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Metric Graphs with Herglotz-Nevanlinna Vertex Interactions

av

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Abstract

Quantum graphs have emerged as a powerful framework for modeling wave propagation on network-like structures. In this work, we construct a class of operators acting as second-order differential operators along the edges of finite compact metric graphs, coupled with finite-dimensional components at the vertices, leading to vertex interactions described by Herglotz–Nevanlinna functions. This interaction mechanism naturally gives rise to spectral parameter–dependent vertex conditions, extending classical coupling models. The operator is first shown to be symmetric, and its self-adjointness is established through a detailed analysis of the resolvent equation. Finally, we investigate the spectrum of the operator in the case of a lasso graph and describe its main spectral properties.

Sammanfattning

Kvantgrafer har vuxit fram som ett kraftfullt ramverk för att modellera vågutbredning på nätverksliknande strukturer. I detta arbete konstruerar vi en klass av operatorer som verkar som andragradens differentialoperatorer längs kanterna i ändliga kompakta metriska grafer, kopplade till ändligdimensionella komponenter vid hörnen, vilket leder till hörninteraktioner beskrivna av Herglotz–Nevanlinna-funktioner. Denna interaktionsmekanism ger naturligt upphov till spektralparameterberoende hörnvillkor, vilka utvidgar klassiska kopplingsmodeller. Operatören visas först vara symmetrisk, och dess självadjungering etableras genom en detaljerad analys av resolventekvationen. Slutligen undersöker vi operatorns spektrum i fallet med en lasso-graf och beskriver dess huvudsakliga spektrala egenskaper.

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Contents

1 Introduction

The origins of graph theory date back to the seminal work of Euler in 1736 on the Königsberg bridge problem [Eul41], which marked the birth of a new mathematical discipline concerned with the study of networks and their structures. Since then, graph theory has evolved significantly, extending far beyond its combinatorial roots to interact with diverse areas of mathematics and applied sciences.

A recent development in this direction is the transition from purely combinatorial graphs to *metric graphs*, where each edge is endowed with a geometric interpretation by identifying it with an interval of the real line. This additional structure allows one to define differential operators along the edges, thereby introducing analytical tools into graph theory. As a result, metric graphs provide a natural framework for modeling various physical and engineering systems where propagation phenomena occur along network-like structures.

Building on this idea, the theory of *quantum graphs* emerges as the study of metric graphs equipped with differential operators—typically of Schrödinger type—acting on the edges, together with suitable conditions at the vertices. Although the terminology originates from quantum mechanics, the scope of quantum graphs is much broader. Indeed, they arise naturally in numerous applications such as quantum wires, photonic crystals, carbon nano-structures, waveguides, and mesoscopic systems, as well as in areas like dynamical systems and number theory [BK13, Kur24, Mug14, Kea08, KN05, KS18, BKKM19, KK02].

Historically, objects now recognized as quantum graphs have appeared in various physical and chemical contexts[RS53]. However, it was only in the last few decades that the subject has experienced rapid development, driven in part by advances in nanotechnology and the need to model wave propagation in thin structures. Today, quantum graphs constitute a rich and active area of research at the intersection of spectral theory, partial differential equations, operator theory, and mathematical physics.

A key feature of quantum graphs is that the behavior of functions at the vertices is not automatic but must be prescribed through *vertex conditions*. These conditions play a role analogous to boundary conditions in classical differential equations and determine how waves propagate and interact across the network, and they also ensure the self-adjointness of the associated differential operator. From both mathematical and physical perspectives, it is essential that the associated differential operator be *self-adjoint*, as this ensures real spectral values and a well-defined dynamical evolution.

Consequently, a significant problem in the study of quantum graphs is the characterization of vertex conditions that yield self-adjoint operators. Standard conditions include Kirchhoff (or Neumann-type) conditions, which enforce continuity of the function and conservation at each vertex. However, more general types of vertex conditions have been introduced and studied extensively[Kur24, MU24].

In particular, there has been increasing interest in *eigenparameter-dependent vertex conditions*, where the coupling at the vertices depends explicitly on the spectral parameter. Such conditions arise naturally in various contexts and can also be interpreted through equivalent formulations in time-dependent problems. This perspective opens the door to a deeper understanding of dynamic interactions on graphs. The works [MU23, ZM26] provide part of the mathematical motivation and inspiration for the present thesis and naturally lead to the study of vertex interactions described through Herglotz–Nevanlinna functions depending on the spectral parameter, which play a fundamental role in the spectral analysis of self-adjoint operators.

The present work is also closely related to the theory of singular perturbations and generalized resolvents of symmetric operators. In particular, models of zero-range potentials with internal structure, introduced by Pavlov in [Pav84], provide an important inspiration for the construction considered in this thesis. In these models, the interaction is realized through an extension of the underlying Hilbert space by an additional finite-dimensional internal space. This approach is deeply connected with the theory of self-adjoint extensions and solvable models developed in [Pav87], as well as with the framework of singular perturbations of differential operators presented in the monograph by Albeverio and Kurasov [AK00]. In particular, the model of generalized δ-interactions in one dimension can be viewed as a prototype for the type of vertex interactions studied in the present work. Our construction extends these ideas from the one-dimensional setting to the framework of metric graphs.

In this thesis, entitled **Metric Graphs with Herglotz–Nevanlinna Vertex Interactions**, we develop and analyze a class of differential operators on metric graphs where the vertex interactions are essentially described by rational Herglotz–Nevanlinna functions. We begin by introducing a model on a metric star graph consisting of three semi-infinite edges, each identified with the half-line $[0, \infty)$ and connected at a central vertex V. The corresponding Hilbert space is constructed as

$$\mathcal{H} := L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+) \oplus \mathbb{C}^n,$$

where n is any natural number and an element $U \in \mathcal{H}$ is written as

$$U = (u|_{e_1},\, u|_{e_2},\, u|_{e_3},\, u).$$

The operator under consideration acts as the negative second derivative along each edge and is coupled with a finite Hermitian matrix at the vertex. Its domain is defined by imposing Sobolev regularity along the edges together with a coupling condition of the form

$$\langle \theta, u \rangle = c\, u(V) + d\, \partial u(V),$$

where $\theta \in \mathbb{C}^n$ is a fixed vector, and $c, d, \in \mathbb{R}$. This construction is then extended to the case of a general finite compact metric graph Γ with vertex set $V = \{V_1, \ldots, V_M\}$, where the Hilbert space is given by

$$\mathcal{H} := L^2(\Gamma) \oplus \bigoplus_{m=1}^{M} \mathbb{C}^{n_m}.$$

In our work, at each vertex V_m, we impose continuity conditions together with coupling conditions of the form

$$\langle \theta_m, u_m \rangle = c_m u(V_m) + d_m \partial u(V_m).$$

Again, here $\theta_m \in \mathbb{C}^{n_m}$ is a fixed vector, and $c_m, d_m \in \mathbb{R}$. The vector u_m denotes the vector related to vertex V_m, while $u(V_m)$ denotes the common vertex value ensured by continuity, and $\partial u(V_m)$ denotes the sum of outgoing derivatives at V_m.

A central part of this thesis is devoted to the analysis of the constructed operator. We first show that the operator is symmetric and then establish its self-adjointness through a detailed study of the associated resolvent equation. This analysis naturally leads to the appearance of a Herglotz–Nevanlinna function $Q(\lambda)$, which describes the vertex interaction and induces spectral parameter dependent vertex conditions. In addition, we investigate the spectral properties of the operator and analyze the corresponding spectrum in the particular case of a lasso graph.

2 Preliminaries

The purpose of this section is to introduce the fundamental notions and tools that will be used throughout this thesis. Our exposition is mainly based on the classical framework of spectral theory and operator theory in Hilbert spaces, with primary reference to [RY08].

Definition 2.1 (Inner product)**.** Let $\mathcal{H}$ be a complex vector space. An *inner product* on $\mathcal{H}$ is a function

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

such that for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$:

(a) $\langle x, x \rangle \in \mathbb{R}$ and $\langle x, x \rangle \geq 0$;

(b) $\langle x, x \rangle = 0$ if and only if $x = 0$;

(c) $\langle \alpha x + \beta y, z \rangle = \overline{\alpha} \langle x, z \rangle + \overline{\beta} \langle y, z \rangle$;

(d) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

Lemma 2.2. *Let $\mathcal{H}$ be an inner product space and let $x, y \in \mathcal{H}$. Then:*

(a) $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$*;*

(b) The function $\| \cdot \| : \mathcal{H} \to \mathbb{R}$ *defined by*

$$\|x\| = \sqrt{\langle x, x \rangle}$$

is a norm on $\mathcal{H}$*.*

The norm $\|x\| = \sqrt{\langle x, x \rangle}$ *is called the norm* induced *by the inner product* $\langle \cdot, \cdot \rangle$*.*

Definition 2.3 (Hilbert space)**.** An inner product space $\mathcal{H}$ is called a *Hilbert space* if it is complete with respect to the metric induced by the norm

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Definition 2.4 (Lebesgue space)**.** Let $\Omega \subset \mathbb{R}^n$. The Lebesgue space $L^2(\Omega)$ is defined as

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{C} \text{ measurable} \;\middle|\; \int_\Omega |f(x)|^2 \, dx < \infty \right\},$$

equipped with the norm

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2 \, dx \right)^{1/2}.$$

Definition 2.5 (Sobolev space W_2^2)**.** Let $\Omega \subset \mathbb{R}$. The Sobolev space $W_2^2(\Omega)$ is defined by

$$W_2^2(\Omega) = \left\{ f \in L^2(\Omega) \;\middle|\; f' \in L^2(\Omega),\ f'' \in L^2(\Omega) \right\}.$$

.

Theorem 2.6. *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $T \in B(\mathcal{H}, \mathcal{K})$(We denote by $B(\mathcal{H}, \mathcal{K})$ the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$). Then there exists a unique operator $T^* \in B(\mathcal{K}, \mathcal{H})$ such that*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for all $x \in \mathcal{H}$ and all $y \in \mathcal{K}$.

