group-like if the set of path components $\pi_0(G)$ forms a group. A canonical example is the based loop space ΩX of a pointed space (X, x), defined as the set of Moore loops based at x, for which $\pi_0 \Omega X = \pi_1(X, x)$ is obviously a group. Omitting the details, the bar construction associates to every group-like topological monoid G a connected space BG such that ΩBG and G are equivalent as topological monoids. The space BG is called the **classifying space** of G. Observe that the relation between $\pi_0 \Omega X$ and $\pi_1(X, x)$ holds also for higher homotopy groups, and thus $\pi_{i-1}(G) = \pi_i(BG)$ for $i > 0$. For simplicity, the relation is stated for path-connected spaces, avoiding the need of choosing base points.

In this case, the relevant topological monoid is the set $\operatorname{aut} X$ of homotopy equivalences $f : X \to X$ of a space X, endowed with the compact-open topology and composition of maps, with unit $\mathbb{1}_X$. Observe that $\pi_0 \operatorname{aut} X$ is the set of homotopy classes of self-homotopy equivalences of X, and by definition of homotopy equivalence, every class $[f]$ has an inverse $[g]$, so $\pi_0 \operatorname{aut} X$ is a group. This allows to construct the space $\operatorname{Baut} X$. Similarly, if we choose a base point $x \in X$, then the set $\operatorname{aut}^\bullet X$ of homotopy equivalences $X \to X$ fixing x is again a group-like topological monoid.

The construction of May provides the universal base for fibrations whose fibre is homotopy equivalent to F (see [DZ79, Prop. 4.1]).

Theorem 2.8. *The map* $\operatorname{Baut}^\bullet F \to \operatorname{Baut} F$ *represents the universal fibration with fibre* F.

2.2 Algebras

We recall that given a ring R, an R-algebra is an R-module M equipped with a compatible ring structure via a map $\psi : M \otimes M \to M$. We assume the details of the definition are known.

In algebraic topology, algebras arise naturally as the underlying structure of the cohomology of a space. Given a space X, its cohomology groups $H^i(X; R)$ with coefficients in a ring R are R-modules, and so is the direct sum $H^*(X; R) = \bigoplus_i H^i(X; R)$. The cup product defines a map $H^i(X; R) \otimes H^j(X; R) \to H^{i+j}(X; R)$, providing $H^*(X; R)$ with a ring structure, and thus converting it into an algebra.

Examples. These are cohomology rings of common spaces over a field $\Bbbk$.

1. By a degree argument, the cohomology of the sphere S^n is isomorphic to the algebra $\Bbbk[x]/(x^2)$, where x generates the n-th cohomology.

2. For the complex projective space [Hat_1, Ex. 3.12], we have $H^*(\mathbb{C}P^n; \Bbbk) \cong \Bbbk[x]/(x^{n+1})$. In infinite dimension, this becomes the ring of polynomials over $\Bbbk$, $H^*(\mathbb{C}P^\infty; \Bbbk) \cong \Bbbk[x]$.

3. If S is a compact connected surface of genus $g \geq 1$, then [Hat_1, Ex. 3.7]
$$H^*(S; \Bbbk) \cong \frac{\Bbbk[x_1, \ldots, x_g, y_1, \ldots, y_g]}{(x_i x_j, y_i y_j, x_i y_j + y_i x_j)}.$$

Later, under certain assumptions on the space X, we will characterize the structure of $H^*(X;\mathbb{Q})$ as a quotient like the ones above.

Next, we introduce two notions in the theory of algebras that are naturally present in the definition of the cohomology ring of a space: grading and differentials. In order to call $H^*(X;R)$ a ring, we have defined it as the direct sum of cohomology groups, allowing to add any pair of elements. In the context of algebraic topology, though, it is rare to add elements of different degrees. Thus, henceforth we adopt the convention that graded structures are simply a collection of objects indexed by their degree, on which we will eventually define a product generalising the cup product.

We use the index notation corresponding to cohomology theories, the homology version being dual.

Definition 2.9. Let R be a ring. A **graded module** over R is a collection of R-modules $A = \{A^n \mid n \in \mathbb{Z}\}$.

The modules A^n are called the graded components of A, and the elements of A^n are called (homogeneous) elements of degree n. If $a \in A^n$, we write $|a| = n$ for the degree. We will denote by A^{even} and A^{odd} the collections of modules of even and odd degree, respectively. A **morphism** $f : A \to B$ of graded modules is a collection of morphisms $f^n : A^n \to B^n$ for each $n \in \mathbb{Z}$. Next, given a graded module A and an integer k, let $A[k]$ denote the graded module having A^{n+k} as the component of degree n for $n \in \mathbb{Z}$.

Definition 2.10. A **differential graded** (dg) **module** is a graded module A together with a morphism $d : A \to A[k]$ such that $d \circ d = 0$, where k is a fixed integer called the **degree** of d. The morphism d is called the **differential**.

Like for homogeneous elements, we will write $|d|$ for the degree of the differential. In most cases, $|d| \in \{-1, 1\}$, and then A is a (co)chain complex of R-modules. We assume $|d| = 1$ from now on. Then, as for any cochain complex, we can form the cohomology of A and obtain a graded module $H(A) = \ker d / \operatorname{im} d$, with a trivial differential structure. A morphism $f : (A, d_A) \to (B, d_B)$ of dg modules is a morphism of graded modules compatible with differentials, that is, $f(A^n) \subseteq B^n$ for $n \in \mathbb{Z}$ and $fd_A = d_B f$.

The introduced structure can be extended to algebras. In the following, all tensor products are taken over the ring R. We define the tensor product of two dg modules (A, d_A) and (B, d_B) as $(A \otimes B, d_\otimes)$, where

$$(A \otimes B)^n = \bigoplus_{p+q=n} A^p \otimes B^q$$

and $d_\otimes$ satisfies the (graded) Leibniz rule, that is $d_\otimes(a \otimes b) = d_A(a) \otimes b + (-1)^{|a|} a \otimes d_B(b)$. One can check that $(A \otimes B, d_\otimes)$ is a dg module.

Next, recall that an algebra structure on a module A is given by a morphism $A \otimes A \to A$ such that the product is associative and multiplication by 1 is identity. We define differential graded algebras in the same fashion.

Definition 2.11. A **differential graded** (dg) **algebra** over a ring R is a differential graded module (A, d), together with a morphism of differential graded modules $\psi : (A \otimes A, d_\otimes) \to (A, d)$ satisfying associativity and the unit property.

In turn, graded algebras are defined in the obvious similar way, ignoring the differential structure. A dg algebra restricted to non-negative degrees is usually called a **cochain algebra**. By construction, the cohomology of a topological space is always of this kind. The **formal dimension** of a cochain algebra is the minimal $n \in \mathbb{N}$ such that $A^k = 0$ for $k > n$, if it exists.

We will denote the image of $a \otimes b$ in A by ab, as the morphism ψ represents the ring product in A. With this, since ψ is a morphism of dg modules, we have the relation $|ab| = |a| + |b|$. Furthermore, with this notation $d(ab) = \psi(d_\otimes(a \otimes b))$ and so the Leibniz rule holds for products in A. This ensures that the product in the graded module $H(A)$ is well-defined, and thus the cohomology of a dg algebra is a graded algebra.

In topological applications, it is common that the product in a graded algebra has the following additional property. For instance, this is the case of the cup product in cohomology. We refer to [FHT01, §3(b)] for more detailed definitions of the next concepts.

Definition 2.12. A graded algebra A is said to be **graded commutative** if $xy = (-1)^{|x||y|} yx$ for all $x, y \in A$.

In the rest of the text, we call these simply commutative algebras. We write cdg algebras for dg algebras which are commutative. Next, suppose that $R = \Bbbk$ is a field. Since we are assuming $\mathrm{char}(\Bbbk) = 0$, if A is a cdg $\Bbbk$-algebra then $x^2 = 0$ for elements x of odd degree. This is a relevant property when working with **free algebras.** To construct these, we start by forming the **tensor algebra** of a vector space V over $\Bbbk$ as

$$TV = \bigoplus_{k=0}^{\infty} T^k V,$$

where $T^0 V = \Bbbk$ and $T^k V = T^{k-1} V \otimes V$ for each $k > 0$. Multiplication $T^k V \otimes T^\ell V \to T^{k+\ell} V$ is given by the tensor product. Observe that an element $v_1 \otimes \cdots \otimes v_k \in T^k V$ has degree $\sum_{i=1}^k v_i$, whereas k is called its **word length**.

In the case of the free algebra, we are interested in the ideal I generated by the elements $v \otimes w - (-1)^{|v||w|} w \otimes v$, with $v, w \in V$.

Definition 2.13. The graded algebra $\Lambda V = TV/I$ is called the **free graded commutative algebra** on V.

The main properties of this construction can be consulted in [FHT01, p. 46]. In particular, ΛV is a graded commutative algebra. We write $\Lambda^k V$ for the elements of word length k, i.e. the vector space generated by products $v_1 \cdots v_k$ with $v_i \in V$. Abusing notation, we denote classes as elements and tensor products as regular products. Then, $\Lambda V = \bigoplus_{k\in\mathbb{N}} \Lambda^k V$. Note that these are not the graded components of ΛV. We also write $\Lambda^{\geq n} V = \bigoplus_{k\geq n} \Lambda^k V$. Given a basis $\{v_i \mid i \in I\}$ of V, we denote $\Lambda V = \Lambda(v_i)$. If $V = \Bbbk\langle v\rangle$ has dimension 1, then a basis of $\Lambda V = \Lambda v$ is given by $\{1, v\}$ if $|v|$ is odd and $\{1, v, v^2, \dots\}$ if $|v|$ is even. In the first case ΛV is an exterior algebra, while in the second it is a polynomial algebra.

To conclude the section, we introduce (differential) *bigraded* algebras, a similar notion to the one of dg algebras that we will need when dealing with spectral sequences. Like before, we first define a **bigraded module** $E^{*,*}$ over a ring R as a collection $\{E^{p,q} \mid p, q \in \mathbb{Z}\}$ such that each $E^{p,q}$ is an R-module. In the context of spectral sequences, we obtain a differential bigraded module by equipping $E^{*,*}$ with an R-linear differential d of bidegree $(s, 1-s)$ and such that $d^2 = 0$. Here s is an integer and the bidegree condition means that d maps $E^{p,q}$ into $E^{p+s,q+1-s}$ for an integer s. The **total degree** of an element in $E^{p,q}$ is the sum $p + q$. Thus, we can regard a bigraded module E as a graded module $\operatorname{tot} E$ with $(\operatorname{tot} E)^n = \bigoplus_{p+q=n} E^{p,q}$. Observe that d increases the total degree by 1, so acts as a differential in $\operatorname{tot} E$.

As in the graded case, we define the tensor product of differential bigraded modules as follows: given $(E^{*,*}, d_E)$ and $(F^{*,*}, d_F)$, we set

$$(E \otimes F)^{p,q} = \bigoplus_{\substack{r+t=p \\ s+u=q}} E^{r,s} \otimes F^{t,u}$$

and $d_\otimes(e \otimes f) = d_E(e) \otimes f + (-1)^{r+s} e \otimes d_F(f)$, for $e \in E^{r,s}$. Then $(E \otimes F)^{*,*}$ is a differential bigraded module.

Definition 2.14. A **differential bigraded algebra** over a ring R is a differential bigraded module $(E^{*,*}, d)$ together with a morphism of differential bigraded algebras $\psi : (E \otimes E)^{*,*} \to E^{*,*}$ satisfying associativity and the unit property.

2.2.1 Derivations of algebras

A derivation in an algebra is an endomorphism that mimics differentiation in a space of functions, namely satisfying the Leibniz rule. We start by defining a more general concept.

Definition 2.15. Let $f : A \to B$ be a morphism of dg algebras, and let k be an integer. A (homogeneous) f**-derivation** of degree k is a morphism of dg algebras $\theta : A \to B[k]$ satisfying the Leibniz rule, that is, for any $a, b \in A$,

$$\theta(ab) = \theta(a)f(b) + (-1)^{|\theta||a|} f(a)\theta(b).$$

Given a map $f : A \to B$, we let $\mathrm{Der}_f^k(A, B)$ denote the set of f-derivations of degree k, which can be checked to be an R-module. Then, the collection $\mathrm{Der}_f(A, B) = \{\mathrm{Der}_f^k(A, B) \mid k \in \mathbb{Z}\}$ forms a graded module. We may define a differential structure on it. Let d_A and d_B be the differentials of A and B, respectively, and define $[\delta, -] : \mathrm{Der}_f(A, B) \to \mathrm{Der}_f(A, B)$ by

$$[\delta, \theta] = d_B\theta - (-1)^{|\theta|}\theta d_A.$$

for each $\theta \in \mathrm{Der}_f(A, B)$. Observe that $[\delta, \theta]$ has degree $|\theta| + 1$, and

$$\begin{aligned}[\delta, [\delta, \theta]] &= d_B(d_B\theta - (-1)^{|\theta|}\theta d_A) - (-1)^{|\theta|+1}(d_B\theta - (-1)^{|\theta|}\theta d_A)d_A \\ &= d_B^2\theta - (-1)^{|\theta|}d_B\theta d_A - (-1)^{|\theta|+1}d_B\theta d_A + \theta d_A^2 = 0\end{aligned}$$

Hence $\mathrm{Der}_f(A, B)$ is a differential graded module, i.e. a cochain complex, and it makes sense to consider its cohomology. Note that cocycles are the derivations which (anti)commute with the differential, $d_B\theta = \pm\theta d_A$.

An example of a derivation is the differential d in a dg algebra, with the additional property that $d^2 = 0$. If $A = B$ and $f = \mathbb{1}_A$, then θ is called a derivation in A. This is the most common situation, but we provide the general definition as we will need it later. In that case, simplifying notation, we write $\mathrm{Der}(A)$ for $\mathrm{Der}_{\mathbb{1}}(A, A)$. Likewise, one can talk of derivations from A to B if it is clear which map f is involved, and write $\mathrm{Der}(A, B)$.

We have mentioned that derivations behave in some sense like derivatives of functions. In particular, if A is a cdg algebra, we have a familiar rule for derivations of powers of elements in A. Indeed, given $\theta \in \mathrm{Der}(A)$ and $x \in A$,

$$\theta(x^k) = \begin{cases} kx^{k-1}\theta(x) & \text{if } |x| \text{ is even,} \\ 0 & \text{if } |x| \text{ is odd.} \end{cases}$$

Note that, in the odd case, all powers x^k with $k > 1$ vanish. The even case can easily be shown by induction.

An advantage of the given definition of f-derivation is the possibility to consider functors $\mathrm{Der}_*(A, -)$ and $\mathrm{Der}_*(-, A)$ for a fixed dg algebra A. First, consider the category $\mathcal{A}$ of pairs (B, f), where B is a dg algebra and $f : A \to B$ a map of dg algebras. Morphisms $(B, f) \to (C, g)$ are maps of dg algebras $\varphi : B \to C$ such that the following diagram commutes

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B \arrow[rr, "\varphi"] & & C \end{tikzcd}$$

Then, $\mathrm{Der}_*(A, -)$ is a covariant functor from $\mathcal{A}$ to the category of dg modules, mapping (B, f) to $\mathrm{Der}_f(A, B)$ and $\varphi : B \to C$ to a morphism $\varphi_* : \mathrm{Der}_f(A, B) \to \mathrm{Der}_{\varphi f}(A, C)$ with $\varphi_*\theta = \varphi\theta$ for a derivation θ. The contravariant functor $\mathrm{Der}_*(-, A)$ is defined in the obvious similar way.

Remark 2.16. Observe that $\mathrm{Der}_f(A, B)$ has the structure of a left B-module, since an element $b \in B$ may act on a derivation θ by left multiplication. Suppose further that A is a free graded commutative algebra over a field $\Bbbk$, with generators $\{v_i \mid i \in I\}$. Using the Leibniz rule, an f-derivation $A \to B$ is determined by its values on generators. In other words, if $V = \Bbbk\langle v_i \rangle$, then we have an isomorphism between $\mathrm{Der}_f(A, B)$ and the space $\mathrm{Hom}(V, B)$ of morphisms of graded vector spaces $V \to B[k]$ for $k \in \mathbb{Z}$. Thus, in this case the space of derivations is a free B-module, and a basis is given by the derivations $\partial/\partial v_i$ mapping $v_i \mapsto 1$ and the remaining generators to 0.

2.3 Spectral sequences

Spectral sequences are a powerful computational tool in homological algebra. In the general setting, one has a mathematical object and wants to compute an algebraic invariant thereof which is difficult to obtain. Here, these will be a topological space and its cohomology. The notion was conceived by J. Leray as a war prisoner in the Second World War, in order to compute sheaf cohomology. The basic idea is to compute the homology of the total complex of a double complex through a perturbative process, by taking successive homologies of suitable chain complexes which eventually converge to the sought homology.

In this thesis we will be working with the Serre spectral sequence, one of the most notable spectral sequences in topology. Before getting into details, let us briefly discuss the construction and state the main result concerning this object. Starting with a fibration $F \to E \to B$ such that the cohomologies of F and B are known, we construct a *grid* denoted $E_2^{*,*}$, by setting $E_2^{p,q} = H^p(B; H^q(F; R))$ for $p, q > 0$ and R a fixed ring. Next, differentials are defined along the line of slope $-1/2$, and the corresponding cohomology allows to define a new grid $E_3^{*,*}$. Then, we have new differentials with slope $-2/3$, allowing to obtain $E_4^{*,*}$, and so on. Later we will call these grids the *pages* of the sequence. The relevance of this construction is due to the following result of Serre. We recall that all spaces are assumed to be path-connected.

Theorem 2.17. *Let $F \to E \to B$ be an orientable fibration. Then, there is a spectral sequence $\{E_r^{*,*}, d_r\}$ with*

$$E_2^{p,q} \cong H^p(B; H^q(F; R))$$

converging to $H^*(E; R)$. This sequence is known as the ***Serre spectral sequence*** *associated to the fibration $F \to E \to B$.*

One of the milestones of Serre's work was to specialize the Leray spectral sequence to singular (co)homology. Acknowledging the previous work of Leray, Serre's sequence is also called the Leray-Serre spectral sequence.

Moreover, the theorem may be stated in a more general version that does not assume that the fibration is orientable, i.e. that the action of $\pi_1(B)$ on $H^*(F;R)$ is trivial. In this general setting, the spectral sequence is given using cohomology with local coefficients, usually denoted by $E_2^{p,q} = H^p(B;\mathcal{H}^q(X;R))$. That said, here we are interested in orientable fibrations, so we can work with constant coefficients.

In the following pages we introduce the concepts involved in Serre's result. In particular, we define the general notion of spectral sequence and explain what it means for a spectral sequence to converge to a certain object. Like this, the inexperienced reader can review the basics of spectral sequences with the specific structure of Serre's sequence as a reference, which is also used in the examples on page 21. Having introduced differential bigraded modules and algebras in the previous section, we will first present spectral sequences as abstract algebraic notions subject to a series of axioms, and then exhibit their relation to topological objects. Over this section we follow McCleary's text [McC01]. A more didactic approach, starting with the topological construction, is carried out by Hatcher in [Hat_2].

Consider a differential bigraded module $(E^{*,*},d)$, where d has bidegree $(s,1-s)$. Given $p,q\in\mathbb{Z}$, there is a sequence

$$E^{p-s,q-1+s} \xrightarrow{d} E^{p,q} \xrightarrow{d} E^{p+s,q+1-s},$$

and we may set the cohomology of $E^{*,*}$ at the position (p,q) as

$$H^{p,q}(E^{*,*}) = {\ker\left(d : E^{p,q} \to E^{p+s,q+1-s}\right)}\Big/{\operatorname{im}\left(d : E^{p-s,q-1+s} \to E^{p,q}\right)}.$$

Definition 2.18. A **cohomology spectral sequence** is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r \mid r \in \mathbb{N}_{>0}\}$, such that the differentials have bidegree $(r,1-r)$ and $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*},d_r)$.

Here, the term spectral sequence will refer to cohomology spectral sequences. The R-module $(E_r^{*,*},d_r)$ is called the r-th term or r-th page, also denoted simply by E_r. Note that knowing a whole page, i.e. $E_r^{*,*}$ and d_r, determines $E_{r+1}^{*,*}$ but not the differential d_{r+1}. Actually, from an algebraic point of view, any differential is allowed at the next page. In practice, as we see in the examples, one can often deduce the differentials thanks to knowledge of some final cohomology. Finally, notice that in the definition the first term is E_1, but we can start at any positive integer n by setting $E_r = E_n$ and zero differentials for $r < n$. A usual setting is to begin with an explicit formula for $E_2^{*,*}$, like in the Serre spectral sequence.

Definition 2.19. A **morphism of spectral sequences** between $\{E_r^{*,*}, d_r\}$ and $\{E_r'^{*,*}, d_r'\}$ is a family of morphisms of differential bigraded modules $f_r : E_r^{*,*} \to E_r'^{*,*}$ such that f_{r+1} is the map induced by f_r in cohomology.

