SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

The Correspondence Between the Tutte and Jones Polynomial with Applications

av

Alexander Bjurström

2026 - No K13

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

The Correspondence Between the Tutte and Jones Polynomial with Applications

Alexander Bjurström

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Thomas Wennink

2026

Abstract

This paper explores the relationship between the Tutte polynomial of graphs and the Jones polynomial of knots. We begin by introducing the necessary background from graph theory and knot theory, including properties of the Tutte and Jones polynomials. We then show that the Jones polynomial of an alternating knot can be obtained as a specialization of the Tutte polynomial of the associated Tait graph.

In the second part of the paper, we apply this correspondence to compute the Jones polynomial of alternating knots in several knot families: twist knots, pretzel knots, torus knots and rational knots.

Sammanfattning

Detta kandidatarbete utforskar relationen mellan grafers Tuttepolynom och knutars Jonespolynom. Först introducerar vi den nödvändiga bakgrunden från grafteori och knutteori, såsom egenskaper hos Tutte- och Jonespolynom. Därefter visar vi att Jonespolynomet av en alternerande knut kan erhållas som en specialisering av Tuttepolynomet av knutens Taitgraf.

I arbetets andra del tillämpar vi detta samband för att beräkna Jonespolynom för alternerande knutar i följande familjer av knutar: twistknutar, pretzelknutar, torusknutar och rationella knutar.

Acknowledgments

I would like to express my gratitude to my supervisor Thomas Wennink for his support and guidance throughout my work. From conversations with other students, I have come to appreciate how rare it is to have a supervisor who is both so available and so engaged as you have been. Thank you for all your help and support.

Contents

1 Introduction

Graph theory and knot theory are both classical areas of combinatorics and topology. In knot theory, invariants such as the Jones polynomial provide a powerful tool for distinguishing between different knots which would otherwise be difficult to tell apart. In graph theory, the Tutte polynomial plays a similar universal role, encoding a wide range of combinatorial information and specializes to the chromatic and flow polynomials.

A remarkable feature of both of these invariants is that they admit similar recursive formulations, but for different objects. For alternating knots, this relationship becomes unambiguous via the Tait graph of a knot. This provides a translation of knot-theoretic computations into graph-theoretic ones. This allows the Jones polynomial to be expressed in terms of a graph invariant.

The purpose of this thesis is to make this connection explicit. To demonstrate how it can be used, we will compute Jones polynomials for several families of knots via the Tutte polynomial.

Chapters 3 to 5 closely follow the book *A Course in Enumeration* by Aigner [Aig07]. In these chapters, we develop the necessary background from graph and knot theory to establish the relationship between two invariants: the Tutte polynomial for graphs and the Jones polynomial for knots. In particular, we show that the Jones polynomial of alternating knots can be computed as a specialization of the Tutte polynomial of the associated Tait graph. While the overall structure and main results are drawn from the reference, certain definitions, arguments and proofs are reorganized or presented in an alternate form to better suit the purpose of this paper.

In Chapter 6 we use the results developed in previous chapters to derive closed formulas for the Jones polynomial of several families of knots. A substantial part of this chapter involves defining and constructing these knot families and then applying the theory developed in the first part of the thesis. We obtain closed formulas for alternating twist knots, pretzel knots, torus knots and rational knots. For rational knots, we are unable to derive a formula for the writhe, and it must therefore be treated as an input to the polynomial.

Although the formulas derived in Chapter 6 are not claimed to be new, many are likely already known or have appeared in similar forms in the literature. The purpose of this chapter is to present explicit derivations of Jones polynomials using the relationship between the Tutte and Jones polynomials. The calculations and derivations presented for these knot families were carried out independently for this thesis, rather than reproducing them from existing formulas in the literature.

2 Conventions and Notation

Definition 2.1. A *graph* is a triple (V, E, ψ) where V is the vertex set, E is the edge set and $\psi : E \to \{\{u, v\} : u, v \in V\}$ is a function that maps each edge to an unordered pair of vertices.

We will sometimes write $E(G)$ or $V(G)$ to specify the edge set or vertex set of G, respectively.

Definition 2.2. Let $G_1 = (V_1, E_1, \psi_1)$ and $G_2 = (V_2, E_2, \psi_2)$ be two graphs. An *isomorphism of graphs* is a pair of bijections

$$\varphi_V : V_1 \to V_2$$
$$\varphi_E : E_1 \to E_2$$

such that for every edge $e \in E_1$,

$$\psi_2(\varphi_E(e)) = \{\varphi_V(u), \varphi_V(v)\} \quad \text{when} \quad \psi_1(e) = \{u, v\}.$$

If there is an isomorphism between G_1 and G_2, we write $G_1 \cong G_2$ and say G_1 and G_2 are *isomorphic*.

Definitions 2.1 and 2.2 may appear somewhat technical, but this level of precision is necessary to allow multiple edges between the same pair of vertices, called *parallel edges*. In some literature, graphs allowing parallel edges are sometimes called *multigraphs* and graphs allowing both parallel edges and loops are referred to as *pseudographs*. In this paper a *graph* will always be allowed to contain both parallel edges and loops unless stated otherwise.

The degree of a vertex v in a graph $G = (V, E, \psi)$ is denoted by $\deg(v)$ and if $\deg(v) = n$ for all $v \in V$ then we call G *n-regular*. We will use the notation $k(G)$ to denote the number of components of a graph G. We write $\cup$ for the union and $\sqcup$ for the disjoint union.

3 The Tutte Polynomial

A central object of this thesis is the Tutte polynomial, a graph invariant that encodes a wide range of combinatorial information, much like the chromatic polynomial. The Tutte polynomial can be viewed as a far-reaching generalization of the chromatic polynomial. For this reason, before introducing the Tutte polynomial, it will be helpful to recall the basic properties and definitions of the chromatic polynomial.

The Chromatic Polynomial

A *proper coloring* of a graph $G = (V, E, \psi)$ is an assignment of colors to the vertices such that no two adjacent vertices are of the same color. We can now define the chromatic polynomial.

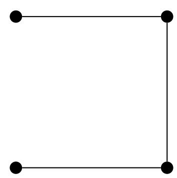
Definition 3.1. The *chromatic polynomial* $\chi_G(\lambda)$ counts the number of proper colorings of the graph G using λ colors.

It is not immediately clear that $\chi_G(\lambda)$ is a polynomial but this will be shown later.

Example 3.2. Let $G = (V, E, \psi)$ be a path graph of 4 vertices, meaning that

$$V = \{v_1, v_2, v_3, v_4\}$$
$$E = \{e_1, e_2, e_3\}$$
$$\psi = \begin{cases} e_1 \mapsto \{v_1, v_2\} \\ e_2 \mapsto \{v_2, v_3\} \\ e_3 \mapsto \{v_3, v_4\} \end{cases}$$

or visually



We will compute its chromatic polynomial. The first vertex may be colored λ ways. Each subsequent vertex may be colored in $\lambda - 1$ ways since it must differ from the color of the previous vertex. By the multiplication principle we obtain

$$\chi_G(\lambda) = \lambda(\lambda - 1)^3.$$

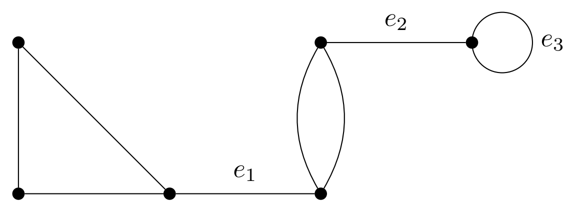
In situations where drawing the graph is sufficient we will omit the formal specification of vertex set, edge set and the map ψ.

To determine how to compute the chromatic polynomial of any given graph we first need some more definitions.

Definition 3.3. Let $G = (V, E, \psi)$ be a graph. A *bridge* in G is an edge $e \in E$ such that removing e increases the number of components of G. A *loop* in G is an edge $e \in E$ incident to one single vertex.

Note that if G contains a loop, then G cannot be properly colored and hence $\chi_G(\lambda) = 0$. Also, if G consists of disjoint components $G = k_1 \sqcup k_2 \sqcup \cdots \sqcup k_n$ then $\chi_G(\lambda) = \chi_{k_1}(\lambda) \cdot \chi_{k_2}(\lambda) \cdot \cdots \cdot \chi_{k_n}(\lambda)$ by the multiplication principle.

Example 3.4. In the following graph e_1 and e_2 are bridges and e_3 is a loop.

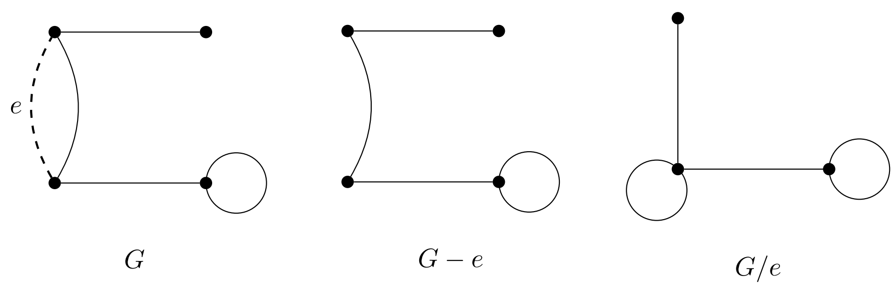


To manipulate graphs we need the following operations.

Definition 3.5. Let G be a graph with edge e adjacent to a and b. We define its *deletion* as removing the edge e from G. The resulting graph is called $G - e$. *Contracting* e means identifying the two endpoints a and b of e and then deleting e. The resulting graph is called G/e.

Edges selected for deletion or contraction are represented by dotted lines.

Example 3.6. The following figure shows deleting and contracting an edge e.



Recursively we can compute the chromatic polynomial using the following theorem.

Theorem 3.7. *Let G be a graph and e an edge in G adjacent to a and b. Then*

$$\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda).$$

Proof. Let us first assume that e is not a loop. Consider the graph $G - e$. Any proper coloring of $G - e$ falls into one of two disjoint classes:

- colorings in which a and b receive the same color,
- colorings in which a and b receive different colors.

If a and b receive the same color, identifying them produces a proper coloring of G/e. Hence, there are $\chi_{G/e}(\lambda)$ such colorings. If a and b receive different colors, then the coloring is also a proper coloring of G. Every proper coloring of G arises in this way and there are $\chi_G(\lambda)$ such colorings.

Therefore,

$$\chi_{G-e}(\lambda) = \chi_{G/e}(\lambda) + \chi_G(\lambda).$$

Let us now assume that e is a loop. Then $\chi_G(\lambda) = 0$, and $G - e = G/e$, so

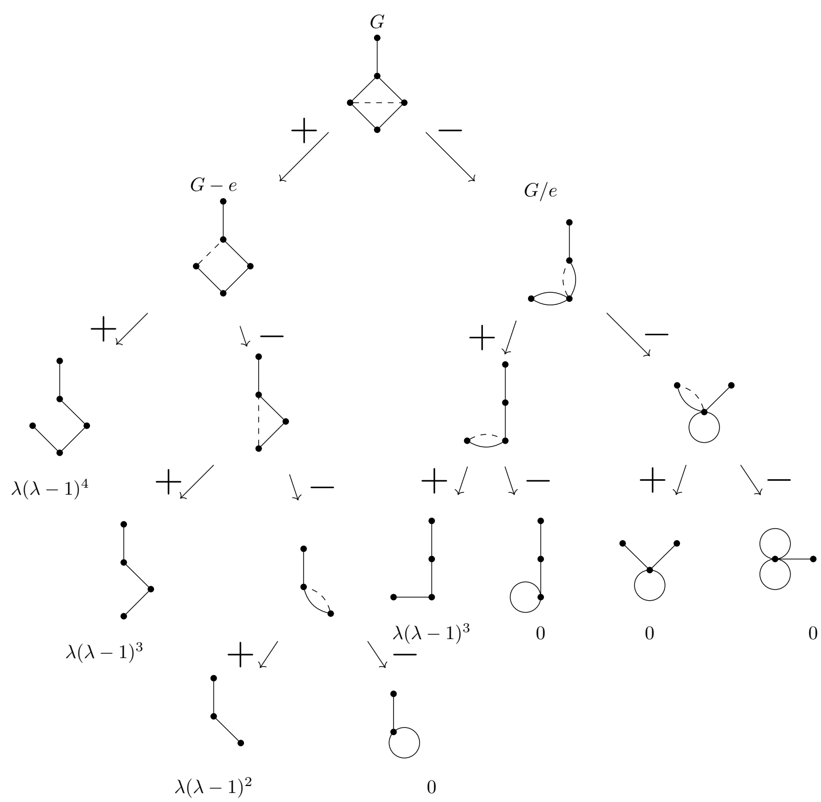
$$\chi_{G-e}(\lambda) - \chi_{G/e}(\lambda) = 0.$$

In both cases,

$$\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda),$$

as claimed. □

Example 3.8. We will now compute the chromatic polynomial for the following graph G. The sequence of deletion-contraction steps can be represented as a binary tree, where each node corresponds to a graph. One generation down to the left represents deleting an edge and one generation down to the right represents contracting an edge.



When we are finished iterating we can determine the chromatic polynomial of every graph on a leaf. If the graph contains a loop, then the chromatic polynomial is 0. Then we find

$$\begin{aligned}\chi_G(\lambda) =&\lambda(\lambda-1)^4 - \lambda(\lambda-1)^3 + \lambda(\lambda-1)^2 - \lambda(\lambda-1)^3\\ =&\lambda^5 - 6\lambda^4 + 13\lambda^3 - 12\lambda^2 + 4\lambda.\end{aligned}$$

The Tutte Polynomial

In the deletion-contraction approach to computing the chromatic polynomial, a loop forces the polynomial to be zero since it is impossible to properly color a graph containing a loop. The Tutte polynomial is a generalization of the chromatic polynomial satisfying a similar recursive relation, but treats loops and bridges as special base cases. We will show that the Tutte polynomial encodes far more combinatorial information than the chromatic polynomial and that every graph polynomial satisfying a suitable contraction-deletion relation can be written as a specialization of the Tutte polynomial.

Definition 3.9. The *Tutte polynomial*, $T_G(x, y)$ of a graph G is defined recursively as follows:

(i) If $E = \emptyset$, then
$$T_G(x, y) = 1.$$

(ii) If $e \in E$ is a bridge, then
$$T_G(x, y) = x \cdot T_{G-e}(x, y).$$

(iii) If $e \in E$ is a loop, then
$$T_G(x, y) = y \cdot T_{G-e}(x, y).$$

(iv) If $e \in E$ is neither a bridge nor a loop, then
$$T_G(x, y) = T_{G-e}(x, y) + T_{G/e}(x, y).$$

This definition produces a polynomial because each step of the recursion decreases the number of edges until we obtain graphs with empty edge sets for which the Tutte polynomial is 1. Since the recursion only involves addition and multiplication of x and y, the resulting expression is a polynomial in $\mathbb{Z}[x, y]$.

Similarly, when computing the chromatic polynomial, the Tutte polynomial may be computed by choosing edges and repeating the recurrence. It is not immediately clear that the resulting polynomial is independent of the order in which the edges are chosen. This can be proven with induction.

Proposition 3.10. *The Tutte polynomial $T_G(x, y)$ is independent of the choice of edge at each step of the recursion.*

Proof. We will provide a proof by induction on the number of edges in G. Let

$$m := |E| .$$

If $m = 0$, then G has no edges and $T_G(x, y) = 1$ as defined. Now assume that for all graphs H with $|E(H)| < m$, the Tutte polynomial is independent of the choice of edge at each step of the recursion.

Now let G be any graph with $|E(G)| = m$ and let $a, b \in E(G)$. Let $T_G^{(a)}$ be the polynomial obtained by first applying the recursion rule to a, then to b and let $T_G^{(b)}$ be the polynomial obtained by first applying the recursion rule to b, then to a. We want to show that

$$T_G^{(a)} = T_G^{(b)}.$$

We distinguish cases according to the types of edges a and b.

Case 1: Neither a nor b are a bridge or a loop.

Applying the recursion first on a, then on b yields

$$T_G^{(a)} = T_{(G-a)-b} + T_{(G-a)/b} + T_{(G/a)-b} + T_{(G/a)/b}.$$

Similarly,

$$T_G^{(b)} = T_{(G-b)-a} + T_{(G-b)/a} + T_{(G/b)-a} + T_{(G/b)/a}.$$

Since deletion and contraction commute up to isomorphism, the corresponding graphs are isomorphic. Then by the induction hypothesis, since every graph has one less edge, the polynomials are equal, hence

$$T_G^{(a)} = T_G^{(b)}.$$

Case 2: Exactly one of a or b is a bridge or a loop. Let us without loss of generality assume that a is neither a bridge nor a loop. Let us furthermore assume that b is a bridge. The case when b is a loop is similar. Then

$$T_G^{(a)} = x \cdot T_{(G-a)-b} + x \cdot T_{(G/a)-b} = x \cdot (T_{(G-a)-b} + T_{(G/a)-b}).$$

Applying to edge b first yields

$$T_G^{(b)} = x \cdot (T_{(G-b)-a} + T_{(G-b)/a}).$$

Every graph has fewer than m edges so the induction hypothesis implies

$$T_G^{(a)} = T_G^{(b)}.$$

Case 3: Both a and b are bridges or loops. Finally, if both are bridges

$$T_G = x^2 \cdot T_{G-\{a,b\}},$$

if both are loops

$$T_G = y^2 \cdot T_{G-\{a,b\}}$$

and if one is a bridge and the other is a loop

$$T_G = xy \cdot T_{G-\{a,b\}}.$$

In every case,

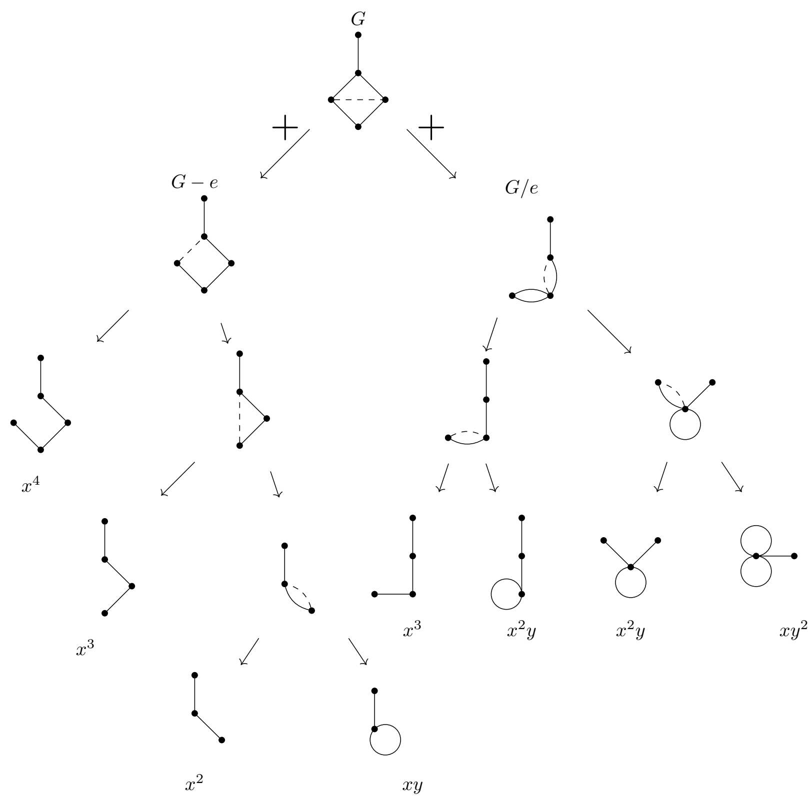
$$T_G^{(a)} = T_G^{(b)}.$$

□

If a graph consists only of bridges and loops, then the recursive formula for the Tutte polynomial is easily computed. Every bridge contributes a factor of x and every loop a factor of y. Thus, if a graph G only consists of n bridges and m loops, then $T_G(x, y) = x^n y^m$.

The Tutte polynomial is a graph invariant. If two graphs G and H are isomorphic, then deleting or contracting an edge in G corresponds to deleting or contracting the corresponding edge in H. Therefore, the recursive computation produces the same polynomial. In other words, $T_G(x, y) = T_H(x, y)$ if $G \cong H$.

Example 3.11. Let us now compute the Tutte polynomial of the same graph as in Example 3.8.



Adding the terms yields

$$T_G(x, y) = x^4 + 2x^3 + x^2 + 2x^2y + xy^2 + xy.$$

Comparing Example 3.8 and Example 3.11 suggests a direct connection from the Tutte polynomial to the chromatic polynomial. We will later prove that the chromatic polynomial is a specialization of the Tutte polynomial. Ignoring the signs in the chromatic polynomial for a moment, one can see that $\lambda \cdot T_G(\lambda - 1, 0)$ roughly captures the same structure as the chromatic polynomial.

Before proving the connection between the Tutte polynomial and the chromatic polynomial we need to develop additional theory concerning the Tutte polynomial. For any graph $G = (V, E, \psi)$ let A be a subset of the edge set E. We define the graph $G_A = (V, A, \psi|_A)$ to be the spanning subgraph of G with edge set A. Intuitively, G_A is obtained by keeping only the edges in the edge set A and all vertices in the vertex set V.

Definition 3.12. The *rank* of A is defined to be

$$r(A) = |V| - k(G_A).$$

For a graph $G = (V, E, \psi)$ when $A = E$ we sometimes write $r(G)$ instead of $r(E)$.

Lemma 3.13. *Let $A \subseteq E$. If $A = \emptyset$, then $r(A) = 0$.*

Proof. Assume that $A = \emptyset$. Then the spanning subgraph G_A contains no edges but all vertices, so $k(G_A) = |V|$. Then $r(A) = |V| - k(G_A) = 0$. □

Proposition 3.14. *For any graph G, every spanning forest of G contains exactly $r(G)$ edges.*

Proof. Let $m := k(G)$ and call the components of $G = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_m$. The number of vertices in each component creates a partition of the number of vertices in G, i.e. $|V| = |V_1| + |V_2| + \cdots + |V_m|$. Every spanning tree of a component G_i contains precisely $|V_i| - 1$ edges. So

$$\begin{aligned}\#\text{edges in spanning forests} =& (|V_1| - 1) + (|V_2| - 1) + \cdots + (|V_m| - 1)\\ =& |V_1| + |V_2| + \cdots + |V_m| - m\\ =& |V| - k(G) = r(G).\end{aligned}$$

□

Definition 3.15. For any graph $G = (V, E, \psi)$, we define the *rank-generating function*

$$R_G(u, v) = \sum_{A \subseteq E} u^{r(E)-r(A)} v^{|A|-r(A)}.$$

Theorem 3.16. *The Tutte polynomial $T_G(x, y)$ is equal to the rank-generating function with a change of variables,*

$$T_G(x, y) = R_G(x-1, y-1) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}.$$

Proof. We show that $R_G(x-1, y-1)$ satisfies the defining recursion of the Tutte polynomial (Definition 3.9). We need to prove

(i) if $E = \emptyset$, then
$$R_G(x-1, y-1) = 1,$$

(ii) if $e \in E$ is a bridge, then
$$R_G(x-1, y-1) = x \cdot R_{G-e}(x-1, y-1),$$

(iii) if $e \in E$ is a loop, then
$$R_G(x-1, y-1) = y \cdot R_{G-e}(x-1, y-1),$$

(iv) if $e \in E$ is neither a bridge nor a loop, then

$$R_G(x-1, y-1) = R_{G-e}(x-1, y-1) + R_{G/e}(x-1, y-1).$$

To prove (i), suppose $E = \emptyset$. Then the only subset A of E is the empty set. By Lemma 3.13, $r(A) = 0$ and $r(E) = 0$. So,

$$R_G(x-1, y-1) = \sum_{A \subseteq E} (x-1)^0 (y-1)^0 = 1$$

which proves (i).

