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Structural and Dynamical Failure in RCSJ Networks

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Abstract

We consider networks of Josephson junctions described by the Resistively and Capacitively Shunted Junction (RCSJ) model. It describes a dynamical current across every junction and a critical current, which if exceeded by the dynamical current, makes a junction non-superconducting. The dynamical currents are driven by a current applied via a source and a sink. We establish an incidence matrix-based formalism to describe the network. Static and dynamical failures are identified as two distinct failure types in an RCSJ network. Using the established formalism, we find a necessary and sufficient condition for the existence of a fully superconducting state in terms of a maximum flow problem. The max-flow-min-cut theorem and Cheeger inequality are used to connect that condition to the spectral properties of the underlying graph Laplacian. We subsequently derive a sufficient condition expressed in terms of the algebraic connectivity of the network. Dynamical failures are analyzed using the Laplace transform of a linearized small perturbation of the network. For small perturbations, the system is found to be asymptotically stable. The analysis further indicates spectral parameter regimes that may result in cascading failures across the network.

Sammanfattning

Vi betraktar nätverk av Josephsonövergångar som beskrivs av Resistively and Capacitively Shunted Junction (RCSJ) modellen. Den beskriver en dynamisk ström genom varje övergång samt en kritisk ström, vilken gör att övergången lämnar det supraledande tillståndet om den överskrids av den dynamiska strömmen. De dynamiska strömmarna drivs av en extern ström som appliceras via en källa och en sänka. Vi etablerar en formalism baserad på incidensmatriser för att beskriva nätverket. Statiska och dynamiska fel identifieras som två distinkta feltyper i ett RCSJ-nätverk. Med hjälp av den etablerade formalismen finner vi ett nödvändigt och tillräckligt villkor för existensen av ett helt supraledande tillstånd, formulerat som ett maxflödesproblem. Max-flow-min-cut-satsen och Cheegers olikhet används för att koppla detta villkor till de spektrala egenskaperna hos den underliggande graf-Laplacianen. Därefter härleder vi ett tillräckligt villkor uttryckt i termer av nätverkets algebraiska konnektivitet. Dynamiska fel analyseras med hjälp av Laplacetransformen av en linjäriserad liten perturbation av nätverket. För små perturbationer visar sig systemet vara asymptotiskt stabilt. Analysen indikerar vidare spektrala parameterområden som kan ge upphov till kaskadfel i nätverket.

Contents

1 Introduction

One of the most striking manifestations of macroscopic quantum behavior is superconductivity, in which electrical current can flow through a material without any resistance below a certain critical temperature [AM76, pp.727–730]. Circuits made up of such superconducting systems form the basis of a range of different modern technologies, like quantum computing devices or ultra-precise measurement equipment. When a material is in such a superconducting state, the absence of resistance implies that the voltage and current are no longer related by a simple expression like Ohm's law $V = RI$.

Instead, the current is described by a macroscopic complex order parameter whose phase evolves in time and becomes the relevant dynamical variable of the system. [AM76, p. 748] Still, the same fundamental rules on the conservation of current, Kirchhoff's law, apply and it is only the mechanism resulting in the macroscopic current that has changed.

When two such superconductors are weakly coupled together, like for example a superconducting wire interrupted by an insulator, the phase difference across that coupling gives rise to the so-called *Josephson effect*. [AM76, p. 751] This effect links the current I across the coupling to the phase difference $\Delta\theta$ through the nonlinear relation

$$I = I_C \sin(\Delta\theta). \tag{1}$$

This is called the *Josephson relation*, while the corresponding coupling is referred to as a so-called *Josephson junction*. [HPD81] Note also that this introduces an additional parameter I_C, referring to the critical current of the junction. Should the current induced by the phase difference $\Delta\theta$ exceed the critical current, no stationary phase difference can satisfy (1) and the junction enters a resistive state again. [AM76, p. 730] Furthermore, since the current depends only on $\sin(\Delta\theta)$, the phase difference is defined modulo 2π. Consequently, the phases themselves are only defined up to integer multiples of 2π. As such, both $\Delta\theta$ and $\Delta\theta + 2\pi k, k \in \mathbb{Z}$ describe the same physical state. The transition between the superconducting and resistive state therewith happens when the phase difference $\Delta\theta \to \frac{\pi}{2}$.

Using the Josephson effect, one can also express the effective voltage V across a junction, such that

$$V = \frac{\hbar}{2e}\frac{d(\Delta\theta)}{dt}, \tag{2}$$

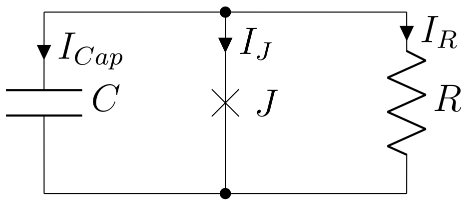

Figure 1: Circuit model of a Josephson junction in the RCSJ model. Illustration inspired by [Sto13].

where $\hbar$ is Planck's constant and e the charge of an electron [HPD81].

1.1 The RCSJ model

The RCSJ model or *Resistively and Capacitively Shunted Junction model* is a simplified circuit model to describe Josephson junctions. It describes the junction as having a resistance R, as well as a capacitance C and a superconducting junction element. [HPD81]

The capacitance can be understood to provide a measure of how much charge can be "stored" in a junction using a phase difference, whereas the superconducting junction element encodes the Josephson relation (1). These individual components can be organized conceptually into a circuit as shown in Figure 1 [HPD81].

So, we can find that every component contributes a current term, giving us

$$\begin{aligned} I &= I_{Cap} + I_J + I_R, \\ I &= C\frac{dV}{dt} + I_C \sin(\Delta\theta) + \frac{V}{R}, \end{aligned} \tag{3}$$

using the standard relations for capacitor and resistor currents [HPD81]. With (2) and its time derivative $\frac{dV}{dt} = \frac{\hbar}{2e}\frac{d^2(\Delta\theta)}{dt^2}$ in mind, we get the RCSJ equation [Tin96, pp.202–204],

$$I(t) = C\frac{\hbar}{2e}\frac{d^2(\Delta\theta)}{dt^2} + I_C \sin(\Delta\theta) + \frac{1}{R}\frac{\hbar}{2e}\frac{d(\Delta\theta)}{dt}.$$

1.2 Dimensionless RCSJ model

To make this model simpler from a mathematical point of view, there are a number of steps one can take to reduce the apparent complexity of the RCSJ equation. For example, when linearizing the equilibrium $\Delta\theta = 0$, such that $\sin(\Delta\theta) \approx \Delta\theta$, we can find that the RCSJ equation becomes a regular damped harmonic oscillator with a certain frequency ω_p. This frequency is called the Josephson plasma frequency [Sto13]

$$\omega_p = \sqrt{\frac{2eI_C}{\hbar C}},$$

and can be used to define the intrinsic timescales of the system. Consequently, it becomes natural to normalize the time variable, such that $\tau = \omega_p t$. Then we can write

$$\frac{d}{dt} = \omega_p \frac{d}{d\tau},$$

such that the RCSJ equation becomes

$$\begin{aligned}
I &= C\frac{\hbar}{2e}\frac{d^2(\Delta\theta)}{dt^2} + I_C \sin(\Delta\theta) + \frac{1}{R}\frac{\hbar}{2e}\frac{d(\Delta\theta)}{dt} \\
&= C\frac{\hbar}{2e}\omega_p^2\frac{d^2(\Delta\theta)}{d\tau^2} + I_C \sin(\Delta\theta) + \frac{1}{R}\frac{\hbar}{2e}\omega_p\frac{d(\Delta\theta)}{d\tau} \\
&= C\frac{\hbar}{2e}\frac{2eI_c}{\hbar C}\frac{d^2(\Delta\theta)}{d\tau^2} + I_C \sin(\Delta\theta) + \frac{1}{R}\frac{\hbar}{2e}\sqrt{\frac{2eI_C}{\hbar C}}\frac{d(\Delta\theta)}{d\tau} \\
\frac{I}{I_C} &= \frac{d^2(\Delta\theta)}{d\tau^2} + \sin(\Delta\theta) + \frac{1}{RI_C}\frac{\hbar}{2e}\sqrt{\frac{2eI_C}{\hbar C}}\frac{d(\Delta\theta)}{d\tau}.
\end{aligned}$$

Thus, we have managed to remove the constant prefactors for two of the three terms. At the same time, we effectively normalized the current across the junction by the critical current. We will refer to that normalized current as $i = \frac{I}{I_C}$. Additionally, all other constants are now collected in front of the resistive term. To simplify their notation, we define the so-called quality factor Q [Sto13],

$$\frac{1}{Q} = \frac{1}{RI_c}\frac{\hbar}{2e}\sqrt{\frac{2eI_C}{\hbar C}} = \frac{1}{R}\sqrt{\frac{\hbar}{2eCI_C}} = \frac{1}{RC\omega_p}. \tag{4}$$

All in all, the RCSJ equation reduces to

$$i = \frac{d^2(\Delta\theta)}{d\tau^2} + \frac{1}{Q}\frac{d(\Delta\theta)}{d\tau} + \sin(\Delta\theta), \tag{5}$$

a much simpler and more usable version of the RCSJ equation. Its dynamics now live in time-like domain τ, the current is normalized by the critical current and all constants have been collected in a dimensionless control parameter Q. This is the version of the RCSJ model that we will utilize moving forward.

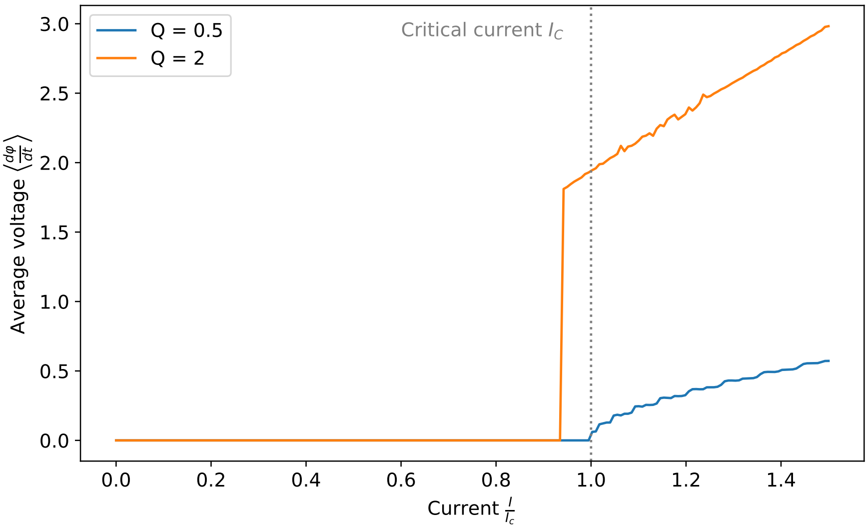

Figure 2: IV curve displaying the qualitative behavior of an RCSJ junction at different damping values. Own simulation.

1.3 Qualitative dynamics

When modeling a superconducting circuit network, it isn't only important to see how the isolated network behaves, but also how it reacts to an externally imposed current. Modeling this interaction with a single junction, one would inject the current at one side and extract it at the other. The phases of the system will then adjust in precisely such a way that (5) will result in that exact current across the junction.

The qualitative behavior of a Josephson junction with respect to the applied currents, as described by the RCSJ model, can be illustrated numerically. By averaging over the voltage obtained at different external current values, one can obtain the IV plot presented in Figure 2.

In Figure 2, three particular current regimes can be identified.

- $I < I_C$: If the current across a junction is *smaller* than the critical current, then the system is in a zero voltage state. This corresponds to the superconducting state discussed before, which means that there exists a stationary solution, such that $i = \sin(\Delta\theta)$ across the junction. Such a stationary state implies $\dot{\Delta\theta} = 0$ and therefore $V = 0$ as we can see. As a consequence, both the capacitive term $C\frac{dV}{dt}$ and the resistive term $\frac{V}{R}$ vanish and the RCSJ equation (5) reduces to the stationary Josephson relation.

- $I \approx I_C$: Here, when $\Delta\theta \approx \frac{\pi}{2}$, we find the transition regime between the superconducting and resistive state where the stationary solution disappears.

- $I > I_C$: With the junction current surpassing the critical current, the junction switches to the resistive state. From here on, phase differences $\Delta\theta$ increase without any bound and we start having a non-zero voltage average, as can be observed in the figure.

Furthermore, we can find that the quality factor Q seems to influence the nature of the transition and transient dynamics in general in the regime where $I \approx I_C$. In fact, a junction with $Q \ll 1$ is referred to as *overdamped*, whereas $Q \gg 1$ corresponds to the *underdamped* regime [Tin96, pp. 205–206]. The quality factor is thereby controlling the *damping* of the junction, meaning how long and far perturbations take to decay in the junction.

For the overdamped junction, perturbations in the phase decay exponentially and the transition to the resistive state can occur smoothly, as shown in Figure 2. In the underdamped case on the other hand perturbations oscillate before decaying and persist for longer, leading to the sharper onset of the finite voltage in the IV plot. These distinctions are not merely qualitative, since in networks of coupled junctions, the damping strength can influence whether perturbations will remain localized or propagate to neighboring junctions.

