SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

The Phragmén-Lindelöf Principle

av

Jacob Thorstenson

2026 - No K26

The Phragmén-Lindelöf Principle

Jacob Thorstenson

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Erik Avelin, Annemarie Luger

2026

Abstract

The Phragmén-Lindelöf principle shows how holomorphic functions in unbounded regions can be bounded by their boundary values, under suitable growth conditions. In this paper we prove the maximum modulus theorem and extend it to unbounded regions using the Phragmen-Lindelöf method, to be applied in right half plane, sectors and general unbounded regions. We also include a proof of the three-lines theorem.

We then consider entire functions of exponential type. Through the indicator function, we show key properties and growth estimates. Finally we prove Carlson's theorem, showing that under suitable growth conditions, an entire function is uniquely determined by its values at the integers.

Sammanfattning

Phragmén-Lindelöfs princip visar hur holomorfa funktioner i obegränsade områden kan vara begränsade av dess värden längst randen, om det föreligger lämpliga tillväxtvillkor. I denna uppsats bevisar vi satsen om maximal modul och visar en utvidgning till obegränsade områden med hjälp av Phragmén-Lindelöfs metod. Vi visar tillämpningar av detta i det öppna högra halvplanet, obegränsade sektorer samt för generella obegränsade områden. Vi inkluderar också ett bevis för satsen om tre linjer.

I senare del betraktar vi hela funktioner av exponentiell typ. Genom att använda indikatorfunktionen visar vi viktiga egenskaper hos hela funktioner och hur vi kan uppskatta deras tillväxt. Slutligen ger vi ett bevis för Carlsons sats, som visar under lämpliga tillväxtvillkor att en hel funktion kan bestämmas entydigt genom att betrakta dess värden vid alla heltal.

1 Introduction

1.1 The Phragmén-Lindelöf Method

This paper is aimed at readers with a background in complex analysis at the undergraduate level. We assume familiarity with holomorphic functions, power series and the Cauchy formula.

The maximum modulus theorem shows that the constants are the only holomorphic functions that attain local maxima. Assume that f is holomorphic in a bounded region Ω and continuous on $\partial\Omega$. If

$$|f(z)| \leq A \qquad (z \in \partial\Omega)$$

for some $A < \infty$, then

$$|f(z)| \leq A \qquad (z \in \Omega).$$

If $|f(z)| = A$ in Ω, the theorem says that f is constant.

For an unbounded region Ω, if $M < \infty$ and

$$|f(z)| \leq M \qquad (z \in \partial\Pi),$$

then we cannot prove that $|f(z)| \leq M$ for all $z \in \Pi$ using the maximum modulus theorem directly, as Π is unbounded.

With the Phragmén-Lindelöf principle we can extend the maximum modulus theorem, to show that $|f| \leq M$ even if Π is unbounded, but only if f is under certain growth conditions. This will be shown through several examples in sections 2 and 3 of the paper.

1.2 Applications

In the last section we will consider entire functions such that

$$|f(z)| \leq Ae^{\tau|z|},$$

with $A, \tau < \infty$. We call this entire functions of exponential type.

We will also define

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r}.$$

This is called the indicator function of an entire function $f(z)$ of exponential type. It describes the growth of the function $f(z)$ along a ray $\{z : \arg z = \theta\}$.

In unbounded sectors, we can use the Phragmén-Lindelöf principle to help us show many interesting consequences. For an entire function f of exponential type, we will show the following.

1. If $h_f(\theta) < 0$ in an interval $[\alpha, \beta]$ of length $> \pi$, then $f = 0$.
2. If $h_f(\theta) = -\infty$ for some θ, then $f = 0$.
3. Unless $f = 0$, $h_f(\theta)$ is continuous.
4. Unless $f = 0$, $h_f(\theta + \pi/2) + h_f(\theta - \pi/2) \geq 0$ for all θ.

These consequences will be stated as propositions in section 4, which will later be used to prove a version of Carlson's theorem. This theorem is the main result of the paper. It shows that if an entire function $f(n) = 0$ at all the integers n, and satisfies a certain growth condition, then f is identically zero. In particular, if two entire functions f and g satisfy the same growth condition and are equal at the integers, then Carlson's theorem implies that

$$f(z) = g(z) \qquad (z \in \mathbb{C}).$$

Acknowledgements

I want to express my deepest gratitude to my advisor Erik Avelin, for his guidance and support throughout the entire process of writing this paper. He inspired and encouraged me to pursue this subject. As an instructor on several occasions, his insights and carefully chosen material have been invaluable to the journey that led me here. Despite my initial limited understanding of the subject and many misunderstandings along the way, he has always taken the time to help me understand the concepts clearly. Without someone as invested in this project as he was, I am certain I would not have come this far.

2 The Maximum Modulus Principle

We begin by denoting the open disk of radius of r centered at a with the notation $D(a; r)$. If f is holomorphic in $D = D(a; R)$, then by Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \qquad (|z-a| < R).$$

For $0 < r < R$, we write $z = re^{i\theta}$, which yields

$$f(a + re^{i\theta}) = \sum_{n=0}^{\infty} c_n r^n e^{in\theta} \qquad (r < R).$$

By the formula, often referred to as Parseval's formula, we have the following consequence.

Theorem 1. *If*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \qquad (|z-a| < R),$$

then, for $0 < r < R$*, we have*

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|^2 d\theta.$$

A proof will not be written here, but can be found in [4], chapter 10, sec. 10.22.

Suppose that f is holomorphic in a bounded region Ω and continuous on its closure, and $D \subset \Omega$. By Theorem 1, we obtain

$$(1) \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \qquad (r < R).$$

Since $\bar{\Omega}$ is compact, a global maximum of $|f|$ is attained at some point a. Assume that a is an interior point of Ω; then a is a local maximum and $|f(a + re^{i\theta})| \leq |f(a)|$ for all θ and $r < R$. Hence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|^2 \, d\theta \leq |f(a)|^2 \qquad (r < R),$$

and by (1), we have

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq |c_0|^2 \qquad (r < R).$$

Since all terms in the sum are nonnegative, $|c_n|^2 r^{2n} = 0$ for $n \neq 0$. Hence, $f(z) = f(a)$ for all $z \in D$. Define

$$A = \{z \in \Omega : f(z) \neq f(a)\}, \qquad B = \{z \in \Omega : f(z) = f(a)\}.$$

The set B is nonempty since it contains D. For $z_0 \in B$, f remains constant in a neighbourhood of z_0 by repeating the above procedure, so z_0 is always an interior point. Hence B is open. The set A is open since f is continuous, and by definition, the region

Ω cannot be written as a disjoint union of two nonempty open sets. Therefore A must be empty. This proves that $z \in \Omega$ if and only if $z \in B$, so f is constant in all of Ω.

