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Maxwell's Conjecture - An Overview

av

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Abstract

Maxwell's Conjecture is a claim made by James Clerk Maxwell regarding the number of critical points of the electric potential generated by point charges. It is also an open problem in mathematics. This paper aims to give the reader an introduction to and overview of Maxwell's Conjecture, and to provide statistics for the distributions of critical points through computer experiments. A basic introduction to the electrostatics necessary to understand the conjecture is given, and the upper bound of $(N-1)^2$ critical points proposed by Maxwell is presented. The paper then outlines and summarises some previous research from other authors, presenting the many inferior upper bounds that have been found, alongside some other relevant results like specific configurations that reach the bound. We then dedicate the rest of the paper to computer experiments that test the conjecture. Random configurations of N point charges are generated for N ranging from 3 to 7, and the number of critical points of each configuration is investigated using repeated attempts at root finding. We then present the data on the number and distribution of critical points. The data clearly indicates that, at the very least, for random configurations generated 'haphazardly', configurations that approach the bound are rare, especially as the number of charges increases. This supports the conjecture's claim, though it does nothing to further any attempts at proving it. These results are approximate given the numerical approach, and there is room for error and undercounting given the method used. Finally, we discuss possible expansions or improvements that can be made to obtain more relevant or reliable results.

Sammanfattning

Maxwells hypotes är ett påstående av James Clerk Maxwell gällande antalet stationära punkter för den elektrostatiska potentialen då den genereras av källpunkter. Det är också ett öppet problem inom matematiken. Syftet med den här uppsatsen är att ge läsaren en introduktion till och överblick av Maxwells hypotes, och att ge statistik för hur de stationära punkterna är fördelade med hjälp av datorexperiment. En grundläggande genomgång av de delar av elektrostatiken som behövs för att förstå hypotesen ges, och den övre gränsen för stationära punkter, $(N-1)^2$, som Maxwell föreslår definieras. Uppsatsen går därefter igenom ett urval av tidigare forskningsresultat från andra upphovspersoner och visar både många av de sämre övre gränser som har hittats och vissa andra relevanta resultat, såsom specifika konfigurationer som når Maxwells gräns. Resten av uppsatsen ägnas åt att genomföra datorexperiment som testar hypotesen. Vi slumpgenererar konfigurationer av N källpunkter i spannet då N går från 3 till 7, och undersöker sedan antalet stationära punkter för varje konfiguration med hjälp av upprepade sökningar med numeriska metoder för att hitta rötter. Datan för antalet stationära punkter och deras fördelning presenteras sedan. Datan tyder tydligt på, åtminstone för vilt slumpgenererade konfigurationer, att konfigurationer vars antal stationära punkter närmar sig den övre gränsen är sällsynta, särskilt då N växer. Detta stödjer hypotesens innehåll, men det bidrar inte till några ytterligare försök att bevisa den. Resultaten är approximativa eftersom vi har använt numeriska metoder, och det finns risk för problem eller underskattning av antalet stationära punkter givet den valda metoden. Slutligen diskuterar vi hur framtida utökningar skulle kunna drivas samt andra förbättringar som kan göras för att nå mer intressanta eller pålitliga resultat.

Contents

1 Introduction

In 1873, James Clerk Maxwell published his book "A Treatise of Electricity and Magnetism"[Max73]. The book is dense, and covers many concepts ranging from describing phenomena to providing many theorems pertaining to the mathematical theory of electricity. Buried in its pages is a statement without any apparent proof, that has since been dubbed Maxwell's Conjecture after it was rediscovered by one of the authors of "A Mystery of Point Charges"[GNS07] almost 130 years later. Proving the conjecture remains an open problem today.

The goal of this paper is to provide the reader with an overview of Maxwell's Conjecture. In Section 2, we cover the very basics of electrostatics, defining the notion of a point charge and a configuration, followed by defining the electric potential and its relation to an electric field. Maxwell's Conjecture is a claim regarding the upper bound of the number of critical points of that electric potential, stating that the number of non-degenerate critical points cannot exceed $(N-1)^2$ where N is the number of point charges in the considered configuration.

In order to give the reader a suitable foundation of knowledge regarding Maxwell's Conjecture, we also summarise and mention a selection of previous research regarding the conjecture, comparing its proposed bound to a variety of other bounds that have stronger theoretical grounding, including some of the results of "A Mystery of Point Charges". We then move on to the actual results of this paper, which is a number of computer experiments made to test the Maxwell bound. The last subsection of Section 2 contains the method for doing so, which is a numerical approach that uses root-finding algorithms to investigate a range of configurations. The experiments are conducted on configurations of $3, 4, 5, 6$, and 7 point charges, and the number of critical points for a few hundred of each of those kinds of configurations are computed.

The results from these computer experiments are presented in Section 3 and discussed in Section 4, where we conclude that the results support the claim of the conjecture. We also note that configurations that even get close to the bound seem rare for large numbers of point charges. Finally, we restate and summarise the results of the paper in Section 5, and discuss what improvements or extensions could be made for any future research on the topic.