Definition 2.7 (Adjoint operator)**.** Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $T \in B(\mathcal{H}, \mathcal{K})$. The operator T^* constructed in Theorem (2.6) is called the *adjoint* of T.

Definition 2.8 (Symmetric operator)**.** Let $\mathcal{H}$ be a Hilbert space and let T be a linear operator with domain $\mathcal{D}(T) \subset \mathcal{H}$. The operator T is called *symmetric* if

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

for all $x, y \in \mathcal{D}(T)$.

Definition 2.9 (Self-adjoint operator)**.** Let $\mathcal{H}$ be a complex Hilbert space and let $T \in B(\mathcal{H})$. The operator T is called *self-adjoint* if

$$T = T^*.$$

Since the operator considered in this thesis is unbounded, we briefly recall some basic notions related to unbounded operators [Mat24].

Let H be a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$, and let

$$A : D(A) \subset H \to H$$

be a densely defined unbounded operator. For a fixed $v \in H$, consider the mapping

$$u \mapsto \langle Au, v \rangle, \qquad u \in D(A).$$

Since the inner product is linear, this mapping defines a linear functional on $D(A)$. We say that v belongs to the domain of the adjoint operator A^* if this functional is bounded with respect to the norm of H, namely if there exists a constant $C_v > 0$ such that

$$|\langle Au, v \rangle| \leq C_v \|u\|, \qquad \forall u \in D(A).$$

The set of all such vectors v is called the domain of the adjoint operator and is denoted by

$$D(A^*) = \{v \in H : \exists\, C_v > 0 \text{ such that } |\langle Au, v \rangle| \leq C_v \|u\|, \; \forall u \in D(A)\}.$$

By the Riesz Representation Theorem, every bounded linear functional on a Hilbert space can be represented uniquely as an inner product with a fixed element of the space. Therefore, for every

$$v \in D(A^*),$$

there exists a unique element $f \in H$ such that

$$\langle Au, v \rangle = \langle u, f \rangle, \qquad \forall u \in D(A).$$

This element is denoted by

$$f = A^* v.$$

Hence, the adjoint operator

$$A^* : D(A^*) \subset H \to H$$

is defined through the identity

$$\langle Au, v \rangle = \langle u, A^* v \rangle, \qquad \forall u \in D(A), \quad v \in D(A^*).$$

An operator A is called symmetric if

$$\langle Au, v \rangle = \langle u, Av \rangle, \qquad \forall u, v \in D(A).$$

Equivalently, the operator A is symmetric if

$$A \subset A^*,$$

that is, the action of A coincides with the action of its adjoint on the domain $D(A)$.

Furthermore, the operator A is called self-adjoint if

$$A = A^*.$$

This means that the actions of A and A^* coincide and, moreover,

$$D(A) = D(A^*).$$

Definition 2.10 (Resolvent set and resolvent operator)**.** Let $\mathcal{H}$ be a complex Hilbert space and let

$$T : D(T) \subset \mathcal{H} \to \mathcal{H}$$

be a densely defined operator.

A complex number $\lambda \in \mathbb{C}$ is said to belong to the resolvent set of T, denoted by $\rho(T)$, if the operator

$$T - \lambda I : D(T) \to \mathcal{H}$$

is bijective and the inverse

$$(T - \lambda I)^{-1} : \mathcal{H} \to \mathcal{H}$$

is bounded. The set

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

is called the spectrum of T. For every $\lambda \in \rho(T)$, the operator

$$R(\lambda, T) := (T - \lambda I)^{-1}$$

is called the resolvent operator of T at λ.

Definition 2.11 (Herglotz–Nevanlinna function [LN17])**.** A function $Q : \mathbb{C}^+ \to \mathbb{C}$ is called a *Herglotz–Nevanlinna function* if it is analytic and

$$\operatorname{Im} Q(z) \geq 0 \quad \text{for all } z \in \mathbb{C}^+.$$

Theorem 2.12 (Nevanlinna representation [LN17])**.** *A function $Q : \mathbb{C}^+ \to \mathbb{C}$ is a*

Herglotz–Nevanlinna function if and only if it admits the representation

$$Q(z) = a + bz + \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t),$$

where $a \in \mathbb{R}$, $b \geq 0$, and μ is a positive Borel measure satisfying

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\, d\mu(t) < \infty.$$

Moreover, a, b, and μ are uniquely determined by Q.

3 Differential Operators on Metric Graphs

In this chapter, we present the basic framework of differential operators on metric graphs, which form the foundation of quantum graph theory. A metric graph is obtained by assigning lengths to the edges of a combinatorial graph, allowing differential operators to act along the edges. These operators typically act as second-order differential expressions and are coupled through vertex conditions that describe the interaction at the vertices. A key aspect of the theory is the choice of vertex conditions ensuring self-adjointness of the operator, which is essential for the analysis of its spectral properties. The material in this chapter relies primarily on the foundational works in [Kur24] and [BK13], which constitute the main references for the framework developed here.

3.1 Metric Graphs

A graph G consists of a set V of vertices and a set $E \subset V \times V$ of edges, where each edge connects two vertices.

For differential operators, the edges play the crucial role, which makes it necessary to turn the standard definition of graphs "upside down" and start the whole construction with the edges. In this framework, each edge is identified with an interval of the real line.

Let $\{E_n\}_{n=1}^N$ be a finite family of intervals, where each edge E_n is either a compact interval

$$E_n = [x_{2n-1}, x_{2n}], \quad n = 1, \dots, N_c,$$

or a semi-infinite interval

$$E_n = [x_{2n-1}, \infty), \quad n = N_c + 1, \dots, N,$$

with $N = N_c + N_i$, where N_c (respectively N_i) denotes the number of compact (respectively infinite) intervals. Compact finite graphs, occur when $N_i = 0$, i.e. all edges are finite closed intervals: $N = N_c$.

Let $V = \{x_j\}$ be the set of all endpoints of these intervals. We define the set of vertices by introducing a partition of V into M disjoint subsets

$$V = \bigcup_{m=1}^{M} V_m, \quad V_{m_1} \cap V_{m_2} = \emptyset \text{ for } m_1 \neq m_2,$$

where each subset V_m is called a vertex. This partition induces an equivalence relation

$\sim$ on the set of endpoints such that

$$x \sim y \quad \text{if and only if} \quad x, y \in V_m \text{ for some } m.$$

The corresponding metric graph Γ is defined as the union of all edges E_n, where endpoints belonging to the same equivalence class are identified. More generally, two points x and y on the graph are equivalent if either they coincide on the same edge or belong to the same vertex, i.e.,

$$x \sim y \Longleftrightarrow \begin{cases} \exists\, E_n : x, y \in E_n \text{ and } x = y, \\ \text{or} \\ \exists\, V_m : x, y \in V_m. \end{cases}$$

A metric graph is called connected if any two points x and y in Γ can be connected by a path—a finite sequence of compact intervals-. The number of elements in a class V_m, denoted by d^m, is called degree of the vertex V_m. In the absence of loops (i.e., edges whose both endpoints belong to the same vertex), the degree of a vertex coincides with the number of edges incident to it.

The distance $d(x, y)$ between two points $x, y \in \Gamma$ is defined as the length of the shortest path connecting them. In particular, for points lying on the same edge, this distance may be strictly smaller than $|x - y|$. If x and y belong to different connected components of the graph, we set $d(x, y) = \infty$. A parametrization of the edges induces a natural measure on the metric graph Γ. We consider complex-valued functions on Γ and define the Hilbert space

$$L^2(\Gamma) = \bigoplus_{n=1}^{N} L^2(E_n).$$

Functions in $L^2(\Gamma)$ are defined edgewise and not pointwise on the graph. In particular, their values at the vertices, which form a set of measure zero, do not affect the space. Integration over the graph is therefore given by

$$\int_\Gamma f(x)\, dx = \sum_{n=1}^{N} \int_{E_n} f(x)\, dx.$$

For functions that are continuous on the edges, it is natural to define their values at the endpoints by taking limits along the edges,

$$u(x_j) = \lim_{x \to x_j} u(x).$$

However, if several endpoints x_j belong to the same vertex V_m, these limits may differ, and thus the value at the vertex is not necessarily well-defined. It becomes well-defined only when the function is continuous across the vertex.

If the function is differentiable along the edges, one defines the normal derivatives at the endpoints by

$$\partial_n u(x_j) = \begin{cases} \lim\limits_{x \to x_j} \dfrac{d}{dx} u(x), & \text{if } x_j \text{ is a left endpoint,} \\ -\lim\limits_{x \to x_j} \dfrac{d}{dx} u(x), & \text{if } x_j \text{ is a right endpoint.} \end{cases}$$

Here, the limits are taken from inside the corresponding interval.

3.2 Differential Operators

We now introduce the main differential operators acting on metric graphs. These operators are defined edgewise and model different physical and mathematical phenomena.

The simplest and most fundamental example is the Laplace operator, given by

$$\tau = -\frac{d^2}{dx^2}.$$

A natural generalization is the Schrödinger operator, which includes a potential term $q(x)$,

$$\tau_q = -\frac{d^2}{dx^2} + q(x).$$

In the presence of a magnetic field, one considers the magnetic Schrödinger operator, defined by

$$\tau_{q,a} = \left(i\frac{d}{dx} + a(x) \right)^2 + q(x),$$

where $a(x)$ represents the magnetic potential.

These operators act along each edge of the graph and will be complemented with suitable vertex conditions in order to define self-adjoint operators on the whole graph. The classical case of differential operators on a single interval (i.e., a graph consisting of one edge) shows that the definition of a metric graph operator is not complete unless its domain is specified. In particular, the domain must include both smoothness conditions along the edges and suitable vertex conditions at the endpoints, which play the role of boundary conditions.

