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Higher-dimensional singularities of rational inner functions

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Abstract

In this thesis, we study singularities of rational inner functions (RIFs) ϕ in three and more variables. Even though substantial research has been done in the two-variable case, much is left unknown in higher dimensions. We focus on curve and higher-dimensional singularities of ϕ, where quite different behaviour is observed compared to isolated singularities. Among other things, we show that the integrability conditions of partial derivatives of ϕ under certain assumptions stay constant along irreducible components of the zero set of the function ρ_ϕ, and that the presence of vertical line singularities does not affect global integrability, therefore refuting Question 2 in [9]. Those results are then generalized to n variables as well. We take a look at slice matrices, which we use to obtain integrability conditions of compositions ϕ^N. Finally, we do a quick observation of level sets and present an overview of interesting original examples, including an RIF with two vertical line singularities admitting a horizontal level surface.

Sammanfattning

I denna avhandling studerar vi singulariteter hos rationella inre funktioner (RIFs) ϕ i tre och fler variabler. Även om omfattande forskning har bedrivits i tvåvariabelsfallet är mycket fortfarande okänt i högre dimensioner. Fokus ligger på kurv- och högredimensionella singulariteter hos ϕ, där ganska olika beteenden observeras. Bland annar visar vi att integrerbarhetsvillkoren för partiella derivator av ϕ, under vissa antaganden, förblir konstanta längs irreducibla komponenter av nollmängden till funktionen ρ_ϕ, samt att förekomsten av vertikala linjesingulariteter inte påverkar global integrerbarhet, vilket därmed vederlägger Fråga 2 i [9]. Dessa resultat generaliseras sedan till n variabler. Vi tittar närmare på slice matrices", som vi använder för att erhålla integrerbarhetsvillkor för kompositioner ϕ^N. Slutligen gör vi en snabb observation av nivåmängder och presenterar en översikt över intressanta originalexempel, inklusive en RIF (rationell inre funktion) med två vertikala linjesingulariteter som tillåter en horisontell nivåyta.

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1 Introduction and overview

A function is called *inner* if it is bounded on a domain, with boundary values of modulus one almost anywhere. In both one and several variables, they map the boundary to the boundary of the unit (poly)disk, as well as the ball.

This thesis deals with a special type of such holomorphic functions, defined on the unit polydisk
$$\mathbb{D}^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n \,:\, |z_j| < 1,\, j = 1, ..., n\} \subset \mathbb{C}^n,$$
called rational inner functions (RIFs). They are precisely the multivariable generalization of finite Blaschke products (see [11] for an extensive overview), which are rational, inner and holomorphic on the unit disk $\mathbb{D}$. There is, of course, a reason for this specific setting, as the unit polydisk possesses an additional structure that, for example, the unit ball $\mathbb{B}_n$ [16] or some other domains in $\mathbb{C}^n$, do not. The unit polydisk $\mathbb{D}^n$ has a product structure, i.e. decomposes as $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$, so automorphisms on it act independently on each coordinate. The reflection symmetries are also specific to that setting, since the reflection of stable polynomials is performed coordinate-wise (see Definition 2.8).

Rational inner functions play a big role in multiple active areas of mathematics, where the most prominent one is in multivariable function and operator theory. They appear as solutions to the Nevanlinna-Pick interpolation problems [2] and approximations of Schur functions on the polydisk [21], playing an important role in the construction of Agler decompositions on the bidisk [5]. Moreover, their denominators, stable polynomials, appear in the study of dynamical systems [18].

The focus of this thesis will be on the behaviour of RIFs on the distinguished boundary of the polydisk, namely the n-torus,
$$\mathbb{T}^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n \,:\, |z_j| = 1,\, j = 1, ..., n\} \subset \partial\mathbb{D}^n,$$
and specifically their singularities on $\mathbb{T}^n$. Such singularities do not exist in the one-variable case of Blaschke products B, since they extend continuously to the boundary of $\mathbb{D}$, are analytic on a neighbourhood of $\overline{\mathbb{D}}$, and possess the property that $|B(z)| = 1$ for every $z \in \mathbb{T}$. Passing to two and more variables, however, brings the possible occurrence of singularities.

These singularities can be examined using different methods. One way would be to study the geometry of the unimodular level sets of the RIF, like in [7], or more recently in [9]. We briefly touch on this approach at the end of the paper, in Section 5.1. We could also look at finite iterations ϕ^N of the RIF ϕ and observe the change of its singular set as we iterate. This is something that is investigated in detail in Section 4, and it's done via the characterization of the partial derivative integrability around the singularities. More specifically, for an RIF ϕ in n-variables, the main question will be to find $\mathfrak{p} \geq 1$ such that $\frac{\partial \phi}{\partial z_j} \in L^{\mathfrak{p}}(\mathbb{T}^n)$, for $j = 1, \ldots, n$, but we will mostly investigate the partial derivative in the variable $j = n$. An extensive analysis of this matter had been done for two-variable RIFs in [8], while [9] dealt with RIFs in three and more variables and displayed behaviour unseen in the two variable case. This is not so surprising, as it usually happens in several complex variables that the move from one to two variables is quite simpler than the one from two to three (and more) variables. In the latter case, the zero set of the denominator of ϕ on the torus can increase in dimension, so instead of point singularities, the problem becomes one of the curve singularities, which are generally harder to work with. Narrowing the question to a special type of RIFs linear in the last variable, and with the singular set comprised purely of isolated points, previously known results from two variables have been extended to more variables in [23].

In Section 2, we give a brief introduction to the theory of several complex variables, and more importantly, formally define rational inner functions and review main results that will be used throughout the thesis.

In the first part of Section 3, we give a brief overview of the main integrability results from [8], [9] and [23]. After that, we move to $(m_1, m_2, 1)$-degree RIFs in three variables with curve singularities and relate the singular set of an RIF ϕ on $\mathbb{T}^3$ to the zero set of an auxiliary function ρ_ϕ on $\mathbb{T}^2$, in a similar manner as it was done in [23]. We present a series of results for partial integrability bounds for such rational inner functions. Further, we extend these results to higher-dimensional RIFs, ones still linear in the last variable, with singular sets of codimension one.

Section 4 turns to an investigation of iterations of such RIFs, by describing the singular set of ϕ^N on $\mathbb{T}^n$ for $N \in \mathbb{N}$, in terms of the singular set of ϕ on $\mathbb{T}^n$. After showing the zero sets are preserved under iteration, we extract the integrability conditions for $\frac{\partial \phi^N}{\partial z_n}$ from those for $\frac{\partial \phi}{\partial z_n}$.

Finally, Section 5 is dedicated to constructing an original and interesting example of a three-variable rational inner function with two vertical line singularities, as well as a brief inspection of the unimodular level sets of the "favourite" example, inspired by the work done in [9]. The RIF with two vertical lines singularities turns out to have interesting level sets of its own as well, admitting a horizontal level surface, a concept unseen so far in literature.

2 Preliminaries

2.1 Several complex variables

We will start by introducing some main concepts of several complex variables. All of the definitions and results in this subsection can be found in [17].

The setting we will work on will be the n-dimensional complex Euclidean space $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$, whose coordinates we denote by $z = (z_1, ..., z_n) = (x_1 + iy_1, ..., x_n + iy_n) \in \mathbb{C}^n$, identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. We will use *multi-index notation* and simply write $z = x + iy$. Analogous to one complex variable, we define the *conjugate* of z by $\overline{z} = (\overline{z_1}, ..., \overline{z_n}) = x - iy$.

Definition 2.1. Let $r = (r_1, ..., r_n) \in \mathbb{R}^n$ for $r_k > 0$, and $a \in \mathbb{C}^n$. The *polydisk* with *center* in a and *polyradius* (or simply radius) of r is

$$\Delta_r(a) := \{z \in \mathbb{C}^n : |z_k - a_k| < r_k, \ k = 1, ..., n\}.$$

Our setting will be the *unit polydisk*

$$\mathbb{D}^n = \{z \in \mathbb{C}^n \ : \ |z_j| < 1, \ j = 1, ..., n\} \subset \mathbb{C}^n,$$

and its distinguished boundary, the *n-torus*

$$\mathbb{T}^n = \{z \in \mathbb{C}^n \ : \ |z_j| = 1, \ j = 1, ..., n\} \subset \mathbb{D}^n.$$

$\mathbb{D}^2$ will be called the unit *bidisk* and $\mathbb{D}^3$ the unit *tridisk*, and sometimes just the bidisk and tridisk when it is clear, for simplicity. Note that $\mathbb{T}^n \subsetneq \partial\mathbb{D}^n$. For instance, the boundary of the unit bidisk is $\partial\mathbb{D}^2 = (\mathbb{T} \times \mathbb{D}) \cup (\mathbb{D} \times \mathbb{T}) \cup \mathbb{T}^2$, which can be seen in Figure 1.

In more than one complex dimension, it is difficult to draw exact pictures, so we can visualize the bidisk in terms of the modulus. See Figure 1 again for reference. In the future, since the modulus of each of the variables is always constant on the torus, we will draw plots with regards to the arguments of the variables. For example, see Figure 2.

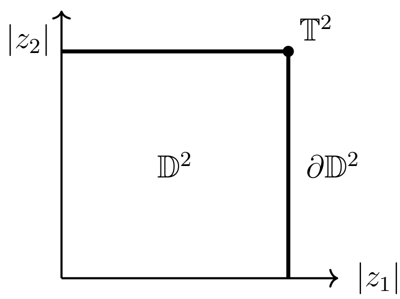


Figure 1: The bidisk

An open connected set $U \subset \mathbb{C}^n$ will be called a *domain*. A function $f : U \to \mathbb{C}$ is *holomorphic* if f is locally bounded (meaning that for every $p \in U$, there is a neighbourhood N of p such that $f|_N$ is bounded) and f is complex-differentiable in each variable separately, i.e. the limit

$$\lim_{\mathbb{C} \ni h \to 0} \frac{f(z_1, ..., z_k + h, ..., z_n) - f(z)}{h}$$

exists for all $z \in U$ and all $k = 1, ..., n$.

A polynomial $p : \mathbb{C}^n \to \mathbb{C}$ is said to be of *polydegree* $m = (m_1, ..., m_n)$ if it has degree m_j in the j-th variable. On the other hand, p is said to be *homogeneous of degree* d if its every term is of total degree d. For example,

$$g(z_1, z_2, z_3) = z_1^3 + 3z_1z_2^2 + 2z_2^2z_3$$

is homogeneous of degree $d = 3$, and has polydegree $m = (3, 2, 1)$.

A function f is holomorphic on U if and only if it is analytic on U, meaning that around each $z \in U$, f has a power series expansion valid in a neighbourhood of z. In that case, for a fixed point $\xi \in \mathbb{C}^n$ around which f is holomorphic and not identically zero, we define

$$\operatorname{ord}_\xi f := \min\{m \in \mathbb{N}_0 \,:\, f_m \not\equiv 0\}$$

where f_m is the homogeneous polynomial of degree m in the series expansion of $f = \sum_{j=0}^{\infty} f_j$. If $f \equiv 0$, we let $\operatorname{ord}_\xi f = \infty$. The number $\operatorname{ord}_\xi f$ is called the *order of vanishing* of f at ξ.

The condition of holomorphicity/analyticity can actually be weakened to define the order of vanishing; precisely, it is enough to ask of a function to be real-analytic. Now taking open $U \subset \mathbb{R}^n$, a function $f : U \to \mathbb{C}$ is *real-analytic* if at each point $p \in U$, the function f has a convergent power series that converges absolutely in some neighbourhood of p.

By identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, every holomorphic function is real-analytic, but not vice versa. In the following chapters, we will work with real-analytic functions a lot, and the following "complexification" property will be of special value:

Proposition 2.2. *Suppose $U \subset \mathbb{R}^n$ is a domain and $f : U \to \mathbb{C}$ is real-analytic. Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the natural inclusion. Then there exists a domain $V \subset \mathbb{C}^n$ such that $U \subset V$ and a unique holomorphic function $F : V \to \mathbb{C}$ such that $F|_U = f$.*

Every real-analytic function is infinitely differentiable (smooth). We will regularly extend real-analytic functions to holomorphic ones, and the reason for that is that holomorphic functions admit a series of special properties that real functions simply do not:

(i) A holomorphic function is infinitely differentiable (smooth).

(ii) **Identity theorem:** If $f : U \to \mathbb{C}$ is holomorphic and vanishes on an open set $V \subset U$, then $f \equiv 0$.

(iii) Let $U \subset \mathbb{C}^n$, $n \geq 2$. The zero set $\mathcal{Z}_f := \{z \in U : f(z) = 0\}$ of a holomorphic $f : U \to \mathbb{C}$ is never compact. In particular, its zeroes are never isolated.

Notice that the property (iii) is specific to multivariable holomorphic functions, and only speaks of the entire zero set of a function. The intersection of $\mathcal{Z}_f$ with a compact set, such as the torus $\mathbb{T}^n$, can indeed be compact, and even finite. See Theorem 3.1.

Sometimes it will be convenient to identify two functions that match on an open set. Let $U, V \subset \mathbb{C}^n$ be two neighbourhoods of $p \in \mathbb{C}^n$. We say that two functions $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ are equivalent if there exists a neighbourhood $W \subset U \cap V$ of p such that $f|_W = g|_W$. The equivalence class $(f, p) := [f]$ of functions under this relation defined in a neighbourhood of p is called the *germ* of a function. If f and g are analytic (holomorphic), we call this equivalence class an *analytic germ*, and in that case, by the identity theorem (property (ii)), $f - g = 0$ on W gives $f \equiv g$ on $U \cup V$. We will denote the ring of germs of holomorphic functions at p by $\mathcal{O}_p$. In the same manner, if f and g are invertible, we call (f, p) an *invertible germ*, or if they are perhaps irreducible, an *irreducible germ*.

However, we can extend the definition to tie it to the set on which two functions are equivalent. More precisely, we say that two sets $A, B \subset \mathbb{C}^n$ containing p are equivalent if there exists a neighbourhood W of p such that $A \cap W = B \cap W$. An equivalence class of sets under this relation is called a *germ of a set* at p, and we denote it by (A, p), or simply A, when there's no confusion. A germ (A, p) is said to be *irreducible* if it cannot be decomposed as $(A, p) = (A_1, p) \cup (A_2, p))$ with analytic sets $(A_j, p) \neq (A, p)$. We will give a precise definition of an analytic set in the next subsection.

2.2 Complex manifolds and analytic varieties

We will give a brief introduction to the main concepts of complex differential and algebraic terms we will use; we refer the reader to [12] and [17] for more details. We start by giving a quick reminder of a definition of a differential manifold.

Definition 2.3. An n-dimensional C^k*-manifold* (k times differentiable, k-differentiable manifold) is a topological space M together with an open covering $\bigcup U_i = M$ and homeomorphisms $\varphi : U_i \to V_i$ onto open subsets of $V_i \subset \mathbb{R}^m$ such that

i. M is Hausdorff,
ii. The topology of M admits a countable basis,
iii. The transition functions $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are C^k-maps.

A *differentiable manifold* is a C^∞-manifold. The set of pairs $\{(U_i, \varphi_i)\}$ is called an *atlas* and each pair (U_i, φ_i) is a *chart*. We say that two atlases $\{(U_i, \varphi_i)\}$ and $\{(U'_j, \varphi'_j)\}$ define the same manifold if all transition functions $\varphi_i \circ (\varphi'_j)^{-1}$ are differentiable.

Clearly, $\mathbb{C}^n$ is a differentiable manifold, and so is $\mathbb{T}^n$. However, $\mathbb{C}^n$ is endowed with an additional structure, making it a complex manifold as well:

Definition 2.4. A *holomorphic atlas* on a differentiable manifold is an atlas $\{(U_i, \varphi_i)\}$ of the form $\varphi_i : U_i \simeq \varphi_i(U_i) \subset \mathbb{C}^n$, such that the transition functions φ_{ij} are holomorphic. The pair (U_i, φ_i) is called a *holomorphic chart*. Two holomorphic atlases $\{(U_i, \varphi_i)\}$ and $\{(U'_j, \varphi'_j)\}$ are equivalent if all maps $\varphi_i \circ (\varphi'_j)^{-1}$ are holomorphic.
A *complex manifold* M of dimension n is a (real) differentiable manifold of dimension $2n$ endowed with an equivalence class of holomorphic atlases.

A complex manifold of dimension one is called a *complex curve*, of dimension two a *complex surface*, of dimension three a *complex threefold*. If a manifold M of dimension m is contained in a manifold N of dimension n, we say M is of *codimension* $n - m$. We can treat real submanifolds analogously. See Theorem 3.12 where we deal with real curves and Theorem 3.23 where we deal with real submanifolds of codimension 1.

Definition 2.5. Let $U \subset \mathbb{C}^n$ be an open subset. An *analytic set (variety)* of U is a closed subset $X \subset U$ such that for any $x \in X$ there exist an open neighbourhood $V \subset U$ of x and holomorphic functions $f_1, ..., f_k : V \to \mathbb{C}$ such that

$$X \cap V = \{z \in V : f_1(z) = \cdots = f_k(z) = 0\}.$$

If $p \in X$ is a point for which, after a permutation of coordinates, X is a graph of a holomorphic function in a neighbourhood of p, we call p a *regular point*. More precisely, if after relabelling the coordinates, there is a neighbourhood $U' \times U'' \subset \mathbb{C}^m \times \mathbb{C}^{n-m}$ of p, for some $0 \leq m \leq n$, and a holomorphic $f : U' \to \mathbb{C}^{n-m}$ such that

$$X \cap (U' \times U'') = \Gamma_f.$$

We say X is of dimension m at p, and of *codimension* $n - m$. The set of regular points is denoted by X_{reg}. Any point of X that is not regular is called *singular*, and the set of those is denoted by X_{sing}.

If all points of X are regular, i.e. if $X = X_{reg}$, X is a complex manifold of dimension m. However, a variety can have regular points of different dimensions. For a singular point q, we define the (complex) dimension of X at q to be the maximum $m \in \mathbb{N}_0$ such that for all neighbourhoods W of q, there exists a regular $p \in W \cap X_{reg}$ such that $\dim_p X = m$. Then the dimension of the entire variety X is defined to be the maximum over all dimensions of regular points, i.e.

$$\dim X := \max_{p \in X_{reg}} \dim_p X.$$

However, if the dimension of X at each regular point p is equal to m, we say that X is of *pure dimension m*.