2 Background

This section covers the basics of electrostatics necessary for understanding Maxwell's Conjecture, the conjecture itself, and provides an overview of some previous results related to the conjecture. It also contains the general methodology used for the computer experiments.

2.1 Electrostatics

The treatment of electrostatics in this section and paper will largely mirror the treatment provided by Lucas Sanz in [LS+23], in places supplemented by notation or definitions from [GNS07].

Definition 2.1. A **point charge** is a pair consisting of a point $p \in \mathbb{R}^n$ denoting the location of the charge in the Euclidean space $\mathbb{R}^n$ ($n \in \mathbb{N}$ and $n \geq 2$) and a value $q \in \mathbb{R}$ ($q \neq 0$) denoting the magnitude of that point charge. A finite set of N such charges ($N \geq 1$) is called a **configuration of point charges** in $\mathbb{R}^n$, and is denoted $\{(p_k, q_k)\}_{k=1}^{N}$. It is usually implicit, and therefore omitted unless necessary.

Definition 2.2. Given a configuration of point charges in n-dimensional Euclidian space, the **electric potential** V (or simply potential) of that configuration evaluated at a point p is denoted $V(p)$, and defined as

$$V(p) = \begin{cases} -\sum_{k=1}^{N} q_k \ln |p - p_k| \text{ if } n = 2, \\ \sum_{k=1}^{N} \frac{q_k}{|p - p_k|^{n-2}} \text{ if } n > 2, \end{cases}$$

where p_k and q_k are the location and value respectively of the k-th point charge in the configuration, and $|p - p_k|$ is the distance between the points p and p_k. Notably, $p \neq p_k$ for any p_k in the configuration. For those familiar with electric potential, the above definition may look a little different because it omits a number of constants normally included in the definition. This is because we are only concerned with the critical points of the potential V, and the exclusion of those constants does not impact the locations of these critical points [LS+23, p. 13]. Therefore, we omit them for the sake of clarity.

Definition 2.3. Finally, we define the corresponding electric field of an electric potential V as

$$E = -\nabla V = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right),$$

or in other words, as the negative gradient of V. The resulting electrostatic force at each point p, once again different from any p_k in the configuration, is thus

$$E(p) = \begin{cases} \sum_{k=1}^{N} q_k \frac{p - p_k}{|p - p_k|^2} & \text{if } n = 2, \\ \sum_{k=1}^{N} (n-2) q_k \frac{p - p_k}{|p - p_k|^n} & \text{if } n > 2. \end{cases}$$

Note here that a point p in $\mathbb{R}^n$ that is not in the configuration of point charges is a critical point of V if $\nabla V(p) = 0$, which is now shown to be equivalent to $E(p) = 0$. This means any critical point of V is a point of equilibrium of E.

Definition 2.4. Before formulating the conjecture, we establish the notion of **non-degenerate** critical points. A critical point p of a function is non-degenerate if the function's Hessian matrix is invertible, or non-singular, at that point p.

With this, we are finally ready to define Maxwell's Conjecture.

Conjecture 2.5. Given any configuration of N point charges in $\mathbb{R}^3$, the number of non-degenerate critical points of that configuration's potential does not exceed $(N-1)^2$ [Max73, Item 113, p.136-137] [GNS07, Conjecture 1.3, p. 471].

The conjecture is just that, a conjecture, and is therefore not proven in its entirety thus far.

2.2 Previous Results

While the conjecture 2.5 has yet to be proven true or false, there have been some advances for configurations with certain constraints, and even certain higher upper bounds. This section aims to provide an overview of some of these results, without delving into too much technical detail.

2.2.1 Two Dimensions by Gauss

Maxwell's Conjecture specifically deals with three dimensions ($\mathbb{R}^3$), but in the case of $\mathbb{R}^2$, a similar bound has been proven by Gauss [GNS07, p.1] [Mar49, p.8]. Finding the reference from Gauss directly has proven difficult, given the age, length, and language of the original work, but Gabrielov et al. do provide the details in [GNS07], though the author of this paper has been unable to verify the reference personally.

Regardless, Lucas Sanz [LS+23, p.15-16] provides a proof in the same style as Gauss by identifying points in $\mathbb{R}^2$ as numbers in the complex plane $\mathbb{C}$. A point (x, y) can then be expressed as $x + iy$, using z and z_k in place of p and p_k respectively for the sake of clarity. Performing this substitution for definition 2.3 yields

$$E(z) = \sum_{k=1}^{N} q_k \frac{z - z_k}{|z - z_k|^2}$$

and, through standard operations on complex numbers, we can show that

$$\sum_{k=1}^{N} q_k \frac{z - z_k}{|z - z_k|^2} = 0 \Leftrightarrow \sum_{k=1}^{N} \frac{q_k}{z - z_k} = 0.$$

By noting that $z \neq z_k$ for all k (since the locations of the point charges in the configuration are not considerations for critical points), Lucas Sanz further simplifies the enumerator of the expression to finally arrive at

$$f = \sum_{j=1}^{N} q_k \prod_{k \neq j} (X - z_k) \tag{1}$$

where z has been replaced by X to emphasise that this is now a polynomial in one variable (f was simply chosen to mimic the source's naming convention). Because it is a polynomial with complex coefficients, we can apply the fundamental theorem of algebra. The product yields a degree of $N-1$, which by the aforementioned theorem means 1 has at most $N-1$ roots. These roots, as shown when defining 2.3, coincide with the critical points of the potential V.