Josephson junctions of this kind, connected by wires, present the basis of any superconducting circuit. Thus, it becomes natural to model superconducting circuits using the same techniques as normal electrical circuits: using graph theory.

1.4 Contributions of the thesis

This thesis starts by developing a formalism to study the behavior of Josephson junctions in a network. The formalism makes use of incidence matrices to calculate phase differences across junctions, encodes individual critical currents and damping factors and enforces current conservation. Using our knowledge of this system, we characterize the transition from superconducting to resistive behavior as the failure of a Josephson junction in Section 3. In Section 4, we study the stationary state of the network and derive a condition allowing us to tell that no failure will occur when

$$I_{ext} \leq I_{max},$$

with I_{max} being the maximum flow of the network and I_{ext} the injected current. Thus, the question of whether the junctions in the network will fail or not is reduced to a maximum-flow problem. We further connect this condition to the spectrum of the underlying graph using a variation of Cheeger's constant, resulting in a lower bound for the maximum current. Finally, in Section 5, we use Laplace transforms to study the effect of small perturbations on the network, confirming that the system is stable with respect to small perturbations when all junctions are superconducting.

2 Network formulation

Having established the general behavior of the RCSJ model, it is now time to apply this framework to more than just a single junction. While it may be possible to derive the exact equations for a small number of connected junctions by hand, this task will become increasingly more complex with the number of junctions and number of connections between them. Thus, it makes sense to use the mathematical framework of *graphs* to describe such a system. In the following, we will work towards such a formalism.

2.1 Fundamental prerequisites

Let's start by defining the absolute basics of graph theory, based on the descriptions in [New18]:

Definition 2.1 (Graph)**.** A finite, simple **graph** $G = (V, E)$ consists of a finite set of vertices V, together with a set $E \subseteq [V]^2$ denoting the edges between those vertices, where $[V]^2$ is the set of all two element subsets of V. Hence, the edges are unordered pairs of vertices.

Definition 2.2 (Connected graph)**.** A graph $G = (V, E)$ is **connected** if for every pair of vertices $u, v \in V$ there exists a finite sequence of vertices $u = v_0, v_1, \dots, v_k = v$, such that $\{v_i, v_{i+1}\} \in E$ for all $i = 0, \dots, k-1$.

Remark 2.3. A graph with real-world properties and applications may also be called a **network**.

With the basic notion of a graph established, we can now extend this definition to allow for a connection strength for every edge.

Definition 2.4 (Weighted graph)**.** A finite, **weighted graph** $G = (V, E, w)$ consists of a finite set of vertices V, a set of edges $E \subseteq [V]^2$ between those vertices and a function $w : E \to \mathbb{R}$, which assigns a weight to each edge.

This is the mathematical object that we will work with. A convenient way to describe such a graph algebraically is by using an *adjacency matrix*.

Definition 2.5 (Adjacency matrix)**.** The **adjacency matrix** A of a weighted graph

$G = (V, E, w)$ is defined to be the $n \times n$ matrix with matrix elements A_{uv} such that

$$A_{uv} = \begin{cases} w(\{u, v\}) & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise,} \end{cases}$$

where $w(\{u, v\}) \in \mathbb{R}$ denotes the weight of the edge $\{u, v\}$.

Furthermore, we can define *the degree matrix*, which holds the aggregate weight associated with each vertex.

Definition 2.6 (Degree matrix)**.** Let $G = (V, E, w)$ be a finite weighted graph with an adjacency matrix A with elements A_{uv}. Then, the **degree** of a vertex is defined by

$$d_u = \sum_{v \in V} A_{uv}$$

and the **degree matrix** by the diagonal matrix

$$D := \operatorname{diag}(d_1, d_2, \dots, d_{|V|}).$$

Using the degree matrix and the adjacency matrix, one can construct the graph Laplacian in the following way.

Definition 2.7 (Graph Laplacian)**.** Let $G = (V, E, w)$ be a finite, weighted graph with an adjacency matrix A and a degree matrix D. The graph Laplacian is defined as

$$L = D - A.$$

Remark 2.8. In fact, we can see immediately from this definition that the entries of the graph Laplacian have to be given by the matrix elements

$$L_{uv} = \begin{cases} d_u & \text{if } u = v, \\ -w(\{u, v\}) & \text{if } u \neq v \text{ and } \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

As the adjacency matrix is symmetric and the degree matrix diagonal, and hence symmetric, the graph Laplacian $L = D - A$ has to be symmetric.

Another way to associate additional information with a network is to define a function on its edges.

Definition 2.9 (Edge function)**.** Let $G = (V, E, w)$ be a weighted graph. An **edge function** is a function $f : E \to \mathbb{R}$ that assigns a real value to each edge of the graph.

Defining such a function allows us to encapsulate possible dynamics on the edges, without incorporating them directly into the network structure. The weight function of the edges is therefore a particular edge function that is incorporated in the network structure through the adjacency and degree matrices.

Remark 2.10*.* Additionally, any edge $e \in E$ can be equally denoted by $\{u, v\}$ for some $u, v \in V$. We therefore write $f(e) = f(\{u, v\})$.

Analogous to the edges being associated with additional properties in the form of functions, vertices of a graph can also carry information.

Definition 2.11 (Vertex function)**.** Let $G = (V, E, w)$ be a weighted graph. A **vertex function** refers to a function g of the form $g : V \to \mathbb{R}$.

Using these definitions, we are now able to construct a network, describe its structure and specify dynamical variables on the same mathematical object. This provides a sufficient basis to finally insert the RCSJ model into this framework.

2.2 Initial formulation of an RCSJ network

First of all, the definition of the dynamical system we are going to implement on the network will be based on the RCSJ model, as discussed previously. To simplify any subsequent calculations, we will make use of the reduced RCSJ equation (5) using the Q-factor simplification:

$$i(\phi, \tau) = \ddot{\phi} + \frac{1}{Q}\dot{\phi} + \sin(\phi).$$

Here, i refers to the normalized current and ϕ represents the *phase difference* across the junctions.

Remark 2.12*.* As in standard physics notation, dots over variables refer to time, or time-like, derivatives. So, $\dot{\phi} = \frac{d\phi}{d\tau}$, whereas two dots imply the second time derivative and similar for higher derivatives.

At this point we have to start thinking about how the Josephson junctions will be placed on the network. In theory, this choice is completely arbitrary, but there are a number of structural considerations that can be made regardless. First of all,

ϕ refers to the phase differences between the phases θ before the junction, and after the junction. This makes it a natural choice to place the junctions on the *edges* of the network, modeling the RCSJ equation as an *edge function.* Consequently, phases θ are defined on the *vertices* as vertex functions. Explicitly, this convention would correspond to the equation

$$i(u, v, \tau) = (\ddot{\theta}(v) - \ddot{\theta}(u)) + \frac{1}{Q}(\dot{\theta}(v) - \dot{\theta}(u)) + \sin(\theta(v) - \theta(u)),$$

where u and v are the two vertices between which the junction lies.

Notation. To circumvent this complex notation, we will write the phase difference as $\phi(u, v) = \theta(v) - \theta(u)$.

Remark 2.13. Keeping in mind that the phases θ are defined modulo 2π, we will only work with real-valued representatives of the physical phase in the following.

Another point to consider on a structural level are the critical currents I_C of the individual junctions. In order to allow for maximum flexibility in what kind of systems can be modeled, each junction should have an individually assigned, static critical current.

Remark 2.14. As the critical current is a static, fundamental property of each junction, it is best stored in terms of the weights A_{uv} of the weighted adjacency matrix A.

Furthermore, we have to ensure that the currents flowing through the Josephson junctions obey Kirchhoff's current laws. This means that all currents have to be conserved. In the current mental model of our network, this means the following:

1. Since the RCSJ equation gives the current across a junction, every junction carries a current $i(u, v, \tau)$.

2. The junctions are connected via the vertices they are adjacent to and multiple junctions can be adjacent to the same vertex. Thus, we can imagine that the current flows through the vertices and into adjacent junctions. This is where we must ensure that the total current flowing in and out of any single vertex adds up to exactly 0.

In other words, it is the current *flowing through the vertices* that has to explicitly be conserved.

Remark 2.15. To formulate current conservation rigorously, we fix an arbitrary orientation for every edge on the graph. For such an edge $\{u, v\}$, the current $i(u, v, \tau)$ flowing from u to v is defined *positive*. Current in the opposite direction changes sign such that

$$i(u, v, \tau) = -i(v, u, \tau).$$

Using this convention, we can now formulate current conservation explicitly.

Condition 2.16. *Let $G = (V, E)$ be a weighted graph with an adjacency matrix A and weights $A_{uv} = I_C(\{u, v\})$ corresponding to critical currents for every edge $\{u, v\} \in E$. For any vertex $v \in V$, the following condition holds:*

$$\sum_{u \in V} A_{uv}\, i(u, v, \tau) = 0.$$

In this condition, we first pick any vertex v from the vertices V. Then, it makes use of the adjacency matrix to check against all available vertices whether there is a junction between them. If there is, the adjacency matrix will be non-zero and is multiplied with the normalized current given by the Q-factor simplification of the RCSJ equation (5). The multiplication cancels out the normalization $i = \frac{I}{I_C}$ and results in the actual physical current I that is then summed over for all edges. Considering that in- and outgoing currents are differentiated by sign, the physical currents are thus explicitly conserved across vertices.

Furthermore, we need to ensure the physical usefulness of the network at hand. Currently, it just enforces current conservation on all vertices, and there is no way to model the act of applying current on the system to see how it evolves. We therefore introduce two special vertices s and t that will act as a *source* and a *sink*, respectively. The source vertex injects an external current I_{ext}, while the sink extracts the same amount. Following the convention introduced in Remark 2.15, the current injected at the source strictly leaves the source vertex. Therefore, it has a negative sign. Similarly, the sign of the current extracted from the sink has to be positive. That way, the sum of currents flowing through any vertex except s and t is 0, making sure that all currents are conserved.

Condition 2.17 (Current conservation)**.** *Let $G = (V, E, w)$ be a weighted graph with an adjacency matrix A, a source $s \in V$ and a sink $t \in V$. The weights correspond* *to the critical currents I_C. For any vertex $v \in V$, the following condition holds:*

$$\sum_{u \in V} A_{uv}\, i(u, v, \tau) = \begin{cases} -I_{ext} & \textit{if } v = s \\ I_{ext} & \textit{if } v = t \\ 0 & \textit{otherwise.} \end{cases} \tag{6}$$

Remark 2.18. For notational simplicity, we will consider the time dependence of the phase $\theta(u, \tau)$ for any $u \in V$ in $G = (V, E)$ to be implicit. Thus, for any term involving phase or a derived quantity it will be assumed that $\theta(u) \doteq \theta(u, \tau)$.

At last, we have to consider the fact that the Q-factor simplification of the RCSJ model contains the factor $1/Q$. Therefore, we will allow this parameter to vary per junction and model it as a static edge function $\mathcal{D} = 1/Q$, the damping parameter.

With these conditions and considerations in mind, we can now define the general structure that the network of Josephson junctions should have.

Definition 2.19 (Initial formulation of the RCSJ Network)**.** Let an **RCSJ Network** be a tuple $\mathcal{N} = (G, A, s, t)$, where G is a finite, simple, weighted and connected graph with an adjacency matrix A and two distinguished vertices $s, t \in V$. The weights of the adjacency matrix A correspond to the positive critical current $I_C(e)$ for any edge $e \in E$. Furthermore, the following functions are defined on the graph G:

- the so-called phase, a vertex function θ,
- an edge function i representing a current and
- a positive, static edge function $\mathcal{D}$, the damping parameter.

This definition strictly only includes information about the objects making up the RCSJ Network and does not include the desired dynamical aspects yet. These are defined separately in the following.

Definition 2.20 (Initial formulation of the RCSJ Network Dynamics)**.** The dynamics of an RCSJ Network $\mathcal{N}$ are given by the edge function

$$i(u, v) = \ddot{\phi}(u, v) + \mathcal{D}(u, v) \cdot \dot{\phi}(u, v) + \sin(\phi(u, v)), \tag{7}$$

where $\phi(u,v) = \theta(v) - \theta(u)$ is the difference of the vertex function θ and $\mathcal{D}$ the damping parameter. The edge function $i(u,v)$ is constrained by current conservation, such that for any $v \in V$,

$$\sum_{u \in V} A_{uv}\, i(u,v) = \begin{cases} -I_{ext} & \text{if } v = s \\ I_{ext} & \text{if } v = t \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

In the following, we will try to use this framework to construct a concrete example network.

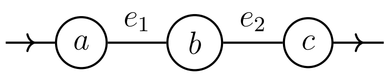

Figure 3: A simple graph G.