Now assume that the point a is any point on $\bar{\Omega}$, and let

$$M = \max_{z \in \partial\Omega} |f(z)|.$$

Since $|f(a)|$ is a global maximum

$$|f(z)| \leq |f(a)| \qquad (z \in \bar{\Omega}).$$

If a is on $\partial\Omega$, then

$$|f(z)| \leq |f(a)| \leq M \qquad (z \in \bar{\Omega}).$$

If a is an interior point then by previous argument f is constant. Since f is continuous on $\bar{\Omega}$, it follows that f is constant with the same value on $\partial\Omega$, so

$$|f(z)| = |f(a)| \leq M \qquad (z \in \bar{\Omega}).$$

Hence we have that $|f(z)| \leq M$ for all $z \in \Omega$.

The above proves the theorem often referred to as the maximum modulus theorem.

Theorem 2. *Suppose that Ω is a bounded region and that f is holomorphic in Ω and continuous on its closure.*

If

$$|f(z)| \leq M \qquad (z \in \partial\Omega),$$

then $|f| \leq M$ in all of Ω. Equality occurs at some point Ω if and only if f is constant.

Consider the unbounded region of the right half-plane, with its boundary given by $\operatorname{Re} z = 0$. The function $f(z) = e^z$ is holomorphic in the right half plane. Along $\operatorname{Re} z = 0$, the function $|e^z| \leq 1$, but for $\operatorname{Re} z > 0$, e^z is unbounded.

This shows that Theorem 2 does not extend directly to unbounded regions. However, in [3] Phragmén and Lindelöf shows that under additional growth restrictions a holomorphic function in an unbounded region which is continuous on its closure can still be bounded by its boundary values. Before we show an example, we require the following lemma.

Lemma 1. *Let $\Omega = \mathbb{C}\backslash(-\infty, 0]$. As Ω is a simply connected domain, there exists a unique holomorphic function g in Ω, such that*

$$g'(z) = \frac{1}{z}, \qquad g(1) = 0.$$

For $\beta \in \mathbb{R}$, define

$$f(z) = e^{\beta g(z)}. \tag{2}$$

For $r > 0$ and $-\pi < \theta < \pi$, we have

$$f(re^{i\theta}) = r^\beta e^{i\beta\theta},$$

and f is holomorphic in Ω.

Proof. Since

$$\left(\frac{e^{g(z)}}{z}\right)' = 0,$$

it follows that $e^{g(z)}/z = C$. Since

$$C = \frac{e^{g(1)}}{1} = 1,$$

we have

$$e^{g(z)} = z. \tag{3}$$

For $z = re^{i\theta}$, with $r > 0$ and $-\pi < \theta < \pi$, by (3) we have

$$e^{g(re^{i\theta})} = re^{i\theta}.$$

Since

$$g(re^{i\theta}) = u + iv, \tag{4}$$

with $u, v \in \mathbb{R}$, it follows that

$$e^{g(re^{i\theta})} = e^u e^{iv} = re^{i\theta}.$$

Therefore

$$e^u = r, \qquad e^{iv} = e^{i\theta}.$$

Hence we have

$$u = \log r \qquad (r > 0), \tag{5}$$

and

$$v = \theta + 2\pi k \qquad (k \in \mathbb{Z},\ -\pi < \theta < \pi). \tag{6}$$

Since g is continuous, k must be constant by (4) and (6). From assumption, we have $g(1) = 0$, thus $v(1) = 0$. So for $re^{i\theta} = 1$, $\theta = 0$, and by (6), we have

$$0 = 2\pi k.$$

Therefore $k = 0$ and $\theta = v$. Using (4), it follows by (5) and (6) that

$$g(re^{i\theta}) = \log r + i\theta.$$

Therefore, by (2), we have

$$f(re^{i\theta}) = e^{\beta \log r} e^{i\beta\theta} = r^\beta e^{i\beta\theta} \qquad (r > 0,\ -\pi < \theta < \pi).$$

Thus, we can define

$$z^\beta := f(z) = e^{\beta g(z)} \qquad (z \in \Omega).$$

Since g is holomorphic in Ω, it follows that $f(z) = e^{\beta g(z)}$ is holomorphic in Ω. □

Here follows the example.

Let Π be the open right half-plane. Let f be holomorphic on Π and continuous on the closure, and suppose there exist constants $A < \infty$ and $\alpha < 1$, such that

$$|f(z)| \leq Ae^{|z|^\alpha} \qquad (z \in \Pi).$$

Assume that $|f(iy)| \leq 1$, along the y-axis. We claim that $|f(z)| \leq 1$ in all of Π.

Let $\beta, \epsilon > 0$, so that $\alpha < \beta < 1$, and define

$$h_\epsilon(z) = e^{-\epsilon z^\beta},$$

where z^β is defined using Lemma 1. Since $\Pi \subset \mathbb{C}\backslash(-\infty, 0]$, z^β is holomorphic on Π. Hence h_ϵ is holomorphic on Π. For $z = re^{i\theta}$, with $-\pi < \theta < \pi$, we have

$$z^\beta = r^\beta e^{i\beta\theta} = r^\beta(\cos(\beta\theta) + i\sin(\beta\theta)).$$

In the right half plane, $|\theta| < \pi/2$, so

$$|\beta\theta| < \beta\frac{\pi}{2} < \frac{\pi}{2}.$$

It follows that

$$\operatorname{Re}(z^\beta) = \operatorname{Re}(r^\beta e^{i\beta\theta}) = r^\beta\cos(\beta\theta) > 0 \qquad (z \in \Pi).$$

Hence

$$|h_\epsilon(z)| = |e^{-\epsilon r^\beta\cos(\beta\theta)}| \leq 1 \qquad (z \in \Pi).$$

Define

$$g_\epsilon(z) = f(z)h_\epsilon(z).$$

Since both $f(z)$ and $h_\epsilon(z)$ are holomorphic in Π, g_ϵ is holomorphic on Π. By the hypothesis and (2), we have

$$|g_\epsilon(z)| \leq A\exp(r^\alpha - \epsilon r^\beta\cos(\beta\pi/2)) \qquad (z \in \Pi). \tag{7}$$

Since $\beta > \alpha$, it follows that

$$\lim_{r\to\infty} r^\alpha - \epsilon r^\beta\cos(\beta\pi/2)) = -\infty.$$

There exists an $R > 0$, so that if $R \leq r$, $|g_\epsilon(z)| \leq 1$. If we let D_h be a half-disk, such that

$$D_h = \Pi \cap D(0; R),$$

then $|g_\epsilon| \leq 1$ along ∂D_h. By Theorem 2, $|g_\epsilon(z)| \leq 1$ in D_h. Since $|g_\epsilon| \leq 1$ for all ϵ. Let $\epsilon \to 0$, so that $|g_\epsilon(z)|$ tends to $|f(z)|$. This shows that $|f| \leq 1$ in D_h. By (7) it follows that $|f| \leq 1$ for all $z \in \Pi$.