Let us partition the subsets $A \subseteq E$ into those containing e and those not containing e. So

$$\begin{aligned} R_G(x-1, y-1) = &\sum_{\substack{A \subseteq E \\ e \in A}} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)} \\ &+ \sum_{\substack{A \subseteq E \\ e \notin A}} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}. \end{aligned}$$

We will now write $r_G(A)$ for the rank of A in G, $r_{G-e}(A)$ for the rank of A in $G-e$ and $r_{G/e}(A)$ for the rank of A in G/e.

We now prove (ii). Assume $e \in E$ is a bridge. Removing e increases the number of components by one so $r_G(E) = r_{G-e}(E - \{e\}) + 1$. If $e \in A$ then $r_G(A) = r_{G-e}(A - \{e\}) + 1$ and $|A| - r_G(A) = |A - \{e\}| - r_{G-e}(A - \{e\})$. If $e \notin A$ then G_A is isomorphic to $(G-e)_A$ since removing e does not affect the edges in A. So, $r_G(A) = r_{G-e}(A)$.

Now let $e \notin A$, then

$$\begin{aligned} &\sum_{\substack{A \subseteq E \\ e \notin A}} (x-1)^{r_G(E)-r_G(A)} (y-1)^{|A|-r_G(A)} \\ &\qquad = \sum_{A \subseteq E-\{e\}} (x-1)^{r_{G-e}(E-\{e\})-r_{G-e}(A)+1} (y-1)^{|A|-r_{G-e}(A)} \\ &\qquad =(x-1) \sum_{A \subseteq E-\{e\}} (x-1)^{r_{G-e}(E-\{e\})-r_{G-e}(A)} (y-1)^{|A|-r_{G-e}(A)}. \end{aligned}$$

Now let $e \in A$ and let $A = B \sqcup \{e\}$ for $B \subseteq E - \{e\}$. Since e is a bridge, $r_G(A) = r_{G-e}(B) + 1$, $|A| - r_G(A) = |B| - r_{G-e}(B)$ and $r_G(E) - r_G(A) = r_{G-e}(E - \{e\}) - r_{G-e}(B)$. Then

$$\sum_{\substack{A\subseteq E\\ e\in A}} (x-1)^{r_G(E)-r_G(A)}(y-1)^{|A|-r_G(A)}$$
$$= \sum_{B\subseteq E-\{e\}} (x-1)^{r_{G-e}(E-\{e\})-r_{G-e}(B)}(y-1)^{|B|-r_{G-e}(B)}$$

So

$$\begin{aligned}
R_G(x-1,y-1)&\\
&=(x-1) \sum_{A\subseteq E-\{e\}} (x-1)^{r_{G-e}(E-\{e\})-r_{G-e}(A)}(y-1)^{|A|-r_{G-e}(A)}\\
&\quad+ \sum_{A\subseteq E-\{e\}} (x-1)^{r_{G-e}(E-\{e\})-r_{G-e}(A)}(y-1)^{|A|-r_{G-e}(A)}\\
&=((x-1)+1) \sum_{A\subseteq E-\{e\}} (x-1)^{r_{G-e}(E-\{e\})-r_{G-e}(A)}(y-1)^{|A|-r_{G-e}(A)}\\
&=x\cdot R_{G-e}(x-1,y-1).
\end{aligned}$$

The proof of (iii) is similar. Let $e \in E$ be a loop. Removing e does not change the number of components. So, when $e \notin A$, then $r_G(E) = r_{G-e}(E - \{e\})$ and $r_G(A) = r_{G-e}(A)$. However, if $e \in A$, then $|A| = |A - \{e\}| + 1$. For $e \notin A$, we obtain

$$\sum_{\substack{A\subseteq E\\ e\notin A}} (x-1)^{r_G(E)-r_G(A)}(y-1)^{|A|-r_G(A)}$$
$$= \sum_{A\subseteq E-\{e\}} (x-1)^{r_{G-e}(E-\{e\})-r_{G-e}(A)}(y-1)^{|A|-r_{G-e}(A)}$$

and if $e \in A$

$$\sum_{\substack{A\subseteq E\\ e\in A}} (x-1)^{r_G(E)-r_G(A)}(y-1)^{|A|-r_G(A)}$$
$$\begin{aligned}
&= \sum_{B\subseteq E-\{e\}} (x-1)^{r_{G-e}(E-\{e\})-r_{G-e}(B)}(y-1)^{|B|-r_{G-e}(B)+1}\\
&=(y-1) \sum_{B\subseteq E-\{e\}} (x-1)^{r_{G-e}(E-\{e\})-r_{G-e}(B)}(y-1)^{|B|-r_{G-e}(B)}.
\end{aligned}$$

So

$$R_G(x-1,y-1) = y \cdot R_{G-e}(x-1,y-1)$$

when e is a loop.

Lastly we will prove that (iv) holds. Let e be an edge in G that is neither a bridge or a loop. If $e \notin A$, then $r_G(A) = r_{G-e}(A)$ and $|A|$ is unchanged. If $e \in A$ let $B = A - \{e\} \subseteq E - \{e\}$. Then the rank of A in G satisfies

$$r_G(A) = r_{G/e}(B) + 1$$

because the rank of A in G is one more than the rank of B in G/e. So if $e \in A$, then $|A| - r_G(A) = |B| - r_{G/e}(B)$. Since e is not a loop, contracting e decreases the number of vertices so $r_G(E) = r_{G/e}(E - \{e\}) + 1$. Therefore, $r_G(E) - r_G(A) = r_{G/e}(E - \{e\}) - r_{G/e}(B)$ when $e \in A$.

Partitioning the summation as before, when $e \notin A$ we have

$$\begin{aligned}
&\sum_{\substack{A \subseteq E \\ e \notin A}} (x-1)^{r_G(E)-r_G(A)}(y-1)^{|A|-r_G(A)} \\
&\qquad = \sum_{A \subseteq E-\{e\}} (x-1)^{r_{G-e}(E-\{e\})-r_{G-e}(A)}(y-1)^{|A|-r_{G-e}(A)} \\
&\qquad = R_{G-e}(x-1,y-1)
\end{aligned}$$

and when $e \in A$, we obtain

$$\begin{aligned}
&\sum_{\substack{A \subseteq E \\ e \in A}} (x-1)^{r_G(E)-r_G(A)}(y-1)^{|A|-r_G(A)} \\
&\qquad = \sum_{B \subseteq E-\{e\}} (x-1)^{r_{G/e}(E-\{e\})-r_{G/e}(B)}(y-1)^{|B|-r_{G/e}(B)} \\
&\qquad = R_{G/e}(x-1,y-1).
\end{aligned}$$

Adding the two cases together, we obtain

$$R_G(x-1,y-1) = R_{G-e}(x-1,y-1) + R_{G/e}(x-1,y-1).$$

By verifying all four cases of Definition 3.9 we conclude that

$$T_G(x,y) = R_G(x-1,y-1)$$

since the Tutte polynomial is uniquely determined by those relations. □

This formula shows that the Tutte polynomial is not only defined recursively, but also admits an explicit combinatorial expression.

With these tools in hand, we establish the following properties of the Tutte polynomial.

Proposition 3.17. *Let $G = (V, E, \psi)$ be any graph. Evaluating the Tutte polynomial of G at the following points gives the following combinatorial information about G:*

(i) $T_G(1, 1)$ *counts the number of spanning forests*

(ii) $T_G(2, 1)$ *counts the number of forests, whether they span G or not*

(iii) $T_G(1, 2)$ *counts the number of spanning subgraphs*

(iv) $T_G(2, 2) = 2^{|E|}$.

Here and throughout, we adopt the convention $0^0 := 1$ as is consistent with viewing the Tutte polynomial as a counting polynomial.

Proof. This proof relies on the identity proven in Theorem 3.16. Let us prove the proposition in order.

(i) At $(x, y) = (1, 1)$, the sum becomes

$$T_G(1, 1) = \sum_{A \subseteq E} 0^{r(E)-r(A)} 0^{|A|-r(A)}.$$

So a term is nonzero only when $r(A) = r(E)$ and $|A| = r(A)$. The first condition means that G_A has the same number of components as G, i.e. G_A is a spanning subgraph of G. The second condition means that G_A does not contain any cycles, meaning that G_A is a forest. When both conditions hold G_A is a spanning forest and each such subgraph contributes one to the sum.

(ii) As in (i), a term is nonzero only when $|A| = r(A)$ which means that G_A is acyclic. Since the factor $(x - 1)^{r(E)-r(A)} = 1$ at $x = 2$ for all $A \subseteq E$, the spanning condition disappears. The function contributes one for every forest G_A, whether G_A is a spanning subgraph or not.

(iii) This proof is similar to (ii), but the roles of spanning and acyclic are reversed.

(iv) Here

$$T_G(2, 2) = \sum_{A \subseteq E} 1^{r(E)-r(A)} 1^{|A|-r(A)}$$

so every subset $A \subseteq E$ contributes one. There are a $2^{|E|}$ subsets of E so $T_G(2, 2) = 2^{|E|}$.

□

Using Theorem 3.16 we can now prove that a well-known fact for the chromatic polynomial also holds for the Tutte polynomial.

Proposition 3.18. *Let G and H be two disjoint graphs. Then*

$$T_{G\sqcup H}(x,y) = T_G(x,y)\cdot T_H(x,y).$$

Proof. Let G and H be disjoint graphs. Being disjoint means that

- $E(G\sqcup H) = E(G)\sqcup E(H)$
- for all $A\subseteq E(G\sqcup H)$, $A = A_G\sqcup A_H$ where $A_G\subseteq E(G)$, $A_H\subseteq E(H)$
- $r_{G\sqcup H}(A) = r_G(A_G) + r_H(A_H)$
- $r_{G\sqcup H}(E(G\sqcup H)) = r_G(E(G)) + r_H(E(H))$

since

$$\begin{aligned} r_{G\sqcup H}(A) =& |V(G\sqcup H)| - k(A)\\ =& |V(G)| + |V(H)| - k(A_G) - k(A_H)\\ =& r_G(A_G) + r_H(A_H). \end{aligned}$$

Then, from Theorem 3.16, we have

$$\begin{aligned} &T_{G\sqcup H}(x,y)\\ &\quad= \sum_{A\subseteq E(G\sqcup H)} (x-1)^{r_{G\sqcup H}(E)-r_{G\sqcup H}(A)}(y-1)^{|A|-r_{G\sqcup H}(A)}\\ &\quad= \sum_{A_G,A_H} \Big((x-1)^{r_G(E(G))-r_G(A_G)}(y-1)^{|A_G|-r_G(A_G)}\\ &\qquad\cdot (x-1)^{r_H(E(H))-r_H(A_H)}(y-1)^{|A_H|-r_H(A_H)}\Big)\\ &\quad= T_G(x,y)\cdot T_H(x,y). \end{aligned}$$

□

Proposition 3.19. *Let G and H be two disjoint graphs with $a_G\in V(G)$ and $a_H\in V(H)$. Contracting $a_G\in V(G)$ and $a_H\in V(H)$ to one vertex yields the graph $(G\sqcup H)/(a_G\sim a_H)$. Then*

$$T_{(G\sqcup H)/(a_G\sim a_H)}(x,y) = T_G(x,y)\cdot T_H(x,y).$$

Proof. We denote $K = T_{(G \sqcup H)/(a_G \sim a_H)}$. Since we are only identifying two vertices every $A \subseteq E(K)$ is the disjoint union A_G and A_H where $A_G \subseteq E(G)$ and $A_H \subseteq E(H)$. Then

$$|A| = |A_G| + |A_H|.$$

Since two vertices are identified $|V(K)| = |V(G)| + |V(H)| - 1$. For a subset A, in G, the vertex a_G lies in exactly one component and the same for a_H in H. After identification, those two components become exactly one. So

$$k(A) = k(A_G) + k(A_H) - 1.$$

Then

$$r_K(A) = r_G(A_G) + r_H(A_H).$$

After the same simplifications as in the proof of Proposition 3.18, the proof is complete. □

The multiplicative property established above mirrors the corresponding property of the chromatic polynomial. Together with the deletion–contraction relation, this further supports the suggested connection between the Tutte polynomial and the chromatic polynomial that we observed earlier. In Example 3.11 we have seen that the expression $\lambda \cdot T_G(\lambda - 1, 0)$ appears to capture the same structure as the chromatic polynomial. We now make this relationship precise by proving that the chromatic polynomial is a specialization of the Tutte polynomial.

Lemma 3.20. *The chromatic polynomial for a graph $G = (V, E, \psi)$ is*

$$\chi_G(\lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{k(G_A)}.$$

Proof. We will use the inclusion-exclusion principle. Let us, for each edge $e \in E$ adjacent to a and b, define

$$S_e = \{\text{colorings where } a, b \text{ have the same color}\}.$$

Proper colorings are those colorings that are not in $\cup_{e \in E} S_e$. Those colorings correspond exactly to

$$\sum_{A \subseteq E} (-1)^{|A|} \left| \bigcap_{e \in A} S_e \right|$$

for $A \subseteq E$. Here, $|\cap_{e \in A} S_e|$ means for every edge in A, its endpoints have the same color. So every connected component in A must have the same color. This can be done in exactly $\lambda^{k(G_A)}$ ways, so

$$\left| \bigcap_{e \in A} S_e \right| = \lambda^{k(G_A)}.$$

Therefore,

$$\chi_G(\lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{k(G_A)}.$$

□

We are now ready to derive the chromatic polynomial from the Tutte polynomial.

Proposition 3.21. *The chromatic polynomial of a graph G can be derived from the Tutte polynomial by*

$$\chi_G(\lambda) = (-1)^{r(E)} \lambda^{k(G)} T_G(1-\lambda, 0).$$

Proof. By Theorem 3.16

$$\begin{aligned} T_G(1-\lambda, 0) &= \sum_{A \subseteq E} ((1-\lambda)-1)^{r(E)-r(A)} (0-1)^{|A|-r(A)} \\ &= \sum_{A \subseteq E} (-1)^{|A|-r(A)} (-\lambda)^{r(E)-r(A)} \\ &= \sum_{A \subseteq E} (-1)^{r(E)-r(A)} (-1)^{|A|-r(A)} \lambda^{r(E)-r(A)} \\ &= \sum_{A \subseteq E} (-1)^{|A|+r(E)-2r(A)} \lambda^{r(E)-r(A)} \\ &= \sum_{A \subseteq E} (-1)^{|A|+r(E)} \lambda^{r(E)-r(A)} \\ &= (-1)^{r(E)} \sum_{A \subseteq E} (-1)^{|A|} \lambda^{r(E)-r(A)}. \end{aligned}$$

Multiplying $T_G(1-\lambda, 0)$ with $(-1)^{r(E)} \lambda^{k(G)}$ yields

$$\begin{aligned} (-1)^{r(E)} \lambda^{k(G)} T_G(1-\lambda, 0) &= (-1)^{r(E)} \lambda^{k(G)} (-1)^{r(E)} \sum_{A \subseteq E} (-1)^{|A|} \lambda^{r(E)-r(A)} \\ &= (-1)^{2r(E)} \sum_{A \subseteq E} (-1)^{|A|} \lambda^{r(E)-r(A)+k(G)} \\ &= \sum_{A \subseteq E} (-1)^{|A|} \lambda^{r(E)-r(A)+k(G)}. \end{aligned}$$

Simplifying the powers of λ

$$r(E) - r(A) + k(G) = (|V| - k(G)) - (|V| - k(G_A)) + k(G) = k(G_A).$$

So

$$(-1)^{r(E)} \lambda^{k(G)} T_G(1-\lambda, 0) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{k(G_A)}$$

which equals the chromatic polynomial by Lemma 3.20. □

Proposition 3.21 shows that the chromatic polynomial arises as a specialization of the Tutte polynomial. In particular, the deletion-contraction relation for the chromatic polynomial is a special case of the more general deletion–contraction recurrence of the Tutte polynomial.

The chromatic polynomial is uniquely determined by the deletion-contraction relation together with multiplicativity over disjoint unions. The Tutte polynomial satisfies the same structural property, but in a two-variable form. The Tutte polynomial is in fact the fundamental object governing all invariants satisfying deletion and contraction as the following theorem shows.

Definition 3.22. Let F be a function from graphs to a commutative ring R. We say that F is a *deletion-contraction invariant* if:

(i) If G contains no edges, then $F(G) = 1$.

(ii) There exist units $a, b \in R$ such that whenever an edge e is neither a loop nor a bridge

$$F(G) = aF(G - e) + bF(G/e).$$

(iii) There exist constants $A, B \in R$ such that

$$\begin{aligned} F(G) =& AF(G - e) && \text{if e is a bridge,} \\ F(G) =& BF(G - e) && \text{if e is a loop.} \end{aligned}$$

Theorem 3.23. *Let F be a deletion-contraction invariant with a, b, A, B as defined above. Any such F can be expressed as*

$$F(G) = a^{|E|-|V|+k(G)} b^{|V|-k(G)} T_G\left(\frac{A}{b}, \frac{B}{a}\right).$$

Proof. We will prove the statement by induction on $|E(G)|$. If G contains no edges then $T_G(x, y) = 1$ and $F(G) = 1$, so the formula holds. Assume that the formula is true for all graphs with edges fewer than G and let $e \in E(G)$.

Case 1: e is a loop. Deleting a loop decreases the number of edges, but not the number of components or vertices.

$$
\begin{aligned}
F(G) =& BF(G-e)\\
=& Ba^{|E(G-e)|-|V(G-e)|+k(G-e)}b^{|V(G-e)|-k(G-e)}T_{G-e}\left(\frac{A}{b},\frac{B}{a}\right)\\
=& Ba^{|E(G)|-1-|V(G)|+k(G)}b^{|V(G)|-k(G)}T_{G-e}\left(\frac{A}{b},\frac{B}{a}\right)\\
=& \frac{B}{a}a^{|E(G)|-|V(G)|+k(G)}b^{|V(G)|-k(G)}T_{G-e}\left(\frac{A}{b},\frac{B}{a}\right)\\
=& a^{|E(G)|-|V(G)|+k(G)}b^{|V(G)|-k(G)}T_{G}\left(\frac{A}{b},\frac{B}{a}\right)
\end{aligned}
$$

since $|E(G-e)| = |E(G)| - 1$.

Case 2: e is a bridge. Deleting a bridge decreases the number of edges and increases the number of components, but the number of vertices is unchanged.

$$
\begin{aligned}
F(G) =& AF(G-e)\\
=& Aa^{|E(G-e)|-|V(G-e)|+k(G-e)}b^{|V(G-e)|-k(G-e)}T_{G-e}\left(\frac{A}{b},\frac{B}{a}\right)\\
=& Aa^{|E(G)|-1-|V(G)|+k(G)+1}b^{|V(G)|-(k(G)+1)}T_{G-e}\left(\frac{A}{b},\frac{B}{a}\right)\\
=& \frac{A}{b}a^{|E(G)|-|V(G)|+k(G)}b^{|V(G)|-k(G)}T_{G-e}\left(\frac{A}{b},\frac{B}{a}\right)\\
=& a^{|E(G)|-|V(G)|+k(G)}b^{|V(G)|-k(G)}T_{G}\left(\frac{A}{b},\frac{B}{a}\right).
\end{aligned}
$$

Case 3: e is not a loop or a bridge. Deleting e decreases the number of edges, but does not change the number of components or the number of vertices. Contracting e decreases the number of vertices and the number of edges, but does not change the number of components.

$$
\begin{aligned}
F(G) =& aF(G-e) + bF(G/e) \\
=& a \cdot a^{|E(G)|-1-|V(G)|+k(G)} b^{|V(G)|-k(G)} T_{G-e}\left(\frac{A}{b}, \frac{B}{a}\right) \\
&+ b \cdot a^{|E(G)|-1-(|V(G)|-1)+k(G)} b^{|V(G)|-1-k(G)} T_{G/e}\left(\frac{A}{b}, \frac{B}{a}\right) \\
=& \left(a^{|E(G)|-|V(G)|+k(G)} b^{|V(G)|-k(G)}\right) \cdot \left(T_{G-e}\left(\frac{A}{b}, \frac{B}{a}\right) + T_{G/e}\left(\frac{A}{b}, \frac{B}{a}\right)\right) \\
=& a^{|E(G)|-|V(G)|+k(G)} b^{|V(G)|-k(G)} T_G\left(\frac{A}{b}, \frac{B}{a}\right).
\end{aligned}
$$

In all three cases, the statement holds. This proves the theorem.

□

Theorem 3.23 shows that the Tutte polynomial is not merely an example of a deletion-contraction invariant. Rather, it is universal. Any deletion-contraction invariant is a specialization of the Tutte polynomial. In this sense, every such invariant factors through the Tutte polynomial.

Note that the chromatic polynomial is not a deletion-contraction invariant. If G contains no edges, then $\chi_G(\lambda) = \lambda^{k(G)}$ which is not necessarily one. However, $\frac{\chi_G(\lambda)}{\lambda^{k(G)}}$ is such an invariant. We let $F = \frac{\chi_G(\lambda)}{\lambda^{k(G)}}$. Then F is a specialization of the Tutte polynomial via Proposition 3.21 by setting $A = \lambda - 1$, $B = 0$, $a = 1$ and $b = -1$. Similar recursive patterns appear in other areas of mathematics as well.

In the next chapter, we will study a class of objects called links. We will distinguish links using invariants. One of these invariants can be defined using a recursive process very similar to deletion and contraction. By translating links into graphs, we will see that this link invariant is closely related to the Tutte polynomial.

4 An Introduction to Knot Theory

Intuitively, one may think of a knot as a tangled rope whose endpoints are glued together. More formally, a *knot* is an embedding of the circle S^1 into $\mathbb{R}^3$. A *link* consists of one or more disjoint knots, that is, a finite collection of disjoint embeddings of S^1 in $\mathbb{R}^3$.

One of the central problems in knot theory is to determine whether two links represent the same embedding up to continuous deformation. In other words, we ask whether one can be untangled into the other without cutting the strands or allowing them to pass through each other.