Moreover, the requirement of self-adjointness imposes additional restrictions on the choice of vertex conditions. As mentioned above, a complete definition of the operator requires a precise description of its domain. Since we consider second-order differential operators (such as τ, τ_q, or $\tau_{q,a}$), this naturally leads to the definition of the Sobolev space on the metric graph Γ as

$$W_2^2(\Gamma) = \bigoplus_{n=1}^{N} W_2^2(E_n),$$

where each component corresponds to the Sobolev space defined on the edge E_n.

We refer to a quantum graph as a Schrödinger operator defined on a metric graph. In particular, the standard Laplacian is uniquely determined by the underlying metric graph[Kur24].

3.3 Vertex Conditions

We begin with the most commonly used class of vertex conditions, known as the *standard vertex conditions*, imposed at each vertex V_m.

Let V_m be a vertex of degree d^m, and denote by

$$\vec{u}(V_m) := \{u(x_j)\}_{j=1}^{d^m}, \qquad \partial\vec{u}(V_m) := \{\partial u(x_j)\}_{j=1}^{d^m},$$

the vectors of function values and outgoing derivatives at the vertex, respectively.

The standard vertex conditions consist of the following two requirements:

$$\begin{cases} x_i, x_j \in V_m \implies u(x_i) = u(x_j), & \text{(continuity condition)}, \\ \displaystyle\sum_{x_j \in V_m} \partial u(x_j) = 0, & \text{(Kirchhoff condition)}. \end{cases} \tag{1}$$

The first condition ensures that the function u is continuous across the vertex, while the second expresses the conservation of flux at the vertex. For each vertex V_m, these conditions provide d^m independent constraints, where d^m denotes the degree of the vertex. In the literature, these conditions are also referred to as *Kirchhoff*, *Neumann*, *natural*, or *free* conditions. In this work, we reserve the term *Kirchhoff condition* specifically for the balance condition on the derivatives.

Another important class of vertex conditions is given by the *Dirichlet condition*, which imposes

$$\vec{u}(V_m) = 0, \tag{2}$$

i.e., the function vanishes at the vertex along all incident edges. More generally, vertex conditions can be formulated as linear relations between the boundary values $\vec{u}(V_m)$ and $\partial\vec{u}(V_m)$. These general conditions will be discussed in theorem (3.1), as they provide a complete description of all self-adjoint realizations of the operator on the graph.

Theorem 3.1. *[BK13] Let Γ be a metric graph with finitely many edges. Consider the operator $\mathcal{A}$ acting as*

$$\mathcal{A}u = -\frac{d^2u}{dx^2}$$

on each edge, with domain consisting of functions $u \in W_2^2(\Gamma)$.
Then the operator $\mathcal{A}$ is self-adjoint if and only if, at each vertex V_m of degree d^m, the vertex conditions can be written in one (and hence any) of the following equivalent forms:

(A) *There exist $d^m \times d^m$ matrices A_m and B_m such that*

- *the $d^m \times 2d^m$ matrix $(A_m\ B_m)$ has maximal rank,*
- *the matrix $A_m B_m^*$ is self-adjoint,*

and the boundary values satisfy

$$A_m\vec{u}(V_m) + B_m\,\partial\vec{u}(V_m) = 0.$$

(B) *There exists a unitary $d^m \times d^m$ matrix U_m such that*

$$i(U_m - I)\vec{u}(V_m) + (U_m + I)\partial\vec{u}(V_m) = 0,$$

where I denotes the $d^m \times d^m$ identity matrix.

(C) *There exist three mutually orthogonal projections $P_{D,m}$, $P_{N,m}$ and*

$$P_{R,m} := I - P_{D,m} - P_{N,m},$$

acting on $\mathbb{C}^{d^m}$, together with an invertible self-adjoint operator Λ_m defined on the subspace

$P_{R,m}\mathbb{C}^{d^m}$*, such that*

$$\begin{cases} P_{D,m}\vec{u}(V_m) = 0, & \textit{(Dirichlet part)},\\ P_{N,m}\partial\vec{u}(V_m) = 0, & \textit{(Neumann part)},\\ P_{R,m}\partial\vec{u}(V_m) = \Lambda_m P_{R,m}\vec{u}(V_m), & \textit{(Robin part)}. \end{cases}$$

3.3.1 Examples of Vertex Conditions

In this subsection, we present some commonly used vertex conditions.

3.3.2 δ-type vertex conditions

We begin with the so-called δ-type (or δ-coupling) conditions.

Let V_m be a vertex of degree d^m. The δ-type condition at V_m is defined by the following requirements:

$$\begin{cases} u(x_i) = u(x_j), \quad \text{for all } x_i, x_j \in V_m,\\ \sum_{x_j \in V_m} \partial u(x_j) = \alpha_m\, u(V_m), \end{cases} \tag{3}$$

where $\alpha_m \in \mathbb{R}$ is a fixed parameter. Since the function u is continuous at the vertex V_m, its value $u(V_m)$ is well-defined. The real parameter α_m describes the strength of the interaction at the vertex. In the special case $\alpha_m = 0$, the δ-condition reduces to the standard Kirchhoff vertex condition:

$$\sum_{x_j \in V_m} \partial u(x_j) = 0.$$

Vertex condition (3) at V_m can be written in the general form

$$A_m\vec{u}(V_m) + B_m\,\partial\vec{u}(V_m) = 0,$$

with appropriately chosen matrices A_m and B_m. More precisely, one can take

$$A_m = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -\alpha_m & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad B_m = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Moreover, one verifies that

$$A_m B_m^* = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -\alpha_m \end{pmatrix},$$

which is a self-adjoint matrix. Hence, the conditions above satisfy the general self-adjointness criterion.

3.3.3 Extended δ-type vertex conditions

We recall that the δ-type vertex condition at a vertex V_m is given by the continuity condition together with

$$\partial u(V_m) = \alpha_m \, u(V_m), \quad \alpha_m \in \mathbb{R}.$$

This condition describes a coupling of strength α_m at the vertex.

An important observation is that the Dirichlet condition can be obtained as a limiting case of the δ-type condition. Indeed, dividing the above relation by α_m and formally passing to the limit $\alpha_m \to \infty$, one obtains

$$u(V_m) = 0,$$

which corresponds to the Dirichlet vertex condition. Therefore, the Dirichlet condition can be interpreted as a particular case of δ-type interactions.

Motivated by this observation, it is convenient to introduce an extended formulation of δ-type conditions that includes both standard and Dirichlet cases in a unified way.

This can be written as

$$\cos(\gamma_m)\,\partial u(V_m) = \sin(\gamma_m)\,u(V_m), \quad \gamma_m \in [0, \pi/2].$$

In this formulation, $\gamma_m = 0$ corresponds to the standard condition, while $\gamma_m = \frac{\pi}{2}$ yields the Dirichlet condition. Intermediate values of γ_m describe general δ-type vertex interactions.

3.4 The graph M-function

Let Γ be a finite compact metric graph consisting of N edges and M vertices $\{V_m\}_{m=1}^{M}$. We fix a subset of vertices $\partial\Gamma \subset \{V_m\}_{m=1}^{M}$, called the *contact vertices*. The remaining vertices are referred to as internal vertices. We denote by

$$D_\partial = \sum_{V_m \in \partial\Gamma} d(V_m)$$

the total degree of the contact vertices.

Let $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, and consider a function $\psi(\lambda, x)$ satisfying the differential equation on each edge

$$-\left(\frac{d}{dx} - ia(x)\right)^2 \psi(\lambda, x) + q(x)\psi(\lambda, x) = \lambda\psi(\lambda, x). \tag{4}$$

We assume that ψ satisfies the following vertex conditions at each internal vertex $V_m \in V \setminus \partial\Gamma$:

$$i\Big(S^m - I\Big)\vec{\psi}(V_m) = \Big(S^m + I\Big)\partial\vec{\psi}(V_m), \tag{5}$$

where S^m are $d_m \times d_m$ unitary matrices. At the contact vertices, only continuity is imposed, and no condition on the derivatives is required. We introduce the vectors of boundary values at the contact vertices:

$$\psi^\partial = \Big\{\psi(V_m)\Big\}_{V_m \in \partial\Gamma}, \qquad \partial\psi^\partial = \left\{\sum_{x_j \in V_m} \partial\psi(x_j)\right\}_{V_m \in \partial\Gamma}.$$

Definition 3.2. The graph's M-function $M_\Gamma(\lambda)$ is the $M_\partial \times M_\partial$ matrix-valued function defined by the map

$$M_\Gamma(\lambda) : \psi^\partial \mapsto \partial\psi^\partial, \quad \operatorname{Im}\lambda \neq 0,$$

where M_∂ denotes the number of contact vertices.

Here ψ^∂ and $\partial\psi^\partial$ are the limiting values for an arbitrary function $\psi(\lambda, x)$ satisfying the differential equation (4), the vertex conditions (5) at internal vertices, and continuity at the contact vertices.

For the existence and uniqueness of the M-function, we refer to [Kur24], Section 17.1.2.

4 Herglotz–Nevanlinna Vertex Interactions

In this chapter, we introduce and study a class of differential operators on metric graphs whose vertex interactions are described through Herglotz–Nevanlinna functions which are often used to describe spectra of self- adjoint operators. The main idea is to couple the differential operator acting on the graph with additional finite-dimensional components concentrated at the vertices. This coupling naturally leads to spectral parameter-dependent vertex conditions and provides a framework extending classical vertex interactions on metric graphs.

To present the construction transparently, we begin with a model problem on a star graph with three infinite edges. This example contains all the essential features of the general case while remaining sufficiently explicit for detailed analysis. After analyzing the star graph case, we extend the construction to the setting of general finite compact metric graphs and investigate the corresponding operator's self-adjointness and spectral properties.

4.1 The Model Problem: A Star Graph with Three Infinite Edges

Throughout this subsection we work on a metric star graph Γ with central vertex v and three edges e_1, e_2, e_3, each identified with the half-line $[0, \infty)$. We use the coordinate $x \in [0, \infty)$ on each edge, where $x = 0$ corresponds to the vertex v.