Example 2.21. We will consider the simple graph G, with $V = \{a, b, c\}$ and $E = \{e_1, e_2\}$ as pictured in Figure 3. Assuming critical currents of $I_C = 1$ for all junctions, the graph would have an adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

with the arrows indicating that node a serves as the injection node, whereas node c is the exit node. A current I_{ext} is injected or exits them respectively. Additionally, the RCSJ junctions between two nodes $u, v \in V$ carry a current $i(u,v)$. In this simple example, we can see with relative ease from Figure 3 that this junction current has to be given by

$$\begin{aligned} i(e_1) = i(a,b) &= \left(\ddot{\theta}(b) - \ddot{\theta}(a)\right) + \mathcal{D}(a,b)\left(\dot{\theta}(b) - \dot{\theta}(a)\right) + \sin\left(\theta(b) - \theta(a)\right) \\ i(e_2) = i(b,c) &= \left(\ddot{\theta}(c) - \ddot{\theta}(b)\right) + \mathcal{D}(b,c)\left(\dot{\theta}(c) - \dot{\theta}(b)\right) + \sin\left(\theta(c) - \theta(b)\right). \end{aligned}$$

Using the Kirchhoff's law, we find just as easily that the physical current I flowing through a vertex $u \in V$ is given by

$$
\begin{aligned}
I(v) &= \sum_{u \in V} A_{uv}\, i(u,v) = \sum_{u \in V} \Big(\ddot{\phi}(u,v) + \mathcal{D}(u,v)\dot{\phi}(u,v) + \sin(\phi(u,v))\Big) \\
&= \sum_{u \in V} A_{uv} \left(\Big(\ddot{\theta}(v) - \ddot{\theta}(u)\Big) + \mathcal{D}(u,v)\left(\dot{\theta}(v) - \dot{\theta}(u)\right) + \sin\left(\theta(v) - \theta(u)\right)\right).
\end{aligned}
$$

Here, the multiplication with the adjacency matrix cancels out the normalization, making the result a physical current I instead of a normalized current i. Since $I_C = 1$ for all junctions, they are the same in this case. This results in

$$
\begin{aligned}
I(a) = \cancel{-}I_{ext} &= 0 \cdot i(a,a) + 1 \cdot i(b,a) + 0 \cdot i(c,a) = -i(a,b) \\
&= \cancel{-}\left(\Big(\ddot{\theta}(b) - \ddot{\theta}(a)\Big) + \mathcal{D}(a,b)\left(\dot{\theta}(b) - \dot{\theta}(a)\right) + \sin\left(\theta(b) - \theta(a)\right)\right) \\
I_{ext} &= (\ddot{\theta}(b) - \ddot{\theta}(a)) + \mathcal{D}(a,b)(\dot{\theta}(b) - \dot{\theta}(a)) + \sin(\theta(b) - \theta(a)),
\end{aligned}
$$

as well as

$$
\begin{aligned}
I(b) = 0 &= 1 \cdot i(a,b) + 0 \cdot i(b,b) + 1 \cdot i(c,b) = i(a,b) - i(b,c) \\
0 &= (2\ddot{\theta}(b) - \ddot{\theta}(a) - \ddot{\theta}(c)) + \mathcal{D}(a,b)(2\dot{\theta}(b) - \dot{\theta}(a) - \dot{\theta}(c)) + \\
&\qquad + \sin(\theta(b) - \theta(a)) + \sin(\theta(b) - \theta(c))
\end{aligned}
$$

and

$$
\begin{aligned}
I(c) = I_{ext} &= 0 \cdot i(a,c) + 1 \cdot i(b,c) + 0 \cdot i(c,c) = i(b,c) \\
I_{ext} &= \Big(\ddot{\theta}(c) - \ddot{\theta}(b)\Big) + \mathcal{D}(b,c)\left(\dot{\theta}(c) - \dot{\theta}(b)\right) + \sin\left(\theta(c) - \theta(b)\right).
\end{aligned}
$$

Since we can easily see the entire graph structure and know the edge currents, it is straightforward to verify that these expanded equations have to be the correct physical equations of motion for current through the vertices. This shows how we can recover the underlying equations describing the graph using the formalism we established.

However, there are some obvious problems with this approach. For example, we implicitly chose an arbitrary direction for the current, as this information is not encoded in the network structure. Therefore, the vertex equations had to be expanded manually. While this approach may be feasible for small graphs, it is not practical for calculations involving more complex graphs. Thus, we introduce a formalism that encodes both connections and directions algebraically in a way that allows us to write both edge and vertex equations compactly.

2.3 Directed RCSJ network

We will make use of *directed graphs* to encode the directional information directly into the network structure.

Definition 2.22 (Directed graph)**.** We define a **directed graph** $G = (V, E_d)$ such that V is the set of vertices and $E_d \subset V \times V$ is the set of *directed* edges. A directed edge $e = (u, v) \in E_d$ points from one vertex $u \in V$ to vertex $v \in V$. We therefore say that the edge e **leaves** vertex u and **enters** vertex v.

Remark 2.23. When talking about connectedness for directed graphs, we still refer to the connectedness of the underlying undirected graph.

Similar to the adjacency matrix, we need to clearly specify the structure of the graph in a way that stores the vertices as well as the edges that are connecting them. For directed graphs, this can be done using an *incidence matrix*.

Definition 2.24 (Incidence matrix)**.** Let $G = (V, E_d)$ be a directed graph. We define the incidence matrix B to have a separate column for every edge $e \in E_d$ and a row for every node $v \in V$, such that

$$B_{ve} = \begin{cases} 1 & \text{if edge } e \text{ enters node } v, \\ -1 & \text{if edge } e \text{ leaves node } v, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

With this definition in mind we can go back to the previous example and express the network in Figure 3 using an incidence matrix.

Example 2.25. The graph in Figure 3 has exactly two edges and three vertices. So, the incidence matrix has two columns and three rows with entries given by:

$$B = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

In fact, we can observe that the transpose of B acting on $\boldsymbol{\theta}$, the vector of phases $\theta(a), \theta(b), \theta(c)$, results in

$$B^{\mathsf{T}}\boldsymbol{\theta} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \theta(a) \\ \theta(b) \\ \theta(c) \end{pmatrix} = \begin{pmatrix} \theta(b) - \theta(a) \\ \theta(c) - \theta(b) \end{pmatrix}.$$

So, the transpose of the incidence matrix produces the phase differences ϕ along the edges.

Remark 2.26. In this example, we made use of vectors to collect all the phases into a single object. Say there is a vector $\boldsymbol{a}$ that is indexed by the set V. To simplify notation, we will refer to any particular element $v \in V$ in $\boldsymbol{a}$ using the notation

$$(\boldsymbol{a})_v.$$

Now, we generalize the observation from the previous example within a lemma.

Lemma 2.27. *Let $G = (V, E_d)$ be a directed, simple graph with an incidence matrix B and a vertex function $\theta(u)$ for all $u \in V$. Let $\boldsymbol{\theta} \in \mathbb{R}^{|V|}$ denote the vector of vertex values. Then, the vector $\Delta\boldsymbol{\theta} \in \mathbb{R}^{|E_d|}$ contains the differences of θ across the directed edges. In particular, for an edge $e = (u, v) \in E_d$,*

$$(\Delta\boldsymbol{\theta})_e = (B^{\mathsf{T}}\boldsymbol{\theta})_e = \theta(v) - \theta(u). \tag{10}$$

Proof. As per the definition of a directed edge in Definition 2.22, any edge e has two neighboring vertices u, v such that it enters v and leaves u. Thus, it follows from Definition 2.24 of the incidence matrix that the column of B corresponding to e has exactly two nonzero entries:

- -1 in the row corresponding to u and
- $+1$ in the row corresponding to v.

All other entries in that column are zero.

After transposition, the row of B^{T} corresponding to e therefore contains the same -1 in the column for u and $+1$ in the column for v, with 0 everywhere else. As a consequence of that:

$$\left(B^{\mathsf{T}}\boldsymbol{\theta}\right)_e = (-1) \cdot \theta(u) + (+1) \cdot \theta(v) = \theta(v) - \theta(u).$$

This is precisely the difference of the vertex function along the edge and thus proves our claim. □

That way, we have identified a neat and simple way to connect the structure of the network to the phase difference used in the RCSJ equations. This means we can now write

$$\boldsymbol{i} = B^\intercal \ddot{\boldsymbol{\theta}} + \frac{1}{Q} B^\intercal \dot{\boldsymbol{\theta}} + \sin\left(B^\intercal \boldsymbol{\theta}\right), \tag{11}$$

where $\boldsymbol{i}$ is the vector containing the currents across all edges $e = (u, v) \in E_d$ and $\boldsymbol{\theta}$ is the vector containing all phases $\theta(u)$ for all $u \in V$.

Here, we exploited the fact that every column in the incidence matrix representing an edge $e = (u, v)$ contains exactly two nonzero entries, corresponding to the vertices u and v. That way, a multiplication with the incidence matrix

$$(B\boldsymbol{i})_u = \sum_{e \in E_d} B_{ue}\, i(e),$$

corresponds directly to a sum over all edges connected to a particular vertex $u \in V$. Moreover, this is the exact kind of operation used in the previously discussed boundary conditions implementing Kirchhoff's law. That's because the values of B_{ue} will only ever be $+1$ or -1, such that we effectively end up with

$$(B\boldsymbol{i})_u = \sum_{e \text{ entering } u} i(e) \; - \sum_{e \text{ leaving } u} i(e). \tag{12}$$

Setting this expression equal to 0 yields Kirchhoff's current conservation in Condition 2.17. Still missing is an adjustment for the fact that i represents the *normalized* current $i = \frac{I}{I_C}$ instead of the physical current. That's because previous current conservation law conserved the *physical* current at the vertices. An easy fix for this problem is to multiply every single row of the $\boldsymbol{i}$ vector by its critical current. To make sure that every junction can have an individual critical current, we can simply define a diagonal *critical current matrix*.

Definition 2.28 (Critical current matrix)**.** Let $G = (V, E_d)$ be a simple, directed graph with a positive critical current $I_C(e)$ assigned to every junction $e \in E_d$. Given the vector of all critical currents $\boldsymbol{I}_C \in \mathbb{R}^{|E_d|}$, we define the **critical current matrix**

to be

$$C = \mathrm{diag}(\boldsymbol{I}_C) = \begin{pmatrix} I_C(e_1) & & \\ & I_C(e_2) & \\ & & \ddots \end{pmatrix}, \tag{13}$$

where $e_1, e_2, \dots \in E_d$.

Multiplying by this matrix will multiply every single normalized current in $\boldsymbol{i}$ with its corresponding critical current and cancel out the normalization.

Thus having a way of recovering the physical currents, we can then apply B in order to sum up all the physical currents entering and exiting the vertices. Therewith, we can adjust our previous condition 2.17 for current conservation in the following way:

Condition 2.29 (Current conservation)**.** *Let $G = (V, E_d)$ be a simple, directed graph with a source $s \in V$ and sink $t \in V$, as well as an incidence matrix B and a vector $\boldsymbol{i} \in \mathbb{R}^{|E_d|}$ containing all normalized junction currents. An external physical current $I_{ext} \in \mathbb{R}$ is applied. In such a graph, the following condition has to hold:*

$$(BC\boldsymbol{i})_v = \begin{cases} -I_{ext} & \textit{if } v = s \\ I_{ext} & \textit{if } v = t \\ 0 & \textit{otherwise.} \end{cases} \tag{14}$$

With this condition, current conservation can be ensured using the formalism introduced for directed graphs.

Remark 2.30. In order to align with the rest of the vector-based notation in the directed formalism, we may rewrite (14) using the standard basis vectors $\boldsymbol{e}_s, \boldsymbol{e}_t \in \mathbb{R}^{|V|}$ corresponding to the source and sink vertices respectively. Then we can write

$$I_{ext} \cdot (\boldsymbol{e}_t - \boldsymbol{e}_s)_v = \begin{cases} -I_{ext} & \text{if } v = s \\ I_{ext} & \text{if } v = t \\ 0 & \text{otherwise.} \end{cases}$$

In the same way as it is for the critical currents, the best way to apply an individual, positive damping parameter $\mathcal{D}(e) = \frac{1}{Q}$ to every single edge $e \in E_d$ is by defining a diagonal damping matrix

$$D = \begin{pmatrix} \mathcal{D}(e_1) & & \\ & \mathcal{D}(e_2) & \\ & & \ddots \end{pmatrix} \tag{15}$$

for $e_1, e_2, \dots \in E_d$.

In order to further analyze the structure and dynamics of a network, a convenient object to make use of again is the *graph Laplacian*. Even in the directed framework, it can be easily formed using the incidence matrix.

Definition 2.31 (Directed Graph Laplacian)**.** Let $G = (V, E_d)$ be a simple, directed graph with an incidence matrix B. We define the **graph Laplacian** L such that

$$L = B \cdot B^{\mathsf{T}}. \tag{16}$$

Note that although the graph is directed, this definition of the graph Laplacian results in it being symmetric and independent of the chosen edge orientation. In fact, we can find the following.

Lemma 2.32. *Let $G = (V, E, w)$ be a simple weighted graph, with critical currents $I_C(e)$ as edge weights for all $e \in E$ and let B be an incidence matrix obtained from any arbitrary orientation of the edges. Let C be the associated critical current matrix. Then, the weighted graph Laplacian satisfies*

$$L = BCB^{\mathsf{T}}, \tag{17}$$

with $L = D - A$, where A is the weighted adjacency matrix of G and D the weighted degree matrix of G, and is independent of the chosen orientation of edges.