Example 2.6. The set $X = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 z_3 = 0\} = \{z_1 = 0\} \cup \{z_2 = 0\} \cup \{z_3 = 0\}$ is the zero set of the holomorphic function $f(z_1, z_2, z_3) = z_1 z_2 z_3$, so it is an analytic variety. Away from the origin $(0, 0, 0)$, X is locally one of the coordinate axes $\{z_1 = 0\}$, $\{z_2 = 0\}$ or $\{z_3 = 0\}$, so every point except the origin is regular, that is $X_{reg} = X \backslash \{(0, 0, 0)\}$ and $X_{sing} = \{(0, 0, 0)\}$. Since every regular point is one-dimensional, the set X is of pure dimension 1 and of codimension 2.

2.3 What are rational inner functions?

To answer the question as simply as possible, we will give a series of definitions and useful properties, most of which can be found in [15].

Definition 2.7. A polynomial $p \in \mathbb{C}[z_1, ..., z_n]$ is *stable* with regards to the unit polydisk $\mathbb{D}^n$ if

$$\mathcal{Z}_p = \{z \in \mathbb{C}^n : p(z) = 0\} \cap \mathbb{D}^n = \varnothing.$$

However, it is possible that $\mathcal{Z}_p \cap \partial\mathbb{D}^n \neq \varnothing$ and we will especially be interested in the case when $\mathcal{Z}_p \cap \mathbb{T}^n \neq \varnothing$. Recall that $\mathbb{T}^n \subsetneq \partial\mathbb{D}^n$. As it turns out, on the bidisk we have a special property that $\mathcal{Z}_p \cap \partial\mathbb{D}^2 \neq \varnothing$ implies $\mathcal{Z}_p \cap \mathbb{T}^2 \neq \varnothing$, as shown in [13].

For a shorter notation, from now we will just say *stable*, when we mean *stable with regards to $\mathbb{D}^n$*, since that will be the setting we will work on.

Definition 2.8. The *reflection polynomial* $\widetilde{p}$ of p is defined as

$$\widetilde{p}(z_1, ..., z_n) = z_1^{m_1} \cdots z_n^{m_n} \overline{p\left(\frac{1}{\overline{z_1}}, ..., \frac{1}{\overline{z_n}}\right)},$$

where $m = \deg p = (m_1, ..m_n)$ is the polydegree of p.

Example 2.9. Let $p(z_1, z_2, z_2) = 3 - z_1 - z_2 - z_3$, sometimes called the "favourite example". Its polydegree is $(1, 1, 1)$, and therefore its reflection polynomial $\widetilde{p}$ is then

$$\widetilde{p}(z_1, z_2, z_2) = z_1 z_2 z_3 \left(3 - \frac{1}{z_1} - \frac{1}{z_2} - \frac{1}{z_3}\right) = 3z_1 z_2 z_3 - z_1 z_2 - z_1 z_3 - z_2 z_3.$$

It is fairly easy to show the following basic properties of such polynomials.

Corollary 2.10. *For $p \in \mathbb{C}^n[z_1, ...z_n]$ stable, and $\widetilde{p}$ its reflection polynomial,*

i) The polydegrees of p and $\widetilde{p}$ are equal.

ii) $\mathcal{Z}_p \cap \mathbb{T}^n = \mathcal{Z}_{\widetilde{p}} \cap \mathbb{T}^n$.

iii) $|p| = |\widetilde{p}|$ on $\mathbb{T}^n$.

Proof. *i)* Let $m = (m_1, \dots, m_n)$ be the polydegree of p. Write p as a finite sum of monomials

$$p(z_1, ..., z_n) = \sum_k a_k z_1^{l_{k,1}} \cdots z_n^{l_{k,n}}$$

for $a_k \in \mathbb{C}$ and each of the degrees $l_{k,j} \in \{0, \dots, m_j\}$. From the definition of polydegree, for each $j \in \{1, ..., n\}$, there exists $l_{k,j} = m_j$. Then

$$\overline{p\left(\frac{1}{\overline{z_1}}, ..., \frac{1}{\overline{z_n}}\right)} = \sum_k \bar{a}_k \frac{1}{z_1^{l_{k,1}}} \cdots \frac{1}{z_n^{l_{k,n}}} = \sum_k \bar{a}_k z_1^{-l_{k,1}} \cdots z_n^{-l_{k,n}},$$

so each term has z_j of degree $-l_{k,j} \in \{-m_j, \dots, 0\}$. After multiplying with the term in front, we get

$$\widetilde{p}(z_1, ..., z_n) = z_1^{m_1} \cdots z_n^{m_n} \sum_k \bar{a}_k z_1^{-l_{k,1}} \cdots z_n^{-l_{k,n}} = \sum_k \bar{a}_k z_1^{m_1 - l_{k,1}} \cdots z_n^{m_n - l_{k,n}},$$

where each $m_j - l_{k,j} \in \{0, \dots, m_j\}$. Therefore, $\deg \widetilde{p} = m$ as well.

ii) Let $z = (z_1, \dots, z_n) \in \mathcal{Z}_p \cap \mathbb{T}^n$. Then for all $j \in \{1, \dots, n\}$, we have $z_j \overline{z_j} = 1$, so $z_j = \frac{1}{\overline{z_j}}$. It follows that

$$\widetilde{p}(z) = z_1^{m_1} \cdots z_n^{m_n} \overline{p\left(\frac{1}{\overline{z_1}}, ..., \frac{1}{\overline{z_n}}\right)} = z_1^{m_1} \cdots z_n^{m_n} \overline{p(z)} = 0.$$

On the other side, if $\widetilde{p}(z) = 0$ for $z \in \mathbb{T}^n$, it holds that $|z_j| = 1 \neq 0$ for all $j \in \{1, \dots, n\}$. Therefore, $p(z) = 0$ as well.

iii) For $z \in \mathbb{T}^n$, the modulus $|z_j| = 1 \neq 0$ for all $j \in \{1, \dots, n\}$. Now it immediately follows that

$$|\widetilde{p}(z)| = |z_1^{m_1}| \cdots |z_n^{m_n}| \, |\bar{p}(z)| = 1 \cdot |p(z)| = |\, p(z)|.$$

□

Finally, we are ready to give a formal definition of our special type of functions.

Definition 2.11. Given polynomials $p, q \in \mathbb{C}[z_1, ..., z_n]$ with no common factors, where p is stable, the function $\phi = \frac{q}{p}$ is a *rational inner function (RIF)* if $|\phi(z)| = 1$ for almost every $z \in \mathbb{T}^n$.

A classic result by Rudin and Stout ([22], also found in [21, Chapter 5]), and independently done by Pfister [20], helps in characterizing RIFs better, as well as in constructing them.

Theorem 2.12. *Given polynomials $p, q \in \mathbb{C}[z_1, ..., z_n]$ with no common factors, where p is stable, the function $\phi = \frac{q}{p}$ is a rational inner function if and only if $q(z)$ is of the form*

$$a z^m \, \widetilde{p}(z)$$

for $a \in \mathbb{T}$ and $m \in \mathbb{Z}^n_{\geq 0}$ (written in multi-index notation). Thus every RIF ϕ is of the form

$$\phi(z) = a z^m \frac{\widetilde{p}(z)}{p(z)}$$

for some stable p that has no common factors with $\widetilde{p}$.

Rational inner functions ϕ are by definition holomorphic on $\mathbb{D}^n$, with $|\phi(z)| < 1$ on $\mathbb{D}^n$, and

$$|\phi^*(z)| = 1 \quad \text{for } z \in \mathbb{T}^n,$$

where $\phi^*(z)$ is the radial limit of ϕ, i.e.

$$\phi^*(z) = \lim_{r\to 1^-} \phi(rz) := \lim_{r\to 1^-} \phi(rz_1, ..., rz_n).$$

This limit indeed exists at all points of the n-torus, by [14, Theorem C]. Unlike the one-variable Blaschke products, even though they have limit points, RIFs need not be defined on the whole $\mathbb{T}^n$. Unless $\widetilde{p}(z) = \text{const} \cdot p(z)$, ϕ will have non-removable boundary singularities at points where $\widetilde{p}(z) = 0$ (or equivalently, where $p(z) = 0$). For example, the "favourite example"

$$\phi(z_1, z_2, z_3) = \frac{3z_1z_2z_3 - z_1z_2 - z_1z_3 - z_2z_3}{3 - z_1 - z_2 - z_3}$$

has a boundary singularity at the point $\zeta = (1, 1, 1)$. Even though both p and $\widetilde{p}$ vanish at the point ζ, it is not possible to factorize and cancel out common terms. This often happens with multivariable rational functions, since the fundamental theorem of algebra is not directly applicable. To make matters simpler, we will be concerned only with irreducible RIFs, i.e. those of the form $\phi = \frac{\widetilde{p}}{p}$. However, p might be reducible, as long as no cancellation of factors of p and $\widetilde{p}$ happens.

Since p and $\widetilde{p}$ are of the same polydegree by Corollary 2.10, we will say that an RIF $\phi = \frac{\widetilde{p}}{p}$ is of polydegree m if p (and equivalently, $\widetilde{p}$) is of polydegree m.

3 Integrability conditions

3.1 Isolated point singularities

This section is dedicated to studying previously mentioned singularities of RIFs that appear on the n-torus $\mathbb{T}^n$. In particular, they will be observed via the partial derivative integrability of ϕ in a $\mathbb{T}^n$-neighbourhood. We will begin by giving an overview of known results, first of two-variable RIFs, and then the general n-variable ones. The reason for this separation of the two cases lies in the following theorem that, written in different terminology, comes from [1, Theorem 2.4]

Theorem 3.1. *For an irreducible RIF $\phi = \frac{\tilde{p}}{p}$ in n variables,* $\dim(\mathcal{Z}_p \cap \mathbb{T}^n) \leq n-2$.

This means that two-variable RIFs can only have point singularities, while the singular sets of three-variable ones might also contain curves. See [9, Example 5.3.] for an example of an RIF with an isolated singularity, as well as a curve of singularities. In the same manner, four-variable RIFs might have surface, curve and/or point singularities and so on. In this section we will review results on RIFs containing only point singularities.

3.1.1 Two-variable RIFs

Singularities of two-variable RIFs have been thoroughly investigated in [8] from multiple viewpoints, but mostly through the geometry of $\mathcal{Z}_p$ on the faces of the bidisk $\mathbb{D}\times\mathbb{T}$ and $\mathbb{T}\times\mathbb{D}$ near a singular point $(\xi_1,\xi_2)\in\mathbb{T}^2$. We discuss these results below.

Without loss of generality, we restrict to $\mathbb{D}\times\mathbb{T}$ by fixing $\zeta_2\in\mathbb{T}$ near ξ_2. Let $\alpha_1(\zeta_2),\dots\alpha_k(\zeta_2)\in\mathbb{D}$ denote the points where $\tilde{p}(\alpha_i(\zeta_2),\zeta_2)=0$, i.e. $(\alpha_i(\zeta_2),\zeta_2)\in\mathcal{Z}_{\tilde{p}}\cap(\mathbb{D}\times\mathbb{T})$. It was shown that $\mathcal{Z}_{\tilde{p}}\cap(\mathbb{D}\times\mathbb{T})$ approaches the singularity (ξ_1,ξ_2) such that there exists an even number K_1 satisfying

$$\min_{1\leq i\leq k}\big(1-|\alpha_i(\zeta_2)|\big)\approx|\xi_2-\zeta_2|^{K_1}$$

for all $\zeta_2\in\mathbb{T}$ sufficiently close to ξ_2. The integer K_1 was called the *local z_1-contact order of ϕ at (ξ_1,ξ_2).*

Taking the maximum K_1 over all of the singularities of ϕ gives us the notion of the *z_1-contact order of ϕ.* This in turn characterizes the *critical integrability index* of $\frac{\partial\phi}{\partial z_1}$, the number $\mathfrak{p}^* := \sup\mathfrak{p}$ such that $\frac{\partial\phi}{\partial z_1}\in L^{\mathfrak{p}}(\mathbb{T}^2)$, in the following way. For $1\leq\mathfrak{p}<\infty$,

$$\frac{\partial\phi}{\partial z_1}\in L^{\mathfrak{p}}(\mathbb{T}^2)\iff K_1<\frac{1}{\mathfrak{p}-1}\iff\mathfrak{p}<1+\frac{1}{K_1}.$$

We can define the z_2-contact order of ϕ in a similar manner, starting with restricting to $\mathbb{T}\times\mathbb{D}$ by fixing $\zeta_1\in\mathbb{T}$ near ξ_1. The integer K_2 would then serve to obtain an analogue result of the partial integrability, i.e. that for $1\leq\mathfrak{r}<\infty$,

$$\frac{\partial\phi}{\partial z_2}\in L^{\mathfrak{r}}(\mathbb{T}^2)\iff K_2<\frac{1}{\mathfrak{r}-1}.$$

As it turns out, we have $\mathfrak{p}^*=\mathfrak{r}^*$, that is, by [7, Theorem 4.3]

$$\frac{\partial\phi}{\partial z_1}\in L^{\mathfrak{p}}(\mathbb{T}^2)\iff\frac{\partial\phi}{\partial z_2}\in L^{\mathfrak{p}}(\mathbb{T}^2).$$

However, in higher dimensions, this result does not hold and critical integrability indices can differ from a variable to another. See [9, Example 3.1.]. Moreover, the notion of contact order hasn't been developed for RIFs with three and more variables. Luckily, there are other ways of characterizing the derivative integrability.

3.1.2 RIFs in three and more variables

Previously in [9], three-dimensional rational inner functions were mostly investigated in the setting where they only had isolated points for singularities, while [23] focused on such RIFs as well, but in the general higher dimensional ($n \geq 3$) setting. Specifically, in order to characterize the integrability of the partial derivative in the last variable $\frac{\partial \phi}{\partial z_n}$, the analysis was restricted to $(m_1, \ldots, m_{n-1}, 1)$-degree irreducible RIFs. When the parametrization exists, the root of the defining polynomial $p(z_1, \ldots, z_{n-1}, \tau)$ can be expressed as a rational function $\tau = \psi(z_1, \ldots z_{n-1})$, yielding a single-valued parametrization of $\tau \in \mathbb{T}$. When the degree in z_n is two or higher, this is no longer necessarily possible.

Now, the linearity in the last variable allows us to write

$$\phi(z) = \frac{\widetilde{p}_1(z_1, \ldots z_{n-1}) \cdot z_n + \widetilde{p}_2(z_1, \ldots z_{n-1})}{p_1(z_1, \ldots z_{n-1}) + p_2(z_1, \ldots z_{n-1}) \cdot z_n} \tag{1}$$

for $z = (z_1, \ldots, z_n)$. By explicitly writing z_n in terms of the other variables, we obtain a parametrization of $\mathcal{Z}_{\widetilde{p}}$,

$$\widetilde{p}_1(z_1, \ldots z_{n-1}) \cdot z_n + \widetilde{p}_2(z_1, \ldots z_{n-1}) = 0 \iff z_n = -\frac{\widetilde{p}_2(z_1, \ldots z_{n-1})}{\widetilde{p}_1(z_1, \ldots z_{n-1})}$$

when $\widetilde{p}_1 \neq 0$. This is actually always true for RIFs with isolated singular sets; see the discussion in the beginning of Section 3.2 for a clear explanation of why. After setting

$$z_n = \psi^0(z_1, \ldots, z_{n-1}) := -\frac{\widetilde{p}_2(z_1, \ldots z_{n-1})}{\widetilde{p}_1(z_1, \ldots z_{n-1})},$$

the function ρ_ϕ is then defined as

$$\begin{aligned}\rho_\phi(z_1, \ldots, z_{n-1}) &:= 1 - |\psi^0(z_1, \ldots, z_{n-1})|^2 \\ &= \frac{|\widetilde{p}_1(z_1, \ldots, z_{n-1})|^2 - |\widetilde{p}_2(z_1, \ldots, z_{n-1})|^2}{|\widetilde{p}_1(z_1, \ldots, z_{n-1})|^2}.\end{aligned}$$

On the torus, the zero set of the function ρ_ϕ is closely connected to the zero set of p. Specifically, for a point $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$, we have that

$$p(\zeta) = 0 \iff \rho_\phi(\zeta_1, \ldots, \zeta_{n-1}) = 0 \iff (|\widetilde{p}_1|^2 - |\widetilde{p}_2|^2)(\zeta_1, \ldots, \zeta_{n-1}) = 0. \tag{2}$$

This comes from [23, Lemma 2], whose series of steps we review presently. By fixing $\hat{\zeta} = (\zeta_1, \ldots, \zeta_{n-1}) \in \mathbb{T}^{n-1}$, the one-variable rational function

$$\phi_{\hat{\zeta}}(z_n) = \phi(\hat{\zeta}, z_n)$$

is obtained, which by [14, Theorem C] has unimodular boundary values at every point $z_n \in \mathbb{T}$. Then it follows that $\phi_{\hat{\zeta}}(z_n)$ has to be either a Möbius transformation or constant, in which case the unimodularity would imply it to be equal to some point on the torus $\mathbb{T}$. Now the Lemma proves that in this latter case when $\phi_{\hat{\zeta}}(z_n) = \alpha \in \mathbb{T}$, we have that $|\widetilde{p}_1(\hat{\zeta})|^2 - |\widetilde{p}_2(\hat{\zeta})|^2 = 0$. This however means that the whole vertical line $\{\hat{\zeta}\} \times \mathbb{T}$ is contained in the level set

$$C_\alpha = \{z \in \mathbb{T} : \tilde{p}(z) - \alpha p(z) = 0\}.$$

Now the Lemma invokes [6, Theorem 3.3, Lemma 3.4], from where the equality $\hat{\zeta} = (\tau_1, ..., \tau_{n-1})$ comes from, for some singularity $(\tau_1, \ldots, \tau_n) \in \mathbb{T}^n$ of ϕ (the result was actually shown in the two variables, but it was nowhere directly used that $\hat{\zeta} \in \mathbb{T}$)). The other direction is proven in detail in Lemma 3.7.

This result turns out to be crucial in the characterization of the partial derivative integrability of ϕ around its singularities (i.e. zeros of p), as it is enough to observe the behaviour of the function ρ_ϕ around the points restricted to their first $n-1$ variables. In the absence of contact order, the following result based on the analysis that can be read about in detail in [9, Theorem 2.1., Section 3.1.] and [23, Lemma 2] serves as an analogue for higher dimensions, but can actually be used to derive the two-dimensional result in the case of bidegree $(m_1, 1)$ as well.