This is, of course, not the full proof given that we have hand-waved the actual calculations, but it should suffice as proof-sketch to convey the general idea.

2.2.2 The Lower Bound

Before we continue to explore the progress made in investigating the upper bound set in 2.5, we establish the lower bound on the number of critical points for the same potential. In [MC14], Morse and Cairns prove that, for configurations in which the sum of all charges in the configuration is non-zero, there are at least $n-1$ critical point. The final case, that of $\sum_{k=1}^{N} q_k = 0$, is treated by Kiang in [Kia32], in which he arrives at a slightly lower bound.

Morse and Morse theory, the branch of mathematics named after him, are discussed more in 2.2.6, and we define the notion of the Morse index in 2.11. Given that definition, the theorem Morse and Cairns, and Kiang, provide that yields the lower bound of $n-1$ is, as formulated in [GNS07, Theorem 1.1].

Theorem 2.6. *Consider a configuration for which the total charge $\sum_{k=1}^{N} q_k$ is negative (respectively positive), consisting of u positive point charges and v negative point charges. If m_1 is the number of critical points of index 1 of the corresponding potential V, and m_2 is the number of critical points of index 2, then $m_2 \geq u$ (respectively $m_2 \geq u-1$), $m_1 \geq v-1$ (respectively $m_1 \geq v$), and $m_1 - m_2 = v - u - 1$.*

The lower bound immediately follows from the theorem by addition of the two equations.

2.2.3 Three Charges of Equal Magnitude by Tsai

The case of entirely unrestricted configurations remains open, but when configurations are constrained to three charges of equal magnitude, advances have been made. In their paper on the topic [Tsa15], Tsai manages to prove that Conjecture 2.5 holds under the conditions specified above. Furthermore, he also proves that the number of critical points of such a configuration is either $2, 3$, or 4.

Tsai has made a number of other contributions to this field of study. Two will be mentioned here, the first of which is proving the conjecture in the case where three point charges are fixed in the plane such that they form an equilateral triangle or an isosceles right triangle in [Tsa11]. The other is in [LT22], where he and Lee together find a configuration of four point charges such that they reach the upper limit of $(N-1)^2$, in this case 9, set by Maxwell's conjecture.

2.2.4 More on Restricted Configurations

There are a number of other contributors who have made advances in investigating Conjecture 2.5 for configurations restricted in some manner, typically either for specific numbers of charges or limiting the placement of these charges to the plane.

Those that explore alternative upper bounds are covered in 2.2.5, but two additional contributions of a different nature will be mentioned here.

In [Per13], Peretz applies the topological argument principle to the system of equations yielded when solving for critical points of the potential V of a configuration of three point charges. While a formulation of the principle would require an introduction to topology beyond the scope of this paper, Peretz obtains several results, including the same lower bound found in 2.2.2. Uteshev and Yashina make a contribution of a different nature in [UY16]. For configurations of three and four positive point charges fixed in the plane, they develop an analytical approach to finding the set of critical points by using symbolic computations. In addition, their method provides both the exact number and locations of these critical points when values are assigned to the parameters used for their computations.

2.2.5 Higher Upper Bounds

Several alternative upper boundaries have been explored and proven to hold, occasionally under specific constraints. While the upper boundary set by Conjecture 2.5 is typically much better, these are still of interest and provide some context to the problem. In their master's thesis [Nun25], Nunes provides an excellent overview of those efforts, which the author recommends the reader reads for detail around how each bound was arrived at. The bounds themselves will be restated here for ease of access and a cursory look at the various results, but are essentially simply lifted from Nunes.

We begin with Solomon Huang and Duy Nguyen's theorems from "Maxwell's Conjecture on four collinear points". Unfortunately, the original text seems inaccessible (and therefore not referenced), so the contents, and the formulation of the two theorems, are taken entirely from Nunes' account in [Nun25]. The theorems are:

Theorem 2.7. *Given a configuration of point charges with N collinear points such that the outermost point charge has the opposite sign of all other charges in the configuration, the potential V has at most $2(N^2 - 1)$ critical points.*

Theorem 2.8. *If there are exactly four collinear point charges in the configuration, and their magnitudes are precisely $-q_1 = q_2 = q_3 = q_4$, the potential V has at most twelve critical points.*

Comparing the boundaries imposed by both theorems to that of Conjecture 2.5, which yields an upper limit of nine critical points for the proposed configuration, they each yield 30 and twelve respectively.