Proof. We start by expanding the matrix elements and find that

$$\left(BCB^{\mathsf{T}}\right)_{uv} = \sum_{e \in E} B_{ue} I_C(e) B_{ve}.$$

Since $B_{ue} = \pm 1$ if e is incident to u and $B_{ue} = 0$ if not, we know that diagonal entries $u = v$ result in

$$\left(BCB^{\mathsf{T}}\right)_{uu} = \sum_{e \in E} B_{ue}^2 I_C(e) = \sum_{\substack{e \in E \\ \text{incident to } u}} I_C(e) = D_{uu}.$$

Here, D_{uu} is a diagonal element of the degree matrix by definition. As for off-diagonal elements $u \neq v$, we know that, given an edge $e = (u, v) \in E$, one of B_{ue} and B_{ve} is $+1$, while the other is -1. This comes from the fact that every edge is in between exactly two vertices. Therefore,

$$\left(BCB^{\mathsf{T}}\right)_{uv} = -I_C(u, v) = -A_{uv}$$

from which it follows that

$$BCB^{\mathsf{T}} = D - A = L.$$

If the orientation of an edge is reversed, the corresponding column of B changes sign. As however $B_{ue} I_C(e) B_{ve} = (-B_{ue}) I_C(e)(-B_{ve})$, the final matrix BCB^T is independent of the orientation of edges. □

Since this weighted graph Laplacian differs slightly from the general Definition 2.31, we will henceforth refer to it as L_C, such that

$$L_C = B \cdot C \cdot B^{\mathsf{T}}.$$

Similarly, we can also include the damping matrix into the weighting and obtain

$$L_D = B \cdot C \cdot D \cdot B^{\mathsf{T}}.$$

Making use of Condition 2.29 and (11), we can therewith write Kirchhoff's law as

$$(L_C \ddot{\boldsymbol{\theta}} + L_D \dot{\boldsymbol{\theta}} + BC \sin(B^{\mathsf{T}} \boldsymbol{\theta}))_v = \begin{cases} -I_{ext} & \text{if } v = s \\ I_{ext} & \text{if } v = t \\ 0 & \text{otherwise.} \end{cases},$$

where we assume a simple, directed graph $G = (V, E_d)$ such that $v, s, t \in V$. By equating it to a vector $\boldsymbol{I} \in \mathbb{R}^{|V|}$ containing the net physical currents I at all vertices in V, this is the equation that also allows us to recover the exact equations of motion of the entire network.

All taken together, we are now in a position to formally define an RCSJ Network in directed-graph-theoretic terms.

Definition 2.33 (RCSJ Network)**.** An **RCSJ Network** is a tuple $\mathcal{N} = (G, B, C, D, s, t)$, where G is a weighted, finite, simple, directed, connected graph $G = (V, E_d)$ with an incidence matrix B and two distinguished vertices $s, t \in V$ acting as a source and a sink. Furthermore, C is the critical current matrix, D the damping matrix and the graph G carries:

- a vertex function θ, called the phase, and
- an edge function i denoting the current.

Let $\boldsymbol{\theta} \in \mathbb{R}^{|V|}$ and $\boldsymbol{i} \in \mathbb{R}^{|E_d|}$ denote the corresponding vectors.

Definition 2.34 (RCSJ Network Dynamics)**.** The dynamics of an RCSJ Network $\mathcal{N}$ are given by

$$\boldsymbol{i} = B^{\mathsf{T}}\ddot{\boldsymbol{\theta}} + DB^{\mathsf{T}}\dot{\boldsymbol{\theta}} + \sin\left(B^{\mathsf{T}}\boldsymbol{\theta}\right). \tag{18}$$

Let $I_{ext} : \mathbb{R} \to \mathbb{R}$ be an applied external current. The currents are constrained by current conservation, such that for any vertex $v \in V$,

$$(BC\boldsymbol{i})_v = \begin{cases} -I_{ext} & \text{if } v = s \\ I_{ext} & \text{if } v = t \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

As mentioned before, this definition should solve the problems we identified with our previous adjacency matrix-based definition. At the same time, it should result in the exact same set of equations of motion for the junctions in the network, as well as for the currents flowing through the vertices. Let's confirm this using the network from the previous example.

Example 2.35*.* We consider the graph depicted in Figure 3. In that example, we set all critical currents and damping parameters equal to 1, which means that the critical current matrix and damping matrix are diagonal matrices diag($\mathbf{1}$). Consequently, they will have no effect on the following calculation. The incidence matrix of Figure 3 was constructed in Example 2.25. Using the incidence matrix form of the RCSJ equation (11) we can obtain the equations of motion for the junctions, such that

$$
\begin{aligned}
\boldsymbol{i} = \begin{pmatrix} i(a,b) \\ i(b,c) \end{pmatrix} &= B^\mathsf{T}\ddot{\boldsymbol{\theta}} + B^\mathsf{T}\dot{\boldsymbol{\theta}} + \sin\left(B^\mathsf{T}\boldsymbol{\theta}\right) \\
&= \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{\theta}(a) \\ \ddot{\theta}(b) \\ \ddot{\theta}(c) \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}(a) \\ \dot{\theta}(b) \\ \dot{\theta}(c) \end{pmatrix} + \sin\left(\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \theta(a) \\ \theta(b) \\ \theta(c) \end{pmatrix} \right) \\
&= \begin{pmatrix} \ddot{\theta}(b) - \ddot{\theta}(a) \\ \ddot{\theta}(c) - \ddot{\theta}(b) \end{pmatrix} + \begin{pmatrix} \dot{\theta}(b) - \dot{\theta}(a) \\ \dot{\theta}(c) - \dot{\theta}(b) \end{pmatrix} + \sin\left(\begin{pmatrix} \theta(b) - \theta(a) \\ \theta(c) - \theta(b) \end{pmatrix} \right),
\end{aligned}
$$

where every row corresponds to a separate equation. Written separately, they are

$$
\begin{aligned}
i(a,b) &= \left(\ddot{\theta}(b) - \ddot{\theta}(a)\right) + \left(\dot{\theta}(b) - \dot{\theta}(a)\right) + \sin\left(\theta(b) - \theta(a)\right) \\
i(b,c) &= \left(\ddot{\theta}(c) - \ddot{\theta}(b)\right) + \left(\dot{\theta}(c) - \dot{\theta}(b)\right) + \sin\left(\theta(c) - \theta(b)\right).
\end{aligned}
$$

So, we can confirm already that this set of equations is identical to the ones presented in Example 2.21. Furthermore, the total current flowing through any vertex in the network is given by

$$
\begin{aligned}
\boldsymbol{I} = \begin{pmatrix} -I_{ext} \\ 0 \\ I_{ext} \end{pmatrix} = B\boldsymbol{i} = &\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{\theta}(b) - \ddot{\theta}(a) \\ \ddot{\theta}(c) - \ddot{\theta}(b) \end{pmatrix} + \\
&+ \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}(b) - \dot{\theta}(a) \\ \dot{\theta}(c) - \dot{\theta}(b) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \sin\left(\begin{pmatrix} \theta(b) - \theta(a) \\ \theta(c) - \theta(b) \end{pmatrix} \right)
\end{aligned}
$$

Performing the matrix multiplications we get

$$
\begin{aligned}
\boldsymbol{I} = \begin{pmatrix} -I_{ext} \\ 0 \\ I_{ext} \end{pmatrix} = &\begin{pmatrix} -\left(\ddot{\theta}(b) - \ddot{\theta}(a)\right) \\ 2\ddot{\theta}(b) - \ddot{\theta}(a) - \ddot{\theta}(c) \\ \ddot{\theta}(c) - \ddot{\theta}(b) \end{pmatrix} + \begin{pmatrix} -\left(\dot{\theta}(b) - \dot{\theta}(a)\right) \\ 2\dot{\theta}(b) - \dot{\theta}(a) - \dot{\theta}(c) \\ \dot{\theta}(c) - \dot{\theta}(b) \end{pmatrix} + \\
&+ \begin{pmatrix} -\left(\sin\left(\theta(b) - \theta(a)\right)\right) \\ \sin\left(\theta(b) - \theta(a)\right) - \sin\left(\theta(c) - \theta(b)\right) \\ \sin\left(\theta(c) - \theta(b)\right) \end{pmatrix},
\end{aligned}
$$

or written line-by-line

$$\begin{aligned}
I(a) = I_{ext} &= (\ddot{\theta}(b) - \ddot{\theta}(a)) + (\dot{\theta}(b) - \dot{\theta}(a)) + \sin(\theta(b) - \theta(a)) \\
I(b) = 0 &= (2\ddot{\theta}(b) - \ddot{\theta}(a) - \ddot{\theta}(c)) + (2\dot{\theta}(b) - \dot{\theta}(a) - \dot{\theta}(c)) + \\
&\quad + \sin(\theta(b) - \theta(a)) - \sin(\theta(c) - \theta(b)) \\
I(c) = I_{ext} &= (\ddot{\theta}(c) - \ddot{\theta}(b)) + (\dot{\theta}(c) - \dot{\theta}(b)) + \sin(\theta(c) - \theta(b)).
\end{aligned}$$

Note that these equations match the previously determined equations of motion for the vertices exactly.

So, we can see that both formalisms encode the exact same set of dynamical equations. At the same time the incidence matrix formalism offers a lot more clarity and less room for errors because it incorporates implicit current directions. Therewith, no implicit sign-bookkeeping based on conventions is required in the calculations. Additionally, this same procedure applies to arbitrary network structures, since the incidence matrix will always collect all currents entering and leaving any vertex.

3 Junction failure

Now that we have established a working formalism for RCSJ Networks, we can take a step back and think about our original goal again: studying failures of junctions on this network. To get started with our discussion around these, we first define a failure in the following way.

Definition 3.1 (Junction failure)**.** We say that a Josephson junction **fails** when it switches out of the superconducting state into its resistive state.

So, why exactly do we define a failure as the junction just exiting the superconducting state? After all, the junction still exists and is even described perfectly fine by the RCSJ model and therewith also by our RCSJ Network.

The reason for that is mostly the physical perspective. In a superconducting circuit, it is understood that all circuit elements have to be superconducting. That's because the current always tends to take the path of least resistance, which will always be other superconducting wires instead of resistive ones [Tin96, p.203]. This will have the same effect as the junction being removed from the circuit, which would have the same the outcome as it breaking physically. Hence, defining a failure as exiting the superconducting state seems to be reasonable.

Thinking about how failures occur, one may find that there seem to be two different kinds of possible situations. Firstly, the junction could be the first junction to fail, in which case the entire network pre-failure is described purely by the Josephson term in the RCSJ equation. The second possibility is that the affected junction is not the first junction to fail, but was instead affected by the failure of another junction. These two cases however turn out to be equivalent mathematically if we disregard the time-dependence of the RCSJ network. That's because if one junction fails, then it becomes non-superconducting and is functionally cut out of the circuit, as described before. Therefore, after the failure occurred, we are left with a network containing *effectively* only superconducting junctions.

That time-dependence is exactly where we can make a clear distinction in our analysis, providing following two perspectives on failures:

- Static failures: Here, we consider just the superconducting state that fulfills the Josephson equation, without taking any time-dependencies into account.

- Dynamic failures: On the other hand, we can also consider the dynamics of the system and try to characterize their behavior as they approach failure.

4 Static failures

We begin by investigating *static failures*, as they allow us to study the structural properties of the network independent of its time evolution. In line with our previous definition, a junction failure is the event in which the current I across a junction becomes larger than the critical current I_C for that junction. In the case of a static failure, the failing edge is the first edge in the network to turn non-superconducting, warranting the following definition:

Definition 4.1 (Static failure)**.** Let $\mathcal{N}$ be a directed RCSJ Network. A **static failure** is the event in which $|I(e)| > I_C(e)$ for an edge $e \in E_d$, while $|I(f)| \leq I_C(f)$ for all other $f \in E_d$ for which $f \neq e$.

So, before a static failure, all junctions in the network are superconducting. This means in this case, that the dynamics of the RCSJ Network reduces to

$$\boldsymbol{i} = \sin\left(B^{\mathsf{T}}\boldsymbol{\theta}\right), \tag{20}$$

where the derivatives vanish since the phase differences are time-independent in the static regime. Moving forward, we will refer to this state as the RCSJ Network *solving*.

Definition 4.2 (Solvability)**.** A directed RCSJ Network $\mathcal{N}$ **solves** when $|I(e)| \leq I_C(e)$ for all edges $e \in E_d$.

Thus, when we have an RCSJ Network that solves, we know that by definition all currents are below the threshold set by their critical currents.

This concept is remarkably similar to another concept in graph theory: The *capacity* in networks of *flows*. In this formalism every edge carries a non-negative flow that is bounded by the capacity of the edge, which is almost exactly the situation we are modeling for a solvable RCSJ Network. Therefore, we will now investigate to what extent a solvable RCSJ Network can be interpreted using the flow formalism, as described in [Cor+14, ch. 26.1].