In order to exploit the power of spectral sequences, it is useful to consider the tower of submodules of the initial module that arises when computing the successive terms. Assume we start at E_2. For the sake of clarity, we omit the bigrading indicators. Consider the submodules

$$Z_2 = \ker d_2, \quad B_2 = \operatorname{im} d_2.$$

Since $d_2^2 = 0$, we have $B_2 \subseteq Z_2 \subseteq E_2$, with $E_3 = Z_2/B_2$. Now, note that $\bar{Z}_3 = \ker d_3$ is a submodule of a quotient of Z_2, and thus there exists a submodule Z_3 of Z_2 such that $\bar{Z}_3 = Z_3/B_2$. In the same way, for $\bar{B}_3 = \operatorname{im} d_3$, there exists $B_3 \subseteq Z_2$ such that $\bar{B}_3 = B_3/B_2$. Iterating this process, we get an infinite tower

$$B_2 \subseteq B_3 \subseteq \cdots \subseteq B_r \subseteq \cdots \subseteq Z_r \subseteq \cdots \subseteq Z_3 \subseteq Z_2 \subseteq E_2$$

where $E_{r+1} = Z_r/B_r$ for each $r \in \mathbb{N}_{\geq 2}$. We set

$$Z_\infty = \bigcup_{r=2}^{\infty} Z_r \quad \text{and} \quad B_\infty = \bigcap_{r=2}^{\infty} B_r,$$

with $B_\infty \subseteq Z_\infty$. The bigraded module $E_\infty := Z_\infty/B_\infty$ that remains after computing the infinite sequence of cohomologies is called the E_∞-page.

Certain spectral sequences, including Serre's, contain non-zero entries only for $p, q \geq 0$ at the starting term, say E_2. Note that this implies that E_r is restricted to non-negative degrees on both coordinates also for $r > 2$. These are called **first quadrant** spectral sequences. In particular, first quadrant spectral sequences are **bounded**, meaning that for every s, E_2 contains only finitely many non-zero entries of total degree s. It is not hard to check that, in a bounded sequence, for every position (p, q) there is a finite $n \in \mathbb{N}$ such that $E_r^{p,q}$ stabilizes at $E_\infty^{p,q}$ for $r \geq n$. Next, we present a stronger notion that is central in this work.

Definition 2.20. A spectral sequence $\{E_r^{*,*}, d_r\}$ **degenerates** at the n-th term (or at E_n) if $d_r = 0$ for all $r \geq n$.

Observe that if a spectral sequence $E_r^{*,*}$ degenerates at the n-th term, then all pages beyond n equal E_n, and we have $E_n = E_\infty$. Most applications involve spectral sequences that degenerate fast. Some authors write that the spectral sequence *collapses* at the n-th term if the previous condition is satisfied. I prefer to reserve the term *collapse* for the stronger condition that E_n has at most one non-zero row or column, which I find more descriptive.

In the final part of this introduction, we address the notion of convergence and state two related results with great importance in the context of spectral sequences. First, Theorem 2.24 shows a natural setting from which a spectral sequence arises. In turn, Theorem 2.25 is a special case of the comparison theorem, a standard tool to find isomorphisms between cohomology objects.

We begin with the concept of filtration.

Definition 2.21. A (decreasing) **filtration** F^* on an R-module A is a family $\{F^pA \mid p \in \mathbb{Z}\}$ of submodules of A such that $F^{p+1}A \subseteq F^pA$ for all p.

Having a filtration, we can obtain an associated graded module by setting

$$E_0^p(A) = F^pA/F^{p+1}A.$$

If A is itself graded, then we may define $F^pA^n = F^pA \cap A^n$, and thus get an associated bigraded module

$$E_0^{p,q}(A) = F^pA^{p+q}/F^{p+1}A^{p+q}.$$

Definition 2.22. A spectral sequence $\{E_r^{*,*}, d_r\}$ **converges** to a graded module H if there is a filtration F^* on H such that

$$E_\infty^{p,q} \cong E_0^{p,q}(H)$$

for each $p, q \in \mathbb{Z}$. It is denoted $E_r^{*,*} \implies H$.

It is important to observe that, in general, the original filtration cannot be recovered from the associated graded module. A close analysis is carried out in [McC01, p. 32]. However, in convenient settings, for instance when R is a field, the dimensions of the filtered components can be determined. In this way, if a spectral sequence over a field is known to converge to certain cohomology, one can deduce whether some position survives to the E_∞ page, as in the examples on page 21.

We define two properties of filtrations needed to state Theorem 2.24.

Definition 2.23. Consider a filtration F^* on a dg module (A, d).

1. The filtration F^* is called **exhaustive** if $A = \bigcup_p F^pA$.
2. The filtration F^* is called **weakly convergent** if, for all $p \in \mathbb{Z}$,

$$F^pA \cap \ker d = \bigcap_{r \in \mathbb{Z}} (F^pA \cap d^{-1}(F^{p+r}A)).$$

With that, we have the following result (see [McC01, Theorem 3.2]).

Theorem 2.24. *Every filtered dg module (A, d, F^*) determines a cohomology spectral sequence $\{E_r^{*,*}, d_r\}$ with*

$$E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+1}A).$$

Furthermore, if the filtration is exhaustive and weakly convergent, then the spectral sequence converges to $H(A)$.

Finally, we give an instance of Zeeman's comparison theorem, originally proved in [Zee57]. The standard formulation of the theorem, although more powerful, has stronger assumptions than the version that we present, and we will only need the latter in Chapter 3. For a proof, see [McC01, Thm. 3.4].

Theorem 2.25 (Comparison theorem)**.** *Consider a morphism of spectral sequences* $\{f_r\} : \{E_r^{*,*}, d_r\} \to \{E_r'^{*,*}, d_r'\}$*, and suppose that* $f_n : E_n \to E_n'$ *is an isomorphism of differential bigraded modules for some* $n \in \mathbb{N}$*. Then, for all* $n \leq r \leq \infty$*, the map* $f_r : E_r \to E_r'$ *is an isomorphism. In particular, if the sequences converge to graded modules* H *and* H'*, respectively,* f_∞ *provides an isomorphism* $H \cong H'$*.*

Examples. The following are examples of cohomology Serre spectral sequences, where the E_2 page is computed with the formula of Theorem 2.17. Here, cohomology is taken over an arbitrary field $\Bbbk$.

1. **The Wang sequence**. Consider a fibration $F \xrightarrow{i} E \xrightarrow{\pi} S^n$ with base a sphere. The E_2 page of the associated Serre spectral sequence has only two non-zero rows, which remain invariant until the E_n page, since all differentials in between vanish by degree reasons.

$$
\begin{array}{c|ccc}
n & H^n(F) & & H^n(F) \\
n-1 & H^{n-1}(F) & d_n & H^{n-1}(F) \\
\vdots & \vdots & & \vdots \\
1 & H^1(F) & & H^1(F) \\
0 & H^0(F) & & H^0(F) \\
\hline
 & 0 & \cdots & n
\end{array}
$$

The $E_{n+1} = E_\infty$ page can then be computed taking the quotients $\ker d_n / \operatorname{im} d_n$ at each position. This yields, for $q \in \mathbb{Z}$, an exact sequence

$$0 \longrightarrow E_\infty^{0,q+n-1} \longrightarrow H^{n-1}(F) \longrightarrow H^q(F) \longrightarrow E_\infty^{n,q} \longrightarrow 0.$$

Moreover, by convergence to $H^*(E)$, we get the short exact sequence

$$0 \longrightarrow E_\infty^{n,q-n} \longrightarrow H^q(E) \longrightarrow E_\infty^{0,q} \to 0$$

for the cohomology of the total space. Together with this, one gets a long exact sequence (see [Wei94, App. 5.3.5])

$$\cdots \to H^{q-1}(E) \to H^{q-1}(F) \xrightarrow{d_n} H^{q-n}(F) \to H^q(E) \to H^q(F) \to \cdots$$

known as the Wang sequence.

2. **The cohomology of $\mathbb{C}P^n$.** The complex projective plane $\mathbb{C}P^n$ admits a fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. The associated Serre spectral sequence has the following E_2 page, where H^i denotes $H^i(\mathbb{C}P^n)$.

$$\begin{tikzcd}
1 & H^0 \arrow[drr] & H^1 \arrow[drr] & H^2 \arrow[drr] & H^3 & \cdots & H^{2n-2} \arrow[drr] & H^{2n-1} \arrow[drr] & H^{2n} & 0 \\
0 & H^0 & H^1 & H^2 & H^3 & \cdots & H^{2n-2} & H^{2n-1} & H^{2n} & 0 \\
 & 0 & 1 & 2 & 3 & \cdots & 2n-2 & 2n-1 & 2n & \cdots
\end{tikzcd}$$

The differentials d_2 have direction $(2,-1)$ as depicted. Note that the higher cohomology groups of $\mathbb{C}P^n$ vanish since it is a manifold of real dimension $2n$. Now, we know that the cohomology of S^{2n+1} is non-trivial only in dimensions 0 and $2n+1$. Moreover, differentials d^r for $r > 2$ are zero by degree reasons, so $E_3 = E_\infty$. Hence, the maps $d_2 : H^k(\mathbb{C}P^n) \to H^{k+2}(\mathbb{C}P^n)$ must all be isomorphisms. The positions $(0,0)$ and $(2n,1)$ remain unchanged, so $H^0(\mathbb{C}P^n) = H^0(S^{2n+1})$ and $H^{2n}(\mathbb{C}P^n) = H^{2n+1}(S^{2n+1})$. We deduce that

$$H^k(\mathbb{C}P^n) = \begin{cases} \Bbbk & \text{if } k \le 2n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

This is a specific case of the Gysin sequence, constructed similarly as the Wang sequence for fibrations with fibre a sphere (see [Wei94, App. 5.3.7]).

3. **The Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$.** In this case, the E_2 page of the Serre spectral sequence has the form, omitting zero rows and columns,

$$\begin{tikzcd}
3 & \Bbbk \arrow[dr] & \Bbbk \\
0 & \Bbbk & \Bbbk \\
 & 0 & 4
\end{tikzcd}$$

There can only be one non-zero differential, namely d_4. Since S^7 has trivial cohomology at degree 4, we deduce that $d_4 : E_4^{0,3} \to E_4^{4,0}$ has to be an isomorphism.

The Serre spectral sequence possesses an additional structure, since the $E_r^{*,*}$ admit a product that allows to regard them as algebras. First, let us define what it means for an algebra structure in the modules $E_r^{*,*}$ to be compatible with the spectral sequence construction. The idea is that the ring product in $E_{r+1}^{*,*}$ should coincide with the one induced when taking cohomology of $E_r^{*,*}$. To have a simpler description, we impose the restriction that $R = \Bbbk$ is a field. Recall that our final goal is to work with coefficients in $\mathbb{Q}$. For a field $\Bbbk$, the Künneth theorem provides a key isomorphism $H^*(E_r \otimes E_r) \cong H^*(E_r) \otimes H^*(E_r)$, which by definition is isomorphic to $E_{r+1} \otimes E_{r+1}$ [McC01, p. 45]. Then, we may define a spectral sequence of algebras as follows.

Definition 2.26. Let $\{E_r^{*,*}, d_r\}$ be a spectral sequence of $\Bbbk$-vector spaces such that each $(E_r^{*,*}, d_r)$ is a differential bigraded $\Bbbk$-algebra, the ring product being denoted by ψ_r. Then, $\{E_r^{*,*}, d_r\}$ is a **spectral sequence of algebras** if, for each r, we can write the product ψ_{r+1} as the composite

$$\psi_{r+1} : E_{r+1} \otimes E_{r+1} \overset{\cong}{\to} H(E_r) \otimes H(E_r) \overset{\cong}{\to} H(E_r \otimes E_r) \overset{H(\psi_r)}{\to} H(E_r) \overset{\cong}{\to} E_{r+1}.$$

With this, the Serre spectral sequence is a spectral sequence of algebras. The $E_2^{*,*}$ page is naturally endowed with a product $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$ obtained as $(-1)^{qs}$ times the cup product

$$H^p(B; H^q(F; \Bbbk)) \times H^s(B; H^t(F; \Bbbk)) \overset{\smile}{\to} H^{p+s}(B; H^{q+t}(F; \Bbbk)).$$

Inductively, for $r \geq 2$, the product in E_r^* induces a well-defined product in E_{r+1}^*. Moreover, the product in the E_∞ page coincides with the cup product in the associated graded components of $H^*(E; \Bbbk)$. For the details, we refer to [Hat$_2$, p. 543].

2.3.1 The Serre spectral sequence

Let us now discuss specifically the Serre spectral sequence. Our goal is to present some important results involving the following property.

Definition 2.27. A fibration $F \overset{i}{\hookrightarrow} E \to B$ is said to be **totally non-cohomologous to zero** (TNCZ) with respect to the ring R if the homomorphism $i^* : H^*(E; R) \to H^*(F; R)$ is surjective.

In particular, Theorem 2.30 shows the relation between this condition and the Halperin conjecture, whereas the Leray-Hirsch theorem (Corollary 2.31) establishes a simple formula for the cohomology groups of the total space of a fibration which is TNCZ.

The TNCZ property is stable under pullbacks of fibrations. Recall that given a fibration $\pi : E \to B$ and a map $f : B' \to B$, we get a fibration $\pi' : E' \to B'$ by taking the pullback of π and f, and the induced map h between fibres is a homeomorphism. We get a commutative diagram in cohomology

$$\begin{tikzcd} H^*(F'; R) & H^*(F; R) \arrow[l, "h^*"', "\cong"] \\ H^*(E'; R) \arrow[u, "i'^*"] & H^*(E; R) \arrow[l, "g^*"] \arrow[u, "i^*"'] \end{tikzcd}$$

Now, if π is TNCZ, then the composition $h^*i^* = i'^*g^*$ is surjective, and thus i'^* as well. Hence π' is TNCZ.

An important property of the Serre spectral sequence, regarded as a functor defined on fibrations, is **naturality**, which can be stated as follows.

Consider a map of fibrations given by maps g and f such that the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, dashed, "h"] & E \arrow[r] \arrow[d, "g"] & B \arrow[d, "f"] \\ F' \arrow[r] & E' \arrow[r] & B' \end{tikzcd}$$

commutes. Denote by E_r and E'_r the Serre spectral sequences associated to each fibration. Then, the following claims hold (see [Hat$_2$, p. 537]):

(1) There are induced maps $f_r^* : E_r'^{p,q} \to E_r^{p,q}$ commuting with differentials and such that f_{r+1}^* is the map induced in cohomology by f_r^*. In particular, $\{f_r^*\}$ is a morphism of spectral sequences.

(2) The induced map $g^* : H^*(E'; R) \to H^*(E; G)$ preserves filtrations, inducing a map on successive quotient groups which coincides with f_∞^*.

(3) The maps $H^p(B'; H^q(F'; R)) \to H^p(B; H^q(F; R))$ induced by f and h coincide with f_2^*.

Using naturality, we can prove the following lemma in our way to the Leray-Hirsch theorem. Notice that $H^0(B; R) = H^0(F; R) = R$ as these are path-connected spaces, and therefore the terms $E_2^{p,0}$ and $E_2^{0,q}$ can be expressed more easily as $H^p(B; R)$ and $H^q(F; R)$.

Lemma 2.28. *Consider an orientable fibration $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$. Then, the composites*

$$H^p(B; R) = E_2^{p,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_p^{p,0} \twoheadrightarrow E_{p+1}^{p,0} = E_\infty^{p,0} \hookrightarrow H^p(E; R),$$
$$H^q(E; R) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \hookrightarrow E_q^{0,q} \hookrightarrow \cdots \hookrightarrow E_3^{0,q} \hookrightarrow E_2^{0,q} = H^q(F; R)$$

are the homomorphisms

$$\pi^* : H^p(B; R) \to H^p(E; R), \quad i^* : H^q(E; R) \to H^q(F; R).$$

Proof. Consider the diagram of fibrations

$$\begin{tikzcd} * \arrow[d] & F \arrow[l] \arrow[d, "i"] & F \arrow[l, "\mathbb{1}"'] \arrow[d, "\mathbb{1}"] \\ B \arrow[d, "\mathbb{1}"] & E \arrow[l, "\pi"'] \arrow[d, "\pi"] & F \arrow[l, "i"'] \arrow[d] \\ B & B \arrow[l, "\mathbb{1}"'] & * \arrow[l] \end{tikzcd}$$

By naturality, we get induced morphisms between the Serre spectral sequences of the vertical fibrations

$$E_r(*, B, B) \xrightarrow{\pi_r^*} E_r(F, E, B) \xrightarrow{i_r^*} E_r(F, F, *),$$

converging to π^* and i^* at E_∞, respectively. Now, notice that $E_2(*, B, B)$ collapses at the row $E_2^{*,0} = H^*(B; R)$, whereas $E_2(F, F, *)$ collapses at the column $E_2^{0,*} = H^*(F; R)$, so both sequences degenerate at the E_2 term.

At the E_∞ page, the 0-th row of the spectral sequence $E_r(F, E, B)$ becomes a quotient of $H^*(B; R)$ through the projections $E_r^{p,0} \twoheadrightarrow E_{r+1}^{p,0}$ in the statement. Hence, by convergence π^* coincides with the composition $H^p(B; R) \to H^p(E; R)$. In turn, the 0-th column becomes a subalgebra of $H^*(F; R)$ through the injections $E_{r+1}^{0,q} \hookrightarrow E_r^{0,q}$, and similarly the composition must coincide with i^*. $\square$

Before proving the main result, let us see that, under suitable assumptions, we can simplify the expression for the E_2 term of the Serre spectral sequence. Once again, we need to assume that $R = \Bbbk$ is a field.

Definition 2.29. A graded vector space V over a field $\Bbbk$ is called of **finite $\Bbbk$-type** if V^n is finite dimensional for all n.

When it is clear over which field $\Bbbk$ we are working, we will write simply *of finite type*. Extending this notion, a topological space X is called of finite $\Bbbk$-type if $H^*(X; \Bbbk)$ is of finite type. Now, given an orientable fibration $F \to E \to B$ such that F and B are of finite type, the universal coefficient theorem allows to show that

$$E_2^{*,*} \cong H^*(B; \Bbbk) \otimes H^*(F; \Bbbk)$$

as bigraded algebras [McC01, Prop. 5.6]. This gives an easy recipe to compute the E_2 term of the Serre spectral sequence for the majority of common topological spaces.

We can now show the following equivalence.

Theorem 2.30. *Let $\Bbbk$ be a field, and consider an orientable fibration $F \overset{i}{\hookrightarrow} E \to B$. The following are equivalent:*

(1) The fibration is totally non-cohomologous to zero with respect to $\Bbbk$.

(2) The Serre spectral sequence with coefficients in $\Bbbk$ associated to the fibration degenerates at the E_2 term.

Proof. Consider the expression for i^* given in Lemma 2.28. If the Serre spectral sequence degenerates at E_2, then $E_\infty^{0,q} = E_2^{0,q}$ and all the maps in between are equalities, so $i^* : H^q(E; R) \twoheadrightarrow E_2^{0,q}$ is surjective.

Conversely, if i^* is surjective, all the inclusions must be equalities, thus $d_r = 0$ on the y-axis. Since $\Bbbk$ is a field and the fibration is orientable, we have $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$ for all p, q. Thus, $E_2^{p,q}$ is generated by elements $a^p \otimes a^q$ with $a^p \in E_2^{p,0}$ and $a^q \in E_2^{0,q}$. Since d_2 is a derivation, we have

$$d_2(a^p \otimes a^q) = d_2(a^p) \otimes a^q \pm a^p \otimes d_2(a^q).$$

Now, a^p lies on the y-axis, so $d_2(a^p) = 0$, while $d_2(a^q) = 0$ as it exits the first quadrant. We deduce that d_2 vanishes on generators and thus $E_2 = E_3$. The same reasoning holds for d_3 and inductively for d_r, so we conclude that the spectral sequence degenerates at E_2. $\square$

Corollary 2.31 (Leray-Hirsch theorem)**.** *Let $F \hookrightarrow E \to B$ be an orientable, TNCZ fibration with respect to $\Bbbk$. Then*

$$H^*(E;\Bbbk) \cong H^*(B;\Bbbk) \otimes H^*(F;\Bbbk)$$

as vector spaces.