We define the Hilbert space

$$\mathcal{H} := L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_+) \oplus \mathbb{C}^n.$$

An element $U \in \mathcal{H}$ will be written as

$$U = (u|_{e_1},\, u|_{e_2},\, u|_{e_3},\, u),$$

where $u|_{e_j} \in L_2(\mathbb{R}_+)$ for $j = 1, 2, 3$ and $u \in \mathbb{C}^n$.

The inner product in $\mathcal{H}$ for any $U, V \in \mathcal{H}$ is defined by

$$\langle U, V \rangle_{\mathcal{H}} = \sum_{j=1}^{3} \int_0^\infty \overline{u|_{e_j}(x)}\, v|_{e_j}(x)\, dx \; + \; \langle u, v \rangle_{\mathbb{C}^n},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ denotes the standard inner product on $\mathbb{C}^n$.

Remark 4.1. Star graphs provide fundamental local models for general finite metric graphs. Indeed, any vertex of degree d in a finite metric graph has a neighbourhood which is naturally identified with a d-edge star graph. Therefore, understanding operator realizations and vertex interactions on star graphs is a key step towards the analysis of operators on arbitrary finite metric graphs.

Now, we introduce the Sobolev space

$$W_2^2(\mathbb{R}_+) := \{f \in L_2(\mathbb{R}_+) : f',\ f'' \in L_2(\mathbb{R}_+)\}\,.$$

We then define

$\partial u(v) := u|'_{e_1}(0) + u|'_{e_2}(0) + u|'_{e_3}(0)$ and $u|_{e_1}(0) = u|_{e_2}(0) = u|_{e_3}(0) := u(v)$. Additionally for real constants c , d and a fixed vector $\theta \in \mathbb{C}^n$ we set

$$\langle \theta, u\rangle = c\,u(v) + d\,\partial u(v). \tag{6}$$

Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix and $a, b \in \mathbb{R}$. For $U \in Dom(\mathcal{A})$ where

$$Dom(\mathcal{A}) = \Big\{U \in \mathcal{H} : u|_{e_j} \in W_2^2(\mathbb{R}_+),\ j = 1,2,3,\ \langle\theta, u\rangle = cu(v) + d\partial u(v)\Big\}\,, \tag{7}$$

we define the operator $\mathcal{A}$ by

$$\mathcal{A}U = \mathcal{A}\begin{pmatrix} u|_{e_1} \\ u|_{e_2} \\ u|_{e_3} \\ u \end{pmatrix} = \begin{pmatrix} -u|''_{e_1} \\ -u|''_{e_2} \\ -u|''_{e_3} \\ Au + \big(a\,u(v) + b\,\partial u(v)\big)\,\theta \end{pmatrix}. \tag{8}$$

Theorem 4.2. *Let $\mathcal{A}$ be the operator defined above in (8) with domain (7), then $\mathcal{A}$ is symmetric in $\mathcal{H}$ if and only if*

$$\det\begin{pmatrix} a & b \\ c & d\end{pmatrix} = 1.$$

Proof. Let $U, V \in Dom(\mathcal{A})$, then we have

$$\begin{aligned}\langle \mathcal{A}U, V\rangle - \langle U, \mathcal{A}V\rangle = &\sum_{j=1}^{3}\int_0^\infty -\overline{u|''_{e_j}}\, v|_{e_j}\,dx + \langle Au, v\rangle_{\mathbb{C}^n} + \Big(a\,\overline{u(v)} + b\,\overline{\partial u(v)}\Big)\langle\theta, v\rangle_{\mathbb{C}^n} \\ &- \sum_{j=1}^{3}\int_0^\infty \overline{u}|_{e_j}(-v|''_{e_j})\,dx - \langle u, Av\rangle_{\mathbb{C}^n} - \Big(c\,v(v) + d\,\partial v(v)\Big)\langle u, \theta\rangle_{\mathbb{C}^n}.\end{aligned}$$

For each edge $j = 1, 2, 3$, integration by parts yields

$$\int_0^\infty -\overline{u|''_{e_j}}\, v|_{e_j}\, dx = \int_0^\infty \overline{u}|'_{e_j}\, v|'_{e_j}\, dx + \overline{u}|'_{e_j}(0) v|_{e_j}(0), \tag{9}$$

and similarly

$$\int_0^\infty \overline{u}|_{e_j}\, (-v|''_{e_j})\, dx = \int_0^\infty \overline{u}|'_{e_j}\, v|'_{e_j}\, dx + \overline{u}|_{e_j}(0) v|'_{e_j}(0). \tag{10}$$

Subtracting (9) and (10) and summing over $j = 1, 2, 3$ gives the boundary contribution

$$\sum_{j=1}^{3} \left(\overline{u}|'_{e_j}(0) v|_{e_j}(0) - \overline{u}|_{e_j}(0) v|'_{e_j}(0) \right). \tag{11}$$

Using the vertex notations $u(v)$ and $\partial u(v)$, the boundary form (11) becomes

$$\overline{\partial u(v)}\, v(v) - \overline{u}(v)\, \partial v(v).$$

Furthermore by using the coupling condition (6) we obtain

$$\begin{aligned}
\langle \mathcal{A}U, V \rangle - \langle U, \mathcal{A}V \rangle &= \overline{\partial u(v)}\, v(v) - \partial v(v)\, \overline{u(v)} + ac\, \overline{u(v)}\, v(v) + ad\, \partial v(v)\, \overline{u}(v) \\
&\quad + bc\, \overline{\partial u(v)}\, v(v) + bd\, \overline{\partial u(v)}\, \partial v(v) - ac\, v(v)\, \overline{u(v)} \\
&\quad - ad\, v(v)\, \overline{\partial u(v)} - bc\, \partial v(v)\, \overline{u(v)} - bd\, \partial v(v)\, \overline{\partial u(v)} \\
&= (1 - ad + bc)\Big(\overline{\partial u(v)}\, v(v) - \overline{u}(v)\, \partial v(v) \Big).
\end{aligned}$$

Therefore, $\langle \mathcal{A}U, V \rangle - \langle U, \mathcal{A}V \rangle$ vanishes for all $U, V \in Dom(\mathcal{A})$ if and only if

$$ad - bc = 1.$$

Hence $\mathcal{A}$ is symmetric if and only if

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1.$$

□

To investigate the self-adjointness of the operator $\mathcal{A}$, we study the resolvent equation

$$(\mathcal{A} - \lambda)U = F, \qquad F \in \mathcal{H} \quad \text{and} \quad U \in Dom(\mathcal{A}), \tag{12}$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.
Let

$$U = (u|_{e_1},\, u|_{e_2},\, u|_{e_3},\, u), \qquad F = (f|_{e_1}, f|_{e_2}, f|_{e_3}, f).$$

Then the equation (12) combined with condition (6) is equivalent to

$$\begin{cases} -u|''_{e_1} - \lambda u|_{e_1} = f|_{e_1}, \\ -u|''_{e_2} - \lambda u|_{e_2} = f|_{e_2}, \\ -u|''_{e_3} - \lambda u|_{e_3} = f|_{e_3}, \\ (A-\lambda)u + (a\,u(v) + b\,\partial u(v))\theta = f. \end{cases}$$

This system consists of three inhomogeneous second-order differential equations on the half-lines $[0,\infty)$ coupled with a finite-dimensional algebraic equation in $\mathbb{C}^n$.

To establish solvability for arbitrary $F \in \mathcal{H}$, we first consider the two particular cases

$$\text{(i)} \quad F = (f|_{e_1}, 0, 0, 0), \qquad \text{(ii)} \quad F = (0, 0, 0, f).$$

Case (i): $F = (f|_{e_1}, 0, 0, 0)$

Choose $k \in \mathbb{C}$ such that $k^2 = \lambda$ (with $\mathrm{Re}(K) \geq 0$ and $\Im k > 0$), then the equation (12) is equivalent to the system

$$\begin{cases} -u|''_{e_1} - \lambda u|_{e_1} = f|_{e_1}, \\ -u|''_{e_2} - \lambda u|_{e_2} = 0, \\ -u|''_{e_3} - \lambda u|_{e_3} = 0, \\ (A-\lambda)u + (a\,u(v) + b\,\partial u(v))\,\theta = 0, \end{cases} \tag{13}$$

together with the vertex relations

$$u|_{e_1}(0) = u|_{e_2}(0) = u|_{e_3}(0) := u(v), \qquad \partial u(v) := u|'_{e_1}(0) + u|'_{e_2}(0) + u|'_{e_3}(0).$$

For $j = 2, 3$ the solutions of $-u|''_{e_j} - \lambda u|_{e_j} = 0$ are of the form

$$u|_{e_2}(x) = m_2 e^{ikx}, \qquad u|_{e_3}(x) = m_3 e^{ikx}.$$

For the defined equation on e_1 we obtain the solution

$$u|_{e_1}(x) = -\int_0^\infty \frac{e^{ik|x-y|}}{2ik}\, f|_{e_1}(y)\, dy + m_1 e^{ikx}.$$

Evaluating at $x = 0$ and imposing continuity gives

$$u|_{e_1}(0) = -\int_0^\infty \frac{e^{iky}}{2ik}\, f|_{e_1}(y)\, dy + m_1, \qquad u|_{e_2}(0) = m_2, \quad u|_{e_3}(0) = m_3,$$

hence

$$m_2 = m_3 =: m, \qquad m_1 = m + \int_0^\infty \frac{e^{iky}}{2ik}\, f|_{e_1}(y)\, dy.$$

Substituting m_1 back yields

$$u|_{e_1}(x) = -\int_0^\infty \frac{e^{ik|x-y|}}{2ik}\, f|_{e_1}(y)\, dy + me^{ikx} + \int_0^\infty \frac{e^{ik(x+y)}}{2ik}\, f|_{e_1}(y)\, dy.$$

On the other hand, from equation (13) we obtain

$$(A-\lambda)u + (a\, u(v) + b\, \partial u(v))\, \theta = 0,$$

hence

$$u = -(a\, u(v) + b\, \partial u(v))\, (A-\lambda)^{-1}\theta. \tag{14}$$

Taking the inner product with θ motivates the definition of the scalar function

$$\Delta(\lambda) := \langle \theta, (A-\lambda)^{-1}\theta \rangle_{\mathbb{C}^n}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{15}$$

Using (14) we obtain

$$\langle \theta, u \rangle_{\mathbb{C}^n} = -(a\, u(v) + b\, \partial u(v))\, \Delta(\lambda).$$

Combining this with the coupling condition (6) yields

$$c\, u(v) + d\, \partial u(v) = -(a\, u(v) + b\, \partial u(v))\, \Delta(\lambda),$$

that is,

$$(d + b\Delta(\lambda))\, \partial u(v) = -(c + a\Delta(\lambda))\, u(v).$$

Thus, we arrive at the effective vertex condition

$$\partial u(v) = -Q(\lambda)\, u(v), \qquad Q(\lambda) := \frac{a\Delta(\lambda) + c}{b\Delta(\lambda) + d}. \tag{16}$$

Remark 4.3. Note that both $Q(\lambda)$ and $\Delta(\lambda)$ are Herglotz-Nevanlinna functions under the determinant condition

$$ad - bc = 1,$$

thus $Q(\lambda)$ yields an energy-dependent vertex condition.