4.1 Flow Interpretation

Definition 4.3 (Flow)**.** Let $G = (V, E_d)$ be a directed graph with two distinguished vertices $s, t \in V$. A **flow** is a function

$$f : E_d \to \mathbb{R}_{\geq 0} \tag{21}$$

that assigns a non-negative value to each directed edge.

Narrowing this definition, we can also describe a *feasible flow*.

Definition 4.4 (Feasible flow). A flow f is **feasible** if:

1. Capacity constraint: $0 \leq f(u,v) \leq c(u,v)$ for any edge $(u,v) \in E_d$, where $c(u,v) \in \mathbb{R}_{\geq 0}$ is the so-called capacity of the edge (u,v),
2. Flow conservation: For any vertex $v \in V \backslash \{s,t\}$, the flow is such that

$$\sum_{(u,v)\in E_d} f(u,v) = \sum_{(v,w)\in E_d} f(v,w).$$

These are the fundamental definitions of a flow and a feasible flow. Apart from the maximum capacity for the edges, they essentially just say that a feasible flow is a conserved *edge function* as defined in Definition 2.9. Note however that currents on our RCSJ Network definition don't meet the lower bound of the capacity constraint. They are only bounded by $|I(e)| \leq I_C(e)$, which allows negative current that is understood to flow in the opposite direction of the edge. Such negative flows however are not permitted by the definition of a flow.

Remark 4.5 (Bidirectional Interpretation). To account for the existence of negative currents, we will now consider every junction as *two* directed edges (u,v) and (v,u) between vertices $u,v \in V$. A signed current $I(e) \in [-I_C(e), I_C(e)]$ on the original edge can then be represented by a non-negative current on exactly one of these two directed edges. This construction allows us to introduce a lower bound for the currents, such that

$$0 \leq I(e) \leq I_C(e) \qquad \forall e \in E_d. \tag{22}$$

In that way, the previously signed current is decomposed into two non-negative currents while preserving the original capacity constraint and Kirchhoff's current conservation. Since the actual physical current can only be in *one* of the two possible directions at once, at most one of the two directed currents is non-zero in any configuration of physical currents. As shown in Lemma 2.32, the graph Laplacian is invariant to the direction of the edges. Therefore, it is perfectly justified to construct the graph Laplacian using one representative edge for each physical junction, making it straightforward to verify that the graph Laplacian constructed that way is equivalent to the original construction detailed in Lemma (2.32).

Thus, our re-interpretation of the junctions in the RCSJ Network allows us to introduce non-negative currents, while being functionally equivalent to our original definition. Using it and the fact that Equation (22) normalizes to $0 \leq i(e) \leq 1$ for all edges $e \in E_d$, we can show the following equivalence.

Lemma 4.6. *Let $\mathcal{N}$ be a directed RCSJ Network. The edge function i in $\mathcal{N}$ defines a feasible flow $f : E_d \to \mathbb{R}_{\geq 0}$, such that*

$$f(e) := i(e),$$

for any edge $e \in E_d$.

Proof. Given a directed RCSJ Network $\mathcal{N} = (G, B, C, D, s, t)$ with $G = (V, E_d)$, we check the defining properties of a feasible flow.

1. Edge function:
   The current i is defined as an edge function $i : E_d \to \mathbb{R}$ and so is $f : E_d \to \mathbb{R}$.

2. Capacity constraint:
   Given that all currents in the bidirectional graph interpretation of an RCSJ Network are non-negative, we write

$$0 \leq i(e) \leq 1 \qquad \forall e \in E_d.$$

   Identifying $f(e) = i(e)$ and $c(e) = 1$, this condition coincides exactly with the capacity constraint

$$0 \leq f(e) \leq c(e) \qquad \forall e \in E_d.$$

3. Flow conservation:
   At all internal vertices, the current conservation $BC\boldsymbol{i} = 0$ ensures that the net current is 0. For any one vertex $u \in V$, the current conservation condition (12) can be expanded to

$$(BC\boldsymbol{i})_u = \sum_{e \text{ entering } u \in E_d} i(e) - \sum_{e \text{ leaving } u \in E_d} i(e) = 0 \Leftrightarrow$$
$$\Leftrightarrow \sum_{(u,v) \in E_d} i(u,v) = \sum_{(v,w) \in E_d} i(v,w) \qquad \forall v \in V \setminus \{s,t\}.$$

   This is precisely the flow conservation condition with $i(u,v) = f(u,v)$.

Consequently, i defines a flow f on $\mathcal{N}$. $\square$

So, the current i in an RCSJ Network $\mathcal{N}$ can be considered a flow. We can even define the *value of a flow* across the entire network.

Definition 4.7 (Value of a flow)**.** Consider a directed graph $G = (V, E_d)$ and a feasible flow f for all $e \in E_d$. The **value** $\mathrm{val}(f) \in \mathbb{R}_{\geq 0}$ of a flow is defined to be

$$\mathrm{val}(f) = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) \tag{23}$$

Remark 4.8. This definition may seem somewhat incomplete since it doesn't account for the flow across any of the internal vertices, but only seems to count the net outflow from the source s. However, we also know that the internal vertices obey strict current conservation, such that

$$\begin{aligned} \sum_{(u,v) \in E_d} f(u, v) &= \sum_{(v,w) \in E_d} f(v, w) \qquad \forall\, v \in V \backslash \{s, t\} \\ \Leftrightarrow 0 &= \sum_{(v,w) \in E_d} f(v, w) - \sum_{(u,v) \in E_d} f(u, v) \qquad \forall\, v \in V \backslash \{s, t\}. \end{aligned}$$

So, the contribution of the internal vertices to the net value of the flow through the network is exactly 0. This makes sense since the flow leaving the source has to pass through all of the internal vertices and is conserved while doing so. Consequently, $\mathrm{val}(f)$ is also the total inflow into the sink t. In fact, an alternative definition for the *value of a flow* could be

$$\mathrm{val}(f) = \sum_{v \in V \backslash \{t\}} \left(\sum_{(v,w) \in E_d} f(v, w) - \sum_{(u,v) \in E_d} f(u, v) \right),$$

where all of the internal nodes would still cancel out and only leave us with the term for $v = s$. We still need to exclude t, since the total inflow into the sink t has to be equal to the total outflow from s, both of which would also cancel. Without the sink however, we will effectively get exactly the same expression as in Definition 4.7.

Intuitively, we can therewith imagine the value of a flow as the total amount of flow that propagates through the network, even if it gets split up along the way. With this insight, we will now investigate how we can use this notion of flow in the context of RCSJ Networks.

4.2 Solvability

We have now learned that when an RCSJ Network is solvable, the current i is a feasible flow f. Simultaneously, there may exist multiple currents satisfying the current conservation constraints. Reversing this logic, we find the following.

Lemma 4.9. *A directed RCSJ Network $\mathcal{N}$ solves if and only if the set of feasible flows $\mathcal{F}$ is non-empty.*

Proof. By definition, an RCSJ Network solves if $|I(e)| \leq I_C(e)$, or in other words $|i(e)| \leq 1$, for every edge $e \in E_d$. Additionally, any current i also has to obey current conservation. If there exists a current that satisfies these criteria, then we know that the RCSJ Network solves. If there exists no such current, then by definition the RCSJ Network cannot solve. At the same time, any such current is a feasible flow on the network. Thus, the RCSJ Network solves if and only if there exists a feasible flow on it. $\square$

This argument will be the backdrop of our static failure analysis. Essentially, the question becomes now whether the RCSJ Network contains at least one such feasible flow. If it does, we know that the network solves and when it solves, then there won't be any failure events. If, on the contrary, there is no feasible flow, then we automatically know that the network has to fail somewhere.

So, what is it that determines if there exists a feasible flow on the network? Conceptually, we could start by trying to find the biggest feasible flow value the network could theoretically support. This is what we will call the *maximum current*.

Definition 4.10 (Maximum current)**.** Let $\mathcal{N}$ be an RCSJ Network and $\mathcal{F}$ its set of feasible flows. The **maximum current** is

$$I_{max} = \max_{f \in \mathcal{F}} (\mathrm{val}(f)) . \tag{24}$$

This maximum current is mostly influenced by any potential bottlenecks in the network. However, since the maximum current is defined as the value of a feasible flow we implicitly ensure that there will always be a feasible flow with the value of the maximum current.

While this conclusion is trivial from the definition of the maximum current, it is less obvious that there also has to be a feasible flow for every value less than the maximum current.

Lemma 4.11. *Let there be a feasible flow f such that val$(f) = i$. Then, for every i' with $0 \leq i' \leq i$, there exists another feasible flow f' such that val$(f') = i'$.*

Proof. Let f be a feasible flow with the value $\mathrm{val}(f) = i$ in an RCSJ Network $\mathcal{N}$. We define a new feasible flow f' on $\mathcal{N}$, such that for every edge $e \in E_d$ and a real number $0 \leq \alpha \leq 1$,

$$f'(e) = \alpha \cdot f(e). \tag{25}$$

From the definition of the value of a feasible flow we then get that

$$\begin{aligned} &\mathrm{val}(f') = \sum_{v \in V} \alpha\, f(s,v) - \sum_{v \in V} \alpha\, f(v,s) \\ \Leftrightarrow \quad &\mathrm{val}(f') = \alpha \left(\sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) \right) = \alpha\, \mathrm{val}(f). \end{aligned}$$

Naming $\mathrm{val}(f') = i'$, we can write $\alpha = \frac{i'}{i}$. This defines a new feasible flow that slightly scales down the flow on every single edge. The definition of a feasible flow demands that $0 \leq f(e) \leq c(e)$ for all edges $e \in E_d$ as well as flow conservation. As $\alpha \leq 1$, we therewith get

$$\begin{aligned} 0 \leq \alpha \cdot f(e) \leq f(e) \leq c(e) \\ 0 \leq f'(e) \leq c(e). \qquad \qquad \forall\, e \in E_d \end{aligned} \tag{26}$$

This shows explicitly that $i(e) \leq 1$, or in other words $I(e) \leq I_C(e)$, for all edges $e \in E_d$, meaning that no junction in f' exceeds the critical current. Current conservation is still fulfilled since f is a feasible flow and the flow on all edges was scaled uniformly. Therefore, we can be sure that f' is a feasible flow. Consequently, there exists a feasible flow for any flow value i' such that $0 \leq i' \leq i$. □

At this point, we have determined that there will always be a maximum current for our network, which will always be a feasible flow. Furthermore, we know from Lemma 4.11 that there exists a feasible flow for every non-negative current value less than the maximum current.

Now, we consider an external current I_{ext} applied on the network, such that $I_{ext} \leq I_{max}$. In that case, we can easily show the following.

Theorem 4.12. *Let $\mathcal{N}$ be an RCSJ Network with an external current I_{ext} and a maximum current I_{max}. If*

$$I_{ext} \leq I_{max}, \tag{27}$$

then the RCSJ Network is solvable.

Proof. Let $\mathcal{N}$ be an RCSJ Network. Then, lemma 4.11 gives that if $I_{ext} \leq I_{max}$, there exists a feasible flow f, such that

$$\text{val}(f) = I_{ext}. \tag{28}$$

Thus, Lemma 4.9 ensures that the RCSJ Network solves. □

Thinking intuitively about this theorem, it certainly makes sense because we know that we can always define a maximum current based on the critical currents in the RCSJ Network. This critical current is defined as the biggest possible value of a feasible flow, and that value of a flow is essentially defined as the net outflow from the source. The definition of an RCSJ Network then considers the external current to be the current injected into the source and consequently the net current leaving the source. So, if the external current makes more current leave the source than it is allowed for the network to be solvable, there is no way that it will solve. This argument also works the other way around.

4.3 Max-flow-Min-cut

In the previous section we found a nice rule that tells quite easily whether an RCSJ Network will solve or not. However, we currently have no simple way to determine I_{max}. To investigate this, we start out by defining a so-called *cut* and its *cut capacity* as in [Cor+14, pp.720–721]

Definition 4.13 (S-T-cut)**.** Let $G = (V, E_d)$ be a directed graph with a source $s \in V$ and a sink $t \in V$. We define an **S-T-cut** (S, T) to be a partition $V = S \cup T$, such that $s \in S$, $t \in T$ and $S \cap T = \varnothing$.