3 Examples of Phragmén-Lindelöf arguments

In this section, we show further examples of the Phragmen-Lindelöf method. The first two theorems are found as exercises 9 and 12 in Rudin [4], chapter 12.

Previously, we chose $\beta < 1$, so that $\operatorname{Re} z^\beta$ is nonnegative in all of Π. Since $|\theta| < 2\pi$ for $re^{i\theta} \in \Pi$, this is equivalent to

$$\beta < \frac{\pi}{2|\theta|}.$$

This inequality is used in the following theorem that applies the method to any sector with an opening angle less than π.

Theorem 3. *Suppose we have the sector* $\Omega = \left\{ re^{i\theta} : |\theta| < \theta_0 < \pi \right\}$. *Let* $f(z)$ *be a holomorphic function in* Ω *which is continuous on its closure. Suppose there exist constants* $A < \infty$ *and* $\alpha < \frac{\pi}{2\theta_0}$, *so that*

$$|f(z)| \le Ae^{|z|^\alpha} \qquad (z \in \Omega).$$

If $|f(z)| \le 1$ *on the boundary rays, then* $|f(z)| \le 1$ *for all of* $z \in \Omega$.

Proof. Let $\beta, \epsilon > 0$ so that $\alpha < \beta < \frac{\pi}{2\theta_0}$ and define

$$h_\epsilon(z) = e^{-\epsilon z^\beta},$$

where z^β is defined using Lemma 1. Since $|\theta| < \theta_0 < \pi$, we have $-\pi < \theta < \pi$, so $\Omega \subset \mathbb{C}\backslash(-\infty, 0]$. It follows that z^β is holomorphic in Ω and thus $h_\epsilon(z)$ is holomorphic in Ω.

Since $|\beta\theta| < \beta\theta_0 < \pi/2$, we have

$$\operatorname{Re} z^\beta = r^\beta \cos(\beta\theta) > 0.$$

Hence

$$|h_\epsilon(z)| = |e^{-\epsilon r^\beta \cos(\beta\theta)}| \le 1 \qquad (z \in \Omega).$$

We define the product $g_\epsilon = f h_\epsilon$. The product function g_ϵ is holomorphic in Ω because the factors f and h_ϵ are holomorphic in Ω, and the remainder of the proof is analogous to that of the example above. □

This theorem will prove useful when we later consider entire functions f bounded by $|f(z)| \le Ce^{\tau|z|}$, with $C, \tau < \infty$. We can divide the entire plane into several sectors and apply Theorem 3 to any entire function f if the conditions of the hypothesis is satisfied.

We can also show the Phragmén-Lindelöf principle in a general region. In comparison to Theorem 3, the next theorem can be applied to any unbounded region, but it requires a much stronger assumption.

Theorem 4. *Let* $\Omega \neq \mathbb{C}$ *be an unbounded region and let* f *be holomorphic in* Ω *and continuous on its closure.*

Assume that

$$|f(z)| \le M \qquad (z \in \partial\Omega), \qquad |f(z)| \le B \qquad (z \in \Omega),$$

for some constants M, B. *Then*

$$|f(z)| \le M \qquad (z \in \Omega).$$

Proof. Let $z_0 \in \Omega$ and choose $a \in \partial\Omega$. Since f is continuous on $\bar{\Omega}$ and $|f(a)| \le M$, for $\epsilon > 0$ there exists $r > 0$ such that

$$|f(z)| \le M + \epsilon \qquad (z \in \bar{\Omega} \cap \bar{D}(a; r)).$$

Choose r small enough so that $z_0 \notin \bar{D}(a; r)$.

Let $\Omega_r = \Omega \backslash \bar{D}(a; r)$. For large $R > 0$, let V be the connected component containing z_0 of

$$\Omega_r \cap D(a; R).$$

This is a bounded region.

Now define

$$g_n(z) = \frac{f^n(z)}{z - a}.$$

Since $a \notin \Omega$, g_n is holomorphic in Ω and hence in V.

The boundary of V consists of three parts.

On the portion of ∂V lying on $\partial\Omega$, we have $|f(z)| \leq M$ and $|z - a| \geq r$, so

$$|g_n(z)| \leq \frac{M^n}{r}.$$

On the portion lying on $\partial D(a; r)$, we have $|f(z)| \leq M + \epsilon$ and $|z - a| = r$, so

$$|g_n(z)| \leq \frac{(M + \epsilon)^n}{r}.$$

On the portion lying on $\partial D(a; R)$, $|f(z)| \leq B$ and $|z - a| = R$, so

$$|g_n(z)| \leq \frac{B^n}{R}.$$

By the maximum modulus principle,

$$\left|\frac{f^n(z_0)}{z_0 - a}\right| \leq \max\left(\frac{M^n}{r}, \frac{(M + \epsilon)^n}{r}, \frac{B^n}{R}\right).$$

Since $M + \epsilon \geq M$ and $B^n/R \to 0$ as $R \to \infty$, we can conclude that

$$\left|\frac{f^n(z_0)}{z_0 - a}\right| \leq \frac{(M + \epsilon)^n}{r}.$$

It follows that

$$|f^n(z_0)| \leq (M + \epsilon)^n \frac{|z_0 - a|}{r}.$$

Taking the n-th root, we obtain

$$|f(z_0)| \leq (M + \epsilon)\left(\frac{|z_0 - a|}{r}\right)^{1/n}.$$

As $n \to \infty$, it follows that

$$|f(z_0)| \leq M + \epsilon.$$

Since this holds for all $\epsilon > 0$, letting $\epsilon \to 0$ we obtain

$$|f(z_0)| \leq M.$$

As $z_0 \in \Omega$ is arbitrary we can conclude that

$$|f(z)| \leq M \qquad (z \in \Omega).$$ □

If f is holomorphic in an unbounded sector such that we can apply Theorem 3. We can consider to apply Theorem 3 to f/g, where g is some suitable function that modify the behaviour of f, which can yield much stronger results. The following theorem, often referred to as the three-lines theorem, is a good example. We will prove this using Theorem 4. A proof not using Theorem 4 is given in Rudin [4], chapter 12, sec. 12.8.