Elementary Knot Theory

Let $L \subset \mathbb{R}^3$ be a link and let

$$\pi : \mathbb{R}^3 \to \mathbb{R}^2$$

be a projection onto the plane. The image $\pi(L)$ is called a *link projection.*

If a component of a link is parametrized by

$$\gamma : S^1 \to \mathbb{R}^3$$

then a *strand* is the image of $\gamma|_I$ for some interval $I \subset S^1$. If distinct points of L are projected onto the same point c, we call c a *crossing.*

Definition 4.1. A projection of a link from $\mathbb{R}^3$ to $\mathbb{R}^2$ is called *regular* if:

(i) the projection has only finitely many crossings,

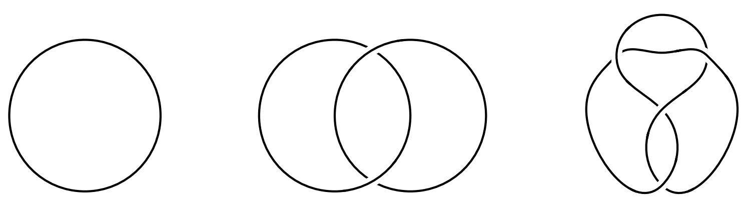
(ii) every crossing consists of exactly two points of the link projected to the same point,

(iii) at each crossing, the corresponding tangent directions are distinct.

When two points p_1 and p_2 on two different strands are projected onto the same point $c \in \mathbb{R}^2$, we preserve crossing information by indicating which of the two strands passes over and which passes under at each crossing. A regular projection of a link together with this crossing information is called a *link diagram.*

Example 4.2. The following picture shows the following link diagrams of links:

- the knot O called the *unknot* which is the trivial embedding of the circle into $\mathbb{R}^3$,
- the link L_1 consisting of two unknots linked together,
- the figure eight knot 4_1.



O L_1 4_1

A natural question is whether we can deform 4_1 into O. That is, if we are given an untangled rope with glued together ends, can we tangle the rope into something that looks like 4_1? To make these notions precise, we need to introduce the concept of ambient isotopy, which formalizes the idea of deforming a knot in $\mathbb{R}^3$ without cutting the strands or letting strands pass through one another. We will introduce two different but related definitions to specify when two links are considered equivalent.

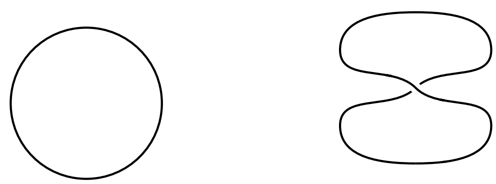
Definition 4.3. Let L_1 and L_2 be two links. We say that the two links are *ambient isotopic* if there is a family of homeomorphisms

$$H_t : \mathbb{R}^3 \to \mathbb{R}^3 \quad t \in [0,1]$$

such that $H_0 = \mathrm{id}_{\mathbb{R}^3}$ and $H_1(L_1) = L_2$. If there is an ambient isotopy we write $L_1 \cong L_2$.

Two knot diagrams may look different but represent the same knot as is shown in the following example.

Example 4.4. The diagrams D_1 and D_2 both represent the unknot O. D_1 can be transformed into D_2 by twisting the top of D_2.



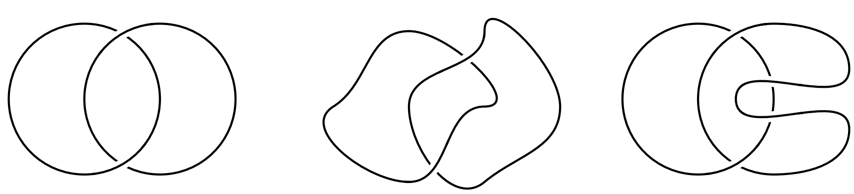
D_1 D_2

Definition 4.5. Two link diagrams D_1 and D_2 are said to be *planar isotopic* if there is a family of homeomorphisms

$$\phi_t : \mathbb{R}^2 \to \mathbb{R}^2 \quad t \in [0,1]$$

such that $\phi_0 = \mathrm{id}_{\mathbb{R}^2}$, $\phi_1(D_1) = D_2$ and for all $t \in [0,1]$, $\phi_t(D_1)$ is a regular projection that preserves crossing information. Such deformations include stretching, bending and sliding strands in the plane.

Example 4.6. Here, L_1, as in Example 4.2, and L_1' are planar isotopic, but L_1'' is not planar isotopic to L_1.



L_1 L_1' L_1''

Note that both being ambient isotopic and planar isotopic are equivalence relations. Being ambient isotopic and planar isotopic are related through the following proposition.

Proposition 4.7. *If two link diagrams D_1 and D_2 are planar isotopic, then the links L_1 and L_2 they represent are ambient isotopic.*

Proof. Assume two link diagrams D_1 and D_2 are planar isotopic and that they represent L_1 and L_2 respectively. Then there exists a family of homeomorphisms

$$\phi_t : \mathbb{R}^2 \to \mathbb{R}^2 \quad t \in [0,1]$$

such that $\phi_0 = \mathrm{id}_{\mathbb{R}^2}$ and $\phi_1(D_1) = D_2$ and the crossing information is preserved throughout the deformations.

To obtain a link diagram, we regularly project $L \subset \mathbb{R}^3$ onto $\mathbb{R}^2$. We may assume that the diagram lies in the plane $z = 0$ and at each crossing, the over-strand lies at $z = \epsilon$ and the under-strand lies at $z = -\epsilon$ for small ϵ. Since $\{\phi_t\}_{t\in[0,1]}$ is a planar isotopy, define

$$H_t : \mathbb{R}^3 \to \mathbb{R}^3 \quad t \in [0,1], \quad H_t(x,y,z) = (\phi_t(x,y), z).$$

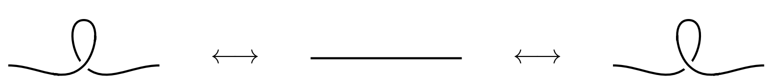
These maps move (x, y) as the planar isotopy but leaves the heights unchanged. Because ϕ_t is a homeomorphism, each H_t is also a homeomorphism. Moreover, $H_0 = \mathrm{id}_{\mathbb{R}^3}$, the family of H_t varies continuously in t and since H_t does not alter the height z the crossing information is preserved. Applying H_1 to the link corresponding to D_1 produces the link corresponding to D_2 so L_1 and L_2 are ambient isotopic.

□

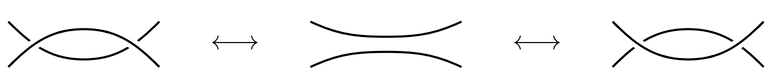
Since we study link diagrams rather than the actual embedded links in $\mathbb{R}^3$, we need methods for manipulating link diagrams that correspond to ambient isotopy. Two links are ambient isotopic up to planar isotopy if and only if their link diagrams are related by finitely many local moves known as Reidemeister moves. With this correspondence, we are able to study link diagrams directly instead of the actual link in $\mathbb{R}^3$.

Definition 4.8. The following local operations on the strands are called *Reidemeister moves*:

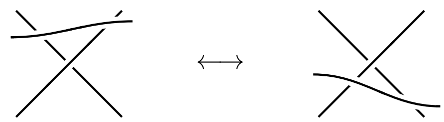
(RI) Introduction or removal of a twist,



(RII) Introduction or removal of two crossings,



(RIII) Sliding a strand over a crossing.



Theorem 4.9. [**Rei26**] Let L_1 and L_2 be two links with diagrams D_1 and D_2 respectively. L_1 and L_2 are ambient isotopic if and only if D_1 can be transformed into D_2 by a finite number of planar isotopies and Reidemeister moves.

This proof is omitted as it is beyond the scope of this thesis.

One central problem in knot theory is to decide whether two links are ambient isotopic. To show that two links L_1 and L_2 are ambient isotopic, we can by Theorem 4.9, show that there are a finite sequence of Reidemeister moves that transform D_1 to D_2. To show that two links are not ambient isotopic we may instead use *link invariants*. A link invariant is a function f with the domain of links such that if $K_1 \cong K_2$, then $f(K_1) = f(K_2)$. With the contrapositive we obtain that if $f(K_1) \neq f(K_2)$, then $K_1 \ncong K_2$.

One example of a link invariant is the number of components of a link. If two links have a different number of components, then they cannot be transformed into each other since that would require breaking and gluing the strands.

The Kauffman Bracket

One of the key consequences of the Reidemeister moves and Theorem 4.9 is that to show that a function f is a link invariant we only need to show that f remains unchanged under the three Reidemeister moves.

Definition 4.10. Let D be a link diagram. The *Kauffman bracket*, denoted $\langle D \rangle$, is defined recursively by

(i) $\langle \bigcirc \rangle = 1$, where $\bigcirc$ = unknot

(ii) $\langle D \sqcup \bigcirc \rangle = -(A^2 + A^{-2}) \langle D \rangle$

(iii) $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$

where A is an indeterminate.

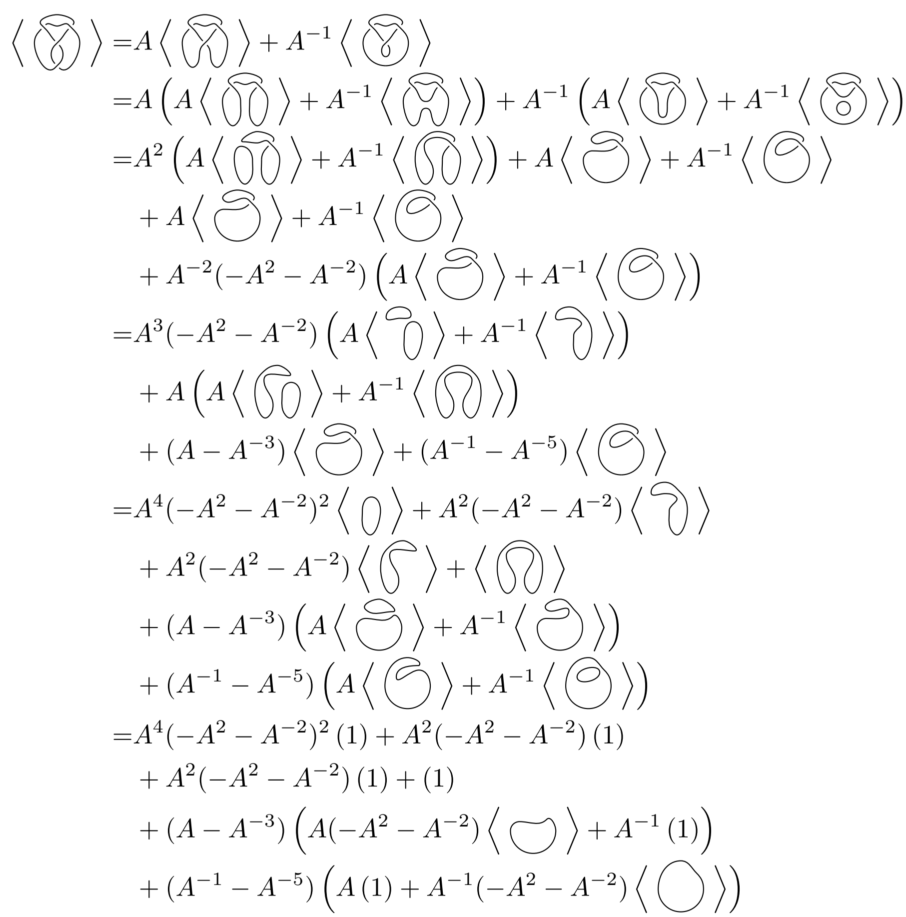
Note that the recursive definition of the Kauffman bracket is independent of any orientation of the diagram. The relation in (iii) replaces a crossing by two planar resolutions called the *A-smoothing* and the *B-smoothing* of the crossing. Thus (iii) is sometimes written as

$$\langle \text{crossing} \rangle = A \langle A\text{-smoothing} \rangle + A^{-1} \langle B\text{-smoothing} \rangle .$$

An A-smoothing is obtained by following the over-strand and, at the crossing, connecting the strands to the left instead of straight across. Conversely, a B-smoothing connects the strands in the alternative way, to the right.

It may not be clear that this recursive definition uniquely determines the Kauffman bracket, since different orders of resolving the crossings could potentially lead to different results. In Proposition 4.13 we show that the value of the Kauffman bracket is independent of the order in which crossings are resolved.

Example 4.11. Let us compute the Kauffman bracket for the figure eight knot 4_1.



$$
\begin{aligned}
&= A^8 + 1 + 2A^4 - A^4 - 1 \\
&\quad - A^4 - 1 + 1 \\
&\quad - A^4 - 1 + 1 + 1 + A^{-4} - A^{-4} \\
&\quad + 1 - 1 - A^{-4} - A^{-4} + A^{-4} + A^{-8} \\
&= A^8 - A^4 + 1 - A^{-4} + A^{-8}.
\end{aligned}
$$

Definition 4.12. Let D be a link diagram with crossing $c_1, c_2, \ldots, c_n$. A *state* of D is a choice at each crossing of either A-smoothing of B-smoothing. Equivalently, a state s is a function

$$s : \{c_1, c_2, \ldots, c_n\} \to \{A, B\}$$

and then applying all chosen smoothings. This produces the diagram D_s consisting only of unknots with no crossings.

Proposition 4.13. *Let s be a state of an unoriented link D and let*

- $\alpha(s) :=$ *number of A-smoothings,*
- $\beta(s) :=$ *number of B-smoothings,*
- $\gamma(s) :=$ *number of unknots (components).*

Then, for any link diagram D,

$$\langle D \rangle = \sum_s A^{\alpha(s)-\beta(s)}(-A^2 - A^{-2})^{\gamma(s)-1}.$$

where the sum runs over all states of D.

Hence, $\langle D \rangle$ is a unique polynomial satisfying the three defining properties of the Kauffman bracket.

Proof. Let D be a diagram with n crossings. Since each crossing admits two possible smoothings, there are 2^n states. For each state s, the expression

$$A^{\alpha(s)-\beta(s)}(-A^2 - A^{-2})^{\gamma(s)-1}$$

is a polynomial. Since the sum is finite, $\langle D \rangle$ is a polynomial.

We will now verify that it satisfies the defining properties of the Kauffman bracket (i)-(iii).

First, suppose D is the unknot. Then D has no crossings so there is only one state where $\alpha(s) = \beta(s) = 0$ and $\gamma(s) = 1$. Hence,

$$\langle \bigcirc \rangle = A^{0-0}(-A^2 - A^{-2})^{1-1} = 1$$

which verifies (i).

Next, consider the diagram $\langle D \sqcup \bigcirc \rangle$. Adjoining a disjoint unknot does not introduce any new crossings. Therefore $\alpha(s)$ and $\beta(s)$ are unchanged for any state, however, $\gamma(s)$ increases by one for every state. It follows that

$$\langle D \sqcup \bigcirc \rangle = \sum_s A^{\alpha(s)-\beta(s)}(-A^2 - A^{-2})^{\gamma(s)} = -(A^2 + A^{-2}) \langle D \rangle$$

satisfying (ii).

Finally, let D be a diagram with crossings and choose one crossing c in D. Let D_A and D_B denote the diagrams obtained by applying the A- and B-smoothing to c respectively.

Now consider the states of D. When choosing an A-smoothing at c, the states of D correspond to the states of D_A. Then $\alpha(s) = \alpha_A(s) + 1$ and $\beta(s) = \beta_A(s)$ where α_A and β_A count smoothings in the corresponding state of D_A.

Similarly, when choosing a B-smoothing, the states of D correspond to the states of D_B. Here $\alpha(s) = \alpha_B(s)$ and $\beta(s) = \beta_B(s) + 1$ where α_B and β_B count the smoothings in the corresponding state of D_B.

Note that in both D_A and D_B the number of components is the same in their associated state of D so $\gamma(s)$ is unchanged in D_A and D_B. Hence,

$$\begin{aligned}\langle D\rangle =&\sum_s A^{\alpha(s)-\beta(s)}(-A^2 - A^{-2})^{\gamma(s)-1}\\ =&\sum_{s \text{ of } D_A} A^{\alpha_A(s)+1-\beta_A(s)}(-A^2 - A^{-2})^{\gamma(s)-1}\\ &+\sum_{s \text{ of } D_B} A^{\alpha_B(s)-(\beta_B(s)+1)}(-A^2 - A^{-2})^{\gamma(s)-1}\\ =&A\langle D_A\rangle + A^{-1}\langle D_B\rangle\end{aligned}$$

which verifies (iii).

It follows that $\langle D\rangle$ is unique because the formula explicitly expresses $\langle D\rangle$ as a finite sum over all states of D. Since the formula is completely determined by the diagram and the definitions of α, β, γ, there is no ambiguity. Any polynomial satisfying (i)-(iii) must agree with this sum. Hence, the Kauffman bracket is uniquely determined. □

The explicit summation to calculate the Kauffman bracket in Proposition 4.13 will be referred to as the *state sum* of diagrams and links.

The Kauffman bracket is usually computed on a connected link but performing A- and B-smoothings will sometimes result in the diagram becoming disconnected. Calculating the Kauffman bracket for one of the two disconnected components will eventually result in an unknot which in turn will contribute $-(A^2 + A^{-2})$ to the Kauffman bracket. The following lemma encapsulates this result.

Lemma 4.14. *Let D denote a link diagram consisting of two disjoint links D_1 and D_2, that is $D = D_1 \sqcup D_2$. Then*

$$\langle D_1 \sqcup D_2\rangle = -(A^2 + A^{-2})\langle D_1\rangle\langle D_2\rangle.$$

Proof. By Proposition 4.13 the Kauffman bracket of D can be expressed as

$$\langle D_1 \sqcup D_2\rangle = \sum_s A^{\alpha(s)-\beta(s)}(-A^2 - A^{-2})^{\gamma(s)-1}.$$

Since D is the disjoint union of D_1 and D_2, a state s of D is equivalent to the pair (s_1, s_2) where s_1 and s_2 are states of D_1 and D_2 respectively. Thus

$$\begin{aligned}\alpha(s) =&\alpha(s_1) + \alpha(s_2),\\ \beta(s) =&\beta(s_1) + \beta(s_2),\\ \gamma(s) =&\gamma(s_1) + \gamma(s_2).\end{aligned}$$

So

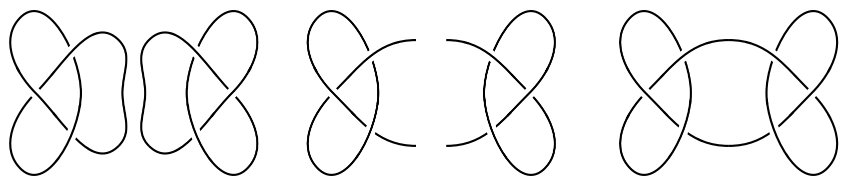
$$\begin{aligned}\langle D_1 \sqcup D_2 \rangle =& \sum_s A^{\alpha(s)-\beta(s)}(-A^2 - A^{-2})^{\gamma(s)-1}\\ =& \sum_{s_1,s_2} A^{\alpha(s_1)+\alpha(s_2)-\beta(s_1)-\beta(s_2)}(-A^2 - A^{-2})^{\gamma(s_1)+\gamma(s_2)-1}\\ =& -(A^2 + A^{-2}) \left(\sum_{s_1} A^{\alpha(s_1)-\beta(s_1)}(-A^2 - A^{-2})^{\gamma(s_1)-1} \right)\\ &\cdot \left(\sum_{s_2} A^{\alpha(s_2)-\beta(s_2)}(-A^2 - A^{-2})^{\gamma(s_2)-1} \right)\\ =& -(A^2 + A^{-2}) \langle D_1 \rangle \langle D_2 \rangle .\end{aligned}$$

□

Performing A- or B-smoothings on a crossing in a link diagram will always completely remove the crossing. Removing the crossing will either connect two regions opposite sides of the crossing or transform a crossing into a local band connecting two portions of the diagram.

Let D_1 and D_2 be two disjoint knot diagrams and choose one strand on each diagram that does not contain a crossing. Then remove small substrands of the chosen strands. The *connected sum* $D_1 \# D_2$ is defined to be the diagram obtained by identifying these endpoints pairwise such that no new crossings are introduced. Note that the connected sum is independent of the choice of strand, up to Reidemeister moves. Intuitively, one may imagine shrinking one of the connected knots and sliding it along the strand of the other knot. So the resulting knot diagrams are equivalent. For link diagrams, however, the connected sum is not uniquely determined unless we specify which components of the links are joined.

Example 4.15. Consider the *left-hand trefoil* T_l and the *right-hand trefoil* T_r. Their connected sum $T_l \# T_r$ is called the *square knot*.



$T_l \sqcup T_r$ Gluing $T_l \# T_r$

The following lemma shows how to calculate the Kauffman bracket for connected sums.

Lemma 4.16. *Let D_1 and D_2 be two disjoint link diagrams. Then*

$$\langle D_1 \# D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle .$$

Proof. We will again use the state sum. Since D_1 and D_2 are disjoint, we can express $\alpha(s) = \alpha(s_1) + \alpha(s_2)$ and $\beta(s) = \beta(s_1) + \beta(s_2)$ where s_1 and s_2 are the states of D_1 and D_2 respectively. When forming the connected sum, D_1 and D_2 are joined along exactly one strand, so $\gamma(s) = \gamma(s_1) + \gamma(s_2) - 1$. Thus

$$\begin{aligned}
\langle D_1 \# D_2 \rangle &= \sum_s A^{\alpha(s)-\beta(s)}(-A^2 - A^{-2})^{\gamma(s)-1} \\
&= \sum_{s_1, s_2} A^{\alpha(s_1)+\alpha(s_2)-\beta(s_1)-\beta(s_2)}(-A^2 - A^{-2})^{\gamma(s_1)+\gamma(s_2)-1-1} \\
&= \left(\sum_{s_1} A^{\alpha(s_1)-\beta(s_1)}(-A^2 - A^{-2})^{\gamma(s_1)-1} \right) \\
&\quad \cdot \left(\sum_{s_2} A^{\alpha(s_2)-\beta(s_2)}(-A^2 - A^{-2})^{\gamma(s_2)-1} \right) \\
&= \langle D_1 \rangle \langle D_2 \rangle .
\end{aligned}$$

□

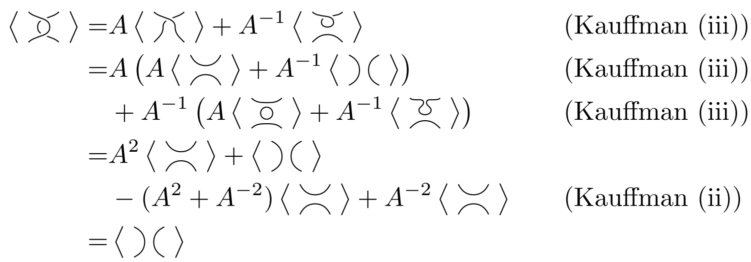
The Kauffman bracket, as defined above, is not invariant under all three Reidemeister moves and does therefore not define a link invariant. However, it is invariant under Reidemeister moves (RII) and (RIII). We first establish these partial invariant properties.

