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From graphs to polynomials: The unit distance problem in the plane

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Nicole Nyberg

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

From graphs to polynomials: The unit distance problem in the plane

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Abstract

The Erdős unit distance problem asks for the maximum number of unit distances determined by n points in the plane. The best known upper bound is $O(n^{4/3})$. We examine the connection between the problem and graph theory. We give two different proofs of the Szemerédi–Trotter theorem, which bounds the number of incidences between points and lines in the plane, using both graph-theoretic and polynomial partitioning methods, and look at generalizations of this theorem. From this, the bound $O(n^{4/3})$ follows. Furthermore, we study the number of unit distances for certain configurations and restrictions on our points. In addition, we describe a configuration, due to [1], that determines at least $n^{1+c/\log\log n}$ unit distances, for some absolute constant $c > 0$ and sufficiently large n. Furthermore, we introduce rigid frameworks for graphs and discuss a related conjecture due to [2], which, if it holds, yields an improvement on the upper bound $O(n^{4/3})$ on the number of unit distances determined by n points in the plane.

Sammanfattning

Erdős enhetsavståndsproblem handlar om det maximala antalet enhetsavstånd mellan n punkter i planet. Den bästa övre gränsen idag är $O(n^{4/3})$. Vi studerar kopplingen mellan problemet och grafteori. Vi introducerar Szemerédi–Trotter-satsen, vilken begränsar antalet incidenser mellan punkter och linjer i planet, och bevisar denna både med en grafteoretisk metod och en polynompartitioneringsmetod, samt beskriver generaliseringar av denna sats. Från satsen följer den övre gränsen $O(n^{4/3})$. Fortsättningsvis studerar vi antalet enhetsavstånd för vissa specifika konfigurationer, samt när vi inför begränsningar på våra punkter. Dessutom visar vi en konfiguration, från [1], som ger minst $n^{1+c/\log\log n}$ enhetsavstånd, för något $c > 0$ och tillräckligt stora n. Fortsättningsvis introducerar vi rigida ramverk och diskuterar en relaterad förmodan från [2] som, givet att den håller, ger en förbättring av den övre gränsen $O(n^{4/3})$ för antalet enhetsavstånd mellan n punkter i planet.

Acknowledgments

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Contents

1 Introduction

In 1946, a mathematician named Paul Erdős published a paper in *The American Mathematical Monthly* [3], concerning distances between points in the plane. To describe Erdős's question, we let $u(n)$ denote the largest possible number of pairs within a set of n points in the Euclidean plane that are some fixed distance apart. We may assume, without loss of generality, that this distance is the unit distance. This may be written as

$$u(n) = \max_{\substack{P \subset \mathbb{R}^2 \\ |P|=n}} |\{\{p, q\} \subset P : \|p - q\| = 1\}|.$$

In particular, Erdős was interested in what happens if we wish to determine or estimate $u(n)$. In 1946, Erdős proved that $cn^{3/2}$, for some absolute constant $c > 0$, is an upper bound on $u(n)$, and conjectured that the order of magnitude of $u(n)$ is approximately equal to $n^{1+c/\log\log n}$, for some absolute constant $c > 0$ [3]. In 1975, Sándor Józsa and Endre Szemerédi improved this upper bound to $o(n^{3/2})$ [4]. In 1984, this bound was improved to $cn^{1.499\cdots}$, for some absolute constant c, by Jozsef Beck and Joel Spencer [5]. Finally, the best known upper bound today of $cn^{4/3}$, for some absolute constant $c > 0$, was achieved by Joel Spencer, Endre Szemerédi, and William T. Trotter, in 1984 [6]. In Section 2 this bound is proved and discussed.

In this thesis, we study the Erdős unit distance problem and examine some of the main ideas used in proving the $O(n^{4/3})$ upper bound on the number of unit distances determined by n points in the plane. We shall also discuss more recent work that gives a conditional improvement on the upper bound. The main purpose of this thesis is to provide an introduction to several aspects of the Erdős unit distance problem and the different techniques used to study it.

In Section 2, we analyze the Erdős unit distance problem using graph theory. We define the crossing number of a graph and provide an inequality for these. Furthermore, we examine how bounds on incidences between points and curves in the plane can be used to bound the number of unit distances determined by n points in the plane. We prove the Szemerédi–Trotter theorem, which bounds the number of incidences between points and lines in the plane, using an argument based on graph theory. From the Szemerédi–Trotter theorem we deduce the upper bound of $O(n^{4/3})$ on the number of unit distances determined by n points in the plane.

In Section 3, we provide an alternative proof of the Szemerédi–Trotter theorem, using polynomial partitioning as well as the polynomial ham sandwich theorem. In preparation for this, we study bivariate polynomials, and prove lemmas concerning the relation between the zero set of a bivariate polynomial and lines in $\mathbb{R}^2$. We then state and prove the polynomial ham sandwich theorem, which asserts the existence of a bivariate polynomial such that, for any finite collection of finite point sets in $\mathbb{R}^2$, the polynomial simultaneously bisects the sets. Additionally, we discuss the concept of an r-partitioning polynomial.

Section 4 explores the maximum number of unit distances among n points in the plane, for small values of n. In the same section we investigate how many unit distances we get among n points in the plane when the points are from a specific triangular pattern and hexagonal pattern, respectively. In addition, we prove an upper bound on the number of unit distances when we impose the restriction that the n points are only allowed to be taken from two given perpendicular lines.

Later in the same section, we let our point set be the $\sqrt{n} \times \sqrt{n}$ grid, with a particular step size. We prove that such a configuration determines at least $n^{1+c/\log\log n}$ unit distances, for some absolute constant $c > 0$, for large values of n. Furthermore, we discuss the concept of nested point sets in connection to different configurations. We also state and prove two theorems concerning the number of unit distances in terms of linear growth in n.

Furthermore, in Section 5, we study graph realizations and frameworks. In order to compare different frameworks, we discuss the notion of equivalent, as well as congruent, frameworks. We characterize certain frameworks by the property of rigidity. Afterwards, we state a conjecture related to the concept of rigid frameworks, proposed in [2]. We then state and prove a theorem that improves the upper bound $O(n^{4/3})$ on the number of unit distances determined by n points in the plane, under the assumption that the conjecture holds.

In the last section, Section 6, we present subjects for future studies concerning the Erdős unit distance problem.

1.0.1 Notation and standing assumptions

In this thesis, we use $f(x) \ll g(x)$ to denote that $f(x) = O(g(x))$, for two functions $f(x)$ and $g(x)$. Similarly, $f(x) \gg g(x)$ means that $g(x) = O(f(x))$. If both $f(x) \ll g(x)$ and $f(x) \gg g(x)$ hold, we will denote this by $f(x) \asymp g(x)$.

Throughout this thesis, unless otherwise specified, the arguments and techniques used in the proofs of theorems and lemmas are inspired by and learned from the proofs in the same reference as the theorem and lemma. Unless otherwise specified, log denotes the natural logarithm. When discussing graphs we always assume that the graph is simple, undirected and finite.

2 Graph theory and incidence geometry

The unit distance problem is closely related to graph theory. In this section, we explain how we can use graph theory to bound the number of unit distances determined by n points in the plane. In particular, we prove the Szemerédi–Trotter theorem. This theorem provides an upper bound on the number of incidences between points and lines in the plane, and we prove it in this section using an argument based on graph theory. From the Szemerédi–Trotter theorem, we deduce the upper bound of $O(n^{4/3})$ on the number of unit distances determined by n points in the plane.

2.1 Graphs and unit distances

We start this section by fixing some notation. For any set P of n points in the Euclidean plane, we define the *unit distance graph* of P as follows:

Definition 2.1 ([7, p. 461])**.** The unit distance graph of P is the graph G_P on the vertex set $V(G_P) = P$ where two vertices are joined by an edge if and only if their distance is 1 .

Hence, we know that for the unit distance graph G_P of a point set P there exists a visual representation of it in the plane such that each edge has length 1. Therefore, to determine the number of unit distances among the points in P, we can simply count the edges in the unit distance graph G_P of P.

Definition 2.2 ([8])**.** Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *graph density* $d(G)$ of G is defined as

$$d(G) = \frac{|E|}{\binom{|V|}{2}}.$$

Since the maximum number of edges of an undirected graph is $\binom{|V|}{2} = \frac{|V|(|V|-1)}{2}$, the graph density is defined as the ratio between the edges present in a graph $G = (V, E)$ and the maximum number of possible edges, and may be written as

$$d(G) = \frac{2|E|}{|V|(|V|-1)}.$$

Using these definitions, the Erdős unit distance problem is equivalently asking for the maximum number of edges in a unit distance graph over all possible point sets P with $|P| = n$. Thus, the unit distance problem means investigating how dense a unit distance graph can be, for a given number of vertices.

We can now establish a trivial upper bound on the number of unit distances determined by n points in the plane. We claim that $\binom{n}{2} = \frac{n(n-1)}{2}$ is an upper bound, although we shall later see that this is far from optimal.

Theorem 2.3 (The trivial upper bound)**.** *A set of n points in the plane can have at most $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of points at unit distance from each other.*

Proof. The bound is trivial since there are only $\binom{n}{2}$ pairs of points among n points, and hence at most $\binom{n}{2}$ pairs of points that are a unit distance apart. We could also deduce the bound using graph theory as follows:

Let $P \subset \mathbb{R}^2$ be a set of points with $|P| = n$. The bound is obtained by defining the unit distance graph G_P of P and using the fact that each unit distance between two points of P corresponds to a unique edge of G_P. Conversely, every edge arises this way. Thus, the upper bound $\binom{|V(G_P)|}{2}$ on the number of edges in G_P is an upper bound on the number of unit distances determined by the points of P. Since $|V(G_P)| = |P| = n$ this completes the proof. □

This trivial upper bound is only sharp for a given n if there exists a unit distance graph with the maximum possible number of edges that a graph on n vertices can have, which means that any pair of vertices is connected by an edge. Using the terminology introduced above, we see that $d(G_P) = 1$ for a unit distance graph G_P of a point set P that has this maximum number of possible edges. However, *existence* of this complete unit distance graph of a set of points P with $|P| = n$ is not guaranteed. As we shall see in Section 4 this graph exists only for $n = 2$ and $n = 3$.

2.2 Crossing numbers

In order to prove the best known upper bound of $O(n^{4/3})$ for the maximum possible number of unit distances determined by n points in the plane, we now introduce the notion of the crossing number of a graph, as well as planar graphs.

Definition 2.4 ([9, p. 285])**.** Let $G = (V, E)$ be a graph. An *embedding* of G in the plane is a planar representation of the graph, where the vertices correspond to points in the plane and each edge $\{u, v\}$ for $u, v \in V$ is represented by a curve joining the points associated with vertex u and v in the plane. To each embedding corresponds a *crossing number*, which is the number of intersection points between curves corresponding to edges with no common endpoint. The *crossing number of* G, denoted $\mathrm{cr}(G)$, is the minimum crossing number among all embeddings of G.

Definition 2.5 ([10, p. 540])**.** A graph $G = (V, E)$ is called *planar* if it has crossing number 0, i.e. $\mathrm{cr}(G) = 0$.

The embedding of the complete graph K_4 in Figure 1 has crossing number 1. The crossing is due to the intersecting edges $\{v_1, v_4\}$ and $\{v_2, v_3\}$. Similarly, the embedding of K_4 provided in Figure 2 has crossing number 1, since we do not count crossings between incident edges. In Figure 3, we see another embedding of K_4 that has crossing number 0. Since a crossing number is a nonnegative integer, we can therefore conclude that $\mathrm{cr}(K_4) = 0$ and that K_4 is planar.

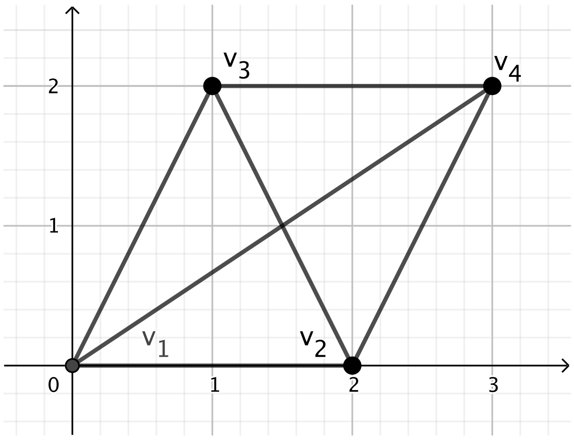

Figure 1: Embedding of K_4 having crossing number 1

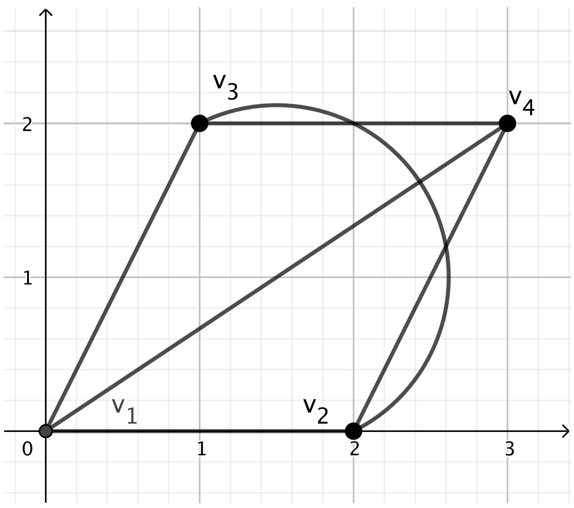

Figure 2: Alternative embedding of K_4 having crossing number 1

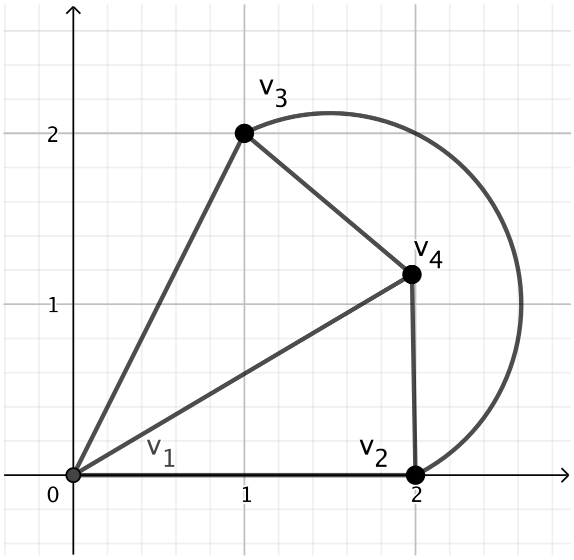

Figure 3: Embedding of K_4 having crossing number 0

The following theorem can be proved using Euler's theorem for planar graphs. However, the proof is of a more technical character, and hence it is omitted here. For details, see [10, p. 547].

Theorem 2.6. *In a connected and planar graph* $G = (V, E)$ *with* $|V| \geq 3$*, we have* $|E| \leq 3|V| - 6$*.*

The following lemma provides a basic lower bound on the crossing number of a graph. It will be of great importance in the deducing of a more sophisticated lower bound on the crossing number of a graph.

Lemma 2.7. *In any graph* $G = (V, E)$*, we have*

$$\mathrm{cr}(G) \geq |E| - 3|V|.$$

Proof. From Theorem 2.6 we know that for any connected and planar graph $G = (V, E)$ with $|V| \geq 3$ we have $|E| \leq 3|V| - 6$. In particular, $|E| \leq 3|V|$. Note that if G is planar but disconnected we limit the number of possible edges even more, hence the bound still holds in this case. If $|V| = 2$ the graph has no more than one edge, and if $|V| = 1$, the graph has no edges. Thus, we may conclude that any planar graph $G = (V, E)$ has at most $3|V|$ edges.