Theorem 3.2. *Near the points of $\mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^{n-1}$ and for $1 \leq \mathfrak{p} < \infty$,*

$$\frac{\partial \phi}{\partial z_n} \in L^{\mathfrak{p}}(\mathbb{T}^n) \iff \int_{\mathbb{T}^{n-1}} (1 - |\psi^0(z_1, \ldots, z_{n-1})|^2)^{1-\mathfrak{p}} \, |dz_1| \cdots |dz_{n-1}| < \infty$$

The theorem can be stated locally as well, that is, on small neighbourhoods around points of the zero set of ρ_ϕ on $\mathbb{T}^{n-1}$. In the following chapters, we will usually be working with this local definition.

3.2 Curve singularities of $(m_1, m_2, 1)$-degree RIFs

This section will study three-variable $(m_1, m_2, 1)$-degree RIFs with curve singularities. Most of the notions introduced in the previous section translate nicely to this setting, with some exceptions. In particular, $\phi = \frac{\widetilde{p}}{p}$ has no singularities inside the tridisk, which together with $|z_3| \leq 1$ on $\mathbb{D}$ gives $|p_1| \geq |p_2|$, by (1). Passing to the boundary, we obtain the following inequality for points on $\mathbb{T}^2$:

$$|\widetilde{p}_1| = |p_1| \geq |p_2| = |\widetilde{p}_2|. \tag{3}$$

Therefore, if p_1 (or equivalently, $\widetilde{p}_1$) vanishes at some point $(\xi_1, \xi_2) \in \mathbb{T}^2$, so does p_2 (and $\widetilde{p}_2$). Then $z_3 \in \mathbb{T}$ can be chosen arbitrarily, so $\mathcal{Z}_p \cap \mathbb{T}^3$ and $\mathcal{Z}_{\widetilde{p}} \cap \mathbb{T}^3$ contain the vertical line $\{(\xi_1, \xi_2)\} \times \mathbb{T}$. Previously (see [9]), their presence has been problematic in the study of integrability of $\frac{\partial \phi}{\partial z_3}$, since the function

$$\rho_\phi(z_1, z_2) = \frac{|\widetilde{p}_1(z_1, z_2)|^2 - |\widetilde{p}_2(z_1, z_2)|^2}{|\widetilde{p}_1(z_1, z_2)|^2}$$

has a singularity at (ξ_1, ξ_2) and is no longer clearly analytic.

Lemma 3.3. *Let $\sigma := |\widetilde{p}_1|^2 - |\widetilde{p}_2|^2$. Then the function σ is real-analytic and non-negative on the closed bidisk $\overline{\mathbb{D}^2}$.*

Proof. Rewrite σ as

$$\begin{aligned}
\sigma(z_1, z_2) &= |\widetilde{p}_1(z_1, z_2)|^2 - |\widetilde{p}_2(z_1, z_2)|^2 \\
&= \widetilde{p}_1(z_1, z_2)\overline{\widetilde{p}_1(z_1, z_2)} - \widetilde{p}_2(z_1, z_2)\overline{\widetilde{p}_2(z_1, z_2)} \\
&= \mathrm{Re}(\widetilde{p}_1(z_1, z_2))^2 + \mathrm{Im}(\widetilde{p}_1(z_1, z_2))^2 - \mathrm{Re}(\widetilde{p}_2(z_1, z_2))^2 - \mathrm{Im}(\widetilde{p}_2(z_1, z_2))^2.
\end{aligned}$$

This is a sum of real-valued polynomials on $\overline{\mathbb{D}^2}$, which is then real-valued as well. Moreover, such polynomials are real-analytic, so their sum is real-analytic as well.

By the discussion in the beginning of Section 3.2, $|\widetilde{p}_1| \geq |\widetilde{p}_2|$. The inequality is preserved with squaring, so it immediately follows that σ is non-negative.

□

In cases when ρ_ϕ turns out to have singularities and ceases to be analytic, the following series of results will help us deal with them. Let's write down a lemma found in [9, Lemma 3.4] first.

Lemma 3.4. *For every irreducible $(m_1, m_2, 1)$-degree RIF $\phi = \frac{\widetilde{p}}{p}$, there are at most finitely many $(\xi_1, \xi_2) \in \mathbb{T}^2$ so that $\mathcal{Z}_p \cap \mathbb{T}^3$ contains the vertical line $(\xi_1, \xi_2) \times \mathbb{T}$.*

Now it immediately follows that

Lemma 3.5. *An irreducible $(m_1, m_2, 1)$-degree RIF has at most finitely many vertical line singularities.*

Proof. By the construction of irreducible $\phi = \frac{\widetilde{p}}{p}$, the zero sets of $\widetilde{p}$ and p on $\mathbb{T}^3$ are equal, thus ϕ contains at most finitely many vertical line singularities. □

Corollary 3.6. *$\mathcal{Z}_{|\widetilde{p}_1|^2} \cap \mathbb{T}^2$ consists of at most finitely many isolated points.*

Proof. For an arbitrary point $(\xi_1, \xi_2) \in \mathbb{T}^2$, $p_1(\xi_1, \xi_2) = 0$ if and only if $\widetilde{p}_1(\xi_1, \xi_2) = 0$. Since $\widetilde{\widetilde{p}}_1 = p_1$, we have that

$$|\widetilde{p}_1(\xi_1, \xi_2)|^2 = 0 \iff \widetilde{p}_1(\xi_1, \xi_2) = 0 \iff p_1(\xi_1, \xi_2) = 0.$$

Also, by (3), we have that if $p_1(\xi_1, \xi_2) = 0$, then $p_2(\xi_1, \xi_2) = 0$ as well.

Now, as we know, $\mathcal{Z}_p \cap \mathbb{T}^3$ contains the vertical line $\{(\xi_1, \xi_2)\} \times \mathbb{T}$ if and only if both p_1 and p_2 vanish at (ξ_1, ξ_2), which happens precisely when $|\widetilde{p}_1(\xi_1, \xi_2)|^2 = 0$. By Lemma 3.5, there are at most finitely many vertical line singularities $\{(\xi_1, \xi_2)\} \times \mathbb{T}$, so the cardinality of

$$\mathcal{Z}_{|\widetilde{p}_1|^2} \cap \mathbb{T}^2 = \{(\xi_1, \xi_2) \in \mathbb{T}^2 \,:\, \{(\xi_1, \xi_2)\} \times \mathbb{T} \subset \mathcal{Z}_p \cap \mathbb{T}^3\}$$

is finite as well. □

We can now classify the zero set of p on $\mathbb{T}^3$ in terms of the zero set of ρ_ϕ on $\mathbb{T}^2$. That is the purpose of the following lemma.

Lemma 3.7. *Let ϕ be an irreducible $(m_1, m_2, 1)$-degree RIF. Then the following holds.*

(i) If ϕ has no vertical line singularities, $\rho_\phi(\zeta_1, \zeta_2) = 0$ if and only if (ζ_1, ζ_2, τ) is a singularity of ϕ for some $\tau \in \mathbb{T}$.

(ii) If ϕ has k vertical line singularities $\{\xi^1\} \times \mathbb{T}, ..., \{\xi^k\} \times \mathbb{T}$, then $\rho_\phi(\zeta_1, \zeta_2) = 0$ if and only if $(\zeta_1, \zeta_2, \tau) \in \mathcal{Z}_p \cap \mathbb{T}^3 \setminus (\{(\xi^1, ..., \xi^k\} \times \mathbb{T})$.

Proof. (i) Since the zero set of p contains no vertical lines, neither does the zero set of $\widetilde{p}$. By the observations made in the beginning of this section, this means that $|\widetilde{p}_1|^2$ is non-vanishing on $\mathbb{T}^2$, and therefore bounded below on $\mathbb{T}^2$. Thus,

$$\rho_\phi(z_1, z_2) = \frac{\sigma(z_1, z_2)}{|\widetilde{p}_1(z_1, z_2)|^2}$$

is analytic on $\mathbb{T}^2$. Therefore, $\mathcal{Z}_{\rho_\phi} = \mathcal{Z}_\sigma$, and we have

$$\begin{aligned}
\widetilde{p}(\zeta_1,\zeta_2,\zeta_3) = 0 &\iff \zeta_3\, \widetilde{p}_1(\zeta_1,\zeta_2) + \widetilde{p}_2(\zeta_1,\zeta_2) = 0 \\
&\iff \zeta_3 = -\frac{\widetilde{p}_2(\zeta_1,\zeta_2)}{\widetilde{p}_1(\zeta_1,\zeta_2)} \\
&\implies 1 = |\zeta_3|^2 = \frac{|\widetilde{p}_2(\zeta_1,\zeta_2)|^2}{|\widetilde{p}_1(\zeta_1,\zeta_2)|^2} \\
&\implies |\widetilde{p}_1(\zeta_1,\zeta_2)|^2 = |\widetilde{p}_2(\zeta_1,\zeta_2)|^2 \\
&\implies |\widetilde{p}_1(\zeta_1,\zeta_2)|^2 - |\widetilde{p}_2(\zeta_1,\zeta_2)|^2 = 0
\end{aligned}$$

so $\rho_\phi(\zeta_1,\zeta_2) = 0$. The other direction follows from the discussion after (2).

(ii) First, notice that $\mathcal{Z}_{|\widetilde{p}_1|^2} \subset \mathcal{Z}_\sigma$. If $|\widetilde{p}_1(\zeta_1,\zeta_2)|^2 = 0$, by the proof of Corollary 3.6, this means that $\widetilde{p}_1(\zeta_1,\zeta_2) = 0$, and therefore $\widetilde{p}_2(\zeta_1,\zeta_2) = 0$ as well. But then $|\widetilde{p}_2(\zeta_1,\zeta_2)|^2 = 0$, so clearly $|\widetilde{p}_1(\zeta_1,\zeta_2)|^2 - |\widetilde{p}_2(\zeta_1,\zeta_2)|^2 = 0$. Thus, $(\zeta_1,\zeta_2) \in \mathcal{Z}_\sigma$.
But then $\mathcal{Z}_{\rho_\phi} = \mathcal{Z}_\sigma \setminus \{\xi^1, ..\xi^k\}$. By (i), $\mathcal{Z}_\sigma$ coincides with the projection of $\mathcal{Z}_p$ onto the first two coordinates. Now the claim follows immediately. □

Example 3.8. We will illustrate the previous results on an example; by looking at the RIF from [9, Example 5.2]

$$\begin{aligned}
\phi(z) &= \frac{1 - z_1 - z_1z_2 + z_1^2z_2 - 2z_3 - z_1z_3 - z_1^2z_3 + z_2z_3 - z_1z_2z_3 + 4z_1^2z_2z_3}{4 - z_1 + z_1^2 - z_2 - z_1z_2 - 2z_1^2z_2 + z_3 - z_1z_3 - z_1z_2z_3 + z_1^2z_2z_3} \\
&= \frac{z_3(-2 - z_1 - z_1^2 + z_2 - z_1z_2 + 4z_1^2z_2) + 1 - z_1 - z_1z_2 + z_1^2z_2}{z_3(1 - z_1 - z_1z_2 + z_1^2z_2) + 4 - z_1 + z_1^2 - z_2 - z_1z_2 - 2z_1^2z_2}.
\end{aligned} \tag{4}$$

In the mentioned paper it has been shown that

$$\mathcal{Z}_p \cap \mathbb{T}^3 = \left\{(0,0,u)\ :\ u \in [-\pi,\pi]\right\} \cup \left\{s, \arg m(e^{is}), \pi)\ :\ s \in [-\pi,\pi]\,,\, m(z) = \frac{3+z^2}{1+3z^2}\right\},$$

meaning that $\mathcal{Z}_p \cap \mathbb{T}^3$ contains the vertical line $\{(1,1)\} \times \mathbb{T}$. In the case of isolated singularities,

$$\rho_\phi(\zeta_1,\zeta_2) = 0 \iff (\zeta_1,\zeta_2,\tau) \text{ is a singularity of } \mathcal{Z}_p \cap \mathbb{T}^3 \text{ for some } \tau \in \mathbb{T}.$$

This example will not behave the same, since ρ_ϕ has a singularity at $\{(1,1)\}$. We expand:

$$\begin{aligned}
\rho_\phi(z_1,z_2) &= \frac{|\widetilde{p}_1(z_1,z_2)|^2 - |\widetilde{p}_2(z_1,z_2)|^2}{|\widetilde{p}_1(z_1,z_2)|^2} \\
&= \frac{\widetilde{p}_1(z_1,z_2)\widetilde{p}_1(\bar{z}_1,\bar{z}_2) - \widetilde{p}_2(z_1,z_2)\widetilde{p}_2(\bar{z}_1,\bar{z}_2)}{\widetilde{p}_1(z_1,z_2)\widetilde{p}_1(\bar{z}_1,\bar{z}_2)} \\
&= \frac{20 + 6\bar{z}_1^2 - 6\bar{z}_2 - 9\bar{z}_1^2\bar{z}_2 + 6z_1^2 - z_1^2\bar{z}_2 - 6z_2 - \bar{z}_1^2z_2 - 9z_1^2z_2}{24 - 2\bar{z}_1 + 6\bar{z}_1^2 - 5\bar{z}_2 - 2\bar{z}_1\bar{z}_2 - 8\bar{z}_1^2\bar{z}_2 - 2z_1 + 6z_1^2 - z_1^2\bar{z}_2 - 5z_2 - \bar{z}_1^2z_2 - 2z_1z_2 - 8z_1^2z_2}.
\end{aligned}$$

The second equality comes from the fact that $\overline{\widetilde{p}_j(z_1,z_2)} = \widetilde{p}_j(\bar{z}_1,\bar{z}_2)$ since both $\widetilde{p}_j$ have real coefficients. We're interested in the zero set of ρ_ϕ on the 2-torus, where

$$z_j\bar{z}_j = |z_j|^2 = 1\,,\ j = 1,2.$$

Therefore, we can write each of the variables z_j in terms of their arguments only, like $z_j = e^{i\theta_j}$ for $\theta_j \in (-\pi,\pi]$. So as Lemma 3.3 stated for the function σ, it is clear that ρ_ϕ is real-valued and non-negative at all points in which it is defined in $\overline{\mathbb{D}^2}$. For $(z_1, z_2) \in \mathbb{T}^2$, we simplify

$$\rho_\phi(z_1,z_2) = \frac{20 + \frac{6}{z_1^2} - \frac{6}{z_2} - \frac{9}{z_1^2 z_2} + 6z_1^2 - \frac{z_1^2}{z_2} - 6z_2 - \frac{z_2}{z_1^2} - 9z_1^2 z_2}{24 - \frac{2}{z_1} + \frac{6}{z_1^2} - \frac{5}{z_2} - \frac{2}{z_1 z_2} - \frac{8}{z_1^2 z_2} - 2z_1 + 6z_1^2 - \frac{z_1^2}{z_2} - 5z_2 - \frac{z_2}{z_1^2} - 2z_1 z_2 - 8z_1^2 z_2} \tag{5}$$

$$= \frac{-9 - 6z_1^2 - z_1^4 + 6z_2 + 20z_1^2 z_2 + 6z_1^4 z_2 - z_2^2 - 6z_1^2 z_2^2 - 9z_1^4 z_2^2}{-(-4 + z_1 - z_1^2 + z_2 + z_1 z_2 + 2z_1^2 z_2)(-2 - z_1 - z_1^2 + z_2 - z_1 z_2 + 4z_1^2 z_2)}$$

$$= \frac{(-3 - z_1^2 + z_2 + 3z_1^2 z_2)^2}{(-4 + z_1 - z_1^2 + z_2 + z_1 z_2 + 2z_1^2 z_2)(-2 - z_1 - z_1^2 + z_2 - z_1 z_2 + 4z_1^2 z_2)}.$$

To see where ρ_ϕ vanishes on the torus, we have to find the zero sets of the numerator and of the denominator, and then remove the latter set, i.e. the set of singularities of ρ_ϕ. The denominator vanishes in two cases:

1)
$$-4 + z_1 - z_1^2 + z_2 + z_1 z_2 + 2z_1^2 z_2 = 0 \iff z_2 = \frac{4 + z_1 + z_1^2}{1 + z_1 + 2z_1^2}.$$

Using the fact that $|z_2| = 1$, and taking the absolute value of the expression above, the equation simplifies to

$$|4 + z_1 + z_1^2| = |1 + z_1 + 2z_1^2|.$$

This is equivalent to

$$(4 + z_1 + z_1^2)(4 + \bar{z}_1 + \bar{z}_1^2) = (1 + z_1 + 2z_1^2)(1 + \bar{z}_1 + 2\bar{z}_1^2).$$

After multiplying it out and using $|z_1| = 1$, we get the equation

$$z_1^4 + z_1^3 + 6z_1^2 + z_1 + 1 = 0,$$

which has no zeros on $\mathbb{T}$.

2)
$$-2 - z_1 - z_1^2 + z_2 - z_1 z_2 + 4z_1^2 z_2 = 0 \iff z_2 = \frac{2 + z_1 + z_1^2}{1 - z_1 + 4z_1^2}.$$

Again, using $|z_1| = |z_2| = 1$ simplifies the above expression to

$$-2z_1^4 + 8z_1^3 + 8z_1 - 12z_1^2 + 8z_1 - 2 = 0,$$

which can be nicely written as
$$-2(z_1 - 1)^4 = 0.$$

Clearly the only solution is $z_1 = 1$, which after substituting into $p = 0$ gives $z_2 = 1$ as well. Thus, as Lemma 3.7 stated, the set of zeros of p on $\mathbb{T}^3$ can precisely be read off the zero set of ρ_ϕ on $\mathbb{T}^2$, without the point $(1, 1)$, i.e.

$$\mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^2 = \left\{ \left(\xi_1, \frac{3 + \xi_1^2}{1 + 3\xi_1^2} \right) \, : \, \xi_1 \in \mathbb{T} \backslash \{1\} \right\}. \tag{6}$$

Theorem 3.9. *Let ϕ be a $(m_1, m_2, 1)$-degree RIF with no vertical line singularities. Then for a point $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathcal{Z}_p \cap \mathbb{T}^3$, we have that $\frac{\partial \phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ at ζ if and only if*

$$\int_{B_\varepsilon(\zeta_1,\zeta_2)} \sigma(z_1, z_2)^{1-\mathfrak{p}} \, dm(z_1, z_2) < \infty$$

if and only if

$$\int_{B_\varepsilon(\zeta_1,\zeta_2)} |\rho_\phi(z_1,z_2)|^{1-\mathfrak{p}}\, dm(z_1,z_2) < \infty$$

for sufficiently small $\varepsilon > 0$.