Next, we look at Vladimir Zolotov's results in [Zol23], where they arrive at two boundaries for the number of critical points of the potential V by using the Thom-Milnor Theorem. While their paper uses this theorem to tackle other problems beyond Maxwell's Conjecture, the result of interest to us one of the two mentioned boundaries, which is:

Theorem 2.9. *Given a configuration of N point charges in $\mathbb{R}^n$, the corresponding potential V has at most $(n+2)(2n+3)^{n+N}$ isolated critical points [Zol23, Section 2.2].*

This is slightly simplified, since the original version is more general than necessary for our purposes. The other boundary is only applicable for even-numbered dimensions higher than 2, and thus not relevant for our exploration of Conjecture 2.5. The above theorem, when plugging in $n = 3$ and comparing it to Maxwell's Conjecture, the upper bound for a configuration of three point charges is $5 \cdot 9^6$, or 2657205, compared to Conjecture 2.5's four.

Finally, we highlight the results of Kenneth Killian in [Kil08], where they studied a subset of configurations that restrict the placement of the charges to the plane ($\mathbb{R}^2$), while still considering the configuration as a configuration in ($\mathbb{R}^3$). The upper bound they arrive at, given the above constraints, is

Theorem 2.10. *Given N point charges in the plane, the maximum number of equilibrium points in the plane is $[2^{N-1}(3N-2)]^2$ assuming a non-degenerate case [Kil08, p.6].*

We make one final comparison, once again applying the theorem to a configuration of three point charges, and thus arrive at the upper bound 784, which while considerably lower than the bound set by Zolotov's theorem is still significantly higher than that of Conjecture 2.5, which remains four.

2.2.6 The Origins of the Mystery by Gabrielov et al.

For our last look at previous results regarding Maxwell's Conjecture, we turn to "A Mystery of Point Charges" by Andrei Gabrielov, Dmitry Novikov, and Boris Shapiro, which could reasonably be considered the origin of research regarding Maxwell's Conjecture. This paper formulated the modern version of Conjecture 2.5 after one of the authors rediscovered it in [Max73] after it had reportedly been lost, or at the very least forgotten about, for 130 years [GNS07, p.2]. Gabrielov et al. also prove a higher upper bound, much like the examples previously discussed, and, furthermore, make several related conjectures. The main tools used by the authors are so called Morse theory and Voronoi diagrams, both of which will be briefly explained conceptually without diverting our focus to the technical aspects of those particular tools.

First, Morse theory. In [Mat02], Matsumoto states that,

> The primary concern of Morse theory is the relation between spaces and functions. The center of interest lies in how the critical points of a function defined on a space affect the topological shape of the space, and conversely, how the shape of a space controls the distribution of the critical points of a function.

which is half of the quote Lucas Sanz chose to describe Morse Theory in [LS+23]. While there is much to say about Morse theory, in the interest of brevity we will simply define the so called Morse Index of critical points of a function.

Definition 2.11. Given a non-degenerate critical point of a function f in n variables, the Morse Index, or simply index, of that critical point is the number of negative eigenvalues of the function's Hessian matrix.

The index of a critical point p informs us of what kind of critical point it is, as stated in [LS+23, Remark 1.1.21]. If the index of p is zero, p is a local minimum. If p's index is n, it is a local maximum. Anything between the two means p is a saddle point.

Next, we look at Voronoi diagrams. The description of Voronoi diagrams provided here is mostly paraphrased from [GNS07]. A Voronoi diagram considers a configuration of points (or sites) in $\mathbb{R}^n$, and divides that space into cells consisting of sets of points based on their distance to these sites. Each cell consists of all points that have the same set of nearest sites. For example, if there are two sites A and B in the plane, we obtain three different Voronoi cells. The cell S_A, which consists of all points whose nearest site is A, the equivalent setup for S_B, and the cell S_{AB}, which consists of all points equidistant from sites A and B. Cells S_A and S_B are each two-dimensional, while S_{AB} is one-dimensional (a line). In cases where a cell is a single point, that cell is zero-dimensional.

Before we look at the results from [GNS07], a slight modification to Definition 2.2 of the potential V needs to be made. Gabrielov et al. studied a generalised version of the potential given by a parameter $\alpha > 0$ given by

$$V_\alpha(p) = \sum_{k=1}^{N} \frac{q_k}{(|p - p_k|^2)^\alpha},$$

from which our original potential V can be extracted by setting the value of α to $\frac{1}{2}$. Now we can state the upper bounds arrived at in [GNS07].

Theorem 2.12. *Denote the maximal number of critical points of the potential $V_\alpha(p)$ for all non-degenerate configurations of N critical points by $M_N(n, \alpha)$ Then, for any $\alpha \geq 0$ and any positive integer n, we have*

$$M_N(n, \alpha) \leq 4^{N^2}(3N)^{2N}$$

and in the case of $N = 3$, we get the much better estimation

$$M_3(n, \alpha) \leq 12.$$

Comparing these to Maxwell's Conjecture for $N = 3$ (which gives a bound of four), the general case above gives a staggering 139314069504.