Hence, the inequality $|E| - 3|V| \leq 0$ holds for planar graphs $G = (V, E)$. Therefore, since the crossing number is always a nonnegative integer, the crossing number is at least $|E| - 3|V|$. For *any* graph $G = (V, E)$, not necessarily planar, we may start with an embedding of G in the plane, which precisely contain $\text{cr}(G)$ crossings of edges. Now, for every crossing, we remove one of the edges that contribute to that crossing so that the resulting graph $G' = (V, E')$, where $|E'| = |E| - \text{cr}(G)$ is planar. Therefore, $|E'| = |E| - \text{cr}(G) \leq 3|V|$. Hence $|E| - 3|V| \leq \text{cr}(G)$. Therefore, the crossing number of any graph is at least $|E| - 3|V|$. □

Before proving the next theorem, we introduce the concept of induced subgraphs.

Definition 2.8 ([10, p. 522]). Let $G = (V, E)$ be a graph. If $\emptyset \neq U \subset V$, the *subgraph of* G *induced by* U is the subgraph whose vertex set is U and which contains all edges from G of the form $\{v_i, v_j\}$ for $v_i, v_j \in U$. We say that a subgraph G' of a graph $G = (V, E)$ is an *induced subgraph* if there exists $\emptyset \neq U \subset V$ such that G' is the subgraph of G induced by U.

In Figure 4, we see an embedding of a graph $G = (V, E)$ with $V = \{v_1, ..., v_7\}$. The subgraph of G induced by $U = \{v_1, v_2, v_4, v_5, v_6\}$ is the graph shown in Figure 5. Note that all edges in the induced subgraph are between vertices in U, while there is no edge in G between vertices U that is not included in the induced subgraph.

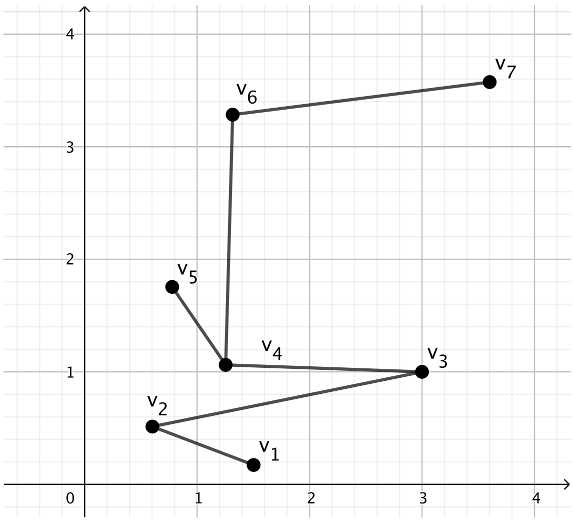

Figure 4: Embedding of a graph $G = (V, E)$ with $V = \{v_1, ..., v_7\}$

Now, the inequality in Lemma 2.7, together with a probabilistic argument, will be used to deduce a more involved lower bound on the crossing number of a graph. This lower bound, stated as an inequality, will be central in our first proof of the Szemerédi–Trotter theorem (Theorem 2.11).

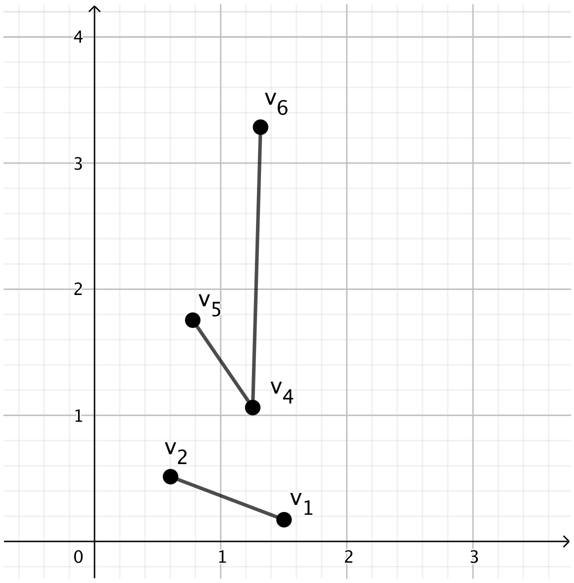

Figure 5: Subgraph of the graph in Figure 4 induced by $U = \{v_1, v_2, v_4, v_5, v_6\}$

Theorem 2.9 (Crossing number inequality [9, p. 285–286])**.** *For any graph $G = (V, E)$ with $|E| \geq 4|V|$, we have*

$$\mathrm{cr}(G) \geq \frac{|E|^3}{64|V|^2}.$$

Proof. Let $G = (V, E)$ be a graph embedded in the plane with $|E| \geq 4|V|$ and $t = \mathrm{cr}(G)$ crossings. We select each vertex from V, randomly and independently, with probability p. Let $H = (V', E')$ be the random induced subgraph of G defined by these randomly selected vertices. The value of p will be determined later.

Note that the expected number of vertices in H is $p|V|$, since each of the $|V|$ vertices is included in H independently with probability p. In addition, the expected number of edges is $p^2|E|$, since an edge of E is included in E' if and only if both vertices incident with this edge are included in V', which has a probability of p^2.

Observe that we already have a given embedding of H in the plane, namely the one induced by the embedding of G. The expected number of crossings in H for this given embedding is p^4t, since a crossing of G is transferred to H if only if we choose both edges that constitute that crossing, which occurs with probability p^4. This implies that the expected value of the crossing number of H, $\mathrm{cr}(H)$, is at most p^4t, since the crossing number of H is the minimum crossing number among *all* embeddings of H and the current embedding of H is merely one of all embeddings.

Since $\mathrm{cr}(H) \geq |E'| - 3|V'|$, by Lemma 2.7, we have

$$\mathbb{E}[\mathrm{cr}(H)] \geq \mathbb{E}[|E'| - 3|V'|] = \mathbb{E}[|E'|] - 3\mathbb{E}[|V'|],$$

using the linearity of expectation. Substituting the expected values computed above yields

$$p^4 t \geq p^2|E| - 3p|V|.$$

Dividing by p^4 on either side of this inequality we get

$$\mathrm{cr}(G) = t \geq \frac{|E|}{p^2} - \frac{3|V|}{p^3}.$$

Now, let $p = 4|V|/|E|$. From the assumption on G this gives $p \leq 1$. The inequality above then becomes

$$\mathrm{cr}(G) \geq \frac{|E|^3}{16|V|^2} - \frac{3|E|^3}{64|V|^2} = \frac{4|E|^3}{64|V|^2} - \frac{3|E|^3}{64|V|^2} = \frac{|E|^3}{64|V|^2}.$$

Hence, we can deduce that

$$\mathrm{cr}(G) \geq \frac{|E|^3}{64|V|^2}.$$

□

2.3 Incidence geometry and the Szemerédi–Trotter theorem

Incidences between points and curves, in particular between points and circles, shall be used later in this subsection to prove that n points in the plane determine at most $O(n^{4/3})$ unit distances. We start by defining the concept of an incidence between a point and a curve.

Definition 2.10 ([11])**.** Let P be a set of n distinct points in $\mathbb{R}^2$, and let Γ be a set of m distinct curves in $\mathbb{R}^2$ from some fixed family of curves. An *incidence* between a point in P and a curve in Γ is an ordered pair (p, γ) with $p \in P$ and $\gamma \in \Gamma$ such that $p \in \gamma$.

For instance, if we let P be a set of n distinct points and L be a set of m distinct lines, an incidence between a point p of P and a line l of L geometrically means that the point p lies on the line l. The following theorem bounds the incidences between points and lines in the plane and was originally proved by Endre Szemerédi and William T. Trotter, but the argument using the crossing number inequality (Theorem 2.9) was developed by László Székely [9, p. 286].

Theorem 2.11 (Szemerédi–Trotter [9, p. 286])**.** *Let P be a set of n distinct points in $\mathbb{R}^2$, and let L be a set of m distinct lines. Then the number of incidences between the elements of P and the elements of L is at most $c(m^{2/3}n^{2/3} + m + n)$, for some absolute constant $c > 0$.*

Proof. We assume that every line $l \in L$ is incident with at least one point $p \in P$. This does not involve a loss of generality, since we are proving an upper bound. Let I denote the number of incidences between the points of P and the lines of L. Let $G = (V, E)$ be the graph with $V = P$ such that any two vertices are connected with an edge if and only if they are consecutive points along some line of L. This will enable us to analyze incidences in terms of the edges of G. Since distinct lines intersect in at most one point, any two vertices are connected by at most one edge, so the resulting graph G is simple.

From the way we have defined G we have $|V| = |P| = n$. Denote the lines of L by l_1,l_2,...,l_m. Let $I(P, l_i)$ denote the number of incidences between the points of P and the line l_i, for $i = 1, 2, ..., m$. By assumption, $I(P, l_i) \geq 1$. Now, the line l_i contributes $I(P, l_i) - 1$ edges in the graph G, because we need consecutive points of P along l_i for an edge to form. Hence, all, except for one, incidence with l_i counts as an edge. So we have

$$|E| = \sum_{i=1}^{m}(I(P, l_i) - 1) = I - m,$$

since $\sum_{i=1}^{m} I(P, l_i) = I$.

Observe that we already have an embedding of G in the plane, where the edges are represented by line segments of the lines in L, so every crossing in this embedding of G comes from two elements of L intersecting. Since any two lines of L intersect in at most one point, this implies that $\mathrm{cr}(G) \leq \binom{m}{2} = \frac{m(m-1)}{2} \leq \frac{m^2}{2}$. Now, we are either in the case $I - m = |E| < 4|V| = 4n$, in which case $I < 4n + m$, or $|E| \geq 4|V|$. In the latter case, we can use the crossing number inequality (Theorem 2.9), together with our upper bound of the crossing number of G, to deduce that

$$\frac{(I - m)^3}{64n^2} \leq \mathrm{cr}(G) \leq \frac{m^2}{2},$$

and, hence,

$$(I-m)^3 \leq 32m^2n^2$$
$$\iff I-m \leq 32^{1/3}m^{2/3}n^{2/3}$$
$$\iff I \leq 32^{1/3}m^{2/3}n^{2/3} + m.$$

Now, $32^{1/3} < 4$, so in both of the above cases $I \leq 4(m^{2/3}n^{2/3} + m + n)$. This completes the proof. □

Reasoning in a similar way to that above, one can conclude that the maximum number of incidences between a set of n distinct points and a set of m distinct unit circles in the plane is also $O(m^{2/3}n^{2/3} + m + n)$. This upper bound on point-circle incidences shall, in Theorem 2.16, be used to deduce the current best bound on the number of unit distances among n points in the plane. We state this as a separate theorem, where again the argument in the proof is due to Székely.

Theorem 2.12 ([7, p. 465])**.** *There is an absolute constant $c > 0$ such that the number of incidences between the elements of a set P of n distinct points and the elements of a set Γ of m distinct unit circles in the plane is at most $c(m^{2/3}n^{2/3} + m + n)$.*

Proof. Let $P = \{p_1, ..., p_n\}$ be a set of n distinct points in the plane and let $\Gamma = \{\gamma_1, ..., \gamma_m\}$ be a set of m distinct unit circles in the plane. Let I denote the number of incidences between the elements of P and Γ. We now construct a graph $G = (V, E)$, so that $V = P$ is the vertex set, in the following way:

For each pair of distinct points $\{p_i, p_j\}$ with $p_i, p_j \in P$ that are consecutive along the circle $\gamma_k \in \Gamma$, for some k, let $\{p_i, p_j\}$ be an edge in E. The edge $\{p_i, p_j\}$ is represented in the plane by the arc of γ_k connecting p_i and p_j that passes through no other points of P than p_i and p_j. If the same two points are consecutive along multiple circles, we add one edge per circle. Hence, this procedure may result in multiple edges between the same two points. For the circles incident to at most two points of P, we do not perform this step, hence these circles do not contribute to any edges in G.

Let $I(P, \gamma_k)$ denote the number of incidences between P and the circle γ_k for $k = 1, 2, ..., m$. If $I(P, \gamma_k) \geq 3$ we form $I(P, \gamma_k)$ edges along the circle γ_k. If $I(P, \gamma_k) \leq 2$ we do not form any edges along the circle γ_k, as mentioned above. Therefore, at most $2m$ incidences between the points and the circles do not form any edges. So we have $|E| \geq I - 2m$.

The graph G is now a multigraph, since we may have multiple edges between the same two vertices. This is because we can have two circles γ_i and γ_j, with $i \neq j$, such that they both have an arc connecting the same two points. If this is the case, we delete one of these arcs, so that we are left with a simple graph with unique edges connecting the same points. Denote this new graph $G' = (V, E')$.

Since the circles in Γ are distinct, any two circles in Γ intersect in at most two points. Since the circles also all have the same radius, the construction above can produce at most two distinct edges between any fixed pair of vertices. At most all points are connected by two different edges in G so we get $|E'| \geq (I - 2m)/2$. Now, every crossing in this embedding of G' comes from two elements of Γ that intersect. This implies that $\mathrm{cr}(G') \leq 2\binom{m}{2} = m(m-1) \leq m^2$, since any two distinct circles intersect in at most two points in the plane.

We are either in the case $|E'| < 4n$ or $|E'| \geq 4n = 4|V|$. In the latter case, we can apply the crossing number inequality (Theorem 2.9) to obtain

$$\frac{((I-2m)/2)^3}{64n^2} \leq \mathrm{cr}(G') \leq m^2$$

from which we can deduce that

$$((I-2m)/2)^3 \leq 64m^2n^2$$

$$\iff I - 2m \leq 2 \cdot 4m^{2/3}n^{2/3}$$

$$\iff I \leq 8m^{2/3}n^{2/3} + 2m.$$

Now, if $|E'| < 4n$ we have $I - 2m < 8n$, so that $I < 8n + 2m$. Hence, in both cases, we have $I \leq 8(m^{2/3}n^{2/3} + m + n)$. The theorem follows.

□

In fact, these results can be generalized even further. We start with a definition. Recall that a simple curve is one that does not cross itself.

Definition 2.13 ([12])**.** Let Γ be a given family of simple curves in $\mathbb{R}^2$. That Γ has k *degrees of freedom* and *multiplicity type* s means that

i) for any k points, a maximum of s curves of Γ passes through all of them and k is minimal with this property, and

ii) any pair of curves of Γ intersect in at most s points.

For example, the family of all unit circles has 2 degrees of freedom and multiplicity type 2. The family of all straight lines has 2 degrees of freedom and multiplicity type 1. The proof of the following theorem requires more theory than that included in this thesis and is therefore omitted here. For example, a more general version of the crossing number inequality (Theorem 2.9) will be needed, that applies also to multigraphs. For a complete proof, see [12].

Theorem 2.14. *Let P be a set of n points and let Γ be a family of m simple curves all lying in the plane. Assume Γ has k degrees of freedom and multiplicity type s. Then*

$$I(P,\Gamma) \leq c(k,s)(n^{k/(2k-1)}m^{(2k-2)/(2k-1)} + n + m),$$

where $I(P,\Gamma)$ is the number of incidences between the points and the curves and $c(k,s)$ is a constant depending only on k and s.

The cases of straight lines and unit circles are both special cases of the above theorem. As mentioned, for straight lines we have $k = 2$ and $s = 1$. Hence $I(P,\Gamma) \leq c(2,1)(n^{2/3}m^{2/3}+n+m)$, which is in accordance with Theorem 2.11. When Γ consists of unit circles, we have $k = s = 2$ and therefore $I(P,\Gamma) \leq c(2,2)(n^{2/3}m^{2/3} + n + m)$, which is precisely the bound that we proved in Theorem 2.12. Note that the multiplicity type s only affects the constant $c(k,s)$.