Now we compute the derivative sum at the vertex v, differentiating $u|_{e_1}$ and evaluating at $x = 0$ gives

$$u|'_{e_1}(0) = \int_0^\infty \frac{e^{iky}}{2}\, f|_{e_1}(y)\, dy + ikm + \int_0^\infty \frac{e^{iky}}{2}\, f|_{e_1}(y)\, dy = \int_0^\infty e^{iky} f|_{e_1}(y)\, dy + ikm.$$

Since $u|'_{e_2}(0) = ikm$ and $u|'_{e_3}(0) = ikm$, we obtain

$$\partial u(v) = \int_0^\infty e^{iky} f|_{e_1}(y)\, dy + 3ikm. \tag{17}$$

Combining (16) with (17) we obtain

$$\int_0^\infty e^{iky} f|_{e_1}(y)\, dy + 3ikm = -Q(\lambda) m,$$

hence

$$m = \frac{\displaystyle\int_0^\infty e^{iky} f|_{e_1}(y)\, dy}{-Q(\lambda) - 3ik}. \tag{18}$$

Substituting (18) into the expressions above yields the following formulas :

$$\begin{cases} u|_{e_1}(x) = \displaystyle\int_0^\infty \left(\frac{-e^{ik|x-y|} + e^{ik(x+y)}}{2ik} + \frac{e^{ik(x+y)}}{-Q(\lambda) - 3ik} \right) f|_{e_1}(y)\, dy, \\ u|_{e_2}(x) = \dfrac{\displaystyle\int_0^\infty e^{ik(x+y)} f|_{e_1}(y)\, dy}{-Q(\lambda) - 3ik}, \\ u|_{e_3}(x) = \dfrac{\displaystyle\int_0^\infty e^{ik(x+y)} f|_{e_1}(y)\, dy}{-Q(\lambda) - 3ik}, \\ \qquad u = \dfrac{a - bQ(\lambda)}{Q(\lambda) + 3ik} \left(\displaystyle\int_0^\infty e^{iky} f|_{e_1}(y)\, dy \right) (A - \lambda)^{-1} \theta. \end{cases}$$

Case (ii): $F = (0,0,0,f)$

We solve

$$(\mathcal{A}-\lambda)U = (0,0,0,f), \qquad k^2=\lambda \quad (\text{with} \quad \mathrm{Re}(K)\geq 0 \quad \text{and} \quad \Im k>0).$$

The equations

$$-u|''_{e_j} - \lambda u|_{e_j} = 0, \quad j=1,2,3,$$

have solutions

$$u|_{e_1}(x) = u|_{e_2}(x) = u|_{e_3}(x) = me^{ikx}.$$

Hence

$$u(v) = m, \qquad \partial u(v) = 3ikm.$$

Therefore we have

$$(A-\lambda)u + (am+3bikm)\theta = f,$$

so that

$$u = (A-\lambda)^{-1}f - (am+3bikm)(A-\lambda)^{-1}\theta.$$

Taking the inner product of u with θ and using (6),we obtain

$$m(c+3dik) = \langle \theta, (A-\lambda)^{-1}f\rangle - m(a+3bik)\Delta(\lambda), \quad \Delta(\lambda) = \langle \theta, (A-\lambda)^{-1}\theta\rangle.$$

Therefore

$$m = \frac{\langle \theta, (A-\lambda)^{-1}f\rangle}{c+3dik+(a+3bik)\Delta(\lambda)}.$$

In a simplified form, we obtain

$$m = \frac{1}{b\Delta(\lambda)+d}\cdot\frac{\langle \theta, (A-\lambda)^{-1}f\rangle}{Q(\lambda)+3ik}.$$

Substituting this value of m determines the following formulas:

$$\begin{cases} u|_{e_1}(x) = u|_{e_2}(x) = u|_{e_3}(x) = \dfrac{e^{ikx}}{b\Delta(\lambda)+d}\dfrac{\langle \theta, (A-\lambda)^{-1}f\rangle}{Q(\lambda)+3ik}, \\ \qquad u = (A-\lambda)^{-1}f - \dfrac{a+3bik}{b\Delta(\lambda)+d}\dfrac{\langle \theta, (A-\lambda)^{-1}f\rangle}{Q(\lambda)+3ik}(A-\lambda)^{-1}\theta. \end{cases}$$

Remark 4.4. Since, every

$$F = (f|_{e_1}, f|_{e_2}, f|_{e_3}, f) \in \mathcal{H}$$

admits the decomposition

$$F = (f|_{e_1}, 0, 0, 0) + (0, f|_{e_2}, 0, 0) + (0, 0, f|_{e_3}, 0) + (0, 0, 0, f). \tag{19}$$

We have treated the case $F = (f|_{e_1}, 0, 0, 0)$, and by the same argument one obtains the corresponding solutions for

$$F = (0, f|_{e_2}, 0, 0) \qquad \text{and} \qquad F = (0, 0, f|_{e_3}, 0),$$

since the differential equations on the three edges have the same form. We have also solved the case $F = (0, 0, 0, f)$.

Now let $U^{(1)}, U^{(2)}, U^{(3)}, U^*$ denote the solutions corresponding to the four components of F in (19). Since the operator $\mathcal{A} - \lambda$ is linear, the sum

$$U := U^{(1)} + U^{(2)} + U^{(3)} + U^*$$

satisfies

$$(\mathcal{A} - \lambda)U = (\mathcal{A} - \lambda)U^{(1)} + (\mathcal{A} - \lambda)U^{(2)} + (\mathcal{A} - \lambda)U^{(3)} + (\mathcal{A} - \lambda)U^* = F.$$

Therefore, the existence of a solution for the above elementary two cases implies the existence of a solution $U \in Dom(\mathcal{A})$ for every $F \in \mathcal{H}$. Hence we ensured the self-adjointness of the operator $\mathcal{A}$.

4.2 General finite compact metric graph case

Let Γ be a finite compact metric graph with N edges and vertex set

$$V = \{V_1, \ldots, V_M\},$$

where M denotes the number of vertices. The graph may contain loops and parallel edges.

We consider the Hilbert space

$$\mathcal{H} = L_2(\Gamma) \oplus \bigoplus_{m=1}^{M} \mathbb{C}^{n_m}.$$

An element $U \in \mathcal{H}$ is written as

$$U = (u, u_1, \ldots, u_M),$$

where $u \in L_2(\Gamma)$ and $u_m \in \mathbb{C}^{n_m}$.
In our setting, we first impose the continuity condition at each vertex V_m, meaning that all the values of component functions meeting at V_m coincide. This common value is denoted by $u(V_m)$ and we also have $\partial u(V_m) = \sum_{x_i \in V_m} \partial u(x_i)$. Additionally, we impose the following coupling condition:

$$\langle \theta_m, u_m \rangle = c_m u(V_m) + d_m \partial u(V_m), \quad m = 1, 2, ..., M. \tag{20}$$

Where $\theta_m \in \mathbb{C}^{n_m}$ is a fixed vector and $c_m, d_m \in \mathbb{R}$.

Let the domain of the operator $\mathcal{A}$ be given by

$$Dom(\mathcal{A}) = \Big\{ U \in \mathcal{H} : u \in W_2^2(\Gamma) \text{ and for each } m = 1, \ldots, M,\ \langle \theta_m, u_m \rangle = c_m u(V_m) + d_m \partial u(V_m) \Big\}. \tag{21}$$

We now define the operator $\mathcal{A} : Dom(\mathcal{A}) \to \mathcal{H}$, which acts as

$$\mathcal{A}U = \Big(-u'',\ A_m u_m + (a_m u(V_m) + b_m \partial u(V_m))\theta_m, \quad m = 1, ..., M \Big), \tag{22}$$

where A_m are Hermitian $n_m \times n_m$ matrices and $a_m, b_m \in \mathbb{R}$.