Proof. From Theorem 2.30, the Serre spectral sequence degenerates at E_2. By convergence, there is an isomorphism of vector spaces $H^*(E;\Bbbk) \cong E_2^{*,*}$, while $E_2^{*,*} \cong H^*(B;\Bbbk) \otimes H^*(F;\Bbbk)$ is an isomorphism of algebras. $\square$

Remark 2.32. The above relation need not be an isomorphism of algebras, as shows the following example of Lupton [Lup98, Ex. 1.2]. Consider the so-called *twistor* fibration $S^2 \hookrightarrow \mathbb{C}P^3 \xrightarrow{\pi} S^4$. Here, the map π is given by identifying S^4 with the projective line of quaternions $\mathbb{H}P^1$, and setting

$$\begin{array}{rccc} \pi: & \mathbb{C}P^3 & \longrightarrow & \mathbb{H}P^1 \\ & \mathbb{C}\langle x\rangle & \longmapsto & \mathbb{H}\langle x\rangle, \end{array}$$

where $x \in \mathbb{C}^4 \cong \mathbb{H}^2$ (as $\mathbb{R}$-algebras) and $\langle x\rangle$ represents the subspace generated by x in the corresponding vector space. The E_2 page of the resulting Serre spectral sequence over a field $\Bbbk$ is given by

$$\begin{array}{c|cc} 2 & \Bbbk & \Bbbk \\ 0 & \Bbbk & \Bbbk \\ \hline & 0 & 4 \end{array}$$

By the distribution of the non-zero terms, the sequence degenerates at E_2, yielding the isomorphism of vector spaces. However, as algebras, $H^*(\mathbb{C}P^3) \cong \Bbbk[x]/(x^4)$ whereas $H^*(S^2) \otimes H^*(S^4) \cong \Bbbk[x,y]/(x^2,y^2)$.

As a consequence of the results above, if the map i^* is surjective then π^* is necessarily injective. The converse is true if the fibration is orientable (or, more generally, *rational* [Hal78, Prop. 4.12]), but does not hold in general, as shows the following example of G. Hirsch (see [McC01, p. 149]).

Consider the Hopf fibration $S^3 \hookrightarrow S^7 \xrightarrow{\nu} S^4$. We have seen in the example of page 22 that the associated Serre spectral sequence over a field $\Bbbk$ has a non-zero differential d_4, which is an isomorphism. Next, we construct a new fibration $E \xrightarrow{\pi} S^2 \times S^2$ as the pullback of ν along the map $S^2 \times S^2 \xrightarrow{q} S^2 \wedge S^2 \cong S^4$. Recall that the smash product $S^2 \wedge S^2$ can be defined as the quotient $S^2 \times S^2/S^2 \vee S^2$, and q is the quotient map. The fibres of π are homeomorphic to S^3. Finally, let p denote the projection of $S_2 \times S_2$ onto the first factor. We get the following commutative diagram

$$\begin{tikzcd}
S^3 \arrow[r, hook] & S^7 \arrow[r, "\nu"] & S^4 & \\
S^3 \arrow[r] \arrow[u, "\mathbb{1}"] & E \arrow[r, "\pi"] \arrow[u] & S^2 \times S^2 \arrow[r, "p"] \arrow[u, "q"'] & S^2
\end{tikzcd}$$

By naturality, the spectral sequence of the bottom fibration also has a non-zero differential d_4. Now, consider the fibration $S^3 \times S^2 \xrightarrow{i} E \xrightarrow{p\pi} S^2$. On the one hand, the associated Serre spectral sequence cannot degenerate at the E_2 page, since we know that $H^*(E;\Bbbk)$ is not isomorphic to $H^*(S^2 \times S^2 \times S^3;\Bbbk)$. Thus, i^* is not surjective.

On the other hand, though, the map $(p\pi)^*$ induced in cohomology is injective. Indeed, p^* maps the generator α of $H^2(S^2;\Bbbk)$ into a generator of $H^2(S^2 \times S^2;\Bbbk)$. The algebra $H^2(E;\Bbbk)$ has dimension 2 (see [BB20, Thm. 4.5]), and is generated by the images through π^* of the generators of $H^2(S^2 \times S^2)$, as inspection of the Serre spectral sequence shows. With that, we have $(p\pi)^*(\alpha) \neq 0$. The argument is also valid if we replace $\Bbbk$ by a ring R, but we stick to coefficients over a field for consistency.

2.4 Rational homotopy equivalences

Rational homotopy theory arises as a simpler theory than homotopy theory that allows to compute topological invariants *up to torsion*. For instance, the k-th rational homotopy group of the sphere S^n is defined as $\pi_k(S^n) \otimes \mathbb{Q}$, becoming a $\mathbb{Q}$-vector space with dimension the rank of $\pi_k(S^n)$ as a $\mathbb{Z}$-module. Notably, tensoring with $\mathbb{Q}$ forgets the torsion part of $\pi_k(S^n)$. Although the resulting theory gathers less information about the space, it is remarkably computational. As an example, there is not even a conjectural description of all the homotopy groups of any sphere S^n with $n > 1$, however the rational homotopy groups of *all* spheres are known. More generally, rational homotopy theory studies the properties of spaces up to rational homotopy type, in the sense defined below.

A fundamental construction in rational homotopy theory, the minimal model of a space, will turn very useful for the purposes of this work. Minimal models are the main algebraic invariant in rational homotopy theory, as they serve to classify spaces precisely up rational homotopy equivalence. Section 2.5 is devoted to their study. In this section, we state a set of basic definitions and results in rational homotopy theory needed to understand future concepts. A full introduction to the topic is provided in [FHT01].

The first elementary result follows from the Whitehead-Serre theorem [FHT01, Thm. 8.6], the analogue of Whitehead's theorem modulo a class of torsion groups. Note that, assuming X has finite $\mathbb{Q}$-type, we have $H_*(X;\mathbb{Q}) \cong H_*(X) \otimes \mathbb{Q}$ since $\mathbb{Q}$ is flat, and likewise in cohomology.

Theorem 2.33. *Let $f : X \to Y$ be a map between simply connected spaces. The following are equivalent:*

(1) The induced map $f_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism.

(2) The induced map $f_* : H_*(X; \mathbb{Q}) \to H_*(Y; \mathbb{Q})$ is an isomorphism.

(3) The induced map $f^* : H^*(Y; \mathbb{Q}) \to H^*(X; \mathbb{Q})$ is an isomorphism.

Note that condition (1) does not make sense if we do not assume that X and Y are simply connected, since the fundamental group need not be abelian and the tensor product with $\mathbb{Q}$ is not defined in a natural way.

Definition 2.34. A map $f : X \to Y$ between simply connected spaces is a **rational homotopy equivalence** if the conditions in Theorem 2.33 are satisfied. We denote $X \xrightarrow{\sim_{\mathbb{Q}}} Y$.

Definition 2.35. Two simply connected spaces X and Y are said to have the same **rational homotopy type**, or to be **rationally homotopy equivalent**, if there is a chain of rational homotopy equivalences

$$X \xleftarrow{\sim_{\mathbb{Q}}} Z_1 \xrightarrow{\sim_{\mathbb{Q}}} \cdots \xleftarrow{\sim_{\mathbb{Q}}} Z_n \xrightarrow{\sim_{\mathbb{Q}}} Y.$$

In this case, we write $X \sim_{\mathbb{Q}} Y$.

Note that if $\pi_*(X)$ and $H_*(X)$ are vector spaces over $\mathbb{Q}$, then they will remain the same after tensoring with $\mathbb{Q}$. Such a space would be a good representative of its rational homotopy type, since its rational homotopy properties can be studied through the usual homotopy theory. Interestingly, for a simply connected space X there is a construction that yields a space $X_{\mathbb{Q}}$ with these characteristics.

The following notions belong to a more general theory defining localizations in abelian groups in terms of divisibility by prime integers. The details can be consulted in [FHT01, §9]. Here we give specific definitions for $\mathbb{Q}$-localizations.

Definition 2.36. A simply connected space is called **rational**, or **$\mathbb{Q}$-local**, if its homotopy groups are vector spaces over $\mathbb{Q}$.

Definition 2.37. Let X be a simply connected space. A **rationalization** of X is a rational space $X_{\mathbb{Q}}$ together with a rational homotopy equivalence $X \to X_{\mathbb{Q}}$.

Theorem 2.38 ([FHT01, Thm. 9.7])**.** *For every simply connected space X there exists a rationalization $X_{\mathbb{Q}}$.*

2.5 Minimal models

Minimal models are an essential construction in rational homotopy theory, as they provide an algebraic tool to classify *nice* spaces up to rational homotopy equivalence. In short, every such space X is associated a dg algebra M_X, with the key property that two spaces X and Y are rationally homotopy equivalent if and only if M_X and M_Y are isomorphic dg algebras. In this section, we review Sullivan's construction of the minimal model [Sul77] and state the main properties that will be used in Chapters 3 and 4. Again, aiming for concision, we omit most technical details and provide references for them. The main texts followed are [Ber, FHT01, FOT08].

Before associating a minimal model to a space, we will construct the Sullivan model of an algebra and define the notion of minimality. The first step is to introduce relative Sullivan algebras, a structure analogue to relative cell complexes in topology, and for that we need analogue versions of spheres and disks. Let $\Bbbk$ be a field and n a positive integer. We define the cochain algebras:

$$\begin{aligned} S(n) &= (\Lambda x, 0), \text{ with } |x| = n, \\ D(n-1) &= (\Lambda(x, sx), d), \text{ with } |x| = n, |sx| = n-1, dx = 0 \text{ and } d(sx) = x. \end{aligned}$$

Observe that there is an inclusion of algebras $S(n) \subseteq D(n-1)$, as with the analogue geometric objects. More generally, given a graded vector space V, we set

$$\begin{aligned} S(V) &= (\Lambda V, 0), \\ D(V) &= (\Lambda(V \oplus sV), d), \text{ with } dv = 0 \text{ and } d(sv) = v \text{ for } v \in V. \end{aligned}$$

When constructing a cell complex, we successively attach cells of higher dimension to the current skeleton via pushout squares. We proceed similarly to form Sullivan algebras. Consider a cochain algebra (A, d) and a cocycle $a \in A$ of degree n. We form the algebra $A \otimes \Lambda y$ with $|y| = n - 1$ and differential determined by $\delta(b \otimes 1) = d(b) \otimes 1$ and $\delta(1 \otimes y) = a \otimes 1$. In this algebra, the cocycle a vanishes in cohomology. We can obtain $(A \otimes \Lambda y, \delta)$ via the following pushout diagram in the category of cochain algebras

$$\begin{tikzcd} S(n) \arrow[r, "a"] \arrow[d] & A \arrow[d] \\ D(n-1) \arrow[r, "y"] & A \otimes \Lambda y \end{tikzcd}$$

Here, the morphism a maps $x \in S(n)$ to $a \in A$ and the morphism y maps $x \in D(n-1)$ to $a \otimes 1$ and $sx \in D(n-1)$ to $1 \otimes y$.

Definition 2.39. A **relative Sullivan algebra** is a pair of cochain algebras (X, A) together with a filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X,$$

such that $X = \bigcup_n X_n$ and each X_n is obtained from X_{n-1} in the following way: for each n there is a graded vector space V_n and a pushout diagram

$$\begin{tikzcd} S(V_n) \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ D(V_n) \arrow[r] & X_n \end{tikzcd}$$

A **Sullivan algebra** is a cochain algebra X such that the pair $(X, \Bbbk)$ admits the structure of a relative Sullivan algebra.

By construction, a Sullivan algebra X is free graded commutative. Namely, it is of the form $X = \Lambda V$, where V is obtained through the filtration

$$0 \subseteq V^{(0)} \subseteq V^{(1)} \subseteq \cdots \subseteq V = \bigcup_n V^{(n)},$$

with $V^{(n)} = \bigoplus_{i \leq n} V_i$.

Definition 2.40. Let $f : A \to B$ be a morphism of cochain algebras. A **Sullivan model** for f is a relative Sullivan algebra (X, A) together with a quasi-isomorphism $\tilde{f} : X \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook] & & B \\ & X \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

A Sullivan model for the cochain algebra B is a Sullivan model for the unit map $\Bbbk \to B$, i.e. a Sullivan algebra X together with a quasi-isomorphism $X \xrightarrow{\sim} B$.

The following crucial result guarantees the existence of a Sullivan model for certain morphisms. For a proof, see [FHT01, Prop. 14.3].

Theorem 2.41. *Let $f : A \to B$ be a morphism of cochain algebras such that $H^0(A) \cong H^0(B) \cong \Bbbk$ and $H^1(f) : H^1(A) \to H^1(B)$ is injective. Then, there is a Sullivan model (X, A) for f.*

This allows to associate a Sullivan model to every cochain algebra A with $H^0(A) \cong \Bbbk$. Furthermore, we show next that such model can be chosen *minimal*, meaning that, in most cases, it is calculable. Then, if A represents a space X in a certain way, the minimal Sullivan model provides a simple algebraic description of the homotopy of X over the field $\Bbbk$.

Thus, consider a cochain algebra of the form $(\Lambda V, d)$, where $V = V^{\geq 1}$ is concentrated in positive degrees. Then the differential can be decomposed into its word length homogeneous components, $d = d_0 + d_1 + d_2 + \ldots$, where $d_r(\Lambda^k V) \subseteq \Lambda^{k+r} V$ [Ber, p. 58]. This allows to define the concept of minimality of a cochain algebra of this form.

Definition 2.42. A cochain algebra of the form $(\Lambda V, d)$ with $V = V^{\geq 1}$ is called **minimal** if $d(V) \subseteq \Lambda^{\geq 2}V$.

Equivalently, it follows from the decomposition of d that $(\Lambda V, d)$ is minimal if and only if $d_0 = 0$.

Theorem 2.43. *Every cdg algebra A with $H^0(A) = \Bbbk$ admits a unique, up to isomorphism, minimal Sullivan model*

$$(\Lambda V, d) \xrightarrow{\sim} A$$

with $V = V^{\geq 2}$. Furthermore, if $H^1(A) = 0$ and $H^*(A)$ is of finite type, then V is also of finite type.

We refer to [FHT01, Prop. 12.2] for a proof. This result will allow us to construct the minimal model of simply connected spaces. Given a space X, there is a natural way to associate a dg algebra to X, namely taking the space of singular cochains $C^*(X; \Bbbk)$ over a field $\Bbbk$. However, $C^*(X; \Bbbk)$ is not commutative in general, so we cannot apply Theorem 2.43. If $X = M$ is a smooth manifold, this can be solved by considering the de Rham algebra $\Omega^*_{dR}(M)$ of real differential forms, which is a cdg algebra. Indeed, the de Rham theorem shows that the cohomology of $\Omega^*_{dR}(M)$ coincides with the real cohomology $H^*(M; \mathbb{R})$.

A similar construction solves the issue in the case of arbitrary path-connected spaces X. Recall that we assume $\mathrm{char}(\Bbbk) = 0$. The idea, that we briefly describe, is to construct an algebra $A^*_{PL}(X; \Bbbk)$ of polynomial differential forms on X with coefficients in $\Bbbk$. For that, we first define A^*_{PL} on simplices by

$$A^*_{PL}(\Delta^n; \Bbbk) = \frac{\Lambda(t_0, \ldots, t_n, dt_0, \ldots, dt_n)}{(\sum t_i - 1, \sum dt_i)},$$

with $|t_i| = 0, |dt_i| = 1$ and $d(t_i) = dt_i$ for each i. Then, A^*_{PL} can be defined globally on X, via the category of simplicial sets. We skip the details since we would need to introduce technical concepts with limited interest for the rest of the work. We refer to [FHT01, §10] for the complete construction of $A^*_{PL}(X; \Bbbk)$ and [Ber, §6] for a didactic review of simplicial objects.

Finally, one can show [FHT01, Cor. 10.10] that there is a chain of quasi-isomorphisms of dg algebras from $C^*(X; \Bbbk)$ to $A^*_{PL}(X; \Bbbk)$. Hence, the cohomology of $A^*_{PL}(X; \Bbbk)$ coincides with the cohomology of X with coefficients in $\Bbbk$.

Definition 2.44. The **minimal model** over $\Bbbk$ of a path-connected space X is defined as the minimal Sullivan model of the cdg algebra $A^*_{PL}(X; \Bbbk)$.

We will denote the minimal model of a space X by (M_X, d), or simply M_X if d is clear. Next, taking $\Bbbk = \mathbb{Q}$, we state the main result of this section, which follows from the previous discussion and [FHT01, §15, §17]. Henceforth, we write $A^*_{PL}(X)$ for $A^*_{PL}(X; \mathbb{Q})$.

Theorem 2.45. *Let X be a space of finite $\mathbb{Q}$-type with $H^0(X) = \mathbb{Q}$ and $H^1(X) = 0$. Then:*

(1) The space X admits a unique, up to isomorphism, rational minimal model $M_X = (\Lambda V, d) \xrightarrow{\sim} A^*_{PL}(X)$, where $V = V^{\geq 2}$ is a $\mathbb{Q}$-vector space of finite type.

(2) If X is simply connected, the rational homotopy groups of X satisfy

$$\pi_k(X) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathbb{Q}}(V^k, \mathbb{Q}).$$

(3) Two simply-connected spaces of finite $\mathbb{Q}$-type X and Y are rationally homotopy equivalent if and only if the minimal models M_X and M_Y are isomorphic as cochain algebras.

Examples. The following are examples of minimal models in [FOT08, p. 72].

1. **The sphere** S^n, with $n > 1$, has rational cohomology $\mathbb{Q}[x]/(x^2)$, where $|x| = n$. Let $\omega \in A^*_{PL}(S^n)$ be a representative of the class x. Consider the morphism of cdg algebras $\varphi : (\Lambda x, 0) \to (A^*_{PL}(S^n), \delta)$ mapping $x \mapsto \omega$.

   If n is odd, then $\Lambda x \cong \mathbb{Q}[x]/(x^2)$ and φ is a quasi-isomorphism, so the minimal model of the odd sphere S^n is $(\Lambda x, 0)$ with $|x| = n$.

   If n is even, Λx is the polynomial algebra $\mathbb{Q}[x]$, which is not isomorphic to $\mathbb{Q}[x]/(x^2)$. To fix it, we introduce a generator y that kills x^2 in cohomology. For degree reasons, ω^2 vanishes in cohomology, and thus there is α of degree $2n-1$ such that $\delta\alpha = \omega^2$. Then, we consider $\Lambda(x, y)$ with $|y| = 2n-1$ and $dy = x^2$, and define $\varphi : (\Lambda(x, y), d) \to A^*_{PL}(X)$ by $\varphi(x) = \omega$ and $\varphi(y) = \alpha$. Now we do have an isomorphism $H(\Lambda(x, y)) \cong \mathbb{Q}[x]/(x^2)$ and conclude that the minimal model of the even sphere S^n is $(\Lambda(x, y), d)$ with $|x| = n$, $|y| = 2n-1$, $dx = 0$ and $dy = x^2$.

2. **The complex projective plane** $\mathbb{C}P^n$ has rational cohomology $\mathbb{Q}[x]/(x^{n+1})$, with $|x| = 2$. Thus, we can proceed similarly as in the case of the even sphere. We choose $\omega \in A^*_{PL}(\mathbb{C}P^n)$ of degree 2 to be a representative of x and $\alpha \in A^*_{PL}(\mathbb{C}P^n)$ of degree $2n+1$ such that $\delta\alpha = \omega^{n+1}$. Next, we consider $(\Lambda(x, y), d)$ with $|x| = 2, |y| = 2n+1, dx = 0$ and $dy = x^{n+1}$, and define a morphism of cdg algebras $\varphi : (\Lambda(x, y), d) \to A^*_{PL}(\mathbb{C}P^n)$ with $\varphi(x) = \omega$ and $\varphi(y) = \alpha$. This induces an isomorphism in cohomology, and we deduce that the minimal model of $\mathbb{C}P^n$ is $(\Lambda(x, y), d)$ as defined.

3. Like for the odd spheres, any space X whose cohomology algebra is free commutative, i.e. $H^*(X; \mathbb{Q}) = \Lambda(x_i)$, has minimal model $(\Lambda(x_i), 0)$.

4. Given path-connected spaces X and Y, there is a quasi-isomorphism $A^*_{PL}(X) \otimes A^*_{PL}(Y) \xrightarrow{\sim} A^*_{PL}(X \times Y)$. Therefore, between minimal models we have the relation $M_{X\times Y} \cong M_X \otimes M_Y$ as graded algebras.

The construction of $A^*_{PL}(X)$ and Theorem 2.43 give quasi-isomorphisms of dg algebras

$$M_X = (\Lambda V, d) \xrightarrow{\sim} A^*_{PL}(X) \xleftarrow{\sim} C^*(X;\mathbb{Q}),$$

and thus there is an isomorphism $H(M_X) \cong H^*(X;\mathbb{Q})$ in cohomology, as we have used in the examples. Here, we can think of $H^*(X;\mathbb{Q})$ as a dg algebra with trivial differential. However, in general we do not have a morphism of dg algebras $M_X \to H^*(X;\mathbb{Q})$ inducing an isomorphism in cohomology. This motivates the following definition.

Definition 2.46. A space X is **formal** if there exists a quasi-isomorphism of cdg algebras from its minimal model to its cohomology algebra,

$$(M_X, d) \xrightarrow{\sim} H^*(X;\mathbb{Q}).$$

We will say that a minimal model M is formal if it is the minimal model of a formal space, i.e. if there exists a map of dg algebras $M \to H(M)$ inducing an isomorphism in cohomology. The existence of this morphism will turn crucial when proving some of the results in Chapters 3 and 4.