Lemma 4.17. *The Kauffman bracket is invariant under Reidemeister moves* (RII) *and* (RIII)*.*

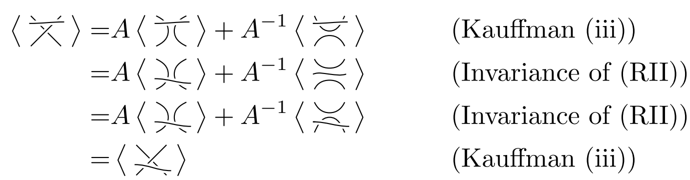
Proof. To check invariance under (RII) and (RIII) we can imagine a local part of a link. We can then perform the relevant Reidemeister move on that specific part since the Reidemeister moves only modify a small portion of the diagram.

Outside of the shown regions we will assume that the diagram is left unchanged. Notice that some of the knots may be redrawn up to planar isotopies.

(RII) We will only consider the removal of two crossings in this proof. The proof of introducing two crossings can be done in a similar way.



(RIII)

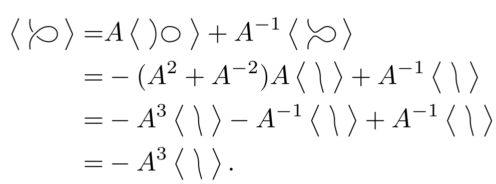


□

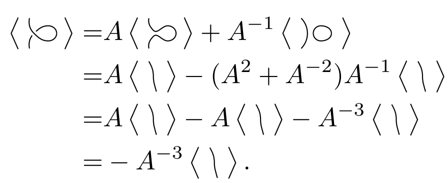
The Kauffman bracket changes when performing the first Reidemeister move as the following lemma shows.

Lemma 4.18. *The Kauffman bracket is not invariant under* (RI)*. Removing a twist will either result in a factor* $-A^3$ *or* $-A^{-3}$ *to the link.*

Proof. We can study performing (RI) to a link the same way as in Lemma 4.17.



Similarly,



□

The failure of invariance under (RI) shows that the Kauffman bracket is sensitive to twisting. By Lemma 4.18, the first Reidemeister move changes the Kauffman bracket by a factor of either $-A^3$ or $-A^{-3}$.

The Jones Polynomial

We will make the Kauffman bracket into a link invariant by solving its failure to be invariant under twists. To compensate for this we need to introduce a correcting term that records the total twisting of a diagram.

An *orientation* of a link is a choice of directions on each component of the link. An *oriented link diagram* is a link diagram of an oriented link. Each crossing of an oriented link diagram D is assigned $+1$ or -1 according to the following convention.

For any knot, the sign assigned to a crossing is independent of the choice of orientation, since reversing the orientation of the entire knot leaves all crossing signs unchanged. For links, however, we must fix an orientation for each component to ensure crossings are assigned signs unambiguously. When comparing two link diagrams, it might be difficult to orient them consistently, especially if we do not know which components in one diagram correspond to which components in the other or whether the two link diagrams even represent the same link. In contrast, for knots, this issue does not arise.

Definition 4.19. Let D be an oriented link diagram. The writhe of D, denoted $w(D)$ is the sum of the signs of all crossings c in D

$$w(D) = \sum_c \mathrm{sgn}(c).$$

A Reidemeister move of type I adds a single crossing. This crossing will either increase or decrease $w(D)$ by one depending on the twist, hence $w(D)$ changes by ± 1 and the writhe is not a link invariant.

Recall that the Kauffman bracket is invariant under (RII) and (RIII) but under (RI) it changes with a factor of $-A^{\pm 3}$ depending on the type of twist. Using the writhe of a link diagram we can compensate for this flaw.

Definition 4.20. Let D be an oriented link diagram. The *normalized Kauffman bracket* of D is

$$f_D(A) = (-A)^{-3w(D)} \langle D \rangle .$$

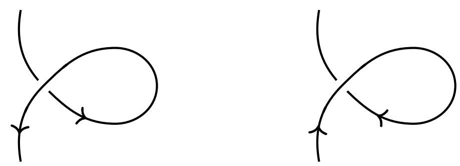
Theorem 4.21. *Let L be an oriented link and D the oriented link diagram of L. The normalized Kauffman bracket*

$$f_D(A) = (-A)^{-3w(D)} \langle D \rangle$$

is invariant under all three Reidemeister moves. In particular, $f_D(A)$ only depends on the orientation of L so we may write $f_L(A)$ for this invariant after choosing a consistent orientation.

Proof. We need to show that the normalized Kauffman bracket is invariant under all three Reidemeister moves.

1. *Reidemeister I:* Suppose D' is obtained from D by adding a twist with positive crossing. Then the writhe of D increases by one no matter what orientation is chosen.



+1 +1

By Lemma 4.18,

$$\langle D' \rangle = -A^3 \langle D \rangle \quad \text{and} \quad w(D') = w(D) + 1.$$

Hence,

$$\begin{aligned} f_{D'}(A) =& (-A)^{-3w(D')} \langle D' \rangle \\ =& (-A)^{-3(w(D)+1)} \cdot (-A^3) \langle D \rangle \\ =& (-A)^{-3w(D)} \langle D \rangle \\ =& f_D(A). \end{aligned}$$

A similar calculation holds for a negative twist.

2. *Reidemeister II and III:* Lemma 4.17 shows that the Kauffman bracket $\langle D \rangle$ is invariant under (RII) and (RIII). Moreover, (RII) and (RIII) do not change the total sum of crossing signs in the diagrams. In (RII), the two new crossings introduced have opposite signs so $w(D)$ is left unchanged. In (RIII), the crossings are locally rearranged so no new crossings are introduced, hence $w(D)$ is unchanged.

Therefore, $f_D(A)$ is invariant under all Reidemeister moves so it defines an oriented link invariant, $f_L(A)$. □

In 1984, Vaughan Jones discovered a new polynomial invariant for links that revolutionized knot theory. Jones originally defined the polynomial using different methods, but the invariant defined below coincides with Jones's original polynomial after a change of variables. [Jon85]

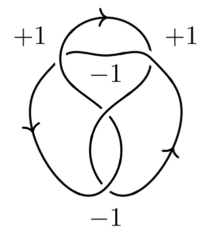
Definition 4.22. Substituting $A = t^{-1/4}$ into $f_L(A)$ gives the *Jones polynomial* $V_L(t)$.

$$V_L(t) = (-1)^{w(D)} t^{\frac{3w(D)}{4}} \langle D \rangle \big|_{A=t^{-1/4}}$$

where D is a diagram of L.

The substitution $A = t^{-1/4}$ is chosen so the resulting invariant agrees with Jones' original normalization of the Kauffman bracket.

Example 4.23. Let us now compute the normalized Kauffman bracket and Jones polynomial of the figure eight knot 4_1. We first choose an orientation to compute the writhe.



We see that the writhe of 4_1 is 0. From Example 4.11 we have

$$\langle 4_1 \rangle = A^8 - A^4 + 1 - A^{-4} + A^{-8}.$$

The normalized Kauffman bracket $f_{4_1}(A)$ and the Jones polynomial $V_{4_1}(t)$ of the figure eight knot are

$$\begin{aligned} f_{4_1}(A) =& (-A)^{-3w(4_1)} \langle 4_1 \rangle \\ =& (-A)^0 \langle 4_1 \rangle = A^8 - A^4 + 1 - A^{-4} + A^{-8}, \\ V_{4_1}(t) =& (-1)^{w(4_1)} t^{\frac{3w(4_1)}{4}} \langle 4_1 \rangle \\ =& (-1)^0 t^{\frac{3 \cdot 0}{4}} \langle 4_1 \rangle = t^{-2} - t^{-1} + 1 - t + t^2. \end{aligned}$$

5 Translating Link Diagrams into Graphs

In knot theory, many link invariants are defined using the diagram of a link, like the normalized Kauffman bracket or the Jones polynomial. Some link invariants, including the normalized Kauffman bracket and the Jones polynomial, can be expressed using graph theory. To show this connection, we will first develop a systematic way to relate a link diagram to a graph called its Tait graph. The recursive definition of the Kauffman bracket mirrors the deletion-contraction recursion of the Tutte polynomial for a special class of links.

Associating Tait Graphs to Link Diagrams

Since the Tutte polynomial is a graph invariant we want to develop techniques for sending knots to graphs. We will associate certain knot diagrams to signed plane graphs which will allow us to translate some knot invariants into graph invariants.

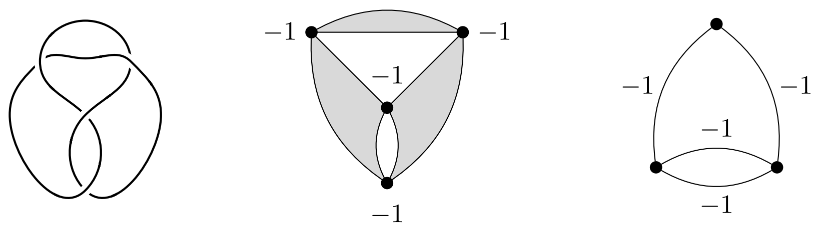
From a regular knot diagram D we construct a graph G as follows: the crossings of D correspond to vertices of G and the strands between crossings in D correspond to the edges of G. Notice that G is a 4-regular planar graph. The faces of G admit a 2-coloring into shaded and unshaded regions such that the exterior of G is unshaded and no two adjacent faces share the same color. To preserve the information of whether one strand crosses over or under the other strand we assign a sign to each vertex in the following way. View the crossing so that the shaded region lies below it. If, in this position, a strand oriented from bottom left to top right crosses over the other strand, we assign it $+1$, otherwise we assign it -1. Coloring the knot ensures that a crossing can be assigned a sign unambiguously. The constructed graph G is called the *plane graph* of D. The coloring convention is as follows.

$+1$ $\qquad$ -1

To relate knot invariants to graph invariants like the Tutte polynomial, it is convenient to encode the crossing information along the edges rather than the vertices. This leads to the following construction.

Definition 5.1. Let G be a planar graph of D. We define the *Tait graph* of G, $\widetilde{G}$ as follows. The set of shaded regions in G become the vertices of $\widetilde{G}$. For each crossing c in G there is a corresponding edge in $\widetilde{G}$ joining the two vertices corresponding to the two shaded regions that are diagonally opposite at c. The sign of the crossing in G is assigned to the corresponding edge in $\widetilde{G}$.

Example 5.2. The following three pictures show the knot diagram of the figure eight knot 4_1, its corresponding 4-regular planar graph G and its Tait graph $\widetilde{G}$.

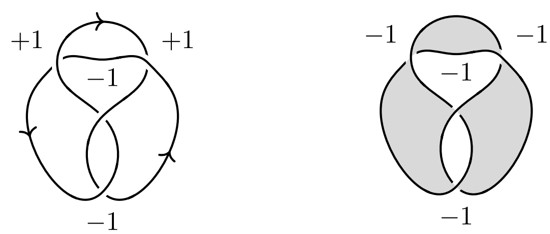


4_1 G $\widetilde{G}$

We will often apply the same 2-coloring convention directly to a link diagram D and construct its Tait graph directly without first explicitly forming its plane graph. When working with multiple link diagrams $D_1, D_2, \ldots$ we write $\widetilde{G}_{D_1}, \widetilde{G}_{D_2}, \ldots$ to indicate the Tait graph associated with each diagram.

Note that the way to sign crossings using oriented links and the way to sign crossings with a shaded link diagram are not equivalent as the following example shows.

Example 5.3. We will sign the crossings of the figure eight knot using an orientation and a shading of the knot.



Orientation Shading

The Jones Polynomial via the Tutte Polynomial

In the preceding example, notice that the crossings alternate over-under as we follow every strand throughout the diagram. Such a link is called an *alternating link*. Both T_l and T_r from Example 4.15 are alternating knots, but the square knot $T_l \# T_r$ is not alternating.

We will now restrict ourselves to alternating links. We call a link diagram *positive* if every crossing is positive under its 2-coloring. A *negative link diagram* contains only negative crossings. Both positive and negative link diagrams are alternating, since along each strand the crossings necessarily alternate between over and under crossings following a strand.

Given a link diagram D, its *mirror image* $\overline{D}$ is obtained by flipping every crossing. That is by making every negative crossing positive and vice versa. Equivalently, the mirror image of a link $L \subset \mathbb{R}^3$ is a reflection of one of the coordinates. Without loss of generality, we define the reflection to be

$$r(x, y, z) = (x, y, -z).$$

The projection $\pi(x, y, z) = (x, y)$ satisfies $\pi \circ r = \pi$ so the planar diagram $\overline{D}$ obtained from $r(L)$ has the same projection as the diagram D of L. However, the z-coordinate determines which strand passes over or under at each crossing and r reverses the z-coordinate. Hence, $\overline{D}$ is obtained from D by flipping every crossing.

Since every crossing now has a different sign under the same orientation, then $w(\overline{D}) = -w(D)$. Thus, $(-A^3)^{-w(\overline{D})} = (-A^{-3})^{-w(D)}$. From (iii) in Definition 4.10, every A-smoothing becomes a B-smoothing and vice versa so $\langle\overline{D}\rangle|_A = \langle D\rangle|_{A\mapsto A^{-1}}$. This reasoning yields the following relation.

Proposition 5.4. *Let D be the link diagram of an oriented link L and $\overline{L}$ be its mirror image. Then*

$$f_{\overline{L}}(A) = f_L(A^{-1}) \quad \textit{and} \quad V_{\overline{L}}(t) = V_L(t^{-1}).$$

We are now able to show how the Tutte and Jones polynomial relate. We will first consider crossings whose corresponding edges in their Tait graph are neither a bridge nor a loop.

Lemma 5.5. *Let D be a 2-colored link diagram with Tait graph $\widetilde{G}$. Let c be a positive crossing in D and $e \in E(\widetilde{G})$ be the corresponding edge in $\widetilde{G}$ such that e is neither a bridge nor a loop. Furthermore, let D_A and D_B be diagrams obtained by A- and B-smoothing c respectively. Then,*

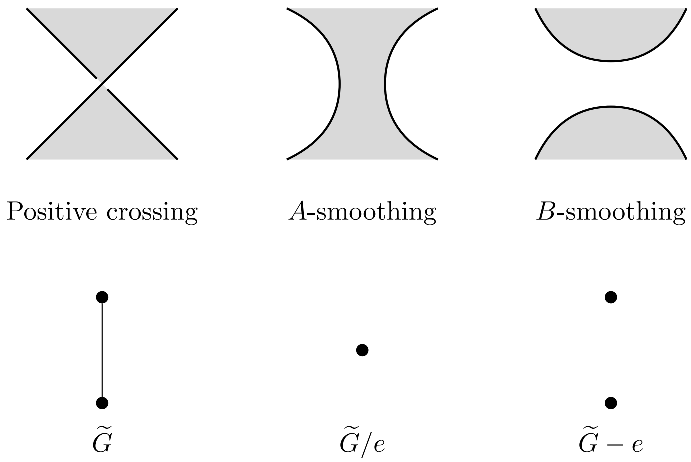
$$\widetilde{G}_{D_A} \cong \widetilde{G}/e \quad \textit{and} \quad \widetilde{G}_{D_B} \cong \widetilde{G} - e.$$

For the corresponding Kauffman bracket of $\widetilde{G}$, this means

$$\langle\widetilde{G}\rangle = A^{-1}\langle\widetilde{G} - e\rangle + A\langle\widetilde{G}/e\rangle.$$

Proof. Assume that c is a positive crossing in D. We have c adjacent to two shaded regions R_1 and R_2. Let us call the associated vertices in $\widetilde{G}$ v_1 and v_2.

Both A- and B-smoothings remove the crossing c so the Tait graph $\widetilde{G}$ does not contain e after the smoothings. When A-smoothing, the two regions R_1 and R_2 become one region, which is equivalent to identifying the vertices with each other $v_1 \sim v_2$. This is exactly the graph $\widetilde{G}/e$. Performing a B-smoothing separates the two regions so they no longer share a crossing. This is precisely the graph $\widetilde{G} - e$.



□

Note that in an alternating diagram, an A-smoothing corresponds uniformly to either deletion or contraction and a B-smoothing corresponds to the other operation. In a non-alternating diagram, this correspondence may vary between crossings which breaks the global identification with the Tutte polynomial. Hence, alternating links ensure consistency.

We will now look at crossings whose corresponding edges in $\widetilde{G}$ is a bridge or a loop.

Lemma 5.6. *Let D be a 2-colored link diagram and let c be a positive crossing of D. Let $\widetilde{G}$ be the Tait graph of D and let $e \in E(\widetilde{G})$ be the edge corresponding to c.*

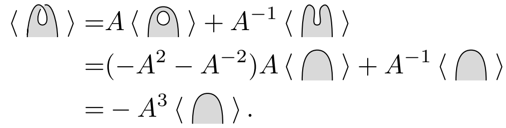
If e is a loop in $\widetilde{G}$, then

$$\langle \widetilde{G} \rangle = (-A^3)\, \langle \widetilde{G} - e \rangle.$$

If e is a bridge in $\widetilde{G}$, then

$$\langle \widetilde{G} \rangle = (-A^{-3})\, \langle \widetilde{G} - e \rangle.$$

Proof. If e is a loop in $\widetilde{G}$, then c is adjacent to exactly one connected shaded region. From the Kauffman relations,



Now, if e is a bridge, denote $D = D_1 \underset{c}{\cup} D_2$. That is, the union of two link diagrams only connected by a positive crossing c. Then c corresponds to a bridge in $\widetilde{G}$ so an A-smoothing will result in $\langle D_1 \# D_2 \rangle$ and a B-smoothing will result in $\langle D_1 \sqcup D_2 \rangle$. Lemma 4.14 and 4.16 yield the following

$$\begin{aligned}
\langle \times \rangle &= A \langle)(\rangle + A^{-1} \langle \asymp \rangle \\
&= A \langle D_1 \# D_2 \rangle + A^{-1} \langle D_1 \sqcup D_2 \rangle \\
&= A \langle D_1 \rangle \langle D_2 \rangle + A^{-1}(-A^2 - A^{-2}) \langle D_1 \rangle \langle D_2 \rangle \\
&= -A^{-3} \langle D_1 \rangle \langle D_2 \rangle .
\end{aligned}$$

□

We are now able to prove one of the central results in this thesis.

For a knot diagram D with Tait graph $\widetilde{G}$ we set $F(\widetilde{G}) = \langle D \rangle$, then Lemma 5.5 implies that $F(\widetilde{G})$ is a deletion-contraction invariant. From Theorem 3.23

$$F(\widetilde{G}) = q^{|E|-|V|+k(\widetilde{G})} r^{|V|-k(\widetilde{G})} T_{\widetilde{G}} \left(\frac{Q}{r}, \frac{R}{q} \right)$$

where

- $F(\widetilde{G}) = qF(\widetilde{G} - e) + rF(\widetilde{G}/e)$ for e neither bridge nor loop
- $F(\widetilde{G}) = QF(\widetilde{G} - e)$ for e bridge
- $F(\widetilde{G}) = RF(\widetilde{G} - e)$ for e loop.

Note that $\widetilde{G}$ contains no edges exactly when D is the unknot.

By Lemma 5.5, $q = A^{-1}$ and $r = A$ and by Lemma 5.6, $Q = -A^{-3}$ and $R = -A^3$. From Theorem 3.23 we obtain the following result.

Lemma 5.7. *Let D be a positive connected link diagram and $\widetilde{G} = (V, E, \psi)$ its Tait graph. Then*

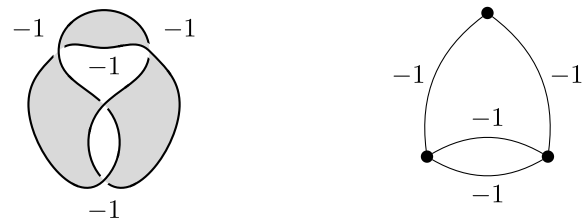
$$\langle D \rangle = A^{2|V|-|E|-2} T_{\widetilde{G}}(-A^{-4}, -A^4).$$

For a negative link diagram $\overline{D}$,

$$\langle \overline{D} \rangle = A^{-2|V|+|E|+2} T_{\widetilde{G}}(-A^4, -A^{-4}).$$

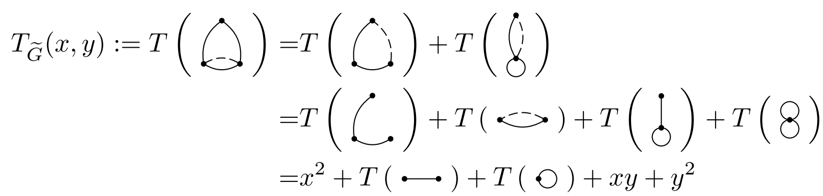
As shown before, the Kauffman bracket is not a link invariant. Recall that the normalized Kauffman bracket $f_L(A)$ is a link invariant and that the Jones polynomial $V_L(A)$ is the normalized Kauffman bracket with a change of variables, $A \mapsto t^{-1/4}$.

Example 5.8. Let us compute the Kauffman bracket of the figure eight knot 4_1 using Lemma 5.7. We will shade the figure eight knot to obtain its Tait graph $\widetilde{G}$.



4_1 $\qquad\qquad$ $\widetilde{G}$

Let us now compute the Tutte polynomial of $\widetilde{G}$.



$$=x + x^2 + xy + y + y^2.$$

Since this figure eight knot is negative, we obtain

$$\begin{aligned}\langle 4_1 \rangle =& A^{-2|V|+|E|+2} T_{\widetilde{G}}(-A^4, -A^{-4})\\ =& A^{-2\cdot 3+4+2} \cdot \left(-A^4 + A^8 + 1 - A^{-4} + A^{-8}\right)\\ =& A^8 - A^4 + 1 - A^{-4} + A^{-8}.\end{aligned}$$

This matches the calculation in Example 4.11.

We are now ready to extend the connection between the Tutte polynomial and the Kauffman bracket to the normalized Kauffman bracket and the Jones polynomial.

Let $\widetilde{G} = (V, E, \psi)$ be the Tait graph of a link diagram D. Each crossing of D corresponds to an edge in $\widetilde{G}$. Let us denote E^+ for the edges corresponding to a positive crossing in D and E^- denote the edges corresponding to negative crossings. Then

$$|E| = \left|E^+\right| + \left|E^-\right|$$

and

$$w(D) = \left|E^+\right| - \left|E^-\right|.$$

Comparing the two we obtain

$$|E| - w(D) = \left(\left|E^+\right| + \left|E^-\right|\right) - \left(\left|E^+\right| - \left|E^-\right|\right) = 2\left|E^-\right|.$$

Thus $|E| - w(D)$ is even so either both $|E|$ and $w(D)$ are even or both are odd so $(-1)^{|E|} = (-1)^{w(D)}$. The previous reasoning, together with Theorem 4.21 and Lemma 5.7 produces the following result.