Proof. Clearly $(\zeta_1,\zeta_2) \in \mathcal{Z}_{\rho_\phi}$. Since ϕ contains no vertical line singularities on $\mathbb{T}^3$, the zero sets of ρ_ϕ and σ are equal. The function $|\widetilde{p}_1|^2$ is bounded above and below on $\mathbb{T}^2$, so the integral

$$\int_{B_\varepsilon(\zeta_1,\zeta_2)} \left(\frac{\sigma(z_1,z_2)}{|\widetilde{p}_1(z_1,z_2)|^2}\right)^{1-\mathfrak{p}} dm(z_1,z_2)$$

converges precisely when

$$\int_{B_\varepsilon(\zeta_1,\zeta_2)} \sigma(z_1,z_2)^{1-\mathfrak{p}}\, dm(z_1,z_2)$$

does. Finally, the first equivalence follows from [9, Theorem 2.1] the same way as Theorem 3.2 does. □

Remark. For a general n-dimensional rational inner function ϕ, it is important to note that around a singularity $\xi \in \mathbb{T}^n$, the partial derivative $\frac{\partial \phi}{\partial z_k} \in L^1_{loc}(\mathbb{T}^n)$ for all $k \in \{1,\dots,n\}$. This was shown for two variables in [8, Proposition 4.7.], in a proof that used a restriction to a Blaschke product, on which the argument principle was then applied. The general n-dimensional case was then proven analogously in [4, Lemma 3].

Lemma 3.10. *Assume ϕ has k vertical line singularities at $\{\xi^1\}\times\mathbb{T},\dots,\{\xi^k\}\times\mathbb{T} \subset \mathbb{T}^3$. Then at all points $\zeta \in \mathcal{Z}_p \cap \mathbb{T}^3$ not contained in $\{\xi^1,\dots,\xi^k\}\times\mathbb{T}$,*

$$\frac{\partial \phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3) \iff \int_{B_\varepsilon(\zeta_1,\zeta_2)} |\rho_\phi(z_1,z_2)|^{1-\mathfrak{p}}\, dm(z_1,z_2) < \infty$$

for sufficiently small $\varepsilon > 0$.

Proof. Such points ζ which are not contained in vertical line singularities, but still contained in $\mathcal{Z}_p \cap \mathbb{T}^3$, are either point singularities or contained in some curve singularity. In both cases, $\widetilde{p}_1(\zeta_1,\zeta_2) \neq 0$, so by the proof of Theorem 3.9, the statement follows. □

The following theorem, that can be found for example in [10, Theorem 6.6], will be of crucial importance.

Theorem 3.11. *Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$, let (A,ξ) be an analytic germ of pure dimension $n-1$ and let A_j, $1 \leq j \leq N$, be its irreducible components. Let $\mathcal{O}_{M,\xi}$ denote the ring of germs of holomorphic functions around $\xi \in M$. Then for every $f \in \mathcal{O}_{M,\xi}$ such that $f^{-1}(0) \subset (A,\xi)$, there is a unique decomposition $f = u g_1^{m_1} \dots g_N^{m_N}$, where u is an invertible germ, g_j are irreducible germs, and m_j is the order of vanishing of f at any point $z \in A_{j,reg} \setminus \bigcup_{k\neq j} A_k$.*

A subtle fact which will be useful to us is that the order of vanishing is constant along irreducible components of analytic germs. Using this we get the following result.

Theorem 3.12. *Let $\mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^2 = \bigcup_{i=1}^N \gamma_i$ for some $N \in \mathbb{N}$, where each γ_i is a real smooth curve. Assume $\frac{\partial \phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ around a point $(\zeta_1,\zeta_2) \in \gamma_j \setminus \bigcup_{k\neq j}\gamma_k$, for some $j \in \{1,\dots,N\}$. Then for that same integrability bound $\mathfrak{p}$, the partial derivative $\frac{\partial \phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ at every other point of the smooth curve γ_j not intersecting other curves, i.e. points $(z_1,z_2) \in \gamma_j \setminus \bigcup_{k\neq j}\gamma_k$.*

Proof. Recall that the function ρ_ϕ is real-analytic at all points except possible singularities, so assume we have found the series expansion for ρ_ϕ around some point $(\zeta_1, \zeta_2) \in \gamma_j$. From the smoothness of the curve, it's clear that ρ_ϕ has no singularities on γ_j, so $|\widetilde{p}_1|$ is non-vanishing on γ_j. By Lemma 3.10,

$$\int_{B_\varepsilon(\zeta_1,\zeta_2)} |\rho_\phi(z_1, z_2)|^{1-\mathfrak{p}}\, dm(z_1, z_2) < \infty$$

for small enough $\varepsilon > 0$. By Theorem 3.9 this also means that for such ε

$$\int_{B_\varepsilon(\zeta_1,\zeta_2)} \sigma(z_1, z_2)^{1-\mathfrak{p}}\, dm(z_1, z_2) < \infty.$$

For $(z_1, z_2) \in \mathbb{T}^2$ we have

$$\begin{aligned} z_1 &= e^{i\theta_1}, \quad \theta_1 \in (-\pi, \pi) \\ z_2 &= e^{i\theta_2}, \quad \theta_2 \in (-\pi, \pi). \end{aligned}$$

Then $\sigma(z_1, z_2) = \sigma(e^{i\theta_1}, e^{i\theta_2})$ is a real-analytic function as a composition of three real-analytic functions σ, e^{it_1} and e^{it_2}, for all $t_1, t_2 \in (-\pi, \pi)$.

Let $A := \mathcal{Z}_\sigma \cap \mathbb{T}^2$. Then $(\gamma_j)_j$ are exactly the sets of regular points of each of the irreducible components A_i of A, i.e.

$$A_{reg} = \bigcup_{i=1}^{N} A_{i,reg} = \bigcup_{i=1}^{N} \gamma_i.$$

By Corollary 3.6, zeros of $|\widetilde{p}_1|$ are isolated, so the set of singular points A_{sing} of A is comprised of finitely many points. The set A is contained in $\mathbb{T}^2$, and since all γ_i are curves, A is of pure real dimension 1.

The domain of σ is $U := (-\pi, \pi)^2$, which is open and connected in $\mathbb{R}^2$, so there exists a domain $V \in \mathbb{C}^2$ such that $U \subset V$, and a unique holomorphic function $F : V \to \mathbb{C}$ such that $F\big|_U = \sigma$. This is a classic complexification property stated in Proposition 2.2; the original statement and proof can be found, for example, in [17, Proposition 3.1.3].

Let $B := \mathcal{Z}_F$. Then F is analytic at all $z = (z_1, z_2) \in V$, which makes $(B, z) \supset F^{-1}(0)$ an analytic germ, meeting one of the conditions of Theorem 3.11. Moreover, F is then holomorphic in all small neighbourhoods of z contained in V, thus $F \in \mathcal{O}_{\mathbb{C}^2,z}$. It's important to notice that $(\zeta_1, \zeta_2) \in V$ as well. By Theorem 3.11, there is a unique decomposition of F as

$$F = v f_1^{l_1} \ldots f_N^{l_N}$$

where l_j is the order of vanishing of F along irreducible component B_j of B, v is invertible and f_j are irreducible germs.

Since A is comprised of a finite union of disjoint real curves, if we take a small enough neighbourhood W of $\mathbb{T}^2$ in $\mathbb{C}^2$, we will see that $\mathcal{Z}_\sigma \cap W$ will be made up of a finite number of (possibly not disjoint) real surfaces, which after complexification, become complex curves. This comes from the uniqueness of F, the fact that holomorphic functions in two variables cannot have isolated zeros, and the identity theorem for holomorphic functions. By taking small enough neighbourhoods W_j of each γ_j, we will have $\gamma_j \subset W_j \subset B_j$ for each $j \in (1, ..., N)$, and the order of vanishing of σ will stay constant along each γ_j the same way it was for F along each B_j. We get the decomposition $f = u g_1^{m_1} \ldots g_N^{m_N}$, for u an invertible and g_j irreducible germs, and m_j the order of vanishing along each γ_j.

From the definition of order of vanishing, this means that the lowest power of a term in the power series expansion of σ around some point $(z_1, z_2) \in \gamma_j$ will be precisely m_j. By applying a real-analytic change of coordinates (see Example 3.13) , without loss of generality we can assume $\gamma_j = \{z_2 = 0\}$. Since every function f can be written as

$$f(z) = \sum_n f_n(z)$$

where f_n are homogeneous functions with all terms of degree n, then for every $(z_1, z_2) \in \gamma_j$ we can write

$$\sigma(z_1, z_2) = z_1^{m_j}(\text{const} + \text{higher degree terms}).$$

Now it immediately follows that

$$\int_{B_\varepsilon(z_1,z_2)} \sigma(z_1, z_2)^{1-\mathfrak{p}}\, dm(z_1, z_2) < \infty$$

and thus

$$\int_{B_\varepsilon(z_1,z_2)} |\rho_\phi(z_1, z_2)|^{1-\mathfrak{p}}\, dm(z_1, z_2) < \infty$$

for all $(z_1, z_2) \in \gamma_j$ and small enough $\varepsilon > 0$. Applying Lemma 3.10 gives the wanted result. $\square$

In practice, the theorem above tells us that for ϕ such that the zero set of ρ_ϕ on $\mathbb{T}^2$ contains no isolated points, we can read off the integrability indexes along entire smooth components of zero curves just by knowing local integrability around "nice" points on them. Here by "nice" points we mean points which are not intersection points of two or more curves and are not first two coordinates of a vertical line in the zero set of ϕ.

Example 3.13. Let's look at Example 3.8 from before. We have

$$\mathcal{Z}_p \cap \mathbb{T}^3 = \left\{(0, 0, u) \,:\, u \in [-\pi, \pi]\} \cup \{s, \arg m(e^{is}), \pi) \,:\, s \in [-\pi, \pi]\,,\, m(z) = \frac{3 + z^2}{1 + 3z^2}\right\}$$

so that

$$\mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^2 = \left\{\left(\xi_1, \frac{3 + \xi_1^2}{1 + 3\xi_1^2}\right) \,:\, \xi_1 \in \mathbb{T}\backslash\{1\}\right\}$$

and $\mathcal{Z}_p \cap \mathbb{T}^3$ contains the vertical line $\{(1, 1)\} \times \mathbb{T}$. On $\mathbb{T}^2$, $\mathcal{Z}_{\rho_\phi}$ consists of a single smooth curve (Figure 2), so the integrability index is the same along all points in $\mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^2$. For example, take $(z_1, z_2) = (-1, 1) \in \mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^3$, or in other words, $(\theta_1, \theta_2) = (\pi, 0)$, for $(z_1, z_2) = (e^{i\theta_1}, e^{i\theta_2})$. Notice that when θ_1 is close to π, the point $\pi - \theta_1$ will be close to 0. Therefore, it's equivalent to look at an expansion of a function around the point (θ_1, θ_2) for θ_1 close to π and θ_2 close to 0, and an expansion of the function around the point $(\pi - \theta_1, \theta_2)$ for θ_1, θ_2 close to 0. We will use the same reasoning for substitutions made in the examples that will follow. Thus, after simplifying the trigonometric identities, around the point $(\pi, 0)$ we have the expansions:

$$\begin{aligned}
&\rho_\phi(\pi - \theta_1, \theta_2) = \\
&= \frac{20 + 12\cos 2\theta_1 - 2\cos(2\theta_1 + \theta_2) - 12\cos\theta_2 - 18\cos(2\theta_1 - \theta_2)}{24 + 4\cos\theta_1 + 12\cos 2\theta_1 - 2\cos(2\theta_1 + \theta_2) - 10\cos\theta_2 + 4\cos(\theta_1 - \theta_2) - 16\cos(2\theta_1 - \theta_2)} \\
&= (\theta_1 - \theta_2)^2 + \mathcal{O}(\|\theta\|^3)
\end{aligned}$$

and

$$\sigma(\pi - \theta_1, \theta_2) = 16(\theta_1 - \theta_2)^2 + \mathcal{O}(\|\theta\|^3) \tag{7}$$

for (θ_1, θ_2) close to $(0, 0)$. The lowest degree of a monomial factor in the expansion is 2, meaning that the order of vanishing of the function $\rho_\phi(\pi - \theta_1, \theta_2)$ around the origin is 2 as well. Precisely because the expansion is done near the origin, the terms following the leading ones will be smaller in comparison, so the convergence of an integral of the function will be determined completely by the leading terms, of degree 2. In particular, this means that $\frac{\partial \phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ around $(\pi, 0, \tau)$, $\tau \in \mathbb{T}$ if and only if

$$2(1-\mathfrak{p}) > -1 \iff 2 - 2\mathfrak{p} > -1 \iff \mathfrak{p} < \frac{3}{2}.$$

Now take $(z_1, z_2) = (-i, -1) \in \mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^2$, i.e. $(\theta_1, \theta_2) = (-\frac{\pi}{2}, \pi)$. After simplifying the expression as above, we get the expansion

$$\begin{aligned}
&\rho_\phi\Big(-\frac{\pi}{2} - \theta_1, \pi - \theta_2\Big) = \\
&= \frac{20 - 12\cos 2\theta_1 - 2\cos(2\theta_1 - \theta_2) + 12\cos\theta_2 - 18\cos(2\theta_1 + \theta_2)}{24 - 4\sin\theta_1 - 12\cos 2\theta_1 - 2\cos(2\theta_1 - \theta_2) + 10\cos\theta_2 + 4\sin(\theta_1 + \theta_2) - 16\cos(2\theta_1 + \theta_2)} \\
&= (4\theta_1 + \theta_2)^2 + \mathcal{O}(\|\theta\|^3)
\end{aligned}$$

for (θ_1, θ_2) close to $(0,0)$. As before, $\frac{\partial \phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ around $(\frac{\pi}{2}, \pi, \tau)$, $\tau \in \mathbb{T}$ if and only if $\mathfrak{p} < \frac{3}{2}$, which is in accordance with Theorem 3.12.

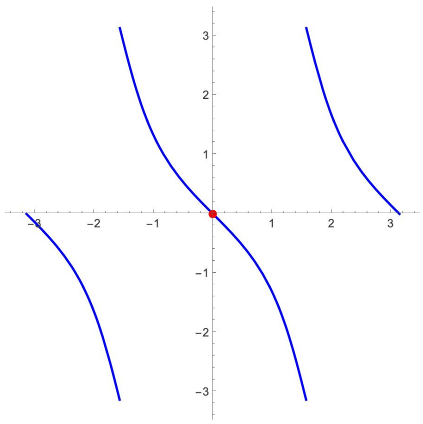


Figure 2: Example 3.8. Zero set of ρ_ϕ on $\mathbb{T}^2$, with the singular point $(0,0)$.

Example 3.14. Let's look at another irreducible (2,1,1)-degree RIF, also investigated in [9, Section 5]:

$$\phi(z_1, z_2) = \frac{1 - 2z_1 + z_1^2 - z_2 - 2z_1z_2 - z_1^2z_2 + 4z_1^2z_2z_3}{4 - z_3 - 2z_1z_3 - z_1^2z_3 + z_2z_3 - 2z_1z_2z_3 + z_1^2z_2z_3}.$$

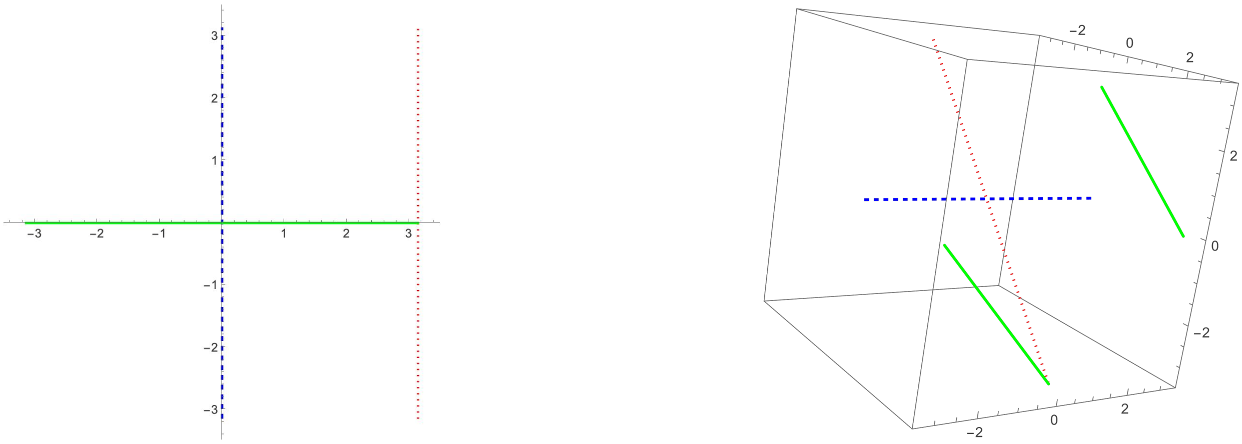
In the same paper it has been shown that ϕ has no vertical line singularities and that

$$\begin{aligned}
\mathcal{Z}_p \cap \mathbb{T}^3 = \{(0,t,0) : t \in [-\pi,\pi]\} \cup \{(t,0,-t) : t \in [-\pi,\pi]\} \cup \\
\cup \{(\pi, t, -\pi - t) : t \in [-\pi, 0]\} \cup \{(\pi, t, \pi - t) : t \in [0,\pi]\}
\end{aligned}$$

if we represent points with their arguments. Now Lemma 3.7 says that we can easily read the zero set of σ on $\mathbb{T}^2$ from $\mathcal{Z}_p \cap \mathbb{T}^3$, so

$$\begin{aligned}
\mathcal{Z}_\sigma \cap \mathbb{T}^2 = \mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^2 = \{(0,t) : t \in [-\pi,\pi]\} \cup \{(t,0) : t \in [-\pi,\pi]\} \cup \\
\cup \{(\pi,t) : t \in [-\pi,\pi]\}\}.
\end{aligned}$$

All of these sets describe smooth real curves (Figure 3). Let's name them γ_1, γ_2 and γ_3, in respective orders of the sets.