Definition 4.14 (Cut capacity)**.** Let $G = (V, E_d)$ be a directed graph with an S-T-cut such that $V = S \cup T$. The **capacity of the cut** $c(S, T)$ is

$$c(S, T) = \sum_{\substack{(u,v) \in E_d \\ u \in S,\ v \in T}} c(u, v). \tag{29}$$

Using these definitions we can actually find that the maximum current we defined before is equal to the minimum cut capacity of the RCSJ Network. This is called the *Max-flow Min-cut theorem.*

As a first step towards getting an intuition why this is true, we can find that it is possible to bound any value of a feasible flow by an S-T-cut. [Cor+14, thm. 26.5]

Lemma 4.15. *Let $\mathcal{N}$ be an RCSJ Network. For any feasible flow f and S-T-cut (S,T), the following is true*

$$val(f) \leq c(S,T). \tag{30}$$

Proof. Let $\mathcal{N}$ be an RCSJ Network with a feasible flow f representing the current i and an S-T-cut (S,T). We will now consider the sum of flows on directed edges crossing the cut from S to T,

$$F = \sum_{\substack{e=(u,v)\in E_d \\ u\in S,\ v\in T}} f(e). \tag{31}$$

From the definition of a feasible flow, each $f(e)$ in (31) has $f(e) \leq 1$, or in other words $I(e) \leq I_C(e)$. Therefore, get immediately that

$$F = \sum_{\substack{e=(u,v)\in E_d \\ u\in S,\ v\in T}} f(e) \leq \sum_{\substack{e=(u,v)\in E_d \\ u\in S,\ v\in T}} c(e) = c(S,T)$$
$$F \leq c(S,T). \tag{32}$$

Furthermore, we can use the version of the value of a flow discussed in Remark 4.8 and notice that the sum over $v \in V\backslash\{t\}$ is just a special case of a sum of S from an S-T-cut. That is because $s \in S$ and $t \in T$ for any S-T-cut. Therefore we can rewrite it as

$$\begin{aligned} \mathrm{val}(f) &= \sum_{v\in S}\left(\sum_{(v,w)\in E_d} f(v,w) - \sum_{(u,v)\in E_d} f(u,v)\right) \\ &= \sum_{v\in S}\sum_{(v,w)\in E_d} f(v,w) - \sum_{v\in S}\sum_{(u,v)\in E_d} f(u,v). \end{aligned}$$

Here, there are a couple of different cases to consider. The contribution from any edges with both vertices in S will cancel because of flow conservation. The edges with both vertices in T are not counted here, but would also cancel if they were. Finally, the edges leaving a vertex in S entering a vertex in T appear in the first term, whereas the edges from T to S appear in the second term. Therefore, we can write

$$\mathrm{val}(f) = \sum_{v\in S} \sum_{\substack{w\in T\\(v,w)\in E_d}} f(v,w) - \sum_{\substack{u\in T\\(u,v)\in E_d}} \sum_{v\in S} f(u,v). \tag{33}$$

Comparing (33) to (31), we can see that the first term is identical to the definition of F and only differs in notation. This is because both F and the first term represent the sum over the flow across the edges from a vertex in S to a vertex in T. Thus, it becomes clear that

$$\begin{aligned} \mathrm{val}(f) = \sum_{\substack{e=(v,w)\in E_d\\ v\in S,\, w\in T}} f(e) - \sum_{\substack{g=(u,v)\in E_d\\ u\in T,\, v\in S}} f(g) \leq \sum_{\substack{e=(v,w)\in E_d\\ v\in S,\, w\in T}} f(e) = F \\ \mathrm{val}(f) \leq F. \end{aligned} \tag{34}$$

Combining (34) with (32) then results in

$$\mathrm{val}(f) \leq c(S,T).$$

$\square$

Since this bound applies to any S-T-cut of the RCSJ Network $\mathcal{N}$ we can sort the cuts by their cut capacities and name them $c_n(S,T)$ with $n = 1, 2, \ldots,$ such that

$$c_1(S,T) \leq c_2(S,T) \leq \ldots.$$

Lemma 4.15 gives for any of those cuts that $\mathrm{val}(f) \leq c_n(S,T)$. In particular, this applies even to the smallest possible cut $\min(c(S,T))$. Therefore, we see quite easily that

$$\mathrm{val}(f) \leq \min(c(S,T)).$$

If we now were to choose the biggest possible $\mathrm{val}(f)$, which we named I_{max} in an RCSJ Network, it can in fact be proven that this upper bound is sharp, such that $\max(\mathrm{val}(f)) = \min(c(S,T))$. This relation is the *Max-flow Min-cut* theorem. [Cor+14, thm. 26.6]

Theorem 4.16. *Let $\mathcal{N}$ be an RCSJ Network with a maximum current I_{max} and the minimal cut capacity* $\min(c(S,T))$. *Then, the following holds:*

$$I_{max} = \min(c(S,T)). \tag{35}$$

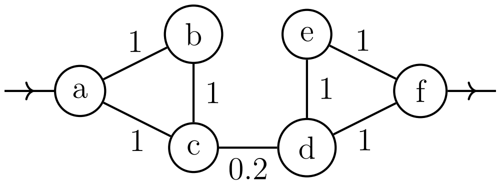

Figure 4: Example of an RCSJ Network $\mathcal{N}$ with a clear bottleneck. Numbers next to the edges $e \in E_d$ denote critical currents $I_C(e)$. An external current I_{ext} is injected into a extracted at f.

If we now combine this insight with our Theorem 4.12, we find that an RCSJ Network solves if

$$I_{ext} \leq \min(c(S,T)). \tag{36}$$

Thus the solvability problem for an RCSJ Network is equivalent to an minimum cut problem at the same time. Furthermore, the Equation (36) provides a necessary and sufficient condition for the solvability of an RCSJ Network. The idea behind it illustrated well by the following example.

Example 4.17. Consider the network shown in Figure 4. In this RCSJ Network, all junctions have a critical current $I_C = 1$, except the edge (c, d) with $I_C = 0.2$. The partition of V into

$$S = \{a, b, c\} \text{ and } T = \{d, e, f\}, \tag{37}$$

then defines an S-T-cut with the cut capacity

$$c(S,T) = I_C(c,d) = 0.2 \tag{38}$$

Since all other junctions have $I_C = 1$, any other S-T-cut has to at least include one edge with a capacity of 1 and therefore has a capacity of at least 1. So, $c(S,T) = 0.2$ is the minimum cut of the network, meaning that $I_{max} = 0.2$ by the Max-flow-Min-cut theorem 4.16. Therefore, the network solves only for external currents satisfying $I_{ext} \leq 0.2$. For larger currents, the junction (c, d) would exceed its critical current and would therefore become the first junction to fail. Therewith, the minimum cut corresponds physically to the smallest set of junctions that have to collectively carry the injected current.

As such, this derived condition is also relatively easy to evaluate since the external current is set explicitly and the minimum cut, or maximum current, depends directly on the structure and critical currents of the network. In fact, there are different algorithms that can calculate the minimum cut from a given network architecture, such as the Ford-Fulkerson algorithm [Cor+14, p.724], the Edmonds-Karp algorithm [Cor+14, p.727] and Dinic's algorithm. All of these algorithms compute the maximum flow in a network with varying time complexities. Dinic's algorithm is the most efficient with a time complexity of $O(V^2E)$ for general networks [Din06].

4.4 Spectral decomposition

Having established and confirmed this method of checking for the solvability of an RCSJ Network, we will now try to find a way to connect it with the actual internal structure of the network. The mathematical object containing all of this information is the graph Laplacian of the network.

One of the motivations for spectral analysis in graphs is the existence of the so-called *Cheeger's inequality* [Spi25, pp.175–176], which relates a graph structure specific constant $h(G)$, the *Cheeger's constant*, to the second smallest eigenvalue λ_2 of the graph Laplacian. Spielman [Spi25, p.173] in particular calls this quantity the *conductance* and defines it as the sum of the weights of the edges leaving a subset S of vertices divided by the total sum of edge weights of edges leaving the vertices in S or $V \backslash S$, depending on which is smaller. In the notation conventions established so far, this sum of weights of edges leaving S, is nothing but the capacity of a cut $c(S,T)$. To be consistent with the flow formulation, we restrict our definition of Cheeger's constant to S-T-cuts.

Definition 4.18 (Cheeger's constant)**.** Let $\mathcal{N}$ be an RCSJ Network with an S-T-cut $S \cup T = V$ and $S \cap T = \varnothing$. For a specific S-T-cut we define

$$h_S = \frac{c(S,T)}{\min\left(\mathrm{d}(S), \mathrm{d}(T)\right)}, \tag{39}$$

with

$$\mathrm{d}(S) = \sum_{\substack{e=(u,v)\in E_d \\ u \in S}} I_C(e). \tag{40}$$

Then, **Cheeger's constant** for the RCSJ Network is

$$h(G) = \min_{S \subset V}(h_S). \tag{41}$$

Theorem 4.19 (Cheeger's inequality)**.** *Let $\mathcal{N}$ be an RCSJ Network with the normalized graph Laplacian*

$$\mathsf{L} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}, \tag{42}$$

based on $L = D - A = BCB^T$. Let $h(G)$ be the Cheeger's constant of the graph and λ_2 denote the second smallest eigenvalue of L. Then the following inequality holds:

$$\frac{\lambda_2}{2} \leq h(G). \tag{43}$$

Remark 4.20. Since $h(G) = \min_{S \subset V}(h_S)$, it follows that for h_S of any subset $S \subset V$,

$$\frac{\lambda_2}{2} \leq h_S. \tag{44}$$

Remark 4.21. All in all, the definition of the Cheeger's inequality that we make use of is technically defined using undirected graphs. There exist different definitions for Cheeger constants on directed graphs, as most prominently the formalism derived by Chung [Chu05]. Since we can interpret our directed graph formulation as a bidirectional graph and use the same Laplacian formulation as for an undirected graph, our system acts for all effects and purposes of the Cheeger constant as an undirected graph. Thus, we can make use of the standard Cheeger constant definitions.

An easy way to find the second smallest eigenvalue of any Laplacian matrix is by one of the corollaries to the so-called *Courant-Fischer mix-max* theorem, which states the following optimization problem. [HJ85, thm. 4.2.2]

Theorem 4.22 (Rayleigh-Ritz)**.** *Let $A \in M_n$ be a real symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then,*

$$\lambda_1 = \min_{x \neq 0} \frac{x^\mathsf{T} A x}{x^\mathsf{T} x}, \tag{45}$$

and if x_1 is an eigenvector corresponding to λ_1, then

$$\lambda_2 = \min_{\substack{x \neq 0 \\ x \perp x_1}} \frac{x^\mathsf{T} A x}{x^\mathsf{T} x}. \tag{46}$$

This characterization allows us to express the second smallest eigenvalues of L and L as

$$\lambda_2(L) = \min_{\substack{u \neq 0 \\ u \perp u_1}} \frac{u^\mathsf{T} L u}{u^\mathsf{T} u} \qquad \text{and} \qquad \lambda_2(\mathsf{L}) = \min_{\substack{v \neq 0 \\ v \perp v_1}} \frac{v^\mathsf{T} \mathsf{L} v}{v^\mathsf{T} v}. \tag{47}$$

Keep in mind that u and v are arbitrary test vectors and can be exchanged. For example, a variable change to $w = D^{-\frac{1}{2}}v$ results in

$$v^\mathsf{T}\mathsf{L}v = v^\mathsf{T}D^{-\frac{1}{2}}LD^{-\frac{1}{2}}v = w^\mathsf{T}Lw.$$

Similarly, we find that the same substitution requires $x = D^{\frac{1}{2}}y$ such that

$$v^\mathsf{T}v = w^\mathsf{T}D^{\frac{1}{2}}D^{\frac{1}{2}}w = w^\mathsf{T}Dw.$$

This gives us a way to rewrite the second smallest eigenvalue of the normalized Laplacian in terms of the non-normalized Laplacian,

$$\lambda_2(\mathsf{L}) = \min_{\substack{w\neq 0\\ w\perp w_1}} \frac{w^\mathsf{T}Lw}{w^\mathsf{T}Dw}.$$

Here we can see that there is an additional term in the denominator. Therefore it could be a reasonable assumption that $\lambda_2(\mathsf{L}) \leq \lambda_2(L)$. This is not quite correct though because when changing the denominator from $w^\mathsf{T}w$ to $w^\mathsf{T}Dw$, we describe the eigenvalues with respect to a different inner product. As such, the eigenvalues cannot be compared directly in that way. In fact, one can show that the following relation holds instead.