Theorem 4.5. *Let $\mathcal{A}$ be the operator defined above in (22) with domain (21), then $\mathcal{A}$ is* *symmetric in $\mathcal{H}$ if and only if*

$$\det\begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} = 1.$$

Proof. Let $U, V \in Dom(\mathcal{A})$. We compute

$$\langle \mathcal{A}U, V\rangle - \langle U, \mathcal{A}V\rangle.$$

Using the definition of the operator $\mathcal{A}$ we obtain

$$\begin{aligned}\langle \mathcal{A}U, V\rangle = &\sum_{n=1}^{N}\int_{x_{2n-1}}^{x_{2n}} \overline{(-u''(x))}v(x)\,dx \\ &+\sum_{m=1}^{M}\langle A_m u_m, v_m\rangle + \Big(a_m\overline{u(V_m)} + b_m\overline{\partial u(V_m)}\Big)\langle \theta_m, v_m\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle U, \mathcal{A}V\rangle = &\sum_{n=1}^{N}\int_{x_{2n-1}}^{x_{2n}} \overline{u(x)}(-v''(x))\,dx \\ &+\sum_{m=1}^{M}\langle u_m, A_m v_m\rangle + \Big(a_m v(V_m) + b_m\partial v(V_m)\Big)\langle u_m, \theta_m\rangle.\end{aligned}$$

Subtracting these expressions gives

$$\begin{aligned}\langle \mathcal{A}U, V\rangle - \langle U, \mathcal{A}V\rangle = &\sum_{n=1}^{N}\int_{x_{2n-1}}^{x_{2n}} \Big(-\overline{u''}v + \overline{u}v''\Big)dx \\ &+\sum_{m=1}^{M}\Big((a_m\overline{u(V_m)} + b_m\overline{\partial u(V_m)})(c_m v(V_m) + d_m\partial v(V_m)) \\ &\quad -(a_m v(V_m) + b_m\partial v(V_m))(c_m\overline{u(V_m)} + d_m\overline{\partial u(V_m)})\Big).\end{aligned}$$

Applying integration by parts on each edge yields

$$\sum_{n=1}^{N}\int_{x_{2n-1}}^{x_{2n}}(-\overline{u''}v + \overline{u}v'')dx = \sum_{m=1}^{M}\Big(\overline{\partial u(V_m)}v(V_m) - \overline{u(V_m)}\partial v(V_m)\Big)$$

Therefore

$$\langle \mathcal{A}U, V\rangle - \langle U, \mathcal{A}V\rangle = \sum_{m=1}^{M} \Big[\overline{\partial u(V_m)}v(V_m) - \overline{u(V_m)}\partial v(V_m)$$
$$+(a_m\overline{u(V_m)} + b_m\overline{\partial u(V_m)})(c_m v(V_m) + d_m\partial v(V_m))$$
$$-(a_m v(V_m) + b_m\partial v(V_m))(c_m\overline{u(V_m)} + d_m\overline{\partial u(V_m)})\Big].$$

Hence

$$\langle \mathcal{A}U, V\rangle - \langle U, \mathcal{A}V\rangle = \sum_{m=1}^{M}(1 + b_m c_m - a_m d_m)\Big(\overline{\partial u(V_m)}v(V_m) - \overline{u(V_m)}\partial v(V_m)\Big).$$

Thus $\mathcal{A}$ is symmetric if and only if

$$1 + b_m c_m - a_m d_m = 0,$$

which is equivalent to

$$\det\begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} = 1.$$

□

In order to investigate the self-adjointness of the operator $\mathcal{A}$, we study the resolvent equation

$$(\mathcal{A} - \lambda)U = F, \quad U \in Dom(\mathcal{A}), F \in \mathcal{H}, \tag{23}$$

where λ is a real number such that $\lambda < 0$, $\lambda \notin \sigma(\mathcal{A})$ and $\lambda \notin \sigma(A_m)$ for $m = 1, \dots, M$. Equation (23) is equivalent to

$$-u'' - \lambda u = f,$$

and for every vertex V_m, $1 \leq m \leq M$,

$$(A_m - \lambda)u_m + (a_m u(V_m) + b_m\partial u(V_m))\theta_m = f_m. \tag{24}$$

We now consider two separate cases for f_m: the case $f_m = 0$ and the case $f_m \neq 0$.

Case 1: $f_m = 0$.

In this case equation (24) becomes

$$(A_m - \lambda)u_m + (a_m u(V_m) + b_m\partial u(V_m))\theta_m = 0. \tag{25}$$

Hence

$$u_m = -(a_m u(V_m) + b_m \partial u(V_m))(A_m - \lambda)^{-1}\theta_m.$$

Taking the inner product with θ_m yields

$$\langle \theta_m, u_m \rangle = -(a_m u(V_m) + b_m \partial u(V_m))\langle \theta_m, (A_m - \lambda)^{-1}\theta_m \rangle.$$

Denote

$$\Delta_m(\lambda) := \langle \theta_m, (A_m - \lambda)^{-1}\theta_m \rangle.$$

Using the condition (20) we obtain

$$c_m u(V_m) + d_m \partial u(V_m) = -(a_m u(V_m) + b_m \partial u(V_m))\Delta_m(\lambda).$$

Therefore

$$\partial u(V_m) = -\frac{a_m \Delta_m(\lambda) + c_m}{b_m \Delta_m(\lambda) + d_m} u(V_m).$$

Denoting

$$Q_m(\lambda) := \frac{a_m \Delta_m(\lambda) + c_m}{b_m \Delta_m(\lambda) + d_m}, \tag{26}$$

we obtain

$$\partial u(V_m) = -Q_m(\lambda) u(V_m). \tag{27}$$

Remark 4.6.

- One can easily verify that the functions $\Delta_m(\lambda)$ and $Q_m(\lambda)$ are Herglotz-Nevanlinna functions provided that $a_m d_m - b_m c_m = 1$,

- For a fixed value of λ, the condition (27) represents δ-type vertex condition at the vertex V_m. Consequently, imposing this condition at each vertex V_m of Γ yields a self-adjoint realization of the operator $\mathcal{A}$.

Case 2: $f_m \neq 0$.

We begin by using the linear decomposition of the vector

$$(f, f_1, f_2, \dots, f_M) = (f, 0, 0, \dots, 0) + (0, f_1, 0, \dots, 0) + \dots + (0, 0, \dots, f_M).$$

Therefore, by linearity of the problem, it is sufficient to treat each component separately. We first consider the component

$$(f, 0, \dots, 0),$$

which has already been treated in detail in the previous case. Hence, we obtaine the corresponding vertex conditions

$$\partial u(V_m) = -Q_m(\lambda)\, u(V_m), \quad m = 1, \ldots, M,$$

yield a self-adjoint realization of the operator. We now consider the contribution corresponding to a fixed vertex, say $m = 1$, i.e.

$$(0, f_1, 0, \ldots, 0).$$

On the vertices V_m, $m = 2, \ldots, M$, we obtain from the previous analysis

$$(A_m - \lambda)u_m + (a_m u(V_m) + b_m \partial u(V_m))\theta_m = 0,$$

and the vertex condition (27)

$$\partial u(V_m) = -Q_m(\lambda)\, u(V_m).$$

We denote this type of condition by $h_m(\lambda)$, which is chosen to ensure the self-adjointness of the operator on these vertices..

We extend this notation and consider $h_m(\lambda)$ as a family of vertex conditions defined on all vertices of the graph, excluding the vertex V_1. Moreover, we recall that the continuity condition is imposed at every vertex, including V_1.

Under these assumptions, we are in the setting of the M-function construction introduced in definition (3.2). Therefore, we can apply the corresponding definition and obtain

$$\partial u(V_1) = M_1\Big(\lambda, h_m(\lambda),\, m = 2, \ldots, M\Big)\, u(V_1). \tag{28}$$

where $M_1\Big(\lambda, h_m(\lambda),\, m = 2, \ldots, M\Big)$ is an M-function and V_1 represents the only contact vertex. Using this relation, we return to the resolvent equation and compute the corresponding component u_1. From equation (24) we get

$$(A_1 - \lambda)u_1 + \Big(a_1 u(V_1) + b_1 \partial u(V_1)\Big)\theta_1 = f_1,$$

and taking the inner product with θ_1, we obtain

$$\langle \theta_1, u_1 \rangle = \langle \theta_1, (A_1 - \lambda)^{-1} f_1 \rangle - \Big(a_1 u(V_1) + b_1 \partial u(V_1)\Big) \langle \theta_1, (A_1 - \lambda)^{-1} \theta_1 \rangle.$$

Using the condition

$$\langle \theta_1, u_1 \rangle = c_1 u(V_1) + d_1 \partial u(V_1),$$

together with M-function relation (28), we obtain a scalar equation for $u(V_1)$:

$$u(V_1) = \frac{\langle \theta_1, (A_1 - \lambda)^{-1} f_1 \rangle}{c_1 + d_1 M_1 + \Big(a_1 + b_1 M_1\Big) \langle \theta_1, (A_1 - \lambda)^{-1} \theta_1 \rangle}.$$

This shows that $u(V_1)$ is uniquely determined, and consequently u_1 is uniquely defined.

Repeating the same argument for each component $m = 2, \ldots, M$, we conclude that the full solution $U \in Dom(A)$ exists and is unique. Hence, The operatpr $\mathcal{A}$ is self-adjoint.

4.3 Spectral Analysis of the Lasso Graph

We now study the spectrum of the operator $\mathcal{A}$ on a finite compact lasso graph $\mathcal{L}$. Our lasso graph consists of one loop e_1 and one edge e_2, attached at a common vertex V_1. The second endpoint of e_2 is denoted by V_2, see figure 1.

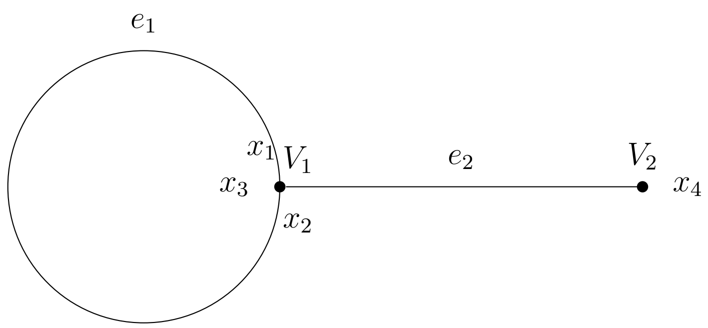

Figure 1: Finite compact lasso graph $\mathcal{L}$.

We write

$$e_1 = [x_1, x_2] = [-\ell, \ell], \qquad e_2 = [x_3, x_4] = [0, \ell],$$

and denote

$$V_1 = \{x_1, x_2, x_3\}, \qquad V_2 = \{x_4\}.$$

We consider the eigenvalue problem for the operator $\mathcal{A}$ defined similarly to (22) with domain (21) as

$$\mathcal{A}\Psi = \lambda \Psi, \qquad \Psi = (\psi, \psi_1), \tag{29}$$

where $\lambda = k^2 \in \mathbb{C}$ is the spectral parameter, $\psi \in W_2^2(\mathcal{L})$ and $\psi_1 \in \mathbb{C}^{n_1}$ is the finite-dimensional vector associated with the vertex V_1.