Theorem 5.9. *Let D be a positive alternating link diagram of a link L and let $\widetilde{G}$ be its Tait graph. Then*

$$\begin{aligned} f_L(A) =& (-1)^{|E|} A^{2|V|-|E|-2-3w(D)} T_{\widetilde{G}}(-A^{-4}, -A^{4}), \\ V_L(t) =& (-1)^{|E|} t^{\frac{-2|V|+|E|+2+3w(D)}{4}} T_{\widetilde{G}}(-t, -t^{-1}). \end{aligned}$$

For a negative alternating link diagram $\overline{D}$ of a link $\overline{L}$ with Tait graph $\widetilde{G}$, then

$$\begin{aligned} f_{\overline{L}}(A) =& (-1)^{|E|} A^{-2|V|+|E|+2+3w(D)} T_{\widetilde{G}}(-A^{4}, -A^{-4}), \\ V_{\overline{L}}(t) =& (-1)^{|E|} t^{\frac{2|V|-|E|-2-3w(D)}{4}} T_{\widetilde{G}}(-t^{-1}, -t). \end{aligned}$$

Example 5.10. Let us compute the Jones polynomial of the figure eight knot using Theorem 5.9. As previously stated, $w(4_1) = 0$ and our knot diagram of 4_1 is negative. Furthermore, the Tait graph $\widetilde{G}$ of 4_1 contains 3 vertices and 4 edges and the Tutte polynomial of $\widetilde{G}$ is $x^2 + x + xy + y + y^2$. Hence,

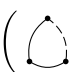
$$\begin{aligned} V_{4_1}(t) =& (-1)^{|E|} t^{\frac{2|V|-|E|-2-3w(D)}{4}} T_{\widetilde{G}}(-t^{-1}, -t) \\ =& (-1)^4 t^{\frac{2\cdot 3-4-2-3\cdot 0}{4}} \cdot \left((-t^{-1})^2 - t^{-1} + (-t^{-1})(-t) - t + (-t)^2\right) \\ =& t^{-2} - t^{-1} + 1 - t + t^2. \end{aligned}$$

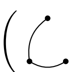
We see that this agrees with the computations in Example 4.23.

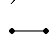
Example 5.11. Let us compute the Jones polynomial of the right-handed trefoil knot T_r in Example 4.15. We denote $\widetilde{G}_{T_r}$ the Tait graph of T_r. We have

$$\left|V(\widetilde{G}_{T_r})\right| = \left|E(\widetilde{G}_{T_r})\right| = 3$$

and the Tutte polynomial of $\widetilde{G}_{T_r}$

$T_{\widetilde{G}_{T_r}}(x, y) = T\Big($  $\Big)$

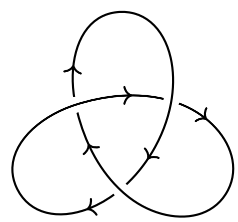
$= T\Big($  $\Big) + T\big($  $\big)$

$= x^2 + T\big($  $\big) + T\big($  $\big)$

$= x + x^2 + y.$

Choosing an orientation, we find that

$$w(T_r) = 3.$$



Using Theorem 5.9 for a positive alternating knot, we obtain

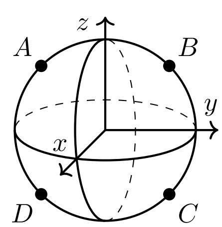
$$V_{T_r}(t) = t + t^3 - t^4.$$

6 Families of Knots

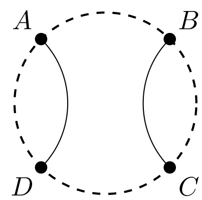
The construction of the tangles, torus knots and rational knots closely follows Murasugi's book *Knot Theory and Its Applications*, although some conventions differ [Mur96].

Using the results we have developed we shall now try to find closed formulas for different types of knots. Even though we have defined the Jones polynomial for links in general, we will mostly restrict ourselves to knots.

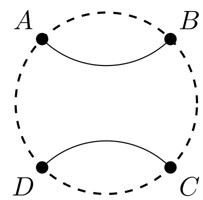
Many of the families in this section are constructed by tangles. Before delving into the specific families, we define what a tangle is. We take the unit sphere in $\mathbb{R}^3$ with points A, B, C and D on its boundary.



We start by connecting A, B, C and D by strands as follows. The diagram obtained by connecting A to D and B to C we call $T(0)$ and connecting A to B and C to D we call $T(\infty)$. Projecting onto the plane $x = 0$ we obtain the following pictures.



$T(0)$

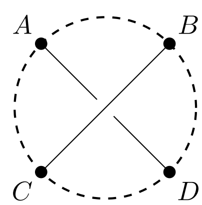


$T(\infty)$

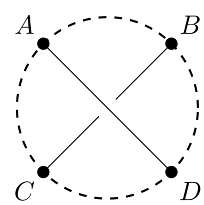
Definition 6.1. A *vertical tangle* is obtained by applying any number of homeomorphisms to $T(0)$ such that each homeomorphism fixes the northern hemisphere but interchanges C and D on the southern hemisphere.

A *horizontal tangle* is obtained by applying any number of homeomorphisms to $T(\infty)$ such that each homeomorphism fixes the western hemisphere but interchanges B and C on the eastern hemisphere.

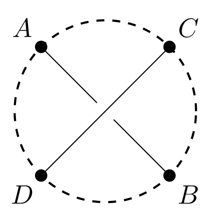
We denote the twisting of the hemisphere half a turn as performing one *positive half-twist* or *negative half-twist* by the following convention.



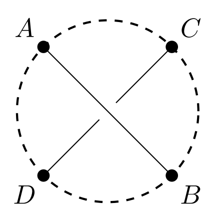
Vertical tangle with positive half-twist



Vertical tangle with negative half-twist



Horizontal tangle with positive half-twist



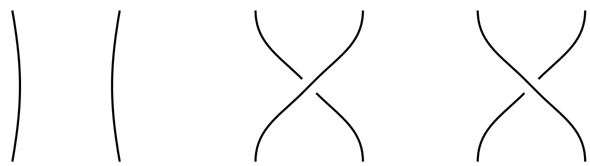
Horizontal tangle with negative half-twist

We are now ready to study individual families of knots.

Twist Knots

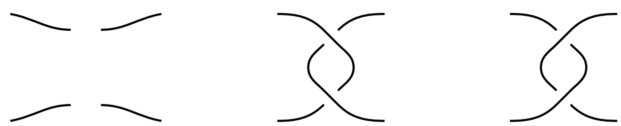
One pleasant family of knots is the family of twist knots. These knots depend on a single integer parameter, which makes them particularly easy to describe and study. In particular, every twist knot is alternating, making twist knots suitable for Theorem 5.9.

In this section, we are only interested in vertical tangles. Thus, we will not repeat this specification in this chapter. We say that t_n is a vertical tangle consisting of n positive half-twists for positive n and n negative half-twists if n is negative.



Untwisted strands   Positive twist   Negative twist

Now, we define a *clasp* of two strands as the local operation as shown below, in which four strands are connected to form a pair of crossings, thus linking them together.



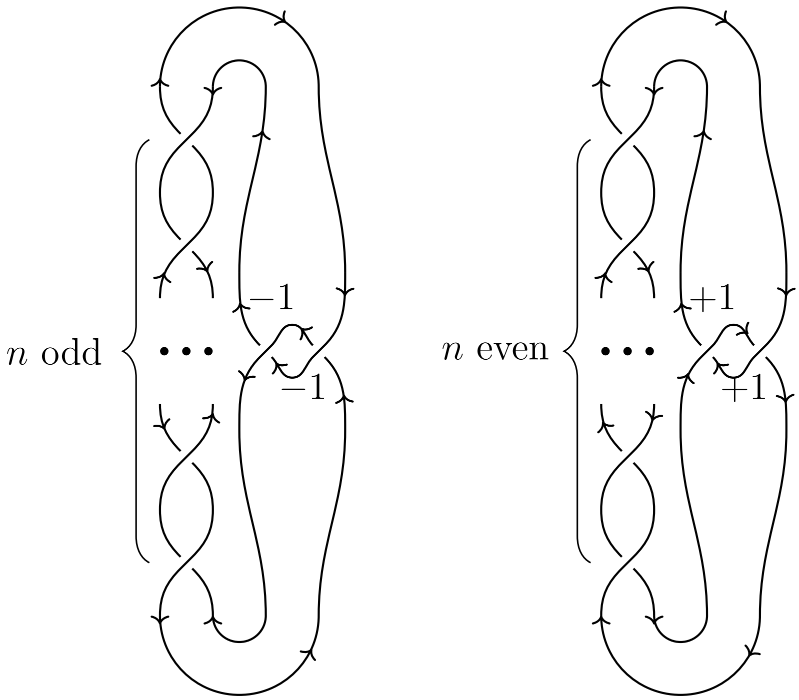
Separate strands   Clasping   Clasping

Note that there are two possible ways to perform a clasp.

Definition 6.2. The *twist knot* τ_n is a knot obtained by taking a tangle t_n as above, and connecting the endpoints by a clasp in such a way that the resulting link diagram is alternating.

- If $n > 0$, τ_n is called a *positive twist knot*, obtained from n positive half-twists.
- If $n < 0$, τ_n is called a *negative twist knot*, obtained from n negative half-twists.
- If $n = 0$, τ_n is the unknot.

Let us restrict ourselves to positive twist knots. A positive twist diagram τ_n consists of n half-twists. Choosing an orientation on τ_n, then every crossing induced by a positive half-twist is assigned -1. The two crossings induced by clasping will differ depending on whether n is even or odd. Observe that one half-twist reverses the relative orientation of one of the strands entering the clasp. Thus the signs assigned to the clasp depend only on the parity of n. If n is even, the strands entering the clasp have the same relative orientation as when $n = 0$. So both crossings contribute $+1$ to the writhe. If n is odd, one strand entering the clasp has the opposite orientation in which each crossing contributes -1 to the writhe.



This reasoning shows that the writhe

$$w(\tau_n) = \begin{cases} \underbrace{-1-1-\cdots-1}_{n} -1-1 = -n-2 & \text{for } n \text{ odd} \\ \underbrace{-1-1-\cdots-1}_{n} +1+1 = -n+2 & \text{for } n \text{ even.} \end{cases} \tag{1}$$

for $n > 0$.

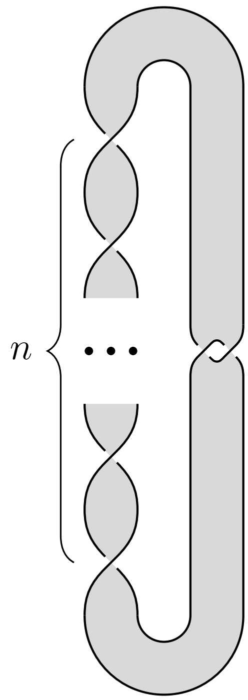
To find the Tait graph $\widetilde{G}_n$ associated to the knot diagram of τ_n we notice that in the 2-shading of τ_n each half-twist adds one more shaded region, hence adding one more vertex in $\widetilde{G}_n$. Since the diagram of τ_0 contains one shaded region, τ_n contains $n+1$ shaded regions, so the vertex set

$$\left|V(\widetilde{G}_n)\right| = n+1. \tag{2}$$

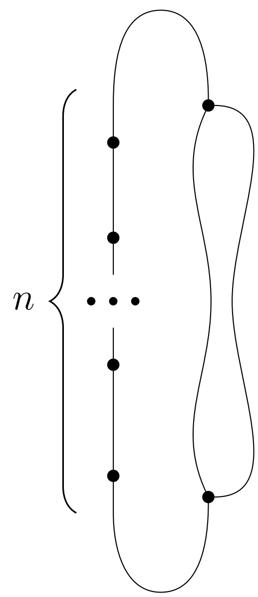
From the positive half-twists we obtain n crossings which correspond to n edges in $\widetilde{G}_n$. Furthermore, the clasp introduces two additional crossings connecting the two vertices with degree one to each other with two parallel edges. Thus

$$\left|E(\widetilde{G}_n)\right| = n+2. \tag{3}$$

The twist knot τ_n with positive half-twists and its Tait graph $\widetilde{G}_n$ are shown below.



2-shaded τ_n



$\widetilde{G}_n$

Notice that the Tait graph of a twist knot diagram is a cycle graph with one additional edge between two adjacent vertices in the cycle graph. A *cycle graph*, $C_n = (V, E, \psi)$ for $n \in \mathbb{Z}$ is a graph with

$$\begin{aligned} V =& \{v_1, v_2, \dots, v_n\} \\ E =& \{e_1, e_2, \dots, e_n\} \\ \psi(e_k) =& \{v_k, v_{k+1}\}, \quad \text{for } k \in [1, n] \text{ and } v_{n+1} = v_1. \end{aligned}$$

The following lemma will be useful to calculate the Tutte polynomial of the Tait graph of a twist knot.

Lemma 6.3. *Let C_n be a cycle graph. Then*

$$T_{C_n}(x, y) = y + \sum_{k=1}^{n-1} x^k.$$

Proof. For $n = 1$, C_1 is a graph with one edge incident to one single vertex, i.e. a loop. So $T_{C_1} = y$ which agrees with the formula. Now let $n \geq 2$ and assume that the formula holds for C_{n-1}. Contracting an edge in C_n gives the graph C_{n-1} and deleting one edge gives the graph P_{n-1} consisting of n vertices and $n-1$ bridges. By the recursive definition of the Tutte polynomial and since no $e \in E(C_n)$ is a bridge or a loop

$$T_{C_n} = T_{C_{n-1}} + T_{P_{n-1}} = y + \sum_{k=1}^{n-2} x^k + x^{n-1} = y + \sum_{k=1}^{n-1} x^k.$$

□

Lemma 6.4. *Let $\widetilde{G}_n$ be the Tait graph of a twist knot τ_n. Then*

$$T_{\widetilde{G}_n}(x,y) = y + y^2 + (1+y)\frac{x(x^{n-1}-1)}{x-1} + x^n.$$

Proof. Recall that the Tait graph $\widetilde{G}_n$ of τ_n contains $n+1$ vertices and $n+2$ edges where two of the edges in $\widetilde{G}_n$ are parallel, call them p_1 and p_2. Then deleting p_2 from $\widetilde{G}_n$ results in the cycle graph of $n+1$ edges. In other words

$$\widetilde{G}_n - p_2 \cong C_{n+1}.$$

Contracting p_2 identifies its endpoints, turning p_1 into a loop. Deleting the loop p_1, the graph is isomorphic to C_n. Hence,

$$(\widetilde{G}_n/p_2) - p_1 \cong C_n.$$

The Tutte polynomial of $\widetilde{G}_n$ is therefore

$$\begin{aligned}
T_{\widetilde{G}_n} &= T_{\widetilde{G}_n - p_2} + T_{\widetilde{G}_n/p_2} \\
&= T_{C_{n+1}} + yT_{(\widetilde{G}_n/p_2)-p_1} \\
&= T_{C_{n+1}} + yT_{C_n} \\
&= y + \sum_{k=1}^{n} x^k + y\left(y + \sum_{k=1}^{n-1} x^k\right) && \text{using Lemma 6.3} \\
&= y + y^2 + x^n + \sum_{k=1}^{n-1} x^k + y\sum_{k=1}^{n-1} x^k \\
&= y + y^2 + x^n + (1+y)\sum_{k=1}^{n-1} x^k \\
&= y + y^2 + (1+y)\frac{x(x^{n-1}-1)}{x-1} + x^n
\end{aligned}$$

when $x \neq 1$. □

We are now ready to derive a formula for the Jones polynomial of positive twist knots.

Theorem 6.5. *Let τ_n be a positive twist knot with n twists. Then the Jones polynomial $V_{\tau_n}(t)$ is*

$$V_{\tau_n}(t) = \begin{cases} \frac{1+t^{-2}-t^{-n-3}+t^{-n}}{1+t} & \text{for } n \text{ odd} \\ \frac{t+t^3-t^{3-n}+t^{-n}}{1+t} & \text{for } n \text{ even.} \end{cases}$$

Proof. Denote $\widetilde{G}_n$ as the Tait graph of τ_n.

If $n = 0$, then

$$V_{\tau_n}(t) = \frac{t + t^3 - t^3 + t^{-0}}{1+t} = 1$$

which is the Jones polynomial for the unknot. Let $n > 0$ and $\widetilde{G}_n$ be the Tait graph of τ_n. So τ_n is a positive knot diagram. From Theorem 5.9

$$V_{\tau_n}(t) = (-1)^{|E|} t^{\frac{-2|V|+|E|+2+3w(\tau_n)}{4}} T_{\widetilde{G}_n}(-t, -t^{-1}).$$

Then, from Lemma 6.4,

$$T_{\widetilde{G}_n}(x, y) = y + y^2 + (1+y)\frac{x(x^{n-1}-1)}{x-1} + x^n.$$

Substituting $x \mapsto -t$ and $y \mapsto -t^{-1}$ and simplifying, we obtain

$$\begin{aligned} T_{\widetilde{G}_n}(-t, -t^{-1}) =& -t^{-1} + t^{-2} + (1 - t^{-1})\frac{(-t)\left((-t)^{n-1} - 1\right)}{-t-1} + (-t)^n \\ =& \frac{t^{-2}(1-t)(1+t)}{1+t} + \frac{t-1}{t} \cdot \frac{t\left((-t)^{n-1}-1\right)}{1+t} + \frac{(-t)^n(1+t)}{1+t} \\ =& \frac{t^{-2}-1}{1+t} + \frac{(t-1)\left((-t)^{n-1}-1\right)}{1+t} + \frac{(-1)^n(t^{n+1}+t^n)}{1+t} \\ =& \frac{t^{-2} - 1 + (-1)^{n-1}\left(t^n - t^{n-1}\right) - t + 1 + (-1)^n(t^{n+1}+t^n)}{1+t} \\ =& \frac{t^{-2} - t + (-1)^{n-1}(-t)^{n-1} + (-1)^n t^{n+1}}{1+t} \\ =& \frac{t^{-2} - t + (-1)^n\left(t^{n+1} + t^{n-1}\right)}{1+t}. \end{aligned}$$

By (1), (2) and (3),

$$\begin{aligned} w(\tau_n) =& \begin{cases} = -n-2 & \text{for } n \text{ odd,} \\ = -n+2 & \text{for } n \text{ even,} \end{cases} \\ \left|V(\widetilde{G}_n)\right| =& n+1, \\ \left|E(\widetilde{G}_n)\right| =& n+2. \end{aligned}$$

Assume n odd. Then

$$(-1)^{|E|} = (-1)^{n+2} = (-1)^n = -1$$

and

$$\frac{-2\,|V| + |E| + 2 + 3w(\tau_n)}{4} = \frac{-2(n+1) + n + 2 + 2 + 3(-n-2)}{4} = -(n+1).$$

Thus

$$\begin{aligned} V_{\tau_n}(t) =& (-1)^n t^{-(n+1)} T_{\widetilde{G}_n}(-t, -t^{-1}) \\ =& (-1)^n t^{-(n+1)} \cdot \frac{t^{-2} - t + (-1)^n \left(t^{n+1} + t^{n-1}\right)}{1+t} \\ =& \frac{1 + t^{-2} - t^{-n-3} + t^{-n}}{1+t}. \end{aligned}$$

Now assume n is even. Then

$$(-1)^{|E|} = (-1)^{n+2} = 1$$

and

$$\frac{-2\,|V| + |E| + 2 + 3w(\tau_n)}{4} = \frac{-2(n+1) + n + 2 + 2 + 3(-n+2)}{4} = 2 - n.$$

This gives

$$\begin{aligned} V_{\tau_n}(t) =& t^{2-n} \cdot \frac{t^{-2} - t + t^{n+1} + t^{n-1}}{1+t} \\ =& \frac{t + t^3 - t^{3-n} + t^{-n}}{1+t}. \end{aligned}$$

□

Corollary 6.6. *Let τ_{-n} be a negative twist knot with n negative half-twists and $\widetilde{G}_{-n}$ be its Tait graph. Then the Jones polynomial $V_{\tau_{-n}}(t)$ is*

$$V_{\tau_{-n}}(t) = \begin{cases} \frac{t+t^3-t^{n+4}+t^{n+1}}{1+t} & \text{for } n \text{ odd} \\ \frac{1+t^{-2}-t^{n-2}+t^{n+1}}{1+t} & \text{for } n \text{ even.} \end{cases}$$

Proof. The mirror image of a positive twist knot changes every crossing sign. The resulting knot is also alternating but with negative half-twists instead of positive half-twist. Hence, it is a negative twist knot, $\overline{\tau_n} \cong \tau_{-n}$. The corollary follows from Proposition 5.4 and Theorem 6.5 by substituting $t \mapsto t^{-1}$. □

Example 6.7. It is readily checked that τ_{-2} yields the figure eight knot 4_1. In Example 4.23 we computed

$$V_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2.$$

Using Corollary 6.6 we compute

$$V_{\tau_{-2}}(t) = \frac{1 + t^{-2} - t^{2-2} + t^{2+1}}{1+t} = \frac{t^{-2} + t^3}{1+t} = \frac{1+t^5}{t^2\,(1+t)}.$$

We can simplify the expression using long division

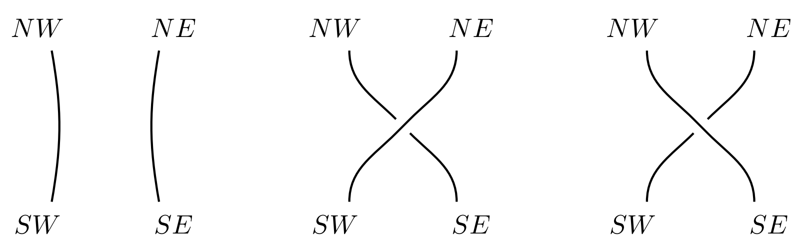
$$t^{-2} \cdot \frac{1+t^5}{1+t} = t^{-2}\left(t^4 - t^3 + t^2 - t + 1\right) = t^2 - t + 1 - t^{-1} + t^{-2}$$

which agrees with the computations done in Example 4.23.

Pretzel Knots

Another family of links is called pretzel links. Recall that a vertical tangle consists of half-twists on two parallel strands in the unit sphere. Informally, a pretzel link is formed by connecting the endpoints of multiple tangles in a cyclical way forming a link diagram.

Given a vertical tangle t_n, we call its top left endpoint NW, top right endpoint NE, bottom left endpoint SW and bottom right endpoint SE.



Untwisted strands Positive twist Negative twist

Definition 6.8. Let $a_1, a_2, \dots, a_k \in \mathbb{Z}$. The *pretzel link* $P(a_1, a_2, \dots, a_k)$ is obtained by taking the tangles $t_{a_1}, t_{a_2}, \dots, t_{a_k}$ and connecting them in the following way. Place the tangles $t_{a_1}, t_{a_2}, \dots, t_{a_k}$ side by side. For each $i \in [1, \dots, k-1]$ identify

$$NE(t_{a_i}) \sim NW(t_{a_{i+1}}), \quad NE(t_{a_k}) \sim NW(t_{a_1})$$

and

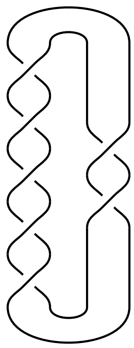
$$SE(t_{a_i}) \sim SW(t_{a_{i+1}}), \quad SE(t_{a_k}) \sim SW(t_{a_1})$$

such that no new crossings are introduced.