Theorem 5. *Suppose* $\Omega = \{x + iy : a < x < b\}$ *and let* f *be holomorphic in* Ω *and continuous on* $\overline{\Omega}$. *Assume for some* $B < \infty$ *that* $|f(z)| \leq B$ *for all* $z \in \overline{\Omega}$. *If*

$$M(x) = \sup\{|f(x+iy)| : -\infty < y < \infty\} \qquad (a \leq x \leq b), \tag{8}$$

then

$$M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a} \qquad (a < x < b). \tag{9}$$

Proof. For $\epsilon > 0$, define the function

$$g_\epsilon(z) = \frac{f(z)}{(M(a)+\epsilon)^{\frac{b-z}{b-a}}(M(b)+\epsilon)^{\frac{z-a}{b-a}}}. \tag{10}$$

We can write the denominator as

$$\begin{aligned}&(M(a)+\epsilon)^{\frac{b-z}{b-a}}(M(b)+\epsilon)^{\frac{z-a}{b-a}}\\ &= \exp\left(\frac{b-z}{b-a}\log(M(a)+\epsilon)\right)\exp\left(\frac{z-a}{b-a}\log(M(b)+\epsilon)\right).\end{aligned}$$

Since

$$M(a)+\epsilon > 0, \qquad M(b)+\epsilon > 0,$$

We know that $\log(M(a)+\epsilon)$ and $\log(M(b)+\epsilon)$ are well defined. Each factor is an exponential function dependent on z, and since exponential functions are entire, the product is entire. We also have that exponential functions are never zero, therefore the expression in the denominator is never zero. Since f is holomorphic in Ω and $(M(a)+\epsilon)^{\frac{b-z}{b-a}}(M(b)+\epsilon)^{\frac{z-a}{b-a}}$ is entire, we have that g_ϵ is holomorphic in Ω. Along $\operatorname{Re} z = a$, we have that,

$$|g_\epsilon(a+iy)| = \frac{|f(a+iy)|}{M(a)+\epsilon} \leq \frac{M(a)}{M(a)+\epsilon} \leq 1,$$

and along $\operatorname{Re} z = b$, we have,

$$|g_\epsilon(b+iy)| = \frac{|f(b+iy)|}{M(b)+\epsilon} \leq \frac{M(b)}{M(b)+\epsilon} \leq 1.$$

The modulus of the denominator is

$$\left|(M(a)+\epsilon)^{\frac{b-z}{b-a}}(M(b)+\epsilon)^{\frac{z-a}{b-a}}\right| = (M(a)+\epsilon)^{\frac{b-x}{b-a}}(M(b)+\epsilon)^{\frac{x-a}{b-a}}. \tag{11}$$

Note that

$$0 \leq \frac{b-x}{b-a}, \qquad 0 \leq \frac{x-a}{b-a}, \qquad \frac{b-x}{b-a} + \frac{x-a}{b-a} = 1$$

for $x \in [a,b]$. Since $M(a)+\epsilon \geq \epsilon$ and $M(b)+\epsilon \geq \epsilon$, we have

$$(M(a)+\epsilon)^{\frac{b-x}{b-a}}(M(b)+\epsilon)^{\frac{x-a}{b-a}} \geq \epsilon^{\frac{b-x}{b-a}}\epsilon^{\frac{x-a}{b-a}} = \epsilon \tag{12}$$

By assumption $|f(z)| \leq B$ for $z \in \bar{\Omega}$. Hence by (11) and (12) the modulus of g_ϵ is

$$|g_\epsilon(z)| = \left|\frac{f(z)}{(M(a)+\epsilon)^{\frac{b-z}{b-a}}(M(b)+\epsilon)^{\frac{z-a}{b-a}}}\right| \leq \frac{B}{\epsilon} \qquad (z \in \bar{\Omega}).$$

The function g_ϵ is holomorphic in Ω and continuous on its closure. Since $|g_\epsilon| \le 1$ along the boundary of Ω, and $|g_\epsilon(z)| \le B/\epsilon$ for $z \in \Omega$, by Theorem 4, it follows that $|g_\epsilon(z)| \le 1$ for all $z \in \Omega$. Hence we have that

$$|f(x+iy)| \le (M(a)+\epsilon)^{\frac{b-x}{b-a}}(M(b)+\epsilon)^{\frac{x-a}{b-a}} \qquad (z \in \Omega).$$

Since the right side of the inequality does not depend on y, we can take the supremum of $|f(x+iy)|$ to obtain

$$M(x) \le (M(a)+\epsilon)^{\frac{b-x}{b-a}}(M(b)+\epsilon)^{\frac{x-a}{b-a}} \qquad (a < x < b).$$

Leting $\epsilon \to 0$, we obtain

$$M(x) \le M(a)^{\frac{b-x}{b-a}} M(b)^{\frac{x-a}{b-a}} \qquad (a < x < b).$$

Raising both sides to the power $(b-a)$ gives the desired conclusion. □

4 The indicator function

This section aims to apply the Phragmén-Lindelöf principle to show deeper results. In particular, we present a proof of Carlson's theorem. Our treatment follows closely that of Erik Avelin [1], but is considerably more detailed.

The goal of Carlson's theorem is to show that an entire function can identically be zero. To prove this, many of the following propositions below show conditions under which an entire function must vanish. A strategy is to first show conditions under which an entire function must be constant. In Rudin [4], chapter 10, sec.10.23, Liouville's theorem is presented, it shows the following consequence for bounded entire functions.

Theorem 6. *Every bounded entire function is constant.*

Proof. Let f be entire and $|f| \le M$. Since f is entire, $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in \mathbb{C}$. By using Parseval's formula we obtain

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \le M^2 \qquad (r > 0).$$

Hence, for $n \ge 1$ we have

$$|c_n|^2 \le \frac{M^2}{r^{2n}} \qquad (r > 0).$$

Let $r \to \infty$ and we obtain $c_n = 0$ for all $n \ge 1$, thus f is constant. □

In the following results of this section, we will only consider functions of exponential type. Some parts, like the indicator function, can be defined for functions of finite order as in Levin [2], sec 8.1. Moreover, the definition in [2] concerns functions holomorphic in a sector, while we consider entire functions.

Definition 1. For an entire function f, if there exist constants $A, \tau < \infty$ such that

$$|f(z)| \le Ae^{\tau|z|}, \tag{13}$$

then we say that the function f is of exponential type.