Examples. We have seen that spaces X with free commutative cohomology algebras satisfy $M_X \cong H^*(X;\mathbb{Q})$. Thus, these spaces are formal. It can also be shown that products and retracts of formal spaces are formal [FOT08, p. 93]. Other relevant examples of formal spaces are positively elliptic spaces (see [BG76, §16] for an explicit construction) and Kähler manifolds [FOT08, Thm. 4.43], both concepts discussed in the following chapters.

To conclude, we present an application of the algebraic power of minimal models to the study of fibrations [Hal78]. The original result was stated for rational fibrations, which include orientable fibrations.

Proposition 2.47. *Let $F \to E \xrightarrow{\pi} B$ be an orientable fibration with B of finite $\mathbb{Q}$-type and such that the induced map in cohomology $H^*(\pi)$ vanishes on $H^1(B;\mathbb{Q})$. Then, there is a sequence of cdg algebras*

$$M_B = (\Lambda V, d) \xrightarrow{\pi^*} (\Lambda V \otimes \Lambda W, D) \xrightarrow{i^*} (\Lambda W, \delta) = M_F,$$

where π^* is the inclusion and i^* the projection onto the quotient of $\Lambda V \otimes \Lambda W$ by the ideal generated by $\Lambda V^{>0}$. Furthermore, $(\Lambda V \otimes \Lambda W, D)$ is a Sullivan model for E.

Remark 2.48. In general, the differential D does not have the trivial form

$$D(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes \delta(w),$$

but it is a *twisted* differential instead. Note that, since we have a sequence of cdg algebras, the differentials satisfy $D\pi^* = \pi^* d$ and $\delta i^* = i^* D$. Thus, for

$v \in \Lambda V$ we have $D(v \otimes 1) = \pi^* d(v) = d(v) \otimes 1$. However, for $w \in \Lambda W$, we only require $\delta i^*(1 \otimes w) = \delta(w) = [D(1 \otimes w)]$. In other words,

$$D(1 \otimes w) = \delta(w) + \sum_i v_i \otimes w_i$$

for $v_i \in \Lambda V^{>0}$ and $w_i \in \Lambda W$. This reflects, in algebraic terms, the fact that the base and the fibre may interact non-trivially to form the total space. Let us look at two specific examples.

Consider a trivial fibration $F \to F \times B \to B$. From Example 4 on page 32, the minimal model M_E of $E = F \times B$ is given as the tensor product of the minimal models $M_B = (\Lambda V, d)$ and $M_F = (\Lambda W, \delta)$. Therefore, the differential D of M_E is precisely $d \otimes 1 \pm 1 \otimes \delta$, where the sign depends on the grading.

In turn, for a setting with a twisted differential, consider the Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$. Recall that we have computed the minimal model of odd and even spheres on page 32. Then, following the proposition, there is a sequence of cdg algebras

$$(\Lambda(x, y), d) \to (\Lambda(x, y) \otimes \Lambda z, D) \to (\Lambda z, 0),$$

where $|x| = 4, |y| = 7$ and $|z| = 3$. Furthermore, $(\Lambda(x, y) \otimes \Lambda z, D)$ must be a Sullivan model for S^7, and thus have $H^*(S^7; \mathbb{Q})$ as cohomology.

If the differential D is not twisted, then there are non-zero elements of degrees 3 and 4 in cohomology, as the elements $1 \otimes z$ and $x \otimes 1$ are cocyles but not coboundaries of D. However, we know that $H^*(S^7; \mathbb{Q})$ is a free algebra on one generator of degree 7. To achieve this, we may set $D(1 \otimes z) = x \otimes 1$, satisfying the condition for D derived above. Then,

$$D(x \otimes z) = D((x \otimes 1)(1 \otimes z)) = (x \otimes 1)(x \otimes 1) = x^2 \otimes 1,$$

which equals $D(y \otimes 1)$. Hence, the element $x \otimes z - y \otimes 1$ is a cocycle, and by degree reasons its cohomology class is non-zero. Since the degree 7 component of $\Lambda(x, y) \otimes \Lambda z$ has only two generators, the resulting cohomology group has dimension 1. In turn, all cohomology groups in degrees other than 0 and 7 vanish. We conclude that the cohomology of $(\Lambda(x, y) \otimes \Lambda z, D)$ coincides with $H^*(S^7; \mathbb{Q})$.

3 Equivalent formulations of the conjecture

In this chapter, we discuss equivalent statements of the Halperin conjecture. In particular, in Section 3.1 we demonstrate the equivalence between the original formulation and the re-phrasing in terms of derivations of algebras, stated as Conjecture 3.6. This result, due to W. Meier [Mei82], has proven to be very valuable for showing the result under specific assumptions. Interestingly, the cited work contained yet another equivalent re-phrasing of the conjecture, that we address in Section 3.2.

In the introduction, we have formulated the conjecture for fibrations of simply connected spaces. Here, we will be slightly more general and extend the statement to orientable fibrations (see Definition 2.3). Note that every fibration of simply connected spaces is orientable. The reason is that all of our results are valid in this setting, and the conjecture is also believed to hold for orientable fibrations. Nevertheless, we continue to assume that the fibre is simply connected, in order to use some concepts in rational homotopy theory that we have defined only for these spaces.

We begin the chapter by defining positively elliptic spaces and their algebraic analogue, in terms of which the different formulations are stated. Later, we introduce a new classifying space that serves to classify orientable fibrations. Although some notions presented in the coming pages make sense for an arbitrary field $\Bbbk$, we will only work over the rationals for consistency.

Definition 3.1. A simply connected space X is called (rationally) **elliptic** if $H^*(X;\mathbb{Q})$ and $\pi_*(X) \otimes \mathbb{Q}$ are finite-dimensional vector spaces over $\mathbb{Q}$.

Here, since all spaces are assumed to be finite CW complexes, the condition on $H^*(X;\mathbb{Q})$ is satisfied automatically. We say that X is **positively elliptic** if it is elliptic and has positive Euler characteristic. Recall that the Euler characteristic is defined as

$$\chi(X) = \sum_n (-1)^n \dim(H^n(X;\mathbb{Q})).$$

With this terminology and the equivalence of Theorem 2.30, the Halperin conjecture has the following form.

Conjecture 3.2 (Halperin conjecture)**.** *Let F be a positively elliptic space. Then every orientable fibration $F \to E \to B$ is TNCZ.*

The property of having positive Euler characteristic can yet be expressed in different ways, following a result of Halperin [Hal77, Thm. 1']. We call $\chi_\pi(X) = \sum_n (-1)^n \dim(\pi_n(X) \otimes \mathbb{Q})$ the *homotopy Euler characteristic* of X. With that, given an elliptic space X, the following properties are equivalent:

(1) The Euler characteristic of X is positive, $\chi(X) > 0$.

(2) The homotopy Euler characteristic of X is zero, $\chi_\pi(X) = 0$.

(3) The rational cohomology of X is evenly graded, $H^{\mathrm{odd}}(X;\mathbb{Q}) = 0$.

The conjecture is often stated in terms of one of these properties. In particular, the third motivates the study of the problem from a purely algebraic point of view. The idea is to consider the class of algebras that serve as the cohomology algebra of a positively elliptic space and formulate the conjecture as an algebraic property of those.

Consider the polynomial ring $\mathbb{Q}[z_1, \ldots, z_k]$, where each z_i has positive, even degree $|z_i|$. Note that $\mathbb{Q}[z_1, \ldots, z_k]$ forms a free graded commutative algebra. We say that a polynomial $f \in \mathbb{Q}[z_1, \ldots, z_k]$ is *homogeneous* if all its terms have the same degree as elements of the graded algebra $\mathbb{Q}[z_1, \ldots, z_k]$.

Definition 3.3. A family of homogeneous polynomials $f_1, \ldots, f_k \in \mathbb{Q}[z_1, \ldots, z_k]$ is said to form a **regular sequence** if $f_1 \neq 0$ and the class of f_i in $\mathbb{Q}[z_1, \ldots, z_k]/(f_1, \ldots, f_{i-1})$ is not a zero divisor for $i \in \{2, \ldots, k\}$.

Definition 3.4. A **positively elliptic algebra** is a cdg algebra of the form

$$H \cong \frac{\mathbb{Q}[z_1, \ldots, z_k]}{(f_1, \ldots, f_k)},$$

where each $|z_i| > 0$ is even and $f_1, \ldots, f_k$ form a regular sequence.

Following [FHT01, Prop. 32.10], positively elliptic algebras are identified with the cohomology algebras of positively elliptic spaces. Thus, this class of algebras fulfils the goal of having algebraic representatives of positively elliptic spaces.

Lastly, let us introduce the classifying space $\mathrm{Baut}_o X$. This space serves as a universal base for orientable fibrations and is crucial in several proofs in this chapter. For an arbitrary space X, recall that $\mathrm{aut}\, X$ denotes the topological monoid of self-homotopy equivalences of X. Since homotopic maps induce the same homomorphism in cohomology, we have a well-defined group homomorphism $\varphi : \pi_0 \,\mathrm{aut}\, X \to \mathrm{Aut}\, H^*(X;\mathbb{Q})$ mapping the homotopy class of a homotopy equivalence f into the induced isomorphism f^* in $H^*(X;\mathbb{Q})$. Then $o := \ker \varphi$ is a subgroup of $\pi_0 \,\mathrm{aut}\, X = \pi_1 \,\mathrm{Baut}\, X$. Thus, it corresponds to the fundamental group of a cover $\mathrm{Baut}_o X$ of $\mathrm{Baut}\, X$. We define $\mathrm{Baut}_o^\bullet X$ as the pullback of the universal fibration by the covering map $p : \mathrm{Baut}_o X \to \mathrm{Baut}\, X$. With that, we have the following classification result for *orientable* fibrations with fibre X [Mei82, Prop. 1.3].

Proposition 3.5. *Let F be a space. Then,*

$$F \to \mathrm{Baut}_o^\bullet F \xrightarrow{u} \mathrm{Baut}_o F$$

represents the universal fibration for orientable fibrations with fibre F. Furthermore, the following properties hold:

(1) *The spaces* $\operatorname{Baut}_o F$ *and* $\operatorname{Baut}_o F_{\mathbb{Q}}$ *are rationally homotopy equivalent.*

(2) *For* $n \geq 2$*, there is an isomorphism*

$$\pi_n \operatorname{Baut} F \otimes \mathbb{Q} \cong \pi_n \operatorname{Baut}_o F_{\mathbb{Q}}.$$

Proof. Consider a fibration $F \to E \xrightarrow{\pi} B$. Using Theorems 2.7 and 2.8, the map π is induced from the universal fibration by a map $f : B \to \operatorname{Baut} F$. If π is orientable, then the image of the induced map $f_* : \pi_1(B) \to \pi_1 \operatorname{Baut} F$ acts trivially on $H^*(F; \mathbb{Q})$. Therefore, we have $\operatorname{im} f_* \subseteq o$, and hence this lifts to a map $\tilde{f} : B \to \operatorname{Baut}_o F$ with $p\tilde{f} \simeq f$. In other words, the fibration is induced from the fibration $u : \operatorname{Baut}_o^\bullet F \to \operatorname{Baut}_o F$ by the map $\tilde{f}$. Conversely, the fibration u is orientable, and thus also any fibration induced from it.

Property (1) follows from [BZ24, Prop. 3.10], replacing the groups G and U in the article by our o. This yields an isomorphism

$$\pi_n \operatorname{Baut}_o F \otimes \mathbb{Q} \cong \pi_n \operatorname{Baut}_o F_{\mathbb{Q}} \otimes \mathbb{Q}.$$

To deduce (2) from here, notice on the one hand that $\operatorname{Baut}_o F \to \operatorname{Baut} F$ is a covering map and thus $\pi_n \operatorname{Baut} F \cong \pi_n \operatorname{Baut}_o F$ for $n \geq 2$. On the other hand, $\pi_n \operatorname{Baut}_o F_{\mathbb{Q}} \otimes \mathbb{Q} \cong \pi_n \operatorname{Baut}_o F_{\mathbb{Q}}$ for all $n > 0$ because $\operatorname{Baut}_o F_{\mathbb{Q}}$ is a rational space. □

3.1 Statement in terms of derivations of algebras

The algebraic formulation is one of the first re-phrasings of the Halperin conjecture, and arguably the most successful in terms of the partial results proved from it. In [Mei82], the author relates the property of a fibration being totally non-cohomologous to zero to the derivations of the cohomology algebra of the fibre. The statement is the following.

Conjecture 3.6 (Algebraic Halperin conjecture)**.** *Positively elliptic algebras do not admit non-zero derivations of negative degree.*

Initially, the algebraic statement served to prove the conjecture in some relevant settings, such as cohomology algebras with three generators or Kähler manifolds. More recently, it has been used in the case of algebras of formal dimension at most 20. We review these results in Chapter 4.

The main result of this section is presented as an equivalence of three statements in Corollary 3.13. We prove the relevant implications separately in Theorems 3.7 and 3.12. For the first, we replicate the proof of [Mei82, Lemma 2.5(a)]. Recall that all spaces are assumed to be path-connected.

Theorem 3.7. *Let* F *be a space of finite type such that* $\operatorname{Der}^{<0} H^*(F; \mathbb{Q}) = 0$*. Then, every orientable fibration* $F \to E \to B$ *is TNCZ.*

Proof. Consider an orientable fibration $F \to E \to B$. If the fibration is not TNCZ, then there exists a minimal $n \geq 1$ such that $d_{n+1} \neq 0$. In this case, $E_i^{p,q} = E_2^{p,q} \cong H^p(B;\mathbb{Q}) \otimes H^q(F;\mathbb{Q})$ for all $i \in \{2, \dots, n+1\}$. Recall that

$$d_{n+1} : H^p(B;\mathbb{Q}) \otimes H^q(F;\mathbb{Q}) \to H^{p+n+1}(B;\mathbb{Q}) \otimes H^{q-n}(F;\mathbb{Q})$$

is a derivation. Since $d_{n+1} \neq 0$, then it cannot vanish simultaneously on all elements of the form $a \otimes 1$ and $1 \otimes u$, where a is a generator of $H^p(B;\mathbb{Q})$ and u of $H^q(F;\mathbb{Q})$.

Now, notice that $d_{n+1}(a \otimes 1) \in H^{|a|+n+1}(B;\mathbb{Q}) \otimes H^{-n}(F;\mathbb{Q}) = 0$. Then, there must exist $v \in H^*(F;\mathbb{Q})$ such that $d_{n+1}(1 \otimes v) \neq 0$. Fix a basis $\{w_i \mid 1 \leq i \leq \ell\}$ of $H^{n+1}(B;\mathbb{Q})$ and define $\theta_i : H^q(F;\mathbb{Q}) \to H^{q-n}(F;\mathbb{Q})$ in such a way that $d_{n+1}(1 \otimes u) = \sum_i w_i \otimes \theta_i(u)$ for $u \in H^q(F;\mathbb{Q})$. Each θ_i defines a derivation in $H^*(F;\mathbb{Q})$ of degree $-n$. Indeed, θ_i is well-defined and $\mathbb{Q}$-linearity is clear, so it is left to check the Leibniz rule: given $u_1, u_2 \in H^*(F;\mathbb{Q})$, we have

$$\begin{aligned}
\sum_{i=1}^{\ell} w_i \otimes \theta_i(u_1u_2) &= d_{n+1}(1 \otimes u_1u_2) \\
&= 1 \otimes d_{n+1}(u_1)u_2 + (-1)^{|u_1|} \cdot 1 \otimes u_1 d_{n+1}(u_2) \\
&= \sum_{i=1}^{\ell} w_i \otimes \theta_i(u_1)u_2 + (-1)^{|u_1|(1+|d(u_2)|)} \sum_{i=1}^{\ell} w_i \otimes \theta_i(u_2)u_1 \\
&= \sum_{i=1}^{\ell} w_i \otimes (\theta_i(u_1)u_2 + (-1)^{|u_1|(1+|d(u_2)|+|\theta_i(u_2)|)} u_1\theta_i(u_2)),
\end{aligned}$$

where, in the exponents, we write d for d_{n+1} applied to the second component of the tensor product. Since

$$|u_1|(1 + |d(u_2)| + |\theta_i(u_2)|) = |u_1|(2 + 2|u_2| + |\theta_i|) \equiv |u_1||\theta_i| \pmod 2,$$

this proves the Leibniz rule for θ_i. To conclude, note that $d_{n+1} \neq 0$ only if some $\theta_i \neq 0$, contradicting the assumption. □

Observe that the proof is valid if we replace $\mathbb{Q}$ by any field of characteristic zero. Moreover, we have only assumed that F is a space of finite type. In this generality, the converse is not true, and we discuss a counterexample by Yamaguchi [Yam05] in Section 4.5. However, it does hold under the assumption that F is positively elliptic, as will follow from Theorem 3.12. In order to get there, we need a series of preliminary results.

The following two lemmas lead us to show that every derivation in the cohomology of a formal minimal model is induced from a derivation in the minimal model itself. The first is an adapted version of [BM20, Lemma 3.5]. Recall from page 16 that given a morphism of dg algebras $f : A \to B$, the space of f-derivations $\mathrm{Der}_f(A, B)$ has a cochain complex structure with differential $[\delta, -]$.

Lemma 3.8. *Let A be a Sullivan algebra, and let B and C be cdg algebras such that $g : B \to C$ is a quasi-isomorphism. Then, for every morphism of cdg algebras $h : A \to B$, the induced chain map*

$$g_* : \mathrm{Der}_h(A, B) \to \mathrm{Der}_{gh}(A, C)$$

is a quasi-isomorphism.

Proof. We show the result by applying the comparison theorem for spectral sequences (Theorem 2.25). We first describe the relevant construction for the space $\mathrm{Der}_h(A, B)$, and then the case of $\mathrm{Der}_{gh}(A, C)$ follows similarly.

By definition, the Sullivan algebra (A, d_A) admits a filtration of the space of generators

$$0 \subseteq V^{(0)} \subseteq V^{(1)} \subseteq \cdots \subseteq V,$$

with $V = \bigcup_p V^{(p)}$, and $d_A(V^{(p)}) \subseteq V^{(p-1)}$. Since A is free graded commutative, we have $\mathrm{Der}_h(A, B) \cong \mathrm{Hom}(V, B)$ following Remark 2.16.

Next, we define a filtration

$$\mathrm{Der}_h(A, B) = F^{-1} \supseteq F^0 \supseteq F^1 \supseteq \dots$$

given by $F^p = \{\theta \in \mathrm{Der}_h(A, B) \mid \theta = 0 \text{ on } V^{(p)}\}$. This filtration is compatible with the differential $[\delta, -]$ on $\mathrm{Der}_h(A, B)$. Indeed, given $\theta \in F^p$ and $x \in V^{(p)}$, we have

$$[\delta, \theta](x) = d_B\theta(x) + (-1)^{|\theta|}\theta d_A(x) = 0,$$

since $d_A(x) \in V^{(p-1)} \subseteq V^{(p)}$.

In addition, observe that the assumptions of Theorem 2.24 are satisfied. The filtration is exhaustive because $F^{-1} = \mathrm{Der}_h(A, B)$, and is weakly convergent because $\bigcap_p F^p = 0$. We deduce that there is a cohomology spectral sequence with

$$E_1^{p,q} = H^{p+q}(F^p/F^{p+1}) \implies H^{p+q}(\mathrm{Der}_h(A, B)).$$

In order to apply the comparison theorem, we will need a more suitable expression for the E_1 page in terms of the spaces of generators. The inclusion $V^{(p)} \hookrightarrow V$ gives rise to a short exact sequence

$$0 \to V^{(p)} \to V \to V/V^{(p)} \to 0.$$

Since we are working with vector spaces, the functor $\mathrm{Hom}(-, B)$ is exact and yields an exact sequence

$$0 \to \mathrm{Hom}(V/V^{(p)}, B) \to \mathrm{Hom}(V, B) \to \mathrm{Hom}(V^{(p)}, B) \to 0.$$

Observe that we can regard F^p as the kernel of $\mathrm{Hom}(V, B) \to \mathrm{Hom}(V^{(p)}, B)$. Hence, there is an isomorphism $F^p \cong \mathrm{Hom}(V/V^{(p)}, B)$. Similarly, we have $F^{p+1} \cong \mathrm{Hom}(V/V^{(p+1)}, B)$.