While interesting, Gabrielov et al. also show that for any configuration of point charges of the same sign, there is an $\alpha_0 > 0$ such that for any $\alpha \geq \alpha_0$, the critical points of the potential $V_\alpha(p)$ correspond to the cells of the Voronoi diagram of the configuration. Furthermore, the Morse Index of the critical points is equal to the dimension of the Voronoi cell [GNS07, Theorem 1.7]. It is this result that, according to the authors of that paper, motivated their main conjecture, which is Conjecture 1.8 in [GNS07]. We will finish this section by stating that conjecture, though also by noting that Edelsbrunner et al. found a counter-example to the conjecture in [EFO25, Section 3.2].

Conjecture 2.13. For any generic configuration of unit point charges and any $\alpha > 0$, one has

$$a_\alpha^j \leq \sharp^j,$$

where a_α^j is the number of critical points of index j of the potential $V_\alpha([p])$ and $\sharp^j$ is the number of all effective Voronoi cells of dimension j in the Voronoi diagram of the considered configuration.[GNS07, Conjecture 1.8]

2.3 Methodology

This section covers how the computer experiments conducted for this paper were set up and performed, as well as some possible shortcomings of the method.

2.3.1 Setup and Execution

The experiments rely on randomly generated configurations of N point charges where the value of N ranges from 3 to 7. Each point charge is generated by selecting a random value between -1 and 1 for q, and a random value following a normal distribution centred around 0 with a standard deviation of 2 for each coordinate of the point p. The configurations are normalised and centred around 0 to make the computations easier, but are denormalised again once the calculations have been performed.

For each configuration, we attempt to find points where the electrostatic force generated by the electric field defined in Definition 2.3 is zero. This is done by repeatedly making qualified guesses about possible locations of critical points, and then using several algorithms[JOP+ a][JOP+ b] to numerically find the closest critical point to that starting guess. For example, for each configuration of three point charges, we make 800 initial guesses. The guesses' locations are determined using a mix of three different methods. First, some number of guesses are made randomly in a cube around the origin. Secondly, some guesses are made near the charges themselves, albeit disturbed by numerical noise. Finally, some guesses are made around the midpoints between each pair of charges, both unweighted and weighted by the q-value of the point charge. This process then yields the (hopefully fairly accurate) number of critical points of the configuration.

The code with which these experiments are performed is written in Python 3.12.3, and was in large part written by the generative A.I Sonnet 4.6. The mathematical framework used in Python is provided by Numpy (version 2.4.4) and Scipy (version 1.17.1), and the optimisation methods used, as mentioned above, are the Powell hybrid method[JOP+ b] and the L-BFSG-B minimisation algorithm[JOP+ a]. Finally, we run the experiments in parallel on eleven workers using Python's multiprocessing library, and the computations are performed on the cloud servers of Hetzner.com, using their CX53 instances.

2.3.2 Shortcomings

The methodology chosen here has some shortcomings. First, there are several cases where numbers are disallowed from being 'too small' for the sake of the computations. This is typically not an issue, but the most notable case is for the values of q, which cannot be closer to 0 than 0.01, which does restrict the possible configurations. For example, the configuration found in [LT22] in Section 2.2.5 is impossible because the desired ratio between values of q is larger than permitted here.

There is also the issue of possible undercounting of critical points. Because the method used only searches for the closest critical point to the initial guess, it is theoretically possible that critical points go undiscovered, especially if the initial guesses are bad. The method of choosing initial guesses and the decently high number of such guesses per configuration should mitigate this, but it nonetheless remains a drawback of the method.

With a sufficiently robust foundation of knowledge regarding the conjecture laid and the details of the computer experiments now outlined, we move on to Sections 3 and 4 to see the results of these experiments.