Lemma 4.23. *Let $\mathcal{N}$ be an RCSJ Network with a weighted graph Laplacian L_C, denoted as L, and a normalized graph Laplacian matrix L. Furthermore, we define the minimum degree d_{min} as the smallest nonzero diagonal matrix element in D and the maximum degree d_{max} as the largest nonzero diagonal matrix element in D. In that case, the following relation holds:*

$$\frac{\lambda_2(L)}{d_{max}} \leq \lambda_2(\mathsf{L}) \leq \frac{\lambda_2(L)}{d_{min}}. \tag{48}$$

Proof. Let $\mathcal{N}$ be an RCSJ Network with the degree matrix D. By definition of the minimum and maximum degree, d_{min} and d_{max}, we can be sure that

$$d_{min} \leq D_{vv} \leq d_{max} \qquad \forall v \in V$$

holds. Multiplication with two vectors x^T and x then results in

$$d_{min}x^\mathsf{T}x \leq x^\mathsf{T}Dx \leq d_{max}x^\mathsf{T}x.$$

Furthermore, we can now divide some constant $x^\intercal Lx$ by that very expression, such that

$$\frac{x^\intercal Lx}{d_{min}x^\intercal x} \geq \frac{x^\intercal Lx}{x^\intercal Dx} \geq \frac{x^\intercal Lx}{d_{max}x^\intercal x},$$

which we subsequently can minimize in the same way as the Rayleigh-Ritz theorem 4.22,

$$\min_{\substack{x\neq 0\\ x\perp x_1}} \frac{x^\intercal Lx}{d_{min}x^\intercal x} \geq \min_{\substack{x\neq 0\\ x\perp x_1}} \frac{x^\intercal Lx}{x^\intercal Dx} \geq \min_{\substack{x\neq 0\\ x\perp x_1}} \frac{x^\intercal Lx}{d_{max}x^\intercal x}$$

By our earlier results in Equation (47) we therefore have

$$\frac{\lambda_2(L)}{d_{max}} \leq \lambda_2(\mathsf{L}) \leq \frac{\lambda_2(L)}{d_{min}}.$$

□

To make use of this result, we first insert the definition of the Cheeger's constant 4.18 into Cheeger's inequality 4.19, such that for some $S \subset V$,

$$\frac{\lambda_2(\mathsf{L})}{2} \leq h_S = \frac{c(S,T)}{\min\left(\mathrm{d}(S), \mathrm{d}(T)\right)}.$$

With the help of Lemma 4.23 we obtain

$$\frac{\lambda_2(L)}{2d_{max}} \cdot \min\left(\mathrm{d}(S), \mathrm{d}(T)\right) \leq \frac{\lambda_2(\mathsf{L})}{2} \cdot \min\left(\mathrm{d}(S), \mathrm{d}(T)\right) \leq c(S,T). \tag{49}$$

This shows us that we can, in fact, relate the cut capacity of an S-T-cut to the eigenvalues of the graph Laplacian of the network. As we are specifically working with an S-T-cut we know by definition that $V = S \cup T$. Therefore, $\mathrm{d}(V) = \mathrm{d}(S) + \mathrm{d}(T)$ has to be true. We can exclude that any edges on the boundary of S will be double-counted by that addition, because by the definition of $\mathrm{d}(S)$, only edges beginning in S are counted. This includes edges starting in S and ending in S, as well as edges starting in S and ending in T. For $\mathrm{d}(T)$, this entire process is reversed, which ensures that every edge is counted exactly once. Therefore, it becomes trivial from elementary algebra that

$$\min(\mathrm{d}(S), \mathrm{d}(T)) \leq \frac{1}{2}\,\mathrm{d}(V).$$

Therefore, a combination with (49) results in

$$\frac{\lambda_2(L)}{2d_{max}} \cdot \frac{1}{2}\,\mathrm{d}(V) \leq c(S,T).$$

This expression is general by definition and holds for any S-T-cut $c(S,T)$ of the RCSJ Network $\mathcal{N}$. Since this includes even the minimum cut we find immediately that

$$\frac{\lambda_2(L)}{4d_{max}} \cdot \mathrm{d}(V) \leq \min(c(S,T)) = I_{max},$$

by application of the Max-flow-Min-cut theorem 4.16. Comparing to the solvability test established in Theorem 4.12, we therewith know that if

$$I_{ext} \leq \frac{\lambda_2(L)}{4d_{max}} \cdot \mathrm{d}(V), \tag{50}$$

then the RCSJ Network has to be solvable. This however is only a sufficient condition, since the case when

$$\frac{\lambda_2(L)}{4d_{max}} \cdot \mathrm{d}(V) \leq I_{ext} \leq I_{max} \tag{51}$$

is not covered, even though the network would still solve. Still, we have successfully found a way to connect the spectrum of the graph Laplacian to the solvability of the network and can make use of it as an indicator as to whether a static failure will occur or not, given a particular RCSJ Network structure and external current.

We demonstrate the efficacy of this bound with the following example.

Example 4.24. Consider the same RCSJ Network as in Example 4.17, illustrated by Figure 4 and with $I_{max} = 0.2$. Using the relation $L = BCB^{\mathsf{T}}$ introduced earlier, the graph Laplacian of the network is given by

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2.2 & -0.2 & 0 & 0 \\ 0 & 0 & -0.2 & 2.2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix},$$

with the eigenvalues $\{0, 0.122, 3, 3, 3, 3.278\}$ and the second smallest eigenvalue therefore being $\lambda_2 = 0.122$. Furthermore, we know that $d_{max} = 1 + 1 + 0.2$, as fulfilled by vertices c and d, while $\mathrm{d}(V) = 12.4$. Using Equation (50), we find that

$$\frac{\lambda_2(L)}{4d_{max}} \cdot \mathrm{d}(V) = \frac{0.122}{4 \cdot 2.2} \cdot 12.4 = 0.172 \leq 0.2 = I_{max}.$$

As expected, our spectral estimate provides a conservative lower bound on the maximum current and could therefore be used as a sufficient condition for solvability whenever $I_{ext} \leq 0.172$.

5 Dynamic failures

In the following section we will try to gain a better understanding of the full dynamical RCSJ system and investigate its behavior in the parameter regimes close to failure. This is in particular the case when many of the phase differences in $B^\mathsf{T}\boldsymbol{\theta}$ are close to $\frac{\pi}{2}$, as discussed in Section 1.3.

Still, this requires us to analyze the system of coupled differential equations provided by our RCSJ Network model. One simple and effective way to analytically analyze differential equations is by applying a Laplace transform on the entire model. In order to proceed, we therefore first review the following preliminaries.

5.1 Laplace transforms

The *Laplace transform* is an operation on a function $f(t)$ with a time-dependence that converts it into a function $F(s)$, where s is a complex parameter, often referred to as the frequency. By transforming differential equations from the time-domain into frequency-domain, it is possible to view and work with them as algebraic equations instead. This particular use of Laplace transforms is what we will focus on in this short overview of Laplace transforms, which is based on [Jam11, pp. 348-380].

Definition 5.1 (Laplace transform)**.** Let $f(t)$ be a function $f : \mathbb{R} \to \mathbb{R}$ of exponential order as $t \to \infty$, meaning that there exist a number σ and positive constants M and T such that

$$|f(t)| < Me^{\sigma t} \tag{52}$$

for all $t > T$. The unilateral **Laplace transform** $\mathcal{L}\{f(t)\} = F(s)$ is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s), \qquad \Re(s) > \sigma. \tag{53}$$

From this definition, we can immediately see that the linearity of the integral implies that the Laplace transform is also a *linear* operation. So, $\mathcal{L}\{f(t) + g(t)\} = F(s)+G(s)$. Furthermore, the simplest Laplace transforms can also be derived easily from the definition. For example, the Laplace transform for a constant function $f(t) = c$ becomes

$$\begin{aligned}\mathcal{L}\{c\} &= \int_0^\infty e^{-st} \cdot c \cdot dt = \lim_{T\to\infty} \left[-\frac{c}{s} e^{-st}\right]_0^T \\ &= \frac{c}{s}\left(-\lim_{T\to\infty} e^{-sT} + 1\right) = \frac{c}{s}(0+1) = \frac{c}{s}.\end{aligned}$$

Similarly, we find by integration by parts that

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t \cdot dt = \frac{1}{s^2}.$$

Integration by parts can even be used to derive the Laplace transform of a derivative $f'(t)$. This results in

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} \cdot f'(t) \cdot dt \\
&= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty \left(-se^{-st}\right) f(t) dt \\
&= \lim_{t\to\infty} e^{-st} f(t) - e^{-s\cdot 0} f(0) + s \int_0^\infty e^{-st} f(t) dt \\
&= -f(0) + sF(s),
\end{aligned}$$

where we make use of the restriction of $f(t)$ to exponential order to conclude that $\lim_{t\to\infty} e^{-st} f(t) \to 0$. As a nested version of this transform of derivative condition, the Laplace transform of a second derivative, becomes

$$\mathcal{L}\{f''(t)\} = -f'(0) + s\mathcal{L}\{f'(t)\} = s^2 F(s) - sf(0) - f'(0).$$

These identities will be used in the following section to transform the dynamical RCSJ equations into an algebraic form.

5.2 Perturbation model

Now, we can use such Laplace transforms in the context of the differential equation,

$$\boldsymbol{i} = B^{\mathsf{T}} \ddot{\boldsymbol{\theta}}(\boldsymbol{\tau}) + DB^{\mathsf{T}} \dot{\boldsymbol{\theta}}(\boldsymbol{\tau}) + \sin(B^{\mathsf{T}} \boldsymbol{\theta}(\boldsymbol{\tau})) \tag{54}$$

in the RCSJ Network. Naively applying the Laplace transform with τ as a time-like domain corresponding to t, would be bound to fail since the sin term is not linear. This means that because $\mathcal{L}\{\sin(f(t))\} \neq \sin(\mathcal{L}\{f(t)\})$, we would be stuck with a part of the system encoding the dynamics that is impossible to transform into an analytical expression. Another approach would be to approximate the state in which all phase differences $\Delta\theta \approx \frac{\pi}{2}$ and linearize the sin term. However, in that case a

Taylor expansion with $\Delta\theta + x$ returns

$$\sin(\Delta\theta + x) = \sin\left(\frac{\pi}{2} + x\right) = \cos(x) \approx 1 - \frac{(x)^2}{2} + \ldots,$$

which contains the quadratic $(x)^2$ term. This term is non-linear again and therefore won't give a clean Laplace transform either.

Thus, we have to take a different approach. Instead of focusing on failures specifically, we will first model how an RCSJ Network reacts to smaller phase changes. These could for example be induced by small changes in the external current. In order to model this, we will therefore introduce a small, time-dependent perturbation $\varphi(\tau)$ for every vertex, such that

$$\boldsymbol{\theta(\tau)} = \boldsymbol{\theta}_0 + \boldsymbol{\varphi(\tau)}. \tag{55}$$

This alone would not yet make the sin term linear. However, a simple Taylor expansion of sin around a fixed baseline x_0 with $x = x_0 + \delta x(t)$ gives

$$\sin(x) \approx \sin(x_0) + \cos(x_0)\delta x(t),$$

such that there is only a time-dependence on the perturbation $\delta x(t)$ and not on the baseline x_0. Thus, the time-dependent part of our equation would be perfectly linear, with only some additional constants to consider. Writing the sin from Equation (54) using the perturbation (55) would therewith result in

$$\begin{aligned}\sin\left(\left(B^\mathsf{T}(\boldsymbol{\theta}_0 + \boldsymbol{\varphi(\tau)})\right)_e\right) &= \sin(\left(B^\mathsf{T}\boldsymbol{\theta}_0\right)_e + \left(B^\mathsf{T}\boldsymbol{\varphi(\tau)}\right)_e) \\ &\approx \sin(\left(B^\mathsf{T}\boldsymbol{\theta}_0\right)_e) + \cos(\left(B^\mathsf{T}\boldsymbol{\theta}_0\right)_e)\left(B^\mathsf{T}\boldsymbol{\varphi(\tau)}\right)_e\end{aligned}$$

for every edge $e \in E_d$ for some RCSJ Network $\mathcal{N}$. This operation is performed edge-wise to ensure the correct multiplication of the individual elements of $\cos(B^\mathsf{T}\boldsymbol{\theta}_0)$ and $B^\mathsf{T}\boldsymbol{\varphi(\tau)}$ because both are vectors. To circumvent this problem we introduce the diagonal matrix

$$E = \operatorname{diag}(\cos(B^\mathsf{T}\boldsymbol{\theta}_0)), \tag{56}$$

which will achieve the same edge-wise multiplication using normal matrix multiplication. So, the linearized version of (54) becomes

$$\boldsymbol{i} = B^\mathsf{T}\ddot{\boldsymbol{\varphi}}(\boldsymbol{\tau}) + DB^\mathsf{T}\dot{\boldsymbol{\varphi}}(\boldsymbol{\tau}) + \sin(B^\mathsf{T}\boldsymbol{\theta}_0) + EB^\mathsf{T}\boldsymbol{\varphi}(\boldsymbol{\tau}), \tag{57}$$

where the $\ddot{\boldsymbol{\theta}}_0 = \dot{\boldsymbol{\theta}}_0 = 0$ since $\boldsymbol{\theta}_0$ is constant. While (57) represents the full perturbation, including the baseline $B^\mathsf{T}\boldsymbol{\theta}_0$, the interesting dynamics of the perturbation are fully contained within the time-dependent terms. In fact, this is the exact steady-state term $\boldsymbol{i}_0 = \sin(B^\mathsf{T}\boldsymbol{\theta}_0)$ governing our notion of static failures. Here, we assume that the stationary baseline configuration satisfies $|(B^\mathsf{T}\boldsymbol{\theta}_0)_e| < \frac{\pi}{2}$ for all edges e, so that $\cos((B^\mathsf{T}\boldsymbol{\theta}_0)_e) > 0$. Therefore, the perturbed current $\delta\boldsymbol{i} = \boldsymbol{i} - \boldsymbol{i}_0$ is

$$\delta\boldsymbol{i} = B^\mathsf{T}\ddot{\boldsymbol{\varphi}}(\boldsymbol{\tau}) + DB^\mathsf{T}\dot{\boldsymbol{\varphi}}(\boldsymbol{\tau}) + EB^\mathsf{T}\boldsymbol{\varphi}(\boldsymbol{\tau}),$$

which is remarkably similar to the original RCSJ equation that we started with. Applying current conservation,

$$\delta I_{ext}(\boldsymbol{e}_t - \boldsymbol{e}_s) = L_C\ddot{\boldsymbol{\varphi}}(\boldsymbol{\tau}) + L_D\dot{\boldsymbol{\varphi}}(\boldsymbol{\tau}) + L_E\boldsymbol{\varphi}(\boldsymbol{\tau}), \tag{58}$$

with a new weighted Laplacian $L_E = BCEB^\mathsf{T}$ and a small perturbation δI_{ext} to the external current. Thus, we have derived a model that should describe a small phase perturbation of an RCSJ Network and has a fully linear phase dependence.