The inclusion $V^{(p+1)} \hookrightarrow V$ gives rise to another short exact sequence

$$0 \to V^{(p+1)}/V^{(p)} \to V/V^{(p)} \to V/V^{(p+1)} \to 0,$$

and subsequently

$$0 \to \mathrm{Hom}(V/V^{(p+1)}, B) \to \mathrm{Hom}(V/V^{(p)}, B) \to \mathrm{Hom}(V^{(p+1)}/V^{(p)}, B) \to 0.$$

It follows that there is an isomorphism $\varphi : F^p/F^{p+1} \xrightarrow{\cong} \mathrm{Hom}(V^{(p+1)}/V^{(p)}, B)$ of graded vector spaces. We claim that it is actually an isomorphism of cochain complexes. Note that the cochain complex $\mathrm{Hom}(V^{(p+1)}/V^{(p)}, B)$ has differential $[\delta, f] = d_B f + (-1)^k f d_A$, where $f : V^{(p+1)}/V^{(p)} \to B[k]$ is a morphism of graded vector spaces. Since $d_A(V^{(p+1)}) \subseteq V^{(p)}$, the differential reduces to composition with d_B. The image of a class of derivations $\theta + F^{p+1}$ is determined by $\varphi(\theta + F^{p+1})(x + V^{(p)}) = \theta(x)$ on elements of $V^{(p+1)}/V^{(p)}$. This is well-defined because both filtrations are compatible with the differentials. For clarity, let us write φ_θ for $\varphi(\theta + F^{p+1})$. Then,

$$\begin{aligned}
\varphi_{[\delta,\theta]}(x + V^{(p)}) &= [\delta, \theta](x) = d_B\theta(x) + (-1)^{|\theta|}\theta d_A(x) \\
&= d_B\varphi_\theta(x + V^{(p)}) + (-1)^{|\theta|}\varphi_\theta(d_A x + V^{(p)}) \\
&= d_B\varphi_\theta(x + V^{(p)}),
\end{aligned}$$

since $d_A x \in V^{(p)}$. Note that the degree of φ_θ is also $|\theta|$.

Hence, we have an isomorphism in cohomology

$$H^{p+q}(F^p/F^{p+1}) \cong H^{p+q}(\mathrm{Hom}(V^{(p+1)}/V^{(p)}, B)).$$

The same reasoning can be done for the filtration

$$\mathrm{Der}_{gh}(A, C) = G^{-1} \supseteq G^0 \supseteq G^1 \supseteq \dots$$

given by $G^p = \{\theta \in \mathrm{Der}_{gh}(A, C) \mid \theta = 0 \text{ on } V^{(p)}\}$. We get a second cohomology spectral sequence

$$E_1'^{p,q} = H^{p+q}(G^p/G^{p+1}) \implies H^{p+q}(\mathrm{Der}_{gh}(A, B))$$

and an isomorphism

$$H^{p+q}(G^p/G^{p+1}) \cong H^{p+q}(\mathrm{Hom}(V^{(p+1)}/V^{(p)}, C)).$$

Now, for fixed p, the map of dg algebras $g : B \to C$ induces a cochain map $g_* : \mathrm{Hom}(V^{(p+1)}/V^{(p)}, B) \to \mathrm{Hom}(V^{(p+1)}/V^{(p)}, C)$. Crucially, since $\mathrm{Hom}(V^{(p+1)}/V^{(p)}, -)$ is an exact functor in this setting, g_* is a quasi-isomorphism just like g. This yields an isomorphism of differential bigraded modules between the first pages of the spectral sequences E and E'. The comparison theorem allows to conclude that there is an isomorphism

$$H^*(\mathrm{Der}_h(A, B)) \cong H^*(\mathrm{Der}_{gh}(A, B)).$$

□

Lemma 3.9. *Let M be a formal minimal model and let H be its cohomology. Consider the chain map*

$$I : H(\mathrm{Der}(M)) \to \mathrm{Der}(H),$$

defined on a cohomology class $[\theta]$ by $I([\theta])[x] = [\theta(x)]$ for $x \in M$. Then, the map I is surjective.

Proof. Since M is formal, there is a map $f : M \to H$ inducing an isomorphism in cohomology. We may assume the induced map is the identity on H. Next, consider the functors $\mathrm{Der}(M, -)$ and $\mathrm{Der}(-, H)$. Then f induces maps $f_* : \mathrm{Der}(M, M) \to \mathrm{Der}(M, H)$ and $f^* : \mathrm{Der}(H, H) \to \mathrm{Der}(M, H)$. In cohomology, we have the diagram

$$\begin{tikzcd} H(\mathrm{Der}(M,M)) \arrow[r, "H(f_*)"] \arrow[dr, "I"'] & H(\mathrm{Der}(M,H)) \arrow[d, bend left, "J"] \\ & \mathrm{Der}(H,H) \arrow[u, "H(f^*)"] \end{tikzcd}$$

where I is defined on a cocycle $[\theta] \in H(\mathrm{Der}(M, M))$ by $I([\theta])[x] = [\theta(x)]$, and similarly $J([\omega])[x] = \omega(x)$ for $[\omega] \in H(\mathrm{Der}(M, H))$. Recall that being a cocycle in this case means commuting with the differential d of M. Abusing notation, we write classes in H as elements since the differential of H is zero. We claim that the following statements hold:

(1) The map f_* is a quasi-isomorphism.

(2) $JH(f^*) = \mathbb{1}$.

(3) $JH(f_*) = I$.

This allows us to conclude that I is surjective, as the composition of an isomorphism with an epimorphism.

The first claim follows from Lemma 3.8, taking g to be the quasi-isomorphism $f : M \to H$ and h the identity on M. Note that we use that M is a Sullivan algebra. For property (2), observe that $f^*\theta = \theta f$ for $\theta \in \mathrm{Der}(H, H)$. Hence, for $[x] \in H$ we have

$$JH(f^*)(\theta)[x] = J([\theta f])[x] = \theta f(x) = \theta[x].$$

Finally, in the case of (3), given $\eta \in \mathrm{Der}(M, M)$, we use $f_*\eta = f\eta$ and obtain

$$JH(f_*)([\eta])[x] = J([f\eta])[x] = f\eta(x) = [\eta(x)] = I([\eta])([x]).$$

$\square$

Next, we will state a powerful result allowing to construct fibrations starting from a purely algebraic setting, namely a suitable map involving Sullivan models. This construction is essential in the proof of Theorem 3.12. To make the construction precise, we would first need to introduce in detail simplicial sets and the abstract notion of (co)fibrations in category theory, which lies out of the scope of this work. Instead, we briefly describe the relevant concepts and then outline the proof of Proposition 3.10 below, which the interested reader should able to reconstruct in detail. For a complete overview, we refer to [FHT01, §17].

Let Δ be the category having as objects the ordered sets $[n] = \{1, \dots, n\}$ for $n \in \mathbb{N}_{>0}$, and as morphisms the non-decreasing maps $[n] \to [m]$. A **simplicial set** is a functor $X : \Delta^{op} \to \mathbf{Set}$, and we denote $X_n = X([n])$. For example, given a topological space Y, we obtain a simplicial set $S_\bullet(Y)$, called the singular simplicial set, by defining

$$S_n(Y) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y)$$

as the set of continuous maps from the n-simplex Δ^n to Y. Here **Top** denotes the category of topological spaces. In turn, every morphism $\varphi : [m] \to [n]$ gives a continuous map $\varphi_* : \Delta^m \to \Delta^n$, and we get $\varphi^* : S_n(Y) \to S_m(Y)$ by setting $\varphi^*(f) = f\varphi_*$ for $f \in S_n(Y)$. The details can be consulted in [Ber, p. 22]. We denote by **sSet** the category of simplicial sets. This category will serve to connect dg algebras with topological spaces, via two adjunctions of functors.

First, one can construct a topological space called the *geometric realization* of a simplicial set X, by setting

$$|X| = \coprod_{n \in \mathbb{N}_{>0}} X_n \times \Delta^n / \sim,$$

where $(\varphi^*(x), \mathbf{t}) \sim (x, \varphi_*(\mathbf{t}))$ for all morphisms $\varphi \in \Delta^n$ and elements $x \in X_n$ and $\mathbf{t} \in \Delta^n$. The geometric realization is a functor $|-| : \mathbf{sSet} \to \mathbf{Top}$, which is left adjoint to $S_\bullet : \mathbf{Top} \to \mathbf{sSet}$. In other words, we have a natural bijection

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, S_\bullet(Y))$$

for every simplicial set X and space Y.

In the second place, given a commutative cochain algebra A, we can form the *spatial realization* of A as the simplicial set

$$\langle A \rangle = \mathrm{Hom}_{\mathbf{CDGA}}(A, A^*_{PL}(\Delta_\bullet)),$$

where $A^*_{PL}(\Delta_\bullet)$ is the functor mapping a simplicial set $[n]$ into the algebra $A^*_{PL}(\Delta_n)$ defined on page 31. We call it the simplicial de Rham algebra. Here **CDGA** denotes the category of cdg algebras. The spatial realization defines a contravariant functor, that is a functor $\langle - \rangle : \mathbf{CDGA} \to \mathbf{sSet}^{op}$, which is left-adjoint to $A^*_{PL}(\Delta_\bullet) : \mathbf{sSet}^{op} \to \mathbf{CDGA}$. Equivalently, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{sSet}}(X, \langle A\rangle) = \mathrm{Hom}_{\mathbf{sSet}^{op}}(\langle A\rangle, X) \cong \mathrm{Hom}_{\mathbf{CDGA}}(A, A^*_{PL}(X))$$

for every simplicial set X and cdg algebra A. Although we will not define the concepts, an important property of the functor $\langle - \rangle$ is that it takes cofibrations of dg algebras, the dual notion of fibrations in the category **CDGA**, to Kan fibrations, the analogue of fibrations in **sSet**.

After this discussion, we are ready to show the following result.

Proposition 3.10. *Let $(\Lambda V, d)$ and $(\Lambda W, \delta)$ be relative Sullivan models and $f : \Lambda V \to \Lambda V \otimes \Lambda W$ a morphism of cdg algebras given by the inclusion. Then, the map of topological spaces*

$$|\langle \Lambda V \otimes \Lambda W\rangle| \to |\langle \Lambda V\rangle|$$

is a fibration whose fibre has $(\Lambda W, \delta)$ as its Sullivan model.

Proof (outline). The map f is a cofibration of algebras, and thus the functor $\langle - \rangle$ maps it into a Kan fibration of simplicial sets $\langle \Lambda V \otimes \Lambda W\rangle \to \langle \Lambda V\rangle$ between the respective spatial realizations. In turn, the geometric realization of a Kan fibration is a fibration (see the remark in [Qui68]), implying that the map $|\langle \Lambda V \otimes \Lambda W\rangle| \to |\langle \Lambda V\rangle|$ is a fibration. By construction, the geometric realizations are rational spaces (see [FOT08, p. 90]), and it can be worked out that a Sullivan model of the fibre is given by $(\Lambda W, \delta)$. □

Finally, we prove a technical lemma allowing to replace an orientable fibration involving the rationalizations of certain spaces with a fibration of the spaces themselves, preserving the rational homotopy type. The underlying idea is to replace the fibration of rational spaces emerging from Proposition 3.10 with a suitable fibration.

Lemma 3.11. *Let $n \geq 2$ and consider an elliptic space F. Then, given an orientable fibration $F_{\mathbb{Q}} \to E' \xrightarrow{p} S^n_{\mathbb{Q}}$, there exists an orientable fibration $F \to E \xrightarrow{\pi} S^n$ such that the following diagram commutes up to homotopy*

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "\sim_{\mathbb{Q}}"'] & S^n \arrow[d, "\sim_{\mathbb{Q}}"] \\ E' \arrow[r, "p"] & S^n_{\mathbb{Q}}. \end{tikzcd}$$

Proof. Recall from Proposition 3.5 that the classifying space $\mathrm{Baut}_o X$ serves as the universal base for orientable fibrations with fibre the space X. Thus, the fibration p corresponds to a homotopy class of maps $S^n_{\mathbb{Q}} \to \mathrm{Baut}_o F_{\mathbb{Q}}$. Let f_p denote a representative of that class. Note that, choosing base points, f_p

defines a class in the set of homotopy classes of *pointed* maps $[S^n_{\mathbb{Q}}, \operatorname{Baut}_o F_{\mathbb{Q}}]^{\bullet}$, which is precisely the rational homotopy group $\pi_n(\operatorname{Baut}_o F_{\mathbb{Q}}) \otimes \mathbb{Q}$.

Next, from Proposition 3.5 we have an isomorphism

$$\pi_n(\operatorname{Baut}_o F_{\mathbb{Q}}) \otimes \mathbb{Q} \cong \pi_n(\operatorname{Baut}_o F) \otimes \mathbb{Q}.$$

This allows to construct the sought fibration. First, we make the following remark: given an algebra A and the canonical map $q : A \to A \otimes \mathbb{Q}$, then for every $x \in A \otimes \mathbb{Q}$ there exists a non-zero integer k such that $kx = q(a)$ for some $a \in A$.

We apply this fact to the map $q : \pi_n(\operatorname{Baut}_o F) \to \pi_n(\operatorname{Baut}_o F) \otimes \mathbb{Q}$. Omitting the underlying isomorphisms, $[f_p]$ is a class in $\pi_n(\operatorname{Baut}_o F) \otimes \mathbb{Q}$, and thus there exists a map $\alpha : S^n \to \operatorname{Baut}_o(F)$ such that $k[f_p] = q[\alpha]$ for some $k \in \mathbb{Z}$. Multiplication by k in $\pi_n(\operatorname{Baut}_o F) \otimes \mathbb{Q}$ is realised by a degree k map $S^n_{\mathbb{Q}} \xrightarrow{k} S^n_{\mathbb{Q}}$, induced by a degree k map on S^n. Note that the map k is a rational homotopy equivalence, and thus also the composition kr, where $r : S^n \to S^n_{\mathbb{Q}}$ denotes the rationalization. With this, the map α represents an orientable fibration $\pi : E \to S^n$ with fibre F.

To complete the proof, note that we have a commutative diagram

$$\begin{tikzcd}
S^n \arrow[r, "r"] \arrow[drr, "\alpha"'] & S^n_{\mathbb{Q}} \arrow[r, "k"] & S^n_{\mathbb{Q}} \arrow[r, "f_p"] & \operatorname{Baut}_o(F_{\mathbb{Q}}) \\
& & \operatorname{Baut}_o(F) \arrow[ur, "\sim_{\mathbb{Q}}"'] &
\end{tikzcd}$$

This can be included in the following diagram commuting up to homotopy:

$$\begin{tikzcd}
& & E' \arrow[r] \arrow[d, "p"] & \operatorname{Baut}^{\bullet}_o(F_{\mathbb{Q}}) \arrow[d] \\
& & S^n_{\mathbb{Q}} \arrow[r, "f_p"] & \operatorname{Baut}_o(F_{\mathbb{Q}}) \\
E \arrow[r] \arrow[d, "\pi"] \arrow[uurr, dashed, bend left] & \operatorname{Baut}^{\bullet}_o(F) \arrow[d] \arrow[uurr, bend right, "s", "\sim_{\mathbb{Q}}"'] & & \\
S^n \arrow[r, "\alpha"] \arrow[uurr, bend left, "kr", "\sim_{\mathbb{Q}}"'] & \operatorname{Baut}_o(F) \arrow[uurr, bend right, "\sim_{\mathbb{Q}}"'] & &
\end{tikzcd}$$

The upper-right square is the pullback square through which the fibration p is induced from the universal orientable fibration, and likewise for the bottom-left square and the fibration π. The rational homotopy equivalence $\operatorname{Baut}^{\bullet}_o(F) \to \operatorname{Baut}^{\bullet}_o(F_{\mathbb{Q}})$ follows similarly as for the base spaces. Therefore, by the universal property of homotopy pullbacks, we get an induced map $E \to E'$ such that the tilted upper-left square commutes up to homotopy. Furthermore, it is a rational homotopy equivalence because so are the maps s and kr, concluding the proof. $\square$

We are now ready to state the following theorem [Mei82, Lemma 2.5(b)]. We provide an alternative argument to Meier's using the lemmas above.

Theorem 3.12. *Let F be a formal, elliptic space, and assume that every orientable fibration with fibre F and base an odd sphere is TNCZ. Then, $H^*(F;\mathbb{Q})$ has no non-trivial derivations of even negative degree.*

Proof. Consider a derivation $\tilde{\theta}$ in $H^*(F;\mathbb{Q})$. Since F is formal, by Lemma 3.9 every such derivation is the cohomology class of a derivation θ in the minimal model (M_F, d). Suppose that θ has even negative degree $-2k$. Consider the map of dg algebras $\pi^* : (\Lambda u, 0) \to (\Lambda u \otimes M_F, D)$, where $D = 1 \otimes d + u \otimes \theta$ and $|u| = 2k + 1$. Note that Λu is the minimal model of S^{2k+1}. By Proposition 3.10, this represents an orientable fibration of rational spaces $F_{\mathbb{Q}} \to E' \to S^{2k+1}_{\mathbb{Q}}$. Then, by Lemma 3.11, we get a fibration $F \to E \to S^{2k+1}$ inducing the same sequence of Sullivan models. In particular, there is a morphism of dg algebras $i^* : (\Lambda u \otimes M_F, D) \to (M_F, d)$ mapping $1 \otimes x \mapsto x$ and $u \otimes x \mapsto 0$. By assumption, such fibrations are TNCZ, hence i^* is surjective in cohomology. We show that then $\tilde{\theta} = 0$.

For that, call $A = \Lambda u \otimes M_F$ and note that $uA = u \otimes M_F$ is a copy of the cochain complex M_F with degrees shifted by $|u| = 2k + 1$, since $D(u \otimes x) = -u \otimes dx$. We have an obvious inclusion $j : uA \hookrightarrow A$, yielding a short exact sequence of cochain complexes $0 \to uA \to A \to A/uA \to 0$. Here, $A/uA \cong 1 \otimes M_F$ and the map $A \to A/uA$ is precisely the map i^* induced by the constructed fibration. We get a long exact sequence of rational cohomology groups

$$\begin{tikzcd}
\cdots \arrow[r] & H^{\ell}(uA) \arrow[r, "j^*"] \arrow[d, "\cong"] & H^{\ell}(A) \arrow[r, "i^*"] \arrow[d, "\cong"] & H^{\ell}(A/uA) \arrow[r, "\partial"] \arrow[d, "\cong"] & H^{\ell+1}(uA) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H^{\ell-2k-1}(M_F) \arrow[r] & H^{\ell}(A) \arrow[r, "i^*"] & H^{\ell}(M_F) \arrow[r] & H^{\ell-2k}(M_F) \arrow[r] & \cdots
\end{tikzcd}$$

where ∂ can be shown to map $[1 \otimes x + uA] \mapsto [u \otimes \theta(x)]$ by diagram chase. Thus, in the bottom line this becomes precisely $\tilde{\theta} : H^{\ell}(M_F) \to H^{\ell-2k}(M_F)$. Since i^* is surjective, exactness implies $\tilde{\theta} = 0$. □

The results above are merged in the next corollary, leading to the algebraic statement of the Halperin conjecture.

Corollary 3.13 (Meier, [Mei82])**.** *Let F be a positively elliptic space. The following are equivalent:*

(1) Every orientable fibration $F \to E \to B$ is TNCZ.

(2) Every orientable fibration $F \to E \to S^{2k+1}$, for $k \geq 1$, is TNCZ.

(3) The cohomology algebra of F has no non-trivial derivations of negative degree, i.e. $\mathrm{Der}^{<0} H^*(F;\mathbb{Q}) = 0$.

Proof. It is clear that (1) implies (2). Next, since F is positively elliptic, $H^*(F;\mathbb{Q})$ is concentrated in even degrees. Therefore, derivations of odd degree can only be trivial. Assuming (2), and given that positively elliptic spaces are formal, it follows from Theorem 3.12 that all derivations of even negative degree are also zero. Hence, $\operatorname{Der}^{<0} H^*(F;\mathbb{Q}) = 0$. Finally, F is of finite type because it is elliptic, so (3) implies (1) by Theorem 3.7. □

3.2 Statement using the space of self-homotopy equivalences

As mentioned above, Meier provides a third equivalent statement in [Mei82]. Meier's result is formulated in terms of the classifying space $\operatorname{Baut}_o F_{\mathbb{Q}}$ of the rationalization of the fibre. Here, instead, we use the space of self-homotopy equivalences $\operatorname{aut} F$. In this way, we obtain the statement as a property of a more familiar space and not involving the rationalization $F_{\mathbb{Q}}$.

Conjecture 3.14. *If F is a positively elliptic space, then $\pi_{2n} \operatorname{aut} F \otimes \mathbb{Q} = 0$ for all $n > 0$.*

The discussion in the coming pages will pave the way for the proof of Proposition 3.18, in which we relate the homotopy groups of the classifying space $\operatorname{Baut}_o F_{\mathbb{Q}}$ to the space of derivations of $H^*(F;\mathbb{Q})$ for a positively elliptic space F. Namely, we find that $\pi_{2n+1} \operatorname{Baut}_o F_{\mathbb{Q}} \cong \operatorname{Der}^{-2n} H^*(F;\mathbb{Q})$ for $n > 0$. The statement of Conjecture 3.14 will follow since

$$\pi_{2n+1} \operatorname{Baut}_o F_{\mathbb{Q}} \cong \pi_{2n+1} \operatorname{Baut} F \otimes \mathbb{Q} \cong \pi_{2n} \operatorname{aut} F \otimes \mathbb{Q}$$

for $n > 0$, as a consequence of Proposition 3.5.