3 Data

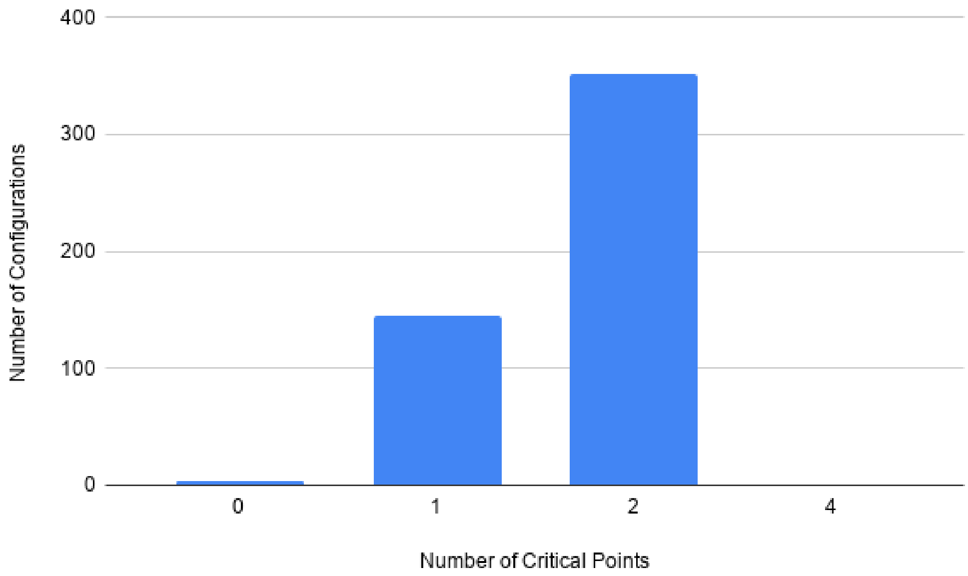

Figure 1: Configurations for $N = 3$. One configuration with four critical points. 500 configurations.

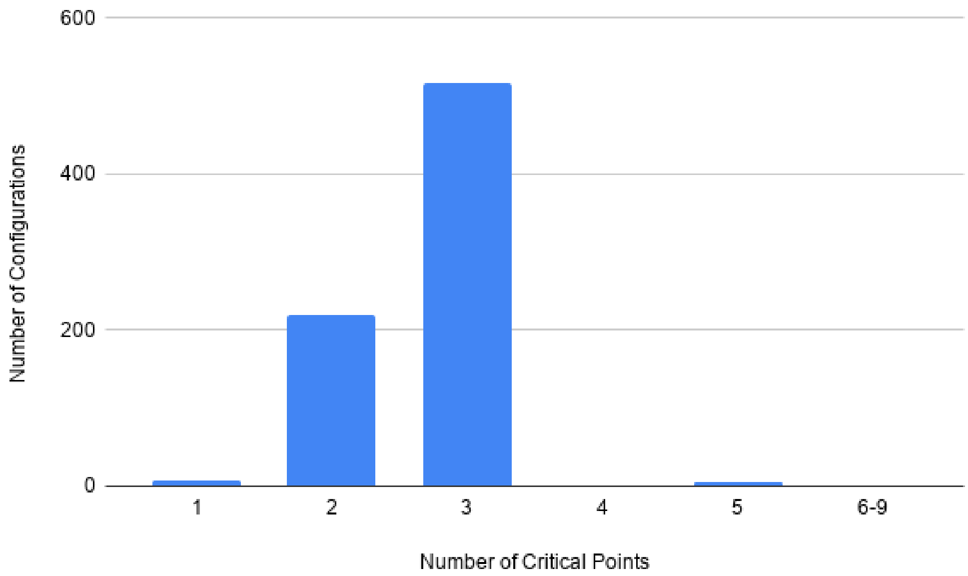

Figure 2: Configurations for $N = 4$. No configurations with more than five critical points. 750 configurations.

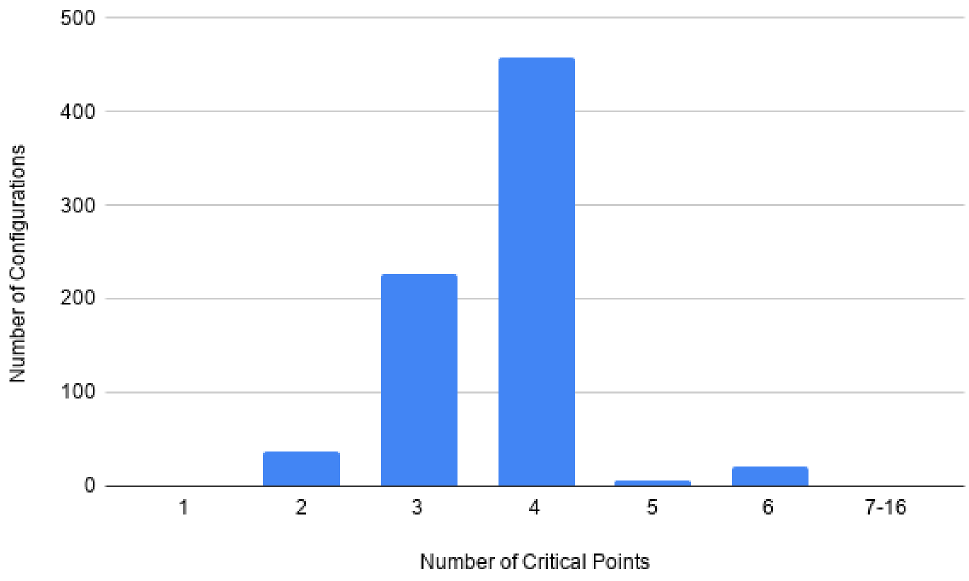

Figure 3: Configurations for $N = 5$. No configurations with more than six critical points. 750 configurations.