2.4 The $O(n^{4/3})$ upper bound for unit distances

Now we state a Lemma which will be useful in deducing the current best bound of $O(n^{4/3})$ on the number of unit distances determined by n points in the plane. The lemma is crucial for connecting incidence geometry to the number of unit distances determined by a set of points in the plane.

Lemma 2.15. *Let $P = \{p_1, p_2, ..., p_n\}$ be a set of n points in the plane. The number of unit distances determined by the points in P is $\frac{1}{2}I(P,C)$, where $C = \{c_1, c_2, ..., c_n\}$ is a set of n unit circles, where c_i is centered at p_i, for $i = 1, 2, ..., n$.*

Proof. Let $P = \{p_1, p_2, ..., p_n\}$ be a set of n points and $C = \{c_1, c_2, ..., c_n\}$ be a set of n unit circles, where c_i is centered at p_i for $i = 1, 2, ..., n$. A pair of points $\{p_i, p_j\}$ from P are a unit distance apart precisely when one of the points, say p_i is on the unit circle centered at p_j, also implying that p_j is on the unit circle centered at p_i. Hence, each unit distance pair corresponds to two point-circle incidences between the points of P and the circles of C. Conversely, the incidences between P and C come in pairs that correspond to two points of P that are a unit distance apart. In other words, the number of unit distances determined by the points in P is $\frac{1}{2}I(P, C)$. □

We now apply the bounds on incidences between curves and points to the unit distance problem and deduce the following theorem:

Theorem 2.16. *The number of unit distances determined by a set of n points in the plane is at most $O(n^{4/3})$.*

Proof. By Lemma 2.15, the maximum number of unit distances determined by a set of n points is bounded up to a constant factor by the maximum possible number of incidences between a set of n points and a set of n unit circles in the plane. So, if we let $m = n$ in Theorem 2.12, it follows that the number of unit distances determined by a set of n points in the plane is at most $O(n^{4/3})$. □

This upper bound, of course, beats the trivial upper bound (Theorem 2.3). In particular, a complete unit distance graph of a set of n points does not exist for large n. The bound in Theorem 2.16 is the best known upper bound for the Erdős unit distance problem, up to the value of the implied constant.

3 Polynomial methods

In recent years, a lot of progress has been made in incidence geometry, due to techniques using polynomials. Among other things, the polynomial method has allowed us to formulate new, simplified proofs of classical theorems. Examples are provided in [13]. For this section, we will discuss polynomials and introduce the concept of polynomial partitioning to provide an alternative proof of the Szemerédi–Trotter theorem (Theorem 2.11) that does not rely on graph-theoretic methods. The proof will be given for the special case where the number of points and the number of lines are the same. We will mostly consider bivariate polynomials, that is polynomials f on the form $f(x,y) = \sum_{i,j} a_{ij}x^iy^j \in \mathbb{R}[x,y]$. First, we introduce some notations and lemmas regarding such polynomials.

3.1 Bivariate polynomials and lines in the plane

Definition 3.1 ([14]). The *degree* of a nonzero polynomial $f(x,y) = \sum_{i,j} a_{ij}x^iy^j \in \mathbb{R}[x,y]$ is defined as $\deg(f) = \max\{i+j : a_{ij} \neq 0\}$. The *zero set* of $f(x,y)$ is defined as $Z(f) = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\}$.

For example, if $f(x,y) = 3x^4y^3$ and $g(x,y) = 10x^{11}+4y^2+7xy$, we have $\deg(f) = 7$ and $\deg(g) = 11$. When $f(x,y) \equiv 0$ we apply the convention that $\deg(f) = -\infty$. Now, we state two lemmas connecting the degree and zero set of a bivariate polynomial to the concept of lines in the plane.

Lemma 3.2 ([15]). *Let l be a line in $\mathbb{R}^2$ and $f \in \mathbb{R}[x,y]$ be such that $\deg(f) \leq D$. Then either $l \subset Z(f)$ or $|l \cap Z(f)| \leq D$.*

Proof. We start by writing l in parametric form as $l = \{(u_1t+v_1, u_2t+v_2) : t \in \mathbb{R}\}$, where $u_i, v_i \in \mathbb{R}$. Then we find that the points of $l \cap Z(f)$ correspond to roots of the univariate polynomial $g(t) = f(u_1t+v_1, u_2t+v_2)$. The degree of $g(t)$ is at most D. Hence g is either identically zero, in which case $l \subset Z(f)$, or it has at most D roots, in which case $|l \cap Z(f)| \leq D$. □

Lemma 3.3 ([15]). *If $f(x,y) \in \mathbb{R}[x,y]$ is not identically zero and $\deg(f) \leq D$, then $Z(f)$ contains at most D distinct lines.*

Proof. Since f is a bivariate polynomial, which is nonzero, f cannot vanish on all of $\mathbb{R}^2$. Fix a point $p \in \mathbb{R}^2$ not in $Z(f)$. Assume $Z(f)$ contains the distinct lines l_1, l_2,...,l_k. Let l be another line that passes through p that is not parallel to any of the lines l_1, l_2,...,l_k and does not pass through any intersection of the form $l_i \cap l_j$ for $1 \leq i < j \leq k$. We know such a line l exists since only a finite number of directions and intersections need to be avoided.

So l is not contained in $Z(f)$, since $p \notin Z(f)$, and l has k intersections with $\bigcup_{i=1}^{k} l_i$, which is precisely when l intersects each of the lines l_i. Hence, by Lemma 3.2 we have $k \leq D$. □

For the same function $f(x, y) = 3x^4y^3$ as above, the line $x = 0$ and $y = 0$, i.e the y-axis and the x-axis, constitute $Z(f)$ while $2 \leq 7 = \deg(f)$, which is in accordance with Lemma 3.3. For $g(x, y) = 10x^{11} + 4y^2 + 7xy$, as defined above, we know that $Z(g)$ contains at most 11 distinct lines, by Lemma 3.3, since $\deg(g) = 11$.

3.2 Polynomial ham sandwich theorem

In order to be able to provide the alternative proof of the Szemerédi–Trotter theorem (Theorem 2.11), mentioned at the beginning of this section, we shall in this subsection look closer into the connection between polynomials and point sets. We start with the definition of a hyperplane.

Definition 3.4. Let $\vec{a}$ be a nonzero vector in $\mathbb{R}^d$. The set $h = \{\vec{x} \in \mathbb{R}^d : \langle \vec{a}, \vec{x} \rangle = c\}$, where c is some real constant, is a *hyperplane* in $\mathbb{R}^d$. In other words, a hyperplane in $\mathbb{R}^d$ is a $(d-1)$-dimensional affine subspace.

Recall that $\langle \cdot, \cdot \rangle$ denotes the scalar product. By this definition, for any nonzero vector $\vec{a} \in \mathbb{R}^d$ and any real constant c, we can define the hyperplane $h = \{\vec{x} \in \mathbb{R}^d : \langle \vec{a}, \vec{x} \rangle = c\}$. For example, in the ambient vector space $\mathbb{R}^3$, let a vector be represented as $\vec{x} = (x_1, x_2, x_3)$ and let $\vec{a} = (1, 2, 3)$. Let $c = 1$. Then $h = \{\vec{x} \in \mathbb{R}^3 : x_1 + 2x_2 + 3x_3 = 1\}$ is a hyperplane in $\mathbb{R}^3$.

The following theorem is proved in full detail in [16, p. 48-49]. However, the proof requires more theory than what is introduced here and is therefore excluded in this thesis.

Theorem 3.5 (Ham sandwich theorem for point sets)**.** *Let $A_1, A_2, ..., A_d$ be finite sets of points of $\mathbb{R}^d$. Then there exists a hyperplane h that simultaneously bisects $A_1, A_2, ..., A_d$, meaning that each of the open half-spaces defined by h contains at most $\lfloor |A_i|/2 \rfloor$ points of A_i, for $i = 1, 2, ..., d$.*

Note that if $|A_i| = 2k+1$ for some integer $k \geq 0$, i.e. A_i has odd cardinality, then $\lfloor |A_i|/2 \rfloor = k$. Hence, each of the open half-spaces contains no more than k points of A_i and, therefore, at least one point of A_i must be on the bisecting hyperplane. In particular, all point sets A_i of cardinality 1 will lie on the hyperplane.

An illustration of the ham sandwich theorem for point sets in $\mathbb{R}^2$ is provided in Figure 6. In this example, A_1 and A_2 are two finite point sets, consisting of the red and blue points, respectively. In $\mathbb{R}^2$ a hyperplane is a line, and for A_1 and A_2 in the figure, the black line h is an example of a hyperplane that simultaneously bisects A_1 and A_2. Observe that $|A_1| = 7$ and one of the red points is on the hyperplane h, as expected. Also $\lfloor |A_1|/2 \rfloor = 3$ and both open half-spaces defined by h contains three points of A_1. Furthermore $|A_2| = 12$, so $\lfloor |A_2|/2 \rfloor = 6$, and each open half-space contains six points of A_2.

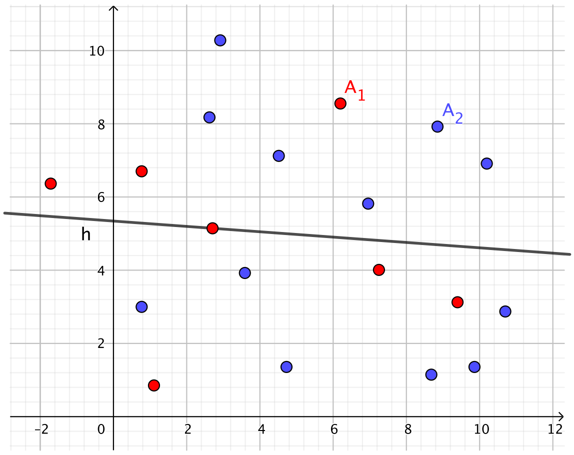

Figure 6: An illustration of Theorem 3.5 in $\mathbb{R}^2$

The ham sandwich theorem was given its name by a specific interpretation when $d = 3$. The three objects being simultaneously bisected are the ingredients of a ham sandwich. Informally, the statement in Theorem 3.5 becomes: For every sandwich made of two slices of bread and a slice of ham, there exists a planar cut that simultaneously halves the bread and the ham [16, p. 47].

We now state and prove the polynomial ham sandwich theorem, in the case of bivariate polynomials.

Theorem 3.6 (Polynomial ham sandwich theorem [15]). *Let $A_1, A_2, ..., A_s \subset \mathbb{R}^2$ be finite sets. Let D be an integer such that $\binom{D+2}{2} - 1 \geq s$. Then there exists a polynomial $f(x, y) \in \mathbb{R}[x, y]$, which is not identically zero, with $\deg(f) \leq D$ such that for every $i = 1, 2, ..., s$ we have $f > 0$ on at most $\lfloor |A_i|/2 \rfloor$ points of A_i and $f < 0$ on at most $\lfloor |A_i|/2 \rfloor$ points of A_i. We say that f simultaneously bisects the sets A_i.*

Proof. First, we note that $\binom{D+2}{2}$ is the number of monomials $x^i y^j$ with $i + j \leq D$. This can be seen by the fact that $\binom{D+2}{2}$ counts the number of integer solutions to $i + j + t = D$ with $i, j, t \geq 0$, which is a standard combinatorial result obtained using stars and bars. See [10, p. 29] for a more in-depth explanation. We then let i represent the exponent of x, j represent the exponent of y and t account for the remaining degree for monomials of degree strictly lower than D. Thus, we can interpret the binomial coefficient as counting pairs (i, j) where $i+j \leq D$ and $i, j \geq 0$, where (i, j) correspond to the monomial $x^i y^j$.

Let $k = \binom{D+2}{2} - 1$. Now, we use $V : \mathbb{R}^2 \to \mathbb{R}^k$, denoting the *Veronese map*, defined by

$$V(x, y) = (x^i y^j)_{(i,j):1 \leq i+j \leq D} \in \mathbb{R}^k.$$

We interpret the above expression as the coordinates of $\mathbb{R}^k$ being indexed by the distinct pairs (i, j) for $1 \leq i + j \leq D$, which is in agreement with how we have defined k. The order among the pairs (i, j) is fixed but arbitrary.

Assume that $s = k$. Set $A'_i = V(A_i)$ for $i = 1, 2, ..., k$ and let h be a hyperplane simultaneously bisecting the sets $A'_1, A'_2, ..., A'_k$. This hyperplane h exists according to the ham sandwich theorem for point sets (Theorem 3.5). Due to the way a hyperplane is defined, h can be represented by an equation of the form $a_{00} + \sum_{i,j} a_{ij} z_{ij} = 0$ where we let $(z_{ij})_{(i,j):1 \leq i+j \leq D}$ represent the coordinates in $\mathbb{R}^k$ fixed in the same order as above. Let $f(x, y) = a_{00} + \sum_{i,j} a_{ij} x^i y^j$. Observe that the degree of f is at most D.

Now, let $p = (x_0, y_0)$ be any point of $\mathbb{R}^2$. Letting $\vec{a} = (a_{ij})_{(i,j):1 \leq i+j \leq D}$ and $\vec{p} = V(x_0, y_0) = (x_0^i y_0^j)_{(i,j):1 \leq i+j \leq D}$ be vectors in $\mathbb{R}^k$, we can write $f(x_0, y_0) = a_{00} + \langle \vec{a}, \vec{p} \rangle$, where we take the scalar product of the vectors. Since h is a hyperplane, $\vec{a}$ is nonzero. Therefore, when writing f this way, we see that f is nonzero, since f has at least one nonzero coefficient. Because h is given by $a_{00} + \sum_{i,j} a_{ij} z_{ij} = 0$, our function f is zero precisely on the points whose image belong to the hyperplane. Thus f is of constant sign on either side of h. In particular, $f(p) > 0$ if $\vec{p} = V(p)$ belongs to one side of h and $f(p) < 0$ if $\vec{p} = V(p)$ belongs to the other side of h.

Each point $p \in A_i$ corresponds to $V(p) \in A'_i$. By the argument above, the sign of $f(p)$ is determined by which side of the hyperplane $V(p)$ belongs to. We know that h bisects A'_i for each i, meaning that neither of the two open half-spaces bounded by h contains more than $\lfloor |A'_i|/2 \rfloor$ points of A'_i. So the points $f(p)$ for each $p \in A_i$ are bisected, with constant sign on each side. Hence f bisects the sets A_i.

Since the conclusion holds for $s = k$ sets, it also holds for $s < k$ sets. Hence, the theorem is proved. □

Similarly as in the ham sandwich theorem for point sets (Theorem 3.5), we have that if $|A_i| = 2k + 1$ for some nonnegative integer k, then at least one point of A_i must be contained in the zero set of the function f. Furthermore, all point sets A_i consisting of only one point will be contained in the zero set.

In Figure 7, we have three finite sets of points in the plane: A_1, consisting of the red points, A_2 consisting of the blue points and A_3 consisting of the pink points. Hence $s = 3$, where s is defined as in the polynomial ham sandwich theorem (Theorem 3.6). If $D = 2$ we have $\binom{D+2}{2} - 1 = \binom{4}{2} - 1 = 5 \geq s$. Therefore, there exists a polynomial $f(x, y) \in \mathbb{R}[x, y]$ with $\deg(f) \leq 2$ such that f simultaneously bisects A_1, A_2 and A_3, by Theorem 3.6.