(a) $\mathcal{Z}_\sigma \cap \mathbb{T}^2$ with γ_1 dashed, γ_2 dotted, γ_3 solid

(b) $\mathcal{Z}_p \cap \mathbb{T}^3$

Figure 3: Example 3.14

It's clear from the plot that γ_1 and γ_2 do not intersect, while γ_3 intersects both of them. Moreover, $\gamma_1 \cap \gamma_3 = \{(0,0)\}$ and $\gamma_2 \cap \gamma_3 = \{(\pi, 0)\}$. Since the curves are closed in $\mathbb{T}^2$, by removing one point, the curves $\gamma_1 \backslash \{(0,0)\}$ and $\gamma_2 \backslash \{(\pi, 0)\}$ remain connected. However, from γ_3 we need to remove two points so the curve $\gamma_3 \backslash \{(0,0), (\pi, 0)\}$ splits into two connected components, $\gamma_{3,1} : [\pi, 0) \to \mathbb{T}^2$ and $\gamma_{3,2} : (0, \pi] \to \mathbb{T}^2$. These four curves satisfy all the conditions of Theorem 3.12, so we calculate the integrability condition for each curve separately.

Around $(\theta_1, \theta_2) = (0, \frac{\pi}{2}) \in \gamma_1$ we get the expansion

$$\begin{aligned}\sigma\left(\theta_1, \frac{\pi}{2} - \theta_2\right) &= 4 - 4\cos 2\theta_1 - 2\sin(2\theta_1 + \theta_2) + 4\sin(\theta_2) + 2\sin(2\theta_1 - \theta_2) \\ &= 8\theta_1^2 \left(1 + \theta_2 - \frac{1}{3}\theta_1^2 + \mathcal{O}(\|\theta\|^3)\right)\end{aligned}$$

for (θ_1, θ_2) close to $(0,0)$. Since the lowest term in the expansion is of degree 2, using Theorem 3.9, $\frac{\partial \phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ around $(0, \frac{\pi}{2}, \tau)$, $\tau \in \mathbb{T}$, and by Theorem 3.12 along all of $\gamma_1 \backslash \{(0,0)\}$ if and only if

$$2(1 - \mathfrak{p}) > -1 \iff \mathfrak{p} < \frac{3}{2}.$$

Now take $(\theta_1, \theta_2) = (\pi, \frac{\pi}{2}) \in \gamma_2$. We have

$$\begin{aligned}\sigma(\pi - \theta_1, \frac{\pi}{2} - \theta_2) &= 4 - 4\cos 2\theta_1 - 2\sin(2\theta_1 + \theta_2) + 4\sin\theta_2 + 2\sin(2\theta_1 - \theta_2) \\ &= 8\theta_1^2 \left(1 + \theta_2 - \frac{1}{3}\theta_1^2 + \mathcal{O}(\|\theta\|^3)\right)\end{aligned}$$

for (θ_1, θ_2) close to $(0,0)$. Same as before, along the curve γ_2 we have $\frac{\partial \phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ along γ_2 if and only if $\mathfrak{p} < \frac{3}{2}$.

In the same manner, we can observe what happens around $(\theta_1, \theta_2) = (-\frac{\pi}{2}, 0) \in \gamma_{3,1}$ and

$(\theta_1, \theta_2) = (\frac{\pi}{2}, 0) \in \gamma_{3,2}$. We get

$$\begin{aligned}\sigma\Big(-\frac{\pi}{2} - \theta_1, \theta_2\Big) &= 4 + 4\cos 2\theta_1 - 2\cos(2\theta_1 + \theta_2) - 4\cos\theta_2 - 2\cos(2\theta_1 - \theta_2) \\ &= 4\theta_2^2\big(1 - \theta_1^2 - \frac{1}{12}\theta_2^2 + \mathcal{O}(\|\theta\|^4)\big)\end{aligned}$$

and

$$\begin{aligned}\sigma\Big(\frac{\pi}{2} - \theta_1, \theta_2\Big) &= 4 + 4\cos 2\theta_1 - 2\cos(2\theta_1 + \theta_2) - 4\cos\theta_2 - 2\cos(2\theta_1 - \theta_2) \\ &= 4\,\theta_2^2\big(1 - \theta_1^2 - \frac{1}{12}\theta_2^2 + \mathcal{O}(\|\theta\|^4)\big)\end{aligned}$$

for (θ_1, θ_2) close to $(0,0)$. The expansions are identical, so $\frac{\partial\phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ along $\gamma_{3,1}$ and $\gamma_{3,2}$ if and only if $\mathfrak{p} < \frac{3}{2}$.

To get the global critical integrability, all that is left to consider is the behaviour around points of intersection. Around $(0,0)$ we have

$$\begin{aligned}\sigma(\theta_1, \theta_2) &= 4 - 4\cos 2\theta_1 + 2\cos(2\theta_1 - \theta_2) - 4\cos\theta_2 + 2\cos(2\theta_1 + \theta_2) \\ &= 4\,\theta_1^2\theta_2^2\big(1 - \frac{1}{3}\theta_1^2 - \frac{1}{3}\theta_2^2 + \mathcal{O}(\|\theta\|^4)\big).\end{aligned}$$

The leading term has bidegree $(2,2)$, but since the following double integral can be separated as such:

$$\iint |4\,\theta_1^2\theta_2^2|^{p-1}\, d\theta_1\, d\theta_2 = 4\int |\theta_2^2|^{p-1}\left(\int |\theta_1^2|^{p-1}\, d\theta_1\right) d\theta_2\,,$$

its convergence will be determined by the convergence of each variable separately. Since the degree of the leading term in each of the variables is two, $\frac{\partial\phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ around $(0,0,\tau)$, $\tau \in \mathbb{T}$ if and only if

$$2(1 - \mathfrak{p}) > -1 \iff \mathfrak{p} < \frac{3}{2}.$$

On the other hand, around $(\pi, 0)$ we have

$$\begin{aligned}\sigma(\pi - \theta_1, \theta_2) &= 4 - 4\cos 2\theta_1 + 2\cos(2\theta_1 + \theta_2) - 4\cos\theta_2 + 2\cos(2\theta_1 - \theta_2) \\ &= 4\theta_1^2\theta_2^2\left(1 - \frac{1}{3}\theta_1^2 - \frac{1}{12}\theta_2^2 + \mathcal{O}(\|\theta\|^4)\right)\end{aligned}$$

so once again, $\frac{\partial\phi}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ if and only if $\mathfrak{p} < \frac{3}{2}$. The global critical integrability is defined as the worst (smallest) of all local critical integrabilities, but as they are all the same, it follows immediately that $\mathfrak{p}^* = \frac{3}{2}$.

The last results of this section will give us a way to make use of vertical lines or, depending on the objective, evade the need to know the precise integrability condition of the partial derivatives around them.

Corollary 3.15. *Let ϕ be an RIF as before with a vertical line $\{(\xi_1, \xi_2)\} \times \mathbb{T} \subset \mathcal{Z}_p \cap \mathbb{T}^3$. Let the zero set of ρ_ϕ on $\mathbb{T}^2$ contain only smooth curves $(\gamma_i)_i$ (as in Theorem 3.12). If there is a smooth curve $\Gamma \in \mathcal{Z}_\sigma \cap \mathbb{T}^2$ such that $\Gamma = \{(\xi_1, \xi_2)\} \cup \gamma_i$ for some i, then for an arbitrary point $(w_1, w_2) \in \Gamma \setminus \big(\bigcup_{k \neq j} \gamma_k \cup \{(\xi_1, \xi_2)\}\big)$ the following are equivalent:*

(i) $\int_{B_\varepsilon(\xi_1, \xi_2)} \sigma(z_1, z_2)^{1-\mathfrak{p}}\, dm(z_1, z_2) < \infty$

(ii) $\int_{B_\varepsilon(w_1, w_2)} \sigma(z_1, z_2)^{1-\mathfrak{p}}\, dm(z_1, z_2) < \infty$

(iii) $\int_{B_\varepsilon(w_1,w_2)} |\rho_\phi(z_1,z_2)|^{1-\mathfrak{p}}\, dm(z_1,z_2) < \infty$

Proof. The equivalence between (ii) and (iii) was shown in Theorem 3.9. The proof of the equivalence between (i) and (ii) however follows the same steps as the proof of Theorem 3.12 since σ is a real-analytic function on a smooth curve Γ, which doesn't intersect any of the other smooth curves γ_k, for $k \neq i$. Thus it's possible to extend it to a holomorphic function, on which we can apply Theorem 3.11. □

Theorem 3.16. *Let ϕ be an RIF that contains a vertical line singularity $\{(\xi_1,\xi_2)\} \times \mathbb{T} \subset \mathcal{Z}_p \cap \mathbb{T}^3$, with the zero set of ρ_ϕ on $\mathbb{T}^2$ as in Theorem 3.12. If*

$$\int_{B_\varepsilon(\xi_1,\xi_2)} \sigma(z_1,z_2)^{1-\mathfrak{p}}\, dm(z_1,z_2) < \infty$$

for some small $\varepsilon > 0$, then

$$\int_{B_\delta(\xi_1,\xi_2)} |\rho_\phi(z_1,z_2)|^{1-\mathfrak{p}}\, dm(z_1,z_2) < \infty$$

for some $0 < \delta \leq \varepsilon$.
Specifically, this means that the local $L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$-integrability of $\frac{\partial \phi}{\partial z_3}$ at (ξ_1,ξ_2,ξ_3) cannot make the global $L^{\mathfrak{p}}(\mathbb{T}^3)$-integrability of $\frac{\partial \phi}{\partial z_3}$ worse.

Proof. The function $|\widetilde{p}_1|^2$ is continuous with a zero in (ξ_1,ξ_2), so there must exist a small neighbourhood of (ξ_1,ξ_2) such that $|\widetilde{p}_1|^2$ is bounded. More precisely, there exists a $\varepsilon' > 0$ such that $|\widetilde{p}_1(z_1,z_2)|^2 < \kappa$, for some $\kappa > 0$ and all $(z_1,z_2) \in B_{\varepsilon'}(\xi_1,\xi_2)$. Take $\delta := \min\{\varepsilon,\varepsilon'\}$. Since $|\sigma|^2 \geq 0$, clearly

$$\int_{B_\delta(\xi_1,\xi_2)} \sigma(z_1,z_2)^{1-\mathfrak{p}}\, dm(z_1,z_2) < \infty$$

as well. Now we get the following

$$\begin{aligned}\int_{B_\delta(\xi_1,\xi_2)} |\rho_\phi(z_1,z_2)|^{1-\mathfrak{p}}\, dm(z_1,z_2) &= \int_{B_\delta(\xi_1,\xi_2)} \frac{\sigma(z_1,z_2)^{1-\mathfrak{p}}}{|\widetilde{p}_1(z_1,z_2)|^{2(1-p)}}\, dm \\ &= \int_{B_\delta(\xi_1,\xi_2)} \sigma(z_1,z_2)^{1-\mathfrak{p}}\, |\widetilde{p}_1(z_1,z_2)|^{2(p-1)}\, dm(z_1,z_2) \\ &\leq \kappa^{\mathfrak{p}-1} \int_{B_\delta(\xi_1,\xi_2)} \sigma(z_1,z_2)^{1-\mathfrak{p}}\, dm(z_1,z_2) < \infty\end{aligned}$$

since $\kappa > 0$ and $\mathfrak{p}$ cannot be less than 1. Moreover, since the global integrability is the smallest among all local integrability exponents, the inequality above says that the one of $\frac{\partial \phi}{\partial z_3}$ at (ξ_1,ξ_2) will not be smaller than the global one, and will in turn not affect the global integrability. □

The above theorem, together with the analysis done in Example 3.13, yields the critical $L^{\mathfrak{p}}(\mathbb{T}^3)$ integrability index of $\frac{\partial \phi}{\partial z_3}$, namely $\mathfrak{p}^* = \frac{3}{2}$. This is easy to check by investigating the function σ around the vertical line. Indeed, we get

$$\sigma(\theta_1,\theta_2) = 16(\theta_1+\theta_2)^2 + \mathcal{O}(\|\theta\|^3),$$

so the order of vanishing of σ matches the previous ones of ρ_ϕ around other points of the curve. This addresses Question 2 in [9, Section 5], answering the first part of the question, while completely disproving the second. Not only does the presence of a vertical line singularity fail to worsen the integrability, but locally the integrability is either improved or remains the same, and it has no effect on the global integrability.

Moreover, this gives another example of a three-variable rational inner function with partial integrability bounds that differ from a variable to another. Here in question those are the partials $\frac{\partial\phi}{\partial z_2}$ and $\frac{\partial\phi}{\partial z_3}$, seeing how it was shown in [9, Example 5.2] that $\frac{\partial\phi}{\partial z_2} \in L^{\mathfrak{p}}(\mathbb{T}^3)$ if and only if $\mathfrak{p} < \frac{5}{4}$. However, further investigation into the nature of $\frac{\partial\phi}{\partial z_1}$ would be needed, since the RIF is not linear in the first variable, and doesn't admit a single-valued parametrization of z_1 (as per the discussion in the beginning of Section 3.1.2).

Remark. If we're in a situation such as Corollary 3.15 suggests, the theorem above tells us that the coordinates (ξ_1, ξ_2) of a vertical line passing through a smooth curve will not affect the integrability of $\frac{\partial\phi}{\partial z_3}$ along that smooth curve as well. The same clearly holds if ϕ has more than one vertical line singularity, as they will all be disjoint, and the argument will hold for each of them separately.

3.3 Higher dimensions

This section will be dedicated to generalizing results of Section 3.2 to higher-dimensional irreducible RIFs of polydegree $(m_1, ..., m_{n-1}, 1)$ with singularities on $\mathbb{T}^n$. Once again, if an RIF ϕ is linear in the last variable, we can write

$$\phi(z) = \frac{\widetilde{p}_1(z_1, ...z_{n-1}) \cdot z_n + \widetilde{p}_2(z_1, ...z_{n-1})}{p_1(z_1, ...z_{n-1}) + p_2(z_1, ...z_{n-1}) \cdot z_n}$$

and therefore,

$$\rho_\phi(z_1, ..., z_{n-1}) = \frac{\sigma(z_1, ..., z_{n-1})}{|\widetilde{p}_1(z_1, ..., z_{n-1})|^2},$$

which once again is analytic at all points of $\mathbb{T}^{n-1}$ except those where $\widetilde{p}_1 = 0$. Since that again happens at points contained in a vertical line, we get the immediate result:

Corollary 3.17. *The function ρ_ϕ is analytic on $\mathbb{T}^n \setminus \bigcup_{\alpha\in A}\{\xi_\alpha\}$, where $\xi_\alpha \in \mathbb{T}^{n-1}$ are points at which ϕ has vertical line singularities $\{\xi_\alpha\} \times \mathbb{T}$. Moreover, σ is analytic on $\mathbb{T}^{n-1}$.*

Proof. The proof follows from the construction made in the beginning of Section 3.2. $\square$

In the corollary above, the set A containing $\alpha \in A$ need not be finite. Unlike the three variable case, general $(m_1, ..., m_{n-1}, 1)$-degree RIFs can have infinitely many vertical lines. This is a consequence of Theorem 3.1. The theorem says that in two variables, singularities of RIFs on $\mathbb{T}^2$, if they have any, can only be points, while in three variables, singularities can be points or at most curves. In four and more variables however, they can be points, curves, surfaces etc. For example, if the zero set of p on $\mathbb{T}^4$, contained a set of the form $\{(\xi, \xi^2, 1)\} \times \mathbb{T}$, where $\xi \in \mathbb{T}$, this would be called a vertical *surface* singularity (Figure 4). Therefore, Theorem 3.5 does not hold in higher dimensions and we obtain:

Corollary 3.18. *For $n \geq 4$, $(m_1, ..., m_{n-1}, 1)$-degree RIF can have infinitely many vertical line singularities. In particular, they are arranged as a finite discrete set and/or as a continuum of vertical lines.*

Proof. Vertical lines appear precisely when $|\widetilde{p}_1|^2 = 0$. Assume $A \subset \mathcal{Z}_{|\widetilde{p}_1|^2} \cap \mathbb{T}^{n-1}$ is a closed discrete set. This set cannot have limit points, as they are points whose all neighbourhood intersect some other points of the set. However, $\mathbb{T}^{n-1}$ is compact in the topology inherited from $\mathbb{R}^{n-1}$, so by [19, Theorem 28.1], every infinite subset of $\mathbb{T}^{n-1}$ has a limit point. Therefore, A must be finite. $\square$

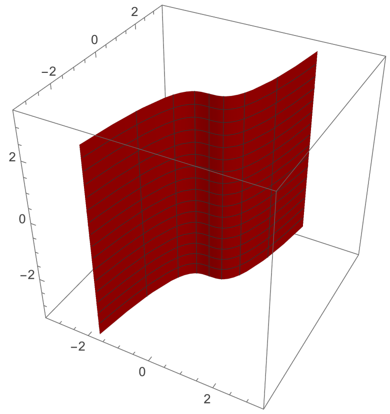


Figure 4: Vertical surface $\{(\xi, \xi^2, 1)\} \times \mathbb{T}$, shown in $\mathbb{T}^3$ by dropping the fixed third coordinate.

However, higher-variable RIFs need not have vertical surfaces, or even vertical lines, contained in their zero sets on the n-torus. The following example will showcase an RIF with a "big" zero set, but with a function ρ_ϕ that's analytic at almost all points of the n-torus.