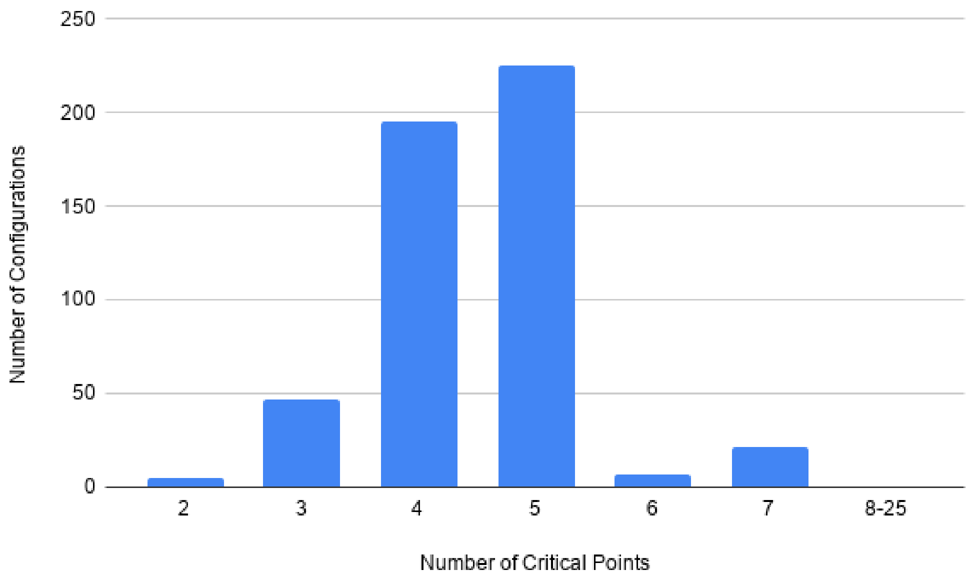

Figure 4: Configurations for $N = 6$. No configurations with more than seven critical points. 500 configurations.

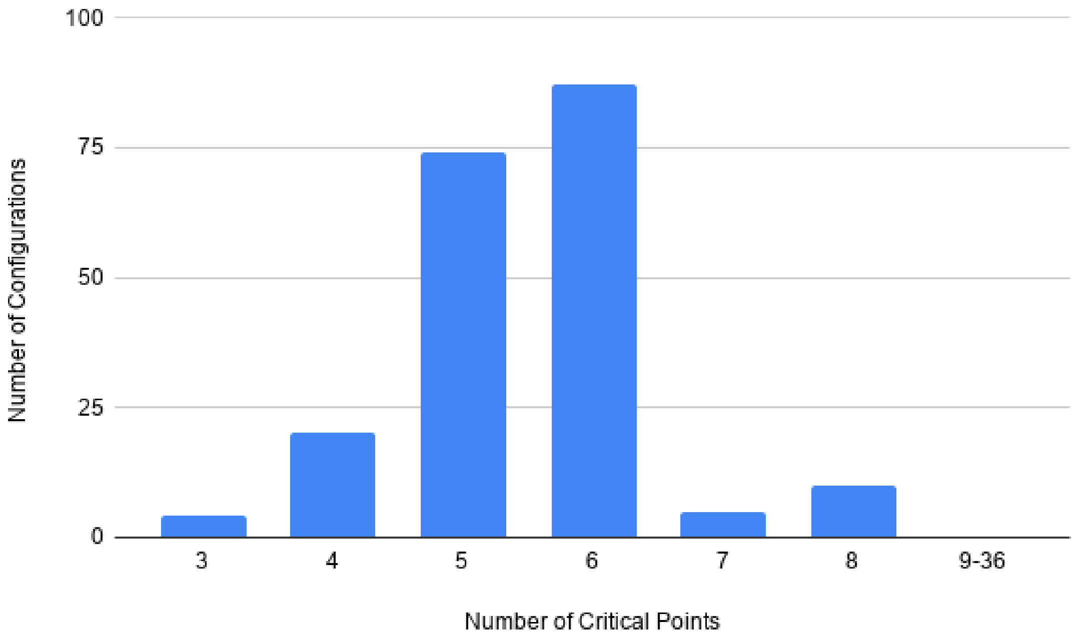

Figure 5: Configurations for $N = 7$. No configurations with more than eight critical points. 200 configurations.

Table 1: Table of Data. N is the number of charges, MXB is the bound set by Maxwell, Config♯ is the number of configurations run, CPMAX is the highest number of critical points found, CPA is the average number of critical points, STD is the standard deviation, CPMe is the median, CPMo is the mode, and RTA is the average runtime for a configuration in seconds.

N	**MXB**	**Config♯**	**CPMAX**	**CPA**	**STD**	**CPMe**	**CPMo**	**RTA**
3	4	500	4	1.70	0.48	2	2	855.33
4	9	750	5	2.71	0.53	3	3	1001.3
5	16	750	6	3.65	0.72	4	4	949.44
6	25	500	7	4.49	0.88	5	5	990.04
7	36	200	8	5.50	0.96	6	6	1083.5

4 Discussion

First, we will state what each of the figures and tables presented in Section 3 contains. Figures 1-5 each contain the distribution of the number of found critical points for $N = 3$-7, respectively. The corresponding figure texts also contain the number of configurations, also found in Table 1, as well as some clarifications on the contents. This proved necessary in the case of $N = 3$, where the single configuration that yielded four critical points is very difficult to see in the diagram. We take this opportunity to make another similar clarification regarding Figure 2, where the two configurations from which four critical points were obtained are likewise difficult to see at a glance. Finally, Table 1 contains the bulk of the statistics gleaned from the data, with its figure text providing a key to the various abbreviations used.