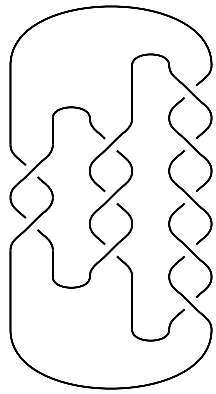
When $a_i = 0$ for some a_i of $P(a_1, \dots, a_k)$, then that tangle contains no half-twists and thus contributes no new crossings to the diagram. We will assume

$$a_i \neq 0, \quad \text{for all } i.$$

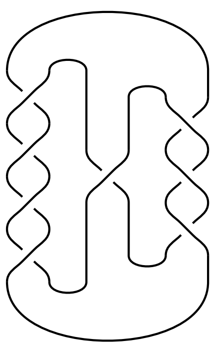
Example 6.9. Here are the diagrams of some pretzel knots.



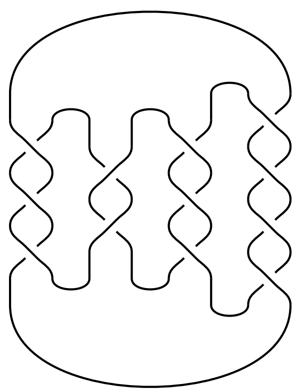
$P(5,2)$



$P(2,3,-5)$



$P(4,1,-3)$



$P(-3,2,-3,-4)$

One may wonder when $P(a_1,\ldots,a_k)$ is a knot. We see that $P(-3,2,-3,-4)$ has two components but $P(5,2)$, $P(2,3,-5)$ and $P(4,1,-3)$ only have one.

Proposition 6.10. *The pretzel link $P(a_1,\ldots,a_k)$ with $a_i \neq 0$ is a knot if and only if either*

- *k is odd and all a_i are odd or*
- *exactly one a_i is even.*

Proof. Note that traversing through an odd tangle swaps left and right, while traversing through an even tangle preserves left and right.

Assume that there are an odd number of odd tangles. From one odd tangle t we traverse from the top right of t to the right. Since there is an even number of tangles that are not t we will eventually reach the top left of t. Then, traveling through t we exit the bottom right of t. Thus there is only one component in P.

Conversely, if there is an even number of odd tangles, we once again start at t and traverse to the right. Now we travel through an odd number of odd tangles and thus eventually reach the bottom left of t which connects to our starting point the top right of t. We have traversed one cycle but never reached the top left of t, hence P is not a knot.

Assume there is exactly one even tangle t. Starting at the top right of t and traversing to the right we will eventually reach the left side of t where we, after going through t, exit on the left. Now, we traverse back through the odd tangles through strands we have not yet traversed. Eventually, we reach the bottom right of t and have passed through every part of P, so P is a knot.

Conversely, if there are at least two even tangles, we start from one even tangle t and traverse it from the top right of t to the right. We then eventually reach one other even tangle t' which we enter through the left and exit through the left. After exiting t', we traverse back to t entering the bottom left and then reaching our starting point, the top right of t. We have completed a cycle, but never reached the top left of t. So P is not a knot.

This exhausts all possibilities so the proof is complete. □

To reduce the number of cases we consider, we make the following observation. For positive n, the tangle t_n only contains n positive crossings and the tangle t_{-n} only contains n negative crossings, observe that t_{-n} is the mirror image of t_n. When identifying the endpoints of multiple tangles to form a pretzel link we are never introducing any new crossings. Taking the mirror image of every crossing will thus result in the mirror image of the pretzel knot. This reasoning yields the following proposition.

Proposition 6.11. *The mirror image of the pretzel link $P(a_1, \dots, a_k)$ is obtained by taking the mirror image of every tangle in $P(a_1, \dots, a_k)$. Equivalently*

$$\overline{P(a_1, \dots, a_k)} \cong P(-a_1, \dots, -a_k).$$

We can also study the necessary conditions for when $P(a_1, \dots, a_k)$ is alternating. In Example 6.9 we see that only $P(5, 2)$ is alternating. The following proposition explains why.

Proposition 6.12. *Let $P(a_1, \dots, a_k)$ be a pretzel link with $a_i \neq 0$ for all i. Then the standard diagram obtained for the pretzel link $P(a_1, \dots, a_k)$ is alternating if and only if all a_i have the same sign for all i.*

Proof. Assume all a_i have the same sign. Furthermore, let us assume a_i is positive for all i. The case when a_i is negative is similar. Every tangle in the pretzel link consists of positive half-twists so every tangle is alternating.

Identifying $NE(t_i)$ with $NW(t_{i+1})$, $\pmod k$ also ensures that the alternating structure is preserved since the strand traversed over before $NE(t_i)$ and travels under after $NW(t_{i+1})$. A similar reasoning holds for $SE(t_i) \sim SW(t_{i+1})$.

Now, assume that not all a_i have the same sign. Since $a_i \neq 0$ for all i there exists at least one i such that a_i and a_{i+1} have different signs. Let us without loss of generality assume that $a_i > 0$ and $a_{i+1} < 0$. Both of the tangles t_{a_i} and $t_{a_{i+1}}$ are alternating by themselves, but when connected this is no longer the case. Following the strand as $SE(t_i) \sim SW(t_{i+1})$ the strand passed under at the crossing before $SE(t_i)$ and will traverse under again at the crossing directly after $SW(t_{i+1})$. □

Now, let us formulate the writhe of oriented pretzel knots.

Lemma 6.13. *Let*

$$I = \{i \in [1, \dots, k] : a_i \text{ is odd}\}.$$

Then, the writhe of an alternating pretzel knot $P(a_1, \dots, a_k)$ *is*

$$w\left(P(a_1, \dots, a_k)\right) = -\sum_{i \in I} a_i.$$

if k *and every* a_i *are odd, or*

$$\sum_{i \in I} a_i + (-1)^{|I|+1} a_j$$

where a_j *is the only even tangle.*

Proof. Let us assume that P is a positive alternating knot.

By Proposition 6.10 $P(a_1, \dots, a_k)$ is a knot precisely if either: both k and every a_s for $s \in \{1, \dots, k\}$ are odd, or if there is only one a_j that is even. We split up the proof in different cases.

Case 1: k and every a_s are odd.

From one tangle a_s traverse the knot from the top right of t_{a_s} to the right. We will eventually return to t_{a_s} from the left, but since there are an odd number of odd tangles we will enter t_{a_s} at its top left. Since a_i is odd, we now exit t_{a_s} from the bottom right eventually returning to the bottom left of t_{a_s}. Every tangle is entered from the left and exited from the right, meaning that every tangle t_{a_i} contributes $-a_i$ to the writhe.

Case 2: Exactly one a_j is even and $|I|$ is odd. Then we want to show

$$w\left(P(a_1, \dots, a_k)\right) = \sum_{i \in I} a_i + a_j.$$

We start from the even tangle t_{a_j}, traversing from its top right to the next tangle which is odd. Since we traverse through an odd number of odd tangles, when we return to t_{a_j} we enter t_{a_j} at the bottom left. We then traverse from the top left of t_{a_j} to the left through the odd tangles now reaching the bottom right of t_{a_j} which connects to our starting point, the top right of t_{a_j}. Thus t_{a_j} is entered from the bottom and exited from the top and every odd tangle is either entered from the bottom and exited from the top or vice versa. So every tangle contributes its number of crossings to the writhe.

Case 3: Exactly one a_j is even and $|I|$ is even. Now, from the formula

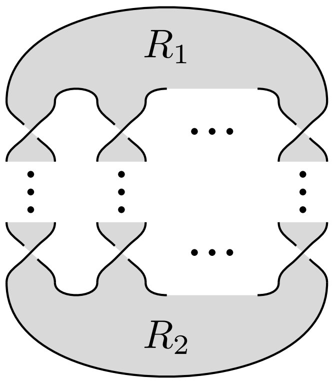
$$w\left(P(a_1, \ldots, a_k)\right) = \sum_{i \in I} a_i - a_j.$$

Starting at t_{a_j} traversing from its top right endpoint to the right we traverse through an even number of odd tangles before returning to t_{a_j}. Since $|I|$ is even, we return to the top left of t_{a_j} and then exit the bottom left of t_{a_j}. Now, traversing to the left through the odd tangles, we eventually return to the bottom right of t_{a_j} connecting to our starting point. Here, t_{a_j} is entered from the left and exited to the right contributing $-a_j$ to the writhe. Every odd tangle is either entered from the bottom and exited from the top or vice versa, each contributing a_i to the writhe.

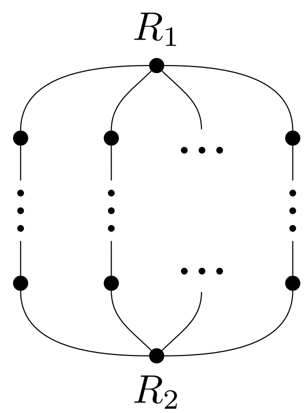
By Proposition 6.11 a negative alternating pretzel knot may be obtained by switching the signs of every a_s for all s. Then, the formula in the lemma still holds because $w(\overline{D}) = -w(D)$.

This completes the proof. □

Let us now find the Tait graph of a pretzel link. Consider an arbitrary pretzel link $P(a_1, a_2, \ldots, a_k)$ with $a_i \neq 0$ for all i. Then every tangle t_{a_i} of P has at least one crossing, hence the region above all tangles is not the same region below the tangles. We call the top region R_1 and the bottom region R_2. In the Tait graph $\widetilde{G}$ of P, R_1 and R_2 become vertices. A tangle t_{a_i} in P separates R_1 and R_2 by a_i crossings, so in $\widetilde{G}$ the given tangle contributes a_i edges between R_1 and R_2, hence $a_i - 1$ vertices. It follows that the Tait graph of P is a graph with two vertices R_1 and R_2 with k disjoint parallel paths between them. Each such path i consists of a_i edges and $a_i - 1$ vertices if we disregard R_1 and R_2. We denote this graph by $\widetilde{G}\left(P(a_1, a_2, \ldots, a_k)\right)$, $\widetilde{G}(a_1, a_2, \ldots, a_k)$ or simply $\widetilde{G}(P)$ if the link P is clear from the context.



$P(a_1, a_2, \ldots, a_k)$



$\widetilde{G}(P)$

To calculate the Tutte polynomial of $\widetilde{G}\,(P(a_1, a_2, \ldots, a_k))$ we first need some more basic results.

Lemma 6.14. *Let G be a graph and let $R_1, R_2 \in V(G)$ be two distinct vertices. For $n \geq 1$, let $G^{(n)}$ be the graph obtained by adding a path of n vertices between R_1 and R_2 whose internal vertices and edges are not in G. Then:*

(i) If R_1 and R_2 are in different connected components of G,

$$T_{G^{(n)}}(x, y) = x^n \cdot T_G(x, y).$$

(ii) If R_1 and R_2 are in the same connected component of G,

$$T_{G^{(n)}}(x, y) = T_{G^{(1)}}(x, y) + \sum_{k=1}^{n-1} x^k \cdot T_G(x, y).$$

Proof. If R_1 and R_2 are in different connected components of G, any new constructed path in $G^{(n)}$ consists only of bridges. Hence,

$$T_{G^{(n)}}(x, y) = x^n \cdot T_G(x, y).$$

If R_1 and R_2 are connected in G, then no edge in the constructed path in $G^{(n)}$ is a bridge so we use the deletion-contraction recursion of the Tutte polynomial. We will use induction on n.

For $n = 1$, the formula clearly holds.

For $n = 2$, deleting one edge in the constructed path makes the other edge in the path into a bridge. Contracting the same edge makes the graph isomorphic to $G^{(1)}$. Hence

$$T_{G^{(2)}}(x, y) = x \cdot T_G(x, y) + T_{G^{(1)}}(x, y)$$

and the formula holds.

Now assume that the formula holds for $n-1$. Let e be an edge in the added path. Since e is not a bridge

$$T_{G^{(n)}}(x,y) = T_{G^{(n)}-e}(x,y) + T_{G^{(n)}/e}(x,y).$$

Contracting e yields $G^{(n-1)}$. Deleting e, the remaining $n-1$ edges in the previous path are now bridges so $T_{G^{(n)}-e}(x,y) = x^{n-1} \cdot T_G(x,y)$. Therefore

$$T_{G^{(n)}}(x,y) = x^{n-1} \cdot T_G(x,y) + T_{G^{(n-1)}}(x,y).$$

By the induction hypothesis

$$T_{G^{(n-1)}}(x,y) = \sum_{k=1}^{n-2} x^k \cdot T_G(x,y) + T_{G^{(1)}}(x,y)$$

so

$$\begin{aligned} T_{G^{(n)}}(x,y) =& x^{n-1} \cdot T_G(x,y) + \sum_{k=1}^{n-2} x^k \cdot T_G(x,y) + T_{G^{(1)}}(x,y) \\ =& \sum_{k=1}^{n-1} x^k \cdot T_G(x,y) + T_{G^{(1)}}(x,y). \end{aligned}$$

□

For $n \geq 1$, we define the *dipole graph* D_n consisting of two vertices connected by n parallel edges between them.

Lemma 6.15. *The Tutte polynomial of the dipole graph D_n is*

$$T_{D_n}(x,y) = x + \sum_{k=1}^{n-1} y^k.$$

Proof. We call the two vertices of D_n R_1 and R_2.

If $n = 1$, then D_n consists of only one edge connecting R_1 and R_2, this edge is a bridge. Therefore

$$T_{D_1}(x,y) = x$$

which agrees with the lemma.

Assume the lemma holds for D_{n-1} for $n \geq 2$. Since D_n has $n \geq 2$ edges, none of them is a bridge. Let e be an edge in D_n. We get

$$T_{D_n}(x,y) = T_{D_n-e}(x,y) + T_{D_n/e}(x,y).$$

Deleting e gives us the graph D_{n-1} which, by the induction hypothesis, has Tutte polynomial

$$T_{D_n - e}(x, y) = T_{D_{n-1}}(x, y) = x + \sum_{k=1}^{n-2} y^k.$$

Contracting e identifies R_1 and R_2, resulting in a single vertex with $n-1$ loops. So

$$T_{D_n/e}(x, y) = y^{n-1}.$$

Thus,

$$\begin{aligned} T_{D_n}(x, y) =& x + \sum_{k=1}^{n-2} y^k + y^{n-1} \\ =& x + \sum_{k=1}^{n-1} y^k \end{aligned}$$

which completes the induction and proves the lemma. □

We are now able to prove a formula for the Tutte polynomial of the Tait graph of a pretzel link.

The general idea is to first apply Lemma 6.14 iterating over every path corresponding to each tangle until only one edge remains of each tangle. Then we can apply Lemma 6.15 to the next tangle in the sequence $(a_1, a_2, \ldots, a_k)$.

Proposition 6.16. *Let $P(a_1, a_2, \ldots, a_k)$ with $a_i \neq 0$ for all i be a positive pretzel link and $\widetilde{G}\,(P(a_1, a_2, \ldots, a_k))$ be its Tait graph.*

Let

$$B = \{i \in \{1, \ldots, k\} : a_i = 1\}$$

and

$$C = \{i \in \{1, \ldots, k\} : a_i \geq 2\}.$$

For any subset $A \subseteq C$ such that $a_i \geq 2$ for all $i \in \{1, \ldots, k\}$, we have:

(i) If $B \neq \emptyset$, then

$$T_{\widetilde{G}(a_1, \ldots a_k)}(x, y) = \sum_{A \subseteq C} \left(\prod_{\substack{1 \leq i \leq k \\ i \notin A \cup B}} \frac{x^{a_i} - x}{x - 1} \right) \cdot \left(x + \sum_{s=1}^{|A|+|B|-1} y^s \right).$$

(ii) If $B = \emptyset$, then

$$T_{\widetilde{G}(a_1,\dots a_k)}(x,y) = x^{a_1} \cdot \prod_{i=2}^{k} \frac{x^{a_i}-x}{x-1}$$
$$+ \sum_{\substack{A\subseteq\{2,\dots,k\}\\ A\neq\emptyset}} \left(\prod_{\substack{2\leq i\leq k\\ i\notin A}} \frac{x^{a_i}-x}{x-1} \right) \cdot \left(\frac{x^{a_1}-x}{x-1} \left(x + \sum_{s=1}^{|A|-1} y^s \right) + \left(x + \sum_{s=1}^{|A|} y^s \right) \right).$$

Proof. The graph $\widetilde{G}(a_1, ..., a_k)$ consists of two vertices R_1 and R_2 with k internally disjoint paths between them, i.e. paths that only share endpoints R_1 and R_2 and no other vertices.

Here, the ith path has length a_i. We apply Lemma 6.14 to each path. For every i, the ith path can either be deleted entirely or contracted to a single edge. These choices are indexed by a subset $A \subseteq \{1, \dots, k\}$, for $a_i \geq 2$ for all $i \in \{1, \dots, k\}$, where $i \in A$ indicates that the ith path is reduced to a single edge and $i \notin A \cup B$ indicates that the ith path is deleted.

Assume $B \neq \emptyset$, then R_1 and R_2 are always connected. For each path of length $a_i \geq 2$, if that path is deleted completely it contributes

$$\frac{x^{a_i}-x}{x-1}.$$

If it is reduced to a single edge between R_1 and R_2, the resulting graph consists of $|A| + |B|$ parallel edges between R_1 and R_2. This is the graph $D_{|A|+|B|}$. By Lemma 6.15

$$T_{D_{|A|+|B|}}(x,y) = x + \sum_{s=1}^{|A|+|B|-1} y^s.$$

Thus, from every subset A, we obtain a contribution of

$$\left(\prod_{i\notin A\cup B} \frac{x^{a_i}-x}{x-1} \right) \left(x + \sum_{s=1}^{|A|+|B|-1} y^s \right).$$

Assume $B = \emptyset$, so that $a_i \geq 2$ for all i. We reserve the first path a_1 to ensure connectivity between R_1 and R_2.

If $A = \emptyset$, we choose to delete every edge. Then every edge in the path corresponding to a_1 is a bridge contributing x^{a_1}, hence we obtain the term

$$x^{a_1} \cdot \prod_{i=2}^{k} \frac{x^{a_i}-x}{x-1}.$$

For $A \neq \emptyset$, the choices of which paths $a_2, \dots, a_k$ are contracted are in bijection with subsets $A \subseteq \{2, \dots, k\}$ such that A is not empty. By Lemma 6.14, every such subset contributes a factor

$$\prod_{\substack{2 \leq i \leq k \\ i \notin A}} \frac{x^{a_i} - x}{x - 1}.$$

Since $A \neq \emptyset$, the edges of a_1 are not bridges. Thus, a_1 is either deleted or contracted. Deleting a_1 contributes a factor

$$\frac{x^{a_1} - x}{x - 1} \left(x + \sum_{s=1}^{|A|-1} y^s \right)$$

by Lemma 6.15. Contracting a_1 makes it into a single edge path, thus increasing the number of single edge paths between R_1 and R_2 by one. This contributes

$$x + \sum_{s=1}^{|A|} y^s$$

to this choice of A.

This completes the proof. $\square$

We are now able to find the Jones polynomial for alternating pretzel knots. This theorem will be split into three different cases depending on under what circumstances the pretzel link is a knot.

Theorem 6.17. *Let $P(a_1, \dots, a_k)$ be a positive pretzel knot and let A be the subset of indices i such that $a_i \geq 2$ and B be the subset of indices i such that $a_i = 1$. Let $V_P(t)$ denote the Jones polynomial of $P(a_1, \dots, a_k)$ and $T_{\widetilde{G}}$ be the Tutte polynomial of the Tait graph of P.*

(i) If k is odd and every a_s is odd, then

$$V_P(t) = -t^{\frac{k-1-2\sum_{s=1}^{k} a_s}{2}} \cdot T_{\widetilde{G}}(-t, -t^{-1}).$$

(ii) If precisely one a_j is even and the number of odd tangles is odd

$$V_P(t) = -t^{\frac{k-1+\sum_{s=1}^{k} a_s}{2}} \cdot T_{\widetilde{G}}(-t, -t^{-1}).$$

(iii) If precisely one a_j is even and the number of odd tangles is even

$$V_P(t) = t^{\frac{k-1+\sum_{s=1}^{k} a_s - 3a_j}{2}} \cdot T_{\widetilde{G}}(-t, -t^{-1}).$$

Proof. Let $\widetilde{G}$ be the Tait graph of a positive pretzel knot P. Every tangle t_{a_s} induces a_s crossings which correspond to a_s edges in $\widetilde{G}$. Also, P consists of two regions R_1 and R_2 which corresponds to two vertices in $\widetilde{G}$. Every tangle t_{a_s} between R_1 and R_2 contains $a_s - 1$ regions. The total number of regions in P, and the total number of vertices in $\widetilde{G}$ is $2 - k + \sum_{s=1}^{k} a_s$. So,

$$\left|E(\widetilde{G})\right| = \sum_{s=1}^{k} a_s,$$

$$\left|V(\widetilde{G})\right| = 2 - k + \sum_{s=1}^{k} a_s.$$

From Proposition 6.16 we have formulas for the Tutte polynomial of the Tait graph of a Pretzel knot.

By Theorem 5.9 we have

$$\begin{aligned} V_P(t) =& (-1)^{\sum_{s=1}^{k} a_s} t^{\frac{-2\left(2-k+\sum_{s=1}^{k} a_s\right)+\sum_{s=1}^{k} a_s+2+3w(P)}{4}} T_{\widetilde{G}}(-t, -t^{-1}) \\ =& (-1)^{\sum_{s=1}^{k} a_s} t^{\frac{-2+2k-\sum_{s=1}^{k} a_s+3w(P)}{4}} T_{\widetilde{G}}(-t, -t^{-1}). \end{aligned}$$

By Proposition 6.10, $P(a_1, \ldots, a_k)$ is a knot if and only if one of the following occurs:

- k is odd and every a_s is odd,
- precisely one a_j is even and $|I|$ is odd,
- precisely one a_j is even and $|I|$ is even,

where $j \in [1, \ldots, k]$ and I is the set of every $i \in [1, \ldots, k]$ such that a_i is odd.

We will derive the Jones polynomial for a pretzel knot given any of these cases using Lemma 6.13 to calculate the writhe.