Example 3.19. We follow the matrix construction as in [9, Example 5.1.], where a new three-variable RIF was created. In order to accommodate for an extra variable in a construction of a four-variable function, we pass to a 5×5 self-adjoint matrix A and four contractions Y_1, Y_2, Y_3 and Y_4. Take the matrices as following.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}.$$

Now the matrix Y_4 is calculated from

$$Y_4 = I - Y_1 - Y_2 - Y_3 = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix}.$$

Finally, we take $v = (1, 1, 0, 0, 1)^T$, after which [3] guarantees that we will obtain a Pick function on the poly-upper half-plane Π^3:

$$\begin{aligned} f(w) &= \langle (A - w_1 Y_1 - w_2 Y_2 - w_3 Y_3)^{-1} v, v \rangle, \quad w \in \Pi^3 \\ &= \frac{1 + z_2 - z_1 z_2 - z_3 - 2z_2 z_3 + z_1 z_2 z_3 - z_4 - z_2 z_4 + z_1 z_2 z_4 + z_2 z_3 z_4}{(1 - z_1)(1 - z_3 - z_2 z_3 - z_4 + z_2 z_3 z_4)}. \end{aligned}$$

By conjugating with Möbius maps

$$m : \mathbb{D}^3 \to \Pi^3, \quad (z_1, z_2, z_2) \mapsto \left(i\,\frac{1 - z_1}{1 + z_1}, i\,\frac{1 - z_2}{1 + z_2}, i\,\frac{1 - z_3}{1 + z_3} \right)$$

and

$$\beta : \Pi \to \mathbb{D}, \quad z \mapsto \frac{1+iz}{1-iz},$$

we obtain a four-variable $(1,1,1,1)$-degree irreducible rational inner function $\phi = \frac{\widetilde{p}}{p}$, where

$$\begin{aligned} p(z_1,z_2,z_3,z_4) =& (-3i + (2-5i)z_1 - (4-i)z_2 - (2+5i)z_1z_2 + (2+i)z_3 + 3iz_1z_3 - (2+3i)z_2z_3 \\ &+ (4-5i)z_1z_2z_3 + (2-i)z_4 + (4+i)z_1z_4 - (2+i)z_2z_4 - 3iz_1z_2z_4 + 3iz_3z_4 \\ &- (2-i)z_1z_3z_4 + (4-5i)z_2z_3z_4 + (10+5i)z_1z_2z_3z_4), \\ \widetilde{p}(z_1,z_2,z_3,z_4) =& \big((-10+5i) - (4+5i)z_1 + (2+i)z_2 + 3iz_1z_2 - 3iz_3 + (2-i)z_1z_3 - (4-i)z_2z_3 \\ &- (2+i)z_1z_2z_3 - (4+5i)z_4 + (2-3i)z_1z_4 + 3iz_2z_4 - (2-i)z_1z_2z_4 \\ &+ (2-5i)z_3z_4 + (4+i)z_1z_3z_4 - (2+5i)z_2z_3z_4 - 3iz_1z_2z_3z_4\big). \end{aligned}$$

The equation $|\widetilde{p}_1|^2 = 0$ is equivalent to $\widetilde{p} = 0$ on $\mathbb{T}^3$. Calculation analogous to previous ones shows $\widetilde{p} = 0$ has a unique solution on $\mathbb{T}^3$, that is, $\mathcal{Z}_{\widetilde{p}_1} \cap \mathbb{T}^3 = \{(i,-1,1)\}$. Therefore, ϕ has only one isolated vertical line singularity on $\mathbb{T}^4$. The complete singular locus is, however, much bigger. In particular,

$$\begin{aligned} \mathcal{Z}_p \cap \mathbb{T}^4 = &\left\{ \left(i, z_2, z_3, \frac{(-5+i) - (1-i)z_2 + (1-i)z_3 - (3+i)z_2z_3}{(1+3i) + (1-i)z_2 - (1-i)z_3 - (1-5i)z_2z_3} \right) \;:\; z_2, z_3 \in \mathbb{T} \right\} \\ &\cup \left\{ \left(z_1, i, z_3, \frac{(4+2i) + 10iz_1 + 2iz_3 - (4-2i)z_1z_3}{-2i + (4-2i)z_1 + (4-2i)z_3 + (8+6i)z_1z_3} \right) \;:\; z_1, z_3 \in \mathbb{T} \right\} \\ &\cup \{(z_1, z_2, 1, i) \;:\; z_1, z_2 \in \mathbb{T}\}. \end{aligned}$$

The sets are shown in $\mathbb{T}^3$ in Figure 5, with the first, second and third variable fixed, respectively. The vertical line $\{(i,-1,1)\} \times \mathbb{T}$ is contained in the first set.

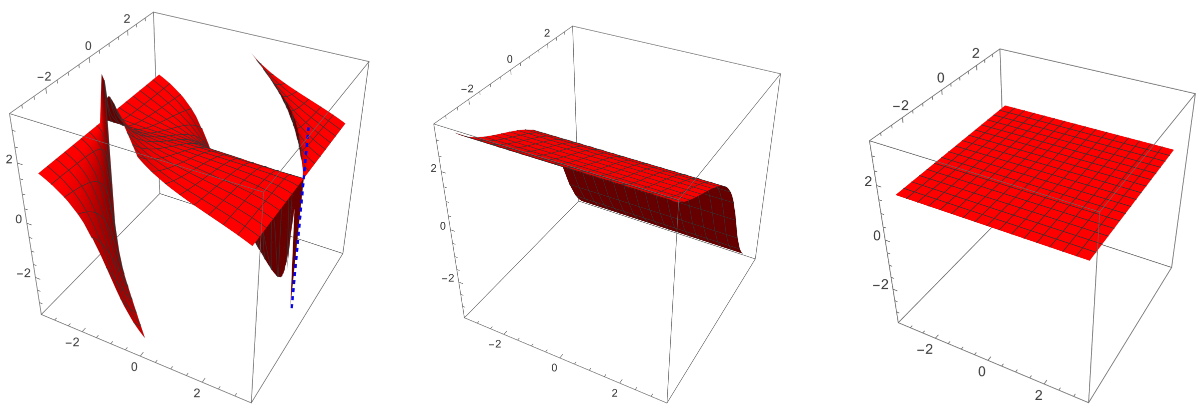


Figure 5: $\mathcal{Z}_\phi \cap \mathbb{T}^4$ from Example 3.19 (pictured after dropping the constant coordinates z_1, z_2 and z_3, respectively)

The complexity of the previous example motivates the extension of previous three-variable results to higher dimensions. Fortunately, this extension is quite straightforward. First, we obtain a generalization of Lemma 3.7 to $n \geq 3$ variables.

Lemma 3.20. *Let ϕ be an irreducible $(m_1, \ldots, m_{n-1}, 1)$-degree RIF. Then the following holds.*

(i) If ϕ has no vertical line singularities, $\rho_\phi(\zeta_1, \ldots, \zeta_{n-1}) = 0$ if and only if $(\zeta_1, \ldots, \zeta_{n-1}, \tau)$ is a singularity of ϕ for some $\tau \in \mathbb{T}$.

(ii) If $\mathcal{Z}_p \cap \mathbb{T}^n \supset \bigcup_{\alpha \in A} \{\xi^\alpha\} \times \mathbb{T}$, for $\alpha \in A$ from some collection A, and $\xi^\alpha \in \mathbb{T}^{n-1}$, then $\rho_\phi(\zeta_1, \ldots, \zeta_{n-1}) = 0$ if and only if $(\zeta_1, \ldots, \zeta_{n-1}, \tau) \in \mathcal{Z}_p \cap \mathbb{T}^n \setminus \big(\{\xi^\alpha\} \times \mathbb{T}\big)$.

Proof. The proof follows the same steps as the proof of Lemma 3.7, since by Corollary 3.17, the function σ is analytic at all points except those possibly contained in vertical line singularities of ϕ. $\square$

Example 3.21. Once again, the above result gives an easier way of finding the zero set of an RIF ϕ, since we have to work with fewer variables. Going back to Example 3.19, we now recover the zero set of σ (Figure 6),

$$\mathcal{Z}_\sigma \cap \mathbb{T}^3 = \{(i, z_2, z_3) : z_2, z_3 \in \mathbb{T}\} \cup \{(z_1, 1, z_3) : z_1, z_3 \in \mathbb{T}\} \cup \{(z_1, z_2, 1) : z_1, z_2 \in \mathbb{T}\},$$

exactly equal to $\mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^3 \backslash \{(i, -1, 1)\}$.

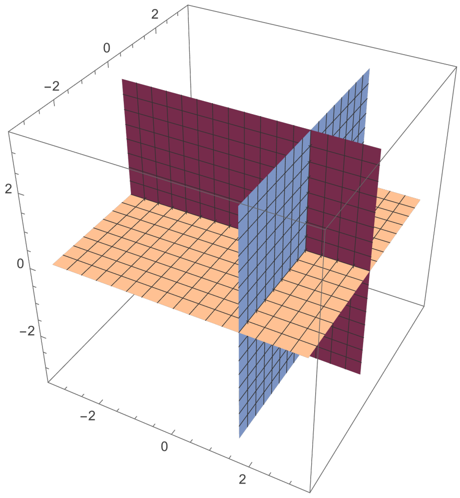


Figure 6: The three sets making up $\mathcal{Z}_\sigma \cap \mathbb{T}^3$

In light of this example and previous research, we are naturally led to the question of integrability in higher dimensions. Consequently, we obtain an analogue of Theorem 3.9 and Lemma 3.10.

Theorem 3.22. *Let ϕ be a $(m_1, .., m_{n-1}, 1)$-degree RIF with possible (not necessarily isolated) vertical line singularities $\bigcup_{\alpha \in A}\{\xi^\alpha\} \times \mathbb{T}$, for some collection $A \ni \alpha$. Then at all points $\zeta \in \mathcal{Z}_p \cap \mathbb{T}^n$ not contained in $\bigcup_{\alpha \in A}\{\xi^\alpha\} \times \mathbb{T}$,*

$$\begin{aligned}\frac{\partial \phi}{\partial z_n} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^n) &\iff \int_{B_\varepsilon(\zeta_1,\dots,\zeta_{n-1})} |\rho_\phi(z_1, ..., z_{n-1})|^{1-\mathfrak{p}}\, dm(z_1, ..., z_{n-1}) < \infty \\ &\iff \int_{B_\varepsilon(\zeta_1,\dots,\zeta_{n-1})} \sigma(z_1, ..., z_{n-1})^{1-\mathfrak{p}}\, dm(z_1, ..., z_{n-1}) < \infty\end{aligned}$$

for sufficiently small $\varepsilon > 0$. In particular, if ϕ contains no vertical line singularities, the above inequalities hold for all $\zeta \in \mathcal{Z}_p \cap \mathbb{T}^n$.

Proof. The equivalences follow by using Lemma 3.20 and repeating the steps of Theorem 3.9, when there is no vertical lines, and afterwards, of Lemma 3.10, when there is. $\square$

And accordingly, we once again make use of Theorem 3.11 to obtain the following result, the generalization of Theorem 3.12.

Theorem 3.23. *Let $(\zeta_1, ..., \zeta_{n-1}) \in \mathbb{T}^{n-1}$ be a point in the zero set of ρ_ϕ such that $\frac{\partial \phi}{\partial z_n} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^n)$. Assume $\mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^{n-1} = \bigcup_{i=1}^N M_i$ for some $N \in \mathbb{N}$, where each M_i is a real manifold* *of dimension $n-2$, and let $(\zeta_1, ..., \zeta_{n-1}) \in M_j \backslash \bigcup_{k\neq j} M_k$ for some $j \in \{1, ..., N\}$. Then $\frac{\partial \phi}{\partial z_n} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^n)$ at every $(z_1, ..., z_{n-1}) \in M_j \backslash \bigcup_{k\neq j} M_k$.*

It is now clear that the integrability results established in the previous section have all translated nicely to higher dimensions as well. To finalize the analysis, we state the n-dimensional extension of Theorem 3.16.

Theorem 3.24. *Let ϕ be an RIF with a vertical line singularity $\{\xi = (\xi_1, ..., \xi_{n-1})\} \times \mathbb{T} \in \mathcal{Z}_p \cap \mathbb{T}^n$ and the zero set of ρ_ϕ as in Theorem 3.23. If*

$$\int_{B_\varepsilon(\xi)} \sigma(z_1, ..., z_{n-1})^{1-\mathfrak{p}}\, dm(z_1, ..., z_{n-1}) < \infty$$

for small $\varepsilon > 0$, then

$$\int_{B_\delta(\xi)} |\rho_\phi(z_1, ..., z_{n-1})|^{1-\mathfrak{p}}\, dm(z_1, ..., z_{n-1}) < \infty$$

for some $0 < \delta \leq \varepsilon$.
Specifically, this means that the local $L^{\mathfrak{p}}_{loc}(\mathbb{T}^n)$-integrability of $\frac{\partial \phi}{\partial z_n}$ at $(\xi, \tau), \tau \in \mathbb{T}$ cannot make the global $L^{\mathfrak{p}}(\mathbb{T}^n)$-integrability of $\frac{\partial \phi}{\partial z_n}$ worse.

Proof. First, notice that the assumption of a pure dimensional $\mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^{n-1}$ dictates that ξ of it is contained in a larger component of the zero set, i.e. that every open neighbourhood around ξ contains infinitely many points of $\mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^{n-1}$. Otherwise, $\{\xi\}$ itself would be a component of the zero set of ρ_ϕ on $\mathbb{T}^{n-1}$, with the dimension $0 < n-2$, for any $n \geq 3$. But this contradicts the statement of Theorem 3.23.

It is possible that $\{\xi\} \times \mathbb{T}$ is contained inside a larger "vertical line set", such as a vertical surface or a vertical threefold. However, if ϕ contains only isolated vertical line singularities (for instance, as in Example 3.19), we proceed by extending arguments made in Theorem 3.16 to higher dimensions as before. Applying Theorem 3.23, and redoing analysis as in Corollary 3.15 proves the statements.

Alternatively, if $\{\xi\} \times \mathbb{T}$ is really a part of a larger vertical line set, that set can't be a "self-standing" component as well. In dimension $n = 4$, this would mean that we have a vertical surface singularity, therefore an isolated curve of points $\{\xi^\alpha\}$ of dimension 1 in a $n-2 = 2$ -dimensional zero set of ρ_ϕ on $\mathbb{T}^{n-1}$. Again, we get a contradiction with the statement of Theorem 3.23. Now, the same argument clearly holds for higher dimensions too. Therefore, reusing the arguments of Corollary 3.15, by swapping the curve Γ containing a smooth component γ_i and the point at which there is a vertical line, with a higher dimensional $\mathcal{M} = M \cup \{\xi^\alpha\}$, for $A \subset \mathcal{Z}_{\rho_\rho} \cap \mathbb{T}^{n-1}$, we prove the statement. □

Example 3.25. We apply the previous result to the RIF from Example 3.21, whose zero set of ρ_ϕ on $\mathbb{T}^3$ is composed of three surfaces $A_1 = \{(i, z_2, z_3) : z_2, z_3 \in \mathbb{T}\}$, $A_2 = \{(z_1, 1, z_3) : z_1, z_3 \in \mathbb{T}\}$, $A_3 = \{(z_1, z_2, 1) : z_1, z_2 \in \mathbb{T}\}$.

Expanding σ into a series around points corresponding to all seven possible cases (points lying in a single component, in the intersection of two components, and in the intersection of all three) gave the same result in each case: the critical integrability of $\mathfrak{p} = \frac{3}{2}$. The details are omitted as the calculation is identical to previous examples.

3.4 Other partial derivatives

It might be interesting to note a possible generalization of previous results on zero sets and integrability of $\frac{\partial \phi}{\partial z_n}$ to other partial derivatives. Really, assume ϕ is an RIF that is linear in

i-th variable, namely, an RIF of polydegree $(m_1, ..., 1, ..., m_n)$, where 1 is in the i-th place, for $i \in \{1, ..., n\}$. Then we can write ϕ as

$$\phi(z) = \frac{q_1(z_1, ..., \hat{z_i}, ..., z_n) + z_i q_2(z_1, ..., \hat{z_i}, ..., z_n)}{\widetilde{q_2}(z_1, ..., \hat{z_i}, ..., z_n) + z_i \widetilde{q_1}(z_1, ..., \hat{z_i}, ..., z_n)},$$

where $\hat{z_i}$ means z_i is omitted from the expression. If $i = n$, polynomials q_i and $\widetilde{q_i}$ are nothing more than previously used p_i and $\widetilde{p_i}$.

It now becomes natural to define functions

$$\sigma_i := |\widetilde{q_1}|^2 - |\widetilde{q_2}|^2$$

and

$$\rho_{i,\phi} := \frac{|\widetilde{q_1}|^2 - |\widetilde{q_2}|^2}{|\widetilde{q_1}|^2} = \frac{\sigma_i}{|\widetilde{q_1}|^2},$$

which can be further investigated in the same manner as σ and ρ_ϕ before. Therefore, the theory developed so far is applicable in this setting as well. Strictly speaking, the main difference will lie in the terminology, since for example, vertical lines would no longer be called "vertical", but instead various axis-parallel lines.

In the following chapters, we continue the analysis on $(m_1, .., m_{n-1}, 1)$-degree RIFs, while noting that the same reasoning applies to any choice of variable in which linearity is assumed.

4 Compositions

4.1 The slice matrix M_ϕ

In this section, we introduce the notion of a slice matrix and investigate to what extent the properties established in [23] remain valid in our setting.

First, recall that all RIFs ϕ that are linear in the last variable can be written as

$$\phi(z) = \frac{\widetilde{p}_1(z_1, ...z_{n-1}) \cdot z_n + \widetilde{p}_2(z_1, ...z_{n-1})}{p_1(z_1, ...z_{n-1}) + p_2(z_1, ...z_{n-1}) \cdot z_n}.$$

It will be useful to give the following "matrix representation" of ϕ using the above $(n-1)$-degree polynomials:

Definition 4.1. The *slice matrix* of a $(m, 1) = (m_1, .., m_{n-1}, 1)$-degree rational inner function $\phi : \mathbb{C}^n \to \mathbb{C}$ is the function $M_\phi : \mathbb{T}^{n-1} \to M_{2,2}(\mathbb{C})$ given by

$$M_\phi(z) = M_\phi(z_1, ..z_{n-1}) = \begin{pmatrix} \widetilde{p}_1(z) & \widetilde{p}_2(z) \\ p_2(z) & p_1(z) \end{pmatrix}.$$

The *slice determinant* of ϕ is the function $P_\phi : \mathbb{T}^{n-1} \to \mathbb{C}$ given by

$$P_\phi(z) = \det M_\phi(z).$$

Example 4.2. The slice matrix of the "favourite" example 2.9

$$\phi(z_1, z_2, z_3) = \frac{3z_1z_2z_3 - z_1z_2 - z_1z_3 - z_2z_3}{3 - z_1 - z_2 - z_3}$$

can be read off quite easily; we have

$$M_\phi(z_1, z_2) = \begin{pmatrix} 3z_1z_2 - z_1 - z_2 & -z_1z_2 \\ -1 & 3 - z_1 - z_2 \end{pmatrix}.$$

Given a finite-singularity RIF with polydegree $(m, 1)$, it has been shown in [23, Lemma 2] that for $\xi \in \mathbb{T}^{n-1}$, $P_\phi(\xi) = 0$ if and only if (ξ, τ) is a singularity of ϕ for some value of $\tau \in \mathbb{T}$. Furthermore,

$$\frac{\partial \phi}{\partial z_n} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^n) \text{ at } (\xi, \tau) \iff \int_{B_\varepsilon(\xi)} |P_\phi(z)|^{1-\mathfrak{p}} dm(z) < \infty.$$

In the setting of RIFs with higher-dimensional zero sets, a similar claim will hold. The only difference lies in the failure of P_ϕ to detect vertical line singularities. In particular, we obtain the following statement.