Equation (29) is equivalent to

$$-\psi'' - \lambda\psi = 0,$$

and for the vertex V_1, we have

$$(A_1 - \lambda)\psi_1 + (a_1\psi(V_1) + b_1\partial\psi(V_1))\theta_1 = 0. \tag{30}$$

Equation (30) is analogous to equation (25). Hence, assuming that $\lambda \notin \sigma(A_1)$ and applying the same argument as in Case 1, we obtain

$$Q_1(\lambda) = \frac{a_1\Delta_1(\lambda) + c_1}{b_1\Delta_1(\lambda) + d_1}, \qquad \partial\psi(V_1) = -Q_1(\lambda)\psi(V_1),$$

and choosing

$$a_1 \neq 0, \qquad d_1 = \frac{1}{a_1}, \qquad b_1 = c_1 = 0,$$

we obtain

$$Q_1(\lambda) = a_1^2\Delta_1(\lambda).$$

Therefore, at the vertex V_1 we get the condition

$$\partial\psi(V_1) = -a_1^2\Delta_1(\lambda)\psi(V_1). \tag{31}$$

Here ψ is assumed to be continuous at each vertex. Hence, at the vertex V_1, all limiting values of ψ along the incident edges coincide and are denoted by $\psi(V_1)$, while at the vertex V_2 we impose the Neumann condition $\partial\psi(V_2) = 0$.

Let T be the reflection operator defined as

$$T : x \mapsto \begin{cases} -x, & x \in e_1, \\ x, & x \in e_2. \end{cases}$$

and acting on functions defined on the graph by

$$(T\psi)(x) = \psi(T^{-1}x),$$

where $T^{-1} = T$. In particular, T satisfies

$$T^2 = I.$$

Since the eigenvalue problem (29) yields

$$-\psi'' = \lambda\psi, \tag{32}$$

it follows that if ψ is a solution of equation (32) then $T\psi$ is also a solution. Indeed, applying T equation (32) yields

$$-(T\psi)'' = \lambda(T\psi).$$

Moreover, $T\psi$ satisfies the same condition (31). Applying T we obtain

$$\partial(T\psi)(V_1) = -a_1^2\Delta_1(\lambda)(T\psi)(V_1).$$

so that $T\psi$ satisfies the same differential equation (32). Hence, odd and even eigenfunctions can be calculated separetly. This allows us to define

$$\psi_+ := \psi + T\psi, \qquad \psi_- := \psi - T\psi,$$

which are even and odd functions satisfying

$$T(\psi_+) = \psi_+, \qquad T(\psi_-) = -\psi_-,$$

respectively. Additionally, the general solution of equation (32) on each edge e_j is given by

$$\psi|_{e_j}(x) = c_j\cos(kx) + d_j\sin(kx), \qquad j = 1, 2,$$

where $c_j, d_j \in \mathbb{C}$.

1- Odd eigenfunctions

The odd eigenfunctions satisfying $T\psi_- = -\psi_-$ are necessarily equal to zero on the edge e_2. Hence

$$\psi_-|_{e_2} = 0 \quad \Rightarrow \quad \psi_-(V_1) = 0$$

On the loop e_1 these functions are given by

$$\psi_-|_{e_1} = \alpha \sin(kx), \qquad \alpha \in \mathbb{C}.$$

Since $\psi_-(V_1) = 0$ then we obtain

$$\psi_-|_{e_1}(\ell) = \alpha \sin(k\ell) = 0 \quad \Rightarrow \quad k\ell = \pi n, \quad n = 1, 2, \ldots$$

Therefore, the corresponding eigenvalues are

$$\lambda = k^2 = \left(\frac{\pi n}{\ell}\right)^2, \quad n = 1, 2, \ldots$$

while the odd eigenfunction can be written as

$$\psi_-(x) = \begin{cases} \alpha \sin\left(\dfrac{\pi n}{\ell} x\right), & x \in e_1, \\ \\ 0, & x \in e_2, \end{cases} \qquad \alpha \in \mathbb{C}, n = 1, 2, \ldots$$

In particular, Using $\psi_-(V_1) = 0$, one can directly verify that ψ_- satisfies the vertex condition (31) at the vertex V_1.

2- Even eigenfunctions

We now consider eigenfunctions satisfying

$$T\psi_+ = \psi_+.$$

On the loop $e_1 = [-\ell, \ell]$, we rewrite the general solution as

$$\psi_+|_{e_1}(x) = A\cos(kx) + B\sin(kx), \qquad A, B \in \mathbb{C}$$

Since ψ_+ is even, i.e. $\psi_+(-x) = \psi_+(x)$, it follows that

$$B = 0,$$

and hence

$$\psi_+|_{e_1}(x) = A\cos(kx).$$

On the edge $e_2 = [0, \ell]$, we write

$$\psi_+|_{e_2}(x) = C\cos(kx) + D\sin(kx), \qquad C, D \in \mathbb{C}. \tag{33}$$

At the vertex V_2, using the Neumann condition and the fact that V_2 corresponds to $x = \ell$ on e_2, this gives

$$-Ck\sin(k\ell) + Dk\cos(k\ell) = 0.$$

Hence, equation (33) can be simplified to

$$\psi_+|_{e_2} = E\cos\Big(k(x-\ell)\Big), \qquad E \in \mathbb{C}$$

At the vertex V_1, continuity yields

$$\psi_+|_{e_1}(\ell) = \psi_+|_{e_2}(0),$$

hence

$$A\cos(k\ell) = E\cos(k\ell).$$

Thus, we obtain $A = E$. For simplicity, we set

$$A = E = 1$$

We now compute the sum of outgoing derivatives at V_1, we obtain

$$\partial\psi_+(V_1) = 3k\sin(k\ell).$$

Using the vertex condition (31) at vertex V_1 we get

$$\partial\psi_+(V_1) = -a_1^2\Delta_1(\lambda)\,\psi_+(V_1),$$

and noting that

$$\psi_+(V_1) = \cos(k\ell),$$

we obtain

$$3k\sin(k\ell) = -a_1^2\Delta_1(\lambda)\cos(k\ell).$$

We finally arrive at the secular equation for even eigenfunctions:

$$3k \sin(k\ell) + a_1^2 \Delta_1(\lambda) \cos(k\ell) = 0.$$

Dividing by $\cos(k\ell)$ (for $\cos(k\ell) \neq 0$), we obtain the equivalent form

$$3k \tan(k\ell) + a_1^2 \Delta_1(\lambda) = 0. \tag{34}$$

To illustrate the previous results, we consider a concrete example of the vertex interaction. In particular, we compute explicitly the function $\Delta(\lambda)$ for a specific choice of the matrix A and the vector θ, and substitute it into the secular equation obtained in (34). This allows us to derive an explicit form of a function $f(k)$ whose zeros determine the eigenvalues of the problem.

Let

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \qquad \theta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Then

$$\Delta(\lambda) = \left\langle \theta, (A - \lambda I)^{-1} \theta \right\rangle.$$

First, we have

$$A - \lambda I = \begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix}.$$

Hence

$$(A - \lambda I)^{-1} = \frac{1}{(1-\lambda)^2 - 1} \begin{pmatrix} 1-\lambda & -i \\ i & 1-\lambda \end{pmatrix}.$$

Since

$$(1-\lambda)^2 - 1 = \lambda^2 - 2\lambda,$$

we obtain

$$(A - \lambda I)^{-1} = \frac{1}{\lambda^2 - 2\lambda} \begin{pmatrix} 1-\lambda & -i \\ i & 1-\lambda \end{pmatrix}.$$

Therefore,

$$(A - \lambda I)^{-1}\theta = \frac{1}{\lambda^2 - 2\lambda} \begin{pmatrix} 1-\lambda-2i \\ i + 2(1-\lambda) \end{pmatrix}.$$

Thus,

$$\Delta(\lambda) = \frac{1 - \lambda - 2i + 2i + 4 - 4\lambda}{\lambda^2 - 2\lambda}.$$

The imaginary parts cancel, and therefore

$$\Delta(\lambda) = \frac{5 - 5\lambda}{\lambda^2 - 2\lambda} = \frac{5(1-\lambda)}{\lambda(\lambda - 2)}, \qquad \lambda \neq 0, 2.$$

Since $\lambda = k^2$, we get

$$\Delta(k^2) = \frac{5(1-k^2)}{k^2(k^2-2)} = \frac{5(k^2-1)}{k^2(2-k^2)}.$$

Taking $\ell = 1$ and $a_1^2 = \frac{1}{5}$, the secular equation for even eigenfunctions becomes

$$3k\tan(k) + \frac{k^2-1}{k^2(2-k^2)} = 0.$$

Hence, we define

$$f(k) := 3k\tan(k) + \frac{k^2-1}{k^2(2-k^2)}. \tag{35}$$

The corresponding eigenvalues are obtained from the zeros of

$$f(k) = 0, \qquad \lambda = k^2.$$

The spectral condition can be analyzed graphically. It is more convenient to rewrite equation (35) in terms of the spectral parameter λ, where

$$\lambda = k^2.$$

For positive values of λ, the parameter k is real and we simply set

$$k = \sqrt{\lambda}.$$

Hence, the function becomes

$$f(\lambda) = 3\sqrt{\lambda}\tan(\sqrt{\lambda}) + \frac{\lambda - 1}{\lambda(2-\lambda)}, \qquad \lambda > 0.$$

The situation is different when $\lambda < 0$. In this case, we write

$$\lambda = -\mu^2, \qquad \mu \in \mathbb{R},$$

which implies

$$k = i\mu.$$

Using the identity

$$\tan(i\mu) = i\tanh(\mu),$$

we obtain

$$k\tan(k) = i\mu\tan(i\mu) = -\mu\tanh(\mu).$$

Therefore, for $\lambda < 0$, the secular function can be written as

$$f(\lambda) = -3\sqrt{|\lambda|}\tanh\Big(\sqrt{|\lambda|}\Big) + \frac{\lambda - 1}{(2-\lambda)\lambda}.$$

The following figures illustrate the relation between the Herglotz–Nevanlinna function

$$g(\lambda) = \frac{\lambda - 1}{(2-\lambda)\lambda},$$

the secular function f and the tangent-type term

$$k(\lambda) = \begin{cases} 3\sqrt{\lambda}\tan(\sqrt{\lambda}), & \lambda > 0, \\ \\ -3\sqrt{|\lambda|}\dfrac{e^{\sqrt{|\lambda|}} - e^{-\sqrt{|\lambda|}}}{e^{\sqrt{|\lambda|}} + e^{-\sqrt{|\lambda|}}}, & \lambda < 0. \end{cases}$$

The green curve represents the Herglotz–Nevanlinna function g, the orange curve corresponds to the secular function f, and the blue dashed curve represents the tangent-type function k. The figures show that the secular function asymptotically follows the tangent-type behavior away from the poles of g.