In Figure 7 we see that $f(x, y) = x - y$ satisfies this property. As illustrated, $Z(f)$ is the line $y = x$ and $f > 0$ where $x > y$ and $f < 0$ where $x < y$. Note that $\deg(f) = 1 \leq D$. Counting the number of red, blue and pink points in Figure 7 we see that $|A_1| = 5$, $|A_2| = 8$ and $|A_3| = 2$. Hence $\lfloor |A_1|/2 \rfloor = \lfloor 5/2 \rfloor = 2$ and $\lfloor |A_2|/2 \rfloor = \lfloor 8/2 \rfloor = 4$. In addition, $\lfloor |A_3|/2 \rfloor = \lfloor 2/2 \rfloor = 1$.

From Figure 7 we can conclude that $f > 0$ on two points of A_1, on four points of A_2 and on one point of A_3. Furthermore, $f < 0$ on two points of A_1, on three points of A_2 and on one point of A_3. Hence, $f(x, y)$ simultaneously bisects the sets A_1, A_2 and A_3 and for this specific example we have thus verified the existence of such polynomial as claimed by the polynomial ham sandwich theorem (Theorem 3.6). Note that, in this example, the ham sandwich theorem (Theorem 3.5) does not provide any information, since we have three point sets in $\mathbb{R}^2$ and, therefore, the theorem does not apply.

3.3 Polynomial partitioning

Now, we introduce the concept of an r-partitioning polynomial, which we will use in the alternative proof of the Szemerédi–Trotter theorem (Theorem 2.11).

Definition 3.7 ([15])**.** Let P be a set of n points in the plane and r be a parameter such that $1 < r \leq n$. The polynomial $f(x, y) \in R[x, y]$ is said to be an *r-partitioning polynomial* for P if no connected component of $\mathbb{R}^2 \backslash Z(f)$ contains more than n/r points of P. We use the term *cell* to refer to a connected component of $\mathbb{R}^2 \backslash Z(f)$.

Note that the cells are open sets and that points belonging to $Z(f)$ are not contained in any cell.

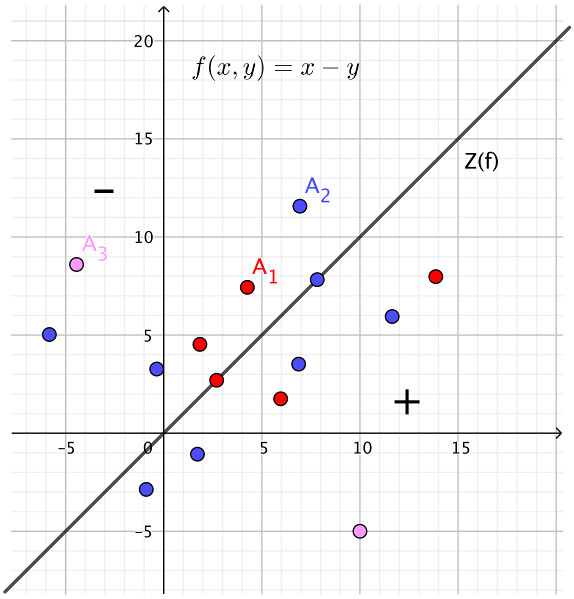

Figure 7: An illustration of Theorem 3.6

Theorem 3.8 (Polynomial partitioning theorem [15])**.** *Every finite point set $P \subset \mathbb{R}^2$ admits an r-partitioning polynomial $f(x, y) \in R[x, y]$, for every r satisfying $1 < r \leq |P|$, with $\deg(f) = O(\sqrt{r})$.*

Proof. We will inductively construct collections of disjoint subsets of P. We denote these collections by $\mathcal{P}_0$, $\mathcal{P}_1$,..., and construct them such that $|\mathcal{P}_j| \leq 2^j$ for each j. First, we define $\mathcal{P}_0 = \{P\}$. Assume we have constructed $\mathcal{P}_j$ for some $j \geq 0$, with $|\mathcal{P}_j| \leq 2^j$. We use the polynomial ham sandwich theorem (Theorem 3.6) to construct a polynomial f_j that bisects each of the sets of $\mathcal{P}_j$.

We can choose f_j so that $\deg(f_j) \leq D = \lceil\sqrt{2^{j+1}}\rceil$. To see this, we note that we must have $\binom{D+2}{2} - 1 \geq s$, according to Theorem 3.6, where D and s are as in the theorem. Note that $|\mathcal{P}_j| \leq 2^j$ so that $s \leq 2^j$. Hence, the inequality is satisfied if D is chosen such that $\binom{D+2}{2} - 1 \geq 2^j$. Furthermore, $(D+2)^2/2 \geq \binom{D+2}{2} - 1$. Hence, if D is chosen such that

$$\frac{(D+2)^2}{2} \geq 2^j$$

the hypotheses of Theorem 3.6 are satisfied. So, letting $D = \lceil\sqrt{2^{j+1}}\rceil$, we see that there exists a bisecting polynomial f_j with $\deg(f_j) \leq \lceil\sqrt{2^{j+1}}\rceil$.

Then, for every subset $Q \in \mathcal{P}_j$, let Q^+ consist of the points of Q for which $f_j > 0$.

Similarly, let Q^- consist of the points of Q for which $f_j < 0$. Put

$$\mathcal{P}_{j+1} = \bigcup_{Q \in \mathcal{P}_j} \{Q^+, Q^-\}.$$

This way $\mathcal{P}_{j+1}$ consists of at most $2 \cdot 2^j = 2^{j+1}$ subsets of P. Also, since we successively bisect sets, the collections $\mathcal{P}_0$, $\mathcal{P}_1$,..., will consist of disjoint subsets of P.

Constructing $\mathcal{P}_j$ this way, it follows that each of the sets in $\mathcal{P}_j$ has size at most $|P|/2^j$. Let r be such that $1 < r \leq |P|$ and let $t = \lceil \log_2(r) \rceil$. Then each set of $\mathcal{P}_t$ has size at most $|P|/r$. Define a function f by $f = f_1 f_2 \cdots f_t$.

No cell from $\mathbb{R}^2 \backslash Z(f)$ can contain points from two different sets of $\mathcal{P}_t$. That is because if two points belong to different sets of $\mathcal{P}_t$, then one of the polynomials f_j must vanish somewhere along any arc connecting these two points. So f also vanishes somewhere along this arc, since f_j is a factor in f. So, the arc crosses $Z(f)$ which means that the points does not belong to the same cell. Therefore, f is an r-partitioning polynomial for P.

It remains to bound the degree of f. We have

$$\deg(f) = \deg(f_1) + \deg(f_2) + ... + \deg(f_t) \leq \sum_{j=1}^{t} \lceil \sqrt{2^{j+1}} \rceil \leq \sum_{j=1}^{t} (\sqrt{2^{j+1}} + 1).$$

Simplifying this even more we obtain

$$\deg(f) \leq t + \sqrt{2} \sum_{j=1}^{t} 2^{j/2} \leq t + \frac{2}{\sqrt{2} - 1} 2^{t/2} \leq c\sqrt{r},$$

for c a large enough constant. Therefore, $\deg(f) = O(\sqrt{r})$. □

3.4 Alternative proof of the Szemerédi–Trotter theorem

We now state and prove the final lemma needed for the alternative proof of the Szemerédi–Trotter theorem (Theorem 2.11) using the method of polynomial partitioning. The lemma gives an upper bound on the number of incidences between points and lines in the plane, different from that provided in the Szemerédi–Trotter theorem.

Lemma 3.9 ([15]). *Let P be a set of n distinct points and L be a set of m distinct lines, both in the Euclidean plane. Then the number of incidences between the points of P and the lines of L is bounded by $m + n^2$.*

Proof. First, we divide the lines of L into two disjoint subsets: Let L' contain the lines that are incident to at most one point of P and L'' contain the lines that are incident to at least two points. Let $I(P, L)$ denote the number of incidences between the elements of P and the elements of L and similarly for L' and L''. Since the lines of L' are incident to at most one point of P the total number of incidences between the points in P and the lines in L' cannot exceed $|L'|$. Therefore, $I(P, L') \leq |L'| \leq m$.

Now, a point $p \in P$ has a maximum of $n - 1$ incidences with the lines of L'' since there are at most $n - 1$ lines passing through p and some other point of P. This is due to the fact that two distinct points uniquely determine a line. Hence $I(P, L'') \leq n(n-1) \leq n^2$. Putting these inequalities together we obtain $I(P, L) = I(P, L') + I(P, L'') \leq m + n^2$. □

We now use the theory introduced in this section to formulate a proof of the Szemerédi–Trotter theorem (Theorem 2.11) for the special case where the number of points and lines are the same, so that $m = n$. The proof is inspired by that given in [15]. What we prove is:

Let P be a set of n distinct points in $\mathbb{R}^2$, and let L be a set of n distinct lines. Then the number of incidences between the elements of P and the elements of L is at most $c(n^{4/3})$, for some absolute constant $c > 0$.

Alternative proof of Theorem 2.11. Let P be a set of n distinct points and let L be a set of n distinct lines in the plane. Define $r = n^{2/3}$. Let f be the r-partitioning polynomial for P that exist by the polynomial partitioning theorem (Theorem 3.8). Then we have $\deg(f) = O(\sqrt{r}) = O(n^{1/3})$. Let $D = \deg(f)$. Label the cells of $\mathbb{R}^2 \backslash Z(f)$ as $C_1, C_2, ..., C_s$. Define $P_0 = P \cap Z(f)$. Let $P_i = P \cap C_i$, for $i = 1, 2, ..., s$, so that the points of P_i are the points of P contained in the cell C_i. Since f is an r-partitioning polynomial we have $|P_i| \leq n/r = n^{1/3}$, for $i = 1, 2, ..., s$. We denote by L_0 the subset of lines of L that are contained in $Z(f)$. Therefore $|L_0| \leq D$, due to Lemma 3.3.

We write $I(P, L)$ as

$$I(P, L) = I(P_0, L_0) + I(P_0, L \backslash L_0) + \sum_{i=1}^{s} I(P_i, L),$$

by decomposing the incidences into contributions from different disjoint subsets.

The first term on the right-hand side can be bounded by

$$I(P_0, L_0) \leq |L_0| \cdot |P_0| \leq Dn = O(n^{4/3}),$$

since $D = O(n^{1/3})$. For the lines in $L \backslash L_0$ we apply Lemma 3.2. Since the lines in L_0 are precisely the lines that are fully contained in $Z(f)$, each line of $L \backslash L_0$ intersects $Z(f)$ and hence also P_0 in at most D points. Since $|L| = n$, we have $|L \backslash L_0| \leq n$. Therefore,

$$I(P_0, L \backslash L_0) \leq Dn = O(n^{4/3}).$$

In order to bound $\sum_{i=1}^{s} I(P_i, L)$ we begin by letting $L_i \subset L$ be the set of lines containing at least one point of P_i for each $i = 1, 2, ..., s$. Observe that the sets L_i are not necessarily disjoint. By Lemma 3.9 we have

$$\sum_{i=1}^{s} I(P_i, L) = \sum_{i=1}^{s} I(P_i, L_i) \leq \sum_{i=1}^{s} (|L_i| + |P_i|^2) = \sum_{i=1}^{s} |L_i| + \sum_{i=1}^{s} |P_i|^2.$$

If a point of P_i is incident to a line $l \in L$, that means that l passes through the cell containing P_i, for $i = 1, 2, ..., s$. Due to how we defined P_i, we have that P_i, for all $i = 1, 2, ..., s$, belong to different cells of $\mathbb{R}^2 / Z(f)$.

By Lemma 3.2 it follows that the line l can cross $Z(f)$ a maximum of D times. This means that l can have points in a maximum of $D + 1$ different cells. Thus, l intersects at most $D + 1$ of the sets P_i and hence can be contained in L_i for at most $D + 1$ distinct indices i. Therefore, we can deduce that

$$\sum_{i=1}^{s} |L_i| \leq (D + 1)n = O(n^{4/3}).$$

Furthermore, we have that

$$\sum_{i=1}^{s} |P_i|^2 \leq (\max_{1 \leq i \leq s} |P_i|) \cdot \sum_{i=1}^{s} |P_i| \leq \frac{n}{r} \cdot n = n^{4/3} = O(n^{4/3}).$$

Combining the preceding results, we have $I(P, L) = O(n^{4/3})$, which finishes the proof.

$\square$

In conclusion, we have now proved the Szemerédi–Trotter theorem (Theorem 2.11) using a technique that does not require graphs and crossing numbers. Instead, we have used theory about polynomials, and, in particular, *polynomial partitioning*. This illustrates the complexity and the many different approaches there are to the study of incidences in the plane, and, as explained in the proof of Theorem 2.16, incidences in the plane is closely related to the Erdős unit distance problem.

The general case $n \neq m$ follows by a similar argument.

4 Special cases

For this section, we look at the maximum number of unit distances among n points in the plane, when $n = 2$, $n = 3$ and $n = 4$. We also determine the number of unit distances among n points in the plane when the points are in what we shall refer to as a triangular pattern and a hexagonal pattern, respectively. In addition, we prove an upper bound for the number of unit distances when our n points are only allowed to be taken from two given perpendicular lines.

We shall also let our point set be the $\sqrt{n} \times \sqrt{n}$ grid, with a carefully chosen step size in order to prove that such a configuration contains at least $n^{1+c/\log\log n}$ unit distances, when n is sufficiently large, for some absolute constant $c > 0$. Later, we discuss the concept of nested point sets and study the number of unit distances determined by n points in the plane in terms of linear growth in n.

This section is meant to illustrate the sophistication of the Erdős unit distance problem and the many different ways in which one can approach the problem. Another purpose of this section is to create a deeper understanding of the behavior of the number of unit distances among n points in the plane, when we have certain restrictions on these points.

4.1 Specific numbers of points

Let us now confine our attention to specific values of n, the size of our point set. When $n = 2$ we can only have one pair of points that are a unit distance apart, and this is achieved simply by choosing any two points in the plane that are a unit distance from each other. When $n = 3$ we can choose our points from the corners of an equilateral triangle in the plane, whose sides have length 1, and obtain three pairs of unit distances. We can view this triangle as an embedding of a unit distance graph of a point set consisting of three points which are all one unit distance from each other. Since $\binom{3}{2} = 3$ and we have precisely three pairs of unit distances, one for every side of the triangle, the trivial upper bound (Theorem 2.3) tells us that we have achieved the maximum number of pairs of points at unit distance from each other. We state this as a theorem.

Theorem 4.1. *Given a set P of points in the plane, with $|P| = 3$, a maximum of three pairs of these points are a unit distance apart and this is achieved only if the points of P are vertices in an equilateral triangle with side length* 1*. In other words* $u(3) = 3$.