5.3 Stability analysis

Now, we can simply apply a Laplace transform to the perturbation model (58). Assuming that there are initially no perturbations, meaning $\varphi(0) = \dot{\varphi}(0) = 0$, the transform yields

$$\begin{aligned}\mathcal{L}\left\{\delta I_{ext}(\boldsymbol{e}_t - \boldsymbol{e}_s)\right\} &= L_C\left(s^2\boldsymbol{\Phi}(s) - s\boldsymbol{\varphi}(0) - \dot{\boldsymbol{\varphi}}(0)\right) + L_D\left(s\boldsymbol{\Phi}(s) - \boldsymbol{\varphi}(0)\right) + L_E\left(\boldsymbol{\Phi}(s)\right)\\ &= s^2L_C\boldsymbol{\Phi}(s) + sL_D\boldsymbol{\Phi}(s) + L_E\boldsymbol{\Phi}(s).\end{aligned}$$

The phases are therewith given by

$$\boldsymbol{\Phi}(s) = \left(s^2L_C + sL_D + L_E\right)^{-1}\mathcal{L}\left\{\delta I_{ext}(\boldsymbol{e}_t - \boldsymbol{e}_s)\right\} = H(s)I(s),$$

where we define

$$G(s) = s^2L_C + sL_D + L_E$$

and $H(s) = G(s)^{-1}$ is the transfer function of the system. We can use the *poles* of $\Phi(s)$ to investigate the stability of our system. These are exactly the values of s where the inverse $G(s)^{-1}$ does not exist, which is equivalent to finding the $G(s)$ for which $\det(G(s)) = 0$. [Jam11, pp. 427, 432] Since the Laplacian matrices are finite dimensional, the poles correspond to those values of s for which $G(s)$ is singular, meaning those s satisfying $\ker(G(s)) \neq 0$. This directly implies that there exists a non-trivial vector $\boldsymbol{x}$, such that

$$G(s)\boldsymbol{x} = 0. \tag{59}$$

Furthermore, because the RCSJ Network can be considered a connected graph, the Laplacians have a kernel spanned by $\mathbf{1}$. This means that we have to restrict the analysis to vectors $\boldsymbol{x} \notin \operatorname{span}(\mathbf{1})$ because otherwise, $G(s)$ would always be singular for any value of s.

With this information, we can now rewrite (59). Multiplying by the Hermitian conjugate $\boldsymbol{x}^*$ then yields

$$\begin{aligned}
0 &= \boldsymbol{x}^* G(s)\boldsymbol{x} \\
0 &= \boldsymbol{x}^*(s^2 L_C + sL_D + L_E)\boldsymbol{x} \\
0 &= s^2\boldsymbol{x}^* L_C\boldsymbol{x} + s\boldsymbol{x}^* L_D\boldsymbol{x} + \boldsymbol{x}^* L_E\boldsymbol{x}.
\end{aligned}$$

As the weighted Laplacian matrices L_C, L_D and L_E are all real-valued and symmetric, we can define three scalars $a, b, c \in \mathbb{R}$, such that $a = \boldsymbol{x}^* L_C\boldsymbol{x}$, $b = \boldsymbol{x}^* L_D\boldsymbol{x}$ and $c = \boldsymbol{x}^* L_E\boldsymbol{x}$. Thus, the problem reduces to the quadratic equation

$$0 = as^2 + bs + c$$

with the solutions

$$s_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{60}$$

To narrow down the possible values of a, b and c, we can use the identity

$$\boldsymbol{x}^* L\boldsymbol{x} = \sum_{e=(u,v)\in E_d} w(u,v)|x_u - x_v|^2, \tag{61}$$

adapted from the real-valued formulation introduced by Spielman [Spi25, ch. 1.2]. It applies to all three Laplacians L_C, L_D, L_E, with their respective weights $w(u, v)$. The weights are represented by critical currents multiplied by scaling factors, where the critical currents can be interpreted as strictly positive. The scaling factors are given by the positive diagonal matrices D and E, depending on the Laplacian. Thus, $\boldsymbol{x}^* L \boldsymbol{x} \geq 0$, meaning that L_C, L_D and L_E are positive semidefinite. Furthermore, given that all weights are positive, we can find that if $\boldsymbol{x}^* L \boldsymbol{x} = 0$, then $|x_u - x_v|^2 = 0$ for all $(u, v) \in E_d$. Therefore, all $x_u = x_v$ and consequently $\boldsymbol{x} \in \text{span}(\mathbf{1})$. However, we restricted $\boldsymbol{x} \notin \text{span}(\mathbf{1})$, giving us that $a > 0$, $b > 0$ and $c > 0$.

In order for the dynamical system to be stable both roots s_+ and s_- have to be on the left half of the imaginary plane [Jam11, def. 5.3], meaning that the system is asymptotically stable if and only if $\Re(s_\pm) < 0$. From our solution (60) we now find that the real part of $s_\pm$ is given by

$$\Re(s_\pm) = -\frac{b}{2a}, \tag{62}$$

and because both a and b are strictly positive, we obtain $\Re(s_\pm) < 0$. Therewith, we have a found that *any small perturbation on the RCSJ Network has to be asymptotically stable.*

Lemma 5.2 (Stability of small perturbations)**.** *Let $\mathcal{N}$ be an RCSJ Network. Assuming all critical currents and damping factors to be positive and all junctions to be superconducting, the system is asymptotically stable with respect to small perturbations.*

Proof. Since poles of the system are given by Equation (60) and $a > 0$, $b > 0$ and $c > 0$, we find $\Re(s_\pm) = -\frac{b}{2a} < 0$. Therefore, all perturbations decay exponentially in time and are asymptotically stable [Jam11, p.432]. □

The decay rate of perturbations is therefore determined by the real part of the poles $s_\pm$,

$$\Re(s_\pm) = -\frac{\boldsymbol{x}^* L_D \boldsymbol{x}}{2\boldsymbol{x}^* L_C \boldsymbol{x}}. \tag{63}$$

We can find that smaller values of $\boldsymbol{x}^* L_D \boldsymbol{x}$ compared to $\boldsymbol{x}^* L_C \boldsymbol{x}$ will lead to slower decay of the perturbation, whereas larger values yield faster decay. The imaginary part of $s_\pm$ on the other hand, described by the discriminant $b^2 - 4ac$ in (60), governs the oscillatory behavior of the decay. Since $\Re(s) < 0$ for all poles, all oscillation modes decay exponentially in time. If the poles are real, the perturbation decays purely exponential. If however they form a conjugate pair, so $\Im(s) \neq 0$, then the imaginary term will give rise to exponentially damped oscillations. [Jam11, fig. 5.43]

Furthermore, the coefficients a, b and c depend on the choice of $\boldsymbol{x}$. Since different $\boldsymbol{x}$ correspond to different perturbation modes of the network, each mode can have a different pair of poles $s_{\pm}$. Therefore, different modes may decay at different rates and exhibit different oscillatory behavior.

5.4 Connections to cascades

Now that we have found that our system is stable with respect to smaller perturbations on top of a stationary phase distribution, we can also think about another phenomenon: cascades. We define a cascade as the process in which one junction fails and forces the phases around the failed junction to redistribute, therewith pushing additional junctions over their critical currents. This then causes more failures in a self-propagating sequence. From pure intuition alone, there are already two conditions that may serve as at least enabling factors for cascades:

1. *Many junction close to failure*: If many junctions carry currents that are relatively close to their critical currents, then the additional current needed to cause a failure will be small. Thus, they are more likely to be pushed over their critical current should a failure occur in their neighborhood.

2. *Weak damping*: If the junctions are only weakly damped, then any perturbation will take longer to decay and can travel further across the network. That way, they reach more junctions and can even amplify for slow dynamics.

When considering the low-frequency limit $s \to 0$, corresponding to slow perturbations, we find that

$$\Phi(0) = (0^2 \cdot L_C + 0 \cdot L_D + L_E)^{-1} I(0) = L_E^{-1} I(0),$$

provided that we still exclude the constant vector direction $\operatorname{span}(\mathbf{1})$. Hence, for slow perturbations, the baseline contribution of $\cos(\Delta\theta_0)$ in L_E dominates the response of the system.

Recall that $L_E = BCEB^\mathsf{T}$ and $E = \operatorname{diag}(\cos(B^\mathsf{T}\theta_0))$. If junctions in the system are close to failure, then $|(B^\mathsf{T}\theta_0)_e| \approx \frac{\pi}{2}$, so that $\cos((B^\mathsf{T}\theta_0)_e)$ becomes small. Consequently, $c = \boldsymbol{x}^* L_E \boldsymbol{x}$ becomes small. Since the system response involves the inverse L_E^{-1}, this can lead to a larger response from any one perturbation and especially slow perturbations.

However, at the same time, weak damping reduces the coefficient $b = \boldsymbol{x}^* L_D \boldsymbol{x}$, which according to Equation (62) reduces the decay rate of perturbations. Therefore, if both effects occur simultaneously, meaning that b is small while the system is close to failures, then perturbations may both decay slowly and produce larger perturbation responses. These are the conditions that may make the RCSJ Network more unstable and prone to cascades.

Thus, the formalism suggests that cascade-like behavior becomes more likely when both $c = \boldsymbol{x}^* L_E \boldsymbol{x}$ and $b = \boldsymbol{x}^* L_D \boldsymbol{x}$ are small compared to $a = \boldsymbol{x}^* L_C \boldsymbol{x}$ in an RCSJ Network.

6 Discussion

All in all, we formulated a graph theoretic-formalism for RCSJ Networks using directed graphs and used it to analyze failure mechanisms in RCSJ Networks. The static failure problem was then shown to be a maximum flow, or minimum cut problem. We found that if the externally applied current exceeds the minimum cut, then the system is not solvable and has to fail. This condition can be checked exactly using max-flow algorithms. Furthermore, we derived a spectral upper bound on the external current using Cheeger's inequality. As for dynamical failures, we derived a linearized model for a perturbation on the RCSJ Network and analyzed its dynamics. In this context, we found that an RCSJ Network is asymptotically stable with respect to small perturbations in non-critical regimes and identified parameters that could potentially influence the decay rate of perturbations, as well as make cascades more likely.

Both through physical intuition and our spectral arguments, we found that static failures directly correspond to the structural bottlenecks in the Network. If a minimal cut of a network now consisted of a single junction, then the network would be highly vulnerable to that junction failing. On the other hand, if the minimal cut consists of many edges, then this vulnerability would get distributed and the network would be much less prone to failures. This observation is also consistent with the notion of algebraic connectivity, meaning second smallest eigenvalues of their graph Laplacians. Networks with larger algebraic connectivity admit larger external currents before reaching the maximum current. Therewith, we can interpret the algebraic connectivity as a robustness indicator for our RCSJ Network.

However, the solvability criterion as a failure indicator has an important limitation. While it guarantees the existence of a feasible flow that satisfies all capacity and conservation constraints, it does not ensure that every feasible flow corresponds to a physically realizable phase configuration. Therefore, the solvability criterion should be interpreted as a structural indicator for failure, rather than a full existence theorem. If one wants to achieve that, then one can attempt to solve the coupled nonlinear RCSJ equations.

As for the stability analysis using Laplace transforms, we assumed a strictly non-critical configuration since at the critical current, the entire linearization argument it is based on breaks down. Still, the fact that the perturbation model shows that that small perturbations will essentially "fizzle out" by decaying as exponentially damped oscillations corresponds direction to the physical intuition of a damped harmonic oscillator-like system. Moreover, we also found that the ratio of $\boldsymbol{x}^* L_D \boldsymbol{x}$ to $\boldsymbol{x}^* L_C \boldsymbol{x}$ controls the damping speed, whereas small values of $\boldsymbol{x}^* L_E \boldsymbol{x}$ will result in a stronger low-frequency response.

Cascades in particular represent a complex process that is hard to study and predict, but we can at the very least make an educated guess as to what may contribute to them. They likely depend on a combination of structural vulnerabilities, for example be small cut capacities, and high sensitivity to perturbations, leading to longer decay times. If these conditions are then combined with a near-critical baseline phase configuration, then we would expect to see cascades.

Outlook

Furthermore, there are still many potential angles to this problem that we could attack. For example, it would be worth investigating whether the Fiedler vector, the eigenvector of the second smallest eigenvalue of the graph Laplacian, carries any information about potential failure locations within the network. Furthermore, a rigorous analysis of the static phase configurations in search of a real existence theorem could also be problem of interest. In particular, a close look at research with Kuramoto-type models may be beneficial, since these describe coupled phase oscillators with a sinusoidal coupling term similar to the one appearing in the RCSJ equation[Ace+05]. Moreover, proper and robust numerical tests to verify the claims and hints we found also present an interesting extension to this problem.

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