If k and every a_s is odd, then $|E|$ is odd. Thus,

$$V_P(t) = -t^{\frac{k-1-2\sum_{s=1}^{k} a_s}{2}} \cdot T_{\widetilde{G}}(-t, -t^{-1}).$$

Now, if precisely one a_j is even with $|I|$ odd, then

$$\sum_{s=1}^{k} a_s = a_j + \sum_{i \in I} a_i$$

is odd, since $|I|$ is odd, so

$$(-1)^{\sum_{s=1}^{k} a_s} = -1.$$

The writhe

$$w(P) = \sum_{i \in I} a_i + a_j = \sum_{s=1}^{k} a_s.$$

Thus

$$V_P(t) = -t^{\frac{k-1+\sum_{s=1}^{k} a_s}{2}} \cdot T_{\widetilde{G}}(-t, -t^{-1}).$$

If exactly one a_j is even and $|I|$ even, then $\sum_{s=1}^{k} a_s$ is even so $(-1)^{\sum_{s=1}^{k} a_s} = 1$. Furthermore,

$$w(P) = \sum_{i \in I} a_i - a_j = \sum_{s=1}^{k} a_s - 2a_j.$$

So

$$V_P(t) = t^{\frac{k-1+\sum_{s=1}^{k} a_s - 3a_j}{2}} \cdot T_{\widetilde{G}}(-t, -t^{-1}).$$

□

Corollary 6.18. *Let $\overline{P(a_1, \ldots, a_k)}$ be a negative pretzel knot and let $\widetilde{G}$ be the Tait graph of its positive mirror image $P(a_1, \ldots, a_k)$. Then the Jones polynomial of $\overline{P(a_1, \ldots, a_k)}$ is*

$$V_{\overline{P}}(t) = -t^{\frac{-k+1+2\sum_{s=1}^{k} a_s}{2}} \cdot T_{\widetilde{G}(a_1,\ldots,a_k)}(-t^{-1}, -t)$$

when k is odd and every a_s is odd,

$$V_{\overline{P}}(t) = -t^{\frac{-k+1-\sum_{s=1}^{k} a_s}{2}} \cdot T_{\widetilde{G}(a_1,\ldots,a_k)}(-t^{-1}, -t)$$

when one a_j is even and the number of odd tangles is odd and

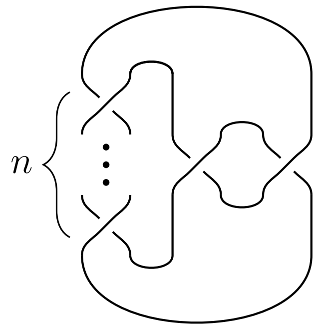
$$V_{\overline{P}}(t) = t^{\frac{-k+1-\sum_{s=1}^{k} a_s + 3a_j}{2}} \cdot T_{\widetilde{G}(a_1,\ldots,a_k)}(-t^{-1}, -t)$$

when one a_j is even and the number of odd tangles is even.

Proof. From Proposition 5.4 and Theorem 6.17, the corollary is proven by sending t to t^{-1}. □

Note that $\sum_{s=1}^{k} a_s$ refers to the sum of the positive pretzel knot as is consistent with Theorem 5.9.

Example 6.19. Drawing the pretzel knot $P(a, 1, 1)$ or $P(-a, -1, -1)$ for $a > 0$ we observe that this is precisely the twist knot τ_a and τ_{-a} respectively.



$P(a,1,1)$

Let us calculate the Jones polynomial of $P(a,1,1)$ as a pretzel knot. We first calculate the Tutte polynomial of the associated Tait graph. Let $a \geq 2$. Then A is either $\{1\}$ or the empty subset, $B = \{2,3\}$. By Proposition 6.16

$$T_{\widetilde{G}(a,1,1)}(x,y) = \sum_{A \subseteq \{1\}} \left(\prod_{\substack{1 \leq i \leq k \\ i \notin A \cup B}} \frac{x^{a_i} - x}{x-1} \right) \cdot \left(x + \sum_{s=1}^{|A|+|B|-1} y^s \right).$$

If $A = \emptyset$, then

$$\frac{x^a - x}{x-1}\,(x+y)$$

is contributed to the sum. If $A = \{1\}$, then

$$x + y + y^2$$

is contributed to the sum. Thus,

$$T_{\widetilde{G}(a,1,1)}(x,y) = \frac{x^a - x}{x-1}\,(x+y) + \left(x + y + y^2\right)$$

which after some further manipulation agrees with Lemma 6.4.

If a is odd, then

$$T_{\widetilde{G}(a,1,1)}(-t,-t^{-1}) = \frac{-t^a + t}{-t-1}\left(-t - t^{-1}\right) + \left(-t - t^{-1} + t^{-2}\right)$$

and if a is even, then

$$T_{\widetilde{G}(a,1,1)}(-t,-t^{-1}) = \frac{t^a + t}{-t-1}\left(-t - t^{-1}\right) + \left(-t - t^{-1} + t^{-2}\right)$$

From Theorem 6.17

$$
\begin{aligned}
& V_{P(a,1,1)}(t) \\
=& -t^{-a-1} \cdot \left(\frac{-t^a + t}{-t - 1} \left(-t - t^{-1}\right) + \left(-t - t^{-1} + t^{-2}\right) \right) \\
=& \frac{1 + t^{-2} - t^{-a-3} + t^{-a}}{1 + t}
\end{aligned}
$$

when a is odd and

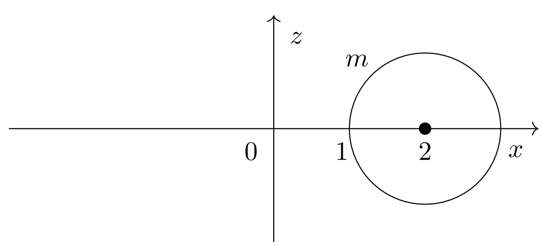
$$
\begin{aligned}
& V_{P(a,1,1)}(t) \\
=& t^{2-a} \cdot \left(\frac{t^a + t}{-t - 1} \left(-t - t^{-1}\right) + \left(-t - t^{-1} + t^{-2}\right) \right) \\
=& \frac{t + t^3 - t^{3-a} + t^{-a}}{1 + t}
\end{aligned}
$$

when a is even.

This matches perfectly the results obtained in Theorem 6.5.

Torus Knots

Let us now study a family of knots known as torus knots. Informally, a *torus knot* is a knot that can be embedded on the surface of a torus without self-intersections. More precisely, a *torus knot* is a knot that is ambient isotopic to a curve lying on the surface of an unknotted torus. The *unknotted torus* is a surface $\mathbb{T} \subset \mathbb{R}^3$ obtained by revolving the circle $m : (x-2)^2 + z^2 = 1$ in the xz-plane around the z-axis. *Torus links* are links lying on the surface of the unknotted torus.



We define the *meridian* as a closed curve on the surface of the torus, moving once around the tube of the torus. The meridian can be thought of as moving along the circle m in the picture above. We define the *equator* of the torus as a closed curve obtained by going once around the hole of the torus. In the picture above, this corresponds to moving once around the z-axis while remaining on the surface of the torus.

One may also construct the unknotted torus as follows. Consider the cylinder $S^1 \times [0,1]$ whose boundary consists of two circles $C_b = S^1 \times \{0\}$ and $C_t = S^1 \times \{1\}$. By identifying C_b and C_t we obtain a topological space homeomorphic to the torus $S^1 \times S^1$. To ensure that the resulting torus is homeomorphic to the unknotted torus we must ensure that the identification is performed without introducing any twists. That is, identifying C_b and C_t, the central axis of the cylinder becomes the unknot. This construction is particularly useful when visualizing curves on the torus. One may think of torus links as separate strands on the cylinder or as separate strands on a rectangular strip with opposite ends identified.

Let us focus on the cylinder. Let the cylinder have its base C_b at the unit circle in the xy-plane and let the cylinder have height 1. We will assign p points to C_b and p points to C_t as follows. The points on C_b are

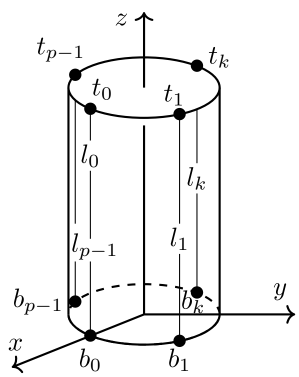
$$b_k = (\cos \frac{2k\pi}{p}, \sin \frac{2k\pi}{p}, 0)$$

and the points on C_t are

$$t_k = (\cos \frac{2k\pi}{p}, \sin \frac{2k\pi}{p}, 1)$$

for $k \in \{0, \ldots, p-1\}$.

Let us now connect the b_k with t_k by a line segment l_k as the following picture shows.



Fixing the base C_b we can twist the whole cylinder by rotating the top by an angle of $\frac{2q\pi}{p}$ counterclockwise as seen from above where $q \in \mathbb{Z}$. Formally, we have the homeomorphism

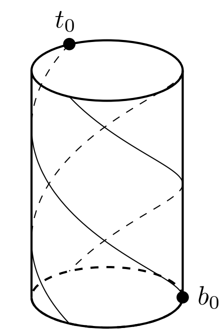
$$f : S^1 \times [0,1] \to S^1 \times [0,1]$$

by

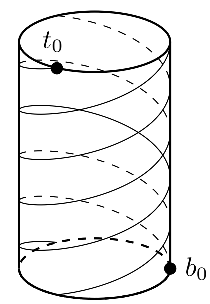
$$f(x,y,z) = \left(x \cos\left(\frac{2q\pi}{p} z\right) - y \sin\left(\frac{2q\pi}{p} z\right), x \sin\left(\frac{2q\pi}{p} z\right) + y \cos\left(\frac{2q\pi}{p} z\right), z \right)$$

for fixed p, q. So no crossings of two line segments are overlapping.

Example 6.20. We will show the previous construction in two cases.



$p = 3$ and $q = -2$



$p = 3$ and $q = 5$

Note that the previous example does not yield links until we identify C_b with C_t. We identify the point $(x, y, 0)$ of C_b with the point $(x, y, 1)$ of C_t so that the center of the cylinder becomes the unknot. This connects the endpoints of the line segments so we are left with a torus link. The link obtained by p line segments and rotating the top C_t by $\frac{2q\pi}{p}$ is called a (p, q)-*torus link* and is denoted $T_{p,q}$.

The unknot traveling only once around the equator is the torus knot $T_{1,0}$. We will extend this definition to include the unknot only traveling once around the meridian as $T_{0,1}$ and the unknot on the surface of the torus traveling no times around the meridian and no times around the equator $T_{0,0}$.

Proposition 6.21. *Assume p and q are not both zero. A torus link $T_{p,q}$ consists of n components where n is the greatest common divisor of p and q,* $\gcd(p, q)$.

Proof. We will use the cylinder model as defined above.

After applying the rotation $\frac{2q\pi}{p}$ on C_t every strand starting at b_k ends up at t_{k+q}. After identifying the top and bottom $C_b \sim C_t$, every t_{k+q} is sent to b_{k+q}. So $b_k \mapsto b_{k+q}$. The components of the link correspond to the orbits of the map

$$k \mapsto k + q \pmod p.$$

Starting from some k we get the sequence

$$k,\ k + q,\ k + 2q,\ k + 3q,\ \dots \pmod p$$

which becomes periodic when

$$k + aq \equiv k \pmod p \iff aq \equiv 0 \pmod p.$$

The smallest integer a where this happens is

$$a = \frac{p}{\gcd(p, q)} = \frac{p}{n}.$$

Each orbit contains p/n distinct points on C_b.

Since there are p distinct starting points on C_b and each orbit contains p/n distinct points, the number of distinct orbits is

$$\frac{p}{p/n} = n.$$

Each orbit corresponds to a component of $T_{p,q}$. Hence $T_{p,q}$ has exactly $n = \gcd(p, q)$ components. □

We can now restrict ourselves to the study of torus knots. This requires p and q to be coprime so $T_{p,q}$ only consists of one component. To restrict ourselves further we will only consider torus knots $T_{p,q}$ where q is non-negative. At first glance it may appear that we are excluding some torus knots. However, the following proposition shows that allowing negative values for q does not produce essentially new examples. Negative values of q only change the knot by taking the mirror image.

Proposition 6.22. *Let $T_{p,q}$ be a torus knot. The mirror image of $T_{p,q}$ is equivalent to $T_{p,-q}$. In other words,*

$$\overline{T_{p,q}} \cong T_{p,-q}$$

Proof. Consider the reflection

$$r(x, y, z) = (x, y, -z).$$

This reflection preserves the projection $\pi(x, y, z) = (x, y)$ but reverses over and under crossings so it produces the mirror image of a knot. From the cylinder method, a torus knot may be represented as strands on a cylinder under the homeomorphism

$$f(x, y, z) = \left(x \cos\big(\frac{2q\pi}{p}z\big) - y \sin\big(\frac{2q\pi}{p}z\big), x \sin\big(\frac{2q\pi}{p}z\big) + y \cos\big(\frac{2q\pi}{p}z\big), z\right).$$

Applying r to f we obtain

$$\begin{aligned}(r \circ f)(x, y, z)&\\ &= \left(x \cos\big(\frac{2q\pi}{p}z\big) - y \sin\big(\frac{2q\pi}{p}z\big), x \sin\big(\frac{2q\pi}{p}z\big) + y \cos\big(\frac{2q\pi}{p}z\big), -z\right)\end{aligned}$$

Setting $-z' = z$ we get

$$\begin{aligned}(r \circ f)(x, y, -z') &= \Big(x \cos\big(-\tfrac{2q\pi}{p}z'\big) - y \sin\big(-\tfrac{2q\pi}{p}z'\big),\\ &\quad x \sin\big(-\tfrac{2q\pi}{p}z'\big) + y \cos\big(-\tfrac{2q\pi}{p}z'\big), z'\Big)\\ &= \Big(x \cos\big(\tfrac{2(-q)\pi}{p}z'\big) - y \sin\big(\tfrac{2(-q)\pi}{p}z'\big),\\ &\quad x \sin\big(\tfrac{2(-q)\pi}{p}z'\big) + y \cos\big(\tfrac{2(-q)\pi}{p}z'\big), z'\Big)\end{aligned}$$

but this is exactly $T_{p,-q}$. Hence,

$$\overline{T_{p,q}} \cong T_{p,-q}.$$

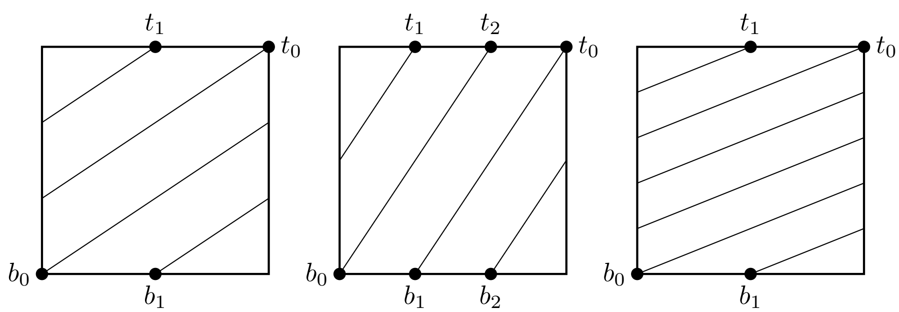
□

Though the cylinder and torus may provide intuition for how the torus knot is constructed in $\mathbb{R}^3$ we are often more interested in knot diagrams since most of our knot invariants apply to the diagrams directly. It is therefore often convenient to represent the torus as a square with opposite edges identified. We consider the quotient space

$$[0,1] \times [0,1]/\sim$$

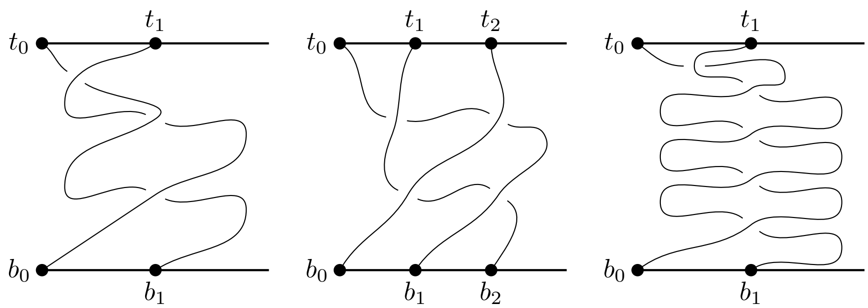
where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$. In this model, a torus knot $T_{p,q}$ can be represented as a straight line from $(0,0)$ with slope p/q in the square.

Example 6.23. Here we represent some torus knots on the square.



To obtain a knot diagram from this representation, we need to interpret the identifications of the boundary. Since $(0,y) \sim (1,y)$, any strand that exits the square through the right edge re-enters through the left edge at the same height. Thus, endpoints on the vertical edges of the square are connected pairwise to form continuous strands. When these connections are drawn in the diagram of the knot, they may intersect strands already present in the interior of the square. Note that these intersections do not correspond to self-intersections of the knot in $\mathbb{R}^3$, they arise only from the planar projection to obtain a knot diagram. At each such intersection, we must choose which strand passes over and which passes under. We adopt the convention that the strands corresponding to these boundary identifications always pass under the existing strands. This choice produces a diagram of a knot isotopic to $T_{p,q}$ by our restriction that q is positive. It is readily checked that choosing the opposite crossing convention produces the mirror image $T_{p,-q}$. Note also that $(0,1) \sim (1,1)$ so t_0 may be identified with $(0,1)$ in our representation. It follows that t_k always lies above b_k in our square representation.

Example 6.24. We can draw the diagrams of $T_{2,3}$, $T_{3,2}$ and $T_{2,5}$ by connecting the vertical edges.

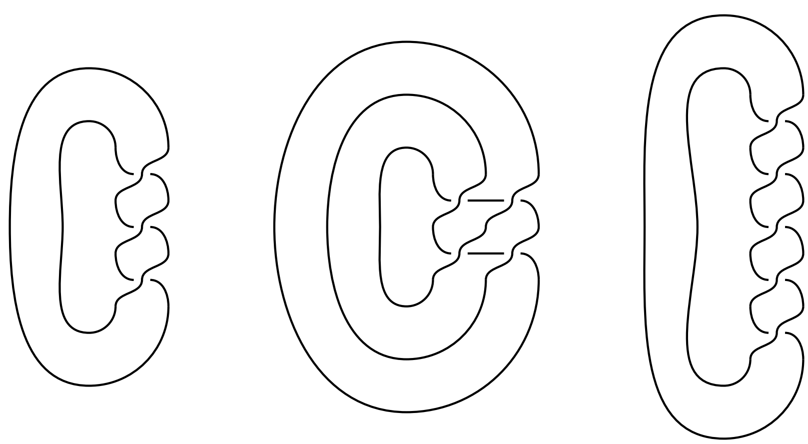


$T_{2,3}$ $T_{3,2}$ $T_{2,5}$

To convert these square representations of the torus knot into actual knot diagrams, we simply connect b_i with t_i such that no new crossings are introduced. This can be done since the previous construction ensures that t_i is above b_i so they can be connected by a strand without inducing any new crossings.

We refer to this construction as the *square model* of the torus knot diagram.

Example 6.25. Constructing $T_{2,3}$, $T_{3,2}$ and $T_{2,5}$ with the square model yields the following diagrams:



$T_{2,3}$ $T_{3,2}$ $T_{2,5}$

From the preceding example, we notice that the square model diagram of $T_{2,3}$ and $T_{2,5}$ are both alternating, whereas $T_{3,2}$ is not. In the square model, there are p strands along the bottom edge of the square. Consider the rightmost strand, which starts at b_{p-1}. As it traverses the diagram, it passes under the $p-1$ other strands in succession. If $p \geq 3$, this produces at least two consecutive under-crossings. So when $p \geq 3$, the diagram is not alternating. When $p = 2$ the rightmost strand first crosses under the other strand as we traverse the square.

As it reaches the point corresponding to the left side in the square, it connects to the strand traveling to the right, which crosses over. Upon reaching the top edge at t_0, its most recent crossing is an under-crossing. Connecting t_0 to b_0 the next crossing is then an over-crossing. This ensures that the diagram is alternating. The following lemma summarizes this observation.

Lemma 6.26. *The square model diagram of a torus knot $T_{p,q}$ is alternating if and only if $p = 2$.*

We also note that each row of crossings in the diagram contains $p-1$ crossings since one of the p strands crosses under the remaining $p-1$ strands. Since there are q such rows of crossings, every diagram contains $q(p-1)$ crossings.

A torus knot $T_{p,q}$ may be oriented such that every strand has direction from b_i to t_i in the square model. Then every crossing has positive writhe by our conventions. This reasoning, together with the fact that there are $q(p-1)$ crossings in each diagram, yields the following lemma.

Lemma 6.27. *Given a square model diagram D of a torus knot $T_{p,q}$, the writhe of D is*

$$w(D) = q(p-1).$$

We can now formulate a closed formula for the Jones polynomial of torus knots $T_{2,q}$.

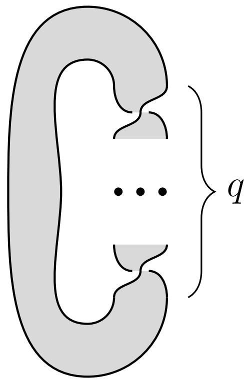
Theorem 6.28. *Let $q > 0$. The Jones polynomial of a torus knot $T_{2,q}$ is*

$$V_{T_{2,q}}(t) = t^{\frac{q-1}{2}} \frac{1 + t + t^2 - t^{q+1}}{1 + t}$$

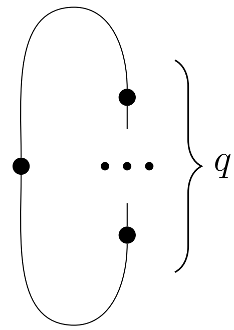
and for $T_{2,-q}$ is

$$V_{T_{2,-q}}(t) = t^{\frac{1-q}{2}} \frac{1 + t + t^{-1} - t^{-q}}{1 + t}.$$

Proof. Assume q is positive. By Lemma 6.26, the square model now provides an alternating diagram D of said torus knot. We find that the number of crossings in D is q. Hence, there are q edges in the Tait graph of D. Furthermore, each shaded region of an alternating knot diagram is only adjacent to two crossings. Hence, the Tait graph of D is isomorphic to the cycle graph C_q.



2-shaded $T_{2.q}$



C_q

Note that C_q contains exactly q edges and q vertices. Together with Lemma 6.27 we get

$$|V| = |E| = w(D) = q$$

Using Lemma 6.3 we obtain

$$T_{C_q}(x, y) = y + \frac{x(1 - x^{q-1})}{1 - x}.$$

Now, by Theorem 5.9

$$V_{T_{2,q}}(t) = (-1)^q t^{\frac{-2q+q+2+3q}{4}} \left(-t^{-1} + \frac{-t(1 - (-t)^{q-1})}{1 - (-t)} \right).$$

Proposition 6.21 ensures that q and 2 are coprime when the torus link is a knot. Hence, q is odd. Then

$$V_{T_{2,q}}(t) = -t^{\frac{-2q+q+2+3q}{4}} \left(-t^{-1} + \frac{-t(1 - t^{q-1})}{1 - (-t)} \right).$$

Simplifying,

$$V_{T_{2,q}}(t) = t^{\frac{q-1}{2}} \frac{1 + t + t^2 - t^{q+1}}{1 + t}.$$

Now, regarding $-q < 0$, by Proposition 6.22

$$\overline{T_{2,-q}} \cong T_{2,q}.$$

Then Theorem 5.9 yields

$$V_{T_{2,-q}}(t) = t^{\frac{1-q}{2}} \frac{1 + t + t^{-1} - t^{-q}}{1 + t}.$$

□

Example 6.29. Since the right-handed trefoil and the torus knot $T_{2,3}$ are isotopic, their Jones polynomial must agree.