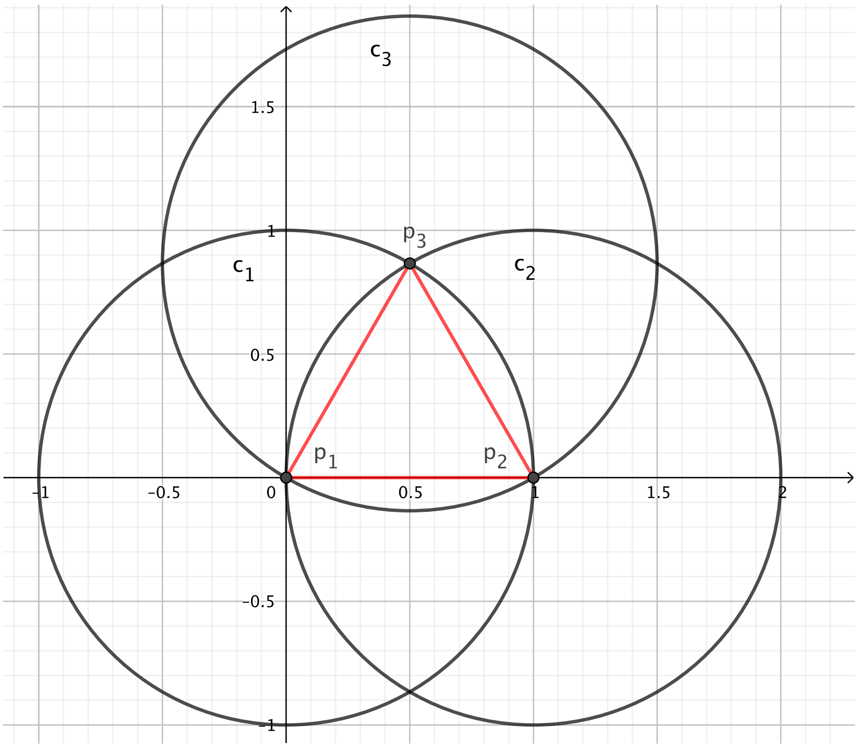

Figure 8: Triangle configuration for $n = 3$

Proof. We have already seen above that we get three pairs of unit distances from the triangular configuration and that this is the maximum, so $u(3) = 3$. Thus, it remains to prove that this is the only configuration for which we have three pairs of unit distances. Hence, assume that we have a point set P, with $|P| = 3$, containing three pairs of points at unit distance from each other, i.e. all pairs of points are at unit distance from each other. Let p_1 be any of these points. Draw a unit circle in the plane centered at p_1, as shown in Figure 8.

Let p_2 be another point of P. Since p_2 and p_1 are a unit distance apart from each other, p_2 is on the circle centered at p_1. Draw a unit circle in the plane centered at p_2. Now, p_1 is also on the unit circle centered at p_2. The unit circles centered at p_1 and p_2 intersect precisely in two points in the plane. In order for the last point p_3 of P to be a unit distance apart from both p_1 and p_2 it must be both on the circle centered at p_1 and on the circle centered at p_2. Hence, p_3 must be one of these two intersection points. Regardless of which of these two points p_3 is, the points of P are vertices in an equilateral triangle with side length equal to 1, as seen in red in Figure 8. □

Now, for the case where $n = 4$, the trivial upper bound (Theorem 2.3) cannot be achieved. If so was the case, all four points would be at unit distance from each other. Assume for contradiction that we have a set $P = \{p_1, p_2, p_3, p_4\}$, such that all points are at unit distance from each other. In particular, this means that if we choose any three points of P, all of these are at unit distance from each other. We can, for instance, look at the points p_1, p_2 and p_3. We know from Theorem 4.1 that the only way three points can be pairwise at unit distance from each other is if these points are the vertices in an equilateral triangle with side length 1. Hence, this holds for p_1, p_2 and p_3. See triangle $p_1p_3p_2$ in Figure 9.

Now, by the same argument as above, p_2, p_3 and p_4 also constitute the vertices of an equilateral triangle. But we see in Figure 9 that in order for these points to be the vertices of an equilateral triangle, the side p_2p_3 must be a side in that triangle, since this side is precisely where p_2 and p_3 are at unit distance from each other. But since our triangle is equilateral, this leaves only two options for our point p_4; either the same point as p_1 or the reflection of p_1 in the line through p_2 and p_3. Since our points are distinct by definition, p_4 must be as in Figure 9.

Using Figure 9, we conclude that p_1 and p_4 are not at unit distance from each other, simply by comparing the length of the black line segment between p_1 and p_4 with the length of the sides of the equilateral triangles. Since the steps leading up to this were forced by maximality, we may conclude that the maximum possible number of unit distances between four points in the plane,

$$\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6,$$

is not attainable. However, what we have found is a configuration that determines five unit distances, between all pairs of points other than $\{p_1, p_4\}$. Therefore, we may conclude that $u(4) = 5$. We state this as a theorem.

Theorem 4.2. *Given a set P of points in the plane, with $|P| = 4$, the maximum number of unit distances between pairs of points is 5, that is $u(4) = 5$.*

From the configuration in Figure 9 we notice another layer of complexity inherent in the Erdős unit distance problem. In particular, having maximized the number of unit distances for n points in the plane, we need not necessarily have the maximum number of unit distances for $n-1$ when we remove an arbitrary point. If we let $n = 4$ and remove p_3 in Figure 9 we see that the remaining configuration contains three points, but only two pairs of unit distances. Hence, the remaining configuration is not optimal for $n - 1$.

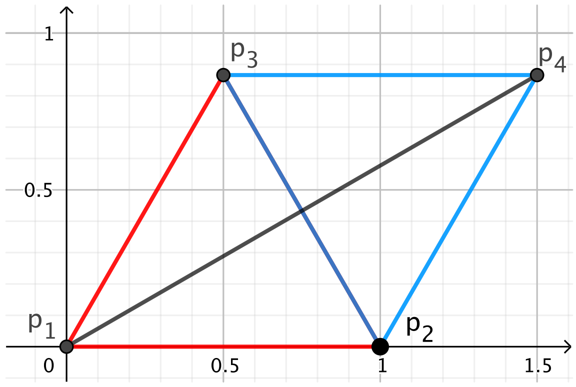

Figure 9: Optimal configuration for $n = 4$ points

Note also that since the trivial upper bound (Theorem 2.3) cannot be achieved when $n = 4$, it cannot be achieved for $n > 4$ either, since sharpness for $n > 4$ would imply sharpness for every subset of four points among the n points, which is a contradiction to the $n = 4$ case.

4.2 Triangular pattern

Continuing the ideas developed above, we now examine what happens when we select our points from a triangular pattern. More specifically, to add a point p to our point set P we choose it so that we obtain an equilateral triangle with side length 1 between p and two other points of P. The new triangle is always placed to the right of the current configuration, as illustrated in Figure 10. Hence, every new point adds two unit distances to our total count. Therefore, we obtain a recurrence relation:

$$\begin{cases} a_3 = 3 \\ a_{n+1} = a_n + 2 \text{ for } n \geq 3, \end{cases},$$

where we let a_n denote the number of unit distances when we have n points placed in the triangular pattern. Since we cannot form triangles with only one or two points, we let the recurrence relation start at $n = 3$. Hence, we get $a_n = 3+2(n-3) = 2n-3$, for $n \geq 3$.

By Theorem 4.1 and Theorem 4.2, it follows that for $n = 3$ and $n = 4$ we obtain $u(3)$ and $u(4)$ unit distances by choosing points in this triangular pattern. However, this is not necessarily the case for all n. For example, if we let $n = 7$, the triangular pattern yields a unit distance graph as in Figure 10. From our formula for a_n, we get

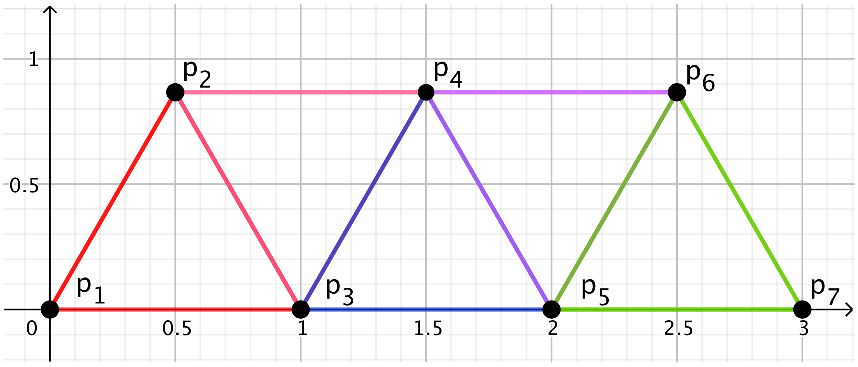

Figure 10: Triangular pattern for $n = 7$

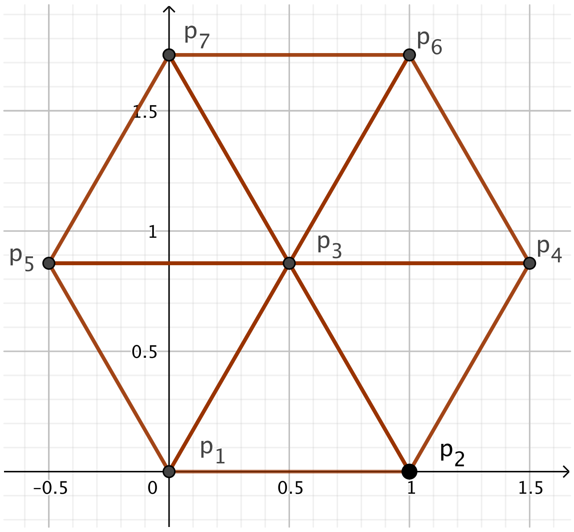

Figure 11: Another configuration for $n = 7$

$a_7 = 2 \cdot 7 - 3 = 11$, so this configuration yields 11 unit distances. We can construct another unit distance graph on a set of seven points as in Figure 11. Since this unit distance graph has 12 edges, we have 12 unit distances among these seven points. Comparing the configuration in Figure 11 with the triangular one in Figure 10, we see that the additional edge arises from completing the hexagon on the boundary.

We can now state a theorem concerning the maximum number of unit distances determined by n points in the plane.

Theorem 4.3. *There exists a constant $C > 0$ such that, for all sufficiently large n, $u(n) \geq Cn$. That is, $u(n) = \Omega(n)$.*

Proof. For the number of unit distances determined by n points in the triangular pattern, we have $a_n = 2n - 3 \geq n$ when $n \geq 3$. Hence, letting $C = 1$, we see that the number of unit distances determined by n points in the triangular pattern is $\Omega(n)$. Since $u(n)$ is the maximum number of unit distances among all possible configurations with n points, the conclusion follows. □

4.3 Hexagonal pattern

The following way of choosing points in the plane for our point set P is inspired by the configuration we found in Figure 11, which provided more unit distances than the triangular pattern when $n = 7$. For what we shall refer to as the *hexagonal pattern*, we start by letting P be the set of all the points in a configuration as in Figure 11. We then add points to P in such a way that we continually increase the size of the hexagon. We do so by adding another layer of the triangular pattern around boundary of our current hexagon, as illustrated in Figure 12.

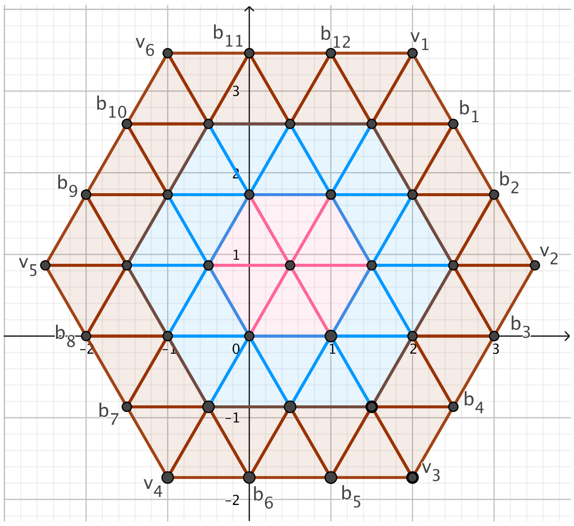

Figure 12: The hexagonal pattern

In order to simplify our analysis of this pattern, we shall only regard configurations for which the layers are complete, meaning that the boundary of the configuration is a hexagon at all times. In Figure 12, we see that the seven points in the pink hexagon constitute the first complete configuration. Furthermore, when adding 12 points around the boundary of the pink hexagon, we obtain the hexagon with blue boundary, which is the second complete configuration. Observe that the pink hexagon is contained in this one. Similarly, all 37 points in Figure 12 constitute the third complete configuration.

Let $\Phi(L) : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be a function of L, where L is the number of complete layers in a hexagonal pattern configuration. In Figure 12, the configuration corresponding to $L = 1$ is the pink hexagonal graph on seven vertices, where unit distances become edges in the graph. The configurations corresponding to $L = 2$ and $L = 3$ are the hexagonal graphs determined by the blue and dark red boundaries, respectively. We denote the graph corresponding to layer L by G_L, so that G_L is the unit distance graph of the underlying point set. Hence, G_1, G_2 and G_3 can be embedded in the plane as in Figure 12. Now, we define $\Phi(L) = |V(G_L)|$, so that $\Phi(L)$ is equal to the number of points in the hexagonal pattern corresponding to layer L. For example, $\Phi(1) = 7$, $\Phi(2) = 19$ and $\Phi(3) = 37$, by Figure 12.

In general, we add $6L$ new points to construct G_L from G_{L-1}, for an arbitrary value of L. Also, $V(G_{L-1})$ is contained in $V(G_L)$, and in constructing G_{L-1} we added $6(L-1)$ points to $V(G_{L-2})$, etc. Letting G_0 be the graph consisting of only the point at the center of the innermost hexagon, this argument holds also for $L = 1$. Hence, we can write

$$\Phi(L) = 6(L + (L-1) + ... + 1) + 1 = 6\frac{L(L+1)}{2} + 1 = 3L^2 + 3L + 1.$$

Furthermore, let $\phi(L) : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be the function that gives the number of unit distances determined by $V(G_L)$, that is $\phi(L) = |E(G_L)|$. Using Figure 12, all points in G_L, except those on the boundary of the hexagon, are contained in six unit distance pairs. We also see that the six corners of the hexagon are contained in three unit distance pairs. In Figure 12, when $L = 3$, these points are $v_1, v_2, ..., v_6$. The remaining $6(L-1)$ points on the boundary, which is denoted $b_1, b_2, ..., b_{12}$ in Figure 12, are contained in four unit distance pairs. Since we only want to count each unit distance pair once, we get

$$\phi(L) = \frac{1}{2}(6 \cdot 3 + 6(L-1) \cdot 4 + 6 \cdot \Phi(L-1)),$$

for $L > 1$ and $\phi(1) = 12$. Using the expression for $\Phi(L)$ derived above, we obtain

$$\phi(L) = \frac{1}{2}(6 \cdot 3 + 6(L-1) \cdot 4 + 6 \cdot (3(L-1)^2 + 3(L-1) + 1)) = 9L^2 + 3L.$$

Since $9 \cdot 1^2 + 3 = 12$, this formula works for all $L > 0$.

Now, we want to express the number of unit distances using the hexagonal pattern as a function of n. Therefore, we define $f : A \to \mathbb{Z}_{>0}$, where A denotes the set of number of values of n for which the pattern is complete. Hence, for example, $7 \in A$, $19 \in A$ and $37 \in A$, which can be verified by Figure 12. We define $f(n)$

for $n \in A$ as the number of unit distances determined by n points in a hexagonal pattern in the plane. We have $n = \Phi(L)$ for some L, since $n \in A$, so we write L in terms of n as

$$3L^2 + 3L + 1 = n$$
$$\iff L = -\frac{1}{2} + \sqrt{-\frac{1}{12} + \frac{n}{3}},$$

since $L > 0$.

Since we only use the above formula for values of n that correspond to complete layers, the value of L computed using this formula will always be a positive integer, as expected. We see, for example, that for $n = 7$, corresponding to the pink hexagonal graph in Figure 12, we get $L = 1$ using the above formula, which is in accordance with how we defined L. Now, we use this expression for L in the terms of n in the formula for $\phi(L)$ to obtain

$$\phi(L) = 9\left(-\frac{1}{2} + \sqrt{-\frac{1}{12} + \frac{n}{3}}\right)^2 + 3\left(-\frac{1}{2} + \sqrt{-\frac{1}{12} + \frac{n}{3}}\right) = 3n - 6\sqrt{-\frac{1}{12} + \frac{n}{3}}.$$

Since we defined $\phi(L)$ to be the number of edges in the unit distance graph G_L, and n is the number of points needed to construct G_L, the formula for $f(n)$ is $f(n) = 3n - 6\sqrt{-\frac{1}{12} + \frac{n}{3}}$.