Thus, consider first an algebra of the form $A = \mathbb{Q}[z_1, \ldots, z_k]/(f_1, \ldots, f_k)$ where $|z_i| > 0$ is even and f_i is a homogeneous polynomial for $i \in \{1, \ldots, k\}$. Let us construct the Sullivan algebra $\Lambda = \Lambda(x_1, \ldots, x_k, y_1, \ldots, y_k)$ such that $|x_i| = |z_i|$, $|y_i| = |f_i| - 1$, $dx_i = 0$ and $dy_i = f_i$ for each i. Then, there is a natural map $q : \Lambda \to A$ with $x_i \mapsto z_i$ and $y_i \mapsto 0$. Note that q induces a map in cohomology. We give an equivalent condition to q being a quasi-isomorphism, which follows from [BH98, Thm. 1.6.17].

Proposition 3.15. *The map $q : \Lambda \to A$ is a quasi-isomorphism if and only if $(f_1, \ldots, f_k)$ is a regular sequence.*

In other words, the map q is a quasi-isomorphism if and only if A is positively elliptic. We may assume that the map induced in cohomology is the identity and regard A as $H(\Lambda)$. Note that, in this case, each z_i represents the class of x_i in cohomology. Thus, abusing notation, we will denote z_i by $[x_i]$, or simply x_i when it is clear that it refers to the cohomology class.

The next result provides a useful choice of generators and a regular sequence of polynomials in the presentation of positively elliptic algebras. For a proof, see [KW23, Thm. 3.1].

Proposition 3.16. *Let H be a non-zero positively elliptic algebra. Then, there exists a presentation*

$$H \cong \frac{\mathbb{Q}[x_1, \dots, x_k]}{(f_1, \dots, f_k)},$$

with the following properties:

(1) Each $|x_i| > 0$ is even and $|x_1| \leq \cdots \leq |x_k|$.

(2) $f_1, \dots, f_k$ is a regular sequence of homogeneous polynomials lying in

$$\mathbb{Q}^{\geq 2}[x_1, \dots, x_k] = \operatorname{span}\{x_1^{a_1} \cdots x_k^{a_k} \mid a_1 + \cdots + a_k \geq 2\},$$

and $|f_1| \leq \cdots \leq |f_k|$.

(3) $|f_i| \geq 2|x_i|$ for all $i \in \{1, \dots, k\}$.

(4) The formal dimension of H satisfies

$$\operatorname{fd} H = \sum_{i=1}^{k} |f_i| - |x_i|.$$

This presentation of H is called a **pure presentation**. It follows that the cohomology of any positively elliptic space X admits a pure presentation. Let us set $A = H^*(X; \mathbb{Q})$. Observe that, since $f_i \in \mathbb{Q}^{\geq 2}[x_1, \dots, x_k]$, then $|f_i| \geq 4$, so the Sullivan algebra $\Lambda = \Lambda(x_1, \dots, x_k, y_1, \dots, y_k)$ has generators of degree at least 2. Furthermore, by Proposition 3.15, we have a quasi-isomorphism $q : \Lambda \to A$. Recall from the example on page 33 that every positively elliptic space is formal. Then, we deduce that Λ serves as the rational minimal model M_X of the space X. Observe that the map q induces a map between the spaces of derivations,

$$q_* : \operatorname{Der} \Lambda \longrightarrow \operatorname{Der}_q(\Lambda, A),$$

given by $q_*(\theta) = q\theta$. Lemma 3.8 implies that q_* is a quasi-isomorphism.

As discussed in Remark 2.16, since Λ is a free graded commutative algebra, the space $\operatorname{Der}(\Lambda)$ is a free Λ-module. A basis is given by the derivations $\partial/\partial z$ mapping z to the unit $1 \in \Lambda$ and the remaining generators to 0, where $z \in \{x_1, \dots, x_k, y_1, \dots, y_k\}$. In other words, any derivation in $\operatorname{Der}(\Lambda)$ is of the form

$$\sum_i g_i \frac{\partial}{\partial x_i} + \sum_i h_i \frac{\partial}{\partial y_i}$$

for $g_i, h_i \in \Lambda$. For instance, we may write the differential in Λ as

$$d = \sum_{i=1}^{k} f_i \frac{\partial}{\partial y_i}.$$

Abusing notation, we use the same expression for the basis elements in the free A-module $\mathrm{Der}_q(\Lambda, A)$. In this case, each $\partial/\partial z$ represents a map into A instead, and the general form holds for $g_i, h_i \in A$.

Since A is evenly graded, any expression of the kind $\sum_i g_i \partial/\partial x_i$ with $g_i \in A$ defines a derivation of even degree, whereas $\sum_i g_i \partial/\partial y_i$ defines a derivation of odd degree. Thus, we can split

$$\mathrm{Der}_q(\Lambda, A) = \mathrm{Der}_q^{\mathrm{even}}(\Lambda, A) \oplus \mathrm{Der}_q^{\mathrm{odd}}(\Lambda, A),$$

where $\mathrm{Der}_q^{\mathrm{even}}(\Lambda, A)$ is spanned by the $\partial/\partial x_i$ and $\mathrm{Der}_q^{\mathrm{odd}}(\Lambda, A)$ by the $\partial/\partial y_i$.

This split reveals an interesting structure of $\mathrm{Der}_q(\Lambda, A)$ as a cochain complex. Recall from page 16 that $\mathrm{Der}_q(\Lambda, A)$ can be equipped with a differential $[\delta, -]$. In this case, A is a cohomology algebra and has trivial differential, so $[\delta, -]$ equals (up to sign) precomposition with d. Given $j \in \{1, \dots, k\}$, we observe that

$$\left[\delta, \frac{\partial}{\partial y_j}\right] = \frac{\partial}{\partial y_j} d = \sum_{i=1}^{k} \frac{\partial}{\partial y_j}\left(f_i \frac{\partial}{\partial y_i}\right) = \sum_{i=1}^{k} q(f_i) \frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i} = 0,$$

using the Leibniz rule and that $q(f_i) = 0$ by construction. Hence, $[\delta, -]$ vanishes on odd derivations. In turn, even derivations are mapped to odd derivations, since $[\delta, -]$ increases the degree by 1. Thus, we have a sequence

$$\mathrm{Der}_q^{\mathrm{even}}(\Lambda, A) \xrightarrow{[\delta,-]} \mathrm{Der}_q^{\mathrm{odd}}(\Lambda, A) \xrightarrow{[\delta,-]} 0,$$

and we can compute the cohomology of the complex $\mathrm{Der}_q(\Lambda, A)$ as

$$H^i(\mathrm{Der}_q(\Lambda, A)) = \begin{cases} \ker[\delta, -] & \text{for } i \text{ even}, \\ \operatorname{coker}[\delta, -] & \text{for } i \text{ odd}. \end{cases}$$

After this discussion, we are ready to prove the main result. Like Meier, we use the space $\mathrm{Baut}_o F_{\mathbb{Q}}$ in the formulation, although we eventually state the conjecture for $\mathrm{aut}\, F$. The interest of this is twofold. First, condition (1) in the proposition cannot be translated in terms of the space $\mathrm{aut}\, F$, as the groups $\pi_1 \mathrm{Baut}\, F$ and $\pi_1 \mathrm{Baut}_o F$ are not isomorphic in general. Hence, we have a more complete statement if we work with $\mathrm{Baut}_o F_{\mathbb{Q}}$. Second, as the universal base for orientable fibrations, this classifying space will allow to establish a direct connection to fibrations without referring to the algebraic statement (Proposition 3.20).

We state first a preparatory lemma.

Lemma 3.17. *Let F be a formal, simply connected space of finite type. Then, every element in $\pi_1 \mathrm{Baut}_o F_{\mathbb{Q}}$ is represented by a morphism of cdg algebras $M_F \to H^*(F; \mathbb{Q})$ inducing the identity in cohomology.*

Proof. Recall that $\pi_1 \operatorname{Baut}_o F_{\mathbb{Q}}$ coincides with the kernel o of the homomorphism $\pi_0 \operatorname{aut} F_{\mathbb{Q}} \to \operatorname{Aut} H^*(F;\mathbb{Q})$.

Since F is formal, we have a morphism of cdg algebras $M_F \to H^*(F;\mathbb{Q})$ inducing an isomorphism in cohomology. Hence, by [FHT01, Prop. 12.9], there is a bijection between the sets $[M_F, M_F]$ and $[M_F, H^*(F;\mathbb{Q})]$ of homotopy classes of cdg algebra maps, in the sense defined in [FHT01, §12].

Now, observe that there is an equivalence between the homotopy category of simply connected spaces of finite type and the homotopy category of simply connected cdg algebras of finite type (see [FHT01, §17]). Thus, each class of maps in $\pi_0 \operatorname{aut} F_{\mathbb{Q}}$ corresponds to precisely one class in $[M_F, M_F]$. In particular, any element in o corresponds to an element in $[M_F, M_F] \cong [M_F, H^*(F;\mathbb{Q})]$ inducing the identity in cohomology, and subsequently is represented by a morphism $g : M_F \to H^*(F;\mathbb{Q})$. $\square$

Proposition 3.18 (Meier, [Mei82])**.** *Let F be a positively elliptic space. Then*

(1) $\pi_1 \operatorname{Baut}_o F_{\mathbb{Q}} = 0$,

(2) $\pi_{2n+1} \operatorname{Baut}_o F_{\mathbb{Q}} \cong \operatorname{Der}^{-2n} H^*(F;\mathbb{Q})$ *for all* $n \geq 1$.

Proof. For (1), it suffices to prove that there is precisely one map of cdg algebras $g : M_F \to H^*(F;\mathbb{Q})$ inducing the identity in cohomology. The claim then follows from Lemma 3.17.

By Proposition 3.16, we can write $H^*(F;\mathbb{Q})$ as $\mathbb{Q}[x_1,\dots,x_k]/(f_1,\dots,f_k)$ and the minimal model M_F as $\Lambda(x_1,\dots,x_k,y_1,\dots,y_k)$ with the differential d described above. Since g induces the identity in $H^*(F;\mathbb{Q})$, we have $g(x_i) = x_i$ for each i. In turn, given that $H^*(F;\mathbb{Q})$ is evenly graded and g is a morphism of graded algebras, each $g(y_i)$ has odd degree and hence is zero. We deduce that o is a singleton.

Now let us prove (2). For $n \geq 1$, we have a chain of isomorphisms

$$H^{-2n} \operatorname{Der} M_F \overset{\text{(a)}}{\cong} \pi_{2n} \operatorname{aut} F \otimes \mathbb{Q} \overset{\text{(b)}}{\cong} \pi_{2n+1} \operatorname{Baut} F \otimes \mathbb{Q} \overset{\text{(c)}}{\cong} \pi_{2n+1} \operatorname{Baut}_o F_{\mathbb{Q}}.$$

Here, (a) is an application of [FLS10, Thm. 4.2] to the fibration $F \to *$, (b) is a property of classifying spaces, and (c) is part of Proposition 3.5. Furthermore, we can compute $H^{2n} \operatorname{Der} M_F \cong H^{2n} \operatorname{Der}_q(\Lambda, A)$ as $\ker[\delta, -]$ following the discussion above. For simplicity, here we use $[\delta, -]$ to refer to the differential restricted to derivations of degree $-2n$. Hence, it only remains to prove that there is an isomorphism between $\ker[\delta, -]$ and $\operatorname{Der}^{-2n} H^*(F;\mathbb{Q})$.

First, notice that given $\theta \in \ker[\delta, -]$ and $i \in \{1,\dots,k\}$, we have

$$0 = [\delta, \theta](y_i) = -\theta d(y_i) = -\theta(f_i).$$

Hence, there is a well-defined derivation $\tilde{\theta}$ in A given by $\tilde{\theta}([x_i]) = \theta(x_i)$. By definition, the association $\theta \mapsto \tilde{\theta}$ defines a morphism of dg modules. We claim that it is an isomorphism. Indeed, if $\tilde{\theta} = 0$ then $\theta(x_i) = 0$ for all i, and this implies $\theta = 0$ because $\ker[\delta, -] \subseteq \mathrm{Der}_q^{\mathrm{even}}(\Lambda, A)$. In turn, given a derivation $\tilde{\theta}$ in A, we find a preimage $\theta \in \mathrm{Der}_q^{-2n}(\Lambda, A)$ by setting $\theta(x_i) = \tilde{\theta}([x_i])$ and 0 otherwise. Then,

$$[\delta, \theta](x_i) = -\theta d(x_i) = 0$$

since $dx_i = 0$, whereas

$$[\delta, \theta](y_i) = -\theta d(y_i) = -\theta(f_i) = 0$$

because $\tilde{\theta}([f_i]) = \tilde{\theta}([0]) = 0$. Therefore $\theta \in \ker[\delta, -]$, and we conclude that we have an isomorphism $\ker[\delta, -] \cong \mathrm{Der}_q^{-2n} A$, completing the proof. □

Remark 3.19. The chain of isomorphisms in the proof can also be used to obtain

$$\pi_{2n} \operatorname{Baut}_o F_{\mathbb{Q}} \cong \operatorname{coker}[\delta, -]$$

for the even homotopy groups. Nonetheless, contrary to the odd case, there is no simple expression describing $\operatorname{coker}[\delta, -]$, and this isomorphism is not relevant in the context of the Halperin conjecture. Hence, we leave the relation out of the statement. Meier's original work included a wrong relation for the even groups, corrected later in [Mei83, Prop. 1].

This result leads to Conjecture 3.14 by observing that all odd homotopy groups of $\operatorname{Baut}_o F_{\mathbb{Q}}$ vanish if and only if $H^*(F; \mathbb{Q})$ has no non-trivial derivations of negative degree. However, one of the directions is quite straightforward at this point of the text, as we show in the next proposition.

In the proof, we use a property of *nilpotent* spaces. A path-connected space X is called **nilpotent** if $\pi_1(X)$ is a nilpotent group and acts nilpotently on the higher homotopy groups. We will not review the details of this action or the properties of nilpotent spaces in this work, and refer to [FOT08, §2.4.1] instead. Nonetheless, this class of spaces is of great importance in rational homotopy theory, namely because most notions defined for simply connected spaces in Sections 2.4 and 2.5 can be extended to nilpotent spaces. In this way, one can obtain a more general theory by studying orientable fibrations with a nilpotent fibre. However, the technicalities needed are out the scope of this work, and we have decided to stay in the simply connected setting.

Here, the interest of nilpotent spaces is that they satisfy property (2) in Theorem 2.45. This follows from [BG76, Thm. 12.8(3)], since the space $Q_\infty X$ in the reference is rationally equivalent to X if X is nilpotent, as discussed in [BS25, Apx. C].

Proposition 3.20. *Let F be a positively elliptic space such that $\pi_{2n+1} \operatorname{Baut}_o F_{\mathbb{Q}} = 0$ for all $n \geq 0$. Then, every orientable fibration $F \to E \to B$ is TNCZ.*

Proof. We use [BZ24, Prop. 3.10] as in Lemma 3.11 and deduce that $\pi_{2n+1} \operatorname{Baut}_o F \otimes \mathbb{Q} = 0$. Now, by [DZ79, Thm. D], the space $\operatorname{Baut}_o F$ is nilpotent, and thus we have $H^{\mathrm{odd}}(\operatorname{Baut}_o F; \mathbb{Q}) = 0$ applying Theorem 2.45 (2). In turn, since F is positively elliptic, we know that also $H^{\mathrm{odd}}(F; \mathbb{Q}) = 0$. Hence, the Serre spectral sequence of the universal fibration

$$F \to \operatorname{Baut}_o^{\bullet} F \to \operatorname{Baut}_o F$$

has zero rows and columns at odd positions. By a degree argument, all differentials vanish and thus the sequence is TNCZ. Finally, every fibration with fibre F is induced from the universal fibration, and thus is TNCZ too as argued in Section 2.3.1. $\square$

4 Known results

The Halperin conjecture is known to hold in various specific settings, as we have anticipated. These may be divided into two classes, depending on whether geometric or algebraic assumptions are imposed. Here, we present the arguments for some of the most notable ones. In the first class, we cover the cases of Kähler manifolds and homogeneous spaces. As examples of algebraic settings, we show the conjecture for algebras with at most three generators or dimension up to 20.

Highlighting the topological significance of the conjecture, we will formulate the results in terms of the TNCZ property, even though we mainly use the algebraic statement in the proofs.

First of all, we introduce the concept of split positively elliptic algebras and state Markl's theorem, which will be useful throughout the chapter.

Definition 4.1. A positively elliptic algebra H is said to **split** if it has a pure presentation

$$H \cong \mathbb{Q}[x_1, \dots, x_k]/(f_1, \dots, f_k)$$

such that, for some $\ell \in \{1, \dots, k-1\}$, the polynomials $f_1, \dots, f_\ell$ only depend on $x_1, \dots, x_\ell$.

Next, we replicate the following observations in [KW23], as they provide a practical description of split positively elliptic algebras. If H splits, then it has a positively elliptic subalgebra

$$K \cong \mathbb{Q}[x_1, \dots, x_\ell]/(f_1, \dots, f_\ell)$$

and a quotient algebra $Q = H/K$ defined by $Q^n = H^n/(K^+H)^n$. Then, Q is also positively elliptic and can be presented as

$$Q \cong \mathbb{Q}[\bar{x}_{\ell+1}, \dots, \bar{x}_k]/[\bar{f}_{\ell+1}, \dots, \bar{f}_k],$$

where the bars denote the images under the projection $H \to Q$. Finally, note that $\operatorname{fd} H = \operatorname{fd} K + \operatorname{fd} Q$, and both H and Q have formal dimension strictly smaller than H.

Theorem 4.2 (Markl, [Mar90])**.** *Let H be a positively elliptic algebra admitting a non-zero derivation of negative degree. If H splits as above, then K or Q also admits a non-zero derivation of negative degree.*

In Sections 4.1 and 4.4, an induction argument is used to prove that certain positively elliptic algebra H does not admit non-trivial derivations of negative degree. Markl's result reduces the problem to the case when H does not split. Indeed, if H splits, the induction hypothesis ensures that neither K nor Q have non-trivial derivations of negative degree, and it follows from the theorem that H does not either.

4.1 Algebras with at most three generators

A first application of the algebraic statement is the proof of the Halperin conjecture for positively elliptic algebras with at most three generators. The case of at most two generators was shown by J.-C. Thomas in 1981 [Tho81]. His original result is prior to Meier's and did not use the reformulation in terms of derivations of algebras. We present it in Theorem 4.7, showing how simple the proof becomes when using the algebraic formulation. Later, in 1990, G. Lupton extended the result to three generators [Lup90]. In Theorem 4.9 we provide a partial proof of the statement.

We start by defining a common property of the objects we will work with. Recall that a bilinear map of finite-dimensional vector spaces $V \times W \to \Bbbk$ is called non-degenerate if the induced map $V \to \mathrm{Hom}_{\Bbbk}(W, \Bbbk)$, or equivalently $W \to \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$, is an isomorphism.

Definition 4.3. Let $H = H^{\geq 0}$ be a graded $\Bbbk$-algebra of finite type. Then, H is said to satisfy **Poincaré duality** if there exists $n \in \mathbb{N}$ such that $H^n \cong \Bbbk$, $H^i = 0$ for $i > n$ and the product map $H^i \times H^{n-i} \to H^n \cong \Bbbk$ is a non-degenerate bilinear map for $i \in \{0, \ldots, n\}$.

Since H is of finite type, the definition implies $H^i \cong H^{n-i}$. It is useful to regard the definition as follows: given a non-zero element x of degree i in an algebra H satisfying Poincaré duality, there exists $y \in H^{n-i}$ such that the product $xy \in H^n$ is non-zero, and thus a generator of the top graded component. It is known that the cohomology of a compact orientable manifold satisfies Poincaré duality. In the context of this work, we have the following relevant result [Hal77, Thm. 3'].

Proposition 4.4. *Positively elliptic algebras satisfy Poincaré duality.*

Next, in our way to prove Theorem 4.7, we establish the following two lemmas (see [AK20, Lemmas 11.1, 11.3]), which give sufficient conditions for a derivation of negative degree to vanish.

Lemma 4.5. *Let H be a positively elliptic algebra, and let θ be a derivation in H of negative degree $-n$, with $n > 0$. Then $\theta = 0$ on H^n. In particular, if all the generators of H have the same degree, then $\theta = 0$.*

Proof. Let $x \in H^n$. Since H is finite dimensional, there is an integer $j_0 > 0$ such that $H^j = 0$ for all $j > j_0$. In particular, there exists a maximal non-zero power x^i of x. Then, since x has even degree, we have

$$0 = \theta(x^{i+1}) = (i+1)x^i\theta(x).$$

Since $\theta(x) \in H^0 = \mathbb{Q}$, the latter is zero only if $\theta(x) = 0$.