Taking logarithms on (13), yields

$$\log|f(z)| \le \tau|z| + \log A.$$

This shows the growth of $\log|f(z)|$ is bounded by a linear function. Dividing by $|z|$, we obtain

$$\frac{\log|f(z)|}{|z|} \le \tau + \frac{\log A}{|z|}. \tag{14}$$

This shows how the growth rate of $|f(z)|$ is bounded.

Definition 2. Let f be of exponential type. The directional growth of f is defined by

$$h(\theta) = h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r}.$$

We say that $h(\theta)$ is the indicator function of f.

Where lim sup is defined by

Definition 3. Let $D \subset \mathbb{R}$ be such that for every $R \in \mathbb{R}$, there exists $x \in D$ with $x \ge R$, and let $f : D \to \mathbb{R}$. The limit superior, as $x \to \infty$, is defined by

$$\limsup_{x\to\infty} f(x) = \lim_{R\to\infty} \sup_{x\ge R} f(x).$$

The limit superior as $x \to \infty$ always exists, since $\sup_{x>R} f(x)$ is decreasing in R.

By the definition of the indicator function, the product of two functions have the following inequality,

$$\begin{aligned} h_{fg}(\theta) &= \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})g(re^{i\theta})|}{r} \\ &\le \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r} + \limsup_{r\to\infty} \frac{\log|g(re^{i\theta})|}{r}. \end{aligned} \tag{15}$$

This will be useful in Theorem 7. We continue with an example of an indicator function.

Let $A, B \in \mathbb{R}$, and define $g(z) = e^{(A-iB)z}$. Since

$$|e^{(A-iB)z}| \le e^{\sqrt{A^2+B^2}|z|},$$

g is of exponential type. By Definition 2, it has the indicator function

$$\begin{aligned} h_g(\theta) &= \limsup_{r\to\infty} \frac{\log|e^{(A-iB)re^{i\theta}}|}{r} \\ &= \limsup_{r\to\infty} \operatorname{Re}(A-iB)e^{i\theta} \\ &= A\cos\theta + B\sin\theta. \end{aligned}$$

By (14), the growth rate of an entire function of exponential type is bounded. The following theorem shows how the indicator function can be bounded in an interval of length less than π.

Theorem 7. *Let f be a function of exponential type and assume that*

$$0 < \beta - \alpha < \pi.$$

If

$$h_f(\alpha) \leq A\cos\alpha + B\sin\alpha, \qquad h_f(\beta) \leq A\cos\beta + B\sin\beta,$$

Then,

$$h_f(\theta) \leq A\cos\theta + B\sin\theta \qquad (\alpha \leq \theta \leq \beta).$$

Proof. We begin by proving the theorem under the stronger assumption

$$h_f(\alpha) < A\cos\alpha + B\sin\alpha, \qquad h_f(\beta) < A\cos\beta + B\sin\beta, \tag{16}$$

then we follow up with the general case.

Let $S = \left\{re^{i\theta} : r \geq 0, \alpha \leq \theta \leq \beta\right\}$, and define the function

$$F(z) = f(z)e^{-(A-iB)z}. \tag{17}$$

Since f is entire, it follows that F is entire. Using the definition 2 and (15), we have

$$h_F(\theta) = \limsup_{r\to\infty} \frac{\log|F(re^{i\theta})|}{r} \leq h_f(\theta) - (A\cos\theta + B\sin\theta) \qquad (\theta = \alpha, \beta).$$

Under the assumption of (16), it follows that

$$h_F(\theta) < 0 \qquad (\theta = \alpha, \beta).$$

Choose $\delta > 0$, such that

$$h_F(\theta) < -2\delta \qquad (\theta = \alpha, \beta).$$

For some large R, if $r \geq R$, we have

$$|F(re^{i\alpha})| \leq e^{-\delta r} \qquad (r > R), \qquad |F(re^{i\beta})| \leq e^{-\delta r} \qquad (r > R).$$

Since F is continuous on the closed intervals $[0, Re^{i\alpha}]$ and $[0, Re^{i\beta}]$, we can choose $C < \infty$ so that F is bounded on these intervals. Hence, we have

$$\max_{r\in\mathbb{R}} \left(|F(re^{i\alpha})|, |F(re^{i\beta})|\right) \leq C.$$

Since the sector S has an opening angle $\beta - \alpha < \pi$ and $|F(z)| \leq C$ on ∂S. We apply Theorem 3, to show that

$$|F(z)| \leq C \qquad (z \in S).$$

By definition 2, we see that

$$h_F(\theta) = \limsup_{r\to\infty} \frac{\log|F(re^{i\theta})|}{r} \leq \lim_{r\to\infty} \frac{\log C}{r} = 0 \qquad (\alpha \leq \theta \leq \beta).$$

Therefore by (15), it follows that

$$h_f(\theta) - (A\cos\theta + B\sin\theta) \leq h_F(\theta) \leq 0 \qquad (\alpha \leq \theta \leq \beta).$$

This only holds true if

$$h_f(\theta) \leq (A\cos\theta + B\sin\theta),$$

which proves the conclusion for a strict inequality.

For the general case, let $\epsilon > 0$, and define

$$\phi_\epsilon(\theta) = A\cos\theta + B\sin\theta + \epsilon[\sin(\theta-\alpha) + \sin(\beta-\theta)].$$

In the endpoint α, we have

$$\phi_\epsilon(\alpha) = A\cos\alpha + B\sin\alpha + \epsilon\sin(\beta-\alpha).$$

In the endpoint β, we have

$$\phi_\epsilon(\beta) = A\cos\beta + B\sin\beta + \epsilon\sin(\beta-\alpha).$$

Since $0 < \beta - \alpha < \pi$, the function $\sin(\beta-\alpha) > 0$, so it follows that

$$A\cos\alpha + B\sin\alpha < \phi_\epsilon(\alpha), \qquad A\cos\beta + B\sin\beta < \phi_\epsilon(\beta).$$

Hence, by the hypothesis, we have

$$h_f(\alpha) < \phi_\epsilon(\alpha) \qquad h_f(\beta) < \phi_\epsilon(\beta).$$

With the strict case already proven, it follows that

$$h_f(\theta) \leq \phi_\epsilon(\theta) \qquad (\alpha < \theta < \beta).$$

This holds for all $\epsilon > 0$. Let $\epsilon \to 0$, so $\phi_\epsilon(\theta)$ tends to $A\cos\theta + B\sin\theta$, and the conclusion follows. □

In the beginning of the proof we define $F(z)$ in (17). This strategy is analogous to how we define $g_\epsilon(z)$ in (10) in Theorem 5.