For example, we have $f(7) = 3 \cdot 7 - 6\sqrt{9/4} = 12$, which we can verify by counting the edges in the pink hexagonal graph G_1 in Figure 12. Furthermore, $f(19) = 42$ and $f(37) = 90$, which can also be seen in Figure 12. Using the formula for the triangular pattern, $a_n = 2n - 3$, we obtain $a_7 = 11$, while $f(7) = 12$. This difference in the number of unit distances is what we showed in Figure 10 and Figure 11. We also see that $a_{19} = 35$ and $a_{37} = 71$, while $f(19) = 42$ and $f(37) = 90$, respectively. Hence, also in these cases we obtain a larger number of unit distances using the hexagonal pattern than the triangular pattern.

For $f(n) = 3n - 6\sqrt{-\frac{1}{12} + \frac{n}{3}}$, the term $3n$ dominates for large values of n. In $a_n = 2n - 3$, the term $2n$ dominates for large n. This is also visible in Figure 13 where we compare $f(n)$ and a_n by plotting the continuous functions $f(x) = 3x - 6\sqrt{-\frac{1}{12} + \frac{x}{3}}$ and $g(x) = 2x - 3$. We can see that for a large number of points we obtain more unit distances using the hexagonal pattern, due to the difference in the coefficient of n in $f(n)$ and a_n. This shows how, perhaps seemingly small, local improvements on the number of unit distances can yield significant global improvements.

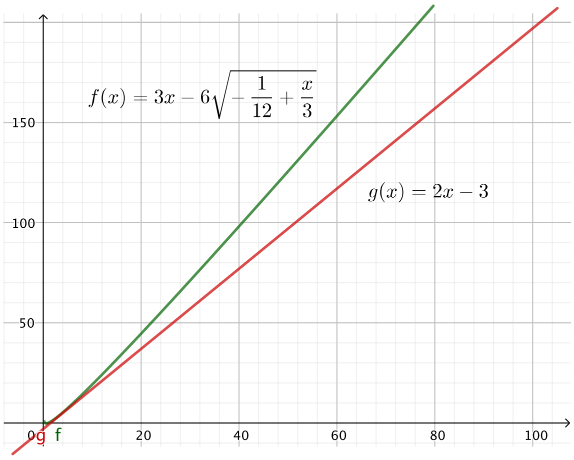

Figure 13: Comparison of the number of unit distances obtained from the triangular pattern and the hexagonal pattern

4.4 Perpendicular lines

Now we investigate what happens if we impose restrictions regarding which points in $\mathbb{R}^2$ our n points are allowed to be. Say we have

$$L = \{(c, y) \in \mathbb{R}^2\} \cup \{(x, k) \in \mathbb{R}^2\}$$

where $c, k \in \mathbb{R}$ are constants. We now impose the restriction that all the selected points must belong to L, i.e. be on the perpendicular lines $x = c$ and $y = k$.

Theorem 4.4. *Let P be a set of n points all belonging to L as defined above. Then the maximum number of unit distances between these points is $O(n)$.*

Proof. An important observation is that we have an upper bound on the number of unit distance pairs to which a specific point of P can belong to. Since a circle can intersect two perpendicular lines in at most four points, one point can belong to at most four unit distance pairs. This is illustrated in Figure 14, where the lines are $y = 2$ and $x = 3$ and the only points that can belong to our point set P that are at unit distance from p_1 are A, B, C and D.

Hence, if we have n points in the plane we can at most have $4n$ pairs of points that are a unit distance apart. Since $4n = O(n)$ the conclusion follows. □

The most significant difference between only selecting our n points from perpendicular lines and selecting them arbitrarily in the plane is the fact that in the former case we have an upper bound on the number of unit distance pairs a specific point

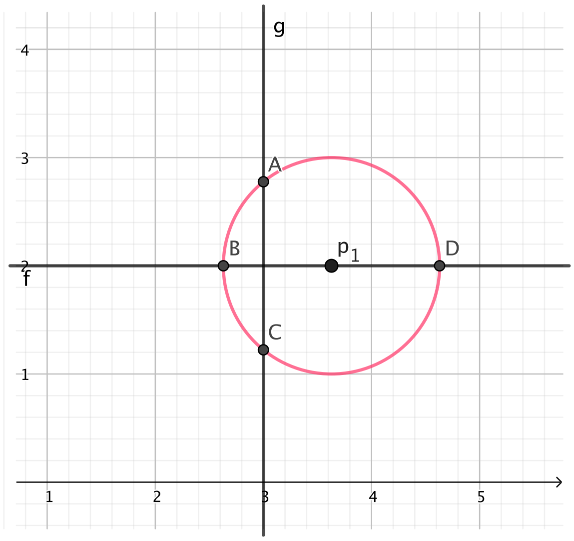

Figure 14: The point p_1 can belong to at most four unit distance pairs

can belong to. As seen above, this is the reason we get an immediate bound on the maximum number of unit distances, and the fact that the number of unit distance pairs each point can belong to is bounded by a constant is what makes the number of unit distances bounded linearly in n.

For the general problem of unit distances in the plane, we do not have such a constant bound. In fact, the number of unit distance pairs a specific point belongs to can be arbitrarily large. Say we have m points belonging to P on the unit circle centered at an arbitrary, but fixed, point p in the plane. If we then add p to P, this point contributes with m unit distance pairs. Since m can be any positive integer, the number of unit distance pairs containing p can be made arbitrary large, and so is unbounded.

4.5 The $\sqrt{n} \times \sqrt{n}$ grid

Let n be a perfect square. For $m > 0$, consider the set $P = \{(mx, my) : x, y \in \mathbb{Z}, 0 \leq x, y, \leq \sqrt{n} - 1\}$. We shall call such a set a $\sqrt{n} \times \sqrt{n}$ grid with step m.

In fact, this configuration, with a suitable value of m, has the asymptotically largest known number of unit distances. The number of possible representations of an integer as a sum of two squares is connected to analyzing the of number unit distances in a $\sqrt{n} \times \sqrt{n}$ grid [1]. We shall use the following two number theoretical results, from [1], without proof.

Lemma 4.5. *Let $p_1 < p_2 < ... < p_r$ be primes of the form $4k + 1$, for some integer $k > 0$. Let $M = p_1 p_2 \cdots p_r$. Then M can be written as a sum of two squares of integers in at least 2^r ways.*

Theorem 4.6. *Let d and a be coprime natural numbers and let $\pi_{d,a}(n)$ be the number of primes of the form $a + kd$, for a nonnegative integer k, not exceeding n. We have*

$$\pi_{d,a}(n) = (1 + o(1))\frac{1}{\varphi(d)} \cdot \frac{n}{\log(n)},$$

where φ denotes Euler's totient function, i.e. $\varphi(d)$ is equal to the number of integers r with $1 \leq r \leq d$ such that r and d are coprime.

Note that, for nonnegative integers k_1 and k_2, we have $a + k_1 d \equiv a + k_2 d \equiv a \pmod{d}$, so the number of primes that are counted in $\pi_{d,a}(n)$ all belong to the same residue class, modulo d. So, if p is a prime counted in $\pi_{d,a}(n)$ we have that $p \equiv a \pmod{d}$.

For every $d \geq 2$ there are $\varphi(d)$ residue classes, modulo d, possibly containing more than one prime [1]. For, assume that $\gcd(a, d) > 1$. Then $\gcd(a, d)$ divides $a + kd$ for all $k = 0, 1, 2, ...$, so $\pi_{d,a}(n)$ counts at most one prime, which happens precisely if $\gcd(a, d)$ is counted in $\pi_{d,a}(n)$. However, if $\gcd(a, d) = 1$, as in the theorem, the sequence $a, a + d, a + 2d, ...$, may contain more primes. So there are $\varphi(d)$ residue classes, modulo d, that may contain more than one prime. Letting a in Theorem 4.6 be such that $1 \leq a \leq d$, the theorem shows that the primes are quite uniformly distributed among the residue classes corresponding to the numbers counted in $\varphi(d)$.

Theorem 4.7 ([1])**.** *There exist configurations of $n \geq n_0$ points in $\mathbb{R}^2$ that determine at least $n^{1+c/\log\log n}$ unit distances, for some $n_0 > 0$ and some $c > 0$.*

Proof. Let us assume that n is a perfect square. Let M be the product of the first $r - 1$ primes of the form $4k + 1$, where r is chosen to be the largest number such that $M \leq n/4$. We let our set P of n points in the plane consist of the points in the $\sqrt{n} \times \sqrt{n}$ grid, with step $1/\sqrt{M}$. This can be written as

$$P = \{(\frac{x}{\sqrt{M}}, \frac{y}{\sqrt{M}}) : x, y \in \mathbb{Z}, 0 \leq x, y, \leq \sqrt{n} - 1\}.$$

Say $M = a^2 + b^2$, with a, b are nonnegative integers. Let $(x/\sqrt{M}, y/\sqrt{M})$ be a point on the grid. We can then view (a, b) as a vector that gives us a point at unit distance from $(x/\sqrt{M}, y/\sqrt{M})$. More precisely, this point is

$$\left(\frac{x+a}{\sqrt{M}}, \frac{y+b}{\sqrt{M}}\right),$$

since then

$$\left\|\left(\frac{x}{\sqrt{M}}, \frac{y}{\sqrt{M}}\right)-\left(\frac{x+a}{\sqrt{M}}, \frac{y+b}{\sqrt{M}}\right)\right\|=\frac{1}{\sqrt{M}}\cdot\sqrt{a^2+b^2}=\frac{1}{\sqrt{M}}\cdot\sqrt{M}=1.$$

This can be done for every point $(x/\sqrt{M}, y/\sqrt{M})$ such that both coordinates remain within the grid after adding the vector $(a/\sqrt{M}, b/\sqrt{M})$. Equivalently, what we require is that $0 \leq x \leq \sqrt{n}-a-1$ and $0 \leq y \leq \sqrt{n}-b-1$. Hence, there are $(\sqrt{n}-a)(\sqrt{n}-b)$ possibilities for x and y in $(x/\sqrt{M}, y/\sqrt{M})$ so that the point provided by a and b remains within the grid, i.e. so that we obtain a unit distance pair within the points of P.

If a and b are nonnegative integers such that $a^2+b^2=M$, we have $a, b \leq \sqrt{n}/2$. Hence, $(\sqrt{n}-a)(\sqrt{n}-b) \geq (\sqrt{n}-\sqrt{n}/2)^2 = n/4$. Thus, the procedure above finds at least $n/4$ unit distances. Therefore, for each representation of M as a sum of two squares of nonnegative integers, we obtain at least $n/4$ unit distance pairs within the points of P.

Now, if (a, b) is such that $M = a^2+b^2$, also $(-a, b)$, $(a, -b)$ and $(-a, -b)$ have this property. Hence, every representation of M as a sum of two squares of nonnegative integers corresponds to at most four representations as a sum of two squares of integers, not necessarily nonnegative. So, by Lemma 4.5, we have at least

$$\frac{n}{4}\cdot\frac{2^{r-1}}{4}=\frac{2^r n}{32},$$

unit distance pairs. Observe that we multiply by the factor $n/4$, since each representation yields at least $n/4$ unit distance pairs.

Let $p_1, p_2, ..., p_r$ denote the first r primes of the form $4k+1$. From the way we chose r, we have $4p_1p_2\cdots p_{r-1} \leq n < 4p_1p_2\cdots p_r$. Thus,

$$p_r > (n/4)^{1/r}. \tag{1}$$

Also,

$$2^r \leq n, \tag{2}$$

since $p_i > 2$ for $i = 1, 2, 3, ..., r-1$.

By Theorem 4.6 we obtain,

$$r = \pi_{4,1}(p_r) = (1 + o(1))\frac{1}{\varphi(4)} \cdot \frac{p_r}{\log(p_r)} = (\frac{1}{2} + o(1)) \cdot \frac{p_r}{\log(p_r)},$$

since $\varphi(4) = 2$. Using this expression, we can conclude that if n is large enough, we have $r \geq \sqrt{p_r} \geq n^{\frac{1}{3r}}$. The first inequality is because $kp_r/\log(p_r)$ grows faster than $\sqrt{p_r}$ does, for any constant $k > 0$, when n tends to infinity. The second inequality holds because Inequality 1 implies that

$$\sqrt{p_r} > \frac{n^{1/2r}}{4^{1/2r}} \geq \frac{n^{1/2r}}{(n^{1/3})^{1/2r}} = \frac{n^{1/2r}}{n^{1/6r}} = n^{\frac{1}{3r}},$$

for sufficiently large n.

So $r^{3r} \geq n$. Taking logarithms base 2 of both sides, we obtain $3r\log_2(r) \geq \log_2(n)$, which implies that

$$r \geq \frac{\log_2(n)}{3\log_2(r)} \geq \frac{\log_2(n)}{3\log_2(\log_2(n))},$$

since $r \leq \log_2(n)$ by Inequality 2. Recall that the number of unit distances is at least $2^r n/32$. Furthermore,

$$\frac{2^r n}{32} = \frac{n}{2^5} \cdot 2^r \geq \frac{n}{2^5} \cdot 2^{\log_2 n/3\log_2\log_2 n} = \frac{n^{1+1/3\log_2\log_2 n}}{2^5} \geq n^{1+c/\log\log n},$$

for large n and a suitably chosen constant $c > 0$. Observe that we use the natural logarithm on the right hand side in the inequality above, which we can do due to the freedom in our choice of c. This proves the theorem when n is a perfect square.

When n is not a perfect square, we define $m = \lfloor\sqrt{n}\rfloor$ and let the corresponding configuration be the $m \times m$ grid with the remaining points placed along the boundary of this grid. Using that this grid determines at least

$$(m^2)^{1+c/\log\log m^2},$$

unit distances, for large m and an absolute constant $c > 0$ as proved above, the theorem can also be proved for this case. □

4.6 Nested sets

For both the triangular pattern and the hexagonal pattern we always include the current configuration in the following configuration. More specifically, we always add points to the underlying point set P when increasing the size of P. For the triangular pattern, this is because we always add a new triangle to the right of the current configuration, when increasing our point set. For the hexagonal pattern this is seen by the fact that we build a new layer around the current hexagon when we increase our point set. This means that the point sets for the different values of n are *nested*.

More specifically, if we let $P_1, P_2, ..., P_s$ denote the sets of points in $\mathbb{R}^2$ corresponding to the first s configurations we have $P_i \subset P_{i+1}$ for $i = 1, ..., s-1$. For example, in the hexagonal pattern we can let P_1 consist of the points in the graph G_1 as defined above, i.e. all points in the pink hexagonal graph in Figure 12. We then let P_2 consist of all points in the graph G_2 and P_3 consist of all points in G_3. Then P_1, P_2, P_3 are nested point sets since $P_1 \subset P_2 \subset P_3$.

However, if we mix configurations from the triangular pattern and the hexagonal pattern when increasing the size of our point set, we do not get nested point sets. In addition, the $\sqrt{n} \times \sqrt{n}$ grids, used in the proof of Theorem 4.7, are generally not nested sets, since the step $1/\sqrt{M}$ changes with n. In conclusion, we see that for the configurations we have analyzed, some create nested point sets, while others do not.