Proposition 4.3. *If ϕ contains vertical line singularities, then for all $\zeta = (\zeta_1, ..., \zeta_{n-1}) \in \mathbb{T}^{n-1}$, the slice determinant $P_\phi(\zeta) = 0$ if and only if (ζ, τ) is a singularity of ϕ not contained in a vertical line singularity, for some $\tau \in \mathbb{T}$. Then around such ζ*

$$\frac{\partial \phi}{\partial z_n} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^n) \iff \int_{B_\varepsilon(\zeta)} |P_\phi(z_1, ..., z_{n-1})|^{1-\mathfrak{p}} dm(z_1, ..., z_{n-1}) < \infty.$$

for sufficiently small $\varepsilon > 0$.

Proof. Since $z_j\bar{z}_j = 1$ on $\mathbb{T}$ for all $j = 1, \dots, n-1$, we have the equalities

$$\begin{aligned} p_i(z_1, \dots, z_{n-1}) &= \widetilde{\widetilde{p}}(z_1, \dots, z_{n-1}) \\ &= z_1^{m_1} \cdots z_{n-1}^{m_{n-1}} \overline{\widetilde{p}_i\Big(\frac{1}{\bar{z}_1}, \dots, \frac{1}{\bar{z}_{n-1}}\Big)} \\ &= z_1^{m_1} \cdots z_{n-1}^{m_{n-1}} \overline{\widetilde{p}_i(z_1, \dots, z_{n-1})} \end{aligned}$$

for $i = 1, 2$. Hence, we can rewrite P_ϕ as

$$P_\phi = \det M_\phi = \widetilde{p}_1 p_1 - p_2 \widetilde{p}_2 = z_1^{m_1} \cdots z_{n-1}^{m_{n-1}}(|\widetilde{p}_1|^2 - |\widetilde{p}_2|^2),$$

and since $z_1^{m_1} \cdots z_{n-1}^{m_{n-1}}$ is non-vanishing on $\mathbb{T}^{n-1}$, we have $\mathcal{Z}_{P_\phi} = \mathcal{Z}_\sigma$ on $\mathbb{T}^{n-1}$. Since $|z_1|^{m_1} \cdots |z_{n-1}|^{m_{n-1}} = 1$ is clearly bounded on $\mathbb{T}^{n-1}$, applying Lemma 3.20 and Theorem 3.22 now proves all the claims in their respective order. □

Example 4.4. Calculation of the slice matrix of the RIF in Example 3.8,

$$M_\phi(z_1, z_2) = \begin{pmatrix} -2 - z_1 - z_1^2 + z_2 - z_1 z_2 + 4z_1^2 z_2 & 1 - z_1 - z_1 z_2 + z_1^2 z_2 \\ 1 - z_1 - z_1 z_2 + z_1^2 z_2 & 4 - z_1 + z_1^2 - z_2 - z_1 z_2 - 2z_1^2 z_2 \end{pmatrix}$$

gives the determinant

$$\begin{aligned} \det M_\phi(z_1, z_2) &= -9 - 6z_1^2 - z_1^4 + 6z_2 + 20z_1^2 z_2 + 6z_1^4 z_2 - z_2^2 - 6z_1^2 z_2^2 - 9z_1^4 z_2^2 \\ &= -(-3 - z_1^2 + z_2 + 3z_1^2 z_2)^2. \end{aligned}$$

Since

$$\det M_\phi(z_1, z_2) = 0 \iff z_2(1 + 3z_1^2) = (3 + z_1^2) \iff z_2 = \frac{3 + z_1^2}{1 + 3z_1^2},$$

the determinant vanishes precisely at the restriction of zero set of $\mathcal{Z}_p \cap \mathbb{T}^3$ to the first two variables, that is, precisely when σ vanishes.

This claim is even stronger than it might seem. The matrix M_ϕ helps determine the local critical integrability around every zero point not contained in a vertical line singularity, but by Theorem 3.23, that might be all we need. If ρ_ϕ has a pure dimensional zero set of dimension $n-2$, the integrability around such points cannot be worse than the integrability of the component of the zero set passing through it, therefore M_ϕ gives us information about the global integrability index $\mathfrak{p}^*$ as well.

4.2 Compositions ϕ^N and M_ϕ^N

In this section, we will study iterations of ϕ using the slice matrix M_ϕ. We will observe how the zero sets of p on $\mathbb{T}^n$ and ρ_ϕ on $\mathbb{T}^{n-1}$ evolve and how the integrability behaves under composition. First, let's define what we mean by composition.

Definition 4.5. For $(m, 1)$-degree RIF ϕ, we define $\phi^N : \mathbb{C}^n \to \mathbb{C}$ inductively as

$$\phi^N := \phi(z_1, \dots, z_{n-1}, \phi^{N-1}(z_1, \dots, z_n)), \quad \text{for } N \geq 2.$$

From the discussion in [23, Section 3], we know that ϕ^N are RIFs as well, and that in the case of finite-singularity RIFs that don't experience a polydegree drop, they preserve the zero set of p. By polydegree drop we mean that the numerator and denominator of ϕ^N share a common factor, which then cancels. The following lemma shows that the same remains true even when ϕ has higher-dimensional singularities.

Lemma 4.6. *Let ϕ be a $(m_1, ..., m_{n-1}, 1)$-degree irreducible RIF and $\phi^N := \frac{\tilde{p}^N}{p^N}$ defined as before, with no polydegree drop. Then ϕ^N has the same singular set on $\mathbb{T}^n$ as ϕ, preserving all the point, curve and higher-dimensional singularities, including the vertical lines, if $\mathcal{Z}_p \cap \mathbb{T}^n$ contains any.*

Proof. Taking the square of the slice matrix M_ϕ gives

$$M_\phi^2 = \begin{pmatrix} \widetilde{p}_1 & \widetilde{p}_2 \\ p_2 & p_1 \end{pmatrix} \begin{pmatrix} \widetilde{p}_1 & \widetilde{p}_2 \\ p_2 & p_1 \end{pmatrix} = \begin{pmatrix} \widetilde{p}_1\widetilde{p}_1 + \widetilde{p}_2 p_2 & \widetilde{p}_1\widetilde{p}_2 + \widetilde{p}_2 p_1 \\ \widetilde{p}_1 p_2 + p_1 p_2 & p_2\widetilde{p}_2 + p_1 p_1 \end{pmatrix},$$

which is precisely the slice matrix of ϕ^2. This can be checked directly by substituting

$$\begin{aligned} \phi^2(z_1, ..., z_n) := \phi(z_1, ..., z_{n-1}, \phi(z_1, ..., z_n)) &= \frac{\widetilde{p}_1 \cdot \frac{z_n \cdot \widetilde{p}_1 + \widetilde{p}_2}{z_n \cdot p_2 + p_1} + \widetilde{p}_2}{p_2 \cdot \frac{z_n \cdot \widetilde{p}_1 + \widetilde{p}_2}{z_n \cdot p_2 + p_1} + p_1} = \frac{\widetilde{p}_1(z_n \cdot \widetilde{p}_1 + \widetilde{p}_2) + \widetilde{p}_2(z_n \cdot p_2 + p_1)}{p_2(z_n \cdot \widetilde{p}_1 + \widetilde{p}_2) + p_1(z_n \cdot p_2 + p_1)} \\ &= \frac{z_n(\widetilde{p}_1\widetilde{p}_1 + \widetilde{p}_2 p_2) + \widetilde{p}_1\widetilde{p}_2 + \widetilde{p}_2 p_1}{z_n(p_2\widetilde{p}_1 + p_1 p_2) + p_2\widetilde{p}_2 + p_1 p_1}. \end{aligned}$$

Then same will follow by induction for all iterations $N \in \mathbb{N}$. Thus,

$$P_{\phi^N} = P_\phi \cdots P_\phi,$$

so the zero sets of P_{ϕ^N} and P_ϕ on $\mathbb{T}^{n-1}$ coincide. If ϕ is a finite-singularity RIF, the claim follows by [23]. On the other hand, if ϕ has higher-dimensional singularities, Lemma 3.20 implies that these are preserved under N iterations as well. This also includes vertical line singularities, which can be seen directly from the form of M_{ϕ^N}; writing $p^N = z_n \cdot p_2^N + p_1^N$, both p_1^N and p_1^N are combinations of p_1 and p_2, and hence vanish at such points. □

This result makes the analysis of integrability conditions of $\frac{\partial \phi^N}{\partial z_n}$ quite simple to do. We recall from the discussion after Theorem 3.9 that the partial derivative $\frac{\partial \phi}{\partial z_n} \in L^1_{loc}(\mathbb{T}^n)$ for all singularities $\xi \in \mathbb{T}^n$ of ϕ. Therefore, if $\mathfrak{p}^*$ is its local z_n-derivative integrability index at a singularity $\xi \in \mathbb{T}^n$, we can write

$$\mathfrak{p}^* := 1 + \mathfrak{q}^*,$$

for some $\mathfrak{q}^* \geq 0$. Now we get the following theorem.

Theorem 4.7. *Let ϕ be a $(m_1, ..., m_{n-1}, 1)$-degree irreducible RIF and $\phi^N = \frac{\tilde{p}^N}{p^N}$ with no polydegree drop. Let $\zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{T}^n$ be a singularity of ϕ not contained in a vertical line. If $\mathfrak{p}^* := 1 + \mathfrak{q}^*$ is the local z_n-derivative integrability index at ζ, then ϕ^N has the local z_n-derivative integrability index equal to $\mathfrak{p}^*_{(N)} = 1 + \frac{\mathfrak{q}^*}{N}$ at ζ.*

Proof. Proposition 4.3 gives

$$\frac{\partial \phi}{\partial z_n} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^n) \iff \int_{B_\varepsilon(\zeta_1, ..., \zeta_{n-1})} |P_\phi(z_1, ..., z_{n-1})|^{1-\mathfrak{p}} dm(z_1, ..., z_{n-1}) < \infty$$

for $\varepsilon > 0$ small enough. By Lemma 4.6, the zero sets of ϕ and ϕ^N are the same on $\mathbb{T}^n$, which together with

$$P_{\phi^N} = P_\phi \cdots P_\phi$$

gives

$$\frac{\partial \phi^N}{\partial z_n} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^n) \iff \int_{B_\varepsilon(\zeta_1, ..., \zeta_{n-1})} |P_\phi(z_1, ..., z_{n-1})|^{N(1-\mathfrak{p})} dm(z_1, ..., z_{n-1}) < \infty$$

for small ε. The expression $\mathfrak{p}^* := 1 + \mathfrak{q}^*$ for ϕ comes from $\frac{1-\mathfrak{p}}{\mathfrak{q}} \geq -1$, which in turn for ϕ^N gives $\frac{N(1-\mathfrak{p})}{\mathfrak{q}} \geq -1$, thus after simplifying, $\mathfrak{p} \leq 1 + \frac{\mathfrak{q}}{N}$, so $\mathfrak{p}^*_{(N)} = 1 + \frac{q^*}{N}$ at ζ. $\square$

Remark. In Theorem 3.16 we have seen how the local $L^{\mathfrak{p}}_{loc}(\mathbb{T}^n)$-integrability of $\frac{\partial \phi}{\partial z_n}$ at a point contained in a vertical line doesn't affect the global $L^{\mathfrak{p}}(\mathbb{T}^n)$-integrability of $\frac{\partial \phi}{\partial z_n}$. In a similar manner, the global integrability of $\frac{\partial \phi^N}{\partial z_n}$ at those points will also not be affected, shown by Theorem 4.7.

Example 4.8. We will apply the previous claims about compositions of ϕ and their integrability on Example 3.8. The rational inner function ϕ^2 has the numerator

$$\begin{aligned}\tilde{p}^2(z_1, z_2) = 17 &- 10z_1 + 10z_1^2 - 2z_1^3 + z_1^4 - 8z_2 - 8z_1z_2 - 12z_1^2z_2 - 4z_1^4z_2 + \\ &+ z_2^2 + 2z_1z_2^2 + 6z_1^2z_2^2 + 2z_1^3z_2^2 + 5z_1^4z_2^2 + \\ &+ z_3(2 - 4z_1 + 2z_1^2 - 4z_1z_2 - 8z_1^2z_2 - 4z_1^3z_2 + 2z_1^2z_2^2 - 4z_1^3z_2^2 + 2z_1^4z_2^2),\end{aligned}$$

which by further calculation gives

$$\sigma^2(z_1, z_2) = \frac{(-3 - z_1^2 + z_2 + 3z_1^2z_2)^4}{z_1^4z_2^2} = P_{\phi^2}(z_1, z_2),$$

so $\mathcal{Z}_{\sigma^2} \cap \mathbb{T}^2 = \mathcal{Z}_\sigma \cap \mathbb{T}^2$ and $\mathcal{Z}_{|\widetilde{p}_1^2|^2} \cap \mathcal{Z}_{\sigma^2} \cap \mathbb{T}^2 = \{(1, 1)\}$. Either by composing ϕ^2 further with ϕ $N - 2$ times, or by simply multiplying P_{ϕ^2} with P_ϕ $N - 2$ times, we get

$$\sigma^N(z_1, z_2) = P_{\phi^N}(z_1, z_2) = \frac{(-3 - z_1^2 + z_2 + 3z_1^2z_2)^{2N}}{z_1^{2N}z_2^N}$$

for all $N \in \mathbb{N}$. Therefore, zero sets of ϕ on $\mathbb{T}^3$ and ρ_ϕ, σ and $|\widetilde{p}_1|^2$ on $\mathbb{T}^2$ are all preserved under iterations, as Lemma 4.6 claims.

To observe how the zero set of ϕ approaches the boundary $\mathbb{T}^3$ with multiple iterations, we observe the integrability of $\frac{\partial \phi^N}{\partial z_3}$ for different N. In Example 3.13 we have shown that the global critical z_3-integrability index of ϕ is $\mathfrak{p}^* = \frac{3}{2} = 1 + \frac{1}{2}$. By Theorem 4.7, it follows that the global $\mathfrak{p}^*_{(N)}$ of ϕ^N equals to $1 + \frac{1}{2N}$, for $N \geq 1$. For example, around $(\theta_1, \theta_2) = (\pi, 0)$, we have the series expansion (7), and further,

$$\sigma^2(\pi - \theta_1, \theta_2) = 256(\theta_1 + \theta_2)^4 + \mathcal{O}(\|\theta\|^6)$$

and

$$\sigma^3(\pi - \theta_1, \theta_2) = 256(\theta_1 + \theta_2)^6 + \mathcal{O}(\|\theta\|^8),$$

thus $\frac{\partial \phi^2}{\partial z_3}, \frac{\partial \phi^3}{\partial z_3} \in L^{\mathfrak{p}}_{loc}(\mathbb{T}^3)$ around $(\pi, 0, \tau)$, $\tau \in \mathbb{T}$ if and only if $\mathfrak{p} < \frac{5}{4}$ and $\mathfrak{p} < \frac{7}{6}$, respectively.

Question. So far, we have seen that all integrability indices $\mathfrak{p}^*$ are rational numbers in the sequence $\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots$, which comes from the fact that $\mathfrak{q}^* = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$. However, it is possible there are indices of different type, and it would be interesting to find an example of a different type of sequence, if such exists.

5 Additional interesting examples

In this section, we will construct and/or investigate some interesting examples. Where applicable, we will shortly demonstrate the usage of results that have been established in the previous sections.

5.1 Unimodular level sets

In this example we turn to a different problem, characterizing the unimodular level sets of three-variable RIFs. As in [9, Section 2.2., Section 4.2], for a $\lambda \in \mathbb{T}$, we define a *unimodular level set* of a general three-variable RIF ϕ as

$$C_\lambda := \overline{\{\zeta \in \mathbb{T}^3 : \phi(\zeta) = \lambda\}} = \{\zeta \in \mathbb{T}^3 : (\widetilde{p} - \lambda p)(\zeta) = 0\}.$$

Moreover, by defining the two-variable polynomial $q_\lambda := \lambda p_1 - \widetilde{p}_2$, we get the parametrization of $C_\lambda \backslash ((\mathcal{Z}_{q_\lambda} \cap \mathbb{T}^2) \times \mathbb{T})$ by

$$z_3 = \Psi_\lambda(z_1, z_2) = \bar{\lambda}\frac{q_\lambda(z_1, z_2)}{\tilde{q}_\lambda(z_1, z_2)}.$$

For two-variable RIFs, the associated Ψ_λ are smooth on $\mathbb{T}$ for all $\lambda \in \mathbb{T}$. For each branch $z_i = \psi_l^0(\zeta_j)$ of $\mathcal{Z}_p \cap \mathbb{T}^2$, there is a notion of a local z_i-contact order K_l^i, an even number satisfying

$$1 - |\psi_l^0(\zeta_j)| \approx |1 - \zeta_j|^{K_l^i}$$

for all $\zeta_j \in \mathbb{T}$ sufficiently close to $(z_1, z_2) \in \mathcal{Z}_p \cap \mathbb{T}^2$. Here $i, j = 1, 2$ and $i \neq j$. See [7, Lemma 2.2] for more details. Denote $K^i_{(z_1,z_2)}$ as the maximum K_l^i of all branches $\psi_l^0(\zeta_j)$ at a point $(z_1, z_2) \in \mathcal{Z}_p \cap \mathbb{T}^2$. Then, in [7, Theorem 2.6.], it was shown that the local derivative integrability $\frac{\partial \phi}{\partial z_j}$ is closely connected to the notion of the local z_j-contact order by the equivalence

$$\int_{B_\epsilon(z_1,z_2)} \left|\frac{\partial \phi}{\partial z_j}(\zeta_1, \zeta_2)\right|^{\mathfrak{p}} |d\zeta_1||d\zeta_2| < \infty \iff K^j_{(z_1,z_2)} < \frac{1}{\mathfrak{p} - 1}$$

for $\epsilon > 0$ small enough. This is quite similar to the results obtained using the order of vanishing of ρ_ϕ we were investigating in RIFs in dimensions three and higher, and since the notion of the contact order is absent in higher dimensions, recall that Theorem 3.9 made a similar analysis of the partial derivatives possible without it. Precisely, the order of vanishing of ρ_ϕ had the same role the contact order here has.