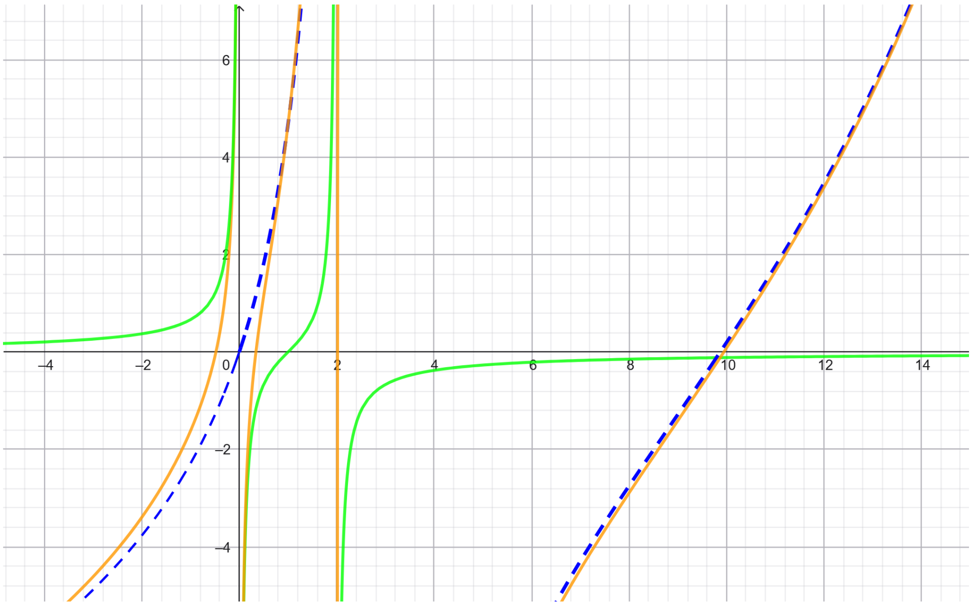

Figure 2: Plots of the functions g, f, and k.

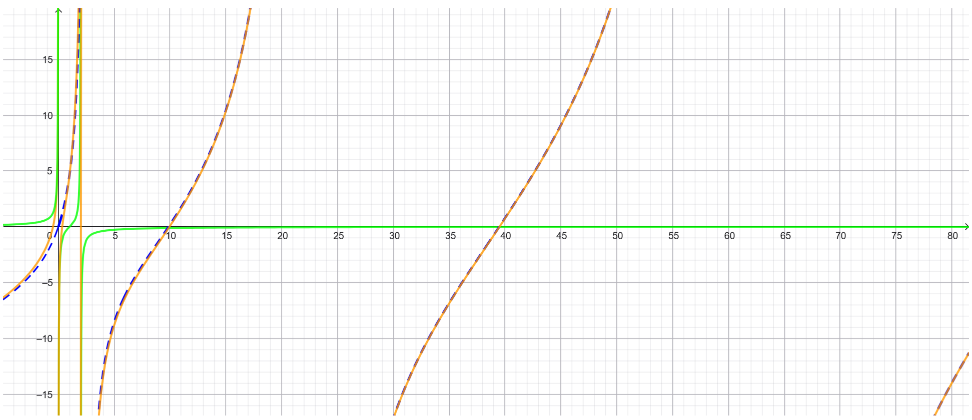

Figure 3: Plots of the functions g, f, and k over a larger domain.

5 Conclusion

In this thesis, we constructed and studied a class of differential operators on metric graphs with vertex interactions described through Herglotz–Nevanlinna functions. The considered model extends the classical framework of quantum graphs by introducing an additional finite-dimensional internal structure at each vertex. More precisely, to every vertex V_m, we associated a finite-dimensional vector space $\mathbb{C}^{n_m}$, and the operator was constructed to act simultaneously as a second-order differential operator along the edges and as a finite-dimensional operator on the internal vertex components.

We first investigated a prototype model consisting of a star graph with three infinite edges. This special case provided the main motivation and intuition for the general construction. By analyzing the corresponding resolvent equation, we show that the constructed operator is symmetric and establish its self-adjointness. A central outcome of this analysis is the appearance of a Herglotz–Nevanlinna function describing the vertex interaction. In particular, the obtained interaction naturally leads to spectral parameter–dependent vertex conditions, which constitute one of the main features of this work.

Motivated by the star graph case, we then extended the construction to the setting of general finite compact metric graphs. In this more general framework, the analysis became substantially richer and led to different types of vertex interactions. In certain cases, the coupling conditions reduced to generalized δ-type vertex conditions, which directly implied the self-adjointness of the operator. In other situations, the analysis naturally involved the graph M-function framework together with the study of the resolvent equation, providing another approach to proving self-adjointness.

Finally, we applied the developed theory to the particular example of a lasso graph. For this graph, we defined the corresponding operator and investigated its spectral properties. The analysis illustrates how the introduced internal vertex structures and the associated Herglotz–Nevanlinna interactions affect the spectral behavior of the operator.

The results obtained in this thesis open several possible directions for future research. One possible continuation is the study of spectral and scattering properties for more complicated classes of graphs equipped with such interactions. Another interesting direction would be the investigation of more general Herglotz–Nevanlinna functions and their relation to inverse spectral problems, resonance phenomena, and singular perturbation theory on metric graphs. Moreover, the extension of the present model to the non-self-adjoint setting remains an interesting open problem.

References

[AK00] Sergio Albeverio and Pavel Kurasov. *Singular Perturbations of Differential Operators*. Cambridge University Press, 2000.

[BK13] Gregory Berkolaiko and Peter Kuchment. *Introduction to Quantum Graphs*. American Mathematical Society, 2013.

[BKKM19] Gregory Berkolaiko, James B. Kennedy, Pavel Kurasov, and Delio Mugnolo. Surgery principles for the spectral analysis of quantum graphs. *Transactions of the American Mathematical Society*, 372(7):5153–5197, 2019.

[Eul41] Leonhard Euler. Solutio problematis ad geometriam situs pertinentis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, 8(53):128–140, 1741.

[Kea08] J. P. Keating. Quantum graphs and quantum chaos. In *Analysis on Graphs and Its Applications*, volume 77 of *Proceedings of Symposia in Pure Mathematics*, pages 279–290. American Mathematical Society, Providence, RI, 2008.

[KK02] Peter Kuchment and Leonid Kunyansky. Differential operators on graphs and photonic crystals. *Advances in Computational Mathematics*, 16(2–3):263–290, 2002.

[KN05] Pavel Kurasov and Marlena Nowaczyk. Inverse spectral problem for quantum graphs. *Journal of Physics A: Mathematical and General*, 38(22):4901–4915, 2005. Corrigendum: ibid. 39, 993 (2006).

[KS18] Pavel Kurasov and Rune Suhr. Schrödinger operators on graphs and geometry. iii. general vertex conditions and counter examples. *Journal of Mathematical Physics*, 59(10):102104, 2018.

[Kur24] Pavel Kurasov. *Spectral Geometry of Graphs*. Springer Nature, 2024.

[LN17] Annemarie Luger and Mitja Nedic. A characterization of Herglotz–Nevanlinna functions in two variables via integral representations. *Complex Analysis and Operator Theory*, 11(7):1609–1629, 2017.

[Mat24] Math StackExchange. Definition of adjoint of unbounded operator, 2024. Accessed: May 2026. URL: https:

//math.stackexchange.com/questions/5072909/definition-of-adjoint-of-unbounded-operator.

[MU23] Gökhan Mutlu and Ekin Uğurlu. On quantum star graphs with eigenparameter dependent vertex conditions. *Analysis and Mathematical Physics*, 13(65), 2023.

[MU24] G. Mutlu and E. Uğurlu. Fractional sturm–liouville operators on compact star graphs. *Demonstratio Mathematica*, 57:20240069, 2024.

[Mug14] Delio Mugnolo. *Semigroup Methods for Evolution Equations on Networks*. Springer, 2014.

[Pav84] B. S. Pavlov. Model of a zero-range potential with internal structure. *Theoretical and Mathematical Physics*, 59(3):544–553, 1984.

[Pav87] B. S. Pavlov. *The Theory of Extensions and Explicitly Solvable Models*. Russian Mathematical Surveys, 1987.

[RS53] K. Ruedenberg and C. W. Scherr. Free-electron network model for conjugated systems. i. theory. *The Journal of Chemical Physics*, 21(9):1565–1581, 1953.

[RY08] B. P. Rynne and M. A. Youngson. *Linear Functional Analysis*. Springer, 2008.

[ZM26] Zineb Zellak and Gökhan Mutlu. Self-adjoint extensions of quantum graphs with eigenparameter-dependent vertex conditions: An operator-theoretic and symplectic geometry approach, 2026. Preprint.