From Theorem 4.7 we know that for all sufficiently large n, there exist configurations of n points in the plane such that the number of unit distances grows superlinearly in n. This implies that the growth rate is strictly faster than kn for any constant $k > 0$. In particular, it implies the following:

Theorem 4.8. *For every constant $k > 0$ there exist configurations of n points in $\mathbb{R}^2$ that determine at least kn unit distances, for large enough values of n.*

Proof. By Theorem 4.7 we know that for sufficiently large n there exist configurations that contain at least $n^{1+c/\log\log n}$ unit distances for some $c > 0$. Let $k > 0$ be a given constant. Since $n^{1+c/\log\log n} = n \cdot n^{c/\log\log n}$ we have

$$kn \leq n \cdot n^{c/\log\log n},$$

for large enough values of n, since $k < n^{c/\log\log n}$ for large enough values of n. Hence, there exist configurations that contain more than kn unit distances, for large enough n. The theorem follows.

□

Now, we can use this theorem to prove that we can get linear growth of the number of unit distances for an arbitrary large constant with the underlying point sets being nested.

Theorem 4.9. *For any fixed constant $k > 0$, there exist a sequence of strictly increasing nested point sets $P_1 \subset P_2 \subset \ldots$, such that the number of unit distances determined by the points of P_i is $k_0|P_i|$, where $k_0 \geq k$.*

Proof. By Theorem 4.8 we know that there exist configurations of n points that determine at least kn unit distances, given that $n \geq n_0$ for n_0 large enough. Say we have precisely n_0 points and that this configuration determines $k_0 n_0$ unit distances, for some constant $k_0 > 0$. Hence $k_0 n_0 \geq k n_0$. Let P_1 contain precisely these n_0 points. Using this configuration for n_0 points, we may increase our point set by forming isomorphic copies at other places in the plane, in such a way that we do not obtain any unit distances between points corresponding to different copies. Let P_i contain the in_0 points corresponding to i such isomorphic copies. Then we have $k_0|P_i| = k_0(n_0 i) = (k_0 n_0)i \geq (kn_0)i = k|P_i|$ unit distances. So we will obtain a linear growth with $k_0 \geq k$ as coefficient. When increasing the size of our point set we include all points from previous constructions and hence these configurations correspond to nested point sets. □

5 Graphs and rigidity

In this section, we approach the Erdős unit distance problem from a more structural perspective. As mentioned earlier, the problem is closely related to graph theory. In this section, we examine this relation more thoroughly and state a conjecture which, if it holds, yields an improvement on the best known upper bound for the number of unit distances among n points in the plane. We begin this section by introducing some relevant concepts.

5.1 Realizations and frameworks

Definition 5.1 ([2]). Let $G = (V, E)$ be a graph with vertex set $V = [n]$[1]. A *realization* $\mathbf{p}$ of G in $\mathbb{R}^2$ is defined as an embedding, not necessarily injective, of the vertex set V of G in $\mathbb{R}^2$. Identifying this embedding with the tuple whose components are the images of the vertices, we obtain

$$\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^2 \times ... \times \mathbb{R}^2 \cong \mathbb{R}^{2n}.$$

A *framework* is a pair $(G, \mathbf{p})$ consisting of a graph G and a realization $\mathbf{p}$.

Definition 5.2 ([2]). Let $G = (V, E)$ be a graph with vertex set $V = [n]$. The *edge function* of G is defined to be the function $f_G : \mathbb{R}^{2n} \to \mathbb{R}^{|E|}$ given by

$$(p_1, ..., p_n) \mapsto (\|p_i - p_j\|^2)_{\{i,j\} \in E},$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^2$. The order on the edges of G is fixed, but arbitrary. Therefore, the image is the vector

$$(\|p_{i_1} - p_{j_1}\|^2, ..., \|p_{i_{|E|}} - p_{j_{|E|}}\|^2)$$

in $\mathbb{R}^{|E|}$. Hence, each component of f_G is a polynomial function in four variables.

Now we introduce the concepts of equivalence and congruence in order to be able to compare different frameworks.

Definition 5.3 ([2]). Let $G = (V, E)$ be a graph with vertex set $V = [n]$ and let $\mathbf{p}$ and $\mathbf{q}$ be realizations of G in $\mathbb{R}^2$. The associated frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$

[1]Throughout this thesis, we use the convention that $[n] = \{1, 2, ..., n\}$.

are said to be *equivalent* if $f_G(\mathbf{p}) = f_G(\mathbf{q})$. This means that $\|p_i - p_j\| = \|q_i - q_j\|$ for every edge $\{i, j\}$ in E. Furthermore, the frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are said to be *congruent* if $\|p_i - p_j\| = \|q_i - q_j\|$ for every subset containing two elements $\{i, j\} \subset V$.

Observe that in order for two frameworks to be equivalent we need only look at edges in E, while congruence requires us to look at *all* subsets of cardinality 2 of V [2]. Therefore, we can conclude that if G is a complete graph, the concepts of equivalent and congruent frameworks coincide.

Definition 5.4 ([17, p. 210])**.** A *rigid motion* of $\mathbb{R}^2$ is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\|f(p) - f(q)\| = \|p - q\|$ for all p and q in $\mathbb{R}^2$. In other words, a rigid motion is a map that preserves distances between points.

Let $a \in \mathbb{R}^2$. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(p) = p + a$. The function f is then a rigid motion of $\mathbb{R}^2$, called a translation by a. Another example of a rigid motion can be obtained by letting $\theta \in \mathbb{R}$ and defining $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$. This rigid motion f is called a rotation of $\mathbb{R}^2$ through θ [17, p. 210].

Geometrically, if a realization $\mathbf{p}$ is injective, we can think of it as determining an embedding of G in the plane, where p_i is the point where vertex i is placed for $i \in V$. We can then represent the edge $\{i, j\} \in E$ by the straight line segment between p_i and p_j in $\mathbb{R}^2$. These points and line segments then form an embedding in the plane of the graph G.

That two frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are congruent can thus be thought of as their associated embeddings of G being congruent as figures. The resulting figures are congruent even if the realizations are not injective. That two frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are equivalent means that their associated embeddings are equal in terms of edge lengths, but not necessarily congruent as figures, since the distance between points that do not form edges is not fixed.

We have that $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are congruent frameworks if and only if there exists a rigid motion f of $\mathbb{R}^2$ such that $f(p_i) = q_i$ for every $i \in V$ [2], which can be seen as follows. If such rigid motion f exists, then $\|p_i - p_j\| = \|f(p_i) - f(p_j)\| = \|q_i - q_j\|$ for every subset containing two elements $\{i, j\} \subset V$. Hence $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are congruent. If $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are congruent, we have $\|p_i - p_j\| = \|q_i - q_j\|$ for every $\{i, j\} \subset V$. We may then define the function f by $f(p_i) = q_i$ for all $i = 1, 2, ..., n$. Without going into detail, the function f then extends to a rigid motion of $\mathbb{R}^2$. Intuitively, this is in agreement with the fact that, for congruent frameworks, their associated embeddings of the graph are congruent as figures, as mentioned earlier.

5.2 Rigid frameworks

Definition 5.5 ([18, p. 6]). Let $G = (V, E)$ be a graph with vertex set $V = [n]$. A framework $(G, \mathbf{p})$ is said to be *rigid* if there exists an $\epsilon > 0$ such that if the frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are equivalent and $\|p_i - q_i\| < \epsilon$ for all $i \in V$, then $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are congruent.

Geometrically we may think of $(G, \mathbf{p})$ being rigid as having the property that if we have another embedding of G from $\mathbf{q}$ such that all edge lengths are the same and the vertices are placed by $\mathbf{q}$ close enough to where the vertices are placed by $\mathbf{p}$, then necessarily the lengths between all vertices are the same and the associated embeddings are congruent as figures.

Thus rigidity depends both on the graph G itself and the specific realization $\mathbf{p}$. So, given a graph G and two realizations $\mathbf{p}$ and $\mathbf{q}$, it might be the case that $(G, \mathbf{p})$ is rigid and $(G, \mathbf{q})$ is not. By the remark about how the definitions of equivalent and congruent frameworks coincide for complete graphs, we may conclude that all frameworks of complete graphs are rigid. Thus, an example of a rigid framework is the framework $(G, \mathbf{p})$ with $G = K_3$, defined by

$$\mathbf{p} = ((0,0), (1,0), (1/2, \sqrt{3}/2)).$$

Example 1. Let $G = ([4], E)$ be a graph where $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\}$. Let $\mathbf{p} = (a_1, a_2, a_3, a_4)$ be a realization of G in the plane, which embeds the vertices into a square. This is illustrated in Figure 15. Define another realization $\mathbf{q}_\epsilon = (a_1, a_2, a_3^\epsilon, a_4^\epsilon)$ by deforming the square into a rhombus by moving the embedding of the vertices 3 and 4, such that a_3^ϵ is within a radius of ϵ from a_3 and that a_4^ϵ is within a radius of ϵ from a_4. See Figure 15.

The frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q}_\epsilon)$ are not congruent, since the distance between non-adjacent vertices is not preserved under the deformation. For example, the distance between a_2 and a_4 is not the same as the distance between a_2 and a_4^ϵ. However, they are equivalent, since all edge lengths are the same. The deformation of the square into a rhombus can be made for any $\epsilon > 0$. So there exist non-congruent

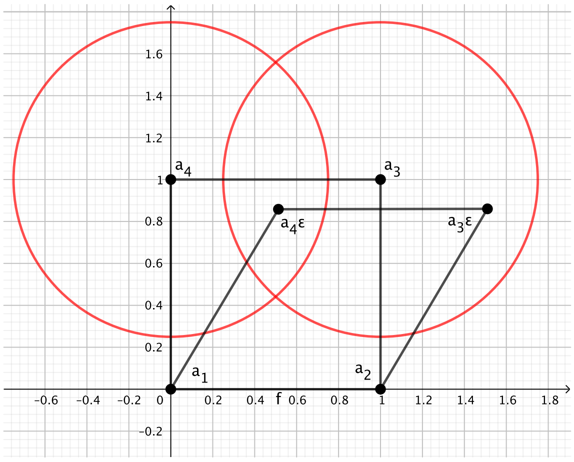

Figure 15: Example of a realization corresponding to a non-rigid framework

equivalent frameworks with realizations arbitrary close to $\mathbf{p}$. Hence, the framework $(G, \mathbf{p})$ is not rigid.

Example 2. Now, we look at another example where $G = ([4], E)$, where E contains edges between all pairs of points, except $\{1, 4\}$. Define the realization $\mathbf{p}$ by $\mathbf{p} = (p_1, p_2, p_3, p_4)$, as in Figure 16. Observe that this is also an embedding of the unit distance graph of the point set $\{p_1, p_2, p_3, p_4\}$, and that it corresponds to the maximum number of unit distances among four points in the plane, as explained in Subsection 4.1 in Section 4.

Let $(G, \mathbf{q})$ be an equivalent framework, denoted by $\mathbf{q} = (q_1, q_2, q_3, q_4)$. Since p_1,p_2 and p_3 are pairwise connected by edges of length 1, the embeddings of these vertices must also form an equilateral triangle by $\mathbf{q}$. In addition, p_2,p_3 and p_4 are pairwise connected by edges of length 1, so vertex 4 must be placed by $\mathbf{q}$ relative to the other points as in Figure 16 or coincide with p_1. Recall that this argument agrees with what was explained in more detail in Subsection 4.1 in Section 4. However, for a sufficiently small radius $\epsilon > 0$ in Definition 5.5, vertex 4 cannot coincide with p_1. So vertex 4 is placed as in Figure 16 by $\mathbf{q}$ and the distance between the embedding of vertex 1 and vertex 4 are thus the same for every equivalent framework $(G, \mathbf{q})$ whose points are placed within this ϵ radius, which implies that the frameworks $(G, \mathbf{q})$ and $(G, \mathbf{p})$ are congruent. Hence, $(G, \mathbf{p})$ is a rigid framework.

Observe the similarities between the graphs in the two examples above. The graphs only differ by an edge and we found a rigid framework for the graph having an additional edge in Example 2 and a non-rigid framework for the graph with

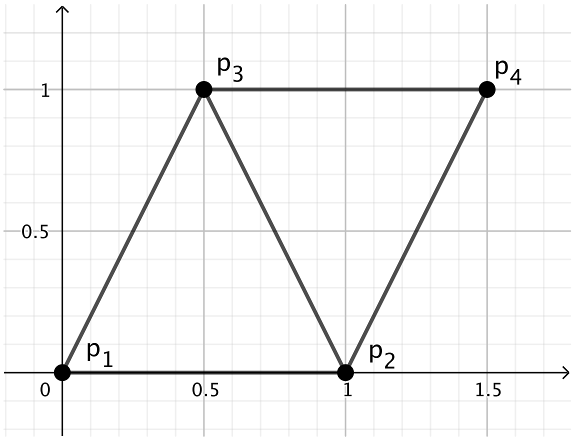

Figure 16: Example of a realization corresponding to a rigid framework

fewer edges in Example 1. Intuitively, the framework $(G, \mathbf{p})$, for some graph G and some realization $\mathbf{p}$, is more likely to be rigid when the graph G has many edges. That is because when adding edges, we impose additional restrictions on possible configurations for equivalent frameworks, since equivalent frameworks preserve edge lengths. In other words, having more edges leaves fewer possibilities for placing the points relative to one another, as a larger amount of distances between pairs of vertices is fixed in advance.

Definition 5.6 ([2])**.** A *unit embedding* of a graph $G = (V, E)$, with $V = [n]$, is a realization $\mathbf{p}$ of its vertex set into $\mathbb{R}^{2n}$ such that

i) $\mathbf{p}$ is injective, and

ii) for every edge $\{i, j\} \in E$ we have $\|p_i - p_j\| = 1$.

Using the terminology developed in Section 2, we see that if we have a unit embedding $\mathbf{p}$ of a graph G such that $\|p_i - p_j\| = 1$ only for points forming edges $\{i, j\} \in E$, we obtain an embedding of the unit distance graph of the point set $\{p_1, p_2, ..., p_n\}$. Observe also that the realization in Figure 16 is a unit embedding of the graph $G = ([4], E)$, where E contains edges between all pairs of points, except $\{1, 4\}$.

5.3 Rigidity conjecture

Now we state a conjecture, which, given that it holds, yields an improvement on the current best upper bound $O(n^{4/3})$ on the number of unit distances determined by n points in the plane. This consequence of the conjecture is proved in Subsection 5.4.

Conjecture 5.7 (Rigidity conjecture [2]). Let $G = ([n], E)$ be a graph such that $|E| \gg n^{7/6}$. Let $\mathbf{p} \in (\mathbb{R}^2)^n$ be a realization of G such that for every vertex $v \in [n]$ the neighbors of v are not mapped into a common line. Then there exists a subgraph $G' \subset G$ for which $|V(G')| \geq 4$ with the property that $(G', \mathbf{p}|_{V(G')})$ is a rigid framework, for n large enough.

This conjecture essentially reflects the idea that, if a graph is sufficiently dense and not too degenerate, locally, then it must contain a rigid subframework. In connection to the Erdős unit distance problem, this suggests that if we have many unit distances among our n points, there are geometric constraints on the corresponding configuration. Hence, having many unit distances locally among our n points, affects the structure of the configuration, globally.

Lending some support to the conjecture is the following theorem, which Conjecture 5.7 assumes holds for any $\alpha \geq 1/6$. In this case, that would mean that the subgraph corresponding to a rigid framework guaranteed by the theorem for graphs with large edge sets, also exists when having less restrictions on the size of our edge set. As mentioned earlier, a framework is more likely to be rigid if the corresponding graph has many edges. Hence, the conjecture is not trivial.

Theorem 5.8 ([2]). *Let $G = ([n], E)$ be a graph with*

$$|E| = \Omega(n^{1+\alpha}),$$

where $\alpha > 1/2$. Let $\mathbf{p} \in (\mathbb{R}^2)^n$ be a realization of G such that for every vertex $v \in [n]$ the neighbors of v are not mapped into a common line. Then there exists a subgraph $G' \subset G$ for which $|V(G')| \geq 4$ with the property that $(G', \mathbf{p}|_{V(G')})$ is a rigid framework, if n is large enough.