By Theorem 6.28 we have

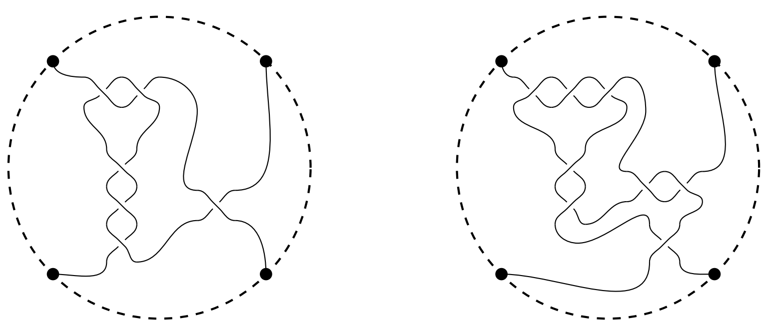
$$V_{T_{2,3}}(t) = t\left(\frac{1+t+t^2-t^4}{1+t}\right) = t + t^3 - t^4.$$

This is the same polynomial as in Example 5.11.

Rational Knots

Definition 6.30. A *rational tangle* is a tangle $T(a_1, \ldots, a_k)$ obtained by first performing a_1 horizontal half-twists on $T(\infty)$ followed by a_2 vertical half-twists, then alternating between horizontal and vertical half-twists until finally performing a_k horizontal or vertical half-twist depending on the parity of the index k. Each block of a_i consecutive half-twists is called a *twist region* of the tangle.

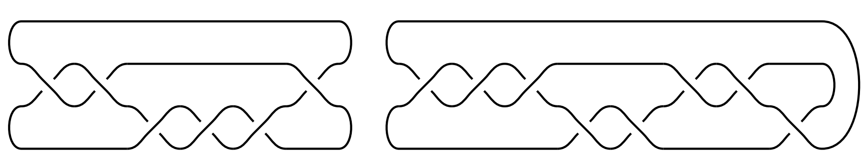
Example 6.31. Here we see some examples of rational tangles.



$T(-2,-3,-1)$ $T(3,2,-2,1)$

Definition 6.32. A *rational link* $R(a_1, \ldots, a_k)$ is constructed by connecting the top left and right endpoints and the bottom left and right endpoints of a rational tangle $T(a_1, \ldots, a_k)$.

Example 6.33. The rational tangles from Example 6.31 can be constructed into rational links. We will also move the twist regions such that every horizontal twist is at the same height and similarly for the vertical twist regions.



$R(-2,-3,-1)$ $R(3,2,-2,1)$

We say that the diagram of a rational link presented as in Example 6.33 is in *tangle block form*. Henceforth, we will refer to rational links in tangle block form.

From our constructions it is straightforward to determine when a rational link is alternating. The argument is similar to the proof of Proposition 6.12. We summarize this in the following proposition.

Proposition 6.34. *A rational link diagram in tangle block form is alternating if every a_i has the same sign.*

The mirror image of a rational knot can also be easily described. Recall that the mirror image of a link is obtained by switching all crossings. This corresponds to switching all crossings in each twist region, which in turn amounts to replacing each a_i with $-a_i$. The following proposition encapsulates this reasoning.

Proposition 6.35. *Let $R(a_1, \dots, a_k)$ be a rational link. Then the mirror image satisfies*

$$\overline{R(a_1, \dots, a_k)} \cong R(-a_1, \dots, -a_k).$$

Let us now leave rational links for a while and focus on number theory. We define the *continued fraction expansion* of a finite sequence $(a_1, \dots, a_k)$ to be the following fraction

$$[a_1, \dots, a_k] = a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_k}}}$$

or equivalently

$$[a_1, \dots, a_k] = a_1 + \frac{1}{[a_2, \dots, a_k]}$$

for all $a_i \in \mathbb{Z}$ for $i \in \{1, \dots, k\}$. We adopt the conventions that

$$\frac{1}{0} = \infty \quad \text{and} \quad \frac{1}{\infty} = 0.$$

Example 6.36. We have

$$[2, -5, 1] = 2 + \frac{1}{-5 + \frac{1}{1}} = \frac{7}{4}$$

and

$$[-3, -5, 9, 7] = -3 + \frac{1}{-5 + \frac{1}{9 + \frac{1}{7}}} = -\frac{1003}{313}.$$

We now prove some useful properties of the continued fraction expansion.

Lemma 6.37. *If $a_i = 0$ for some $i \in \{2, \dots, k-1\}$, then*

$$[a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_k] = [a_1, \dots, a_{i-1} + a_{i+1}, \dots, a_k].$$

Proof. We have

$$[a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_k] = [a_1, \dots, a_{i-1} + \frac{1}{[0, a_{i+1}, \dots, a_k]}]$$

by the recursive definition applied to position $i - 1$. Then

$$[0, a_{i+1}, \dots, a_k] = \frac{1}{[a_{i+1}, \dots, a_k]}$$

so

$$\frac{1}{[0, a_{i+1}, \dots, a_k]} = [a_{i+1}, \dots, a_k].$$

Therefore,

$$[a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_k] = [a_1, \dots, a_{i-1} + a_{i+1}, \dots, a_k]$$

proving the lemma. □

Lemma 6.38. *If*

$$[a_1, \dots, a_k] = \frac{p}{q}$$

then

$$[-a_1, \dots, -a_k] = -\frac{p}{q}.$$

Proof. We will provide a proof by induction. For $k = 1$ we have $[a_1] = a_1$ and $[-a_1] = -a_1$. Assume the formula holds for $k - 1$. Let

$$[a_1, \dots, a_k] = \frac{p}{q}.$$

By the recursive definition of the continued fraction

$$[-a_1, \dots, -a_k] = -a_1 + \frac{1}{[-a_2, \dots, -a_k]}$$

but $[-a_2, \dots, -a_k]$ only contains $k - 1$ elements so we can apply the induction hypothesis

$$[-a_1, \dots, -a_k] = -a_1 - \frac{1}{[a_2, \dots, a_k]} = -\left(a_1 + \frac{1}{[a_2, \dots, a_k]}\right) = -\frac{p}{q}$$

completing the induction. □

Using the previous properties, we obtain the following result.

Proposition 6.39. *Every rational number r, such that $|r| \geq 1$, together with the Euclidean algorithm admits a unique continued fraction*

$$[b_1, \ldots, b_m]$$

such that either

$$b_i \geq 1 \text{ for all } i, \text{ and either } m = 1 \text{ or } b_m \geq 2$$

or

$$b_i \leq -1 \text{ for all } i, \text{ and either } m = 1 \text{ or } b_m \leq -2.$$

Proof. If r is positive, the Euclidean algorithm produces r as a unique continued fraction

$$[c_1, \ldots, c_n]$$

such that every c_i is positive and either $m = 1$ or $c_n \geq 2$. We then take

$$[b_1, \ldots, b_m] = [c_1, \ldots, c_n].$$

If r is negative we use the Euclidean algorithm on $-r$ to obtain a unique continued fraction

$$[c_1, \ldots, c_n]$$

such that every c_i is positive and either $m = 1$ or $c_n \geq 2$. We then take

$$[b_1, \ldots, b_m] = [-c_1, \ldots, -c_n]$$

by Lemma 6.38. □

Given a continued fraction written as in Proposition 6.39 we can make it of odd length. If m is even, then

$$\begin{aligned}
[b_1, \ldots, b_m] &= b_1 + \frac{1}{\cdots + \frac{1}{b_m}} \\
&= b_1 + \frac{1}{\cdots + \frac{1}{b_m - 1 + 1}} \\
&= b_1 + \frac{1}{\cdots + \frac{1}{b_m - 1 + \frac{1}{1}}} = [b_1, \ldots, b_m - 1, 1]
\end{aligned}$$

for every b_i positive and

$$[b_1, \ldots, b_m] = [b_1, \ldots, b_m + 1, -1]$$

for every b_i negative. If this results in $b_m \pm 1 = 0$ we use Lemma 6.37. The resulting continued fraction has odd length. This ensures that every b_i still has the same sign.

We say that a continued fraction in this unique form where every term has the same sign and is of odd length is in *standard form.*

The name "Rational Knots" may not seem like a natural name. The following theorem proven by John Conway in 1970 shows a beautiful connection between the equivalence classes of rational links to the rational numbers.

Theorem 6.40. [**Con70**] Let

$$R(a_1, \ldots, a_k) \quad \text{and} \quad R(b_1, \ldots, b_m)$$

be two rational links. Then

$$R(a_1, \ldots, a_k) \cong R(b_1, \ldots, b_m)$$

if and only if

$$[a_1, \ldots, a_k] = [b_1, \ldots, b_m].$$

Hence, the equivalence class of rational links is completely determined by the value of its continued fraction.

We define $R(p/q)$ to be the rational link associated to any continued fraction expansion of p/q. This is well-defined by Theorem 6.40.

The correspondence established in Theorem 6.40 allows us to study rational links using number theory, instead of purely relying on geometric and topological arguments. The following result shows this usefulness.

Given a rational link $R(a_1, \ldots, a_k)$ we evaluate $[a_1, \ldots, a_k] = p/q$. By Proposition 6.39 we now write p/q in the standard form $[b_1, \ldots, b_m]$ where m is odd and every b_i has the same sign. By Theorem 6.40

$$R(a_1, \ldots, a_k) \cong R(b_1, \ldots, b_m)$$

which is alternating by Proposition 6.34 and where m is odd. This representation of a rational link is called its *standard form.* This reasoning yields the following proposition.

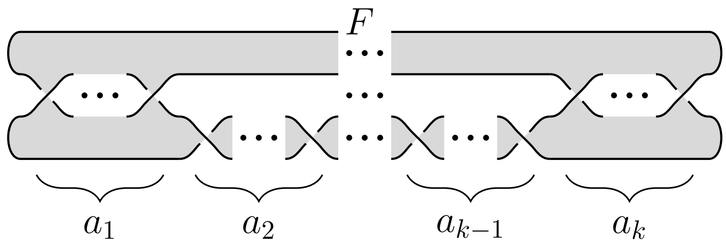
Proposition 6.41. *Every rational link R admits a standard form.*

Proposition 6.42. [**Con70**] Let $R(a_1, \ldots, a_k)$ be a rational link with $[a_1, \ldots, a_k] = p/q$. Then

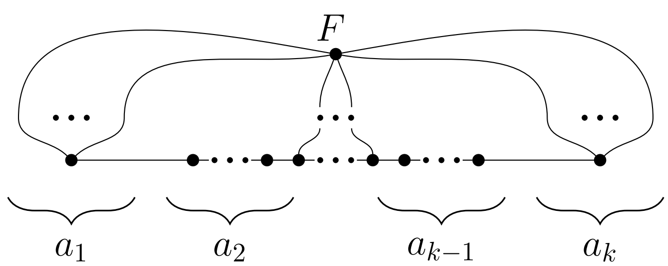
$$\begin{aligned} R \text{ is a knot} &\iff p \text{ is odd,} \\ R \text{ has two components} &\iff p \text{ is even.} \end{aligned}$$

Let us now study the connection between a rational knot R in standard form and its Tait graph.

When R is in standard form and tangle block form we obtain its Tait graph $\widetilde{G}(R)$ by shading. The top shaded region corresponds to one vertex in $\widetilde{G}(R)$ which we denote by F. Each twist region contributes vertices and edges depending on the value of a_i and the parity of i. If i is odd, the twist region corresponding to a_i produces a single shaded region beneath it, giving one vertex. The a_i crossings correspond to a_i parallel edges between this vertex and the top vertex F. If i is even, the a_i crossings correspond to a_i edges between the vertices corresponding to a_{i-1} and a_{i+1}. In this case, no edges are incident to F.



R



$\widetilde{G}\,(R)$

Lemma 6.43. *Let $R(a_1, \ldots, a_k)$ be a rational knot in standard form and $\widetilde{G}(R)$ its Tait graph. Then*

$$\left|E(\widetilde{G}(R))\right| = \sum_{i=1}^{k} |a_i|\,,$$
$$\left|V(\widetilde{G}(R))\right| = 2 + \sum_{\substack{1 \le i \le k \\ i \text{ even}}} |a_i|\,.$$

Proof. The number of edges follows directly from the reasoning above.

Regarding the number of vertices, every Tait graph of R contains one top vertex F contributing 1 to the count. For every i odd, there exists one vertex and for every i even there exists $a_i - 1$ vertices. Since k is odd, there are a $\frac{k+1}{2}$ odd indices i and $\frac{k-1}{2}$ even i. Thus

$$\begin{aligned}\left|V(\widetilde{G}(R))\right| =& 1 + \frac{k+1}{2} + \sum_{\substack{1\le i\le k\\ i \text{ even}}} (a_i - 1)\\ =& 2 + \sum_{\substack{1\le i\le k\\ i \text{ even}}} a_i.\end{aligned}$$

□

Proposition 6.44. *Let $\widetilde{G}$ be the Tait graph of $R(a_1, \ldots, a_k)$ where all a_i are positive and let $E = \{i \in \{1, \ldots, k\} : i \text{ even}\}$. For each subset $A \subseteq E$, we define $\mathcal{B}(A)$ to be a partition of the odd indices obtained by joining two odd indices whenever every even index between them belongs to A. Then*

$$T_{\widetilde{G}}(x,y) = \sum_{A\subseteq E} \left(\prod_{\substack{i\notin A\\ i \text{ even}}} \frac{x^{a_i}-1}{x-1} \right) \cdot \left(\prod_{B\in\mathcal{B}(A)} \left((x-1) + \frac{y^{\sum_{j\in B} a_j} - 1}{y-1} \right) \right).$$

Example 6.45. We note that

$$\mathcal{B}(\emptyset) = \{\{1\}, \{3\}, \ldots, \{k\}\}$$

so no dipole graphs are merged, while

$$\mathcal{B}(E) = \{\{1, 3, \ldots, k\}\}$$

merges all dipole graphs into one

$$D_{a_1+a_3+\cdots+a_k}.$$

For example, if $A = \{2\}$

$$\mathcal{B}(\{2\}) = \{\{1, 3\}, \{5\}, \ldots, \{k\}\}$$

so only the first two dipoles are merged.

Proof. If $k = 1$, then the Tait graph is isomorphic to the dipole graph. The result follows from Lemma 6.15.

If $k \geq 3$, we perform the Tutte recursion on the edges induced by a_i for even i. By Lemma 6.14 deleting the path a_i for even i we obtain a factor

$$x + x^2 + \cdots + x^{a_i - 1}$$

with the Tutte polynomial for the graph consisting of deleting the path completely together with the Tutte polynomial for the graph where the remaining path contains only one edge. Performing the recursion on this last edge we obtain the graph where the path is completely deleted and the graph obtained by identifying the vertices a_{i-1} with a_{i+1}. In other words, the path induced by an a_i can either be deleted completely resulting in a factor of

$$1 + x + \cdots + x^{a_i - 1}$$

or be fully contracted resulting in the vertices produced by a_{i-1} and a_{i+1} being identified.

The choice of whether the paths are deleted or contracted is in bijection with every subset of E.

We fix a subset $A \subseteq E$. For every $i \notin A$ we perform the recursion until those paths are completely deleted resulting in a factor

$$\prod_{i \notin A} (1 + x + \cdots + x^{a_i - 1}) = \prod_{i \notin A} \frac{x^{a_i} - 1}{x - 1}.$$

For every $i \in A$ we perform the recursion until a_{i-1} is identified with a_{i+1}. For every $i \in A$ the sets of $\mathcal{B}$ around i are joined. After contracting all paths corresponding to indices in A, each maximal interval of indices separated only by elements of A contracts into a single vertex. The resulting graph consists of dipole graphs where one vertex of each graph is identified to F. So all dipoles D_{a_j} with j in the same block $B \in \mathcal{B}(A)$ become one dipole with $\sum_{j \in B} a_j$ parallel edges. By Lemma 6.15 this contributes a factor of

$$x + y + \cdots + y^{(\sum_{j \in B} a_j) - 1}.$$

Since every such dipole is identified to one vertex, we use Proposition 3.19 to obtain the product

$$\prod_{B \in \mathcal{B}(A)} \left(x + y + \cdots + y^{(\sum_{j \in B} a_j) - 1} \right) = \prod_{B \in \mathcal{B}(A)} \left((x - 1) + \frac{y^{\sum_{j \in B} a_j} - 1}{y - 1} \right).$$

The Tutte polynomial is then obtained by summing over every subset $A \subseteq E$. □

I was not able to derive a general formula for the writhe of a rational link. I tried to compute the writhe explicitly for the cases $k = 1, 3, 5$, but these calculations did not reveal a clear pattern that could be extended to arbitrary values of k. Thus, in the final formula, the writhe must be computed individually for each link.

We have now developed the necessary theory to find a closed formula for the Jones polynomial of a rational knot that takes the writhe as an input parameter.

Theorem 6.46. *Let $R(a_1,\dots,a_k)$ be a positive alternating rational knot with diagram D in standard form, that is, k is odd and all a_i are positive. Let $\widetilde{G}$ be the Tait graph of D. We denote*

$$S_{odd} := \sum_{\substack{1\le i\le k\\ i\ odd}} a_i,$$
$$S_{even} := \sum_{\substack{1\le i\le k\\ i\ even}} a_i.$$

Then the Jones polynomial for R is

$$V_R(t) = (-1)^{S_{odd}+S_{even}} t^{\frac{-2+S_{odd}-S_{even}+3w(D)}{4}} \cdot T_{\widetilde{G}}(-t,-t^{-1})$$

and

$$V_{\overline{R}}(t) = (-1)^{S_{odd}+S_{even}} t^{\frac{2-S_{odd}+S_{even}-3w(D)}{4}} \cdot T_{\widetilde{G}}(-t^{-1},-t).$$

Here, the Tutte polynomial is obtained from Proposition 6.44.

Proof. By Theorem 5.9 we have

$$V_R(t) = (-1)^{\left|E(\widetilde{G})\right|} t^{\frac{-2\left|V(\widetilde{G})\right|+\left|E(\widetilde{G})\right|+2+3w(D)}{4}} T_{\widetilde{G}}(-t,-t^{-1}).$$

Together with Lemma 6.43 we simplify this to

$$(-1)^{S_{\text{odd}}+S_{\text{even}}} t^{\frac{-2+S_{\text{odd}}-S_{\text{even}}+3w(D)}{4}}.$$

Thus

$$V_R(t) = (-1)^{S_{\text{odd}}+S_{\text{even}}} t^{\frac{-2+S_{\text{odd}}-S_{\text{even}}+3w(D)}{4}} \cdot T_{\widetilde{G}}(-t,-t^{-1}).$$

Applying Proposition 5.4 we get the Jones polynomial of the mirror image $R(-a_1,\dots,-a_k)$ by mapping $-t$ to $-t^{-1}$,

$$V_{\overline{R}}(t) = (-1)^{S_{\text{odd}}+S_{\text{even}}} t^{\frac{2-S_{\text{odd}}+S_{\text{even}}-3w(D)}{4}} \cdot T_{\widetilde{G}}(-t^{-1},-t).$$

□

Example 6.47. We will use our formula to calculate the Jones polynomial of the figure eight knot using Theorem 6.46.

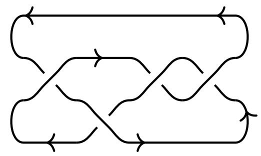
It is readily checked that the figure eight knot is isotopic to the rational knot $R(-1,-1,-2)$. Using Proposition 6.35

$$\overline{R}(-1,-1,-2) \cong R(1,1,2).$$

Thus

$$S_{\text{odd}} = 3,$$
$$S_{\text{even}} = 1.$$

Finding the writhe we chose an orientation on the diagram of $R(1,1,2)$.



Oriented $R(1,1,2)$

The total writhe of this oriented diagram is

$$w(D) = -1 - 1 + 2 = 0.$$

Hence,

$$(-1)^{S_{\text{odd}}+S_{\text{even}}} t^{\frac{2-S_{\text{odd}}+S_{\text{even}}-3w(D)}{4}} = t^{\frac{2-3+1-3\cdot 0}{4}} = 1.$$

Thus

$$V_{R(-1,-1,-2)}(t) = T_{\widetilde{G}}(-t^{-1}, -1)$$

where $\widetilde{G}$ is the Tait graph of $R(1,1,2)$. By Proposition 6.44 we sum over the subsets of $\{2\}$, i.e. the empty subset and $\{2\}$. The empty subset contributes

$$(1) \cdot (x) \cdot (x+y) = x^2 + xy$$

and the subset $\{2\}$ contributes

$$(1) \cdot (x + y + y^2) = x + y + y^2.$$

Evaluating at $(-t^{-1}, -t)$ we obtain the Jones polynomial

$$V_{R(-1,-1,-2)}(t) = t^{-2} - t^{-1} + 1 - t + t^2.$$

This is the same polynomial as we derived in Example 5.10.

Example 6.48. The right-handed trefoil T_r in Example 4.15 belongs to each of the families studied in this section. Using Reidemeister moves, it is readily checked that

$$T_r \cong \tau_{-1} \cong P(-1,-1,-1) \cong T_{2,3} \cong R(-3).$$

Since they are all isotopic, their Jones polynomials must agree. We will compute them case by case.

(i) *Twist knot* τ_{-1}. Applying Corollary 6.6 we obtain

$$V_{\tau_{-1}}(t) = \frac{t + t^3 - t^5 + t^2}{1+t} = t + t^3 - t^4.$$

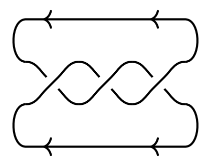
(ii) *Pretzel knot* $P(-1,-1,-1)$. Let $\widetilde{G}(1,1,1)$ be the Tait graph of the mirror image of $P(-1,-1,-1)$. Using Corollary 6.18 we obtain

$$V_{P(-1,-1,-1)}(t) = -t^{\frac{-3+1+6}{2}}(-t^{-1} - t + t^2) = t + t^3 - t^4.$$

(iii) *Torus knot* $T_{2,3}$. By Theorem 6.28 we have

$$V_{T_{2,3}}(t) = t\left(\frac{1 + t + t^2 - t^4}{1+t}\right) = t + t^3 - t^4.$$

(iv) *Rational knot* $R(-3)$. Choosing an orientation on the diagram D of $R(3)$ we find that $w(D) = -3$



Oriented $R(3)$

By Theorem 6.46 we obtain

$$V_{R(-3)}(t) = -t^{\frac{-2+3-9}{4}} \cdot \left(-t^{-1} - t + t^2\right) = t + t^3 - t^4.$$

In each case, the same Jones polynomial is obtained, consistent with Example 5.11.

References

[Aig07] Martin Aigner. *A Course in Enumeration.* Springer-Verlag, 2007, pp. 393–400, 434–443.

[Con70] John H. Conway. "An Enumeration of Knots and Links, and Some of Their Algebraic Properties". In: *Computational Problems in Abstract Algebra.* Pergamon Press, 1970, pp. 329–358.

[Jon85] Vaughan Jones. "A polynomial invariant for knots via von Neumann algebras". In: *Bulletin of the American Mathematical Society* 12.1 (1985), pp. 103–111.

[Mur96] Kino Murasugi. *Knot Theory and Its Applications.* Birkhäuser, 1996, pp. 132–137, 171–182.

[Rei26] Kurt Reidemeister. "Elementare Begründung der Knotentheorie". German. In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 5 (1926).