Defining a function $A\cos\theta + B\sin\theta$ to help us satisfy the hypothesis of Theorem 7 can be done using the following definition.

Definition 4. Let $0 < \beta - \alpha < \pi$, and let the constants a, b be real. We define

$$H(\theta) = \frac{a\sin(\beta-\theta) + b\sin(\theta-\alpha)}{\sin(\beta-\alpha)} \qquad (\alpha \leq \theta \leq \beta).$$

The function $H(\theta)$ satisfies

$$H(\alpha) = a, \qquad H(\beta) = b,$$

It follows that $H(\theta)$ is of the form

$$H(\theta) = A\cos\theta + B\sin\theta.$$

In the case that $h(\alpha), h(\beta) > -\infty$, we can define

$$H_{\alpha,\beta}(\theta) = \frac{h(\alpha)\sin(\beta-\theta)h(\beta)\sin(\theta-\alpha)}{\sin(\beta-\alpha)} \qquad (\alpha \leq \theta \leq \beta).$$

The following propositions (with the exception of Proposition 3) show that if a function of exponential type is small (in some certain sense), it must vanish identically. They are defined to help us prove Carlson's theorem, which also assumes that f is small, in the sense that $f(n) = 0$ at all the integers.

Proposition 1. *Let f be of exponential type. If $h_f(\theta) < 0$ in an interval $[\alpha, \beta]$, with $\beta - \alpha > \pi$, then $f = 0$.*

Proof. Let $\gamma = (\alpha + \beta)/2$, so that $\gamma \in [\alpha, \beta]$. It follows that

$$\gamma - \alpha < \pi, \qquad \beta - \gamma < \pi, \qquad \alpha + 2\pi - \beta < \pi.$$

By the assumption, we have

$$h(\theta) < 0, \qquad (\theta = \alpha, \gamma, \beta)$$

Since $h(\theta)$ is periodic it follows that $h(\alpha) = h(\alpha + 2\pi)$.

As in Theorem 7, we can show that

$$|f(re^{i\theta})| \le e^{-\epsilon r} \qquad (\theta = \alpha, \gamma, \beta), \tag{18}$$

For some $\epsilon > 0$. Hence, we choose $C < \infty$, so that

$$\max_{r \in \mathbb{R}} \left(|f(re^{i\alpha})|, |f(re^{i\gamma})|, |f(re^{i\beta})|\right) \le C.$$

Define the sectors

$$S_1 = \left\{z = re^{i\theta} \,:\, r \ge 0, \alpha \le \theta \le \gamma\right\},$$
$$S_2 = \left\{z = re^{i\theta} \,:\, r \ge 0, \gamma \le \theta \le \beta\right\},$$
$$S_3 = \left\{z = re^{i\theta} \,:\, r \ge 0, \beta \le \theta \le \alpha + 2\pi\right\}.$$

Each sector has an opening angle $< \pi$. The function f is entire, thus f is holomorphic and continuous on the closure of each sector, and

$$|f(z)| \le C \qquad (z \in \partial S_1, \partial S_2, \partial S_3).$$

By Theorem 3, it follows that

$$|f(z)| \le C \qquad (z \in S_1, S_2, S_3).$$

The union of the sectors is

$$S_1 \cup S_2 \cup S_3 = \mathbb{C},$$

so $|f(z)| \le C$ for all $z \in \mathbb{C}$. Since f is bounded in the complex plane, it must be constant by Theorem 6. Therefore, for some c, $f(z) = c$.

By (18), we have

$$|c| \le e^{-\epsilon r} \qquad (r > R).$$

As $r \to \infty$, $|e^{-\epsilon r}|$ tends to zero. so $|c| = 0$. Therefore, we have

$$f = 0.$$

□

By Theorem 5, if $M(a) = 0$ then by (9), we have $M(x) = 0$. In (8), the supremum of $|f(x + iy)|$ is equal to zero for $|y| < \infty$ and x in the closed interval $[a, b]$. As f is continuous, f must vanish identically in the strip. The proof of the following proposition is analogous to the argument above.

Proposition 2. *Let f be of exponential type. If $h(\theta) = -\infty$ for some θ, then $f = 0$.*

Proof. Assume $h(\alpha) = -\infty$ for some α. Choose β such that

$$\frac{\pi}{2} < \beta - \alpha < \pi$$

Fix θ in (α, β). Let H be as in Definition 4, with $a > h(\alpha)$, $b > h(\beta)$.

Since $\pi/2 < \beta - \alpha < \pi$, we have

$$\sin(\beta - \theta) > 0, \qquad \sin(\theta - \alpha) > 0, \qquad \sin(\beta - \alpha) > 0.$$

By Theorem 7, it follows that

$$h(\theta) \leq H(\theta).$$

Let $a \to -\infty$, so that the term $a \sin(\beta - \theta)$ forces $H(\theta) \to -\infty$. This yields

$$h(\theta) = -\infty \qquad (\alpha < \theta < \beta).$$

Choose β_1 so that

$$\alpha + \frac{\pi}{2} < \beta_1 < \beta,$$

and choose γ, such that

$$\frac{\pi}{2} < \gamma - \beta_1 < \pi.$$

Since $h(\beta_1) = -\infty$, we can repeat the argument above in the interval (β_1, γ), to show that

$$h(\theta) = -\infty \qquad (\beta_1 < \theta < \gamma).$$

We have $(\alpha, \beta) \cup (\beta_1, \gamma) = (\alpha, \gamma)$. Since

$$h(\theta) = -\infty < 0 \qquad (\alpha < \theta < \gamma),$$

and $\gamma - \alpha > \pi$, it follows that $f = 0$ by Proposition 1. □

Before we can prove of Proposition 4, we need to show that $h(\theta)$ is continuous, hence the following proposition.