If Conjecture 5.7 holds, we would be able to deduce the following:

Theorem 5.9 (Unit distance bound under the rigidity conjecture [2]). *Given that Conjecture 5.7 is true, the maximum number of unit distances, $u(n)$, determined by n points in $\mathbb{R}^2$ satisfies*

$$u(n) = O\left(\frac{n^{4/3}}{\log^{1/12}(n)}\right).$$

5.4 Proof of upper bound under the rigidity conjecture

As stated earlier, given that the Conjecture 5.7 holds, we obtain an improvement of the current best upper bound of $u(n)$, which is $O(n^{4/3})$. In this subsection, we prove Theorem 5.9, which conditionally asserts that

$$u(n) = O\left(\frac{n^{4/3}}{\log^{1/12}(n)}\right).$$

For the proof we will use the following notation:

Definition 5.10 ([2])**.** Let P be a set of points in the plane. We denote by $u(P)$ the number of unit distance pairs determined by the points of P. That is

$$u(P) = |\{\{p, q\} \subset P : \|p - q\| = 1\}|.$$

We will also be using the following theorem, which is proved in [2] using similar techniques as in Section 3:

Theorem 5.11. *Let $h(n)$ be a function tending to infinity as n tends to infinity. Let P be a set of n points in $\mathbb{R}^2$ such that $u(P) \geq \frac{n^{4/3}}{h(n)}$. Suppose n is sufficiently large. Then there exist:*

i) a subset $P' \subset P$ with $P' \asymp n^{1/3}h(n)^4$;

ii) bipartite graphs $G_i = (U_i \cup V_i, E_i)$ for every i with $1 \leq i \leq k$ where $k \gg n^{2/3}/h(n)^5$ with $2 \leq |U_i|, |V_i| \ll h(n)^6$ and $|E_i| \gg h(n)^7$;

iii) unit embeddings $\mathbf{p}^{(i)}$ of G_i into $(\mathbb{R}^2)^{|U_i|+|V_i|}$ for every i with $1 \leq i \leq k$ such that

(a) the sets $\mathbf{p}^{(1)}(U_1), ..., \mathbf{p}^{(k)}(U_k)$ are pairwise disjoint subsets of P,

(b) the sets $\mathbf{p}^{(1)}(V_1), ..., \mathbf{p}^{(k)}(V_k)$ are subsets of P'.

The intuition behind the above theorem is as follows: Say we have a point set P of n points in the plane, where n is large, such that the points in P determine nearly $n^{4/3}$ unit distances, up to division by a slowly growing loss factor. Then a substantial part of its unit distance structure must be organized around a relatively small "reservoir" P', with many small, pairwise disjoint outside groups each making many unit-distance connections into that reservoir.

We can now prove Theorem 5.9.

Proof of Theorem 5.9. Let P be a set of n points in $\mathbb{R}^2$. Assume for contradiction that $u(P) \geq \frac{n^{4/3}}{h(n)}$ for some function $h(n)$ which tends to infinity as n tends to infinity, and that function will be specified later in the proof, where we see what it needs to fulfill to get a contradiction. Also, assume that n is sufficiently large. Let $P' \subset P$, k, $G_i = (U_i \cup V_i, E_i)$ for $1 \leq i \leq k$ and $\mathbf{p}^{(1)}, ..., \mathbf{p}^{(k)}$ be as in Theorem 5.11.

The realizations $\mathbf{p}^{(1)}, ..., \mathbf{p}^{(k)}$ are thus unit embeddings. Therefore, letting v be an arbitrary vertex in one of the graphs G_i, all neighbors of v are mapped to distinct points on the unit circle centered at $\mathbf{p}^{(i)}(v)$. Since a line intersects a circle in at most two points, at most two neighbors of v can lie on the same line. So, if all neighbors of v lie on the same line, v has at most two neighbors.

Now, we iteratively remove the vertices from G_i which has at most two neighbors so that we are left with a graph H_i for which every vertex has at least three neighbors. We also remove the edges incident with these vertices. Thus

$$|E(H_i)| \geq |E(G_i)| - 2|V(G_i)|,$$

since we remove at most two edges for each vertex. Now, since $|E(G_i)| \gg h(n)^7$ and $|V(G_i)| \ll h(n)^6$, by Theorem 5.11, we can deduce that $|E(H_i)| \gg h(n)^7$. Also, the inequality $|V(G_i)| \ll h(n)^6$ implies that $|V(H_i)| \ll h(n)^6$. Since $|V(H_i)|^{7/6} \ll h(n)^7$ we have $|E(H_i)| \gg |V(H_i)|^{7/6}$. Also, since the neighbors of a vertex in H_i are not mapped into a common line under the realization $\mathbf{p}^{(i)}|_{V(H_i)}$, all the assumptions in Conjecture 5.7 are satisfied for H_i.

Assuming Conjecture 5.7 holds, we may apply it to the graphs H_i, for $1 \leq i \leq k$. However, since H_i is a subgraph of $G_i = (U_i \cup V_i, E_i)$, the subgraph of H_i which we obtain from the conjecture is also a subgraph of the graph G_i, for $1 \leq i \leq k$. Therefore, there exists a subgraph $G'_i = (U'_i \cup V'_i, E'_i)$ of G_i such that G'_i has at least four vertices and the framework $(G'_i, \mathbf{p}^{(i)}|_{U'_i \cup V'_i})$ is rigid. Since $\mathbf{p}^{(i)}$ is a unit embedding of G_i, $\mathbf{p}^{(i)}|_{U'_i \cup V'_i}$ is a unit embedding of G'_i.

Also, G'_i is bipartite and thus $|U'_i|, |V'_i| \geq 2$ is implied, for otherwise $\mathbf{p}^{(i)}|_{U'_i \cup V'_i}$ would not be rigid. Because if we assume, without loss of generalization, that $|V'_i| \leq 1$, then $|U'_i| \geq 3$. Due to bipartiteness, edges can only exist between vertices of U'_i and V'_i. If $|V'_i| = 1$ we obtain a star graph with unit distance edges, and hence the framework is not rigid. That is because no matter how small we choose ϵ in Definition 5.5, an equivalent framework need not be congruent, due to the ability of the neighbors of the one vertex in V'_i to rotate around this vertex. If $|V'_i| = 0$ the graph contains no edges, so the framework is certainly not rigid.

Now, by Theorem 5.11 ii) we have $2 \leq |U_i|, |V_i| \ll h(n)^6$ for each $1 \leq i \leq k$. Since our $h(n)$ is not given, we can choose to define it such that $2 \leq |U_i|, |V_i| \leq h(n)^6$, by multiplication by a suitable constant. Hence also $2 \leq |U_i'|, |V_i'| \leq h(n)^6$.

The number of possible edges in the bipartite graphs G_i' is $|U_i'| \cdot |V_i'| \leq h(n)^6 \cdot h(n)^6 = h(n)^{12}$. In addition, each possible edge is either in G_i' or not, so we can determine the edges of G_i' in at most $2^{h(n)^{12}}$ ways. Therefore, there are at most $2^{h(n)^{12}}$ ways to construct G_i'. Thus, we also have $2^{h(n)^{12}}$ as an upper bound on the number of isomorphism types for bipartite graphs satisfying the above conditions. Note that we consider bipartite isomorphisms, i.e. graph isomorphisms that preserve the bipartition classes U and V. Now, we have k such bipartite graphs G_i'. Hence, by the pigeonhole principle, at least $k/2^{h(n)^{12}}$ of the graphs G_i' have the same isomorphism type.

We let $H = (U \cup V, E)$ denote a bipartite graph that has this isomorphism type. Thus $2 \leq |U|, |V| \leq h(n)^6$. A rigid framework in the plane on n vertices has at most 9^n distinct non-congruent embeddings that induce the same edge lengths (See Theorem 2 [19]). Since each G_i has $|U_i| + |V_i| \leq 2h(n)^6$ vertices, we may apply the pigeonhole principle again. This time we conclude that for at least

$$\frac{k}{2^{h(n)^{12}} 9^{2h(n)^6}}$$

indices i, the graphs G_i' are isomorphic, and the corresponding embeddings are all congruent. So, for these indices, the frameworks $(G_i', \mathbf{p}^{(i)}|_{U_i' \cup V_i'})$ are pairwise congruent. Letting I denote the set of such indices, we have $|I| \geq k/2^{h(n)^{12}} 9^{2h(n)^6}$.

Let v_1 and v_2 be arbitrary distinct vertices in V. Then, due to congruence, there exists a number $a > 0$ such that v_1 and v_2 are embedded by the realization $\mathbf{p}^{(i)}$ to points of P' that are at distance a apart, for each $i \in I$. Here we use the fact that H is isomorphic to G_i' for $i \in I$ so we may identify H with G_i', after fixing some labeling on the vertices of H, and apply the realization defined on G_i' also on H. Therefore, since the sets $\mathbf{p}^{(1)}(V_1), ..., \mathbf{p}^{(k)}(V_k)$ are subsets of P', by Theorem 5.11, we may say the same for $\mathbf{p}^{(i)}(V)$, for each $i \in I$.

Now we claim that for every $p, q \in P'$ there is a maximum of two indices $i \in I$ such that v_1 is embedded to p and v_2 is embedded to q by $\mathbf{p}^{(i)}$. Observe that when we have fixed the points in P' that v_1 and v_2 are mapped to, $\mathbf{p}^{(i)}$ must induce one of at most two possible realizations of H, due to congruence. These are the realizations that can be obtained from each other by reflection through the line incident with p and q. Hence, we are left with two possibilities on how U_i' is embedded.

We now use Theorem 5.11 iii) which claims that the sets $\mathbf{p}^{(1)}(U_1), ..., \mathbf{p}^{(k)}(U_k)$

are pairwise disjoint subsets of P. Hence, i has to be one of at most two indices in I, corresponding to each of the reflections through the line incident with p and q. Analogously, there are at most two indices in I such that v_1 is embedded to q and v_2 is embedded to p by $\mathbf{p}^{(i)}$.

By the argument above, we have that for every pair of points $p, q \in P'$ there are at most $2 \cdot 2 = 4$ indices $i \in I$ such that $\mathbf{p}^{(i)}$ embeds the pair v_1, v_2 into the pair p, q. Since this holds for each pair $p, q \in P'$ and each pair has a fixed distance, $\|p - q\|$, we may conclude that the number of pairs in P' determining an arbitrary, but fixed, distance a is at least

$$\frac{1}{4} \cdot \frac{k}{2^{h(n)^{12}} 9^{2h(n)^6}}.$$

Using the Szemerédi–Trotter theorem (Theorem 2.11) we can say that the same distance among the points of P' can occur at most $O(|P'|^{4/3})$ times. Thus we have

$$\frac{1}{4} \cdot \frac{k}{2^{h(n)^{12}} 9^{2h(n)^6}} \ll |P'|^{4/3} \asymp (n^{1/3} h(n)^4)^{4/3}, \tag{3}$$

where the cardinality of P' is given by Theorem 5.11. We also know that $k \gg n^{2/3}/h(n)^5$, from Theorem 5.11, so that

$$\frac{1}{4} \cdot \frac{n^{2/3}/h(n)^5}{2^{h(n)^{12}} 9^{2h(n)^6}} \ll \frac{1}{4} \cdot \frac{k}{2^{h(n)^{12}} 9^{2h(n)^6}}. \tag{4}$$

Putting the Inequality 3 and 4 together and simplifying, we obtain

$$\frac{n^{2/3}}{h(n)^5 2^{h(n)^{12}} 9^{2h(n)^6}} \ll n^{4/9} h(n)^{16/3}$$

Thus, we get that $n^{2/9} \ll h(n)^{31/3} 2^{h(n)^{12}} 9^{2h(n)^6}$.

Assume that $h(n) \asymp \log_2^{1/12}(n)$. Let $h(n) = c \log_2^{1/12}(n)$ for some constant $c > 0$. Then we have

$$n^{2/9} \ll (c \log_2^{1/12}(n))^{31/3} 2^{(c \log_2^{1/12}(n))^{12}} 9^{2(c \log_2^{1/12}(n))^6},$$

which implies that

$$n^{2/9} \ll n^{c^{12}} \cdot (c^{31/3} \log_2^{31/36}(n) 9^{2c^6 \log_2^{1/2}(n)}).$$

Now, if $c < (2/9)^{1/12}$ the asymptotic inequality above does not hold, because the parentheses on the right hand side grows asymptotically slower than any power of

n. So, our assumption that $u(P) \geq \frac{n^{4/3}}{h(n)}$, does not hold for all $h(n) \asymp \log_2^{1/12}(n)$. Therefore, we may conclude that

$$u(n) = O\left(\frac{n^{4/3}}{\log^{1/12}(n)}\right).$$

□

6 Summary and future work

Despite significant progress, the Erdős unit distance problem remains unsolved. The polynomial method, as well as the concept of rigid frameworks, has provided new tools, but further effort is required to improve the best known upper bound or to prove that the this bound is not optimal.

In future work, it would be interesting to analyze the problem from perspectives different from those considered in this thesis. For example, one could examine whether there exists an algorithm for choosing points for our point set P in such a way that we obtain the maximum number of unit distances among these points for every n.

Similarly, the question arises whether one can always find an optimal configuration for $n + 1$ points, starting from an optimal configuration for n points. In other words, does there exist a *greedy* algorithm for creating optimal configurations. This question might be answered by examining possible optimal configurations for small values of n. Closely related to this is also the question of whether there exists a formula, possibly a very intricate one, for the sequence of maximum number of unit distances determined by n points in the plane, $u(n)$.

For each value of n, one could also study the number of distinct configurations that have $u(n)$ pairs of points at unit distance from each other. More specifically, one could examine the number of non-isomorphic unit distance graphs on a set P, with $|P| = n$, such that $E(G_P) = u(n)$, using the notation introduced in Section 2.

Like we did in Section 4 with the triangular and hexagonal pattern, there are many other patterns and configurations that one could analyze and potentially find an upper bound for. With the triangular pattern we found that for seven points, for instance, this pattern was not optimal, since we found another configuration containing more unit distances. However, it would be interesting to study whether some patterns have regularity in regards to which values of n, if any, the configuration is optimal.

In Section 4 we also studied an example in which we imposed restrictions on which points in the plane were allowed in our point set P, namely only those from two given perpendicular lines. One could impose other restrictions and, for example, analyze how the number of unit distances determined by the points in P behaves if x and y in a point $(x, y) \in P$ have to be algebraic numbers over $\mathbb{Q}$ of at most a certain degree.

The configurations used for maximizing the number of pairs of unit distances, in Section 4, among two, three and four points in the plane corresponded to rigid frameworks of the associated graphs. Hence, naturally the question arises if we, for each n, can always find a configuration corresponding to a rigid framework containing $u(n)$ unit distances. It would also be interesting to examine the number of unit distances if we confine ourselves to only allowing configurations that correspond to rigid frameworks.

Throughout this paper we analyze the Erdős unit distance problem using the standard Euclidean metric on $\mathbb{R}^2$, but one could also analyze and potentially solve the problem using other metrics.

As mentioned in Section 4, the point sets for the triangular and hexagonal patterns correspond to nested point sets, while the $\sqrt{n} \times \sqrt{n}$ grids with changing step does not. In the two former cases the number of unit distances grow linearly in n, while in the latter case the number of unit distances has superlinear growth. It would be interesting to analyze whether the number of unit distances determined by n points can grow at most linearly in n, when looking at configurations corresponding to nested point sets.

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