However, for two dimensional RIFs, the study of the contact order is interchangeable with the study of their unimodular level sets. Explicitly, if $z_2 = \Psi_\lambda(z_1)$ has multiple branches ψ_i^λ, all of them are analytic, for all $\lambda \in \mathbb{T}$, so taking a difference $\psi_i^{\lambda_1} - \psi_j^{\lambda_2}$ results in an analytic function as well. By [7, Theorem 3.1.], the maximal order of vanishing of $\psi_i^{\lambda_1} - \psi_j^{\lambda_2}$ (for most pairs λ_1, λ_2) is precisely equal to the z_1-contact order of ϕ at a singularity $\tau \in \mathbb{T}^2$. The same applies to the z_1-coordinate.

The question we pose now: is the order of vanishing of ρ_ϕ generally equal to the order of vanishing of $\psi_i^{\lambda_1} - \psi_j^{\lambda_2}$ for RIFs of dimension three and higher?

Example 5.1. We will observe the so called "favourite example" in three variables

$$\phi(z) = \frac{3z_1z_2z_3 - z_1z_2 - z_1z_3 - z_2z_3}{3 - z_1 - z_2 - z_3},$$

which has the zero set $\mathcal{Z}_p \cap \mathbb{T}^2 = \{(1, 1, 1)\}$. That means that the global derivative integrability is quite easy to calculate, and we proceed as before, by looking at the series expansion of the function ρ_ϕ around the point $(\theta_1, \theta_2) = (0, 0)$. From

$$\rho_\phi(z_1, z_2) = \frac{3z_1 - z_1^2 + 3z_2 - 10z_1z_2 + 3z_1^2z_2 - z_2^2 + 3z_1z_2^2}{(-3 + z_1 + z_2)(-z_1 - z_2 + 3z_1z_2)} \tag{8}$$

we get the series expansion

$$\rho_\phi(\theta_1, \theta_2) = 2(\theta_1^2 + \theta_1\theta_2 + \theta_2^2) + \mathcal{O}(\|\theta\|^3)$$

around the point $(\theta_1, \theta_2) = (0, 0)$. The level curves of this example were first studied in [9, Example 4.9], where it was shown that each unimodular level surface C_λ is described by

$$z_3 = \Psi_\lambda(z_1, z_2) = \bar{\lambda}\frac{q_\lambda(z_1, z_2)}{\tilde{q}_\lambda(z_1, z_2)} = \frac{3\lambda - \lambda z_1 - \lambda z_2 + z_1z_2}{\lambda - z_1 - z_2 + 3z_1z_2}.$$

For each $\lambda \neq -1$, surfaces C_λ are smooth, while C_{-1} has a singularity at $(1, 1)$. For each $\lambda \neq -1$, this means that Ψ_λ is analytic around the point $(1, 1)$ and we can look at the series expansions for selected Ψ_λ, $\lambda \neq -1$. For example, around $(\theta_1, \theta_2) = (0, 0)$,

$$\begin{aligned}(\Psi_1 - \Psi_i)(\theta_1, \theta_2) &= -i(\theta_1^2 + \theta_1\theta_2 + \theta_2^2) + \mathcal{O}(\|\theta\|^3),\\ (\Psi_1 - \Psi_{-i})(\theta_1, \theta_2) &= i(\theta_1^2 + \theta_1\theta_2 + \theta_2^2) + \mathcal{O}(\|\theta\|^3),\\ (\Psi_i - \Psi_{-i})(\theta_1, \theta_2) &= 2i(\theta_1^2 + \theta_1\theta_2 + \theta_2^2) + \mathcal{O}(\|\theta\|^3),\end{aligned}$$

and in general, for $\lambda \neq -1$, we have

$$(\Psi_{\lambda_1} - \Psi_{\lambda_2})(\theta_1, \theta_2) = \frac{2(\lambda_1 - \lambda_2)}{(\lambda_1 + 1)(\lambda_2 + 1)}(\theta_1^2 + \theta_1\theta_2 + \theta_2^2) + \mathcal{O}(\|\theta\|^3), \tag{9}$$

so the order of vanishing of each of these functions matches the order of vanishing of ρ_ϕ around the point $(1, 1)$. Now, substituting the equations

$$(\Psi_{\lambda_1} - \Psi_{\lambda_2})(z_1, z_2) = -\frac{(\lambda_1 - \lambda_2)(3z_1 - z_1^2 + 3z_2 - 10z_1z_2 + 3z_1^2z_2 - z_2^2 + 3z_1z_2^2)}{(\lambda_1 - z_1 - z_2 + 3z_1z_2)(\lambda_2 - z_1 - z_2 + 3z_1z_2)}$$

and (8) into

$$(\Psi_{\lambda_1} - \Psi_{\lambda_2})(z_1, z_2) = \mathfrak{C}_{\lambda_1,\lambda_2}(z_1, z_2)\,\rho_\phi(z_1, z_2) \tag{10}$$

gives the expression

$$\mathfrak{C}_{\lambda_1,\lambda_2}(z_1, z_2) = -\frac{(\lambda_1 - \lambda_2)(-3 + z_1 + z_2)(-z_1 - z_2 + 3z_1z_2)}{(\lambda_1 - z_1 - z_2 + 3z_1z_2)(\lambda_2 - z_1 - z_2 + 3z_1z_2)}.$$

This gives $\mathfrak{C}_{\lambda_1,\lambda_2} \to \infty$ on $\mathbb{T}^3$ whenever $\lambda_1 \to -1$ or $\lambda_2 \to -1$. Looking at the value of this function at the point $(1, 1)$, we get

$$\mathfrak{C}_{\lambda_1,\lambda_2}(1, 1) = \frac{\lambda_1 - \lambda_2}{(\lambda_1 + 1)(\lambda_2 + 1)},$$

which is in accordance with (9) for $\lambda \neq -1$. Therefore, we get a result analogous to the one in two variables, i.e. that the order of vanishing of ρ_ϕ is the same as the order of vanishing of $\Psi_{\lambda_1} - \Psi_{\lambda_2}$ around some point $(\zeta_1, \zeta_2) \in \mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^2$ whenever (ζ_1, ζ_2) is not a singularity of λ_1 or λ_2.

5.2 Example with two vertical line singularities

We construct an example of a type that, to the best of our knowledge, has not appeared previously in literature. In what follows we will see an RIF with two vertical line singularities and a horizontal level surface. Following the matrix construction as in [9, Example 5.1.], we set

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \quad Y_3 = I - Y_1 - Y_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}.$$

Here it was necessary for A to be self-adjoint, and Y_1 and Y_2 to be contractions. By choosing vector $v = (1,0,0,1)^T$, by [3] we obtain the following Pick function on the poly-upper half-plane Π^3:

$$\begin{aligned} f(w) = &\ \langle (A - w_1Y_1 - w_2Y_2 - w_3Y_3)^{-1}v, v\rangle \\ = &\ \frac{2 - w_1 - w_2 - 2w_2^2 + w_1w_2^2 - w_3 + w_2^2w_3}{(1 - w_1 - w_2 - w_2^2 + w_1w_2^2)(1 - w_3)}, \quad w = (w_1, w_2, w_3) \in \Pi^3. \end{aligned}$$

By conjugating with Möbius maps

$$m : \mathbb{D}^3 \to \Pi^3, \quad (z_1, z_2, z_2) \mapsto \left(i\,\frac{1 - z_1}{1 + z_1}, i\,\frac{1 - z_2}{1 + z_2}, i\,\frac{1 - z_3}{1 + z_3} \right)$$

and

$$\beta : \Pi \to \mathbb{D}, \quad z \mapsto \frac{1 + iz}{1 - iz},$$

we obtain a new RIF

$$\begin{aligned} \phi(z_1, z_2, z_3) = &\ \frac{\widetilde{p}(z_1, z_2, z_3)}{p(z_1, z_2, z_3)} = \\ = &\ \frac{-1 - 4i + (3 - 4i)z_1 + (1 - 4i)z_2^2 + (5 - 4i)z_1z_2^2 + z_3(3 - 6i + (7 + 2i)z_1 + (5 - 2i)z_2^2 + (9 + 6i)z_1z_2^2)}{9 - 6i + (5 + 2i)z_1 + (7 - 2i)z_2^2 + (3 + 6i)z_1z_2^2 + z_3(5 + 4i + (1 + 4i)z_1 + (3 + 4i)z_2^2 - (1 - 4i)z_1z_2^2)}, \end{aligned}$$

which contains two vertical line singularities, namely

$$\{(-1, i)\} \times \mathbb{T} \cup \{(-1, -i)\} \times \mathbb{T} \ \subset \mathcal{Z}_p \cap \mathbb{T}^3.$$

Specifically, the zero set of ρ_ϕ on $\mathbb{T}^2$ is of the form $(\mathcal{Z}_\sigma \cap \mathbb{T}^2) \cup \{(-1, i)\} \times \mathbb{T} \cup \{(-1, -i)\} \times \mathbb{T}$, where

$$\mathcal{Z}_\sigma \cap \mathbb{T}^2 = \left\{ \left(\frac{-2 + 3i + (i - 2)z_2^2}{2 + i + (2 + 3i)z_2^2}, z_2 \right) : z_2 \in \mathbb{T} \right\}.$$

The entirety of $\mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^2$ is composed of two smooth curves; see Figure 7.

In order to determine the global derivative integrability of this RIF, we perform an expansion of σ around two arbitrary points of each of these smooth curves, analogous to the various calculations done before. The vertical lines do not affect the global integrability by Theorem 3.16, so they need not be observed now. At the end, we get the same local critical integrability index along both components, so finally $\frac{\partial \phi}{\partial z_3} \in L^{\mathfrak{p}}(\mathbb{T}^3)$ for $\mathfrak{p} < \frac{3}{2}$.

To study the iterations ϕ^N, we proceed by raising the function P_ϕ to the N-th power, for

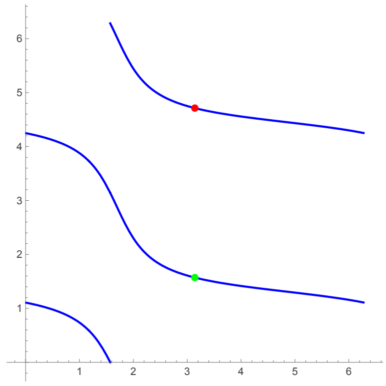


Figure 7: Zero set of ρ_ϕ on $\mathbb{T}^2$ parametrized on $(0,2\pi]^2$, with singular points $(\pi, \frac{\pi}{2})$ and $(\pi, \frac{3\pi}{2})$.

an arbitrary $N \in \mathbb{N}$. Since

$$P_\phi(z_1,z_2) = \frac{(2-3i+(2+i)z_1+(2-i)z_2^2+(2+3i)z_1z_2^2)^2}{z_1z_2^2},$$

we obtain

$$P_{\phi^N}(z_1,z_2) = \frac{(2-3i+(2+i)z_1+(2-i)z_2^2+(2+3i)z_1z_2^2)^{2N}}{(z_1z_2^2)^N}.$$

The zero set of σ, that is, of P_ϕ, stays preserved under the composition of the function with itself by Lemma 4.6. Using the notation of Theorem 4.7, we have $\mathfrak{q} = \frac{1}{2}$. Then, that same theorem will give us the global critical integrability index of ϕ^N, notably $\mathfrak{p}_N^* = 1 + \frac{1}{2N}$.

Finally, we will take a look at the level sets of this example. Following the construction in Section 5.1, we have that each unimodular level surface C_λ is described by

$$\Psi_\lambda(z_1,z_2) = \bar{\lambda}\frac{q_\lambda(z_1,z_2)}{\tilde{q}_\lambda(z_1,z_2)} = \bar{\lambda}\frac{\lambda p_1 - \tilde{p}_2}{\bar{\lambda}\tilde{p}_1 - p_2},$$

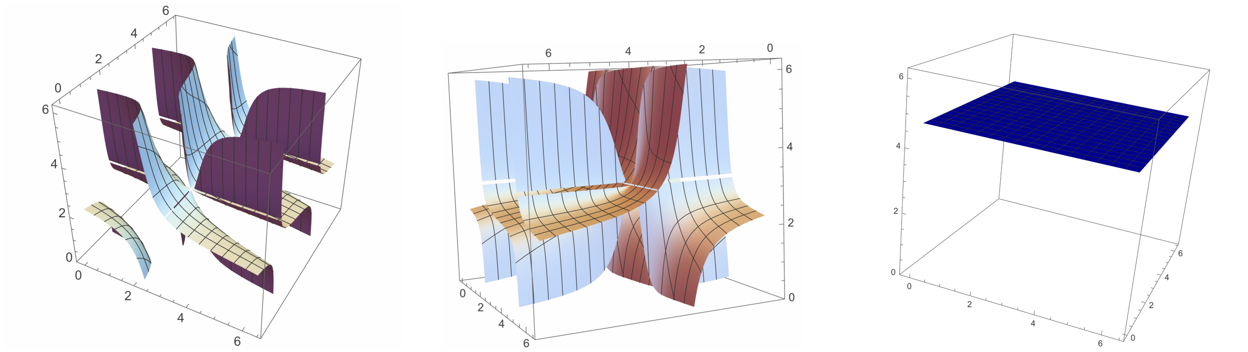
where

$$\begin{aligned} q_\lambda(z_1,z_2) &= -1-4i+(9-6i)\lambda+(3-4i)z_1+(5+2i)\lambda z_1+(1-4i)z_2^2 \\ &\quad + (7-2i)\lambda z_2^2+(5-4i)z_1z_2^2+(3+6i)\lambda z_1z_2^2 \\ \frac{\tilde{q}_\lambda(z_1,z_2)}{\lambda} &= -3+6i-(5+4i)\lambda-(7+2i)z_1-(1+4i)\lambda z_1-(5-2i)z_2^2 \\ &\quad -(9+6i)z_1z_2^2+(1-4i)\lambda z_1z_2^2. \end{aligned}$$

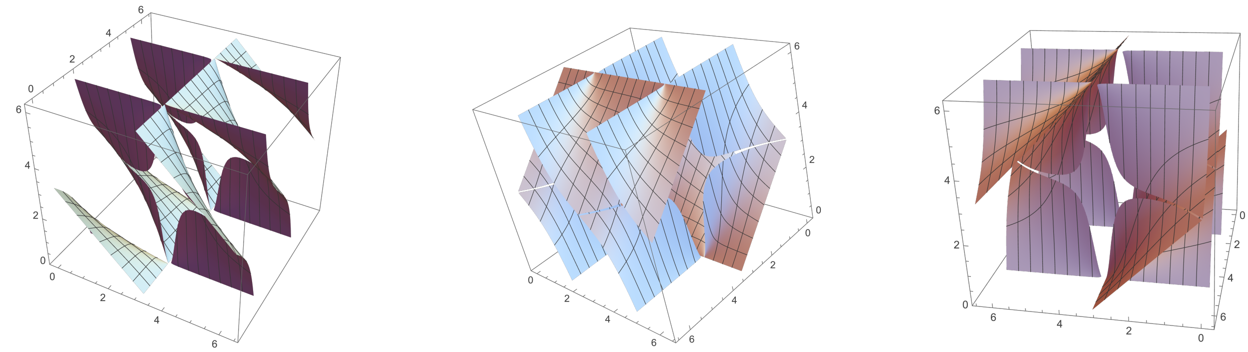
Direct calculation shows that all C_λ have singularities at points $(-1,i)$ and $(-1,-i)$, which is precisely where vertical line singularities of ϕ appear. However, for C_{-1} they are removable, since the whole level set simplifies to just

$$C_{-1} = \frac{-10+2i-(2+6i)z_1-(6+2i)z_2^2+(2-10i)z_1z_2^2}{2+10i-(6-2i)z_1-(2-6i)z_2^2-(10+2i)z_1z_2^2} = -i$$

for all $(z_1,z_2) \in \mathbb{T}^2$. Therefore, we get a smooth vertical surface level set. See Figure 8b. This is an interesting, previously unseen behaviour, which seems to be a higher-dimensional generalization of the straight line level curves that were common in two-variable RIFs. See [7, Section 7.4] for an example of a such two-variable rational inner function.



(a) C_1 from different angles (b) The horizontal surface C_{-1}



(c) C_i from different angles

Figure 8: Level surfaces

Around the point $(i, 1) \in \mathcal{Z}_{\rho_\phi} \cap \mathbb{T}^2$, the function ρ_ϕ has the expansion

$$\rho_\phi\left(\frac{\pi}{2} - \theta_1, \pi - \theta_2\right) = 2\theta_1^2 + 2\theta_1\theta_2 + \frac{1}{2}\theta_2^2 + \mathcal{O}(\|\theta\|^3).$$

Taking for example $\lambda_1 = i$ and $\lambda_2 = -i$, we see that the level surfaces C_i and C_{-i} are well-defined and smooth around the point $(i, 1)$. Therefore we examine

$$(\Psi_i - \Psi_{-i})(\theta_1, \theta_2) = -2\theta_1^2 - 2\theta_1\theta_2 - \frac{1}{2}\theta_2^2 + \mathcal{O}(\|\theta\|^3).$$

The order of vanishing is once again the same as the one of ρ_ϕ, as seen before in Example 5.1. Here $\mathfrak{C}_{i,-i} = -1$. In general, $\mathfrak{C}_{\lambda_1,\lambda_2}$ is of the form $\mathfrak{C}_{\lambda_1,\lambda_2}(z_1, z_2) = \frac{A}{B}$, where

$$\begin{aligned} A = {} & (\lambda_1 - \lambda_2)((4-5i) + (4-i)z_1 + (4-3i)z_2^2 + (4+i)z_1z_2^2)\cdot \\ & \cdot((4-i) + (4+3i)z_1 + (4+i)z_2^2 + (4+5i)z_1z_2^2), \\ B = {} & ((-6-3i) + (4-5i)\lambda_1 + (2-7i)z_1 + (4-i)\lambda_1z_1 - (2+5i)z_2^2 + (4-3i)\lambda_1z_2^2 \\ & + (6-9i)z_1z_2^2 + (4+i)\lambda_1z_1z_2^2)((-6-3i) + (4-5i)\lambda_2 + (2-7i)z_1 \\ & + (4-i)\lambda_2z_1 - (2+5i)z_2^2 + (4-3i)\lambda_2z_2^2 + (6-9i)z_1z_2^2 + (4+i)\lambda_2z_1z_2^2). \end{aligned}$$

However, it is unknown whether this connection between the orders of vanishing seen in these two examples is a general occurrence, or if they are an exception. Further research is needed to investigate more complicated cases, when multiple level sets have singularities, and at different points. See [9, Example 5.1.] for an example of such a case.

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