Proposition 3. *Let f be of exponential type. Unless $f = 0$, h is continuous.*

Proof. Let $\alpha < \theta < \varphi < \beta$ and $\beta - \alpha < \pi$. In case $h(\theta) = -\infty$ for some θ, then $f = 0$ by Proposition 2. Hence for $f \neq 0$, $h(\theta) > -\infty$. Define $H_{\alpha,\varphi}(\theta)$, $H_{\theta,\beta}(\varphi)$ as in definition 4. By Theorem 7, we have

$$h(\theta) \leq H_{\alpha,\varphi}(\theta), \qquad h(\varphi) \leq H_{\theta,\beta}(\varphi).$$

From the definition of H it follows that the first inequality is equivalent to

$$H_{\alpha,\theta}(\varphi) \leq h(\varphi).$$

Therefore we have

$$H_{\alpha,\theta}(\varphi) \leq h(\varphi) \leq H_{\theta,\beta}(\varphi). \tag{19}$$

If we let $\varphi \to \theta^+$, we have

$$\lim_{\varphi \to \theta^+} H_{\alpha,\theta}(\varphi) = h(\theta), \qquad \lim_{\varphi \to \theta^+} H_{\theta,\beta}(\varphi) = h(\theta).$$

Thus (19) proves that

$$\lim_{\varphi \to \theta^+} h(\varphi) = h(\theta).$$

Hence h is continuous from the right. The left-continuity is proven similarly. □

The hypothesis of Carlson's theorem assumes that

$$h(-\pi/2) + h(\pi/2) < 2\pi. \tag{20}$$

In the next proposition, we use the continuity of h to obtain an inequality tailored to the assumption of (20).

Proposition 4. *Let f be of exponential type. Unless $f = 0$, $h(\theta+\pi/2)+h(\theta-\pi/2) \geq 0$ for all θ.*

Proof. For $\epsilon > 0$ choose

$$\alpha = \theta + \epsilon - \pi/2, \qquad \beta = \theta + \pi/2,$$

so that $\beta - \alpha = \pi - \epsilon < \pi$ and $\theta \in (\alpha, \beta)$. Assume that $f \neq 0$. By Proposition 2, it follows that $h(\theta) \neq -\infty$. Define $H_{\alpha,\beta}(\phi)$ as in definition 4. We apply Theorem 7, to show that

$$h(\phi) \leq H_{\alpha,\beta}(\phi) \qquad (\alpha \leq \phi \leq \beta).$$

In particular, since $\theta \in (\alpha, \beta)$, we have

$$h(\theta) \leq H_{\alpha,\beta}(\theta). \tag{21}$$

As we compute $H_{\alpha,\beta}(\theta)$ we have

$$\sin(\beta - \theta) = 1, \qquad \sin(\theta - \alpha) = \cos\epsilon, \qquad \sin(\beta - \alpha) = \sin\epsilon.$$

Hence, the inequality (21) can be written as

$$h(\theta) \leq \frac{h(\theta + \epsilon - \pi/2) + h(\theta + \pi/2)\cos(\epsilon)}{\sin(\epsilon)}.$$

Multiply both sides by $\sin\epsilon$ to obtain

$$h(\theta)\sin\epsilon \leq h(\theta + \epsilon - \pi/2) + h(\theta + \pi/2)\cos(\epsilon). \tag{22}$$

Since $f \neq 0$ by assumption, h is continuous by Proposition 3. Therefore we can let $\epsilon \to 0^+$, so that

$$h(\theta + \epsilon - \pi/2) \to h(\theta - \pi/2), \qquad \cos\epsilon \to 1, \qquad \sin\epsilon \to 0.$$

Therefore by (22), we have $h(\theta)\sin\epsilon \to 0$, showing that

$$0 \leq h(\theta - \pi/2) + h(\theta + \pi/2).$$

Since θ is arbitrary, the conclusion follows. □

In the Phragmén-Lindelöf method, we commonly use an auxiliary function h_ϵ. In Carlson's theorem we assume that $f(n) = 0$ at all the integers. The auxiliary function for Carlson's theorem is

$$\frac{1}{\sin(\pi z)},$$

where $\sin(\pi n) = 0$ at all the integers.

Lemma 2. *If f is entire and $f(n) = 0$ for all integers n, then*

$$\frac{f(z)}{\sin(\pi z)},$$

is entire.

Proof. The function $f(z)/\sin(\pi z)$ is holomorphic on $\mathbb{C}\backslash\mathbb{Z}$, since $\sin(\pi z) \neq 0$ on $\mathbb{C}\backslash\mathbb{Z}$. Hence the conclusion follows if we show that, for each $n \in \mathbb{Z}$, $f(z)/\sin(\pi z)$ is holomorphic at $z = n$.

The power series expansion of sin shows that $\sin(\pi z)/z$ is entire and does not vanish at 0. We can choose $r > 0$, so that

$$\sin(\pi z)/z \neq 0 \qquad (|z| < r).$$

Therefore the function

$$\frac{z}{\sin(\pi z)} = \frac{1}{\sin(\pi z)/z},$$

is holomorphic for $|z| < r$. For $n \in \mathbb{Z}$, write $z = w + n$. It follows that

$$\frac{z-n}{\sin(\pi z)} = \frac{w}{\sin(\pi(w+n))}.$$

Since

$$\sin(\pi(w+n)) = (-1)^n \sin(w\pi),$$

we have

$$\frac{z-n}{\sin(\pi z)} = (-1)^n \frac{w}{\sin(\pi w)}.$$

The function $w/\sin(\pi w)$ is holomorphic for $|z| < r$, so it follows that $(z-n)/\sin(\pi z)$ is holomorphic in a neighbourhood of n.

Since f is entire, around n, we have the Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z-n)^k.$$

Since $f(n) = 0$, we have that $a_0 = 0$. Therefore, the sum can start at $k = 1$. We factor out $(z-n)$, to obtain

$$f(z) = (z-n) \sum_{k=1}^{\infty} a_k (z-n)^{k-1}.$$

Define the function

$$\varphi(z) = \sum_{k=1}^{\infty} a_k (z-n)^{k-1}.$$

This function is holomorphic in a neighbourhood of n. Thus, we have

$$f(z) = (z-n)\varphi(z).$$

Since

$$\frac{f(z)}{\sin(\pi z)} = \frac{(z-n)}{\sin(\pi z)}\varphi(z),$$

and $(z-n)/\sin(\pi z)$ and $\varphi(z)$ is holomorphic in a neighbourhood of n, the conclusion follows. □

Theorem 8. *Let f be an entire function of exponential type. Suppose*

$$h(-\pi/2) + h(\pi/2) < 2\pi,$$

and that

$$f(n) = 0 \qquad (n \in \mathbb{Z}).$$

Then $f(z) = 0$ for all z.