The most basic observation to be made is that no configuration was found that breaks the bound set by Maxwell. Even when considering configurations that reach the bound, only one such setup was found, and that was for three-point configurations, the 'simplest' case. Beyond that, the greatest number of critical points found strays further and further from the upper bound as we add more point charges to the configurations. This supports the claim made by the conjecture, given that no counter-examples were found. One might even be tempted to propose a different bound, given that the highest number of critical points found for each N seems to be $N + 1$, but recall the specific configuration mentioned in 2.2.3 that does reach the bound. The existence of such configurations means that one should not be too quick to propose an even lower bound, despite the results shown here. It seems more probable that, for configurations of many point charges, highly engineered configurations are required to reach (or possibly exceed) the Maxwell bound. Both the configuration from 2.2.3 and the paper containing the counter-example mentioned in 2.2.6 explore this avenue.

Returning to the data, the number of critical points seem to most commonly sit right beneath the number of point charges in the configuration, as seen in the CPMo, with the vast majority of configurations having either $N-1$ or $N-2$ critical points. This is reflected in the average, as the CPA sits between those two values for all N investigated in this paper. The low variation is also reflected in the standard deviation, which remains below 1 for all N, though this would likely continue slowly increasing as N grows as seen in 1.

It is, however, crucial to note that a large number of configurations, for every

N, fall below the lower bound on the number of critical points established in 2.2.2. Had this only been the case for a small number of configurations, this could have been explained by the risk of undercounting mentioned in 2.3, or possibly by the configuration being one in which the sum of the charges is zero. However, the great number of problematic cases instead suggests that there is a systematic problem with the method used to find the critical points. This means that it is indeed an issue of undercounting, but that it exceeds levels that can be considered acceptable for any conclusions drawn from the data to be reliable. The likely culprit is the method with which initial guesses are made, that some common locations of critical points are not represented well by the strategy used for finding starting locations from the algorithms are run. For example, it is possible that critical points regularly fall outside the cube within which random guesses are distributed, and that those same points are located further away from the origin than the point charges themselves. Regardless, this result means the usefulness of the data is dubious, since we do not know for certain how the uncounted critical points are distributed. If they were, for example, provably uniform, we could still draw conclusions by simply shifting the results by the appropriate number. As it stands, however, it is difficult to make any inferences regarding the true CPA or other numbers, beyond what has been stated above.

The average runtime for each configuration is perhaps the least interesting number mathematically speaking, but it does highlight the computational intensity of the problem, and why even this relatively meagre number of configurations takes so long. The lack of difference between the average runtimes may seem surprising at first, but this is one advantage of the method chosen for this paper. Finding the closest critical point to a given guess is a relatively similar task even in larger configurations, which means the factor that increases the computation time most is simply the number of guesses made. Since that number was left similar for all N, the average runtime remains comparable between the different values of N.

5 Conclusion

The data presented in this paper supports Maxwell's bound, and even getting close to the bound seems very difficult with random configurations. We end this paper by briefly discussing some alternate routes or possible extensions that could have or can be made.

An easy extension to make is simply to run more configurations. The author of this paper has relatively limited resources with which to run the computer experiments or purchase computing power in the manner done for this thesis, and throwing more computing power (or money) at the problem is one way of potentially getting more reliable results. This, however, is likely fruitless unless some of the parameters used for finding critical points are altered, such as making more initial guesses for each configuration, introducing better or additional ways of generating starting locations, and loosening some of the restrictions imposed for computational feasibility. While this is a promising strategy, a better way of utilising a drastic increase in computing power is likely to change the method however, from the root-finding approach used here to numerically solving systems of equations. Anecdotally from the test done when exploring options for this paper, this is much more expensive and takes a lot longer, but it does increase the reliability of any obtained results as well, since the risk of undercounting critical points is eliminated. This was not feasible for this paper as, even with the method used, the computations took a significant amount of time.

A better direction for future research might be to explore what was briefly mentioned in 4 regarding specific configurations. Instead of using truly random configurations, generating configurations where the point charges are placed according to pattern, like on the vertices of a platonic solid or along the curves of some chosen function might yield results that get closer to the Maxwell bound or that have a greater spread or a higher average of critical points.

Finally, it is not unlikely that the brute force approach is simply not well-suited to tackling Maxwell's Conjecture, and that more careful theoretical pursuits, whether directly or in tangentially related fields, have a greater likelihood of yielding any clarity in regards to the truth of the conjecture. Until then, proving or disproving Maxwell's Conjecture remains an open problem.

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