For the second claim, observe that $\theta = 0$ if and only if $\theta(x) = 0$ on each generator x of H. If all the generators have the same degree, then $\theta(x)$ lies either in H^0 or a zero group, and thus vanishes in any case. $\square$

Lemma 4.6. *Let H be a positively elliptic algebra, and let θ be a derivation of negative degree in H such that $\theta(x_i) = 0$ for $k-1$ of the k generators x_i of H. Then $\theta = 0$.*

Proof. Reorder the generators such that $\theta(x_i) = 0$ for $i \geq 2$. We need to show that $\theta(x_1) = 0$. Otherwise, by Poincaré duality, there is an element $x \in H$, that we can assume monomial in the x_i, such that $\theta(x_1)x \neq 0$ generates the top cohomology group. We write $x = x_1^\ell x'$ with x' monomial in $x_2, \dots, x_k$. In particular $\theta(x') = 0$ because $\theta(x_i) = 0$ for $i \geq 2$. Furthermore, by degree reasons we have $x_1^{\ell+1}x' = 0$. Hence, using the power rule,

$$0 = \theta(x_1^{\ell+1}x') = \theta(x_1^{\ell+1})x' = (\ell+1)\theta(x_1)x_1^\ell x' = (\ell+1)\theta(x_1)x.$$

But this contradicts the fact that $\theta(x_1)x$ is a generator. $\square$

Theorem 4.7 (Thomas, [Tho81])**.** *Let F be a positively elliptic space such that $H^*(F;\mathbb{Q})$ has at most two generators. Then every orientable fibration $F \to E \to B$ is TNCZ.*

Proof. Using Theorem 3.7, it suffices to show that $H^*(F;\mathbb{Q})$ has no non-trivial derivations of negative degree. Assume $H^*(F;\mathbb{Q})$ has precisely two generators x and y, and let θ be a derivation of negative degree. Without loss of generality $|x| \leq |y|$. Then $\theta(x)$ can only land in $H^0(F;\mathbb{Q})$ or a zero group, and thus by Lemma 4.5 we have $\theta(x) = 0$. It follows from Lemma 4.6 that $\theta = 0$.

A similar argument works if $H^*(F;\mathbb{Q})$ has one generator, whereas the result is obvious if $H^*(F;\mathbb{Q})$ has no generators. $\square$

Next, we extend the result to cohomology algebras with three generators. Lupton's proof uses several preparatory lemmas and becomes too technical for the scope of this work, especially since its methods have not succeeded further. Instead, we provide a short argument under the assumption that the formal dimension of $H^*(F;\mathbb{Q})$ is at most 16, following the proof of [AK20, Lemma 11.5]. We need first a preliminary result [AK20, Lemma 11.4].

Lemma 4.8. *Let $H = \mathbb{Q}[x_1, \dots, x_k]/[f_1, \dots, f_k]$ be a pure presentation of a positively elliptic algebra. If there exists $\ell \in \{1, \dots, k-1\}$ such that $|f_\ell| < |x_1| + |x_{\ell+1}|$, then H splits into positively elliptic algebras K and Q of ℓ and $k-\ell$ generators, respectively.*

Proof. Under these assumptions, the regular sequence $f_1, \dots, f_\ell$ lies in $\mathbb{Q}[x_1, \dots, x_\ell]$. Indeed, assume otherwise that there exist $j \in \{1, \dots, \ell\}$ and $m \in \{\ell+1, \dots, k\}$ such that f_j has a non-zero term in x_m. Then, since the $f_i \in \mathbb{Q}^{\geq 2}[x_1, \dots, x_k]$, we would have

$$|x_1| + |x_{\ell+1}| \leq |x_1| + |x_m| \leq |f_j| \leq |f_\ell| < |x_1| + |x_{\ell+1}|,$$

a contradiction. It follows from the claim that the algebra splits. $\square$

Theorem 4.9 (Lupton, [Lup90])**.** *Let F be a positively elliptic space such that $H^*(F;\mathbb{Q})$ has at most three generators. Then every orientable fibration $F \to E \to B$ is TNCZ.*

Proof. Let us denote $H = H^*(F;\mathbb{Q})$. As discussed before, we assume that the formal dimension of H is at most 16.

Following Theorem 4.7, we may suppose that H has precisely three generators. A pure presentation is then given by

$$H \cong \mathbb{Q}[x_1, x_2, x_3]/[f_1, f_2, f_3]$$

for homogeneous polynomials $f_i \in \mathbb{Q}^{\geq 2}[x_1, x_2, x_3]$ and the x_i and f_i in increasing order of degrees. Using once more the algebraic statement of the conjecture, we assume by contradiction that H admits a non-trivial derivation θ of negative degree. We show that then H splits.

We make the following observations:

- $\theta([x_1]) = 0$ by Lemma 4.5.
- $\theta([x_2]) \neq 0$ and $\theta([x_3]) \neq 0$ by Lemma 4.6.
- $|x_1| < |x_2|$ by Lemma 4.5.
- $|x_2| < |x_3|$. Note that θ maps $H^{|x_2|}$ linearly into a space of dimension 0 or 1. If $|x_2| = |x_3|$, then $\theta([x_2]) = 0$ possibly after a change of basis.
- $|f_1| \geq |x_1| + |x_2|$ by Lemma 4.8. We have actually $|f_1| \geq |x_1| + |x_3|$. Indeed, otherwise $f_1 \in \mathbb{Q}[x_1, x_2]$. If $f_1 \in \mathbb{Q}[x_1]$, then H splits. In turn, if $f_1 \notin \mathbb{Q}[x_1]$, then applying θ to $[f_1] = 0$ gives a relation between $[x_1]$ and $[x_2]$ of degree smaller than f_1, which contradicts the ordering of the f_i.
- $|f_2| \geq |x_1| + |x_3|$ by Lemma 4.8.

Note that we also have $|f_i| \geq 2|x_i|$ by Proposition 3.16. Then, the expression

$$16 \geq \operatorname{fd} H = \sum_{i=1}^{3} |f_i| - |x_i|,$$

leaves $(2, 4, 6; 8, 8, 12)$ as the only possibility for the degrees of the x_i and f_i, respectively.

As reasoned in the proof of the fifth item above, f_1 contains a non-zero term involving x_3. Then, by degree reasons we have $f_1 = x_1x_3 + p(x_1, x_2)$ up to scaling. Similarly, we may assume $f_2 = x_1x_3 + q(x_1, x_2)$. Replacing f_1 by $f_1 - f_2$, we have $f_1 \in \mathbb{Q}[x_1, x_2]$. Like before, if $f_1 \notin \mathbb{Q}[x_1]$ we get a contradiction to the minimality of $|f_1|$. Thus we have $f_1 \in \mathbb{Q}[x_1]$ and conclude that H splits into algebras K^* and Q^* having strictly less generators. By Markl's theorem, then K^* or Q^* has a non-trivial derivation of negative degree, but this contradicts the two generators case of Theorem 4.7. □

4.2 Kähler manifolds

Next, we show that the Halperin conjecture holds when the fibre belongs to a family of spaces with great relevance in algebraic and differential geometry: Kähler manifolds. We provide a brief overview of these objects and prove the result in terms of the derivations of their cohomology in Theorem 4.12.

First, we recall some basic notions in differential geometry necessary to introduce Kähler manifolds. In the following, by *manifold* we mean a Hausdorff, second-countable space endowed with a finite-dimensional differentiable structure of class C^∞ over $\mathbb{R}$. A manifold M is called *symplectic* if it is equipped with a closed, non-degenerate differential 2-form ω. In turn, an *almost-complex* structure J on a manifold M is a smooth tensor field of degree $(1,1)$ such that $J_x^2 = -\mathbb{1}$ for each $x \in M$. The form J is called *integrable* if it is induced from a complex manifold structure. Lastly, a manifold is called *Riemannian* if the tangent space at each point is equipped with a symmetric, positive definite bilinear form.

Definition 4.10. A **Kähler manifold** is a symplectic manifold (M, ω) equipped with an integrable almost-complex structure J, such that the bilinear form

$$g(u, v) = \omega(u, Jv),$$

where u and v are tangent vectors at a given point, is symmetric and positive definite.

Equivalently, a Kähler manifold admits three compatible structures: complex, Riemannian and symplectic. The symplectic form ω is called the **Kähler form** of the manifold M. Examples of Kähler manifolds include $\mathbb{C}^n$, Riemann surfaces, complex projective spaces $\mathbb{C}P^n$ and smooth complex projective varieties (see [FOT08, p. 157]).

Definition 4.11. Let $H = H^{\geq 0}$ be a graded algebra. Then, H is said to satisfy the **hard Lefschetz property** if there exists $\omega \in H^2$ such that

$$\begin{array}{rccc} L^k: & H^{n-k} & \to & H^{n+k} \\ & x & \mapsto & \omega^k x \end{array}$$

is an isomorphism for all k.

It follows that H has formal dimension $2n$, with ω^n a generator of the top cohomology group. As a notable example, the complex cohomology of a compact Kähler manifold M satisfies the hard Lefschetz property, by taking ω to be the class of the Kähler form of M, abusing notation (see [Wel08, Chapter V, Cor. 4.13] for a proof). This observation is the starting point for proving that fibrations with fibre a compact, simply connected Kähler manifold satisfy the Halperin conjecture. The fundamental theorem is stated next. We replicate the proof of [FOT08, Thm. 4.36].

Theorem 4.12 (Blanchard, [Bla56]). *Let M be a compact manifold whose complex cohomology algebra satisfies the hard Lefschetz property. If θ is a derivation of negative degree on $H^*(M;\mathbb{C})$ such that the restriction of θ to $H^1(M;\mathbb{C})$ is zero, then θ is the zero derivation.*

Proof. Let us denote $H = H^*(M;\mathbb{C})$. Let $\omega \in H^2$ be the class for which the hard Lefschetz property is stated and let θ be a derivation of negative degree in H. Recall that ω^n is a generator of the top cohomology, where n is the complex dimension of M.

First, we claim that $\theta(\omega) = 0$. Otherwise, we only have two possibilities for the degree of θ. If $|\theta| = -2$, then $\theta(\omega) = \lambda \in \mathbb{C}$. Since $\omega^{n+1} = 0$, we have $0 = \theta(\omega^{n+1}) = (n+1)\lambda\omega^n$, and hence $\lambda = 0$ as ω^n is a generator. In turn, if $|\theta| = -1$, then $\theta(\omega) = \alpha \in H^1$. Let $\{\alpha_i\}$ be a basis of H^1. By degree reasons, $\alpha_i\omega^n = 0$ for all i, and so $0 = \theta(\alpha_i\omega^n) = n\alpha\alpha_i\omega^{n-1}$ using that $\theta = 0$ on H^1. Thus, the cup product $\alpha \smile \alpha_i\omega^{n-1}$ vanishes for all i. By the hard Lefschetz property, $\{\alpha_i\omega^{n-1}\}$ forms a basis of H^{2n-1}. The two previous claims together with Poincaré duality imply that $\alpha = 0$.

Now, we can use an inductive argument to finish the proof. Let $r \geq 1$ be such that $\theta = 0$ on $H^{\leq r}$. We show that then $\theta = 0$ on H^{r+1}. Notice that the base case is covered by the assumption in the statement. Denote by $-s$ the negative degree of θ.

We claim that $\theta = 0$ on $H^{\geq 2n-r}$. Indeed, if $y \in H^{2n-p}$ with $p \leq r$, then by the hard Lefschetz property there exists $z \in H^p$ such that $y = z\omega^{n-p}$. Hence, $\theta(y) = \theta(z)\omega^{n-p} + z\theta(\omega^{n-p}) = 0$ because $p \leq r$ and $\theta(w) = 0$.

Finally, given $x \in H^{r+1}$, we have $\theta(x) \in H^{r+1-s}$. Then, for any $z \in H^{2n-r-1+s}$, since $s > 0$ we have $xz = 0$ by degree reasons, and so $\theta(xz) = \theta(x)z = 0$. Poincaré duality then implies that $\theta(x) = 0$. □

Corollary 4.13. *Let M be a compact Kähler manifold such that $H^1(M;\mathbb{C}) = 0$. Then, every orientable fibration $M \to E \to B$ is TNCZ.*

Proof. Following Theorem 3.7, it suffices to show that $\operatorname{Der}^{<0} H^*(M;\mathbb{Q}) = 0$. By Theorem 4.12, we have $\operatorname{Der}^{<0} H^*(M;\mathbb{C}) = 0$ for M as in the statement. Thus, we only need to change from complex to rational coefficients.

Since $\mathbb{C}$ is flat, then $H^*(M;\mathbb{C}) = H^*(M;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. Thus, there is a map

$$\varphi : \operatorname{Der} H^*(M;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \to \operatorname{Der} H^*(M;\mathbb{C})$$

such that, for a derivation θ in $H^*(M;\mathbb{Q})$ and $\lambda \in \mathbb{C}$, the image $\varphi(\theta \otimes \lambda)$ of $\theta \otimes \lambda$ maps $x \otimes 1$ to $\theta(x) \otimes \lambda$ for $x \in H^*(M;\mathbb{Q})$. It is easy to check that φ is an isomorphism of graded $\mathbb{C}$-vector spaces. It follows that $\operatorname{Der}^{<0} H^*(M;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = 0$ and thus $\operatorname{Der}^{<0} H^*(M;\mathbb{Q}) = 0$ as wished. □

In particular, the Halperin conjecture holds when the fibre is a Kähler manifold. Notice that the proved result is more general, as M need not be positively elliptic. In fact, the claim is valid for any compact space satisfying Poincaré duality and the hard Lefschetz property.

4.3 Homogeneous spaces

Homogeneous spaces are another important family of spaces for which the Halperin conjecture holds. We review the definition and elementary properties of these objects and then state the main result in Theorem 4.17. We follow [FOT08] for the basic notions in this section.

Recall that manifolds are assumed to be Hausdorff, second-countable spaces with a finite-dimensional differentiable structure of class C^∞ over $\mathbb{R}$.

Definition 4.14. A **Lie group** is a set G that is both a group and a manifold, and such that the group operation $m : G \times G \to G$ and the inverse map $i : G \to G$ are smooth maps.

Basic examples of Lie groups are $(\mathbb{R}, +)$ and S^1 with the complex product. It is easy to check that products of Lie groups are Lie groups, so $\mathbb{R}^n$ and the torus $T^n = (S^1)^n$ are Lie groups as well. Other classical examples include the groups of matrices $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $SO(n)$ and $SU(n)$ (see [FOT08, §1.5] for details).

A Lie subgroup of a Lie group G is a subgroup of G which is also a submanifold. It is known that a subgroup H of G is a Lie subgroup if and only if it is a closed subgroup of G. Now, given a Lie group G and a closed subgroup H, we have an equivalence relation on G given by

$$x \sim y \quad \text{if and only if} \quad xy^{-1} \in H,$$

yielding as a quotient space G/H the set of left cosets of H.

Definition 4.15. A **homogeneous space** is a quotient G/H where G is a Lie group and H a closed subgroup of G.

A Lie group is called abelian if it is an abelian group. Apart from $\mathbb{R}^n$, the tori T^n are prominent examples of abelian Lie groups. Actually, any connected abelian Lie group is isomorphic to $T^p \times \mathbb{R}^q$ for $p, q \in \mathbb{N}$ (see [Pro07, §4.2]). Hence, if G is a compact connected Lie group, any abelian Lie subgroup of G is a torus T. A subtorus $T \subseteq G$ is called maximal if it is not properly contained in any other torus. We have the following result.

Proposition 4.16. *Every element of a compact connected Lie group is contained in a maximal torus. Any two maximal tori are conjugate subgroups.*

Therefore, the dimension of a maximal torus is an invariant of a Lie group G, which we call the **rank** of G. Let T be a maximal torus in G and let $N(T)$ denote its normalizer subgroup. The quotient $W(G) = N(T)/T$ is called the **Weyl group** of G, and is independent, up to isomorphism, of the maximal torus T. The Weyl group is always finite, and its cardinal coincides with the Euler characteristic of G/T (see [FOT08, Prop. 3.31]). Furthermore, note that $W(G)$ acts by conjugation on T. Indeed, since T

is abelian, the restriction of the conjugation action of $N(T)$ on T to the subgroup T is trivial, and thus the action is well-defined on $W(G)$. Any isomorphism $T \to T$ induces an isomorphism in cohomology, and thus the action of $W(G)$ on T induces an action on $H^*(T; \Bbbk)$.

The cohomology of Lie groups is well understood, and we cite a couple of results providing its structure. We use complex coefficients as $\mathbb{C}$ is the relevant field in the proof of Theorem 4.17. A compact connected Lie group G has for cohomology algebra the free algebra

$$H^*(G; \mathbb{C}) \cong \Lambda(z_1, \ldots, z_k),$$

where $|z_i|$ is odd and k is the rank of G. This result is known as Hopf's theorem (see [FOT08, Thm. 1.34]). In turn, the cohomology of the classifying space BG is the polynomial algebra

$$H^*(BG; \mathbb{C}) \cong \mathbb{C}[x_1, \ldots, x_k],$$

where $|x_i| = |z_i| + 1$ (see [FOT08, Thm. 1.81]). Furthermore, if we let T be a maximal torus of G, then we have an injection $H^*(BG; \mathbb{C}) \hookrightarrow H^*(BT; \mathbb{C})$ and the cohomology of BG coincides with the invariant set of $H^*(BT; \mathbb{C})$ under the action of the Weyl group $W(G)$.

We are now ready to state the principal result. The proof is technical and giving a complete argument would require to dedicate a whole chapter to Lie groups and homogeneous spaces. Instead, with the concepts introduced above, we outline its structure.

Theorem 4.17 (Shiga-Tezuka, [ST87])**.** *Let G be a compact connected Lie group and let U be a closed subgroup of the same rank. Then, every orientable fibration $G/U \to E \to B$ is TNCZ.*

Proof (outline). Let T be a common maximal torus of G and U, and let n denote its dimension. The classifying space BT has cohomology $H^*(BT, \mathbb{C}) \cong \mathbb{C}[x_1, \ldots, x_k]$, whereas the cohomology of BG is of the form $H^*(BG, \mathbb{C}) \cong \mathbb{C}[f_1, \ldots, f_k]$, where the f_i are elements in $\mathbb{C}[x_1, \ldots, x_k]$ which are invariant under the action of the Weyl group.

Now, consider the Jacobian J of the function $(f_1, \ldots, f_k)$ of the x_i, that is $J = \det \partial f_i / \partial x_j$. The authors show first [ST87, §3] that J is not contained in any prime ideal of the ideal generated by $f_1, \ldots, f_{k-1}$, where the f_i are in increasing order of degree. Later, they note that the polynomials $f_1, \ldots, f_k$ form a regular sequence and deduce that $H^*(BT, \mathbb{C})$ is a free module over $H^*(BG, \mathbb{C})$ [ST87, Lemma 4.3]. These two assertions allow them to derive that $\mathrm{Der}^{<0}(G/U, \mathbb{C}) = 0$, and subsequently $\mathrm{Der}^{<0}(G/U, \mathbb{Q}) = 0$ like in Corollary 4.13. The result follows from Theorem 3.7. $\square$

Interestingly, the authors do not follow the reasoning of Theorem 3.7 in order to conclude that any fibration $G/U \to E \to B$ is TNCZ. Instead, they use a result equivalent to Proposition 3.18 to obtain $\pi_{2n+1} \operatorname{Baut}_o F_{\mathbb{Q}} = 0$, and then deduce that the fibration is TNCZ like in Proposition 3.20.

4.4 Algebras of dimension at most 20

Recently, the attempts to prove partial instances of the Halperin conjecture have focused on the formal dimension of positively elliptic algebras. A first result in this regard was obtained by Amann and Kennard [AK20], who showed the conjecture for algebras of dimension at most 16. Later, Kennard and Wu [KW23] extended the statement to dimensions up to 20.

Theorem 4.18 (Kennard-Wu, [KW23]). *The Halperin conjecture holds for positively elliptic algebras with formal dimension at most 20.*

Both articles use a similar idea, based on the induction scheme following from Markl's result and the analysis of the degree type.

Definition 4.19. Let $H = \mathbb{Q}[x_1, \dots, x_k]/(f_1, \dots, f_k)$ be a positively elliptic algebra. The **degree type** of H is the sequence of even, positive integers

$$(|x_1|, \dots, |x_k|; |f_1|, \dots, |f_k|).$$

Examples. We compute the degree type of the cohomology algebras of the example on page 12. Note that the definition of degree type does not make sense for the third class of spaces in the example, since they have Euler characteristic $2 - 2g \leq 0$.

1. Given n even, the cohomology of the sphere S^n has degree type $(n; 2n)$.
2. The cohomology of the projective space $\mathbb{C}P^n$ has degree type $(2; 2n+2)$.

Crucially, as a consequence of Proposition 3.16, a given formal dimension admits only finitely many degree types. Indeed,

$$\operatorname{fd} H = \sum_{i=1}^{k} |f_i| - |x_i| \geq \sum_{i=1}^{k} |x_i| \geq 2k.$$

Then, the number $k \leq \operatorname{fd} H/2$ is bounded, and so are the possible degrees of the x_i and therefore those of the f_i.