Proof. Begin by defining the function

$$g(z) = \frac{f(z)}{\sin(\pi z)}.$$

By Lemma 2, it follows that g is entire. Consider first the function $1/\sin(\pi z)$. By the definition of sin, we have

$$\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Write $z = x + iy$, to obtain

$$e^{i\pi(x+iy)} = e^{i\pi x}e^{-\pi y}, \qquad e^{-i\pi(x+iy)} = e^{-i\pi x}e^{\pi y}$$

By the triangle inequality, we have

$$|e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}| \geq |e^{\pi y} - e^{-\pi y}| = e^{\pi|y|} - e^{-\pi|y|}.$$

The right side of the equality is equal to

$$e^{\pi|y|}(1 - e^{-2\pi|y|}). \tag{23}$$

We can clearly see that

$$1 - e^{-2\pi|y|} \geq 1 - e^{-2\pi} \qquad (|y| > 1).$$

Therefore, there exists a constant C, such that

$$|\sin(\pi z)| \geq Ce^{\pi|y|} \qquad (|y| > 1).$$

Hence, we have

$$\frac{1}{|\sin(\pi z)|} \leq \frac{1}{Ce^{\pi|y|}} \qquad (|y| > 1).$$

Thus, we conclude that $1/|\sin(\pi z)|$ is bounded for $|y| > 1$.

Let $0 < \delta < 1/2$, and consider for fixed $n \in \mathbb{Z}$ the compact set

$$D_n = \{z = x + iy : |y| \leq 1,\ n - 1/2 \leq x \leq n + 1/2,\ |z - n| \geq \delta\}.$$

In D_0, $\sin(\pi z)$ is non-zero, therefore $\sin(\pi z)$ is continuous on D_0, thus $|\sin(\pi z)|$ is continuous on D_0. Since D_0 is compact, $|\sin(\pi z)|$ attains a minimum on D_0. This minimum is strictly positive since $|\sin(\pi z)|$ is non-zero on D_n. Hence there exists a constant c_0 such that

$$|\sin(\pi z)| \geq c_0 \qquad (z \in D_0).$$

It follows that

$$\frac{1}{|\sin(\pi z)|} \leq \frac{1}{c_0} \qquad (z \in D_0).$$

Since $|\sin(\pi z)|$ is periodic with period 1, the same bound holds for all D_n. Therefore, we conclude that $1/|\sin(\pi z)|$ is bounded for $|y| \leq 1$ outside the disks $|z - n| < \delta$ for all $n \in \mathbb{Z}$.

Since $1/|\sin(\pi z)|$ also is bounded by $1/C$ for $|y| > 1$, there exists a constant B, such that

$$B = \max\left(\frac{1}{C}, \frac{1}{c_0}\right).$$

Therefore, we have

$$\frac{1}{|\sin(\pi z)|} \leq B \qquad (|z - n| \geq \delta).$$

For fixed $N \in \mathbb{Z}$, consider circles

$$C_N = \left\{z : |z| = N + \frac{1}{2}\right\}.$$

On these circles, we have

$$|z - n| \geq ||z| - |n|| \qquad (z \in C_N).$$

As $|z| = N + 1/2$, we obtain

$$|z - n| \geq |N + \frac{1}{2} - |n|| \qquad (z \in C_N).$$

Since $|n|$ and N are both integers, it follows that

$$|N + \frac{1}{2} - |n|| \geq \frac{1}{2} \geq \delta \qquad (z \in C_N).$$

Hence, we have that $|z - n| \geq \delta$ for all $z \in C_N$.

As f is of exponential type, there exist constants A and τ such that

$$|f(z)| \leq Ae^{\tau|z|}.$$

It follows for $|f(z)|$, that

$$|f(z)| \leq Ae^{\tau|N+1/2|} \qquad (z \in C_N).$$

For $1/|\sin(\pi z)|$, we have

$$\frac{1}{|\sin(\pi z)|} \leq B \qquad (z \in C_N).$$

Therefore for $|g(z)|$, we have

$$|g(z)| \leq ABe^{\tau(N+1/2)}.$$

By the maximum modulus theorem, it follows that

$$|g(z)| \leq ABe^{\tau(N+1/2)} \qquad (|z| \leq N + 1/2).$$

Let $z \in \mathbb{C}$ and choose N such that $|z| \leq N + 1/2 \leq |z| + 1$. It follows that

$$|g(z)| \leq ABe^{\tau(N+1/2)} \leq ABe^{\tau(|z|+1)}.$$

Hence, let $C' = ABe^{\tau}$, so that

$$|g(z)| \le C' e^{\tau|z|} \qquad (z \in \mathbb{C}).$$

It follows that $g(z)$ is of exponential type.

To compute the indicator function in the directions $\theta = \pm\pi/2$, recall from (23), that

$$|\sin(i\pi y)| = \frac{1}{2}e^{\pi|y|}(1 - e^{-2\pi|y|}).$$

For $y > 0$, $|y| = y$, so

$$|\sin(i\pi y)| = \frac{1}{2}e^{\pi y}(1 - e^{-2\pi y}) \qquad (y > 0).$$

Take logarithms and divide by y, to obtain

$$\frac{\log\left|\frac{1}{2}\, e^{\pi y}(1 - e^{-2\pi y})\right|}{y} = \pi + \frac{\log\left(\frac{1}{2}\right)}{y} + \frac{\log(1 - e^{-2\pi y})}{y}.$$

Take the limit as $y \to \infty$, so that

$$\lim_{y\to\infty} \pi + \frac{\log(1/2)}{y} + \log(1 - e^{-2\pi y})y = \pi.$$

By the definition of the indicator function, we have that

$$h_g(\pi/2) = \limsup_{y\to\infty} \frac{\log|g(iy)|}{y} = \limsup_{y\to\infty} \left(\frac{\log|f(iy)|}{y} - \frac{\log|\sin(\pi i y)|}{y} \right).$$

By (15), it follows that

$$h_g(\pi/2) \le h_f(\pi/2) - \pi.$$

By symmetry, we have $|\sin(-\pi i y)| = |\sin(\pi i y)|$. Hence

$$h_g(-\pi/2) \le h_f(-\pi/2) - \pi.$$

By the hypothesis, we have

$$h_g(\pi/2) + h_g(-\pi/2) \le h_f(\pi/2) + h_f(-\pi/2) - 2\pi < 0.$$

By Proposition 4, it follows that $g = 0$, so

$$0 = \frac{f(z)}{\sin(\pi z)} \qquad (z \notin \mathbb{Z}).$$

Thus $f(z) = 0$ for all $z \notin \mathbb{Z}$, and since we already know f is zero at the integers, we conclude that

$$f = 0 \qquad (z \in \mathbb{C}). \qquad \square$$

References

[1] Erik Avelin. Indicator functions and Carlson's theorem. Lecture notes, 2026.

[2] B. Ya. Levin. *Lectures on entire functions*, volume 150 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translated from the Russian manuscript by Tkachenko.

[3] E. Phragmén and Ernst Lindelöf. Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier. *Acta Math.*, 31(1):381–406, 1908.

[4] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.