Next, we describe the strategy used in the cited works. We consider a positively elliptic algebra H of formal dimension $\operatorname{fd} H$ bounded by 16 or 20 in each case. We proceed by induction over $\operatorname{fd} H$. Note that the base case is covered by the results in Section 4.1, since for small $\operatorname{fd} H$ we have $k \leq 3$.

By contradiction, let us assume that H admits a non-trivial derivation of negative degree. We may suppose that H does not split, following the reasoning from the beginning of the chapter. Now, using the restrictions of Proposition 3.16 and Lemmas 4.5, 4.6 and 4.8, one can find a finite list of possible degree types for the algebra H. Then, each specific case is addressed individually to show that such an algebra does not admit non-trivial derivations of negative degree.

Let us first work out the details in the case $\operatorname{fd} H \leq 16$. From Theorem 4.9 we have $k > 3$, while Lemmas 4.5 and 4.6 imply that the first or last $k-1$ generators cannot have the same degree. Moreover, since H does not split, the relation $|f_i| \geq |x_1| + |x_{i+1}|$ must hold for $i < k$ by Lemma 4.8. Now, if $\operatorname{fd} H < 14$ there is no degree type satisfying these conditions. If $\operatorname{fd} H = 14$, then:

- $k \leq 7$, and equality is not possible because all generators would have the same degree.
- The cases $k \in \{5, 6\}$ are also not possible. Indeed, since $|f_i| \geq 2|x_i|$, the formal dimension formula of Proposition 3.16 can only be satisfied if $k = 5$ and the degrees of the generators are $(2, 2, 2, 4, 4)$. But we must have $|f_3| \geq 2 + 4 = 6$, and hence $\operatorname{fd} H \geq 16$.
- If $k = 4$ and some x_i has degree 6, then $\operatorname{fd} H \geq 16$ as well. Hence, we have $(2, 2, 4, 4)$ for the generators, and $(4, 6, 8, 8)$ is the only possible sequence for the f_i. We deduce that $(2, 2, 4, 4; 4, 6, 8, 8)$ is the only allowed degree type.

Similar arguments lead to the following possibilities when $\operatorname{fd} H = 16$:

$$(2, 2, 4, 4; 4, 8, 8, 8),\ (2, 2, 4, 4; 6, 6, 8, 8),$$
$$(2, 2, 4, 6; 4, 6, 8, 12),\ (2, 2, 2, 4, 4; 4, 4, 6, 8, 8).$$

Finally, for each of the five possible degree types, arguments like the ones in Theorem 4.9 allow to deduce that any derivation of negative degree is trivial. The details can be found in [AK20, Thm. 11.6].

For the sake of completeness, we give the specific formulations of the main results in [KW23], covering all cases with $\operatorname{fd} H \leq 20$. Here, the authors carry out the reasoning for three different cases, namely

(1) $|x_{k-1}| + |x_k| \leq 8$,

(2) $|x_{k-1}| + |x_k| = 10$,

(3) $|x_{k-1}| + |x_k| \geq 12$.

In addition, we consider the families of degree types

$$\begin{aligned}
\mathcal{F}_1 &= \{(2, 2, 4, 4; 4, 6, 8, 12), (2, 2, 4, 4; 4, 8, 8, 12), (2, 2, 2, 4, 4; 4, 4, 6, 8, 12)\},\\
\mathcal{F}_2 &= \{(2, 2, 2, 4, 6; 4, 4, 6, 10, 12)\},\\
\mathcal{F}_3 &= \{(2, 4, 6, 6; 6, 8, 12, 12), (2, 2, 6, 6; 4, 8, 12, 12)\}.
\end{aligned}$$

Then, Theorem 4.18 is a consequence of the following series of results.

Proposition 4.20. *Let H be a positively elliptic algebra that does not split and satisfies condition (i), with $i \in \{1, 2, 3\}$. Assume that H admits a non-zero derivation of negative degree. Then, either* $\operatorname{fd} H > 20$ *or the degree type of H lies in $\mathcal{F}_i$.*

Proposition 4.21. *Let H be a positively elliptic algebra that does not split. If the degree type of H lies in any of the $\mathcal{F}_i$, then H admits no non-zero derivations of negative degree.*

Corollary 4.22. *Let $H = \mathbb{Q}[x_1, \ldots, x_k]/(f_1, \ldots, f_k)$ be a positively elliptic algebra that does not split and satisfies either of the conditions* (1), (2) *or* (3). *If H admits a non-zero derivation of negative degree, then* $\operatorname{fd} H > 20$.

4.5 Counterexamples

To conclude the work, we present a series of examples showing that the assumptions of the Halperin conjecture are sharp, that is, examples of fibrations lacking one of the assumptions and for which the statement does not hold.

Non-positive Euler characteristic

A straightforward example is given by the Hopf fibrations. Keeping the assumption that the fibre is simply connected, we consider only the Hopf fibrations in higher dimensions. Namely, we show that the fibration $S^3 \hookrightarrow S^7 \to S^4$ is not TNCZ. The reasoning for $S^7 \hookrightarrow S^{15} \to S^8$ is analogous. Observe that odd spheres are simply connected elliptic spaces, but their Euler characteristic is zero.

Recall from the Leray-Hirsch theorem (Corollary 2.31) that if a fibration $F \to E \to B$ is TNCZ, we have

$$H^*(E; \mathbb{Q}) \cong H^*(B; \mathbb{Q}) \otimes H^*(F; \mathbb{Q}).$$

This relation does not hold for the Hopf fibration with fibre S^3. Indeed, the cohomology of S^7 is an exterior algebra on one generator of degree 7. In turn, $H^*(S^4; \mathbb{Q})$ and $H^*(S^3; \mathbb{Q})$ have one generator in degrees 3 and 4, respectively, and therefore the tensor product also has non-zero components in those degrees. Notice that, as a consequence, $H^*(S^3; \mathbb{Q})$ must have a non-zero derivation of negative degree, namely $\partial/\partial z$.

The work by T. Yamaguchi [Yam05] provides a more intricate example. The author constructs a simply connected, elliptic space F such that every orientable fibration with fibre F is TNCZ, but for which $H^*(F; \mathbb{Q})$ admits a non-trivial derivation of negative degree. This shows that the assumption on the Euler characteristic of F is needed to prove the converse of Theorem 3.7. Let us review the details of Yamaguchi's example.

Consider the free algebra $A = (\Lambda(x, y, z, a, b, c), d)$ over $\mathbb{Q}$ with

$$\begin{array}{llll} |x| = 2, & |y| = 3, & dx = 0, & dy = 0, \\ |z| = 3, & |a| = 4, & dz = x^2, & da = xy, \\ |b| = 5, & |c| = 7, & db = xa + yz, & dc = a^2 + 2yb. \end{array}$$

Observe that A is a Sullivan algebra ΛV, where $V = \bigoplus_{i=1}^{4} V_i$ for

$$V_1 = \langle x, y \rangle, \quad V_2 = \langle z, a \rangle, \quad V_3 = \langle b \rangle \quad \text{and} \quad V_4 = \langle c \rangle.$$

The cohomology of A is of the form

$$H^i(A) = \begin{cases} \mathbb{Q} & \text{if } i \in \{0, 2, 3, 11, 12, 14\}, \\ \mathbb{Q}^2 & \text{if } i = 7, \\ 0 & \text{otherwise.} \end{cases}$$

The set of generators of $H(A)$ as a $\mathbb{Q}$-algebra is

$$\{x, y, e, f, g, h\},$$

where $e = [ya]$, $f = [xb - za]$, $g = [x^2c - xab + yzb]$ and $h = [3xyz + a^3]$. Abusing notation, we write x and y for the cohomology classes $[x]$ and $[y]$. It can be checked that $H(A)$ satisfies Poincaré duality, and the only non-trivial products between generators are xh, yg and ef in $H^{14}(A)$. With this, there is a well-defined derivation θ of degree -8 mapping g to y and the remaining generators to 0. Let us check the Leibniz rule in the relevant cases.

- If $m \notin \{y, g\}$, then $\theta(gm) = 0$ and so are $\theta(g)m$ and $g\theta(m)$.
- If $m = y$, then $\theta(gy) = \theta(g)y = y^2 = 0$, in accordance with $\theta(xh) = \theta(ef) = 0$ for the other generators of $H^{14}(A)$.
- If $m = g$, then $\theta(g^2) = 0 = \theta(g)g + g\theta(g)$, since $|g|$ is odd. Note that this coincides with the Leibniz rule because $|\theta|$ is even.

We conclude that the space of derivations of negative degree $\operatorname{Der}^{<0} H^*(A; \mathbb{Q})$ is non-zero.

Finally, by a result of Sullivan [Sul77, Thm. 13.2], the Sullivan algebra A is realized by a manifold F of dimension 14, that is, there is a space F with $H^*(F; \mathbb{Q}) \cong H^*(A)$ as dg algebras. Yamaguchi shows in [Yam05, Thm. 2.2] that every orientable fibration with fibre F is TNCZ.

Non-elliptic space

The *path space* fibration provides a counterexample in which the fibre is not an elliptic space. For $n \geq 2$, consider a base point $x_0 \in S^n$. There is a fibration π associated to the inclusion $\{x_0\} \hookrightarrow S^n$ (see page 10), and the total space coincides with the path space

$$PS^n = \{\gamma : I \to S^n \mid \gamma(0) = x_0\}.$$

Namely, $\pi : PS^n \to S^n$ maps every path γ to its endpoint $\gamma(1)$. Hence, the fibre $\pi^{-1}(x_0)$ is the space ΩS^n of loops based on γ.

Let us consider the Serre spectral sequence associated to the fibration $\Omega S^n \hookrightarrow PS^n \to S^n$. Note that this is a special case of the Wang sequence (see page 21). Now, observe that the space PS^n is contractible, as we can shrink paths into x_0. Thus, the Wang sequence reads, for $q > 1$,

$$0 = H^{q-1}(PS^n) \to H^{q-1}(\Omega S^n) \overset{\cong}{\to} H^{q-n}(\Omega S^n) \to H^q(PS^n) = 0,$$

where cohomologies are taken over $\mathbb{Q}$. Moreover, since S^n is simply-connected, any two loops are homotopic and therefore ΩS^n is path-connected, i.e. $H^0(\Omega S^n; \mathbb{Q}) \cong \mathbb{Q}$. Alternatively, this follows from the Wang sequence at level $q = 1$. In any case, we have

$$H^i(\Omega S^n; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } i \equiv 0 \pmod{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The multiplicative structure is worked out in [Hat_2, Ex. 5.17]. Here, it suffices to notice that ΩS^n is not an elliptic space. Furthermore, the isomorphism in the Wang sequence coincides with the differential d_n in the n-th term of the Serre spectral sequence. Therefore, the fibration $\Omega S^n \to PS^n \to S^n$ is not TNCZ.

Lastly, we observe that all three spaces are simply connected if $n \geq 3$. Note that the Euler characteristic is not well-defined in this case. However, for n odd we can still assert that the algebra $H^*(\Omega S^n; \mathbb{Q})$ is evenly graded. Hence, if $n \geq 3$ is odd, ellipticity is the only missing assumption.

Non-orientable fibration

To conclude, we present a non-orientable fibration with positively elliptic fibre which is not TNCZ. In this case, the Serre spectral sequence (with local coefficients) does collapse at the E_2 page, though. Hence, the example shows that Theorem 2.30 does not hold for general fibrations.

Let $n > 0$ be an even integer and consider the inclusion $f : \mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$, which represents a fibration $\pi : E_f \to \mathbb{R}P^\infty$ as discussed in Section 2.1. We show first that the fibre F of this fibration is homotopy equivalent to S^n. In other words, the sphere S^n is the homotopy fibre of the map f.

Using the universal cover S^∞, it is proved in [Hat_1, Ex. 1B.3] that

$$\pi_k(\mathbb{R}P^\infty) = \begin{cases} \mathbb{Z}/2 & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\mathbb{R}P^\infty$ is an *Eilenberg-MacLane* space $K(\mathbb{Z}/2, 1)$, i.e. a space with exactly one non-zero homotopy group $\pi_1 = \mathbb{Z}/2$. Then, from the long exact sequence in homotopy arising from the fibration π, we deduce that F is a simply connected space and the map $i : F \to \mathbb{R}P^n$ induces isomorphisms in homotopy for $k \geq 2$. Recall that this map is obtained as the composition of the inclusion $F \hookrightarrow E_f$ with the homotopy inverse of $\mathbb{R}P^n \hookrightarrow E_f$.

Next, let $p : S^n \to \mathbb{R}P^n$ be the universal covering map. The composite fp is null-homotopic, since $[S^n, \mathbb{R}P^\infty] \cong H^1(S^n; \mathbb{Z}/2) = 0$ using that $\mathbb{R}P^\infty$ is an Eilenberg-Maclane space (see [Hat$_1$, Thm. 4.57]). Then, the covering map p factors through the homotopy fibre F, as in the diagram

$$\begin{tikzcd} S^n \arrow[d, "g"] \arrow[dr, "p"] & & \\ F \arrow[r, "i"] & \mathbb{R}P^n \arrow[r, "f"] & \mathbb{R}P^\infty \end{tikzcd}$$

This is because, for any fibration $E \to B$, we may represent the (homotopy) fibre as a (homotopy) pullback along the inclusion $* \to B$, as a consequence of the definition.

Observe that p also induces isomorphisms in homotopy for all $k \geq 2$, since the corresponding fibre $\mathbb{Z}/2$ is discrete. Furthermore, both S^n and F are simply connected. Thus, the map g induces isomorphisms in homotopy for all $k \geq 1$, and a bijection for $k = 0$. By Whitehead's theorem, we deduce that g is a homotopy equivalence.

Finally, we note that the fibration is not orientable, since $\pi_1(\mathbb{R}P^\infty) = \mathbb{Z}/2$ acts on S^n as the antipodal map, of degree $(-1)^{n+1}$. Hence, when n is even, the induced map in cohomology is $-\mathbb{1}$ and the action is not trivial. In turn, the fibration is not TNCZ. Indeed, if n is even then $H^0(\mathbb{R}P^n; \mathbb{Q}) = \mathbb{Q}$ and $H^k(\mathbb{R}P^n; \mathbb{Q}) = 0$ for all $k > 0$. In particular, the induced map $p^* : H^*(\mathbb{R}P^n; \mathbb{Q}) \to H^*(S^n; \mathbb{Q})$ cannot be surjective by dimensional reasons.

5 Conclusions

In this thesis we have carried out a survey of the Halperin conjecture, an important open conjecture in algebraic topology. More concretely, the conjecture was listed in [FHT01, §39] as the first of a series of selected open problems in rational homotopy theory. The authors being some of the most notable names in the field, this choice is a first indicator of the relevance of the problem.

From a more technical point of view, the conclusion of the Halperin conjecture is a strong assertion. As a reminder, it states that the rational Serre spectral sequence associated to a certain type of fibrations degenerates at the E_2 page. We show in Section 2.3.1 that some interesting properties follow when the Serre spectral sequence degenerates. In particular, the cohomology of the total space can be obtained directly as the tensor product of the cohomologies of the base and the fibre. Recall that spectral sequences were conceived as a tool for (co)homology computations, and the degenerate case is thus an elementary example of its use.

Multiple efforts have been put on solving the conjecture since Halperin proposed it in 1976, confirming that the problem has attracted the interest of the community. In this way, although the conjecture remains open, the statement has been demonstrated for several restricted cases, as we discuss in Chapter 4. Among these lie spaces with "simple" cohomology, low-dimensional spaces and familiar spaces such as Kähler manifolds or homogeneous spaces. Therefore, the statement may be used for most spaces one would encounter in mathematical or real-world applications.

The central part of this thesis discusses two equivalent reformulations of the Halperin conjecture. First, it is necessary to consider the algebraic statement in terms of derivations of positively elliptic algebras, as a vast majority of the known results for specific settings has been proved using this formulation. We do so in Section 3.1. With little extra effort, we are also able to reformulate the conjecture in terms of the space of self-homotopy equivalences of the fibre in Section 3.2. These sections contain the main contribution of this work as a relevant source about the Halperin conjecture. Indeed, we present complete proofs for Theorem 3.12 and Proposition 3.18, which had been shown in [Mei82] skipping some arguably non-trivial steps. Let us see how our results differ from the previously published arguments.

In Corollary 3.13, we show the equivalence between Halperin's original formulation of the conjecture and the algebraic re-phrasing. Meier presents the same conclusion in [Mei82, Lemma 2.5]. In particular, we reproduce part (a) of Meier's result in Theorem 3.7. As we discuss in the text, this is a fairly straightforward result that holds in great generality, namely for any field and any space of finite type over it. In contrast, in order to give a complete argument for part (b), whose statement coincides with our Theorem 3.12, we need three preparatory results demonstrating some of Meier's claims. First, the author affirms that, given a formal space F, any derivation of $H^*(F;\mathbb{Q})$ is induced from a derivation θ of the minimal model M_F. This follows from the statement of Lemma 3.9, which in turn makes use of Lemma 3.8. Then, Meier uses the fact that the dg algebra map $\Lambda u \to (\Lambda u \otimes M_F, D)$ represents a fibration with base S^{2k+1} and fibre rationally equivalent to F. Although this may be rather direct for experts in rational homotopy theory, we provide a detailed proof as the combination of Proposition 3.10 and Lemma 3.11, using some fundamental tools in the field. Finally, the author asserts that if $\theta \neq 0$ then it is *easy* to verify that the map of dg algebras above is not injective in cohomology. Then, from Section 2.3.1, it follows that i^* is not surjective. In our case, we show directly in the proof of Theorem 3.12 that surjectivity of i^* implies $\theta = 0$. We have not been able to prove Meier's stronger claim. Furthermore, in order to restrict ourselves to even degree derivations, we are forced to state the result just for positively elliptic spaces. Meier may have forgotten to include this assumption, which is anyway considered in his Theorem A collecting the three formulations of the Halperin conjecture.

Regarding Proposition 3.18, its statement coincides with parts (i) and (ii) of [Mei82, Prop. 2.6]. In part (1) of the proposition we replicate Meier's (i), using Lemma 3.17 to identify $\pi_1 \operatorname{Baut}_o F_{\mathbb{Q}}$ with a suitable set of maps of cdg algebras $M_F \to H^*(F;\mathbb{Q})$. Then, in part (2), we follow a different scheme for the proof. Concretely, we use a couple of results in rational homotopy theory and the previous discussion on the cohomology of the space of derivations to find the sought relation between $H^{-2n} \operatorname{Der} M_F$ and $\pi_{2n+1} \operatorname{Baut}_o F_{\mathbb{Q}}$. In this way, we exhibit the structure of positively elliptic algebras and the space $\operatorname{Der}(\Lambda, A)$ on our way to an elegant proof of the result. Moreover, the technique is equally valid to obtain the corresponding expression for the even groups $\pi_{2n} \operatorname{Baut}_o F_{\mathbb{Q}}$.

To conclude, let me include a personal assessment of this thesis. Overall, I believe that the text fulfils the goal of serving as an introductory survey of the Halperin conjecture. The prerequisites assumed match those of a master's student having followed general courses in algebraic topology and complementary areas like commutative algebra and category theory. In this way, Chapter 2 aims to provide the reader with the knowledge of fibrations, algebras, spectral sequences and rational homotopy theory necessary to follow the arguments in Chapters 3 and 4, where the Halperin conjecture comes into play. Personally, I had little experience with some of these subjects, and I have tried to present them in the way that seemed most didactic to me when learning about them myself.

A more exhaustive survey should include additional equivalent formulations and a longer list of partial results. We have already mentioned the work of Lupton, where a re-phrasing is given in terms of formality relations between the base and total spaces [Lup98, Thm. 3.4]. In the same paper, the author proposes and then proves a weaker version of the conjecture involving numerical invariants of the spaces in the fibration [Lup98, Thm. 4.7]. Here, after discussing the unavoidable algebraic statement, we have preferred to stick to Meier's article and review the formulation in terms of the space of self-equivalences $\operatorname{aut} F$, which has an explicit topological interpretation and has not required the introduction of additional concepts.

Concerning the missing partial results, the works of Papadima-Păunescu [PP96] and Chen-Yau-Zuo [CYZ19] are likely the most notable, as they describe large families of positively elliptic algebras for which the conjecture holds. A thorough analysis of the literature would surely give rise to other specific cases, although of limited interest in an introductory review of the topic. Either way, the cases featured in Chapter 4 are the ones allowing to apply the statement in the most common settings, and therefore provide a suitable family of examples for the purpose of this work.

Finally, regarding future work on the conjecture, the algebraic statement seems to be the right candidate in potential attempts of a general proof. Although no clear belief about its truth has been expressed, references like [Lup98] or [BK03] treat the conjecture with confidence and explore possible consequences, suggesting that it is widely expected to hold. So far, the restrictions on the number of generators [Tho81, Lup90] or the formal dimension [AK20, KW23] have yielded the broadest families of spaces satisfying the statement. The work by Kennard-Wu is very recent and a similar strategy might be used to increase the dimension bound in coming years. However, it is not straightforward that such methods will serve in a general proof, as they use combinatorial arguments relying on the low number of possible degree types given a (small) dimension bound.

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